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ON AN EXTENSION OF PEXIDER'S EQUATION

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1. Introduction

Some results of Z. Daróczy and L. Losonczi [2] on the extensions of additive functions seem to have important applications in the theory of functional equations (cp. e. g. K. Lajkó [3], L. Székelyhidi [4]).

In connection with the above-mentioned results, in this paper we shall deal with the extensions of the equation

$$f(x+y)=g(x)+h(y).$$

2. Definitions and notations

We shall use the following notations and definitions.

Let $D\subset \mathbf{R}^2$ be an arbitrary non-empty set ($\mathbf{R}$ is the set of real numbers) and

$$D_x=\{x \mid \exists\, y,\quad (x, y)\in D\},$$
$$D_y=\{y \mid \exists\, x,\quad (x, y)\in D\},$$
$$D_{x+y}=\{x+y \mid (x, y)\in D\}$$

Throughout the paper E denotes an Abelian group (written additively).

D e f i n i t i o n 1. Let $D\subset \mathbf{R}^2$ $(D\neq \varnothing)$, $f\colon D_{x+y}\to E$, $g\colon D_x\to E$, $h\colon D_y\to E$ be functions such that

$$f(x+y)=g(x)+h(y),\quad (x, y)\in D.$$

If there exists an ordered triple of functions (F, G, H) such that

(i) $\qquad F, G, H\colon \mathbf{R}\to E,$

(ii) $\qquad F(x+y)=G(x)+H(y)$ for all $(x, y)\in \mathbf{R}^2$

and

$$\text{(iii)}\qquad \begin{aligned} F(x)&=f(x) \quad \text{for all} \quad x\in D_{x+y},\\ G(x)&=g(x) \quad \text{for all} \quad x\in D_x,\\ H(x)&=h(x) \quad \text{for all} \quad x\in D_y,\end{aligned}$$

then (F, G, H) is called an *extension* of (f, g, h) from the set D.

Definition 2. Let $D \subset \mathbf{R}^2$ $(D \neq \varnothing)$, $f: D_{x+y} \to E$, $g: D_x \to E$, $h: D_y \to E$ be functions such that

$$f(x+y) = g(x) + h(y), \quad (x, y) \in D.$$

If there exists an ordered triple of functions (F^*, G^*, H^*) and a point $(u, v) \in D$ such that

$$1^\circ \qquad F^*, G^*, H^*: \mathbf{R} \to E,$$

$$2^\circ \qquad F^*(x+y) = G^*(x) + H^*(y) \qquad \text{for all } (x, y) \in \mathbf{R}^2$$

and

$$3^\circ \qquad \begin{aligned} F^*(x) - F^*(u+v) &= f(x) - f(u+v) &&\text{for all } x \in D_{x+y}, \\ G^*(x) - G^*(u) &= g(x) - g(u) &&\text{for all } x \in D_x, \\ H^*(x) - H^*(v) &= h(x) - h(v) &&\text{for all } x \in D_y, \end{aligned}$$

then (F^*, G^*, H^*) is called a *quasi-extension* of (f, g, h) from the set D.

3. Results

Let $D = K_r = \{(x, y) \mid x^2 + y^2 < r^2\}$ $(r > 0$ is a constant) be an open disk. Then we have

Theorem 1. (*cp.* [2], *Satz* 2.) *Let* $f: (K_r)_{x+y} \to E$, $g: (K_r)_x \to E$ *and* $h: (K_r)_y \to E$ *be functions such that*

$$f(x+y) = g(x) + h(y), \quad (x, y) \in K_r.$$

Then (f, g, h) *has one and only one extension* (F, G, H) *from the set* K_r.

