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APPLICATIONS OF THE SIDE CONDITION METHOD

Abstract. Stevo Todorčević introduced a general approach to constructing proper forcing notions, so called the side condition method. This method is powerful and is widely used. This article summarizes some techniques of the side condition method, specifically, forcing notions for the Open Coloring Axiom, the P-ideal Dichotomy, the failures of $\square(\kappa)$ and Weak Club Guessing.

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Contents

1. Introduction

Stevo Todorčević introduced a general approach to constructing proper forcing notions, so called the side condition method (see the discussion on [36, p. 212]). The side condition method deals with forcing notions equipped with models as side conditions. Todorčević's first motivation for this method is to add an uncountable set with some property to some given structure by a proper forcing notion. This method enables us to obtain lots of consequences from the Proper Forcing Axiom. The first application of this method in the literature is the failure of Jensen's $\square_\kappa$ for every uncountable regular cardinal κ in Todorčević [36].

The side condition method is also applied for other purposes. Aspero and Mota [3] introduced a forcing iteration of proper forcing notions with finite support which is equipped with systems of models. They showed, by their forcing iteration, a consistency result of a consequence of the Proper Forcing Axiom with the size of the continuum greater than $\aleph_2$. Neeman [31] introduced a forcing iteration with models of two types. He gave a new proof of the consistency of the Proper Forcing Axiom by his finite support forcing iteration, under the existence of a supercompact cardinal.

This article summarizes some techniques of the side condition method. In Section 2, the most basic and simple forcing notion of the side condition method is investigated. This is called the $\in$-collapse. The $\in$-collapse has properties stronger than the properness: the strong properness and Y-properness. This observation presents some possibilities of the side condition method. For example, a strongly proper forcing notion preserves a Suslin tree, so some Asperó and Mota's forcing iterations can preserve a Suslin tree (see [51]). Chodounský and Zapletal [8, §6] constructed a Y-proper forcing iteration by Neeman's method which forces the forcing axiom for Y-proper forcing notions. Section 3 gives some preservation theorems of strongly proper forcing notions, and Section 4 gives some preservation theorems of Y-proper forcing notions.

Section 5 gives some applications of the side condition method, specifically, the Open coloring Axiom (Section 5.1), the P-ideal Dichotomy (Section 5.2), the failure of $\square(\kappa)$ (Section 5.3), and the failure of Weak Club Guessing (Section 5.4). Each of introductions of these topics is in the begining of each subsection. In each subsection, it will be proved not only that a forcing notion defined in each subsection is proper but also that it has an extra property which the $\in$-collapse has. Each subsection is self-contained.

Our notation in this article is fairly standard. We refer readers to [16, 19, 20] for standard notations and forcing theory. Many theorems on the side condition method in this article can be found in [30, 42, 43]. For a cardinal κ, $H(\kappa)$ denotes the set of all sets of hereditarily cardinality less than κ, $[X]^\kappa$ denotes the set of all subsets of a set X of cardinality κ, and $[X]^{<\kappa}$ denotes the set of all subsets of a set X of cardinality less than κ. $H(\kappa)$ is always considered as the structure equipped with the membership relation $\in$ and a fixed well-order.

2. The $\in$-collapse

The side condition method deals with forcing notions which are equipped with elementary submodels as side conditions. In this section, we will introduce necessary basics of elementary submodels of countable size, and will see the properties of the most basic forcing notion equipped with countable elementary submodels.

Definition 2.1 (E.g. [16, Ch. 12])**.** Let κ be an uncountable regular cardinal, N a subset of $H(\kappa)$. N is called an elementary submodel of $H(\kappa)$ if and only if, for any formula $\varphi(v_1, \dots, v_n)$ with free variables $v_1, \dots, v_n$ and any $a_1, \dots, a_n$ in N, $\varphi(a_1, \dots, a_n)$ holds in N if and only if $\varphi(a_1, \dots, a_n)$ holds in $H(\kappa)$.

Proposition 2.2 (E.g. [14, §4.2], [18, Ch. 24])**.** *Let κ be an uncountable regular cardinal, and N a countable elementary submodel of $H(\kappa)$.*

(1) *For any $s \in H(\kappa)$, there exists a countable elementary submodel of $H(\kappa)$ which contains s as a member.*
(2) *For any regular cardinal λ and any elementary submodel M of $H(\lambda)$, if $H(\kappa)$ belongs to M, then $M \cap H(\kappa)$ is an elementary submodel of $H(\kappa)$.*
(3) *Any finite subset of N belongs to N.*
(4) *Any countable set in N is a subset of N.*
(5) *$\omega_1 \cap N$ is a countable limit ordinal.*

The following is the most basic forcing notion equipped with countable elementary submodels.

Definition 2.3 (Todorčević, e.g. [36, p. 212] and [43, §7.1])**.** Let κ be an uncountable regular cardinal.

(1) A finite set p of countable elementary submodels of $H(\kappa)$ is called an *$\in$-chain* if and only if p is totally ordered by the membership relation $\in$.
(2) *The $\in$-collapse $\mathbb{P}_\in$ of $H(\kappa)$* is the partial order which consists of all finite $\in$-chains of countable elementary submodels of $H(\kappa)$, and, for all conditions p and q in $\mathbb{P}_\in$, $q \leqslant_{\mathbb{P}_\in} p$ if and only if $q \supseteq p$.

$\mathbb{P}_\in$ adds a chain of countable elementary submodels of $H(\kappa)$ such that the union of all coordinates of its chain is the whole $H(\kappa)$. We will show that $\mathbb{P}_\in$ does not collapse $\aleph_1$, so $\mathbb{P}_\in$ collapses κ to $\aleph_1$.

Shelah introduced the notion of the properness of forcing notions [33, 34]. Let $\mathbb{P}$ be a forcing notion, λ a regular cardinal with $\mathcal{P}(\mathbb{P}) \in H(\lambda)$, N a countable elementary submodel of $H(\lambda)$, and p a condition of $\mathbb{P}$. p is called an *$(N, \mathbb{P})$-generic* provided that, for any dense subset D of $\mathbb{P}$, if $D \in N$, then $D \cap N$ is predense below p in $\mathbb{P}$. A forcing notion $\mathbb{P}$ is *proper* if and only if, for any regular cardinal λ with $\mathcal{P}(\mathbb{P}) \in H(\lambda)$, there is a closed unbounded set of countable elementary submodels N of $H(\lambda)$ with $\mathbb{P} \in N$ such that every condition of $\mathbb{P}$ in N has an extension which is $(N, \mathbb{P})$-generic. Typical proper forcing notions are ccc forcing notions and σ-closed forcing notions. The Proper Forcing Axiom (PFA) is the assertion that, for any proper forcing $\mathbb{P}$ and $\aleph_1$ many dense subsets $\{D_\alpha : \alpha \in \omega_1\}$ of $\mathbb{P}$, there exists a filter G of $\mathbb{P}$ which meets all D_α. Todorčević proved that $\mathbb{P}_\in$ is proper. As seen below, $\mathbb{P}_\in$ has more stronger properties. Baumgartner [5, §3] proved that it is consistent that PFA holds if it is consistent that the supercompact cardinal exists.

Definition 2.4 (Shelah [34, Ch. IX, 2.6 Definition])**.** (1) Let $\mathbb{P}$ be a forcing notion, λ a regular cardinal with $\mathcal{P}(\mathbb{P}) \in H(\lambda)$, N a countable elementary submodel of $H(\lambda)$ with $\mathbb{P} \in N$, and p a condition of $\mathbb{P}$. p is called *strong $(N, \mathbb{P})$-generic in the sense of Shelah* if and only if, for any countable sequence $\langle D_n : n \in \omega \rangle$ with $D_n \subseteq \mathbb{P} \cap N$ dense in $\mathbb{P} \cap N$, there exists $q \leqslant_{\mathbb{P}} p$ which is generic for $\{D_n : n \in \omega\}$, that is, for all $n \in \omega, D_n$ is predense below q in $\mathbb{P}$.
(2) A forcing notion $\mathbb{P}$ is *strongly proper* provided that, for any regular cardinal λ with $\mathcal{P}(\mathbb{P}) \in H(\lambda)$, there is a closed unbounded set of countable elementary submodels N of $H(\lambda)$ with $\mathbb{P} \in N$ such that every condition of $\mathbb{P}$ in N has an extension which is strong $(N, \mathbb{P})$-generic in the sense of Shelah.

As seen below, Todorčević's proof that the $\in$-collapse $\mathbb{P}_\in$ is proper in [36] and [43] actually shows that it is in fact strongly proper.

Definition 2.5 (Chodounský and Zapletal [8, §1])**.** For a forcing notion $\mathbb{P}$, RO($\mathbb{P}$) denotes the regular open algebra of $\mathbb{P}$ (see e.g. [19, Ch. II 3.3. Lemma], [20, Lemma III.4.8]).

(1) Let $\mathbb{P}$ be a forcing notion, λ a regular cardinal with $\mathcal{P}(\mathbb{P}) \in H(\lambda)$, N a countable elementary submodel of $H(\lambda)$ with $\mathbb{P} \in N$, and p a condition of $\mathbb{P}$. p is called *$(N, \mathbb{P})$-Y-generic* if and only if, for any $r \leqslant_{\mathbb{P}} p$, there exists a filter $F \in N$ on RO($\mathbb{P}$) such that the set $\{s \in \mathrm{RO}(\mathbb{P}) \cap N : r \leqslant_{\mathrm{RO}(\mathbb{P})} s\}$ is included in the set F as a subset.

(2) A forcing notion $\mathbb{P}$ satisfies *Y-proper* condition provided that, for any regular cardinal λ with $\mathcal{P}(\mathbb{P}) \in H(\lambda)$, there is a closed unbounded set of countable elementary submodels N of $H(\lambda)$ with $\mathbb{P} \in N$ such that every condition of $\mathbb{P}$ in N has an extension which is $(N, \mathbb{P})$-generic and $(N, \mathbb{P})$-Y-generic.

A forcing notion $\mathbb{P}$ is called *Y-cc* provided that, for any regular cardinal λ with $\mathcal{P}(\mathbb{P}) \in H(\lambda)$, and any countable elementary submodel N of $H(\lambda)$ with $\mathbb{P} \in N$, every condition of $\mathbb{P}$ is $(N, \mathbb{P})$-Y-generic. Chodounský and Zapletal proved that a Y-cc forcing notion is ccc. It has not been known yet whether a Y-proper ccc forcing notion is Y-cc [8, Question 4.13].

Lemma 2.6. *Let κ be an uncountable regular cardinal, and $\mathbb{P}_\in$ the $\in$-collapse of $H(\kappa)$. $\mathbb{P}_\in$ is strongly proper and is Y-proper.*

Proof. Let λ be a regular cardinal with $\mathcal{P}(\mathbb{P}_\in) \in H(\lambda)$, N a countable elementary submodel of $H(\lambda)$ with $\{\mathbb{P}_\in, H(\kappa)\} \in N$, and $p \in \mathbb{P}_\in \cap N$. Define $p^+ := p \cup \{N \cap H(\kappa)\}$. Then $p^+ \in \mathbb{P}_\in$ and $p^+ \supseteq p$, hence $p^+ \leqslant_{\mathbb{P}_\in} p$.

To show that $\mathbb{P}_\in$ is strongly proper, let us show that p^+ is strong $(N, \mathbb{P}_\in)$-generic in the sense of Mitchell [24, Definition 2.3], that is, for any dense subset D of $\mathbb{P}_\in \cap N$, D is predense below p^+ in $\mathbb{P}_\in$. Let D be a dense subset of $\mathbb{P}_\in \cap N$ and $q \leqslant_{\mathbb{P}_\in} p^+$. Then $q \cap N$ is in $\mathbb{P}_\in \cap N$, so there exists $r \in D$ such that $r \leqslant_{\mathbb{P}_\in} q \cap N$. Then, since $D \subseteq N$ and r is a finite subset of $H(\kappa)$, r belongs to $N \cap H(\kappa)$. Since $N \cap H(\kappa) \in q$, $r \cup q$ is an $\in$-chain, hence is in $\mathbb{P}_\in$, and so is an extension of r and q in $\mathbb{P}_\in$.

To show that $\mathbb{P}_\in$ is Y-proper, let us show that p^+ is $(N, \mathbb{P})$-Y-generic. This proof is a prototype of the proof of the Y-genericity. To show this, we introduce the following notion. For a finite subset a of $H(\kappa)$, a subset $\mathcal{A}$ of $\mathbb{P}_\in$ is called *a-large* if and only if, for any set b in $H(\kappa)$, there are $r \in \mathcal{A}$ and $M \in r$ such that $r \cap M = a$ and $b \in M$.

We claim that, for any condition p of $\mathbb{P}_\in$, the set $\{\bigvee \mathcal{A} : \mathcal{A} \subseteq \mathbb{P}_\in \text{ is } p\text{-large}\}$ is a centered subset of $\mathrm{RO}(\mathbb{P}_\in)$. Let $p \in \mathbb{P}_\in$, $n \in \omega$, and $\mathcal{A}_i$, $i \in n$, p-large subsets of $\mathbb{P}_\in$. We will prove that the Boolean calculation $\bigwedge_{i \in n}(\bigvee \mathcal{A}_i)$ in $\mathrm{RO}(\mathbb{P}_\in)$ is non-zero. It suffices to prove that there are $p_i \in \mathcal{A}_i$, $i \in n$, such that $\{p_i : i \in n\}$ has a common extension in $\mathbb{P}_\in$. Since each $\mathcal{A}_i$ is p-large, by induction on $i \in n$, we can find $p_i \in \mathcal{A}_i$ and $M_i \in p_i$ such that $p_i \cap M_i = p$ and $\{p_j : j < i\} \in M_i$. Then $\bigcup_{i \in n} p_i$ is a finite $\in$-chain, so is in $\mathbb{P}_\in$, and is a common extension of all p_i.

