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REMARKS ON THE ENTROPY EQUATION

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1. In the paper [4] A. Kamiński and J. Mikusiński have proved the following

Theorem. *If a function* $H(x, y, z)$ *is continuous, symmetric and positively homogeneous (of order* 1) *in the domain*

$$D=\{(x, y, z) \mid x, y, z \geq 0, \ xy+yz+zx>0\}$$

and satisfies in the interior of D the functional equation

$$H(x, y, z)=H(x+y, 0, z)+H(x, y, 0) \tag{1}$$

then

$$H(x, y, z)=c[(x+y+z) \ln (x+y+z)-x \ln x-y \ln y-z \ln z], \tag{2}$$

where c is a real constant and $0 \ln 0 \stackrel{\text{def.}}{=} 0$.

From this theorem the authors have deduced a proof for the Faddeev's theorem [2] on the Shannon entropy.

In this note I shall give some remarks to the above result. In section 2. I shall determine the *general* solution of the problem of A. Kamiński and J. Mikusiński. This is a simple remark by a theorem of B. Jessen – J. Karpf. – A. Thorup [3]. By help of the general solution of the problem, in section 3., I prove a generalization of [4]. In this proof I use a known result of N. G. de Bruijn [1] on the difference functions.

2. Let $\mathbf{R}_+$ be the set of nonnegative real numbers and let $\mathbf{R}$ be the set of real numbers. Let $\mathcal{H}$ denote the set of all functions $H^*: \mathbf{R}_+^3 \to \mathbf{R}$ with following properties:

(i) For all $x, y, z \in \mathbf{R}_+$

$$H^*(x, y, z)=H^*(x+y, 0, z)+H^*(x, y, 0);$$

(ii) For all $x, y, t \in \mathbf{R}_+$

$$H^*(tx, ty, 0)=t H^*(x, y, 0).$$

Theorem 1. *If $H^* \in \mathcal{H}$ is a symmetric function, then there exists a function $f: \mathbf{R}_+ \to \mathbf{R}$ such that*

$$f(x y) = x f(y) + y f(x) \tag{3}$$

for all $x, y \in \mathbf{R}_+$ and

$$H^*(x, y, z) = f(x+y+z) - f(x) - f(y) - f(z) \tag{4}$$

for all $x, y, z \in \mathbf{R}_+$.

Proof. Let the function $F: \mathbf{R}_+^2 \to \mathbf{R}$ be defined by

$$F(x, y) = H^*(x, y, 0) \qquad (x, y \in \mathbf{R}_+). \tag{5}$$

By the symmetry of H^*

$$F(x, y) = F(y, x) \qquad (x, y \in \mathbf{R}_+). \tag{A 1}$$

From (i), by the symmetry of H^*, we have

$$F(x+y, z) + F(x, y) = H(x, y, z) = H(y, z, x) = F(y+z, x) + F(y, z),$$

from which by $(A\,1)$ it follows

$$F(x, y, z) + F(x, y) = F(x, y+z) + F(y, z) \tag{A 2}$$

for all $x, y, z \in \mathbf{R}_+$. From (ii) it follows

$$F(tx, ty) = tF(x, y) \tag{A 3}$$

for all $x, y, t \in \mathbf{R}_+$. In the paper [3] of B. Jessen—J. Karpf—A. Thorup it can be found the following assertion: If $F: \mathbf{R}_+^2 \to \mathbf{R}$ satisfies the functional equations $(A\,1)$, $(A\,2)$ and $(A\,3)$ then there exists a function $f: \mathbf{R}_+ \to \mathbf{R}$ for which (3) is true and

$$F(x, y) = f(x+y) - f(x) - f(y) \tag{6}$$

for all $x, y \in \mathbf{R}_+$. From (6) and (i) we have

$$H^*(x, y, z) = F(x+y, z) + F(x, y) = f(x+y+z) - f(x+y) -$$

$$-f(z) + f(x+y) - f(x) - f(y) = f(x+y+z) - f(x) - f(y) - f(z)$$

for all $x, y, z \in \mathbf{R}_+$, where the function $f: \mathbf{R}_+ \to \mathbf{R}$ satisfies the equation (3). Thus the theorem is proved.

Theorem 2. *If a function $H: D \to \mathbf{R}$ is symmetric in D and satisfies the equation* (1) *in the interior of D and $H(x, y, 0)$ $(x>0, y>0)$ is positively homogeneous (of order* 1), *then there exists a function $f: \mathbf{R}_+ \to \mathbf{R}$ such that* (3) *holds for all $x, y \in \mathbf{R}_+$ and*

$$H(x, y, z) = f(x+y+z) - f(x) - f(y) - f(z) \tag{7}$$

for all $(x, y, z) \in D$.

