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A CLASS OF BALANCED LAWS ON QUASIGROUPS (I)

Branka P. Alimpić

Let $w_1 = w_2$ be a balanced law of the I kind [2], in the form

$$A(u_1, \ldots, u_m) = B(v_1 \ldots, v_n) \quad m \geqslant 2, \quad n \geqslant 2, \tag{1}$$

where u_i $(i = 1, \ldots, m)$ is either a variable or a term $A_i(x_{i_1}, \ldots, x_{i_\alpha})$, $x_{i_1}, \ldots, x_{i_\alpha}$ being variables, analogously v_j $(j = 1, \ldots, n)$ is either a variable or a term $B_j(x_{j_1}, \ldots, x_{j_\beta})$, $x_{j_1}, \ldots, x_{j_\beta}$ being variables. A, B, A_i and B_j are function letters. Let p be the number of different variables occuring in the law (1).

For (1) we suppose the following conditions hold:

(i) For any two terms u_i and v_j there is at most one variable occuring in each of them.

(ii) If there are terms u_i, u_{i+1} (or v_j, v_{j+1}) in each of them occurs exactly one variable, these variables occur in different terms v_j, v_{j+1} (or u_i, u_{i+1}) respectively.

For example, such is the law

$$A(A_1(x_1, x_2), x_3, x_4, x_5, A_5(x_6, x_7, x_8), x_9) = \tag{2}$$
$$= B(x_1, B_2(x_2, x_3), x_4, B_4(x_5, x_6), x_7, B_6(x_8, x_9)).$$

Let $|P|$ denote the length of function letter P. From (i) and (ii) it follows $|A_i| \leqslant 3$, $|B_j| \leqslant 3$. Indeed, if e. g. $u_i = A_i(x, y, z, u)$, the variables x, y, z, u occur in different terms $v_j, v_{j+1}, v_{j+2}, v_{j+3}$ respectively, and in each of terms v_{j+1}, v_{j+2} occurs exactly one variable. According to (ii) that is impossible, hence, $|A_i| \leqslant 3$.

We define a relation of the set $T = \{u_1, \ldots, u_m, v_1, \ldots, v_n\}$. For any two terms $t_1, t_2 \in T$ we say t is connected with t_2 iff tere are terms $s_1, \ldots, s_k \in T$ such that $t_1 = s_1$, $t_2 = s_k$ and s_i, s_{i+1} $(i = 1, \ldots, k-1)$ are terms occuring on opposite sides of (1) and having a common variable. This relation is an equivalence of the set T. Let r denote the number of equivalence classes.

For example, in (2) we have $r = 3$, and the classes are

$$C_1 = \{u_1, u_2, v_1, v_2\}, \quad C_2 = \{u_3, v_3\}, \quad C_3 = \{u_4, u_5, u_6, v_4, v_5, v_6\}.$$

In this paper we consider the law (1) as a functional equation of unknown functions A, B, A_i, B_j and we give its general solution, provided A, B, A_i, B_j are quasigroups defined on a nonempty set S.

We give some definitions.

If t is a term, let $[t]$ denote the set of variables occuring in t [2]. If t is a subterm either of w_1 or of w_2, $a \in S$ a fixed element, and $\tau \subset [t]$, let $t\big|_a^\tau$ denote the term obtained from t substituting all $x_i \in \tau$ by a. If e. g. $t = A_i(x, y, z)$, $\tau = \{x, z\}$, then $t\big|_a^\tau = A_i(a, y, a)$.

Let $t = P(t_1, \ldots, t_\alpha)$ be a subterm either of w_1 or of w_2 and $(t_\mu, \ldots, t_\nu)$ be some nonempty subsequence of the sequence $(t_1, \ldots, t_\alpha)$ of the length $k \leqslant \alpha$. We define an operation of the set S, derived from the quasigroup P

$$L^P_{\mu\ldots\nu} : S^k \to S,$$

putting

$$L^P_{\mu\ldots\nu}(t_\mu, \ldots, t_\nu) \overset{\text{def}}{=} t\big|_a^{[t]\setminus([t_\mu]\cup\cdots\cup[t_\nu])}.$$

So, for the law (2) we have

$$L^A_{13}(u_1, u_3) = A(u_1, a, u_3, a, A_5(a, a, a), a),$$

$$L^A_{23}(x_7, x_8) = A_5(a, x_7, x_8),$$

$$L^B_{25}(v_2, v_5) = B(a, v_2, a, B_4(a, a), v_5, B_6(a, a)), \text{ etc.}$$

These operations depend on the choice of a and on the form of the law (1). Since A, B, A_i, B_j are quasigroups, the operations, derived from them, are quasigroups too.

Let $\mathcal{T}$ be the set of all quasigroups derived from A and B. We define the relation $\to$ of the set $\mathcal{T}$ on the following way: For quasigroups $L^P_{\alpha\ldots\beta}$ and $L^Q_{\mu\ldots\nu}(P, Q \in \{A, B\}, P \neq Q)$ we put

$$L^P_{\alpha\ldots\beta} \to L^Q_{\mu\ldots\nu}$$

iff there are variables $x_\gamma, \ldots, x_\delta$ of the law (1) so that by substitution of all variables in (1), except $x_\gamma, \ldots, x_\delta$ we get

$$L^P_{\alpha\ldots\beta}(\varphi_\alpha x_\gamma, \ldots, \varphi_\beta x_\delta) = L^Q_{\mu\ldots\nu}(\psi_\mu(x_\gamma, \ldots, x_{\gamma'}), \ldots, \psi_\nu(x_{\delta'}, \ldots, x_\delta)),$$

where $\varphi_\alpha, \ldots, \varphi_\beta$ are bijections of the set S, $\psi_\mu, \ldots, \psi_\nu$ are either derived quasigroups or bijections of the set S. In this case we write also

$$L^P_{\alpha\ldots\beta} \xrightarrow{x_\gamma\ldots x_\delta} L^Q_{\mu\ldots\nu}.$$

