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ON EXTENDING OF SOLUTIONS OF FUNCTIONAL EQUATIONS IN A SINGLE VARIABLE

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In this talk I want to present two results regarding the problem of the unique extension of solutions of the functional equation

$$\varphi(x) = h\big(x, \underset{s \in S}{\triangle} \varphi \circ f_s(x)\big) \tag{1}$$

in which

$$h: X \times Y^S \to Y \text{ and } f_s: X \to X, \ s \in S,$$

where X, Y and S are arbitrary sets, are given functions. Here and in the sequel Y^S denotes the set of all functions from S into Y with the Tychonoff topology in the case where Y is a topological space whereas $\underset{s \in S}{\triangle} g_s$ denotes the diagonal of a family of transformations $\{g_s : s \in S\}$ (i.e. if g_s map X into Y, $s \in S$, then $\underset{s \in S}{\triangle} g_s$ is a map from X into Y^S such that for the projection map p_s

$$p_s \circ \underset{s \in S}{\triangle} g_s = g_s, \qquad s \in S).$$

Theorem 1. *Let $U \subset X$ be an arbitrary set such that*

$$f_s(U) \subset U, \qquad s \in S. \tag{2}$$

If

(i) *for every $x \in X$ there exists a positive integer k such that for every $s_1, \ldots, s_k \in S$*

$$f_{s_1} \circ \cdots \circ f_{s_k}(x) \in U,$$

then for every solution $\varphi_0 : U \to Y$ of the equation (1) *there exists exactly one solution $\varphi : X \to Y$ of it such that $\varphi|_U = \varphi_0$.*

Moreover, if X and Y are topological spaces, U is open, h, f_s, $s \in S$, and φ_0 are continuous functions and

(ii) *for every open set V such that $U \subset V \subset X$ we have $\cap\{f_s^{-1}(V) : s \in S\}$ open,*

then φ is also continuous.

The hypothesis (i) in this theorem cannot be replaced (an example may be given) by

(iii) for every $x \in X$ there exists a positive integrer k such that for every $s \in S$, $f_s^k(x) \in U$.

On the other hand the hypothesis

(iv) X is a closed subset of a finite dimensional Banach space and $\{f_s : s \in S\}$ is a locally equicontinuous family such that for a certain $\xi \in X$

$$\sup\{\|f_s(x) - \xi\| : s \in S\} < \|x - \xi\|, \qquad x \in X \setminus \{\xi\}, \tag{3}$$

implies (i) whenever U is open (in X) and $\xi \in U$.

Theorem 2. *Let X be a closed and convex subset of a finite dimensional Banach space, $U \subset X$ an open set (in X) such that condition* (2) *is satisfied. If $\{f_s : s \in S\}$ is a locally equicontinuous family such that* (3) *holds for a certain $\xi \in U$, then for every solution $\varphi_0 : U \to Y$ of* (1) *there exists exactly one solution $\varphi : X \to Y$ of it such that $\varphi|_U = \varphi_0$.*

Moreover, if Y is a topological space, h and φ_0 are continuous functions, then φ is also a continuous function.

In view of the above mentioned connexion between hypotheses (iv) and (i) the first part of Theorem 2 follows from Theorem 1. However, in the other part of Theorem 2 the restrictive hypothesis (ii) does not occur.

On the other hand the proof of Theorem 1 is effective contrary to the proof of Theorem 2 where the Kuratowski-Zorn Lemma is used.

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