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A FUNCTIONAL EQUATION WITH DIFFERENCES

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1. Introduction

Let **R** denote the set of real numbers. Let us determine all functions $f:\mathbf{R}\to\mathbf{R}$ such that

$$\frac{f(tx+ty)-f(tx)}{f(tx)-f(tx-ty)}=\frac{f(x+y)-f(x)}{f(x)-f(x-y)} \tag{1}$$

for all $x, y, t\in\mathbf{R}$, $yt\neq 0$. This problem is due to P. Drăgilă [1].

Without loss of generality we can assume that $f(0)=0$ and $f(1)=1$. In the following we employ the notation

$$\mathcal{F}=\left\{f\,\middle|\,\begin{array}{l} f:\mathbf{R}\to\mathbf{R};\ f \text{ satisfies the equation (1) for all } x,\ y,\ t\in\mathbf{R};\ yt\neq 0\\ f(0)=0;\ f(1)=1\end{array}\right\}$$

The purpose of this paper is to give a necessary and sufficient condition for $f\in\mathcal{F}$. A sufficient condition can easily be given: Let $f:\mathbf{R}\to\mathbf{R}$ be invertible with the properties $f(0)=0$ and $f(1)=1$, and additive or multiplicative. Then $f\in\mathcal{F}$. We shall show that these conditions are essentially also necessary.

2. Necessary conditions

The form of the equation (1) suggests a simple necessary condition for $f\in\mathcal{F}$. It can be formulated as follows:

Lemma 1. *Let $f\in\mathcal{F}$, then f is invertible and an odd function.*

Proof. If the function f were not invertible we could find real numbers x_0, y_0, $y_0\neq 0$ so that $f(x_0)=f(x_0-y_0)$, therefore the equation (1) could not hold for $x=x_0$, $y=y_0$, $t\neq 0$. In order to prove that f is odd, we must put $x=0$, $t=\frac{1}{y}$ in the equation (1). Thus we obtain $f(-y)=f(-1)f(y)$ for all $y\in\mathbf{R}$. Hence $f^2(-1)=1$, but f is invertible, thus $f(-1)=-1$. ∎

We guess that, if $f\in\mathcal{F}$ then either f is multiplicative or f is an additive function. The following lemma allows to distinguish these two cases.

Lemma 2. *Let $f\in\mathcal{F}$, then the set*

$$L=\{\alpha\mid\alpha\in\mathbf{R};\quad f(\alpha x)=f(\alpha)f(x)\ \text{for all}\ x\in\mathbf{R}\}$$

is a field.

Proof. It is obvious that $0, 1\in L$, furthermore if $\alpha\in L$ then $(-\alpha)\in L$, for f is odd by Lemma 1. Now, let $\alpha\in L$ and $\alpha\neq 0$, then the equation $f(x)=f(\alpha)f\left(\frac{x}{\alpha}\right)$, $(x\in\mathbf{R})$ implies that

$$f\left(\frac{1}{\alpha}x\right)=\frac{1}{f(\alpha)}f(x)=f\left(\frac{1}{\alpha}\right)f(x)\qquad(x\in\mathbf{R}),$$

therefore $\frac{1}{\alpha}\in L$. Putting $x=y$, $t=\frac{1}{x}$ in the equation (1), we have $2\in L$. Thus if we write $x+2y$ instead of y in the equation (1) we get

$$\text{(2)}\qquad \begin{aligned}&[f(2)f(tx+ty)-f(tx)][f(x)+f(2)f(y)]=\\&=[f(2)f(x+y)-f(x)][f(tx)+f(2)f(ty)]\end{aligned}$$

for all $x, y, t\in\mathbf{R}$, and therefore — with the substitution $x=\alpha_1\in L$, $y=\alpha_2\in L$ — we obtain that $\alpha_1+\alpha_2\in L$, provided that $\alpha_1\neq-2\alpha_2$. If $\alpha_1=-2\alpha_2$ then $\alpha_1+\alpha_2=-\alpha_2\in L$ too holds. Finally, if $\alpha_1, \alpha_2\in L$ then $f(\alpha_1\alpha_2x)=f(\alpha_1)f(\alpha_2x)=f(\alpha_1)f(\alpha_2)f(x)=f(\alpha_1\alpha_2)f(x)$ for all $x\in\mathbf{R}$ and therefore $\alpha_1\alpha_2\in L$.∎

The following lemma yields an equation coming from the equation (2) which comfirms our above mentioned hypothesis.