For example, from the law (2) we have

$$L^A_{125}(L^{A_1}_2 x_2, x_2, x_5) = L^B_{23}(B_2(x_2, x_3), L^{B_4}_1 x_5),$$

namely

$$L^A_{125} \xrightarrow{x_2 x_3 x_5} L^B_{23}.$$

Let $\Rightarrow$ be the minimal transitive relation of the set $\mathcal{T}$ containing the relation $\to$. Relations $\Rightarrow$ and $\to$ of the set $\mathcal{T}_k$ of all derived quasigroups of the length k $(1 \leqslant k \leqslant \max(m, n))$ are symmetric and we denote them $\Leftrightarrow$ and $\leftrightarrow$ respectively. The relation $\Leftrightarrow$ of the set $\mathcal{T}_k$ is an equivalence.

We prove several lemmas.

Lemma 1. *If all terms u_i, v_j of the law* (1) *are connected, for any two binary quasigroups $L^P_{\alpha\beta}$ and $L^Q_{\mu\nu}$ $(P, Q \in \{A, B\})$ we have $L^P_{\alpha\beta} \Leftrightarrow L^Q_{\mu\nu}$.*

Proof. First we prove $L^A_{ij} \Leftrightarrow L^A_{ij+1}$, where $1 \leqslant i < j < m$. Let x be the first variable occuring in the term u_i, y the last variable occuring in the term u_j and z the first variable occuring in the term u_{j+1}. Since all terms are connected, the variables y and z occur in the same term v_k, and x occurs in a term v_h, $h < k$. Indeed, there is at least one variable x' occuring between x and y. If x occurs in v_k, in v_k occur at least four variables x, x', y, z, which is impossible. Therefore $L^A_{ij} \overset{xy}{\leftrightarrow} L^B_{hk} \overset{xz}{\leftrightarrow} L^A_{ij+1}$, namely $L^A_{ij} \Leftrightarrow L^A_{ij+1}$. Then we have

$$L^A_{ij} \Leftrightarrow L^A_{ik}, \quad 1 \leqslant i < m, \quad i < j < k \leqslant m. \tag{3}$$

Analogously it yields

$$L^A_{ki} \Leftrightarrow L^A_{ji}, \quad 1 \leqslant k < j < i, \quad 1 < i \leqslant m. \tag{4}$$

Using (3) and (4), for any two $L^A_{\alpha\beta}$ and $L^A_{\gamma\delta}$ we get $L^A_{\alpha\beta} \Leftrightarrow L^A_{\gamma\beta} \Leftrightarrow L^A_{\gamma\delta}$.

For every $L^B_{\mu\nu}$ there are some L^A_{jk} so that $L^A_{jk} \leftrightarrow L^B_{\mu\nu}$. Using $L^A_{\alpha\beta} \Leftrightarrow L^A_{jk}$, we have $L^A_{\alpha\beta} \Leftrightarrow L^B_{\mu\nu}$. The lemma is proved.

Lemma 2. *If all terms u_i, v_j of the law* (1) *are connected, for every derived quasigroup $L^P_{\alpha\beta\ldots\gamma}$ $(P \in \{A, B\})$ of the length $\geqslant 3$ holds:*

$$L^P_{\alpha\beta\ldots\gamma} \Rightarrow L^A_{12}$$

and

$$L^P_{\alpha\beta\ldots\gamma} \Rightarrow L^A_1.$$

Proof. As all terms are connected, there is at least one term u_μ with two variables. The variables x and y of the term u_μ occur in two different terms v_j, v_k respectively, hence we get $L^B_{jk} \to L^A_\mu$. By lemma 1, we have $L^P_{\alpha\beta} \Leftrightarrow L^B_{jk}$ and therefore $L^P_{\alpha\beta} \Rightarrow L^A_\mu$. Hence, there is a quasigroup $L^A_{\mu\ldots\nu}$, so that $|L^P_{\mu\ldots\nu}| < |L^P_{\alpha\beta\ldots\gamma}|$ and $L^P_{\alpha\beta\ldots\gamma} \Rightarrow L^A_{\mu\ldots\nu}$. On this way we get a quasigroup $L^Q_{\lambda\rho}$ of the length 2 $(Q \in \{A, B\})$ such that $L^P_{\alpha\beta\ldots\gamma} \Rightarrow L^Q_{\lambda\rho}$, and according to Lemma 1 we have $L^P_{\alpha\beta\ldots\gamma} \Rightarrow L^A_{12}$. Since $L^A_{12} \Rightarrow L^A_1$, we have $L^P_{\alpha\beta\ldots\gamma} \Rightarrow L^A_1$.

