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INFINITARY QUASIGROUPS

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In this paper we give a short survey of the communication on infinitary quasigroups which took place during the Symposium on quasigroups and functional epuations in Belgrade in September 1974. Since one part of the results of this work is published in [7] in extensive form, and the other part is going to be published in [8], here we shall restrict ourselves only to the main results, definitions and theorems without proofs.

The notion of infinitary operation in implicit form can be found in different fields of mathematics. There are some attempts to consider those operations but up to now the notion of infinitary quasigroup was not submitted to any investigations.

We shall use the following notations. The sequence $x_m, x_{m+1}, \ldots, x_n$ will be denoted by x_m^n or $\{x_i\}_{i=m}^n$. If $m>n$, x_m^n will be considered empty, and if $m=n$ then x_m^n is the element x_m. By $\overset{n}{x}$ we denote the sequence $x, x, \ldots, x$ where x is repeated n times. The symbol $\overset{0}{x}$ denotes the empty sequence.

The infinite sequence $x_m, x_{m+1}, \ldots, x_n, \ldots$ (of order type ω) will be denoted by x_m^∞ or $\{x_i\}_{i=m}^\infty$ (m finite natural number). The infinite sequence $x, x, \ldots, x, \ldots$ we denote by $\overset{\infty}{x}$.

If Q is a nonempty set and α any ordinal by Q^α we denote the set of all sequences of order type α, of elements form Q. A mapping $A: Q^\alpha \to Q$, $(\alpha \geqslant \omega)$ we call an infinitary operation of the type α defined on Q. The type of A we shall denote by $|A|$. The set Q together with the infinitary operation A we call an infinitary operative and denote by $Q(A)$. The infinitary operative $Q(A)$ such that the set Q contains only one element will be called trivial.

The notion of quasigroup in infinitary case can be introduced in a natural way.

Definition 1. A set Q together with an infinitary operation A of the type ω we call an infinitary quasigroup of type ω (briefly ω-quasigroup), if the equation

$$A(a_1^{i-1}, x, a_{i+1}^\infty) = b$$

has a unique solution x for all a_1^∞, $b \in Q$ and for every positive integer i.

D e f i n i t i o n 2. The element e of the infinitary operative $Q(A)$ of the type ω is called a unity if

$$A(\overset{i-1}{e}, x, \overset{\infty}{e}) = x,$$

for all $x \in Q$ and every $i = 1, 2, \ldots, n, \ldots$.

If an infinitary quasigroup $Q(A)$ of the type ω contains at least one unity then $Q(A)$ is called an infinitary loop (ω-loop).

First we have shown the existence of nontrivial infinitary quasigroups and loops. In [7] are given the constructions, based on the axiom of choice, of infinitary quasigroups of the type ω defined on the set of all real numbers, on the set of all integers and on the set with n elements, n arbitrary natural number. In a similar manner is shown the existence of nontrivial ω-loops, defined on the set D of all real numbers, in which the set of all unities is an arbitrary subset $M \subseteq D$. These constructions can be extended to the infinitary quasigroups of the type $\omega+k$ (definition 6).

Now we shall consider (i, j)-associative infinitary quasigroups.

D e f i n i t i o n 3. An infinitary operative $Q(A)$ of the type ω is called (i, j)-associative if it satisfies the identity

$$A(x_1^{i-1}, A(x_i^\infty), y_1^\infty) = A(x_1^{j-1}, A(x_j^\infty), y_1^\infty), \tag{1}$$

for all $x_1^\infty, y_1^\infty \in Q$.

(Of course, we suppose $i \neq j$).

D e f i n i t i o n 4. An infinitary operative $Q(A)$ of the type ω is called an infinitary semigroup of the type ω (ω-semigroup) if it satisfies the identity (1) for all i and j.

D e f i n i t i o n 5. An ω-quasigroup which is ω-semigroup is called an infinitary group of the type ω (ω-group).

Examples of ω-semigroups are given in [1]. Also it is proved there that nontrivial ω-groups do not exist.

T h e o r e m 1. There does not exist a nontrivial (i, j)-associative infinitary quasigroup.

Now we introduce infinitary quasigroup of the type $\omega+k$ and consider functional equation (2) of generalized associativity on infinitary quasigroups.

