Зборник радова Математичког института, Нова серија књ. 1 (9), 1976.
Recueil des travaux de l'Institut Mathématique, Nouvelle série, № 1 (9), 1976.

Симпозијум КВАЗИГРУПЕ И ФУНКЦИОНАЛНЕ ЈЕДНАЧИНЕ
Symposium en QUASIGROUPES ET EQUATIONS FONCTIONNELLES
Београд—Нови Сад, 18—21. 9. 1974.

SOLUTIONS OF BOUNDED VARIATION OF A LINEAR HOMOGENEOUS FUNCTIONAL EQUATION

Marek Cezary Zdun

This paper aims at giving a survey of some results related to the functional equation

$$\varphi(f(x)) = g(x)\,\varphi(x) \tag{1}$$

where, f and g are given functions and φ is unknown function.

Let J be an interval in R. Let us put

$$\operatorname{Var} g \,|\, J = \sup \{\operatorname{Var} g \,|\, \langle c, d\rangle \mid \langle c, d\rangle \subset J\}.$$

Let us denote by $BV[J]$ the class of real-valued functions of bounded variation on the interval J, i.e.

$$BV[J] = \{g : J \to R \mid \operatorname{Var} g \,|\, J < \infty\}.$$

We assume the following general hypothesis:

(H) f is continuous and strictly increasing in $J = (a, b\rangle$ (we admit $a = -\infty$) and inequality $a < f(x) < x$ holds for $x \in J$ moreover $g \in BV[J]$, $\lim_{x \to a+} g(x| = 1$ and $\inf \langle \{g(x) \mid x \in J\} \rangle 0$.

Let us consider the following sequence of functions

$$G_n(x) = \prod_{i=0}^{n-1} g(f^i(x)) \qquad n \geqslant 1, \quad x \in J,$$

where f^i denotes the i-th iterate of the function f.

There are three posibilities regarding the behaviour of the sequence G_n.

(A) there exists an $x_0 \in J$ such that there exists a finite limit $\lim_{n\to\infty} G_n(x_0) \neq 0$.

(B) there exists an $x_0 \in J$ such that $\lim_{n\to\infty} G_n(x_0) = 0$.

(C) neither of the cases (A) and (B) occurs.

Theorem 1. *Let hypothesis* (H) *be fulfilled and suppose that case* (A) *occurs. Then equation* (1) *has at most one-parameter family of solutions in the class* $BV[J]$. *If a function* $\varphi \in BV[J]$ *satisfies equation* (1) *then* φ *is given by the formula*

$$\varphi(x) = \eta / \prod_{i=0}^{\infty} g(f^i(x)), \quad x \in J.$$

The product $\prod_{i=0}^{\infty} g(f^i(x))$ *converges uniformly on any compact* $K \subset J$.

In this case there need not exist solution $\varphi \in BV[J]$ and $\varphi \not\equiv 0$.

In the case (B) we have the following theorems.

Theorem 2. *Let hypothesis* (H) *be fulfilled. If* $\sum_{n=1}^{\infty} G_n(n) < \infty$ *for an* $x \in J$, *then equation* (1) *has a solution in the class* $BV[J]$ *depending on an arbitrary function. More exactly: for any* $y \in J$ *and for any function* $\varphi_0 \in BV[f(y), y\rangle]$ *there exists exactly one function* $\varphi \in BV[J]$ *satisfying equation* (1) *such that* $\varphi(x) = \varphi_0(x)$ *for* $x \in (f(y), y\rangle$.

Theorem 3. *If hypothesis* (H) *is fulfilled and case* (B) *occurs and* $\sum_{n=1}^{\infty} G_n(x) = \infty$ *for an* $x \in J$, *then equation* (1) *has at most one-parameter family of solutions in the class* $BV[J]$. *If a function* $\varphi \in BV[J]$ *satisfies equation* (1), *then* φ *is given by the formula*

$$\varphi(x) = \eta \lim_{n \to \infty} G_n(x) / G_n(\bar{x}), \quad \text{for an } \bar{x} \in J. \tag{2}$$

Sequence (2) *converges uniformly on any compact* $K \subset J$.

Under the above assumption there need not exists a solution $\varphi \in BV[J]$ and $\varphi \not\equiv 0$. The example is given in paper [1].

In the case (C) the only solution in the class $BV[J]$ is a function $\varphi \equiv 0$.

We give some conditions of the existance of solutions in the class $BV[J]$ non identically equal to zero.

Theorem 4. *Let hypothesis* (H) *be fulfilled and suppose that case* (A) *or* (B) *occurs and* $\sum_{n=1}^{\infty} G_n(x) \operatorname{Var} g \,|\, (a, f^n(x)\rangle < \infty$ *for an* $x \in J$. *Then equation* (1) *has a one-parameter family of solutions in the class* $BV[J]$.

Theorem 5. *Let hypothesis* (H) *be fulfilled. If case* (A) *or* (B) *occurs and* g *is monotonic in a neighbourhood of* a, *then equation* (1) *has a one-parameter family of solutions in the class* $BV[J]$.

Finally we shall consider the case where $g(x) < 0$ for $x \in J$ and $\lim_{x \to a+} g(x) = -1$.

We have the following.

Theorem 6. *If the functions* f *and* $-g$ *satisfy hypothesis* (H), *then equation* (1) *has a solution* $\varphi \in BV[J]$ *and* $\varphi \not\equiv 0$ *if and only if* $\sum_{n=1}^{\infty} |G_n(x)| < \infty$ *for an* $x \in J$. *The solution* $\varphi \in BV[J]$ *and* $\varphi \not\equiv 0$ *if it exists then* φ *depends on an arbitrary function.*

The proofs of the above theorems are presented in paper [1]. The solutions of bounded variation of a general linear equation

$$\varphi(f(x)) = g(x)\varphi(x) + h(x)$$

are considered in papers [2], [3]. In particular, the case where $\lim_{x \to a+} |g(x)| \neq 1$ is investigated in paper [3].

REFERENCES

[1] Zdun M. C., *Solutions of bounded variation of a linear homogeneous functional equation in the indeterminate case*, Aequationes Math. (to appear).

[2] Zdun M. C., *Note on solutions of bounded variation of a linear non-homogeneous functional equation*, Mathematica, Cluj (to appear).

[3] Zdun M. C., *Solutions of bounded variation of a linear functional equation and some their interpretation for recurrent sequences*, Prace Matematyczne Uniwersytetu Slaskiego (to appear).