Proof. Clearly $(K_r)_x = (K_r)_y = (-r, r)$ and $(K_r)_{x+y} = (-r\sqrt{2}, r\sqrt{2})$. Every $x \in \mathbf{R}$ can be written in one and only one way in the form

$$x = n\frac{r}{2} + t,$$

where $n \in \mathbf{Z} = \{0, \pm 1, \pm 2, \dots\}$ and $t \in \left[0, \frac{r}{2}\right)$. Let us define the functions $F: \mathbf{R} \to E$, $G: \mathbf{R} \to E$, $H: \mathbf{R} \to E$ as follows:

$$F(x) = n f\left(\frac{r}{2}\right) + f(t) - n(a+b),$$

$$G(x) = n g\left(\frac{r}{2}\right) + g(t) - na,$$

$$H(x) = n h\left(\frac{r}{2}\right) + h(t) - nb,$$

where $a = g(0)$ and $b = h(0)$. We show that (F, G, H) is the unique extension of (f, g, h) from K_r.

A) First we prove that (ii) holds. If $(x, y)\in \mathbf{R}^2$, then

$$\begin{aligned} x&=n\frac{r}{2}+t_1,\\ y&=m\frac{r}{2}+t_2, \end{aligned} \qquad \left(n, m\in \mathbf{Z},\ t_1, t_2\in\left[0, \frac{r}{2}\right)\right).$$

1) If $t_1+t_2\in\left[0, \frac{r}{2}\right)$, then we have

$$F(x+y)=F\left((n+m)\frac{r}{2}+t_1+t_2\right)=$$

$$=(n+m)f\left(\frac{r}{2}\right)+f(t_1+t_2)-(n+m)(a+b)=$$

$$=n\left(g\left(\frac{r}{2}\right)+b\right)+m\left(a+h\left(\frac{r}{2}\right)\right)+g(t_1)+h(t_2)-(m+n)(a+b)=$$

$$=ng\left(\frac{r}{2}\right)+g(t_1)-na+mh\left(\frac{r}{2}\right)+h(t_2)-mb=G(x)+H(y).$$

2) If $t_1+t_2\in\left[\frac{r}{2}, r\right)$, then we can write $t_1+t_2=\frac{r}{2}+t$ with $t\in\left[0, \frac{r}{2}\right)$ and

$$F(x+y)=F\left((n+m+1)\frac{r}{2}+t\right)=(n+m+1)f\left(\frac{r}{2}\right)+f(t)-(n+m+1)\ (a+b)=$$

$$=n\left(g\left(\frac{r}{2}\right)+b\right)+m\left(a+h\left(\frac{r}{2}\right)\right)+g\left(\frac{r}{2}\right)+b+a+h(t)-$$

$$-(n+m+1)(a+b)=ng\left(\frac{r}{2}\right)-na+mh\left(\frac{r}{2}\right)-mb+f\left(\frac{r}{2}+t\right)=$$

$$=ng\left(\frac{r}{2}\right)-na+mh\left(\frac{r}{2}\right)-mb+f(t_1+t_2)=$$

$$=ng\left(\frac{r}{2}\right)+g(t_1)-na+mh\left(\frac{r}{2}\right)+h(t_2)-mb=G(x)+H(y).$$

Thus $F(x+y)=G(x)+H(y)$ for all $(x, y)\in\mathbf{R}^2$.

B) Now we show that (iii) also holds.

1) If $x\in\left[0, \frac{r}{2}\right)$, then $x=0\cdot\frac{r}{2}+t\ \left(t\in\left[0, \frac{r}{2}\right)\right)$ and

$$F(x)=0\cdot f\left(\frac{r}{2}\right)+f(t)-0\cdot(a+b)=f(t)=f(x) \text{ and}$$

similarly $G(x)=g(x)$, $H(x)=h(x)$.

2) If $x\in\left[\frac{r}{2}, r\right)$, then $x=\frac{r}{2}+t$ and thus

$$F(x)=f\left(\frac{r}{2}\right)+f(t)-a-b=g\left(\frac{r}{2}\right)+b+a+h(t)-a-b=$$

$$=g\left(\frac{r}{2}\right)+h(t)=f\left(\frac{r}{2}+t\right)=f(x),$$

$$G(x)=g\left(\frac{r}{2}\right)+g(t)-a=g\left(\frac{r}{2}\right)+h(t)-(a+h(t))+g(t)=$$

$$=f\left(\frac{r}{2}+t\right)-f(t)+g(t)=f(x)-f(t)+g(t)=$$

$$=g(x)+b-g(t)-b+g(t)=g(x) \text{ and similarly } H(x)=h(x).$$

3) If $x\in[r, r\sqrt{2})$, then $\frac{x}{2}\in[0, r)$ and so by (ii), B./1. and B./2. we have

$$F(x)=F\left(\frac{x}{2}+\frac{x}{2}\right)=G\left(\frac{x}{2}\right)+H\left(\frac{x}{2}\right)=g\left(\frac{x}{2}\right)+h\left(\frac{x}{2}\right)=f(x).$$