For example, in the law

$$A\big(A_1(x_1, x_2), x_3, A_3(x_4, x_5, x_6), A_4(x_7, x_8)\big) = B\big(x_1, B_2(x_2, x_3, x_4), x_5, B_4(x_6, x_7), x_8\big)$$

all terms u_i, v_j are connected, and we have

$$A \xrightarrow{x_2x_3x_4x_7} L^B_{24} \xrightarrow{x_2x_6} L^A_{13} \xrightarrow{x_1x_4} L^B_{12} \xrightarrow{x_1x_2} L^A_1,$$

$$B \xrightarrow{x_1x_4x_5x_6x_8} L^A_{134} \xrightarrow{x_2x_4x_7} L^B_{24}, \text{ etc.}$$

L e m m a 3. *For every* $L^{A_i}_{\alpha\beta}$ *and* $L^{B_j}_{\mu\nu}$ *there are quasigroups* $L^{B}_{\gamma\delta}$ *and* $L^{A}_{\lambda\rho}$ *so that* $L^{A}_{i} L^{A_i}_{\alpha\beta}(x, y) = L^{B}_{\gamma\delta}(\varphi_\gamma x, \varphi_\delta y)$, *and* $L^{B}_{j} L^{B_j}_{\mu\nu}(x, y) = L^{A}_{\lambda\rho}(\varphi_\lambda x, \varphi_\rho y)$, *where* φ_γ, φ_δ, φ_λ *and* φ_ρ *are bijections of the set* S.

P r o o f. These equalities are obtained by substitution of all variables of the law (1), except, x and y, by $a \in S$.

T h e o r e m 1. *Let all terms* u_i, v_j *of the law* (1) *be connected. There exists a group* $(S, \circ)$ *so that the following equalities hold*:

$$A(x_1, \ldots, x_m) = L^{A}_{1} x_1 \circ \cdots \circ L^{A}_{m} x_m,$$

$$B(x_1, \ldots, x_n) = L^{B}_{1} x_1 \circ \cdots \circ L^{B}_{n} x_n,$$

$$L^{A}_{i} A_i(x_1, \ldots, x_\alpha) = L^{A}_{i} L^{A_i}_{1} x_1 \circ \cdots \circ L^{A}_{i} L^{A_i}_{\alpha} x_\alpha,$$

$$L^{B}_{i} B_i(x_1, \ldots, x_\alpha) = L^{B}_{i} L^{B_i}_{1} x_1 \circ \cdots \circ L^{B}_{i} L^{B_i}_{\alpha} x_\alpha, \qquad (1 \leqslant \alpha \leqslant 3).$$

P r o o f. We define a binary operation $\circ$ of the set S as follows:

$$L^{A}_{12}(x, y) = L^{A}_{1} x \circ L^{A}_{2} y. \tag{6}$$

Since L^{A}_{12} is a quasigroup. the operation $\circ$ is a loop [1], mainly isotopic with L^{A}_{12}. We prove the operation $\circ$ is isotopic with all binary quasigroups derived from A, B, A_i, B_j.

Let be $L^{B}_{\alpha\beta} \overset{xy}{\leftrightarrow} L^{A}_{12}$, namely

$$L^{B}_{\alpha\beta}(\varphi_\alpha x, \varphi_\beta y) = L^{A}_{12}(\psi_1 x, \psi_2 y). \tag{7}$$

If we put in (7) $y = a$, we get

$$L^{B}_{\alpha} \varphi_\alpha x = L^{A}_{1} \psi_1 x, \tag{8}$$

and, if we put $x = a$, we get

$$L^{B}_{\beta} \varphi_\beta y = L^{A}_{2} \psi_2 y. \tag{9}$$

Further, we have

$$\begin{aligned} L^{B}_{\alpha\beta}(\varphi_\alpha x, \varphi_\beta y) &= L^{A}_{1} \psi_1 x \circ L^{A}_{2} \psi_2 y && \text{(from (6) and (7))} \\ &= L^{B}_{\alpha} \varphi_\alpha x \circ L^{B}_{\beta} \varphi_\beta y && \text{(from (8) and (9))}. \end{aligned}$$

Since φ_α, φ_β are bijections, it follows

$$L^{B}_{\alpha\beta}(x, y) = L^{B}_{\beta} x \circ L^{B}_{\beta} y. \tag{10}$$

Let $L^{P}_{\mu\nu}$ be any quasigroup derived from either A or B. According to lemma 1, we get $L^{P}_{\mu\nu} \Leftrightarrow L^{A}_{12}$, hence there is a chain of quasigroups $K_1, \ldots, K_s$ so that $L^{P}_{\mu\nu} = K_1$, $K_1 \leftrightarrow K_2 \leftrightarrow \cdots \leftrightarrow K_s$, $K_s = L^{A}_{12}$. By induction on s, we can prove

$$L^{P}_{\mu\nu}(x, y) = L^{P}_{\mu} x \circ L^{P}_{\nu} y.$$

With respect to lemma 3, for every quasigroup $L^{P_i}_{\mu\nu}$ derived from either A_i or B_i, it yields

(11) $$L^P_i L^{P_i}_{\mu\nu}(x, y) = L^P_i L^{P_i}_\mu x \circ L^P_i L^{P_i}_\nu y.$$

We prove the operation $\circ$ is associative. Substituting all variables of the law (1) except x_1, x_2, x_3, we get either

$$L^A_{12}(A_1(x_1, x_2), L^{A_2}_1 x_3) = L^B_{12}(B_1(x_1), L^{B_2}_{12}(x_2, x_3)),$$

or

$$L^A_{12}(A_1(x_1), L^{A_2}_{12}(x_2, x_3)) = L^B_{12}(B_1(x_1, x_2), L^{B_2}_1 x_3).$$

In both cases, using (10), and (11) and equalities of the form $L^A_i L^A_\alpha x_k = L^B_j L^B_\beta x_k$, $k = 1, \ldots, p$, obtained from (1) by substitution of all variables except x_k, we have $(x \circ y) \circ z = x \circ (y \circ z)$, that is, $(S, \circ)$ is a group.