Let $r \in \mathbb{P}_\in$ be such that $r \leqslant_{\mathbb{P}_\in} p^+$, and define F as the filter on $\mathrm{RO}(\mathbb{P}_\in)$ that is generated by the set $\{\bigvee \mathcal{A} : \mathcal{A} \subseteq \mathbb{P}_\in \text{ is } r \cap N\text{-large}\}$. Since $r \cap N$ belongs to N, F also belongs to N. We will prove that, for any $s \in \mathrm{RO}(\mathbb{P}_\in) \cap N$, if $r \leqslant_{\mathrm{RO}(\mathbb{P}_\in)} s$, then $s \in F$. Let $s \in \mathrm{RO}(\mathbb{P}_\in) \cap N$ be such that $r \leqslant_{\mathrm{RO}(\mathbb{P}_\in)} s$, and define $\mathcal{A}$ as the set of all t in $\mathbb{P}_\in$ such that $t \leqslant_{\mathrm{RO}(\mathbb{P}_\in)} s$. Then $\mathcal{A} \in N$ and $\bigvee \mathcal{A} = s$. Since $N \cap H(\kappa) \in r \in \mathbb{P}_\in$, $r \leqslant_{\mathrm{RO}(\mathbb{P}_\in)} s$ and $s \in N$, $r \cap N$ is an extension of s in $\mathrm{RO}(\mathbb{P}_\in)$. So by Proposition 2.2(1), $\mathcal{A}$ is $r \cap N$-large. Therefore, $s = \bigvee \mathcal{A} \in F$. □

The $\in$-collapse can be modified to the forcing notion of finite matrices of countable elementary submodels, e.g. [37, §4]. This has the $(2^{\aleph_0})^+$-chain condition, and is used to add some structures. For example, Miyamoto proved that some modification of the $\in$-collapse may add a simplified $(\omega_2, 1)$-morass with linear limits [26], and Kuzeljevic and Todorčević proved that some modification of the $\in$-collapse may add a Kurepa tree which is almost Suslin [22].

3. Preservation theorems for strongly proper forcing notions

Shelah proved that a countable support iteration of strongly proper forcing notions is strongly proper [34, IX.2.7A Remark]. In this section, it is proved that a strongly proper forcing notion preserves a Suslin tree and the equality $\mathfrak{a} = \mathsf{cov}(\mathcal{M}) = \aleph_1$. Therefore, if it is consistent that a supercompact cardinal exists, then the forcing axiom for strongly proper forcing notions is consistent with Suslin Hypothesis, $\mathfrak{a} = \aleph_1$ and $\mathsf{cov}(\mathcal{M}) = \aleph_1$.

Theorem 3.1 (Miyamoto [25])**.** *A strongly proper forcing notion adds no uncountable antichains through a Suslin tree.*

Proof. Let $\mathbb{P}$ be a strongly proper forcing notion and T a Suslin tree. Assume that $p \in \mathbb{P}$ and $\dot{A}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}}$ "$\dot{A}$ is a maximal antichain in T". Let λ be an uncountable regular cardinal with $\mathcal{P}(\mathbb{P}) \in H(\lambda)$, M a countable elementary submodel of $H(\lambda)$ such that M contains T, $\mathbb{P}$, p and $\dot{A}$ as members, and let $\delta := \omega_1 \cap M$.

Denote T_δ by the set of δ-th elements of T. For $t \in T_\delta$, D_t is defined as the set of all conditions q in $\mathbb{P} \cap M$ such that, for some $s \in T_{<\delta}$, $s <_T t$ and $q \Vdash_{\mathbb{P}}$ "$s \in \dot{A}$". Each D_t may *not* be in M. We claim that each D_t is dense below p in $\mathbb{P} \cap M$. To show this, Let $r \in \mathbb{P} \cap M$ be a stronger condition than p in $\mathbb{P}$. Then, (inside M) the set $\{s \in T : r \nVdash_{\mathbb{P}} \text{“} s \notin \dot{A} \text{”}\}$ is predense in T, so we can find a maximal antichain A' in this set. By elementarity of M, we may assume that $A' \in M$. Since T is a Suslin tree, A' is countable, hence $A' \subseteq M$. Then, (outside M) since A' is a maximal antichain in T, there exists $s \in A'$ compatible with t in T. Since $M \models$ "$s \in A'$", there exists $q \leqslant_{\mathbb{P}} r$ in M such that $q \Vdash_{\mathbb{P}}$ "$s \in \dot{A}$". Since $T \cap M = \bigcup_{\alpha<\delta} T_\alpha =: T_{<\delta}$, $s \in T_{<\delta}$ and so $s <_T t$ holds, hence $q \in D_t$.

Since T_δ is countable, by the strong properness of $\mathbb{P}$, there exists $q \leqslant_{\mathbb{P}} p$ such that D_t is predense below q for every $t \in T_\delta$. Then $q \Vdash_{\mathbb{P}}$ "$\forall t \in T_\delta \, \exists s \in \dot{A}(s <_T t)$", therefore $q \Vdash_{\mathbb{P}}$ "$\dot{A} \subseteq T_{<\delta}$, which is countable". □

In [25], Miyamoto proved that, for any Suslin tree T and any proper forcing notion $\mathbb{P}$, $\mathbb{P}$ preserves the countable chain condition of T if and only if, for any regular cardinal λ with $\mathcal{P}(\mathbb{P}) \in H(\lambda)$, any countable elementary submodel N of $H(\lambda)$ with $\{T, \mathbb{P}\} \in N$, any $(N, \mathbb{P})$-generic condition p of $\mathbb{P}$ and any $t \in T$ of level $\omega_1 \cap N$, the pair $\langle p, t \rangle$ is $(N, \mathbb{P} \times T)$-generic. The last assertion is different from the assertion that $\mathbb{P} \times T$ is proper. In fact, for a free Suslin tree T, $T \times T$ is proper[1], but T destroys the countable chain condition of T.

[1] For any countable elementary submodel N of $H(\lambda)$ with $T \in N$ and $s, t \in T \cap N$, there are $s', t' \in T_{\omega_1 \cap N}$ such that $s <_T s'$, $t <_T t'$ and $s' \perp_T t'$. Then, since $T \restriction s' \times T \restriction t'$ is ccc (which follows from the freeness of T), $\langle s', t' \rangle$ is $(N, T \times T)$-generic.

A subset $\mathcal{A}$ of the set $[\omega]^{\aleph_0}$ is called almost disjoint if and only if the intersection of any two elements of $\mathcal{A}$ is finite, and an almost disjoint family $\mathcal{A}$ on ω is called a mad family if and only if $\mathcal{A}$ is infinite and is maximal with respect to almost disjointness, that is, any infinite subset of ω has an infinite intersection with some element of $\mathcal{A}$. For a forcing notion $\mathbb{P}$, a mad family $\mathcal{A}$ is called $\mathbb{P}$-indestructible if and only if $\mathbb{P}$ forces that $\mathcal{A}$ is still a mad family. A Cohen forcing is denoted by $\mathbb{C}$ in this article.

Theorem 3.2 (Brendle and Yatabe [7, Theorem 2.4.8], Hrušák [15, Theorem 5], Kurilić [21, Theorem 2])**.** *A mad family $\mathcal{A}$ is $\mathbb{C}$-indestructible if and only if, for any function f from $\mathbb{C}$ into ω, there exists $a \in \mathcal{A}$ such that $f^{-1}[a]$ is somewhere dense in $\mathbb{C}$.*

Hrušák showed that, if $\mathfrak{b} = 2^{\aleph_0}$, then there exists a $\mathbb{C}$-indestructible mad family [15, Proposition 6(2)]. So the Continuum Hypothesis (CH) implies the existence of a $\mathbb{C}$-indestructible mad family.

Theorem 3.3. *A strongly proper forcing notion preserves the maximality of a $\mathbb{C}$-indestructible mad family.*

Proof. Let $\mathbb{P}$ be a strongly proper forcing notion, $\mathcal{A}$ a $\mathbb{C}$-indestructible mad family, $p \in \mathbb{P}$, and $\dot{x}$ a $\mathbb{P}$-name for an infinite subset of ω. Let us find $q \leqslant_{\mathbb{P}} p$ and $a \in \mathcal{A}$ such that $q \Vdash_{\mathbb{P}}$ "$\dot{x} \cap a$ is infinite".

Denote $\lambda := (2^{|\mathbb{P}|})^+$. Take a countable elementary submodel M of $H(\lambda)$ such that M contains $\mathbb{P}$, $\mathcal{A}$, p and $\dot{x}$ as members. If there exists $q \leqslant_{\mathbb{P}} p$ such that the set $b_q := \{k \in \omega : q \Vdash_{\mathbb{P}} \text{“} k \in \dot{x} \text{”}\}$ is infinite, then the maximality of $\mathcal{A}$ follows the existence of our desired $a \in \mathcal{A}$. So we assume that any extension q of p in $\mathbb{P}$ satisfies that b_q is finite.

For each $r \leqslant_{\mathbb{P}} p$, k_r denotes the maximal number of the set $b_r \cup \{0\}$. Let C be a subset of $\mathbb{P} \cap M$ which is dense below p in $\mathbb{P} \cap M$. Since C is a dense subset of the countable forcing notion $\mathbb{P} \cap M$, by shrinking C if necessary, we may assume that there exists an order-isomorphism h of (a dense subset of) $\mathbb{C}$ onto C. Define the function f from $\mathbb{C}$ into ω such that, for each $\sigma \in \mathbb{C}$, $f(\sigma) = k_{h(\sigma)}$. Since $\mathcal{A}$ is $\mathbb{C}$-indestructible, there are $a \in \mathcal{A}$ and $\sigma \in \mathbb{C}$ such that $f^{-1}[a]$ is dense below σ in $\mathbb{C}$. Then $h(\sigma) \leqslant_{\mathbb{P}} p$ and $h(\sigma) \in M$.

For each $n \in \omega$, D_n denotes the set of all conditions q in $\mathbb{P} \cap M$ such that $n \leqslant k_q$ and $k_q \in a$. Let us show that each D_n is dense below $h(\sigma)$ in $\mathbb{P} \cap M$. To show this, let $n \in \omega$ and $s \in \mathbb{P} \cap M$ be such that $s \leqslant_{\mathbb{P}} h(\sigma)$. Since $\dot{x}$ is a $\mathbb{P}$-name for an infinite subset of ω and belongs to M, by elementarity of M, there exists $r \in C$ such that $r \leqslant_{\mathbb{P}} s$ and $k_r \geqslant n$. Since $h^{-1}(r) \leqslant_{\mathbb{C}} \sigma$, there is $\tau \in f^{-1}[a]$ such that $\tau \leqslant_{\mathbb{C}} h^{-1}(r)$. Then $f(\tau) \in a$, $h(\tau) \leqslant_{\mathbb{P}} r \leqslant_{\mathbb{P}} h(\sigma)$, and $n \leqslant k_r \leqslant k_{h(\tau)} = f(\tau)$, hence $h(\tau) \in D_n$.

By the strong properness of $\mathbb{P}$, there exists $q \leqslant_{\mathbb{P}} h(\sigma)$ such that, for all $n \in \omega$, D_n is predense below q in $\mathbb{P}$. Then $q \Vdash_{\mathbb{P}}$ "$\dot{x} \cap a$ is infinite". □

Theorem 3.4. *A strongly proper forcing notion preserves non-meager sets of reals.*

Proof. Let $\mathbb{P}$ be a forcing notion. For each $\sigma \in \omega^{<\omega}$, denote $[\sigma] := \{f \in \omega^\omega : \sigma \subseteq f\}$. For $r \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{F}$ for a nowhere dense subset of ω^ω, define $G(r, \dot{F})$ as the set of all f in ω^ω such that, for any $k \in \omega$, $r \Vdash_{\mathbb{P}}$ "$\dot{F} \cap [f \restriction k] \neq \emptyset$". We claim that $G(r, \dot{F})$ is nowhere dense. To show this, let $\sigma \in \omega^{<\omega}$. Then there are $s \leqslant_{\mathbb{P}} r$ and $\tau \in \omega^{<\omega}$ such that $\sigma \subseteq \tau$ and $s \Vdash_{\mathbb{P}}$ "$\dot{F} \cap [\tau] = \emptyset$". Then $G(r, \dot{F}) \cap [\tau]$ is empty. Because, if $f \in G(r, \dot{F}) \cap [\tau]$, then there is $k \in \omega$ such that $\tau \subseteq f \restriction k$, and then

$$r \Vdash_{\mathbb{P}} \text{“}\dot{F} \cap [f \restriction k] \neq \emptyset\text{” and } s \Vdash_{\mathbb{P}} \text{“}\dot{F} \cap [\tau] = \emptyset\text{”}.$$

This contradicts to the fact that $[f \restriction k] \subseteq [\tau]$ and $s \leqslant_{\mathbb{P}} r$.

Suppose that $\mathbb{P}$ is strongly proper, X is a non-meager subset of ω^ω, $p \in \mathbb{P}$, and $\{\dot{F}_n : n \in \omega\}$ is a set of $\mathbb{P}$-names for nowhere dense subsets of ω^ω. Let us show that $p \not\Vdash_{\mathbb{P}}$ "$X \subseteq \bigcup_{n\in\omega} \dot{F}_n$".

