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ON MODULAR LAW FOR TERNARY *GD*-GROUPOIDS

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1. Let X_1, X_2, X_3, X_4 be four nonempty sets, and

$$A: X_1 \times X_2 \times X_3 \to X_4$$

then the ordered fivefold $(X_1, X_2, X_3, X_4; A)$ we call *G*-groupoid (generalized ternary groupoid).

Let us introduce following notations

$$L_1^A x = A(x, a_2, a_3), \quad L_2^A y = A(a_1, y, a_3), \quad L_3^A z = A(a_1, a_2, z)$$

where $a_i \in X_i$ $(i = 1, 2, 3)$ are some fixed elements. The mapping L_k^A we call *G*-translation in relation to the fixed elements.

If L_k^A are surjections for arbitrary fixed elements, then the ternary *G*-groupoid we call *GD*-groupoid (*G*-groupoid with division).

For *G*-groupoids and *GD*-groupoids we introduce the notion of homotopy.

Definition: For ternary *G*-groupoid $(Y_1, Y_2, Y_3, Y_4; B)$ we say that it is the homotopic image of *G*-groupoid $(X_1, X_2, X_3, X_4; A)$ if there exist fourfold $H = [\alpha, \beta, \gamma]$ surjection

$$\alpha: X_1 \to Y_1 \quad \beta: X_2 \to Y_2 \quad \gamma: X_3 \to Y_3 \quad \delta: X_4 \to Y_4$$

such that is fulfilled

$$\delta A(x_1, x_2, x_3) = B(\alpha x_1, \beta x_2, \gamma x_3)$$

for arbitrary $x_i \in X_i$ $(i = 1, 2, 3)$.

If $\alpha, \beta, \gamma, \delta$ are bijections, then the homotopy H we call an isotopy.

2. This paper is the enlargement (supplement) of the paper [2], therefore, we shall only formulate certain assertions without detailed proofs, as the proofs are similar to those in [2].

Theorem 1. *If four GD-groupoids A, B, C, D where*

$$B: Y_1 \times Y_2 \times Y_3 \to Q_i \quad D: X_1 \times \cdots \times X_{i-1} \times Y_j \times X_{i+1} \times \cdots \times X_3 \to Q_j$$

(1) $$A: X_1 \times \cdots \times X_{i-1} \times Q_i \times X_{i+1} \times \cdots \times X_3 \to Q$$

$$C: Y_1 \times \cdots \times Y_{j-1} \times Q_j \times Y_{j-1} \times \cdots \times Y_3 \to Q$$

satisfy the equation

$$(2)\quad \begin{aligned} &A(x_1, \ldots, x_{i-1}, \ B(y_1, y_2, y_3), \ x_{i+1}, \ldots, x_3) \\ &= C(y_1, \ldots, y_{j-1}, \ D(x_1, \ldots, x_{i-1}, y_j, x_{i+1}, \ldots, x_3), \ y_{j+1}, \ldots, y_3) \end{aligned}$$

for every $x_k \in X_k$, $y_m \in Y_m$ $(k, m = 1, 2, 3)$ *and if*

$$L_i^A : Q_i \to Q, \quad L_j^C : Q_j \to Q$$

are bijections (for arbitrary elements) then there exists the group $(Q, \circ)$ *that*

$$(3)\quad \begin{aligned} &A(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_3) = L_i^A z \circ K(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_3) \\ &B(y_1, y_2, y_3) = (L_i^A)^{-1} (P(y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_3) \circ L_j^C L_i^D y_j) \\ &C(y_1, \ldots, y_{j-1}, z, y_{j+1}, \ldots, y_3) = P(y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_3) \circ L_j^C z \\ &D(x_1, \ldots, x_{i-1}, y_j, x_{i+1}, \ldots, x_3) \\ &= (L_j^C)^{-1} (L_j^C L_i^D y_j \circ K(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_3)) \end{aligned}$$

where $L_i^D : Y_j \to Q_j$ *and* K *and* P *are arbitrary binary GD-groupoids (Here, by definition for* $i, j \in \{1, 2, 3\}$ *we omit* X_0, Y_0, X_4, Y_4 *from* (1), *as well as* x_0, y_0, x_4, y_4 *from* (2).

By introducing the relation of equivalence in the set of surjection [1] [2] then is valid

Theorem 2. *If four ternary GD-groupoids A, B, C, D satisfy the conditions of Theorem 1, then the general solution of equation* (2)

$$\begin{aligned} &A(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_3) = \alpha z \circ K(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_3) \\ &B(y_1, y_2, y_3) = \alpha^{-1} (P(y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_3) \circ \beta y_j) \\ &C(y_1, \ldots, y_{j-1}, z, y_{j+1}, \ldots, y_3) = P(y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_3) \circ \gamma z \\ &D(x_1, \ldots, x_{i-1}, y_j, x_{i+1}, \ldots, x_3) = \gamma^{-1} (\beta y_j \circ K(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_3)) \end{aligned}$$

where $(Q, \circ)$ *is the group determined up to isomorphism and the mappings* α, β, γ *are determined up to on the equivalence and K and P are arbitrary binary GD-groupoids.*

For the proof of theorem 1. respectively of the theorem 2. is used the same method as in the paper [2], at it, we obtain the *GD*-groupoids S isotopic A, D and T arbitrary for B, C and they satisfy following modular laws for different i, j

1) $(1, 1)$ — modular law

$$S(T(t, y_2, y_3), \ x_2, x_3) = T(S(t, x_2, x_3), \ y_2, y_3)$$

2) $(1, 2)$ — modular law

$$S(T(y_1, t, y_2), \ x_2, x_3) = T(y_1, S(t, x_2, x_3), \ y_2)$$

3) $(1, 3)$ — modular law

$$S(T(y_1, y_2, t), \ x_2, x_3) = T(y_1, y_2, S(t, x_2, x_3))$$

4) (2, 2) — modular law

$$S(x_1, T(y_1, t, y_3), x_3) = T(y_1, S(x_1, t, x_3), y_3)$$

5) (2, 3) — modular law

$$S(x_1, T(y_1, y_2, t), x_3) = T(y_1, y_2, S(x_1, t, x_3))$$

6) (3, 3) — modular law

$$S(x_1, x_2, T(y_1, y_2, t)) = T(y_1, y_2, S(x_1, x_2, t)).$$

All these equations are solved similarly as the equation under 4) (see [2]).

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