Finally, we prove for every quasigroup $L^P_{\alpha\ldots\beta}$ $(P \in \{A, B\})$ of the length k holds

(12) $$L^P_{\alpha\ldots\beta}(x_1, \ldots, x_k) = L^P_\alpha x_1 \circ \cdots \circ L^P_\beta x_k.$$

Let us remark the following. If for quasigroups derived from A and B holds (12), then for quasigroups derived from A_i and B_j hold analogous equalities, too. If, e. g. $|A_i| = 3$, there is a quasigroup $L^B_{\mu\nu\lambda}$ such that

$$L^A_i A_i(x, y, z) = L^B_{\mu\nu\lambda}(\varphi_\mu x, \varphi_\nu y, \varphi_\lambda z) = L^B_\mu \varphi_\mu x \circ L^B_\nu \varphi_\nu y \circ L^B_\lambda \varphi_\lambda z.$$

Using $L^A_i L^{A_i}_1 = L^B_\mu \varphi_\mu$, $L^A_i L^{A_i}_2 = L^B_\nu \varphi_\nu$, $L^A_i L^{A_i}_3 = L^B_\lambda \varphi_\lambda$, we get

$$L^A_i A_i(x, y, z) = L^A_i L^{A_i}_1 x \circ L^A_i L^{A_i}_2 y \circ L^A_i L^{A_i}_3 z.$$

By induction on k, we prove the equality (12).

If $|L^P_{\alpha\beta}| = 2$, the equality (12) holds with respect to lemma 1.

If $|L^P_{\alpha\ldots\beta}| > 2$, there exists $L^Q_{\mu\ldots\nu}$, such that $|L^Q_{\mu\ldots\nu}| < |L^P_{\alpha\ldots\beta}|$, and

$$L^P_{\alpha\ldots\beta}(\varphi_\alpha x_1, \ldots, \varphi_k x_k) = L^Q_{\mu\ldots\nu}(\psi_{\mu\ldots\mu'}(x_1, \ldots, x_i), \ldots, \psi_{\nu\ldots\nu'}(x_j, \ldots x_k)).$$

By induction's hypothesis, operations $\psi_{\mu\ldots\mu'}$, $\psi_{\nu\ldots\nu'}$, $L^Q_{\mu\ldots\nu}$ are expressible by $\circ$, hence it yields

$$\begin{aligned} L^P_{\alpha\ldots\beta}(\varphi_\alpha x_1, \ldots, \varphi_\beta x_k) &= L^Q_\mu \psi_{\mu\ldots\mu'}(x_1, \ldots x_i) \circ \cdots \circ L^Q_\nu \psi_{\nu\ldots\nu'}(x_j, \ldots x_k) \\ &= L^Q_\mu \psi_\mu x_1 \circ \cdots \circ L^Q_\mu \psi_{\mu'} x_i \circ \cdots \circ L^Q_\nu \psi_\nu x_j \circ \cdots \circ L^Q_\nu \psi_{\nu'} x_k. \end{aligned}$$

Using the equalities $L^P_\alpha \varphi_\alpha = L^Q_\mu \psi_\mu, \ldots, L^P_\beta \varphi_\beta = L^Q_\nu \varphi_{\nu'}$, we get (12). The theorem is proved.

Now, we suppose for the relation of connectness of the set $T = \{u_1, \ldots, u_m, v_1, \ldots, v_n\}$ there is $r\,(r > 1)$ equivalence classes $C_i = \{u_{\alpha_i}, \ldots, u_{\beta_i}, v_{\gamma_i}, \ldots, v_{\delta_i}\}$, $i = 1, \ldots, r$. In that case the law (1) has the form

(13) $$A(u_{\alpha_1}, \ldots, u_{\beta_1}, \ldots, u_{\alpha_r}, \ldots, u_{\beta_r}) = B(v_{\gamma_1}, \ldots, v_{\delta_1}, \ldots, v_{\gamma_r}, \ldots, v_{\delta_r}).$$

Let $\pi: S^r \to S$ be the operation of the set S defined by

$$\pi(L^A_{\alpha_1} x_1, \ldots, L^A_{\alpha_r} x_r) = L^A_{\alpha_1 \ldots \alpha_r}(x_1, \ldots, x_r).$$

The operation π is a loop, mainly isotopic to $L^A_{\alpha_1 \ldots \alpha_r}$.

Lemma 4. *For any two derived quasigroups $L^A_{\mu_1 \ldots \mu_r}$ and $L^B_{\nu_1 \ldots \nu_r}$ with* $u_{\mu_i} \in C_i$, $v_{\nu_i} \in C_i$, $i = 1, \ldots, r$,

$$L^A_{\mu_1 \ldots \mu_r}(x_1, \ldots, x_r) = \pi(L^A_{\mu_1} x_1, \ldots, L^A_{\mu_r} x_r) \tag{14}$$

and

$$L^B_{\nu_1 \ldots \nu_r}(x_1, \ldots, x_r) = \pi(L^B_{\nu_1} x_1, \ldots, L^B_{\nu_r} x_r). \tag{15}$$