D e f i n i t i o n 6. An infinitary operative of the type $\omega+k$ is called infinitary quasigroup of the type $\omega+k$ if the equations

$$A(a_1^{i+1}, x, a_{i+1}^\infty, b_1^k) = c, \quad A(a_1^\infty, b_1^{i-1}, x, b_{i+1}^k) = c,$$

have a unique solution x for all a_p, b_q, $c \in Q$ and for all positive integers i in the first equation and for all $i = 1, 2, \ldots, k$ in the second equation.

T h e o r e m 2. All solutions of the equation

$$A(x_1^{i-1}, B(x_i^\infty, y_1^r), y_{r+1}^\infty) = C(x_1^{j+1}, D(x_j^\infty, y_1^s), y_{s+1}^\infty), \tag{2}$$

where A, B, C, D are infinitary quasigroups of types $|A|=\omega$, $|B|=\omega+r$, $|C|=\omega$, $|D|=\omega+s$, defined on the same nonempty set Q, i, j, some fixed natural numbers and r, s non-negative integers, are given by the following relations:

I_1 $(i=j,\quad r=s\geqslant 0)$:

(3) $$D=\theta B,\qquad A(x_1^{i-1},\ z,\ y_{r+1}^{\infty})=C(x_1^{i-1},\ \theta z,\ y_{r+1}^{\infty}),$$

where θ is an arbitrary permutation of the set Q, B and C arbitrary infinitary quasigroups of types $|B|=\omega+r$, $|C|=\omega$ and D and A are determined by the equations (3).

I_2 $(i=j,\quad s>r\geqslant 0)$:

(4) $$D(x_i^{\infty},\ y_1^{s})=K(B(x_i^{\infty},\ y_1^{r}),\ y_{r+1}^{s}),$$

(5) $$A(x_1^{i-1},\ z,\ y_{r+1}^{\infty})=C(x_1^{i-1},\ K(z,\ y_{r+1}^{s}),\ y_{s+1}^{\infty}),$$

where B and C are arbitrary infinitrary quasigroups of types $|B|=\omega+r$, $|C|=\omega$, K arbitrary quasigroup of arity $s-r+1$ and A and D are determined by the equations (4) and (5).

II_1 $(i<j,\quad r\geqslant s\geqslant 0)$:

(6) $$B(x_{i+1}^{\infty},\ y_1^{r})=K(x_i^{j-1},\ D(x_j^{\infty},\ y_1^{s}),\ y_{s+1}^{r}),$$

(7) $$C(x_1^{j-1},\ z,\ y_{s+1}^{\infty})=A(x_1^{i-1},\ K(x_i^{j-1},\ z,\ y_{s+1}^{r}),\ y_{r+1}^{\infty}),$$

where A and D are arbitrary infinitary quasigroups of types $|A|=\omega$, $|D|=\omega+s$, K arbitrary quasigroup of arity $j-i+r-s+1$ and B and C are determined by the equations (6) and (7).

II_2 $(i<j,\quad s>r\geqslant 0)$:

$$A(x_1^{i-1},\ x,\ y_{r+1}^{\infty})=K(x_1^{i-1},\ \alpha x\circ\beta(y_{r+1}^{s}),\ y_{s+1}^{\infty}),$$

$$B(x_i^{\infty},\ y_1^{r})=\alpha^{-1}(\gamma(x_i^{j-1})\circ\delta(x_j^{\infty},\ y_1^{r})),$$

$$C(x_1^{j-1},\ y,\ y_{s+1}^{\infty})=K(x_1^{i-1},\ \gamma(x_i^{j-1})\circ\varphi y,\ y_{s+1}^{\infty}),$$

$$D(x_j^{\infty},\ y_1^{s})=\varphi^{-1}(\delta(x_j^{\infty},\ y_1^{r})\circ\beta(y_{r+1}^{s})),$$

where α, φ are permutations of the set Q, $Q(\circ)$ binary group, β and γ quasigroups of arities $|\beta|=s-r$, $|\gamma|=j-i$, and δ, K infinitary quasigroups of types $|\delta|=\omega+r$, $|K|=\omega$.

The isotopy of two quasigroups can be generalized to infinitary case.

