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A CLASS OF BALANCED LAWS ON QUASIGROUPS (II)

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In this paper we enlarge the results, obtained in [1] for a class of balanced laws of the I kind on an analogous class of balanced laws of the II kind. In both cases the operations, satisfying the laws, are quasigroups, defined on a nonempty set S.

Let $w_1 = w_2$ be a balanced law of the II kind in the form

$$A(u_1, \dots, u_m) = B(v_1, \dots, v_n),\ m > 2,\ n > 2, \tag{1}$$

where u_i $(i = 1, \dots, m)$ is either a variable or a term $A_i(x_{i_1}, \dots, x_{i_\alpha})$, $x_{i_1}, \dots, x_{i_\alpha}$ being variables, analogously v_j $(j = 1, \dots, n)$ is either a variable or a term $B_j(x_{j_1}, \dots, v_{j_\beta})$, $x_{j_1}, \dots, x_{j_\beta}$ being variables. A, B, A_i, B_j are functions letters.

For (1) we suppose the following conditions hold:

(i) For any two terms u_i and v_j there is at most one variable occuring in each of them.

(ii) If in each of two terms u_i and u_k (v_j, v_h) occurs exactly one variable, these variables occur in different terms v_j, v_h (u_i, u_k) respectively.

(iii) The order of occurence of the variables in any term u_i (v_j) is equal to the order of occurence of these variables in the term w_2 (w_1).

For example, such is the law

$$A(A_1(x, y, z, u), A_2(v, w), t) = B(x, B_2(y, v), B_3(z, t), B_4(u, w)).$$

In the set of terms $T = \{u_1, \dots, u_m, v_1, \dots, v_n\}$ we introduce the relation of connectness defined in [1], and in the set of all quasigroups derived from A and B we introduce the relations $\Rightarrow$ and $\Leftrightarrow$, defined in [1], too.

For the laws of the II kind, hold the lemmas, analogous to the lemmas 1, 2, 3 from [1]. The proofs of the lemmas 2 and 3 rest unchanged, for the lemma 1 we need a new proof.

L e m m a 1. *If all terms u_i, v_j of the law (1) are connected, for any two binary quasigroups $L^P_{\alpha\beta}$ and $L^Q_{\mu\nu}$ $(P, Q \in \{A, B\})$, derived from A and B, we have $L^P_{\alpha\beta} \Leftrightarrow L^Q_{\mu\nu}$.*

P r o o f. First, if $L^A_\alpha \leftrightarrow L^B_\mu$, and $\beta > \alpha$, there exists $\nu \neq \mu$ such that $L^A_{\alpha\beta} \leftrightarrow L^B_{\overline{\mu\nu}}$ ($L^B_{\overline{\mu\nu}} = L^B_{\mu\nu}$, for $\mu < \nu$, and $L^B_{\overline{\mu\nu}} = L^B_{\nu\mu}$, for $\nu < \mu$). Indeed, if $L^A_\alpha \overset{x}{\longleftrightarrow} L^B_\mu$, then x occurs in the terms u_α and v_μ. The term u_β either contains a variable y occuring in some term v_ν, $\nu \neq \mu$, or contains only one variable y occuring in the term v_μ. In the first case we get $L^A_{\alpha\beta} \overset{xy}{\longleftrightarrow} L^B_{\overline{\mu\nu}}$, and in the second case, in view of (ii), there exists a variable z occuring in the term u_α and in a term v_ν, $\nu \neq \mu$, and we get $L^A_{\alpha\beta} \overset{zy}{\longleftrightarrow} L^B_{\overline{\mu\nu}}$.

Since for every two L^P_α and L^Q_μ holds $L^P_\alpha \Leftrightarrow L^Q_\mu$, it follows for every $L^P_{\alpha\beta}$ and L^Q_μ ($P, Q \in \{A, B\}$) there exists ν so that $L^P_{\alpha\beta} \Leftrightarrow L^Q_{\overline{\mu\nu}}$.

Further, for every α, β, μ, $\alpha < \beta$, $\beta \neq \mu$ we get $L^P_{\alpha\beta} \Leftrightarrow L^P_{\overline{\mu\beta}}$ ($P \in \{A, B\}$). Let us consider a sequence

$$L^P_\alpha \leftrightarrow L^Q_\rho \leftrightarrow L^P_\sigma \leftrightarrow \cdots \leftrightarrow L^Q_\tau \leftrightarrow L^P_\mu$$

defining $L^P_\alpha \leftrightarrow L^P_\mu$. There exists an index ν so that $L^P_{\alpha\beta} \leftrightarrow L^Q_{\overline{\rho\nu}}$. If $\sigma \neq \beta$, then $L^Q_{\overline{\rho\nu}} \leftrightarrow L^P_{\overline{\sigma\beta}}$, if $\sigma = \beta$, then $L^Q_{\overline{\rho\nu}} \leftrightarrow L^P_{\alpha\sigma} = L^P_{\alpha\beta}$. On this way, we get finally $L^P_{\overline{\alpha\beta}} \leftrightarrow L^P_{\overline{\mu\beta}}$,