Suppose not. Denote $\lambda := (2^{|\mathbb{P}|})^+$, and take a countable elementary submodel M of $H(\lambda)$ such that M contains $\mathbb{P}$, X, p, and $\{\dot{F}_n : n \in \omega\}$ as members. Since X is non-meager, we can take f in the set $X \setminus \left(\bigcup_{n\in\omega} \bigcup_{r\in\mathbb{P}\cap M} G(r, \dot{F}_n)\right)$.

For each $n \in \omega$, D_n is defined as the set of all conditions s in $\mathbb{P} \cap M$ such that, for some $k \in \omega$, $s \Vdash_{\mathbb{P}}$ "$\dot{F}_n \cap [f \restriction k] = \emptyset$". Each D_n may not be in M. We claim that D_n is dense in $\mathbb{P} \cap M$. To show this, let $r \in \mathbb{P} \cap M$. Then $f \notin G(r, \dot{F}_n)$, which means that there exists $k \in \omega$ such that $r \not\Vdash_{\mathbb{P}}$ "$\dot{F}_n \cap [f \restriction k] \neq \emptyset$". Since M contains r, $\mathbb{P}$, $\dot{F}_n$ and $f \restriction k$ as members, by elementarity of M, there exists $s \in \mathbb{P} \cap M$ such that $s \leqslant_{\mathbb{P}} r$ and $s \Vdash_{\mathbb{P}}$ "$\dot{F}_n \cap [f \restriction k] = \emptyset$". Then $s \in D_n$.

By the strong properness of $\mathbb{P}$, there exists $q \leqslant_{\mathbb{P}} p$ such that each D_n is predense below q in $\mathbb{P}$. Then $q \Vdash_{\mathbb{P}}$ "$f \in X \subseteq \bigcup_{n\in\omega} \dot{F}_n$", so there are $s \leqslant_{\mathbb{P}} q$ and $n \in \omega$ such that $s \Vdash_{\mathbb{P}}$ "$f \in \dot{F}_n$". Since D_n is predense below p in $\mathbb{P}$, there are $r \in D_n$, $k \in \omega$ and $t \leqslant_{\mathbb{P}} s$ such that $r \Vdash_{\mathbb{P}}$ "$\dot{F}_n \cap [f \restriction k] = \emptyset$" and $t \leqslant_{\mathbb{P}} r$. But then $t \Vdash_{\mathbb{P}}$ "$f \in \dot{F}_n$ and $f \notin \dot{F}_n$", which is a contradiction. □

4. Preservation theorems for Y-proper forcing notions

Chodounský and Zapletal [8, §6] proved that it is consistent relative to the existence of a supercompact cardinal that the forcing axiom for Y-proper forcing notions is consistent, by applying Neeman's forcing iteration with two types of models as side conditions. Their forcing iteration is Y-proper. So the forcing axiom for Y-proper forcing notions is consistent with the existences of some mathematical structures which are preserved by Y-proper forcing extensions.

Chodounský and Zapletal presented many preservation theorems of Y-proper forcing notions [8, §2]. In this section, two preservation theorems are proved.

Theorem 4.1 (Chodounský and Zapletal [8, Corollary 2.9])**.** *A Y-proper forcing notion adds no new uncountable branches into ω_1-trees.*

Proof. Suppose that $\mathbb{P}$ is a Y-proper forcing notion, T is an ω_1-tree, $p \in \mathbb{P}$, and $\dot{b}$ is a $\mathbb{P}$-name such that p forces that $\dot{b}$ is an uncountable branch through T. Let λ be a regular cardinal with $\mathcal{P}(\mathbb{P}) \in H(\lambda)$, and N a countable elementary submodel of $H(\lambda)$ containing $\mathbb{P}$, T, p and $\dot{b}$ as members. Since $\mathbb{P}$ is Y-proper, there are $(N, \mathbb{P})$-generic $q \leqslant_{\mathbb{P}} p$ and a filter $F \in N$ on $\mathrm{RO}(\mathbb{P})$ such that $\{s \in \mathrm{RO}(\mathbb{P}) \cap N : q \leqslant_{\mathrm{RO}(\mathbb{P})} s\}$ is included in F as a subset.

Since q is $(N, \mathbb{P})$-generic, for any $\alpha \in \omega_1 \cap N$, there are $t \in T$ of level α and $r \in \mathbb{P} \cap N$, which is compatible with q in $\mathbb{P}$, such that r forces that t is in $\dot{b}$. It follows that the Boolean value $\|t \in \dot{b}\|$ belongs to N and is weaker than r in $\mathrm{RO}(\mathbb{P})$. Hence $\|t \in \dot{b}\|$ is in F. Let c be the set of all $t \in T$ such that $\|t \in \dot{b}\|$ is in F. Since F is a filter, c forms a chain through T. It follows from the above observation that $q \Vdash_{\mathbb{P}}$"$c \cap N = \dot{b} \cap N$". By elementarity of N, c belongs to N, and so $q \Vdash_{\mathbb{P}}$"$c = \dot{b}$". □

The author provided examples of forcing notions which add no random reals in [27, 47–50]. Right after [49, 50], Chodounský and Zapletal introduced Y-ccness and Y-properness, and presented a general theorem which implies the following theorem. The proof below is direct.

Theorem 4.2 (Chodounský and Zapletal [8, Corollary 2.6])**.** *A Y-proper forcing notion adds no random reals.*

Proof. Suppose that $\mathbb{P}$ is a Y-proper forcing notion, $p \in \mathbb{P}$, and $\dot{x}$ is a $\mathbb{P}$-name for a real in 2^ω. Let λ be a regular cardinal with $\mathcal{P}(\mathbb{P}) \in H(\lambda)$, and take a countable elementary submodel N of $H(\lambda)$ containing $\mathbb{P}$, p and $\dot{x}$ as members. Since N is countable, there exists a countable set $\{U_n : n \in \omega\}$ of open subsets of 2^ω such that $2^\omega \cap N$ is included in the measure zero set $\bigcap_{n\in\omega} U_n$. Since $\mathbb{P}$ is Y-proper, there exists an extension q of p in $\mathbb{P}$ such that q is $(N, \mathbb{P})$-generic and $(N, \mathbb{P})$-Y-generic.

Let us show that $q \Vdash_{\mathbb{P}}$"$\dot{x} \in \bigcap_{n\in\omega} U_n$". Assume not. Then there are $r \leqslant_{\mathbb{P}} q$ and $m \in \omega$ such that $r \Vdash_{\mathbb{P}}$"$\dot{x} \notin U_m$". Let F be a filter on $\mathrm{RO}(\mathbb{P})$ such that $F \in N$ and $\{s \in \mathrm{RO}(\mathbb{P}) \cap N : r \leqslant_{\mathrm{RO}(\mathbb{P})} s\} \subseteq F$.

Define S as the set of $v \in 2^{<\omega}$ such that the Boolean value $\|\dot{x} \restriction |v| \neq v\|$ is not in F. Then the order structure $(S, \subseteq)$ forms a tree, and by elementarity of N, S belongs to N. We claim that S is infinite. Because, if S is finite, then there exists $k \in \omega$ such that $S \subseteq 2^{<k}$, but then, since F is a filter, the Boolean value $\bigwedge_{v\in 2^k} \|\dot{x} \restriction |v| \neq v\|$ is a non-zero element of $\mathrm{RO}(\mathbb{P})$ and forces in $\mathrm{RO}(\mathbb{P})$ that $\dot{x} \restriction k$ does not belong to 2^k, which is a contradiction. Therefore, by elementarity of N, there exists $u \in 2^\omega \cap N$ such that $u \restriction k \in S$ for all $k \in \omega$. Let $l \in \omega$ be such that the open set $[u \restriction l]$, which consists of all $y \in 2^\omega$ such that $u \restriction l \subseteq y$, is a subset of U_m. Then the Boolean value $q \wedge \|\dot{x} \restriction l = u \restriction l\|$ is non-zero, and

$$q \wedge \|\dot{x} \restriction l = u \restriction l\| \Vdash_{\mathrm{RO}(\mathbb{P})} \text{"}\dot{x} \in [\dot{x} \restriction l] = [u \restriction l] \subseteq U_m\text{"},$$

which is a contradiction. □

5. Forcing notions equipped with models as side conditions

5.1. Todorčević's Open Coloring Axiom. There are two different Open Coloring Axioms. One is due to Abraham, Rubin and Shelah [1], and the other is due to Todorčević [39, §8],[42, Part 2]. Both axioms may not be equivalent, but their motivation is the same: Baumgartner's theorem on $\aleph_1$-dense sets of reals in [4]. Moore proved that the conjunction of two Open Coloring Axioms implies that the size of the continuum is equal to $\aleph_2$ [28].

Todorčević's Open Coloring Axiom has been studied by many people, and has a lot of applications, for example, [42, Part 2],[9–11, 13, 17, 28, 35, 41, 44, 45]. In order to distinguish two Open Coloring Axioms, Todorčević's Open Coloring Axiom is sometimes called a different name, for example, Todorčević's Axiom [12], or the Open Graph Axiom [42, Part 2], [43, §7.2]. In this subsection, we consider Open Coloring Axiom due to Todorčević.

For a separable metric space X, an open graph $\mathcal{G}$ on X means a symmetric irreflexive open subset of the product space X^2. For a subset Y of X, we denote $Y^{[2]} := \{\langle x, y\rangle, \langle y, x\rangle : \{x, y\} \in [Y]^2\}$. A subset Y of X is called a $\mathcal{G}$-homogeneous set, or a $\mathcal{G}$-clique, if and only if $Y^{[2]} \subseteq \mathcal{G}$. A subset Z of X is called $\mathcal{G}$-independent if and only if $Z^{[2]} \cap \mathcal{G} = \emptyset$. The Open Coloring Axiom (OCA) is the assertion that, for any separable metric space X and any open graph $\mathcal{G}$ on X, either $\mathcal{G}$ is countably chromatic (that is, X can be decomposed into countably many $\mathcal{G}$-independent subsets), or there exists an uncountable $\mathcal{G}$-clique [39, §8].

A typical example of OCA is about the structure of gaps in $\mathcal{P}(\omega)$. For subsets a and b of ω, we denote $a \subseteq^* b$ if and only if a is almost included in b, that is, $a \setminus b$ is finite. A pair (A, B) of families of infinite subsets of ω is called a pregap if and only if, for any $a \in A$ and any $b \in B$, $a \subseteq^* b$. For ordinals κ and λ, a pregap (A, B) is called a (κ, λ)-pregap if and only if A and B can be enumerated by $\{a_\alpha : \alpha \in \kappa\}$ and $\{b_\beta : \beta \in \lambda\}$ respectively such that, for any $\alpha, \alpha' \in \kappa$ and any $\beta, \beta' \in \lambda$, if $\alpha < \alpha'$ and $\beta < \beta'$, then

$$a_\alpha \subseteq^* a_{\alpha'} \subseteq^* b_{\beta'} \subseteq^* b_\beta.$$

A subset d of ω splits a pregap (A, B), or d is an interpolation of (A, B), if and only if, for any $a \in A$ and any $b \in B$, $a \subseteq^* d \subseteq^* b$. A pregap is called a gap if and only if it has no interpolations. A (κ, λ)-gap is a (κ, λ)-pregap which forms a gap. For example, $\diamondsuit$ implies that there exists an (ω_1, ω_1)-gap which has an interpolation in some ccc forcing extension (e.g. [51, 52]).

Suppose that $(A, B) = (\{a_\alpha : \alpha \in \omega_1\}, \{b_\alpha : \alpha \in \omega_1\})$ is an (ω_1, ω_1)-gap such that, for all $\alpha \in \omega_1$, $a_\alpha \subseteq b_\alpha$. Let X be the set of all pairs $\langle a_\alpha, b_\alpha\rangle$. X is considered as a subspace of the product topology $2^\omega \times 2^\omega$ of two Cantor spaces. Define the graph $\mathcal{G}$ as the set of all sets $\{\langle a, b\rangle, \langle a', b'\rangle\}$ in $X^{[2]}$ such that $a \not\subseteq b'$ or $a' \not\subseteq b$. $\mathcal{G}$ is an open graph on X. Since (A, B) forms a gap, $\mathcal{G}$ is not countably chromatic. So if OCA holds, then $\mathcal{G}$ has an uncountable $\mathcal{G}$-clique Y. It is known that Y guarantees that (A, B) forms a gap in any extension without collapsing ω_1.

In this section, we consider the forcing notion which plays a role in the following theorem.

Theorem 5.1 (Todorčević [39, Ch. 8])**.** PFA *implies* OCA*.*

To prove the theorem, suppose that X is an uncountable separable metric space and $\mathcal{G}$ is an open graph on X which is not countably chromatic.

Definition 5.2. Fix a regular cardinal κ such that $X \in H(\kappa)$, and $\mathfrak{M}$ denotes the set of all countable elementary submodels of $H(\kappa)$ which contain X and $\mathcal{G}$ as members.

$\mathbb{P}$ is defined as the set of all pairs $p = \langle H_p, \mathcal{M}_p\rangle$ such that

- (working part) H_p is a finite $\mathcal{G}$-clique,
- (side-condition part) $\mathcal{M}_p$ is a finite $\in$-chain of members of $\mathfrak{M}$,

- H_p is separated by $\mathcal{M}_p$, that is, for each $\{x, x'\} \in [H_p]^2$, there exists $M \in \mathcal{M}_p$ such that $M \cap \{x, x'\}$ is a singleton, and
- for each $M \in \mathcal{M}_p$ and each $\mathcal{G}$-independent $Z \in \mathcal{P}(X) \cap M$, $Z \cap H_p \subseteq M$.