Proof. Since all terms of the set C_i $(i = 1, \ldots, r)$ are connected, for any two derived quasigroups $L^P_{\mu_1 \ldots \mu_r}$ and $L^Q_{\nu_1 \ldots \nu_r}$ there is a finite sequence of derived quasigroups $K_1, \ldots, K_n$, such that

$$L^P_{\mu_1 \ldots \mu_r} = K_1 \leftrightarrow K_2 \cdots \leftrightarrow K_n = L^Q_{\nu_1 \ldots \nu_r},$$

that is

$$L^P_{\mu_1 \ldots \mu_r} \Leftrightarrow L^Q_{\nu_1 \ldots \nu_r} (P, Q \in \{A, B\}).$$

If, for example, $L^A_{\mu_1 \ldots \mu_r} \leftrightarrow L^B_{\nu_1 \ldots \nu_r}$ and

$$L^A_{\mu_1 \ldots \mu_r}(x_1, \ldots, x_r) = \pi(L^A_{\mu_1}(x_1, \ldots, L^A_{\mu_r} x_r),$$

then

$$L^B_{\nu_1 \ldots \nu_r}(x_1, \ldots, x_r) = \pi(L^B_{\nu_1} x_1, \ldots, L^B_{\nu_r} x_r). \tag{16}$$

Indeed, from the law (13) it follows

$$L^A_{\mu_1 \ldots \mu_r}(\varphi_1, \ldots, \varphi_r x_r) = L^B_{\nu_1 \ldots \nu_r}(\psi_1 x_1, \ldots, \psi_r x_r)$$

and

$$L^A_{\mu_i} \varphi_i x_i = L^B_{\nu_i} \psi_i x_i \qquad i = 1, \ldots, r,$$

where φ_i and ψ_i are certain bijections of the set S, and so we have

$$L^B_{\nu_1 \ldots \nu_i}(\psi_1 x_1, \ldots, \psi_r x_r) = \pi(L^A_{\mu_1} \varphi_1 x_1, \ldots, L^A_{\mu_r} \varphi_r x_r) = \pi(L^B_{\nu_1} \psi_1 x_1, \ldots, L^B_{\nu_r} \psi_r x_r).$$

Since ψ_i are bijections, we get (16).

Analogously we can prove if $L^P_{\mu_1 \ldots \mu_r}$ is expressible by π, and

$$L^P_{\mu_1 \ldots \mu_r} \Leftrightarrow L^Q_{\nu_1 \ldots \nu_r}, \quad \text{then} \quad L^Q_{\nu_1 \ldots \nu_r}$$

is expressible by π, too

According to definition of π, it yields (14) and (15).

Lemma 5. *For quasigroups A and of the law* (13) *it holds*

$$A \Rightarrow L^A_{\alpha_1 \ldots \alpha_r} \quad \text{and} \quad B \Rightarrow L^A_{\alpha_1 \ldots \alpha_r}.$$

Proof. By substitution of all variables, occuring in (13), except the variables occuring in the terms of C_i ($i \in \{1, \ldots, r\}$), we get the law

$$L^A_{\alpha_i \ldots \beta_i}(u_{\alpha_i}, \ldots, u_{\beta_i}) = L^B_{\gamma_i \ldots \delta_i}(v_{\gamma_i}, \ldots, v_{\delta_i}),$$

in which all terms $u_{\alpha_i}, \ldots, u_{\beta_i}, v_{\gamma_i}, \ldots, v_{\delta_i}$ are connected.

By lemma 2 we have

$$L^A_{\alpha_i \ldots \beta_i} \Rightarrow L^A_{\alpha_i},$$

and

$$L^B_{\gamma_i \ldots \delta_i} \Rightarrow L^A_{\alpha_i}.$$

Hence,

$$A = L^A_{\alpha_1 \ldots \beta_1 \ldots \alpha_r \ldots \beta_r} \Rightarrow L^A_{\alpha_1 \ldots \alpha_r}$$

and

$$B = L^B_{\gamma_1 \ldots \delta_1 \ldots \gamma_r \ldots \delta_r} \Rightarrow L^A_{\gamma_1 \ldots \gamma_r}.$$

The lemma is proved.

Theorem 2. *Let A, B, A_i, B_j are quasigroups satisfying the law* (13). *There exists a loop (S, π) of the length r, and for every class C_i with card $C_i > 2$, there exists a group $(S, \circ_i)$, $i = 1, \ldots, r$, so that*

$$\begin{aligned}(17)\quad & A(x_{\alpha_1}, \ldots, x_{\beta_1}, \ldots, x_{\alpha_r}, \ldots, x_{\beta_r}) \\ &\quad = \pi(L^A_{\alpha_1} x_{\alpha_1} \circ_1 \cdots \circ_1 L^A_{\beta_1} x_{\beta_1}, \ldots, L^A_{\alpha_r} x_{\alpha_r} \circ_r \cdots \circ_r L^A_{\beta_r} x_{\beta_r}). \\ & B(x_{\gamma_1}, \ldots, x_{\delta_1}, \ldots, x_{\gamma_r}, \ldots, x_{\delta_r}) \\ &\quad = \pi(L^B_{\gamma_1} x_{\gamma_1} \circ_1 \cdots \circ_1 L^B_{\delta_1} x_{\delta_1}, \ldots, L^B_{\gamma_r} x_{\gamma_r} \circ_r \cdots \circ_r L^B_{\delta_r} x_{\delta_r}).\end{aligned}$$

If $u_\alpha \in C_i$, and $u_\alpha = A_\alpha(x_1, \ldots, x_k)$, where $k = 2$ or $k = 3$, then

$$(18)\qquad L^A_\alpha A_\alpha(x_1, \ldots, x_k) = L^A_\alpha L^{A_\alpha}_1 x_1 \circ_i \cdots \circ_i L^A_\alpha L^{A_\alpha}_k x_k.$$

If $v_\beta \in C_i$, and $v_\beta = B_\beta(x_1, \ldots, x_k)$, where $k = 2$ or $k = 3$, then

$$(19)\qquad L^B_\beta B_\beta(x_1, \ldots, x_k) = L^B_\beta L^{B_\beta}_1 x_1 \circ_i \cdots \circ_i L^B_\beta L^{B_\beta}_k x_k.$$

Proof. By theorem 1 there exist groups $(S, \circ_i)$ so that the equalities (18) and (19) hold.