Let $L^P_{\alpha\beta}$ and $L^Q_{\mu\nu}$ be two arbitrary quasigroups. There exists an index λ so that $L^P_{\alpha\beta} \Leftrightarrow L^Q_{\overline{\mu\lambda}}$, and $L^Q_{\overline{\mu\lambda}} \Leftrightarrow L^Q_{\mu\nu}$. Hence we get $L^P_{\alpha\beta} \Leftrightarrow L^Q_{\mu\nu}$, and the lemma is proved.

Let $L^P_{\alpha\beta}$ and $L^P_{\mu\nu}$ ($P, Q \in \{A, B, A_i, B_j\}$) be two derived quasigroups so that $L^P_{\alpha\beta} \overset{xy}{\longleftrightarrow} L^Q_{\mu\nu}$. Since the law (1) is of the II kind, we have either

$$\varphi L^P_{\alpha\beta} (\varphi_1 x, \varphi_2 y) = \psi L^Q_{\mu\nu} (\psi_1 x, \psi_2 y), \text{ or}$$

$$\varphi L^P_{\alpha\beta} (\varphi_1 x, \varphi_2 y) = \psi L^Q_{\mu\nu} (\psi_2 y, \psi_1 x).$$

In the second case we say the relation $L^P_{\alpha\beta} \leftrightarrow L^Q_{\mu\nu}$ is an inversion.

Let $\approx$ be the following equivalence relation of the set I_2 of all quasigroups $L^P_{\alpha\beta}$ ($P \in \{A, B, A_i, B_j\}$). For $L^P_{\alpha\beta}$, $L^Q_{\mu\nu} \in I_2$ we put $L^P_{\alpha\beta} \approx L^Q_{\mu\nu}$, iff $L^P_{\alpha\beta} \Leftrightarrow L^Q_{\mu\nu}$ and there exists at least one sequence defining $L^P_{\alpha\beta} \Leftrightarrow L^Q_{\mu\nu}$ with an even number of inversions.

The relation $\approx$ is containing in the relation $\Leftrightarrow$, more precisely, each class $C_\Leftrightarrow$ of the relation $\Leftrightarrow$ is the union of at most two classes $C'_\approx$ and $C''_\approx$ of the relation $\approx$.

Let all terms u_i, v_j of the law (1) be connected. By the lemmas 1 and 3, for every two operations $L^P_{\alpha\beta}$ and $L^Q_{\mu\nu}$ ($P, Q \in \{A, B, A_i, B_j\}$) we have $L^P_{\alpha\beta} \Leftrightarrow L^Q_{\mu\nu}$. We distinguish two cases:

1. $C_\Leftrightarrow = C_\approx$,

2. $C_\Leftrightarrow = C'_\approx \cup C''_\approx$.

In the case 1. there are two possibilities:

1.′ From the law (1) does not yield any inversion, that is the law (1) is of the I kind.

1.'' From the law (1) yields at least one inversion, that is, the law (1) is of the II kind.

In the case 1'' for every quasigroup $L^P_{\alpha\beta}$ there exists at least one sequence defining $L^P_{\alpha\beta} \Leftrightarrow L^P_{\alpha\beta}$ with an odd number of inversions. Let, for example, the relation $L^P_{\gamma\delta} \leftrightarrow L^Q_{\mu\nu}$ be an inversion. Since $L^P_{\alpha\beta} \approx L^P_{\gamma\delta}$ and $L^Q_{\mu\nu} \approx L^P_{\alpha\beta}$, there exists a sequence $L^P_{\alpha\beta} \Leftrightarrow L^P_{\gamma\delta} \leftrightarrow L^Q_{\mu\nu} \Leftrightarrow L^P_{\alpha\beta}$ with an odd number of inversions.

In the case 2 each sequence defining $L^P_{\alpha\beta} \Leftrightarrow L^P_{\alpha\beta}$ has an even number of inversions. Indeed, let, for example, be $L^P_{\alpha\beta} \in C'_{\approx}$, and let exist a sequence defining $L^P_{\alpha\beta} \Leftrightarrow L^P_{\alpha\beta}$ with an odd number of inversions. Let $L^P_{\mu\nu} \in C'_{\approx}$, and $L^Q_{\rho\lambda} \in C''_{\approx}$. Then there exists a sequence $L^P_{\mu\nu} \Leftrightarrow L^P_{\alpha\beta} \Leftrightarrow L^P_{\alpha\beta} \Leftrightarrow L^Q_{\lambda\rho}$ with an even number of inversions, that is, $L^P_{\mu\nu} \approx L^Q_{\lambda\rho}$, what is in contradiction with the assumption about $L^P_{\mu\nu}$ and $L^Q_{\lambda\rho}$.