The order is defined by $q \leqslant_{\mathbb{P}} p$ if and only if $H_q \supseteq H_p$ and $\mathcal{M}_q \supseteq \mathcal{M}_p$.

We will prove that $\mathbb{P}$ is proper. If PFA holds, then there exists a filter G on $\mathbb{P}$ such that the set $\bigcup_{p \in G} H_p$ is uncountable. Then $\bigcup_{p \in G} H_p$ forms an uncountable $\mathcal{G}$-clique. The following lemma implies the properness of $\mathbb{P}$.

Lemma 5.3. *Suppose that λ is a regular cardinal such that $H(\kappa) \in H(\lambda)$, N is a countable elementary submodel of $H(\lambda)$ such that N contains X, $\mathcal{G}$ and $H(\kappa)$ as members. Then a condition p of $\mathbb{P}$ is $(N, \mathbb{P})$-generic if $N \cap H(\kappa) \in \mathcal{M}_p$.*

Since X is separable, any member of $\mathfrak{M}$ contains all basic open subsets of X as members.

Proof. To show that p is $(N, \mathbb{P})$-generic, let $q \leqslant_{\mathbb{P}} p$ and $D \in N$ a dense subset of $\mathbb{P}$. By extending q if necessary, we may assume that $q \in D$ and $H_q \setminus N$ is not empty. Let k be the size of the set $H_q \setminus N$, and take an increasing sequence $\langle M_i : i < k \rangle$ of members of $\mathcal{M}_q$ and the enumeration $\{x_i^q : i < k\}$ of the set $H_q \setminus N$ such that for each $i < k$, $\omega_1 \cap N \leqslant \omega_1 \cap M_0$, $\{x_j^q : j < i\} \in M_i$, and $x_i^q \notin M_i$. Since $\mathcal{G}$ is an open graph on the separable metric space X and H_q is a $\mathcal{G}$-clique, we can take pairwise disjoint basic open subsets σ_i of X, $i < k$, such that

- for each $i < k$, $x_i^q \in \sigma_i$ and $(H_q \cap N) \cap \sigma_i = \emptyset$,
- for each $x \in H_q \cap N$ and each $i < k$, $\{x\} \times \sigma_i \subseteq \mathcal{G}$, and
- for each $\{i, j\} \in [k]^2$, $\sigma_i \times \sigma_j \subseteq \mathcal{G}$.

Define Q as the set of all members r of D such that there are an increasing sequence $\langle M_i^r : i < k \rangle$ of members of $\mathcal{M}_r$ and an enumeration $\langle x_i^r : i < k \rangle$ of $H_r \setminus M_0^r$ such that

- $H_r \cap M_0^r = H_q \cap N$,
- $\mathcal{M}_r \cap M_0^r = \mathcal{M}_q \cap N$, and
- for each $i < k$, $x_i^r \in \sigma_i$, $\{x_j^r : j < i\} \in M_i^r$ and $x_i^r \notin M_i^r$,

and $F := \{\langle x_i^r : i < k \rangle : r \in Q\}$. We note that

- for any $r \in \mathbb{P} \cap N$, if $H_r \cup H_q$ is a $\mathcal{G}$-clique, then r and q are compatible in $\mathbb{P}$ (because then $\langle H_r \cup H_q, \mathcal{M}_r \cup \mathcal{M}_q \rangle$ is a condition of $\mathbb{P}$),
- $\{Q, F\} \in N$, and
- $\langle x_i^q : i < k \rangle \in F$.

By reverse induction on $i < k$, we will build basic open subsets τ_i^0 and τ_i^1 of σ_i such that

- $x_i^q \in \tau_i^0$, $\tau_i^0 \cap \tau_i^1 = \emptyset$, $\tau_i^0 \times \tau_i^1 \subseteq \mathcal{G}$, and
- $F \cap \left(\{\langle x_j^q : j < i \rangle\} \times \prod_{j \in [i,k)} \tau_j^1\right) \neq \emptyset$.

Suppose that we have built τ_j^0 and τ_j^1 for each $j \in [i+1, k)$. Define Y as the set of all members y of σ_i such that there exists a tuple $\langle x_j^q : j < i \rangle ^\frown \langle y_j : j \in [i, k) \rangle$ in the product space $\prod_{j \leqslant i} \sigma_j \times \prod_{j \in [i+1,k)} \tau_j^1$ such that $\langle x_j^q : j < i \rangle ^\frown \langle y_j : j \in [i, k) \rangle \in F$

and $y_i = y$, and define Z as the set of all members y of Y such that the set $\{y\} \times (Y \setminus \{y\})$ does not meet $\mathcal{G}$. Then $Y, Z \in M_i$, Z is $\mathcal{G}$-independent, and $x_i^q \in Y$. So by the definition of $\mathbb{P}$, $x_i^q \notin Z$, and hence there exists $y \in Y \setminus \{x_i^q\}$ such that $\{x_i^q, y\} \in \mathcal{G}$. Since $\mathcal{G}$ is open, there exists basic open subsets τ_i^0 and τ_i^1 of σ_i such that $x_i^q \in \tau_i^0$, $y \in \tau_i^1$, $\tau_i^0 \cap \tau_i^1 = \emptyset$ and $\tau_i^0 \times \tau_i^1 \subseteq \mathcal{G}$.

Since $\{\tau_i^1 : i < k\} \in N$, by elementarity of N, there exists $r \in \mathbb{P} \cap N$ such that $\langle x_i^r : i < k \rangle \in F \cap \prod_{i<k} \tau_i^1$. Then by the choice of σ_i, τ_i^0 and τ_i^1, $i < k$, $H_r \cup H_q$ is a $\mathcal{G}$-clique, hence r is compatible with q in $\mathbb{P}$. □

Remark 5.4. If CH holds, then we can modify $\mathbb{P}$ to have the countable chain condition. Suppose that CH holds, X is a separable metric space of size $\aleph_1$ and $\mathcal{G}$ is an open graph on X. Let $\langle M_\alpha : \alpha \in \omega_1 \rangle$ be a continuous increasing sequence of countable elementary submodels of $H(\aleph_2)$ such that M_0 contains X, $\mathcal{G}$ and some fixed enumeration of the reals as members. Define $\mathbb{Q}$ as the set of all pairs $p = \langle H_p, O_p \rangle$ such that

- (working part) H_p is a finite $\mathcal{G}$-clique,
- (side-condition part) O_p is a finite subset of ω_1,
- H_p is separated by O_p, that is, for each $\{x, x'\} \in [H_p]^2$, there exists $\alpha \in O_p$ such that $M_\alpha \cap \{x, x'\}$ is a singleton, and
- for each $\alpha \in O_p$ and each $\mathcal{G}$-independent $Z \in \mathcal{P}(X) \cap M_\alpha$, $Z \cap H_p \subseteq M_\alpha$.

The order is defined by $q \leqslant_{\mathbb{P}} p$ if and only if $H_q \supseteq H_p$ and $O_q \supseteq O_p$.

$\mathbb{Q}$ is of size $\aleph_1$. By a similar argument to the proof of Lemma 5.3, if N is a countable elementary submodel of $H((2^{\aleph_1})^+)$ such that N contains $\mathbb{Q}$ and the sequence $\langle M_\alpha : \alpha \in \omega_1 \rangle$ as members, then the condition $\langle \emptyset, \{\omega_1 \cap N\} \rangle$ is $(N, \mathbb{Q})$-generic. Moreover, if $p \in \mathbb{Q}$ and $H_p \cap (M_{\min(O_p \setminus N)} \setminus M_{\max(O_p \cap N)}) = \emptyset$, then $\langle H_p, O_p \cup \{\omega_1 \cap N\} \rangle$ is a condition of $\mathbb{Q}$. These observations imply that $\mathbb{Q}$ is ccc.

Chodounský and Zapletal proved that Y-proper forcing notions cannot add an instance of OCA in a clopen graph [8, Corollary 2.5]. So $\mathbb{P}$ cannot be Y-proper. However $\mathbb{P}$ has some properties of strongly proper forcing notions, namely, $\mathbb{P}$ preserves a Suslin tree. The following is proved by Todorčević. Farah [10] proved that OCA is consistent with the failure of Suslin Hypothesis in a different way.

Theorem 5.5. *$\mathbb{P}$ adds no uncountable antichains through a Suslin tree.*

Proof. Assume that T is a Suslin tree, $p \in \mathbb{P}$ and $\dot{A}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}}$ "$\dot{A}$ is an uncountable antichain through T". Let λ be as in Lemma 5.3, N a countable elementary submodel of $H(\lambda)$ such that N contains T, $\mathbb{P}$, $H(\kappa)$, p and $\dot{A}$ as members. Define $p^+ = \langle H_p, \mathcal{M}_p \cup \{N \cap H(\kappa)\} \rangle$. By Lemma 5.3, p^+ is $(N, \mathbb{P})$-generic.

Let $t \in T \setminus N$ and $q \leqslant_{\mathbb{P}} p^+$ be such that $q \Vdash_{\mathbb{P}}$ "$t \in \dot{A}$". Let k be the size of the set $H_q \setminus N$, and take an increasing sequence $\langle M_i : i < k \rangle$ of members of $\mathcal{M}_q$, the enumeration $\{x_i^q : i < k\}$ of the set $H_q \setminus N$, and pairwise disjoint basic open subsets σ_i of X, $i < k$, such that, for each $i < k$,

- $\omega_1 \cap N \leqslant \omega_1 \cap M_0$,
- $\{x_j^q : j < i\} \in M_i$ and $x_i^q \notin M_i$,
- $x_i^q \in \sigma_i$ and $(H_q \cap N) \cap \sigma_i = \emptyset$,

- for each $x \in H_q \cap N$, $\{x\} \times \sigma_i \subseteq \mathcal{G}$, and
- for each $j \in k \setminus \{i\}$, $\sigma_i \times \sigma_j \subseteq \mathcal{G}$.

Let $\dot{G}_T$ be a canonical T-name for a generic filter. Define the T-names $\dot{Q}$ and $\dot{F}$ such that

$\Vdash_T$ " $\dot{Q}$ consists of all $r \in \mathbb{P}$ (this $\mathbb{P}$ is the same object to the one in the ground model) such that there are an increasing sequence $\langle M_i^r : i < k \rangle$ of members of $\mathcal{M}_r$ and an enumeration $\langle x_i^r : i < k \rangle$ of $H_r \setminus M_0^r$ such that
  - $H_r \cap M_0^r = H_q \cap N$,
  - $\mathcal{M}_r \cap M_0^r = \mathcal{M}_q \cap N$,
  - for each $i < k$, $x_i^r \in \sigma_i$, $\{x_j^r : j < i\} \in M_i^r$ and $x_i^r \notin M_i^r$, and
  - $r \Vdash_{\mathbb{P}}$ "$s \in \dot{A} \cap \dot{G}_T$" for some $s \in T$,

  and $\dot{F} := \{\langle x_i^r : i < k \rangle : r \in \dot{Q}\}$".

Then both $\dot{Q}$ and $\dot{F}$ belong to N and t forces that q is in $\dot{Q}$. Since T adds no new reals, T adds no new closed subsets of X. So by a similar argument as in Lemma 5.3 applied to the $\in$-chain $\langle M_i^r[\dot{G}_T] : i < k \rangle$, whose members are elementary submodels of $H(\lambda)[\dot{G}_T]$ (which is the extension with T), there are $t' \geqslant_T t$ (t' is an extension of t in the forcing notion T) and basic open subsets τ_i^0 and τ_i^1 of σ_i such that t' forces that, for each $i < k$,

- $x_i^q \in \tau_i^0$, $\tau_i^0 \cap \tau_i^1 = \emptyset$, $\tau_i^0 \times \tau_i^1 \subseteq \mathcal{G}$, and
- $\dot{F} \cap \left(\{\langle x_j^q : j < i \rangle\} \times \prod_{j \in [i,k)} \tau_j^1\right) \neq \emptyset$.

By elementarity of N, there are $t'' \geqslant_T t'$, $r \in \mathbb{P} \cap N$ and $s \in T \cap N$ such that t'' forces that $r \in \dot{Q}$, $\langle x_i^r : i < k \rangle \in F \cap \prod_{i<k} \tau_i^1$, and r forces that s is in $\dot{A} \cap \dot{G}_T$. Since t'' forces that $s \in \dot{G}_T$, s and t are compatible in T. As seen in the proof of Lemma 5.3, q and r are compatible in $\mathbb{P}$. A common extension of q and r in $\mathbb{P}$ forces that both t and s are in the antichain $\dot{A}$, which is a contradiction. □

5.2. The P-ideal Dichotomy and the ideal-based forcings. The P-ideal Dichotomy (PID) is introduced by Todorčević [2, 40]. The origin of the P-ideal dichotomy is an analysis of the problem whether every hereditarily separable regular space is Lindelöf (i.e. there are no S-spaces) [42, §23]. Todorčević proved that PFA implies PID. If PID holds and the pseudo-intersection number $\mathfrak{p}$ is greater than $\aleph_1$, then there are no S-spaces [42, §23]. Todorčević asked under PID, whether $\mathfrak{p} > \aleph_1$ is equivalent that there are no S-spaces [42, Question 23.8]. Raghavan and Todorčević gave some consequences of PFA which are equivalent to the assertion that some cardinal invariants are greater than $\aleph_1$ under PID [32].