Let $\mu_i, \ldots, \nu_i$ be a subsequence of the sequence $\alpha_i, \ldots, \beta_i$, and $\lambda_i, \ldots, \rho_i$ a subsequence of the sequence $\gamma_i, \ldots, \delta_i$, $i = 1, \ldots, r$.

If $L^A_{\mu_1 \ldots \nu_1 \ldots \mu_r \ldots \nu_r} \to L^B_{\lambda_1 \ldots \rho_1 \ldots \lambda_r \ldots \rho_r}$, and if

$$L^B_{\lambda_1 \ldots \rho_1 \ldots \lambda_r \ldots \rho_r}(x_{\lambda_1}, \ldots, x_{\rho_r}) = \pi(L^B_{\lambda_1} x_{\lambda_1} \circ_1 \cdots \circ_1 L^B_{\rho_1} x_{\rho_1}, \ldots, L^B_{\lambda_r} x_{\lambda_r} \circ_r \cdots \circ_r L^B_{\rho_r} x_{\rho_r}),$$

then, by definition of $\to$, it is easy to see that

$$L^A_{\mu_1 \ldots \nu_1}(x_{\mu_1}, \ldots, x_{\nu_r}) = \pi(L^A_{\mu_1} x_{\mu_1} \circ_1 \cdots \circ_1 L^A_{\nu_1} x_{\nu_1}, \ldots, L^A_{\mu_1} x_{\mu_r} \circ_r \cdots \circ_r L^A_{\nu_r} x_{\nu_r}).$$

Using such equalities, by lemma 5 and by definition of π we prove the equalities (17). This completes the proof.

Theorem 3. *General solution of the functional equation* (13) *on unknown quasigroups* A, B, A_α, B_β *is given by*

$$(20)\quad \begin{cases} A(x_{\alpha_1}, \ldots, x_{\beta_1}, \ldots, x_{\alpha_r}, \ldots, x_{\beta_r}) \\ \qquad = \pi(\varphi_{\alpha_1} x_{\alpha_1} \circ_1 \cdots \circ_1 \varphi_{\beta_1} x_{\beta_1}, \ldots, \varphi_{\alpha_r} x_{\alpha_r} \circ_r \cdots \circ_r \varphi_{\beta_r} x_{\beta_r}), \\ B(x_{\gamma_1}, \ldots, x_{\delta_1}, \ldots, x_{\gamma_r}, \ldots, x_{\delta_r}) \\ \qquad = \pi(\psi_{\gamma_1} x_{\gamma_1} \circ_1 \cdots \circ_1 \psi_{\delta_1} x_{\delta_1}, \ldots, \psi_{\gamma_r} x_{\gamma_r} \circ_r \cdots \circ_r \psi_{\delta_r} x_{\delta_r}), \\ \varphi_\alpha A_\alpha(x_j, \ldots, x_{j+k}) = \varphi_\alpha \varepsilon_j x_j \circ_i \cdots \circ_i \varphi_\alpha \varepsilon_{j+k} x_{j+k}, \\ \qquad (u_\alpha \in C_i,\ u_\alpha = A_\alpha(x_j, \ldots, x_{j+k})), \qquad k \in \{0, 1, 2\}, \\ \psi_\beta B_\beta(x_j, \ldots, x_{j+k}) = \psi_\beta \eta_j x_j \circ_i \cdots \circ_i \psi_\beta \eta_{j+k} x_{j+k}, \\ \qquad (v_\beta \in C_i,\ v_\beta = B_\beta(x_j, \ldots, x_{j+k})), \qquad k \in \{0, 1, 2\}, \end{cases}$$

where (S, π) *is an arbitrary loop of the length* r, $(S, \circ_i)$ *are arbitrary groups, and* φ_α, ψ_γ, ε_i, η_i *are bijections of the set* S *satisfying the following conditions:*

For every $i \in \{1, \ldots, p\}$, *if the variable* x_i *occurs in the terms* u_α *and* v_γ, *then*

$$(21)\qquad \varphi_\alpha \varepsilon_i = \psi_\gamma \eta_i.$$

Proof. By theorem 2, for the law (13) there exist a loop (S, π), groups $(S, \circ_i)$ and bijections φ_α, ψ_γ, ε_i, η_i so that the equalities (20) and (21) hold.

Conversely, let for quasigroups A, B, A_α, B_β the equalities (20) and (21) hold. It is easy to verify in this case A, B, A_α, B_β satisfy the law (13).

We give an example.