Let for the law (1) hold 2. We change all operations $L^P_{\alpha\beta}$ of one of the classes, say $C''_{\approx}$, with the operations $L^{P*}_{\alpha\beta}$ ($L^{P*}_{\alpha\beta}(x, y) \overset{\text{def}}{=} L^P_{\alpha\beta}(y, x)$). The obtained law $w^*_1 = w^*_2$ is of the I kind. Indeed, from so obtained law $w^*_1 = w^*_2$ it yields $C_{\approx} = C_{\Leftrightarrow}$, and for every operation $L^P_{\alpha\beta}$ each sequence defining $L^P_{\alpha\beta} \Leftrightarrow L^P_{\alpha\beta}$ has an even number of inversions.

Hence, we can consider only the laws of the II kind for which the relation $\Leftrightarrow$ and $\approx$ on the set I_2 are the same.

Theorem 1. *Let all quasigroups derived from quasigroups satisfying the law (1) be in the relation $\approx$. Then there exist a commutative group (S, o) so that the following equalities hold:*

$$A(x_1, \ldots, x_m) = L^A_1 x_1 \circ \cdots \circ L^A_m x_m$$

$$B(x_1, \ldots, x_n) = L^B_1 x_1 \circ \cdots \circ L^B_n x_n$$

$$L^A_i A_i(x_1, \ldots, x_\alpha) = L^A_i L^{Ai}_1 x_1 \circ \cdots \circ L^A_i L^{Ai}_\alpha x_\alpha$$

$$L^B_j B_j(x_1, \ldots, x_\beta) = L^B_j L^{Bj}_1 x_1 \circ \cdots \circ L^B_j L^{Bj}_\beta x_\beta$$

Proof. In the set S we introduce the binary operation $\circ$ defined by

$$L^A_{12}(x, y) = L^A_1 x \circ L^A_2 y.$$

Let $L^P_{\alpha\beta}$ and $L^Q_{\mu\nu}$ $(P, Q \in \{A, B\})$ be two quasigroups derived from the law (1) so that $L^P_{\alpha\beta} \leftrightarrow L^Q_{\mu\nu}$ holds. If this relation is an inversion, and if $L^P_{\alpha\beta}(x, y) = L^P_\alpha x \circ L^P_\beta y$, we have $L^Q_{\mu\nu}(x, y) = L^Q_\nu y \circ L^Q_\mu x$.

Since all quasigroups $L^P_{\alpha\beta}$ are in the relation $\approx$, and for every $L^P_{\alpha\beta}$ there exists a sequence with an odd number of inversions, defining $L^P_{\alpha\beta} \Leftrightarrow L^P_{\alpha\beta}$, we get

$$L^P_{\alpha\beta}(x, y) = L^P_\alpha x \circ L^P_\beta y, \text{ and}$$

$$L^P_{\alpha\beta}(x, y) = L^P_\beta y \circ L^P_\alpha x.$$

Hence, we get for every $x, y \in S$

$$L^P_\alpha x \circ L^P_\beta y = L^P_\beta y \circ L^P_\alpha x.$$

Since L_α^p and L_β^p are bijections, we have

$$x \circ y = y \circ x,$$

that is, the operation $\circ$ is commutative.

The proof of the rest of the theorem is analogous to the proof of the theorem 1 in [1].

Now we suppose the relation of connectness of the set $T=\{u_1, \ldots, u_m, v_1, \ldots, v_n\}$ of the law (1) has r $(r>1)$ equivalence classes $C_i=\{u_{\alpha_i}, \ldots, u_{\beta_i}, v_{\gamma_i}, \ldots, v_{\delta_i}\}$, $i=1, \ldots, r$. Introducing quasigroups A and B conjugated with A and B respectively, from (1) we obtain a law in the form

$$(2) \qquad A(u_{\alpha_1}, \ldots, u_{\beta_1}, \ldots, u_{\alpha_r}, \ldots, u_{\beta_r}) = B(v_{\gamma_1}, \ldots, v_{\delta_1}, \ldots, v_{\gamma_r}, \ldots, v_{\delta_r}).$$

For the law (2) hold all results obtained in [1] for the analogous law of the I kind.

Finally, let us consider an arbitrary law in the form

$$(3) \quad A(A_1(x_1, \ldots, x_\alpha), \ldots, A_m(x_\beta, \ldots, x_p)) = B(B_1(y_1, \ldots, y_\gamma), \ldots, B_n(y_\delta, \ldots, y_p)),$$

where the sequence $y_1, \ldots, y_p$ is a permutation of $x_1, \ldots, x_p$, and A, B, A_i and B_j are quasigroups on a set S. Such a law can be transformed into a law (1) satisfying conditions (i), (ii) and (iii) by substitution of some quasigroups by GD-groupoids.