For an uncountable set S, an ideal $\mathcal{I}$ on $[S]^{\leqslant\aleph_0}$ (so $\mathcal{I} \subseteq [S]^{\leqslant\aleph_0}$) is called a P-ideal if and only if, for any countable subset $\mathcal{A}$ of $\mathcal{I}$, there exists $b \in \mathcal{I}$ such that, for each $a \in \mathcal{A}$, $a \subseteq^* b$ (that is, $a \setminus b$ is finite). A subset X of S is called orthogonal to $\mathcal{I}$ if and only if $X \cap a$ is finite for every $a \in \mathcal{I}$. The P-ideal dichotomy (PID) is the assertion that, for every index set S and every P-ideal $\mathcal{I}$ on $[S]^{\leqslant\aleph_0}$, either there exists a decomposition $S = \bigcup_{n\in\omega} S_n$ of countably many sets orthogonal to $\mathcal{I}$, or there exists an uncountable subset H of S such that $[H]^{\leqslant\aleph_0} \subseteq \mathcal{I}$.

A typical application of PID is Suslin Hypothesis. Let T be an ω_1-tree which is well-pruned, that is, every node of T has successors in any level of T larger than its own. Define $\mathcal{I}$ as the set of all countable subsets a of T such that, for any $t \in T$, the set $\{s \in a : s <_T t\}$ is finite.

We claim that $\mathcal{I}$ is a P-ideal. To show this, let $\mathcal{A} = \{a_n : n \in \omega\}$ be a countable subset of $\mathcal{I}$. We will find $b \in \mathcal{I}$ such that $a \subseteq^* b$ for every $a \in \mathcal{A}$. Let $\alpha \in \omega_1$ be such that $\bigcup \mathcal{A} \subseteq \bigcup_{\beta<\alpha} T_\beta$, and let $\{t_i : i \in \omega\}$ be an enumeration of T_α. For each $s \in \bigcup \mathcal{A}$, denote by n_s the minimal number n with the property that $s \in a_n$. Define b as the set of all members s of $\bigcup \mathcal{A}$ such that $s \not<_T t_i$ for all $i \leqslant n_s$. Then, for any $n \in \omega$, since $n_s \leqslant n$ for every $s \in a_n$, $a_n \setminus \bigcup_{i \leqslant n}\{s \in a_n : s <_T t_i\} \subseteq b$. Since each set $\{s \in a_n : s <_T t_i\}$ is finite, $a_n \subseteq^* b$. Moreover, for each $i \in \omega$, the set $\{s \in b : s <_T t_i\}$ is a subset of the set $\bigcup_{n<i}\{s \in a_n : s <_T t_i\}$, which is finite. Therefore $b \in \mathcal{I}$.

Since any countable antichain in T belongs to $\mathcal{I}$, any subset of T orthogonal to $\mathcal{I}$ does not have an infinite antichain, so is a union of countably many chains through T. So if T has a decomposition $\bigcup_{n\in\omega} S_n$ such that each S_n is orthogonal to $\mathcal{I}$, T has an uncountable chain through T. If there exists an uncountable subset A of T such that any countable subset of A is in $\mathcal{I}$, then the subtree $(A, <_T\restriction A)$ is of height $\leqslant \omega$, and so A has an uncountable antichain, which is also an uncountable antichain in T. Therefore, PID implies Suslin Hypothesis.

Baumgartner proved that Martin's Axiom implies that every Aronszajn tree is special, which is a strong form of Suslin Hypothesis. Kuzeljevic and Todorčević proved that it is consistent that PID holds and there exists a non special Aronszajn tree by use of Neeman's iteration [23].

Theorem 5.6 (Todorčević [40]). *PFA implies PID.*

To show this, there are two options. One is forcing by countable approximations [40], and the other is forcing by finite approximations [37, p. 722], [42, Theorem 20.6], [30, §5.2]. The former forcing notion enables one to show that it is consistent that both PID and $2^{\aleph_0} = \aleph_1$ hold. In this section, we consider the later forcing notion in Definition 5.7.

Suppose that S is an uncountable set, $\mathcal{I}$ is a P-ideal on $[S]^{\leqslant\aleph_0}$, and S is not covered by countably many subsets of S orthogonal to $\mathcal{I}$.

Definition 5.7. Fix a regular cardinal κ such that $\mathcal{P}(S) \in H(\kappa)$, and $\mathfrak{M}$ denotes the set of all countable elementary submodels of $H(\kappa)$ which contains S and $\mathcal{I}$ as members. We fix an assingment of all members M of $\mathfrak{M}$ to members b_M of $\mathcal{I}$ such that $b_M \subseteq S \cap M$ and $a \subseteq^* b_M$ for every $a \in \mathcal{I} \cap M$.

$\mathbb{P}$ is defined as the set of all pairs $p = \langle H_p, \mathcal{M}_p \rangle$ such that

- (working part) H_p is a finite subset of S,
- (side-condition part) $\mathcal{M}_p$ is a finite $\in$-chain of members of $\mathfrak{M}$,
- H_p is separated by $\mathcal{M}_p$ (see Definition 5.2),
- for any $M \in \mathcal{M}_p$, $H_p \cap M \subseteq b_M$,
- for any $M \in \mathcal{M}_p$ and any $Y \in \mathcal{P}(S) \cap M$ which is orthogonal to $\mathcal{I}$, $H_p \cap Y \subseteq M$.

The order is defined by $q \leqslant_{\mathbb{P}} p$ if and only if $H_q \supseteq H_p$ and $\mathcal{M}_q \supseteq \mathcal{M}_p$.

Proposition 5.8. *Define $\mathfrak{J}$ as the ideal which consists of the subsets of S orthogonal to $\mathcal{I}$. $\mathbb{P}$ and $\mathfrak{J}$ have the following properties.*

(A) *For any $p \in \mathbb{P}$ and any $H \subseteq H_p$, the pair $\langle H, \mathcal{M}_p \rangle$ is a condition of $\mathbb{P}$, and if conditions p and q of $\mathbb{P}$ are compatible, then the pair $\langle H_p \cup H_q, \mathcal{N} \rangle$ is their common extension in $\mathbb{P}$ for some $\mathcal{N}$ which includes $\mathcal{M}_p$ and $\mathcal{M}_q$.*

(B) *$\mathfrak{J}$ is a nonprincipal ideal on S such that every $\mathfrak{J}$-positive set has a countable $\mathfrak{J}$-positive subset, and the σ-ideal $\sigma\mathfrak{J}$ generated by $\mathfrak{J}$ is a proper ideal.*

(C) *For each $p \in \mathbb{P}$, there exists a $\sigma\mathfrak{J}$-positive subset Z of S such that, for any $x \in Z$, the pair $\langle H_p \cup \{x\}, \mathcal{M}_p \rangle$ is a condition of $\mathbb{P}$.*

(D) *For any condition $p \in \mathbb{P}$, any $M \in \mathcal{M}_p$ and any $\mathfrak{J}$-positive subset $Z \in M$ of S, the set $Z \cap \bigcap_{N \in \mathcal{M}_p \smallsetminus M} b_N$ is not empty.*

Proof. (A) directly follows from the definition of $\mathbb{P}$. To show (B), let A be a $\mathfrak{J}$-positive subset of S, that is, A does not belong to $\mathfrak{J}$, namely, $A \cap a$ is infinite for some $a \in I$. Let $M \in \mathfrak{M}$ contain A, $\mathcal{I}$ and $\mathfrak{J}$ as members. Then, by elementarity of M, $A \cap M$ is $\mathfrak{J}$-positive. So $A \cap M$ is a countable $\mathfrak{J}$-positive subset of A. By our assumption of $\mathcal{I}$, $\sigma\mathfrak{J}$ is a proper ideal. To show (C), let $p \in \mathbb{P}$. Define $Z = S \smallsetminus \left(\bigcup_{M \in \mathcal{M}_p} \bigcup (\mathfrak{J} \cap M) \right)$. Then Z is $\sigma\mathfrak{J}$-large, hence is $\sigma\mathfrak{J}$-positive, and any $x \in Z$ satisfies that $\langle H_p \cup \{x\}, \mathcal{M}_p \rangle \in \mathbb{P}$. (D) directly follows from the definition of $\mathfrak{J}$ and the choice of $\{b_N : N \in \mathfrak{M}\}$. □

We will prove that $\mathbb{P}$ is proper. If PFA holds, then, by (C), there exists a filter G on $\mathbb{P}$ such that the set $\bigcup_{p \in G} H_p$ is uncountable. By the definition of $\mathbb{P}$, any countable subset of $\bigcup_{p \in G} H_p$ belongs to $\mathcal{I}$.

Lemma 5.9. *Suppose that λ is a regular cardinal such that $H(\kappa) \in H(\lambda)$, N is a countable elementary submodel of $H(\lambda)$ such that N contains S, $\mathcal{I}$ and $H(\kappa)$ as members. Then a condition p of $\mathbb{P}$ is $(N, \mathbb{P})$-generic if $N \cap H(\kappa) \in \mathcal{M}_p$.*

Proof. To show that p is $(N, \mathbb{P})$-generic, let $q \leqslant_{\mathbb{P}} p$ and $D \in N$ a dense subset of $\mathbb{P}$. By extending q if necessary, we may assume that $q \in D$ and $H_q \smallsetminus N$ is not empty. Let l be the size of the set $H_q \smallsetminus N$, $\langle K_j : j < l \rangle$ an $\in$-subchain of $\mathcal{M}_q$ and $\langle x_j^q : j < l \rangle$ an enumeration of $H_q \smallsetminus N$ such that $K_0 = N \cap H(\kappa)$ and, for each $i < l$, $\langle x_j^q : j < i \rangle \in K_i$ and $x_i^q \notin K_i$.

Define T_l as the set of all sequences $\sigma = \langle \sigma(i) : i < l \rangle$ of S of length l such that there exists $s \in D$ such that, for some $M \in \mathcal{M}_s$, $\mathcal{M}_s \cap M = \mathcal{M}_q \cap N$, $H_s \cap M = H_q \cap N$, $H_s \smallsetminus M = \operatorname{ran}(\sigma)$, and for any $i < l$, there exists $K \in \mathcal{M}_s \smallsetminus M$ such that $\{\sigma(j) : j < i\} \in K$ and $\sigma(i) \notin K$. Then $\langle x_j^q : j < l \rangle \in T_l \in K_0$. So T_l is in K_i for all $i < l$. By the downward induction on $j < l$, we define T_j such that

$$T_j := T_{j+1} \smallsetminus \{\sigma \in T_{j+1} : \{\tau(j) : \tau \in T_{j+1} \text{ and } \tau \restriction j = \sigma \restriction j\} \in \mathfrak{J}\}.$$

We claim that $\langle x_j^q : j < l \rangle$ is in T_i for all $i < l$, especially $\langle x_j^q : j < l \rangle$ is in T_0. To show this, suppose that $i < l$ and $\langle x_j^q : j < l \rangle \in T_{i+1}$. Then by the definition of $\mathbb{P}$, $x_i^q \notin \bigcup (\mathfrak{J} \cap K_i)$, and also that

$$x_i^q \in \{\tau(i) : \tau \in T_{i+1} \text{ and } \tau \restriction i = \langle x_j^q : j < i \rangle\} \in K_i.$$

Therefore the set $\{\tau(i) : \tau \in T_{i+1}$ and $\tau \restriction i = \langle x^q_j : j < i\rangle\}$ is $\mathfrak{J}$-positive and hence $\langle x^q_j : j < l\rangle \in T_i$.

We consider T_0 as a tree which consists of all initial segments of members of T_0. Then T_0 has a cofinal branch (which is of length l) and each non-terminal node has $\mathfrak{J}$-positive many successors in the tree T_0. We will take $y_\nu \in S \cap N = S \cap K_0$ by induction on $\nu < l$ such that $\langle y_\mu : \mu < \nu\rangle$ is an initial segment of some member of T_0 such that each y_μ belongs to b_K for all $K \in \mathcal{M}_q \setminus K_0$. Given $\langle y_\mu : \mu < \nu\rangle$, define Z as the set $\{\tau(i) : \tau \in T_{i+1}$ and $\tau \restriction \nu = \langle y_j : j < \nu\rangle\}$. Z is $\mathfrak{J}$-positive and in K_0. By (B) and (D) in Proposition 5.8, there exists y_ν in the set $Z \cap K_0 \cap \bigcap_{K \in \mathcal{M}_q \setminus K_0} b_K$.

Since T_0 is a subset of T_l, the sequence $\langle y_\nu : \nu < l\rangle$ is in T_l. By elementarity of N, there exists $r \in D \cap N \cap H(\kappa)$ which witnesses that $\langle y_\nu : \nu < l\rangle$ is in T_l. Then $\langle H_r \cup H_q, \mathcal{M}_r \cup \mathcal{M}_q\rangle$ is a conditon of $\mathbb{P}$, and so is a common extension of r and q in $\mathbb{P}$. □

In [49], the author proved that $\mathbb{P}$ adds no random reals. Chodounský and Zapletal extend it by proving the following [8, Corollary 2.6, Theorem 4.6].

Theorem 5.10. *$\mathbb{P}$ is Y-proper.*

Proof. This proof follows the one of Lemma 2.6. The difference is the notion of the largeness for this forcing notion $\mathbb{P}$.

Let λ be a regular cardinal such that $\mathcal{P}(\mathbb{P})$ is in $H(\lambda)$. For a condition $p \in \mathbb{P}$, a subset $\mathcal{A}$ of $\mathbb{P}$ is p-large if and only if, for any $b \in H(\kappa)$, there are $q \in \mathcal{A}$ and a countable elementary submodel M of $H(\lambda)$ such that M contains $H(\kappa)$ as a member, $M \cap H(\kappa) \in \mathcal{M}_q$, $H_q \cap M = H_p$, $\mathcal{M}_q \cap M = \mathcal{M}_p$, and $b \in M$.