Two infinitary quasigroups of the type ω defined on the same set Q are called isotopic if there exists a sequence $T=\alpha_0^{\infty}$ of permutations of Q such that $B(x_1^{\infty})=\alpha_0^{-1}A(\{\alpha_i x_i\}_{i=1}^{\infty})$ for all $x_1^{\infty}\in Q$. This we shall denote by $B=A^T$. The usual theorems for isotopy of finitary quasigroups are also valid for infinitary case.

Definition 7. Let Q be a nonempty set, $Q(+)$ abelian group and $Q(R)$ infinitary quasigroup of the type ω, such that

$$R(\{x_i+y_i\}_{i=1}^{\infty})=R(x_1^{\infty})+R(y_1^{\infty}),$$

for all x_1^{∞}, $y_1^{\infty}\in Q$, then ω-quasigroup $Q(R)$ is called additive over abelian group $Q(+)$.

The existence of such nontrivial ω-quasigroups additive over abelian group is proved in [8].

It is also proved there that every infinitary quasigroup $Q(R)$ of the type ω additive over abelian group $Q(+)$ can be represented in the form

$$R(z_1^{\infty})=\alpha_1 z_1+\alpha_2 z_2+\cdots+\alpha_n z_n+R_n(z_{n+1}^{\infty}), \tag{8}$$

where n is an arbitrary natural number, α_i, $i=1, 2, \ldots, n$ automorphisms of the group $Q(+)$ and $Q(R_n)$ an infinitary quasigroup of the type ω additive over $Q(+)$.

Theorem 3. All solutions of the functional equation

$$A(B_1(x_1^{\infty}),\ B_2(y_1^{\infty}))=C(\{D_i(x_i,\ y_i)\}_{i=1}^{\infty}),$$

where A, D_i, $i=1, 2, \ldots$ are binary quasigroups and B_1, B_2, C infinitary quasigroups of the type ω, all defined on the same nonempty set Q, are given by

$$A(x, y)=\alpha x+\beta y,$$

$$D_i(x, y)=\varphi_i^{-1}(\gamma_i x+\theta_i y),$$

$$B_1(x_1^{\infty})=\alpha^{-1}(R(\{\gamma_i x_i\}_{i=1}^{\infty})+b),$$

$$B_2(x_1^{\infty})=\beta^{-1}(R(\{\theta_i x_i\}_{i=1}^{\infty})+a),$$

$$C(x_1^{\infty})=R\{\varphi_i x_i\}_{i=1}^{\infty})+c,$$

where $Q(+)$ is an arbitrary abelian group, $Q(R)$ arbitrary infinitary quasigroup of the type ω additive over $Q(+)$, α, β, γ_i, θ_i, φ_i, $i=1, 2, \ldots$ arbitrary permutations of the set Q, a, b, c arbitrary elements from Q such that $a+b=c$.

At the end we shall make some remarks concerning further investigations on infinitary quasigroups.

The notion of infinitary quasigroup of an arbitrary type α can also be introduced. Let X_α be a linear ordered sequence of variables of the type α, and let x be an arbitrary variable from X_α. Let X_{α_1} be the set of all variables from X_α which are less than x and X_{α_2} the set of all elements from X_α which are greater than x. Both X_{α_1} and X_{α_2} are linear ordered and they have the order types α_1 and α_2 respectively. Hence α can be represented as $\alpha_1+1+\alpha_2$. The operative $Q(A)$ of the type α is an infinitary quasigroup if the equation

$$A(C_{\alpha_1},\ x,\ C_{\alpha_2})=b,$$

where C_{α_1} and C_{α_2} are arbitrary linear ordered sequences from Q^{α_1} and Q^{α_2} respectively, b is an arbitrary element from Q, has a unique solution.

The definitions of the infinitary quasigroups of the type ω and $\omega+k$ are particular cases of the definition given above.

The definition of the isotopy of ω-quasigroups is already given and the notions of principal isotopy, isomorphism, autotopy etc. can be introduced as in finitary case. The inverse operations of the infinitary quasigroup operation of the type α are also defined in [7]. The notion of parastrophy can be extended to finitary case in two ways — one which directly generalizes the parastrophy of finitary quasigroups and the other which is specific for infinitary case, in which the type of the quasigroup is changed.

The insertion algebra for finitary quasigroups, which one of the authors considered in [4], [5], can also be extended to infinitary case.

At the end some problems on infinitary quasigroups were given. The precise formulation of these problems can be found in [7].

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