4) It is easy to see that

$$\begin{aligned} f(-x)&=-f(x)+2(a+b), && x\in D_{x+y},\\ g(-x)&=-g(x)+2a, && x\in D_x,\\ h(-x)&=-h(x)+2b, && x\in D_y, \end{aligned}$$

and similarly for functions F, G, H for all $x\in\mathbf{R}$. On the basis of the above

$$F(x)=f(x) \quad \text{for all } x\in(-r\sqrt{2}, 0),$$

$$\left.\begin{aligned} G(x)&=g(x)\\ H(x)&=h(x)\end{aligned}\right\} \text{ for all } x\in(-r, 0)$$

and thus (iii) is proved.

C) Finally we show that (F, G, H) is the unique extension of (f, g, h) from K_r.

Namely if (F_1, G_1, H_1) is also an extension of (f, g, h) from K_r, then by (iii)

$$\begin{aligned} F(0)&=F_1(0)=f(0)=a+b,\\ G(0)&=G_1(0)=g(0)=a, \qquad (1)\\ H(0)&=H_1(0)=h(0)=b. \end{aligned}$$

Let t be an arbitrary real number. There exist $x\in\left[0, \frac{r}{2}\right)$ and $n\in\mathbf{Z}$ such that $t=nx$, furthermore one easily proves that for all $x\in\mathbf{R}$ and for all $n\in\mathbf{Z}$

$$\begin{aligned} F(nx)&=nF(x)-(n-1)(a+b),\\ G(nx)&=nG(x)-(n-1)a, \qquad (2)\\ H(nx)&=nH(x)-(n-1)b,\end{aligned}$$

and similarly for functions F_1, G_1, H_1.
By virtue of (1), (2) and (iii) we have

$$\begin{aligned} F(t)=F(nx)&=nF(x)-(n-1)(a+b)=nf(x)-(n-1)(a+b)=\\ &=nF_1(x)-(n-1)(a+b)=F_1(nx)=F_1(t)\end{aligned}$$

and $G(t)=G_1(t)$, $H(t)=H_1(t)$ for all $t\in\mathbf{R}$, q. e. d.

Before formulating Theorem 2. we note the following:

Let $D\subset\mathbf{R}^2$ $(D\neq\varnothing)$, $f\colon D_{x+y}\to E$, $g\colon D_x\to E$, $h\colon D_y\to E$ be functions such that

$$f(x+y)=g(x)+h(y), \quad (x, y)\in D.$$

If (F^*, G^*, H^*) is a quasi-extension of (f, g, h) from the set D and

$$F_1^*(x)\equiv F^*(x)+c_1, \quad G_1^*(x)\equiv G^*(x)+c_2, \quad H_1^*(x)\equiv H^*(x)+c_3 \ (x\in\mathbf{R}),$$

where $c_1, c_2, c_3\in E$ and $c_1-c_2-c_3=0$, then (F_1^*, G_1^*, H_1^*) is also a quasi-extension of (f, g, h) from D.

In the sequel the quasi-extensions of the two above types of (f, g, h) will be regarded as equivalent.

Define the set $K_r(u, v)\subset\mathbf{R}^2$ as follows:

$K_r(u, v)=\{(x, y)\mid (x-u)^2+(y-v)^2<r^2\}$ ($r>0$ is a constant and $(u, v)\in\mathbf{R}^2$). Then we have

Theorem 2. (cp. [2], Satz 3) *Let $f\colon(K_r(u, v))_{x+y}\to E$, $g\colon(K_r(u, v))_x\to E$, $h\colon(K_r(u, v))_y\to E$ be functions such that*

$$f(x+y)=g(x)+h(y), \ (x, y\in K_r(u, v)).$$

Then (f, g, h) has a quasi-extension (F^*, G^*, H^*) from the set $K_r(u, v)$ which is unique up to equivalence.