Let $p \in \mathbb{P}$. We will show that $\{\bigvee \mathcal{A} : \mathcal{A} \subseteq \mathbb{P}$ is p-large$\}$ is a centered subset of $\mathrm{RO}(\mathbb{P})$. Let $n \in \omega$ and $\mathcal{A}_i$, $i \in n$, p-large subsets of $\mathbb{P}$. It suffices to find $q_i \in \mathcal{A}_i$, $i \in n$, such that $\{q_i : i \in n\}$ has a common extension in $\mathbb{P}$. To do this, let λ^* be a regular cardinal with $H(\lambda) \in H(\lambda^*)$, and countable elementary submodels N_i, $i \in n$, of $H(\lambda^*)$ such that each N_i contains $\mathcal{I}$, $\mathfrak{J}$, $H(\kappa)$, p, $\mathbb{P}$, $H(\lambda)$, $\{N_j : j < i\}$ and $\mathcal{A}_i$ as members. By reverse induction on $i \in n$, as in the proof of Lemma 5.9, we can find $q_i \in \mathcal{A}_i \cap N_i$ and a countable elementary submodel M_i of $H(\lambda)$ such that

- M_i contains $H(\kappa)$ and $\{N_j : j < i\}$ as members, and $M_i \cap H(\kappa) \in \mathcal{M}_{q_i}$,
- $H_{q_i} \cap M_i = H_p$,
- $\mathcal{M}_{q_i} \cap M_i = \mathcal{M}_p$, and
- $H_{q_i} \setminus M_i$ is a subset of the set $\bigcap_{j=i+1}^{n-1} \bigcap_{K \in \mathcal{M}_{q_j} \setminus M_j} b_K$.

Since $\mathbb{P}$ is a subset of $H(\kappa)$, each q_i is in $M_i \cap H(\kappa)$. Therefore, the pair $\langle \bigcup_{i\in n} H_{q_i}, \bigcup_{i\in n} \mathcal{M}_{q_i}\rangle$ is a condition of $\mathbb{P}$, and is a common extension of q_i's.

To show that $\mathbb{P}$ is Y-proper, let N be a countable elementary submodel of $H(\lambda)$ such that N contains S, $\mathcal{I}$ and $H(\kappa)$ as members, and let $p \in \mathbb{P}$ be such that $N \cap H(\kappa) \in \mathcal{M}_p$. Let us show that p is $(N, \mathbb{P})$-Y-generic. Let r be an extension of p in $\mathbb{P}$. Denote $r \restriction N = \langle H_r \cap N, \mathcal{M}_r \cap N\rangle$, and define F as the filter on $\mathrm{RO}(\mathbb{P})$ that is generated by the set $\{\bigvee \mathcal{A} : \mathcal{A} \subseteq \mathbb{P}$ is $r \restriction N$-large$\}$. Then F belongs to N. We will show that, for any $s \in \mathrm{RO}(\mathbb{P}) \cap N$, if $r \leqslant_{\mathrm{RO}(\mathbb{P})} s$, then $s \in F$. Let

$s \in \mathrm{RO}(\mathbb{P}) \cap N$ be such that $r \leqslant_{\mathrm{RO}(\mathbb{P})} s$, and define $\mathcal{A}$ as the set of all $q \in \mathbb{P}$ such that $q \leqslant_{\mathrm{RO}(\mathbb{P})} s$. Then $\mathcal{A}$ is in N, and $\bigvee \mathcal{A} = s$. By elementarity of N and the fact that $N \cap H(\kappa) \in \mathcal{M}_r$, $\mathcal{A}$ is $r \restriction N$-large. Therefore $s = \bigvee \mathcal{A} \in F$. □

Remark 5.11. In [53, §3], Zapletal introduced a wide class of forcing notions with models as side conditions, which are clossly related to the forcing notion $\mathbb{P}$ in this subsection.

A triple $\langle A, \sqsubseteq, \mathfrak{J} \rangle$ is called an ideal-based triple if and only if

(A) $A \subseteq [\omega_1]^{<\aleph_0}$ and $\sqsubseteq$ is a transitive relation on A which refines the set-inclusion $\subseteq$ such that
  - for each $a \in A$ and $\beta \in \omega_1$, $a \cap \beta \in A$ and $a \cap \beta \sqsubseteq a$, and
  - for each $a, b \in A$, if a and b are $\sqsubseteq$-compatible (i.e. there exists $c \in A$ such that $a \sqsubseteq c$ and $b \sqsubseteq c$), then $a \cup b$ is in A and is a $\sqsubseteq$-upper bound of a and b,

(B) $\mathfrak{J}$ is a nonprincipal ideal on ω_1 such that every $\mathfrak{J}$-positive set has a countable $\mathfrak{J}$-positive subset, and the σ-ideal $\sigma\mathfrak{J}$ generated by $\mathfrak{J}$ is a proper ideal,

(C) for each $a \in A$, there exists a $\sigma\mathfrak{J}$-positive set Z such that, for every $\beta \in Z$, $a \cup \{\beta\}$ is in A and $a \sqsubseteq a \cup \{\beta\}$, and

(D) for each $a \in A$, there exists a $\mathfrak{J}$-large set Y such that, for every $\beta \in Y$, if $(a \cap \beta) \cup \{\beta\}$ is in A and $a \cap \beta \sqsubseteq (a \cap \beta) \cup \{\beta\}$, then $a \cup \{\beta\}$ is in A and $a \sqsubseteq a \cup \{\beta\}$.

For an ideal-based triple $\langle A, \sqsubseteq, \mathfrak{J} \rangle$, the forcing notion $\mathbb{P}(A, \sqsubseteq, \mathfrak{J})$ is defined as the set of all pairs $p = \langle H_p, \mathcal{M}_p \rangle$ such that

- (working part) H_p is in A,
- (side-condition part) $\mathcal{M}_p$ is a finite $\in$-chain of countable elementary submodels of $H((2^{\aleph_1})^+)$ which contain $\langle A, \sqsubseteq, \mathfrak{J} \rangle$ as a member,
- H_p is separated by $\mathcal{M}_p$,
- for any $M \in \mathcal{M}_p$ and any $Y \in \mathfrak{J} \cap M$, $H_p \cap Y \subseteq M$.

The ordering is defined by $q \leqslant_{\mathbb{P}(A, \sqsubseteq, \mathfrak{J})} p$ if and only if $H_p \sqsubseteq H_q$ and $\mathcal{M}_q \supseteq \mathcal{M}_p$.

For example, shooting an uncountable discrete subspace into a right-separated hereditarily separable regular space is one of the ideal-based forcings [49, Example 2.3]. In [53, §3], Zapletal proved that $\mathbb{P}(A, \sqsubseteq, \mathfrak{J})$ keeps the additivity of the null sets small. In [49, §4], the author proved that $\mathbb{P}(A, \sqsubseteq, \mathfrak{J})$ does not add random reals, and in [8, Theorem 4.4], Chodounský and Zapletal proved that $\mathbb{P}(A, \sqsubseteq, \mathfrak{J})$ is Y-proper.

5.3. The failure of $\square(\kappa)$.

For a set S of ordinals, $\mathsf{Lim}(S)$ denotes the set of all limit ordinals in S. In [38, §1], Todorčević defined a $\square(\kappa)$-sequence, for an uncountable cardinal κ, as a sequence $\langle C_\alpha : \alpha \in \kappa \rangle$ such that

(i) $C_{\alpha+1} = \{\alpha\}$, and if α is a limit ordinal, then C_α is a closed unbounded subset of α, and

(ii) if α is a limit point of C_β, then $C_\alpha = C_\beta \cap \alpha$.

Jensen's $\square_\kappa$ is the assertion that there exists a $\square(\kappa^+)$-sequence such that, if the cofinality of α is less than κ, then the cardinality of C_α is also less than κ. The following is a weakening of $\square_\kappa$.

Definition 5.12 (Todorčević [6, Ch. 4], [38, §1])**.** For an uncountable regular cardinal κ, $\square(\kappa)$ is the assertion that there exists a $\square(\kappa)$-sequence such that, for every club subset C of κ, there exists a limit point α of C so that $C_\alpha \neq C \cap \alpha$.

It can be proved that $\square_\kappa$ implies $\square(\kappa^+)$. In [36], Todorčević proved that PFA implies that $\square_\kappa$ fails for any uncountable cardinal κ. This is the first application of the side condition method in the literature. His proof is essentially a proof of the following theorem.

Theorem 5.13 (Todorčević [36])**.** PFA *implies that* $\square(\kappa)$ *fails for any regular cardinal* $\kappa > \omega_1$*.*

After that, Todorčević [40, §4] proved that PID implies the failure of $\square(\kappa)$ for any regular cardinal $\kappa > \omega_1$. In this subsection, we investigate the forcing notion for the proof of Theorem 5.13 in [36] because this is a different fashion from the one in Section 5.2.

To prove the theorem, suppose that PFA holds, κ is an uncountable regular cardinal greater than ω_1, $\square(\kappa)$ holds, and $\vec{C} = \langle C_\alpha : \alpha \in \kappa \rangle$ is a $\square(\kappa)$-sequence which witnesses $\square(\kappa)$. We will show a contradiction as follows.

For $\alpha, \beta \in \mathsf{Lim}(\kappa)$, define $\alpha \prec \beta$ if and only if α is a limit point of C_β. Then, by (ii) above, $\prec$ forms a tree order of $\mathsf{Lim}(\kappa)$. If E is a chain of the tree $(\mathsf{Lim}(\kappa), \prec)$, then, by (i) and (ii), $\bigcup_{\alpha \in E} C_\alpha$ is a club subset of $\sup(E)$. Therefore, since $\vec{C}$ is a witness of $\square(\kappa)$, there are no chains of $(\mathsf{Lim}(\kappa), \prec)$ of size κ. A function f from a subset A of $\mathsf{Lim}(\kappa)$ into ω is called a specializing map if, for any $\alpha, \beta \in A$ with $\alpha \prec \beta$, $f(\alpha) \neq f(\beta)$. We will present a proper forcing $\mathbb{P}$ which adds a countably closed subset E of $(\mathsf{Lim}(\kappa), \prec)$ of order type ω_1 and a specializing function g on E. Then it follows from PFA that there are such E and g. Since $\sup(E)$ is in $\mathsf{Lim}(\kappa)$ and $C_{\sup(E)}$ is a closed unbounded subset of $\sup(E)$, there exists a closed unbounded subset E' of E such that each member of E' is a limit point of $C_{\sup(E)}$. Then by (ii), E' is a chain with respect to $\prec$, and hence $g \restriction E'$ is an injection from the uncountable set E' into ω, which is a contradiction, and finishes the proof.

The following is one of such forcing notion. The following formulation is a little different from the original one in [36], however these are essentially same.

Definition 5.14. For a countable elementary submodel N of $H(\kappa^+)$, denote

$$\delta_N := \sup(\kappa \cap N),$$

and $\mathfrak{M}$ denotes the set of all countable elementary submodels of $H(\kappa^+)$ which contains $\vec{C}$ as a member.

$\mathbb{P}$ is defined as the set of all pairs $p = \langle f_p, h_p \rangle$ such that

- (side-condition part)
  - $\mathrm{dom}(h_p)$ is a finite $\in$-chain of elements of $\mathfrak{M}$,
  - for each $N \in \mathrm{dom}(h_p)$, $h_p(N)$ is in $\kappa \smallsetminus (\delta_N + 1)$,
  - for any N_0, N_1 in $\mathrm{dom}(h_p)$, if $N_0 \in N_1$, then $h_p(N_0) \in N_1$, and hence $h_p(N_0) < \delta_{N_1}$,
- (working part) f_p is a specializing map from the set $\{\delta_N : N \in \mathrm{dom}(h_p)\}$.

The order is defined by $q \leqslant_{\mathbb{P}} p$ if and only if $f_q \supseteq f_p$ and $h_q \supseteq h_p$.

We will prove the following two lemmata.

Lemma 5.15. *Suppose that λ is a regular cardinal such that $H(\kappa^+) \in H(\lambda)$, N is a countable elementary submodel of $H(\lambda)$ such that N contains $\vec{C}$, $\mathbb{P}$ and $H(\kappa^+)$ as members. Then a condition p of $\mathbb{P}$ is $(N, \mathbb{P})$-generic if $N \cap H(\kappa^+) \in \operatorname{dom}(h_p)$.*

Lemma 5.16. $\Vdash_{\mathbb{P}}$ *"$\dot{E} := \{\delta_N : N \in \bigcup_{p \in \dot{G}} \operatorname{dom}(h_p)\}$ is countably closed".*

Lemma 5.15 implies that $\mathbb{P}$ is proper. So it follows from PFA, by use of the cardinal-collapsing trick as in [5, §4], that there exists a filter G of $\mathbb{P}$ such that the set $E := \{\delta_N : N \in \bigcup_{p \in G} \operatorname{dom}(h_p)\}$ is uncountable. Then E is of order type ω_1 and $\bigcup_{p \in G} f_p$ is a specializing function from E, which implies a contradiction as mentioned before. So the rest of the proof of Theorem 5.13 is to show Lemmata 5.15 and 5.16. Before proceeding to the proof of Lemma 5.15, we will show one preliminary proposition.

Proposition 5.17. *Let λ and N be as in the assumption of Lemma 5.15.*

(1) *Let I be a subset of $\mathsf{Lim}(\kappa)$ of size κ in N and $\varepsilon \in \mathsf{Lim}(\kappa) \setminus \delta_N$. Then there exists a subset J of I of size κ in N such that every element of J is incomparable to ε with respect to $\prec$.*

(2) *Let I be a subset of $[\mathsf{Lim}(\kappa)]^{<\aleph_0}$ of size κ in N and $\sigma \in [\mathsf{Lim}(\kappa) \setminus \delta_N]^{<\aleph_0}$. Then there exists a subset J of I of size κ in N such that every element of $\bigcup J$ is incomparable to any element of σ with respect to $\prec$.*

Proof. (2) follows from (1). We will show (1). Let I and ε be as in the assertion of the proposition. Define A as the set of all α in I such that the set $\{\beta \in I : \alpha \prec \beta\}$ is of size κ. By elementarity of N, A belongs to N. We consider the following two cases.