We give an example.

Let

$$A(A_1(x, y), z, A_3(u, v), A_4(w, t)) = B(x, B_2(y, z), B_3(u, w), B_4(v, t))$$

be the functional equality by unknown quasigroups $A, B, A_1, \ldots, B_4$ of a set S. A general solution of this equality is given by

$$A(x, y, z, u) = \pi(\alpha x \circ \beta y, \mu z + \nu u),$$

$$B(x, y, z, u) = \pi(\gamma x \circ \varepsilon y, \sigma z + \tau u),$$

$$\alpha A_1(x, y) = \gamma x \circ \delta y,$$

$$\varepsilon B_2(x, y) = \delta x \circ \beta y,$$

$$\mu A_3(x, y) = \lambda x + \rho y,$$

$$\nu A_4(x, y) = \omega x + \pi y,$$

$$\sigma B_3(x, y) = \mu x + \omega y,$$

$$\tau B_4(x, y) = \rho x + \pi y,$$

where π is an arbitrary loop, $\circ$ is an arbitrary group, $+$ is an arbitrary commutative group, and $\alpha, \beta, \ldots, \pi$ are arbitrary bijections of the set S.

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ON EXTENDING OF SOLUTIONS OF FUNCTIONAL EQUATIONS IN A SINGLE VARIABLE

Karol Baron

In this talk I want to present two results regarding the problem of the unique extension of solutions of the functional equation

$$\varphi(x) = h\left(x, \underset{s \in S}{\triangle} \varphi \circ f_s(x)\right) \tag{1}$$

in which

$$h: X \times Y^S \to Y \text{ and } f_s: X \to X, \ s \in S,$$

where X, Y and S are arbitrary sets, are given functions. Here and in the sequel Y^S denotes the set of all functions from S into Y with the Tychonoff topology in the case where Y is a topological space whereas $\underset{s \in S}{\triangle} g_s$ denotes the diagonal of a family of transformations $\{g_s : s \in S\}$ (i.e. if g_s map X into Y, $s \in S$, then $\underset{s \in S}{\triangle} g_s$ is a map from X into Y^S such that for the projection map p_s

$$p_s \circ \underset{s \in S}{\triangle} g_s = g_s, \qquad s \in S).$$

Theorem 1. *Let $U \subset X$ be an arbitrary set such that*

$$f_s(U) \subset U, \qquad s \in S. \tag{2}$$

If

(i) *for every $x \in X$ there exists a positive integer k such that for every $s_1, \ldots, s_k \in S$*

$$f_{s_1} \circ \cdots \circ f_{s_k}(x) \in U,$$

then for every solution $\varphi_0 : U \to Y$ of the equation (1) *there exists exactly one solution $\varphi : X \to Y$ of it such that $\varphi|_U = \varphi_0$.*

Moreover, if X and Y are topological spaces, U is open, h, f_s, $s \in S$, and φ_0 are continuous functions and

(ii) *for every open set V such that $U \subset V \subset X$ we have $\bigcap\{f_s^{-1}(V) : s \in S\}$ open,*

then φ is also continuous.

The hypothesis (i) in this theorem cannot be replaced (an example may be given) by

(iii) for every $x \in X$ there exists a positive integrer k such that for every $s \in S$, $f_s^k(x) \in U$.

On the other hand the hypothesis

(iv) X is a closed subset of a finite dimensional Banach space and $\{f_s : s \in S\}$ is a locally equicontinuous family such that for a certain $\xi \in X$

$$\sup\{\|f_s(x) - \xi\| : s \in S\} < \|x - \xi\|, \qquad x \in X \setminus \{\xi\}, \tag{3}$$

implies (i) whenever U is open (in X) and $\xi \in U$.

Theorem 2. *Let X be a closed and convex subset of a finite dimensional Banach space, $U \subset X$ an open set (in X) such that condition* (2) *is satisfied. If $\{f_s : s \in S\}$ is a locally equicontinuous family such that* (3) *holds for a certain $\xi \in U$, then for every solution $\varphi_0 : U \to Y$ of* (1) *there exists exactly one solution $\varphi : X \to Y$ of it such that $\varphi|_U = \varphi_0$.*

Moreover, if Y is a topological space, h and φ_0 are continuous functions, then φ is also a continuous function.

In view of the above mentioned connexion between hypotheses (iv) and (i) the first part of Theorem 2 follows from Theorem 1. However, in the other part of Theorem 2 the restrictive hypothesis (ii) does not occur.

On the other hand the proof of Theorem 1 is effective contrary to the proof of Theorem 2 where the Kuratowski-Zorn Lemma is used.

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