Proof. Put $x=X+u$ and $y=Y+v$, where $(X, Y)\in K_r$. Then $(x, y)\in K_r(u, v)$ and

$$f(X+Y+u+v)=g(X+u)+h(Y+v), \ (X, Y)\in K_r. \qquad (3)$$

Setting $Y=0$ and $X=0$ in (3) we obtain

$$f(X+u+v)=g(X+u)+h(v), \quad X\in(K_r)_x \qquad (4)$$

and

(5) $$f(Y+u+v)=g(u)+h(Y+v), \quad Y\in(K_r)_y$$

respectively.

Define functions f^*, g^*, h^* by

$$f^*(X)=f(X+u+v), \qquad X\in(K_r)_{x+y},$$

$$g^*(X)=f(X+u+v)-g(u), \quad X\in(K_r)_x \text{ and}$$

$$h^*(X)=f(X+u+v)-h(v), \quad X\in(K_r)_y.$$

By virtue of equations (3), (4) and (5) we have

$$f^*(X+Y)=g^*(X)+h^*(Y) \text{ for all } (X, Y)\in K_r.$$

By virtue of Theorem 1. (f^*, g^*, h^*) has one and only one extension (F^*, G^*, H^*) from the set K_r.

Obviously,

$$F^*(x+y)=G^*(x)+H^*(y) \text{ for all } (x, y)\in\mathbf{R}^2.$$

Now we prove that 3° also holds. First choose $x\in(K_r(u, v))_x$. Then

$$G^*(x)-G^*(u)=G^*(X+u)-G^*(u)=F^*(X+u+v)-H^*(v)-G^*(u)=$$

$$=G^*(X)+H^*(u+v)-H^*(v)-G^*(u)=G^*(X)-G^*(0)=$$

$$=g^*(X)-g^*(0)=f(X+u+v)-g(u)-f(u+v)+g(u)=$$

$$=f(X+u+v)-g(u)-h(v)=g(X+u)-g(u)=g(x)-g(u).$$

In a similar manner we can prove that

$$H^*(x)-H^*(v)=h(x)-h(v) \text{ for all } x\in(K_r(u, v))_y.$$

Finally if $t\in(K_r(u. v))_{x+y}$, then we can write $t=x+y$, where $x\in(K_r(u, v))_x$ and $y\in(K_r(u, v))_y$. Thus

$$F^*(t)-F^*(u+v)=F^*(x+y)-F^*(u+v)=$$

$$=G^*(x)-G^*(u)+H^*(y)-H^*(v)=g(x)-g(u)+$$

$$+h(y)-h(v)=f(x+y)-f(u+v)=f(t)-f(u+v).$$

By a simple calculation it can be shown that (F^*, G^*, H) is the unique quasi-extension of (f, g, h) from $K_r(u, v)$, apart from equivalence, q. e. d.

The following lemma has fundamental importance for the proof of the main result of the present paper:

Lemma. (cp. [2], Hilfssatz) *Let $D\subset\mathbf{R}^2$ be a set, $D=D^1\cup D^2$, where D^1, D^2 are open sets and $D^1\cap D^2\neq\emptyset$. Furthermore let $f\colon D_{x+y}\to E$, $g\colon D_x\to E$, $h\colon D_y\to E$ be functions such that*

$$f(x+y)=g(x)+h(y), \quad (x, y)\in D.$$

Assume that (f, g, h) has a quasi-extension (F_i, G_i, H_i) unique up to equivalence from the set D^i $(i=1, 2)$. Then

$$F_1(x)\equiv F_2(x)+c_1, \quad G_1(x)\equiv G_2(x)+c_2, \quad H_1(x)\equiv H_2(x)+c_3 \quad (x\in\mathbf{R})$$

and $c_1 - c_2 - c_3 = 0$ $(c_1, c_2, c_3 \in E)$ *and with the notations* $F = F_1$, $G = G_1$, $H = H_1$ (F, G, H) *is a quasi-extension of* (f, g, h) *from the set* D, *which is unique up to equivalence.*

P r o o f. First we note that the point $(u, v) \in D$ in Definition 2. can be replaced by an arbitrary point $(c, d) \in D$.