Suppose that A is bounded in κ. Then by induction on $\xi \in \kappa$, we can take $\alpha_\xi \in I$ which is greater than the supremum of the set

$$A \cup \{\beta \in I : \exists \eta < \xi (\alpha_\eta \prec \beta)\} \cup \{\alpha_\eta + 1 : \eta < \xi\}.$$

By elementarity of N, we may assume that the set $J = \{\alpha_\xi : \xi \in \kappa\}$ is in N. J forms an antichain with respect to $\prec$. We divide J into two disjoint subsets J_0 and J_1 of size κ in N. Then, since $J_0 \cup J_1$ forms an antichain in the tree $(\mathsf{Lim}(\kappa), \prec)$ and $\delta_N \leqslant \varepsilon$, for some $i \in \{0, 1\}$, every element of J_i is incomparable to ε with respect to $\prec$.

Suppose that A is cofinal in κ. Since the tree $(\mathsf{Lim}(\kappa), \prec)$ has no chain of size κ, there are two elements α_0 and α_1 in A which are incomparable with respect to $\prec$. By elementarity of N, we may assume that α_0 and α_1 are in N. Then, since both α_0 and α_1 are less than ε, for some $i \in \{0, 1\}$, α_i is incomparable to ε with respect to $\prec$. Then $J := \{\beta \in I : \alpha_i \prec \beta\}$ belongs to N, and any element of the set J is incomparable to ε with respect to $\prec$. □

Proof of Lemma 5.15. Let λ, N and p be as in the assumption of the lemma, and let D be a dense subset of $\mathbb{P}$ in N. We will show that $D \cap N$ is predense below p.

To do this, let q be an extension of p in $\mathbb{P}$. By extending q if necessary, we may assume that q belongs to D. Denote $n := |f_q \setminus N|$. Define S as the set of all elements σ of $[\mathsf{Lim}(\kappa)]^n$ such that there exists r in D which satisfies that

- $f_r \restriction \min(\sigma) = f_q \cap N$,
- for some $M \in \mathrm{dom}(h_r)$, $h_r \restriction M = h_q \cap N$, and
- $\mathrm{dom}(f_r) = \mathrm{dom}(f_q \cap N) \cup \sigma$.

By elementarity of N and the fact that κ is regular, S is in N and is of size κ. By Proposition 5.17, there exists $\sigma \in S \cap N$ such that every element of σ is incomparable to any element of $\mathrm{dom}(f_q)$. By elementarity of N, there exists $r \in \mathbb{P} \cap N$ which witnesses that σ is in S. Then $r \in D$ and r and q are compatible in $\mathbb{P}$, because $\langle f_r \cup f_q, h_r \cup h_q \rangle$ is a condition of $\mathbb{P}$ and is a common extension of r and q in $\mathbb{P}$. $\square$

Proof of Lemma 5.16. Let $p \in \mathbb{P}$, and $\langle \dot\alpha_n : n \in \omega \rangle$ a sequence of $\mathbb{P}$-names such that $p \Vdash_{\mathbb{P}}$ "$\{\dot\alpha_n : n \in \omega\}$ is a strictly increasing subset of $\dot E$". Let us show that $p \nVdash_{\mathbb{P}}$ "$\sup_{n\in\omega} \dot\alpha_n \notin \dot E$".

Assume not. Let $q \leqslant_{\mathbb{P}} p$ and a limit ordinal β be such that $q \Vdash_{\mathbb{P}}$ "$\sup_{n\in\omega} \dot\alpha_n = \beta$". By extending q if necessary, we may assume that there exists $N \in \mathrm{dom}(h_q)$ such that $q \Vdash_{\mathbb{P}}$ " N is the least element of $\bigcup_{r\in\dot G} \mathrm{dom}(h_r)$ with the property that $\beta \leqslant \delta_N$". By our assumption, it follows that $\beta < \delta_N$ because $q \Vdash_{\mathbb{P}}$ "$\beta \notin \dot E$ and $\delta_N \in \dot E$". Then there are $q' \leqslant_{\mathbb{P}} q$, $m \in \omega$ and $M \in \mathrm{dom}(h_{q'})$ such that $q \cap N \in M$ and $q' \Vdash_{\mathbb{P}}$ " $\dot\alpha_m = \delta_M$". Then $\mathrm{dom}(h_q) \cup \{M\}$ is an $\in$-chain and $\delta_M < \beta < \delta_N$. Let $q'' = \langle f_{q''}, h_{q''} \rangle$ be such that $\mathrm{dom}(h_{q''}) = \mathrm{dom}(h_q) \cup \{M\}$, $h_{q''}(M) = \beta + 1$ and $f_{q''} = f_{q'} \restriction \{\delta_N : N \in \mathrm{dom}(h_{q''})\}$. Then q'' is a condition of $\mathbb{P}$. q'' may not be compatible with q' in $\mathbb{P}$ (hence q'' may not force that $\dot\alpha_m = \delta_M$), but is an extension of q in $\mathbb{P}$. Then $q'' \Vdash_{\mathbb{P}}$ " the interval of the ordinal from $\delta_M + 1$ to $\beta + 1$ is disjoint from $\dot E$, $\{\dot\alpha_n : n \in \omega\} \subseteq \dot E$, and $\sup_{n\in\omega} \dot\alpha_n = \beta$". This is a contradiction. $\square$

Theorem 5.18. *$\mathbb{P}$ is Y-proper.*

Proof. This proof follows the one of Lemma 2.6 and Theorem 5.10. Let λ be a regular cardinal such that $\mathcal{P}(\mathbb{P})$ is in $H(\lambda)$. For a condition $p \in \mathbb{P}$, a subset $\mathcal{A}$ of $\mathbb{P}$ is p-large if and only if, for any $b \in H(\kappa^+)$, there are $q \in \mathcal{A}$ and a countable elementary submodel M of $H(\lambda)$ such that M contains $H(\kappa^+)$ as a member, $M \cap H(\kappa^+)$ is in $\mathrm{dom}(h_q)$, $h_q \restriction M = h_p$, $f_q \restriction \delta_{M\cap H(\kappa^+)} = f_p$, and $b \in M$.

Let $p \in \mathbb{P}$. We will show that $\{\bigvee \mathcal{A} : \mathcal{A} \subseteq \mathbb{P}$ is p-large$\}$ is a centered subset of $\mathrm{RO}(\mathbb{P})$. Let $n \in \omega$ and $\mathcal{A}_i$, $i \in n$, p-large subsets of $\mathbb{P}$. It suffices to find $q_i \in \mathcal{A}_i$, $i \in n$, such that $\{q_i : i \in n\}$ has a common extension in $\mathbb{P}$. To do this, let λ^* be a regular cardinal with $H(\lambda) \in H(\lambda^*)$, and countable elementary submodels N_i, $i \in n$, of $H(\lambda^*)$ such that each N_i contains $H(\kappa^+)$, p, $\mathbb{P}$, $H(\lambda)$, $\{N_j : j < i\}$ and $\mathcal{A}_i$ as members. By reverse induction on $i \in n$, we will find $q_i \in \mathcal{A}_i \cap N_i$ and a countable elementary submodel M_i of $H(\lambda)$ such that

- M_i contains $H(\kappa^+)$ and $\{N_j : j < i\}$ as members and $M_i \cap H(\kappa^+)$ is in $\mathrm{dom}(h_{q_i})$,
- $h_{q_i} \restriction M_i = h_p$,
- $f_{q_i} \restriction \delta_{M_i \cap H(\kappa^+)} = f_p$, and

- every member of $\mathrm{dom}(f_{q_i} \smallsetminus f_p)$ is incomparable to any element of $\bigcup_{i<j<n} \mathrm{dom}(f_{q_j} \smallsetminus f_p)$ with respect to $\prec$.

After finding all q_i's, the pair $\langle \bigcup_{i\in n} h_{q_i}, \bigcup_{i\in n} f_{q_i} \rangle$ is a condition of $\mathbb{P}$, and hence is a common extension of all q_i.

Suppose that we have found $q_j \in \mathcal{A}_j \cap N_j$ for all j with $i < j < n$. To find q_i, define I_i as the set of all $\mathrm{dom}(f_q \smallsetminus f_p)$ for all $q \in \mathcal{A}_i$. Then I_i is in N_i and, since $\mathcal{A}_i$ is p-large and κ is regular, I_i is of size κ. So by Proposition 5.17, there exists $\tau \in I_i \cap N_i$ such that every member of τ is incomparable to any element of $\bigcup_{i<j<n} \mathrm{dom}(f_{q_j} \smallsetminus f_p)$ with respect to $\prec$. By elementarity of N_i, there exists $q_i \in \mathcal{A}_i \cap N_i$ that witnesses that τ is in I_i. This finishes the choice of q_i.

The rest of the proof is similar to the one in the proof of Theorem 5.10. To show that $\mathbb{P}$ is Y-proper, let N be a countable elementary submodel such that N contains $\vec{C}$, $\mathbb{P}$ and $H(\kappa^+)$ as members, and let $p \in \mathbb{P}$ be such that $N \cap H(\kappa^+) \in \mathrm{dom}(h_p)$. Let us show that p is $(N, \mathbb{P})$-Y-generic. Let r be an extension of p in $\mathbb{P}$. Denote $r \restriction N = \langle f_r \cap N, h_r \cap N \rangle$, and define F as the filter on $\mathrm{RO}(\mathbb{P})$ that is generated by the set $\{\bigvee \mathcal{A} : \mathcal{A} \subseteq \mathbb{P} \text{ is } r \restriction N\text{-large}\}$. Then F belongs to N. We will show that, for any $s \in \mathrm{RO}(\mathbb{P}) \cap N$, if $r \leqslant_{\mathrm{RO}(\mathbb{P})} s$, then $s \in F$. Let $s \in \mathrm{RO}(\mathbb{P}) \cap N$ be such that $r \leqslant_{\mathrm{RO}(\mathbb{P})} s$, and define $\mathcal{A}$ as the set of all $q \in \mathbb{P}$ such that $q \leqslant_{\mathrm{RO}(\mathbb{P})} s$. Then $\mathcal{A}$ is in N, $\bigvee \mathcal{A} = s$. By elementarity of N and the fact that $N \cap H(\kappa^+) \in \mathrm{dom}(h_r)$, $\mathcal{A}$ is $r \restriction N$-large. Therefore $s = \bigvee \mathcal{A} \in F$. □

5.4. The failure of weak club guessing and the Mapping Reflection Principle.

A ladder system is a sequence $\langle C_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim} \rangle$ such that, for each $\alpha \in \omega_1 \cap \mathsf{Lim}$, C_α is a cofinal subset of α of order type ω. We say that a ladder system $\langle C_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim} \rangle$ weak club guesses a club subset C of ω_1 if and only if $C_\alpha \cap C$ is infinite for some α in C. Weak Club Guessing, denoted by WCG, is the assertion that there exists a ladder system which weak club guesses all club subsets of ω_1. PFA implies the negation of WCG, and it is known that a finite support iteration of ccc forcing notions of infinite length forces WCG (see introduction of [3]). Asperó and Mota introduced the following property of forcing notions and its forcing axiom.

Definition 5.19 (Asperó and Mota [3]).

- A forcing notion $\mathbb{P}$ is called *finitely proper* if and only if, for any large enough regular cardinal λ, any finite set $\{N_i : i < m\}$ of countable elementary submodels of $H(\lambda)$ which contain $\mathbb{P}$ as a member, and any condition p of $\mathbb{P}$ in all N_i, there exists an extension of p which is $(N_i, \mathbb{P})$-generic for every $i < m$.
- $\mathsf{PFA}^{\mathrm{fin}}(\omega_1)$ denotes the forcing axiom for the class of finitely proper forcing notions of size $\aleph_1$ and for families of $\aleph_1$ many dense sets.

Asperó and Mota proved in [3] that $\mathsf{PFA}^{\mathrm{fin}}(\omega_1)$ implies the failure of WCG, and, for any regular cardinal κ greater than $\aleph_1$, there exists a forcing iteration, called Asperó–Mota iteration, which forces $\mathsf{PFA}^{\mathrm{fin}}(\omega_1)$ and $2^{\aleph_0} = \kappa$.

In this section, we give the forcing notion for showing the following theorem.

Theorem 5.20 (Asperó and Mota [3]). $\mathsf{PFA}^{\mathrm{fin}}(\omega_1)$ *implies the nagation of* WCG.

To show the theorem, let $\vec{C} = \langle C_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim} \rangle$ be a ladder system.

Definition 5.21. $\mathbb{P}$ is defined as the set of all finite functions p such that

- $\mathrm{dom}(p)$ is a finite set of countable limit ordinals, and, for each $\alpha \in \mathrm{dom}(p)$, denote $p(\alpha) = \langle p_0(\alpha), p_1(\alpha)\rangle$ which is in $\alpha \times \omega_1$,
- (working part) for each $\alpha \in \mathrm{dom}(p)$, $C_\alpha \cap \mathrm{dom}(p) \subseteq p_0(\alpha)$,
- (side-condition part) for any α and β in $\mathrm{dom}(p)$, if $\alpha<\beta$, then $\alpha<p_1(\alpha)<\beta$.