Lemma 3. *Let $f\in\mathcal{F}$, then*

$$\text{(3)}\qquad [(f(2)-1)f(x+y)-f(x)-f(y)][f(tx)f(y)-f(ty)f(x)]=0$$

for all $x, y, t\in\mathbf{R}$.

Proof. By equation (2)

$$f(tx+ty)=\frac{f(tx)[f(x+y)+f(y)]+f(ty)[f(2)f(x+y)-f(x)]}{f(x)+f(2)f(y)}$$

for all $x, y, t\in\mathbf{R}$, $x\neq-2y$. Interchanging x and y, we have

$$\frac{f(tx)[f(x+y)+f(y)]+f(ty)[f(2)f(x+y)-f(x)]}{f(x)+f(2)f(y)}=$$
$$=\frac{f(ty)[f(x+y)+f(x)]+f(tx)[f(2)f(x+y)-f(y)]}{f(y)+f(2)f(x)}$$

for all $x, y, t\in\mathbf{R}$, $x\neq-2y$, $y\neq-2x$. Hence, after a simple calculation we obtain the equation (3).

3. The main result

Combining the statements of Lemma 1 and Lemma 2 we get the following theorem (in which we use the notations of the lemmas mentioned).

Theorem 1. *Let* $f\in\mathcal{F}$

a) *if* $L=\mathbf{R}$ *then* $f(xy)=f(x)f(y)$ *for all* $x, y\in\mathbf{R}$,

b) *if* $L\neq\mathbf{R}$ *then* $f(x+y)=f(x)+f(y)$ *and* $f(\alpha x)=f(\alpha)f(x)$ *for all* $x, y\in\mathbf{R}$ *and* $\alpha\in L$.

Proof. By definition of L the proof of part a) is trivial, furthermore in the case b) it will be sufficient to show that the equation

$$f(x+y)=f(x)+f(y) \tag{4}$$

holds for all $x, y\in\mathbf{R}$. Equation (3) implies that

$$[f(2)-1]f(x+y)=f(x)+f(y)$$

for all $x\in\mathbf{R}-L$, $y\in L$, $y\neq 0$. By Lemma 2 L is a field and therefore $x+y\in\mathbf{R}-L$, $(-y)\in L$. Since f is odd, it follows that

$$[f(2)-1]f(x)=f(x+y)-f(y)$$

for all $x\in\mathbf{R}-L$, $y\ni L$, $y\neq 0$. Thus we obtain

$$[f(2)-2]\,[f(x+y)+f(x)]=0$$

for all $x\in\mathbf{R}-L$, $y\in L$, $y\neq 0$. Since L is a field $x+y\neq -x$, furthermore — using that f is invertible — we get $f(2)=2$ and therefore the equation (4) holds for all $x\in\mathbf{R}-L$, $y\in L$. The equation (3) also implies that the equation (4) holds for all $x, y\in\mathbf{R}-L$. Finally, if $x, y\in L$ and $x_0\in\mathbf{R}-L$ then

$$f(x+y)=f(x+x_0+y-x_0)=f(x+x_0)+f(y-x_0)=$$
$$=f(x)+f(x_0)+f(y)-f(x_0)=f(x)+f(y).\ \blacksquare$$

We can summarize our results as follows:

Theorem 2. $f\in\mathcal{F}$ *if and only if* f *satisfies the following conditions:*

(i) $f:\mathbf{R}\to\mathbf{R}$, $f(0)=0$, $f(1)=1$

(ii) *the function* f *is invertible and odd*

(iii) *either*

$$f(xy)=f(x)f(y)$$

for all $x, y\in\mathbf{R}$ *or there exists a field* L, $L\subset\mathbf{R}$, $L\neq\mathbf{R}$ *such that*

$$f(x+y)=f(x)+f(y) \quad \text{and} \quad f(\alpha x)=f(\alpha)f(x)$$

for all $x, y\in\mathbf{R}$, $\alpha\in L$.

Finally, we make some simple remarks:

1. If L is a field of the rational numbers then the equation of additivity implies the equation

$$f(\alpha x)=f(\alpha)f(x)\ (\alpha\in L,\ x\in\mathbf{R}).$$

2. If $L\neq\mathbf{R}$ but it has a measurable set of positive measure then $f(x)=x$ $(x\in\mathbf{R})$. In this case there exists a measurable set A of positive measure such that $f(x)\geqq 0$ for all $x\in A$.

3. The continuous (or measurable) solutions of the equations

$$f(xy)=f(x)f(y) \quad \text{and} \quad f(x+y)=f(x)+f(y)$$

are well-known thus the continuous (or measurable) solutions of the equation (1) can easily be given.

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