It is known (see e. g., [1]) that there exist additive functions $\varphi_1 : \mathbf{R} \to E$ and $\varphi_2 : \mathbf{R} \to E$ such that

$$(6) \qquad F_i(x) = \varphi_i(x) + a_i + b_i, \quad G_i(x) = \varphi_i(x) + a_i, \quad H_i(x) = \varphi_i(x) + b_i \ (x \in \mathbf{R})$$
$$(i = 1, 2), \text{ where } a_i = G_i(0), \ b_i = H_i(0).$$

Let $(c, d) \in D^1 \cap D^2$ be an arbitrary point. Since D^1 and D^2 are open, $D_x^1 \cap D_x^2$ contains an open interval I_x and by our conditions we obtain

$$G_1(x) - G_1(c) = g(x) - g(c) \text{ and } G_2(x) - G_2(c) = g(x) - g(c), \ x \in I_x.$$

From this we have

$$G_1(x) - G_1(c) = G_2(x) - G_2(c) \text{ for all } x \in I_x \text{ and by (6)}$$
$$\varphi_1(x) + a_1 - \varphi_1(c) - a_1 = \varphi_2(x) + a_2 - \varphi_2(c) - a_2, \ x \in I_x,$$

i. e.

$$\varphi_1(x - c) = \varphi_2(x - c) \text{ for all } x \in I_x.$$

Thus $\varphi_1(x) \equiv \varphi_2(x)$ $(x \in \mathbf{R})$ and with the notation $\varphi(x) = \varphi_1(x)$ we obtain

$$G_1(x) = \varphi(x) + a_1 \text{ and } G_2(x) = \varphi(x) + a_2 \text{ for all } x \in \mathbf{R},$$

i. e.

$$G_1(x) = G_2(x) + a_1 - a_2 \text{ for all } x \in \mathbf{R}.$$

Similarly we can prove that

$$H_1(x) = H_2(x) + b_1 - b_2 \text{ and}$$
$$F_1(x) = F_2(x) + a_1 + b_1 - a_2 - b_2 \text{ for all } x \in \mathbf{R}.$$

With the notations $c_1 = a_1 + b_1 - a_2 - b_2$, $c_2 = a_1 - a_2$, $c_3 = b_1 - b_2$ one indeed has $c_1 - c_2 - c_3 = 0$.

By a simple calculation we obtain that $(F = F_1, G = G_1, H = H_1)$ is a quasi-extension of (f, g, h) from D unique up to equivalence.

By the lemma and by theorem 2. one can prove the following

T h e o r e m 3. (cp. [2], Satz 4.) *Let* $D \subset \mathbf{R}^2$ $(D \neq \varnothing)$ *be an arbitrary open connected set and* $f : D_{x+y} \to E$, $g : D_x \to E$, $h : D_y \to E$ *be functions such that*

$$f(x + y) = g(x) + h(y), \quad (x, y) \in D.$$

Then (f, g, h) *has a quasi-extension* (F, G, H) *from the set* D, *which is unique up to equivalence.*

P r o o f. Since D is open and connected, there exist open disks $K^1, K^2, \ldots, K^n, \ldots$ such that $D = \bigcup_{i=1}^{\infty} K^i$ and $(K^1 \cup K^2 \cup \ldots \cup K^n) \cap K^{n+1} \neq \varnothing$ $(n = 1, 2, \ldots)$.

By virtue of Theorem 2. (f, g, h) has a quasi-extension (F_n, G_n, H_n) from the set K^n $(n=1, 2, \ldots)$, which is unique up to equivalence. From this with the aid of the Lemma, the statement of Theorem 3. already follows.

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