The order is defined by $q \leqslant_{\mathbb{P}} p$ if and only if $q \supseteq p$.

This forcing notion seems to be different from the one in [3], but these are essentially same. By use of the side condition method, the proof of the properness may be simpler than the one in [3].

We will prove that $\mathbb{P}$ is proper. If PFA holds, then there exists a filter G on $\mathbb{P}$ such that $\omega_1 \cap \mathsf{Lim}$ is included in the union of the intervals $[\alpha, p_1(\alpha))$ of ordinals for all α in the set $\bigcup_{p\in G} \mathrm{dom}(p)$. Since the open intervals $(\alpha, p_1(\alpha))$ of ordinals for all α in the set $\bigcup_{p\in G} \mathrm{dom}(p)$ are pairwise disjoint, the set

$$(\omega_1 \cap \mathsf{Lim}) \setminus \Bigg(\bigcup_{p\in G} \bigcup_{\alpha\in \mathrm{dom}(p)} (\alpha, p_1(\alpha)) \Bigg)$$

is equal to $\bigcup_{p\in G} \mathrm{dom}(p)$. $\bigcup_{p\in G} \mathrm{dom}(p)$ is uncountable. Since $\omega_1 \cap \mathsf{Lim}$ is closed in the ordered topology of ω_1, $\bigcup_{p\in G} \mathrm{dom}(p)$ is club in ω_1. Therefore, by the definition of $\mathbb{P}$, $\vec{C}$ does not weak club guess the club set $\bigcup_{p\in G} \mathrm{dom}(p)$.

Lemma 5.22. *Suppose that λ is a regular cardinal such that $\mathcal{P}(\mathbb{P}) \in H(\lambda)$, λ^* is a regular cardinal such that $H(\lambda) \in H(\lambda^*)$, N^* is a countable elementary submodel of $H(\lambda^*)$ such that N^* contains $\vec{C}$ and $H(\lambda)$ as members. Then a condition p of $\mathbb{P}$ is $(N^*, \mathbb{P})$-generic (then p is also $(N^* \cap H(\lambda), \mathbb{P})$-generic), if $\mathrm{dom}(p)$ contains $\omega_1 \cap N^*$ as a member.*

Proof. Let N^* and p be as in the assumption of the lemma, and let D be a dense subset of $\mathbb{P}$ in N^*. We will show that $D \cap N^*$ is predense below p.

To do this, let q be an extension of p in $\mathbb{P}$. By extending q if necessary, we may assume that q belongs to D. The point of the proof is that, for all $\alpha \in \mathrm{dom}(q)$ which is greater than $\omega_1 \cap N^*$, $C_\alpha \cap N^*$ belongs to N^* (because then $C_\alpha \cap N^*$ is finite), however $C_{\omega_1\cap N^*} \cap N^* = C_{\omega_1\cap N^*}$, which does not belong to N^*. Take a countable elementary submodel M of $H(\lambda)$ in N^* such that M contains $\vec{C}$, $\mathbb{P}$, D, $q \cap N^*$ (which is equal to $q \restriction N^*$ because $\omega_1 \cap N^* \in \mathrm{dom}(q)$), $q_0(\omega_1 \cap N^*)$, and the set $\{C_\alpha \cap N^* : \alpha \in \mathrm{dom}(q) \setminus ((\omega_1 \cap N^*) + 1)\}$ as members. Then, for any $\alpha \in \mathrm{dom}(q) \setminus ((\omega_1 \cap N^*) + 1)$, $C_\alpha \cap M = C_\alpha \cap N^*$, and $C_{\omega_1\cap N^*} \cap M$ is finite. Let $\delta \in \omega_1 \cap M$ be such that, for all $\alpha \in \mathrm{dom}(q) \setminus N^*$, $C_\alpha \cap M \subseteq \delta$. By elementarity of M, there exists $r \in D \cap M$ such that $r \restriction \delta = q \cap N^*$ (which is equal to $q \restriction M$) and $r \leqslant_{\mathbb{P}} q \cap N^*$. Then q and r are compatible in $\mathbb{P}$. □

Theorem 5.23. *$\mathbb{P}$ is Y-proper.*

Proof. This proof follows the ones of Lemma 2.6, Theorems 5.10 and 5.18. But here, we need to take more care of checking the largeness of the set $\mathcal{A}$ than ones in Theorems 5.10 and 5.18.

For a condition $p \in \mathbb{P}$ and a subset $\mathcal{A}$ of $\mathbb{P}$, $E(p, \mathcal{A})$ denotes the set of all countable ordinals δ such that there exists $q \in \mathcal{A}$ such that $q \restriction (\delta + 1) = q \restriction \delta = p$ (hence $\delta \notin \mathrm{dom}(q)$) and $q \leqslant_{\mathbb{P}} p$, and define that a subset $\mathcal{A}$ of $\mathbb{P}$ is p-large if and only if $E(p, \mathcal{A})$ is stationary in ω_1.

Let λ and λ^* be as in Lemma 5.22, and $p \in \mathbb{P}$. We will show that $\{\bigvee \mathcal{A} : \mathcal{A} \subseteq \mathbb{P}$ is p-large$\}$ is a centered subset of $\mathrm{RO}(\mathbb{P})$. Let $n \in \omega$ and $\mathcal{A}_i$, $i \in n$, p-large subsets of $\mathbb{P}$. It suffices to find $q^i \in \mathcal{A}_i$, $i \in n$, such that $\{q^i : i \in n\}$ has a common extension in $\mathbb{P}$. To do this, take a sequence $\langle \lambda_i : i \in n+1 \rangle$ of regular cardinals such that $\lambda_0 = \lambda$ and, for each $i \in n$, $\lambda_{i+1} = (2^{\lambda_i})^+$. Denote $M_n = H(\lambda_n)$. By reverse induction on $i \in n$, we will find a countable elementary submodel M_i of $H(\lambda_i)$ in M_{i+1} and $q^i \in \mathcal{A}_i \cap M_{i+1}$ such that

(1) M_i contains $\vec{C}$, $\mathbb{P}$, p, $\{[H(\lambda_j)]^{\aleph_0}, \mathcal{A}_j : j \in i\}$ and $\{C_\alpha \cap M_j : j \in n \smallsetminus (i+1), \alpha \in \mathrm{dom}(q^j) \smallsetminus M_j\}$ as members,

(2) $q^i \restriction ((\omega_1 \cap M_i) + 1) = p$ (hence $\omega_1 \cap M_i \notin \mathrm{dom}(q^i)$) and $q^i \leqslant_{\mathbb{P}} p$.

Then, as seen in the proof of the properness, we can conclude that $\bigcup_{i \in n} q^i$ is a condition of $\mathbb{P}$, and so $\{q^i : i \in n\}$ has a common extension in $\mathbb{P}$. To find M_i and q^i as above, we assume that we have $\{M_j : j \in n \smallsetminus (i+1)\}$. Since $[H(\lambda_i)]^{\aleph_0}$ and $\mathcal{A}_i$ are in M_{i+1} and $\mathcal{A}_i$ is p-large, by elementarity of M_{i+1}, we can take a countable elementary submodel M_i of $H(\lambda_i)$ in M_{i+1} such that $\omega_1 \cap M_i \in E(p, \mathcal{A}_i)$ and M_i satisfies (1) above. Then, by elementarity of M_{i+1} again, we take $q^i \in \mathcal{A}_i \cap M_{i+1}$ which satisfies (2) above, which finishes the constructions of M_i and q^i.

To show that $\mathbb{P}$ is Y-proper, suppose that N^* is a countable elementary submodel of $H(\lambda^*)$ such that N^* contains $\vec{C}$ and $[H(\lambda)]^{\aleph_0}$ as members, $p \in \mathbb{P}$, and $\omega_1 \cap N^* \in \mathrm{dom}(p)$. By Lemma 5.22, p is $(N^*, \mathbb{P})$-generic. Let us show that p is $(N^*, \mathbb{P})$-Y-generic. Let r be an extension of p in $\mathbb{P}$. Define F as the filter on $\mathrm{RO}(\mathbb{P})$ that is generated by the set $\{\bigvee \mathcal{A} : \mathcal{A} \subseteq \mathbb{P}$ is $(r \restriction N^*)$-large$\}$. Then F belongs to N^*. We will show that, for any $s \in \mathrm{RO}(\mathbb{P}) \cap N^*$, if $r \leqslant_{\mathrm{RO}(\mathbb{P})} s$, then $s \in F$. Let $s \in \mathrm{RO}(\mathbb{P}) \cap N^*$ be such that $r \leqslant_{\mathrm{RO}(\mathbb{P})} s$, and define $\mathcal{A}$ as the set of all $q \in \mathbb{P}$ such that $q \leqslant_{\mathrm{RO}(\mathbb{P})} s$. Then $\mathcal{A}$ is in N^*, and $\bigvee \mathcal{A} = s$. So it suffices to show that $\mathcal{A}$ is $(r \restriction N^*)$-large, because then $s = \bigvee \mathcal{A} \in F$.

We will show that $\mathcal{A}$ is $(r \restriction N^*)$-large, that is, $E(r \restriction N^*, \mathcal{A})$ is stationary in ω_1. Since N^* contains $E(r \restriction N^*, \mathcal{A})$ as a member, it suffices to show that N^* satisfies that $E(r \restriction N^*, \mathcal{A})$ is stationary in ω_1. To do this, let I be a club subset of ω_1 in N^*. By elementarity of N^*, there exists a countable elementary submodel N of $H(\lambda)$ in N^* such that N contains $\vec{C}$, $\mathbb{P}$, $r \restriction N^*$, $\mathcal{A}$ and I. Then $\omega_1 \cap N$ belongs to I. r witnesses the assertion that $\omega_1 \cap N$ belongs to $\mathcal{A}$. Therefore, $I \cap E(r \restriction N^*, \mathcal{A})$ is not empty in N^*. □

These techniques can be applied to show that the negation of $\mho$ is implied by both $\mathsf{PFA}^{\mathrm{fin}}(\omega_1)$ and YPFA. But we need more care to show this. For more details, see [46].

The methods in Sections 5.3–5.4 can extend to Moore's Mapping Reflection Principle [29]. The Mapping Reflection Principle is the key notion to show that the Bounded Proper Forcing Axiom implies that the size of the continuum is $\aleph_2$.

The Ellentuck topology on the set $[X]^{\aleph_0}$ is the topology generated by the sets of the form

$$[x, Z] := \{Y \in [X]^{\aleph_0} : x \subseteq Y \subseteq Z\}$$

for some finite subset x of X and some infinite subset Z of X.

Definition 5.24 (Moore [29])**.** Σ is called an open stationary set mapping when there are an uncountable set X_Σ and a regular cardinal θ_Σ with $[X_\Sigma]^{\aleph_0} \in H(\theta_\Sigma)$ such that

- $\mathrm{dom}(\Sigma)$ is a club subset of the set of countable elementary submodels of $H(\theta_\Sigma)$,
- for every $M \in \mathrm{dom}(\Sigma)$,
  - $\Sigma(M)$ is an open subset of the space $[X_\Sigma]^{\aleph_0}$ equipped with the Ellentuck topology, and
  - $\Sigma(M)$ is M-stationary, i.e. for any club subset E of $[X_\Sigma]^{\aleph_0}$, if $E \in M$, then $E \cap \Sigma(M) \cap M \neq \emptyset$.

The Mapping Reflection Principle (MRP) is the assertion that, for any open stationary set mapping Σ, there exists a reflecting sequence for Σ, which means a continuous $\in$-chain $\langle N_\nu : \nu \in \omega_1 \rangle$ in $\mathrm{dom}(\Sigma)$ such that, for all limit ordinals $\nu \in \omega_1$, there exists $\nu_0 < \nu$ such that, for any $\xi \in (\nu_0, \nu)$, $N_\xi \cap X_\Sigma \in \Sigma(N_\nu)$.

Moore proved that PFA implies MRP, and MRP implies the equation $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$, the failure of $\square(\kappa)$ for all regular cardinal $\kappa > \omega_1$ [29], and the failure of both WCG and $\mho$. To prove that PFA implies MRP, Moore used a σ-distributive forcing notion. The following forcing notion is different from Moore's one. In [27], Miyamoto and the author proved that the following forcing notion is a Y-proper forcing notion which adds a reflecting sequence for a given open stationary set mapping.

Definition 5.25 (Miyamoto–Y. [27])**.** Let Σ be an open stationary set mapping. Define the forcing notion $\mathbb{P}$ which consists of finite subsets p of $\omega_1 \times \mathrm{dom}(\Sigma) \times \mathrm{dom}(\Sigma)$ such that

- for any $\langle \varepsilon, M_0, M_1 \rangle \in p$, $\varepsilon \in M_0 \in M_1$ and M_0 is a closure point of $\mathrm{dom}(\Sigma)$, that is, $M_0 = \bigcup(\mathrm{dom}(\Sigma) \cap M_0)$,
- for any different $\langle \varepsilon, M_0, M_1 \rangle$ and $\langle \varepsilon', M_0', M_1' \rangle$ in p, $\omega_1 \cap M_0 \neq \omega_1 \cap M_0'$ holds, and moreover,
  - if $\omega_1 \cap M_0 < \omega_1 \cap M_0'$, then $M_1 \in M_0'$, and
  - if $\varepsilon' < \omega_1 \cap M_0 < \omega_1 \cap M_0'$, then $M_0 \cap X_\Sigma \in \Sigma(M_0')$.

The order is defined by $q \leqslant_{\mathbb{P}} p$ if and only if $q \supseteq p$.

This forcing notion is proved to be Y-proper. However, this proof is more difficult than the ones in Sections 5.3 and 5.4.

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