Proof. We define the function $H^*: \mathbf{R}^3_+ \to \mathbf{R}$ by the equations $H^*(x, y, z) = H(x, y, z)$ for all $(x, y, z) \in D$, $H^*(x, 0, 0) = H^*(0, x, 0) = H^*(0, 0, x) = 0$ for $x \in \mathbf{R}_+$. It is easy to see that the function H^* is an element of $\mathcal{H}$ and it is symmetric in $\mathbf{R}^3_+$. By Theorem **1.** it follows our theorem.

3. In this section I shall give a generalization of the theorem of A. Kamiński and J. Mikusiński [4].

Theorem 3. *Let $H: D \to \mathbf{R}$ be a symmetric function fulfilling the equation* (1) *in the interior of D and let $H(x, y, 0)$ $(x>0, y>0)$ be a positively homogeneous (of order* 1) *function. If for each $y>0$ the function*

$$x \to H(x, y, 0) \qquad (x>0)$$

is continuous, then (2) *holds for all $(x, y, z) \in D$, where $c \in \mathbf{R}$ is a constant and* $0 \ln 0 \overset{\text{def.}}{=} 0$.

Proof. By Theorem **2.** there exists a function $f: \mathbf{R}_+ \to \mathbf{R}$ such that (3) holds for all $x, y \in \mathbf{R}_+$ (from which we obtain $f(0) = 0$) and

$$H(x, y, 0) = f(x+y) - f(x) - f(y)$$

for all $x>0$, $y>0$. By our assumption for each $y>0$ the difference function

$$x \to f(x+y) - f(x)$$

is *continuous*. N. G. de Bruijn [1, Theorem 1.3.] has proved the following result: If $f(x)$ is defined in an interval I, and if $f(x)$ is such that, for each y, $x \to f(x+y) - f(x)$ is continuous for all $x \in I \cap (I-y)$, then we have $f(x) = g(x) + A(x)$, where $g(x)$ is continuous in I, and $A: \mathbf{R} \to \mathbf{R}$ is an additive function, i.e. $A(x+y) = A(x) + A(y)$ for all $x, y \in \mathbf{R}$. In our case we know, that $x \to f(x+y) - f(x)$ is continuous in $I = \{x \mid x>0\}$ for each $y \in I$. If $y \notin I$, then $-y \geq 0$ and $I \cap (I-y) = (-y, \infty)$. Let $x \in (-y, \infty)$ be arbitrary and let $x_n \in (-y, \infty)$ be a sequence with $x_n \to x$. Then we have

$$f(x_n+y) - f(x_n) = -[f(x_n+y-y) - f(x_n+y)]$$

from which by $x_n + y > 0$ and $-y \geq 0$ we obtain

$$\lim_{n\to\infty} [f(x_n+y) - f(x_n)] = -[f(x+y-y) - f(x+y)] = \\ = f(x+y) - f(x),$$

i.e. $x \to f(x+y) - f(x)$ is continuous in $I \cap (I-y)$. From this result we have

(8) $$f(x) = g(x) + A(x) \qquad (x>0),$$

where $g(x)$ is a continuous function in I and $A: \mathbf{R} \to \mathbf{R}$ is an additive function. By Theorem **2.** we know, that the function $f: \mathbf{R}_+ \to \mathbf{R}$ satisfies the equation (3) for all $x, y \in \mathbf{R}_+$, from which by (8) we obtain

(9) $$g(xy) + A(xy) = x[g(y) + A(y)] + y[g(x) + A(x)]$$

for all $x, y>0$. We put $x=r$ and $y=s$ in (9), where r and s are arbitrary positive rational numbers. Then by $A(r)=rA(1)$ we obtain from (9)

$$g(rs)=rg(s)+sg(r)+rsA(1) \tag{10}$$

from which by the continuity of g it follows by taking limits $r\to x$, $s\to y$ that

$$g(xy)=xg(y)+yg(x)+xyA(1) \tag{11}$$

holds for all $x, y>0$. From (11) we have that the function

$$m(x)=\frac{g(x)}{x}+A(1) \qquad (x>0) \tag{12}$$

satisfies the functional equation

$$m(xy)=m(x)+m(y)$$

for all $x>0$, $y>0$. It is clear that m is continuous, i.e.

$$m(x)=c\ln x \qquad (x>0),$$

where $c\in\mathbf{R}$ is a constant. From (12) we obtain

$$g(x)=cx\ln x-xA(1)$$

for all $x>0$, from which it follows by (8)

$$f(x)=cx\ln x-xA(1)+A(x) \qquad (x>0) \tag{13}$$

where $A:\mathbf{R}\to\mathbf{R}$ is an additive function. By (3) $f(0)=0$, from which we obtain that (13) is true for $x=0$, if $0\ln\overset{\text{def.}}{=\!=}0$. From Theorem **2.** by (13) it follows (2) for all $(x, y, z)\in D$. Thus the theorem is proved.

It is clear, that Theorem **3.** is a generalization of the result of A Kamiński and J. Mikusiński [4].

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