Let us consider the functional equation

$$(22)\qquad A(A_1(x_1, x_2), x_3, x_4, x_5, A_5(x_6, x_7, x_8), A_6(x_9, x_{10})) = B(x_1, B_2(x_2, x_3), x_4, B_4(x_5, x_6), x_7, B_6(x_8, x_9), x_{10}).$$

Here is $r=3$, and

$$C_1 = \{u_1, u_2, v_1, v_2\},\ C_2 = \{u_3, v_3\},\ C_3 = \{u_4, u_5, u_6, v_4, v_5, v_6, v_7\}.$$

General solution of (22) is given by

$$A(y_1, y_2, y_3, y_4, y_5, y_6) = \pi(\varphi_1 y_1 \circ_1 \varphi_2 y_2, \varphi_3 y_3, \varphi_4 y_4 \circ_3 \varphi_5 y_5 \circ_3 \varphi_6 y_6),$$
$$B(y_1, \ldots, y_7) = \pi(\psi_1 y_1 \circ_1 \psi_2 y_2, \psi_3 y_3, \psi_4 y_4 \circ_3 \psi_5 y_5 \circ_3 \psi_6 y_6 \circ_3 \psi_7 y_7),$$
$$\varphi_1 A_1(x_1, x_2) = \varphi_1 \varepsilon_1 x_1 \circ_1 \varphi_1 \varepsilon_2 x_2,$$
$$\varphi_5 A_5(x_6, x_7, x_8) = \varphi_5 \varepsilon_6 x_6 \circ_3 \varphi_5 \varepsilon_7 x_7 \circ_3 \varphi_5 \varepsilon_8 x_8,$$
$$\varphi_6 A_6(x_9, x_{10}) = \varphi_6 \varepsilon_9 x_9 \circ_3 \varphi_6 \varepsilon_{10} x_{10},$$
$$\psi_2 B_2(x_2, x_3) = \psi_2 \eta_2 x_2 \circ_1 \psi_2 \eta_3 x_3,$$
$$\psi_4 B_4(x_5, x_6) = \psi_4 \eta_5 x_5 \circ_3 \psi_4 \eta_6 x_6,$$
$$\psi_6 B_6(x_8, x_9) = \psi_6 \eta_8 x_8 \circ_3 \psi_6 \eta_9 x_9,$$

where (S, π) is an arbitrary loop. $(S, \circ_1)$ and $(S, \circ_3)$ are arbitrary groups and $\varphi_\alpha, \psi_\mu, \varepsilon_i, \eta_j$ are bijections of the set S so that

$$\varphi_1\varepsilon_1=\psi_1,\ \varphi_1\varepsilon_2=\psi_2\eta_2,\ \varphi_2=\psi_2\eta_3,\ \varphi_3=\psi_3,\ \varphi_4=\psi_4\eta_5,\ \varphi_5\varepsilon_6=\psi_4\eta_6,$$

$$\varphi_5\varepsilon_7=\psi_5,\ \varphi_5\varepsilon_8=\psi_6\eta_8,\ \varphi_6\varepsilon_9=\psi_6\eta_9,\ \varphi_6\varepsilon_{10}=\psi_7.$$

Now, let us consider an arbitrary law of the I kind, in the form (1), on quasigroups. Such is, e. g. the law

$$A(x_1, A_2(x_2, x_3, x_4, x_5), A_3(x_6, x_7, x_8), x_9, x_{10})$$
$$=B(B_1(x_1, x_2, x_3), x_4, x_5, x_6, B_5(x_7, x_8), B_6(x_9, x_{10})).$$

We prove, that such a law can be transformed into a law satisfying conditions (i) and (ii), substituting some quasigroups by GD-groupoids [3].

Let for two terms u_i and v_j there are more than one variable occuring in both of them, e. g. $u_i=A_i(x_1, \ldots, x_k, \ldots, x_{k+s})$ and $v_j=B_j(x_k, \ldots, x_{k+s}, \ldots, x_{k+m})$. Then we define GD-groupoids

$$\bar{A}_i(x_1, \ldots, x_{k-1}, (x_k, \ldots, x_{k+s})) \overset{\text{def}}{=} A_i(x_1, \ldots, x_{k+s}),$$

$$\bar{B}_j((x_k, \ldots, x_{k+s}), x_{k+s+1}, \ldots, x_{k+m}) \overset{\text{def}}{=} B_j(x_k, \ldots, x_{k+m}).$$

So we get a law on GD-groupoids satisfying the condition (i).

If in such obtained law occurs a sequence of terms $u_k, \ldots, u_{k+s}$ (or $v_k, \ldots, v_{k+s}$) in each of them occurs exactly one variable $x_k, \ldots, x_{k+s}$, respectively, and if these variables occur in the same term v_j (or u_j) on the opposite side of law, we define GD-groupoids

$$\bar{A}(\ldots, (u_k, \ldots, u_{k+s}), \ldots) \overset{\text{def}}{=} A(\ldots, u_k, \ldots, u_{k+s}, \ldots),$$

and

$$\bar{B}_j(\ldots, (x_k, \ldots, k_{k+s}), \ldots) \overset{\text{def}}{=} B_j(\ldots, x_k, \ldots, x_{k+s}, \ldots),$$

or

$$\bar{B}(\ldots, (v_k, \ldots, v_{k+s}), \ldots) \overset{\text{def}}{=} B(\ldots, v_k, \ldots, v_{k+s}, \ldots),$$

and

$$\bar{A}_j(\ldots, (x_k, \ldots, x_{k+s}), \ldots) \overset{\text{def}}{=} A_j(\ldots, x_k, \ldots, x_{k+s}, \ldots).$$

In such a way we obtain the law (13) on GD-groupoids satisfying the conditions (i) and (ii).

We note that for so obtained law holds the condition:

(iii) If a term u_i (or v_j) is not a variable, then L_i^A (or L_j^B) is a bijection.

We prove in this case that the operation π is well defined.

If all functions $L_{\alpha_1}^A, \ldots, L_{\alpha_r}^A$ are bijections, the operation π is well defined.

If, for some i, function $L_{\alpha_i}^A$ is not a bijection, then u_{α_i} is a variable, but v_{γ_i} is not a variable, namely $L_{\gamma_i}^B$ is a bijection.

We assume, functions $L_{\alpha_1}^A, \ldots, L_{\alpha_s}^A$ $(1 \leqslant s < r)$ are bijections and $L_{\alpha_{s+1}}^A, \ldots, L_{\alpha_r}^A$ are surjections. The sequence $C_1, \ldots, C_r$ can be reordered so that it holds.

Let $z_1, \ldots, z_r$ be arbitrary choiced elements of the set S. Since $L_{\alpha_1}^A, \ldots, L_{\alpha_s}^A$ are bijections, for every $z_i\,(i=1, \ldots, s)$ there is exactly one x_i so that $L_{\alpha_1}^A x_1=z_1, \ldots, L_{\alpha_s}^A x_s=z_s$.

Since functions $L^A_{\alpha_{s+1}}, \ldots, L^A_{\alpha_r}$ are surjections, for every z_i $(i=s+1, \ldots, r)$ there is some x_i so that $L^A_{\alpha_{s+1}} x_{s+1} = z_{s+1}, \ldots, L^A_{\alpha_r} x_r = z_r$.

We prove that from the equalities

$$L^A_{\alpha_j} x'_j = L^A_{\alpha_j} x''_j \qquad (j=s+1, \ldots, r) \tag{23}$$

follows the equality

$$L^A_{\alpha_1 \ldots \alpha_r}(x_1, \ldots, x_s, x'_{s+1}, \ldots, x'_r) = L^A_{\alpha_1 \ldots \alpha_r}(x_1, \ldots, x_s, x''_{s+1}, \ldots, x''_r).$$

Substituting all variables except x_j $(j=s+1, \ldots, r)$ by $a \in S$, from (13) we get

$$L^A_{\alpha_j} x_j = L^B_{\gamma_j} \psi_j x_j. \tag{24}$$

From (23) and (24) it follows

$$L^B_{\gamma_j} \psi_j x'_j = L^B_{\gamma_j} \psi_j x''_j \qquad (j=s+1, \ldots, r).$$

As $L^B_{\gamma_j}$ is bijection, we have

$$\psi_j x'_j = \psi_j x''_j, \qquad (j=s+1, \ldots, r). \tag{25}$$

Substituting all variables, except one in each of sets C_i, $i=1, \ldots, r$, from (13) we get

$$L^A_{\alpha_1 \ldots \alpha_r}(\varphi_1 y_1, \ldots, \varphi_s y_s, y_{s+1}, \ldots, y_r)$$
$$= L^B_{\gamma_1 \ldots \gamma_r}(\psi_1 y_1, \ldots, \psi_s y_s, \psi_{s+1} y_{s+1}, \ldots, \psi_r y_r).$$

Since φ_i are surjections, we can choose $y_1, \ldots, y_s$ so that

$$\varphi_1 y_1 = x_1, \ldots, \varphi_s y_s = x_s.$$

From (25) we have

$$L^B_{\gamma_1 \ldots \gamma_r}(\psi_1 y_1, \ldots, \psi_s y_s, \psi_{s+1} x'_{s+1}, \ldots, \psi_r x'_r)$$
$$= L^B_{\gamma_1 \ldots \gamma_r}(\psi_1 y_1, \ldots, \psi_s y_s, \psi_{s+1} x''_{s+1}, \ldots, \psi_r x''_r)$$

and therefore, by the choice of $y_1, \ldots, y_s$, it follows

$$L^A_{\alpha_1 \ldots \alpha_r}(x_1, \ldots, x_s, x'_{s+1}, \ldots, x'_r) = L^A_{\alpha_1 \ldots \alpha_r}(x_1, \ldots, x_s, x''_{s+1}, \ldots, x''_r),$$

that is

$$\pi(L^A_{\alpha_1} x_1, \ldots, L^A_{\alpha_s} x_s, L^A_{\alpha_{s+1}} x'_{s+1}, \ldots, L^A_{\alpha_r} x'_r)$$
$$= \pi(L^A_{\alpha_1} x_1, \ldots, L^A_{\alpha_s} x_s, L^A_{\alpha_{s+1}} x''_{s+1}, \ldots, L^A_{\alpha_r} x''_r),$$

hence $\pi(z_1, \ldots, z_r)$ is well defined. It is easy to see that the operation π is a loop.

Analogously we can prove that the groups $\circ_i$ are well defined. Hence, we can apply theorem 3, too.

We give an example.

Let

$$A\left(x,\ A_2(y,\ z),\ A_3(u,\ v)\right)=B\left(B_1(x,\ y,\ z,\ u),\ v\right)$$

be functional equation by unknown quasigroups A, B, A_2, A_3, B_1 of a set S. According to theorem 3, general solution of this equation is given by

$$A(x,\ y,\ z)=M(x,\ y)\circ\alpha z,$$

$$A_2(x,\ y)=P(x,\ y),$$

$$\alpha A_3(x,\ y)=\beta x\circ\gamma y,$$

$$\delta B_1(x,\ y,\ z,\ u)=M\left(x,\ P(y,\ z)\right)\circ\beta u,$$

$$B(x,\ y)=\delta x\circ\gamma y,$$

where M, P are arbitrary quasigroups, α, β, γ, δ are arbitrary bijections, and $\circ$ is an arbitrary group of the set S.

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