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L SPACES WITH ADDITIONAL STRUCTURE

Abstract. We give a survey of L spaces with additional structure and several related open questions.

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1. Introduction

L space araised naturally in the investigation of the following topological basis problem.

Question 1.1. Which class of topological spaces has a 3-element basis? Or more specifically, what uncountable regular space contains no subspace homeomorphic to an uncountable subspace of reals, Sorgenfrey space or discrete space?

Then Todorcevic proved the following partial positive result in [26].

- (Todorcevic) Assume PFA. A regular space contains no uncountable discrete subspace iff it is hereditarily Lindelöf.

Topological spaces in this paper are all regular Hausdorff.

A space is *hereditarily Lindelöf* (or HL) if every subspace is Lindelöf and a space is *Lindelöf* if every open cover has a countable subcover.

Another topological property that is closely related to the topological basis problem is hereditary separability. A space is *hereditarily separable* (or HS) if every subspace is separable.

A natural attempt to construct a counterexample to the topological basis problem is to construct a space distinguishing HL and HS since none of the 3 fundamental spaces distinguishes HL and HS (see [28]). More precisely, an S space or an L space is a counterexample to the topological basis problem.

A space is an *S space* if it is HS but not HL. A space is an *L space* if it is HL but not HS.

The investigation of S and L spaces has a long history and has its own independent interest. A natural consistent example of an L space is a Suslin line. Then Rudin constructed in [20] an S space from a Suslin tree.

After that, S and L spaces with various properties were constructed from CH, $\diamondsuit$ and by forcing. Then Todorcevic's above result proves that S space does not exist under PFA. So S space can not be a counterexample to the topological basis problem.

For L spaces, various L spaces with additional properties were ruled out, e.g., strong L spaces [11] and first countable L spaces [25].

A space is *strong S (L) space* if all of its finite powers are S (L) spaces.

Then Moore constructed in [12] an L space and showed that the topological basis problem has a negative answer in the class of all regular spaces.

Moore's construction is based on Todorcevic's technique of minimal walks introduced in [27] which is a powerful tool in ZFC construction and has many applications (see [29]). Then Moore's discovery of the lower trace function leads to the ingenious construction of an L space.

It is natural to expect a positive answer to Question 1.1 in some restricted nice subclass, in the more specific sense if some fundamental space is not in the subclass. For example, positive results in specific subclasses are proved under PFA in [3] and [16]. Then the following question arise naturally.

Question 1.2. Which class of topological spaces contains no L space?

L spaces also arise naturally in different contexts other than the topological basis problem, e.g., in the context of cardinal functions (see [8]).

The paper is organized as follows. Section 2 reviews the minimal walk method. Section 3 reviews earlier results and Moore's L space construction. Section 4 reviews constructions of L spaces with additional algebraic structure. Section 5 reviews combinatorial properties of osc introduced in [12] which plays essential roles in constructing L spaces. Section 6 reviews two longstanding open problems on L space.

2. The minimal walk method

In this section, we introduce the minimal walk method introduced by Todorcevic in [27] (see also [29]). Most definitions and notations originate in [27] and [29] except for the lower trace function L which originates in [12]. For the following sections, we assume that readers are familiar with the combinatorial properties of osc in [12] and [17].

Definition 2.1. (1) $[X]^\kappa$ is the set of all subsets of X of size κ. Moreover, if X is a set of ordinals, $k < \omega$ and $b \in [X]^k$, then $b(0), b(1), \ldots, b(k-1)$ is the increasing enumeration of b and throughout this paper, we will use $(b(0), \ldots, b(k-1))$ to denote b.

(2) Suppose that a, b are both finite sets of ordinals but neither is an ordinal. Say $a < b$ if $\max a < \min b$.

Definition 2.2. A *C-sequence* (or a *ladder system*) is a sequence $\langle C_\alpha : \alpha < \omega_1 \rangle$ such that $C_{\alpha+1} = \{\alpha\}$ and C_α is a cofinal subset of α of order type ω for limit α's.

Roughly speaking, the *minimal walk* from β towards a smaller ordinal α is the sequence $\beta = \beta_0 > \beta_1 > \cdots > \beta_n = \alpha$ such that for each $i < n$, $\beta_{i+1} = \min(C_{\beta_i} \setminus \alpha)$. Here the *weight* of the step from β_i to α is $|C_{\beta_i} \cap \alpha|$.

Definition 2.3 ([27])**.** For a C-sequence $\langle C_\alpha : \alpha < \omega_1 \rangle$, the *maximal weight* of the walk is the function $\varrho_1 \colon [\omega_1]^2 \to \omega$, defined recursively by

$$\varrho_1(\alpha, \beta) = \max\{|C_\beta \cap \alpha|, \varrho_1(\alpha, \min(C_\beta \setminus \alpha))\}$$

with boundary value $\varrho_1(\alpha, \alpha) = 0$. $\varrho_{1\beta} \colon \beta \to \omega$ is defined by $\varrho_{1\beta}(\alpha) = \varrho_1(\alpha, \beta)$ for $\alpha < \beta$.

Intuitively, the function ϱ_1 is the function constructed from the C-sequence which records the maximal weight of all steps in one walk.

We also need the following splitting function.

Definition 2.4. For $\alpha < \beta < \omega_1$, $\Delta(\alpha, \beta) = \min(\{\xi < \alpha : \varrho_1(\xi, \alpha) \neq \varrho_1(\xi, \beta)\} \cup \{\alpha\})$.

The following two trace functions will be needed.

Definition 2.5 ([29])**.** For a given C-sequence, the *upper trace* $\mathrm{Tr} \colon [\omega_1]^2 \to [\omega_1]^{<\omega}$ is recursively defined for $\alpha \leqslant \beta < \omega_1$ as follows:

- $\mathrm{Tr}(\alpha, \alpha) = \{\alpha\}$;
- $\mathrm{Tr}(\alpha, \beta) = (\mathrm{Tr}(\alpha, \min(C_\beta \setminus \alpha)) \cup \{\beta\})$.

Definition 2.6 ([12])**.** For a given C-sequence, the *lower trace* $L \colon [\omega_1]^2 \to [\omega_1]^{<\omega}$ is recursively defined for $\alpha \leqslant \beta < \omega_1$ as follows:

- $L(\alpha, \alpha) = \emptyset$;
- $L(\alpha, \beta) = (L(\alpha, \min(C_\beta \setminus \alpha)) \cup \{\max(C_\beta \cap \alpha)\}) \setminus \max(C_\beta \cap \alpha)$.

We recall the following properties of these functions:

Definition 2.7. A function $a \colon [\omega_1]^2 \to \omega$ (or a sequence $\langle a_\beta : \beta < \omega_1 \rangle$ where $a_\beta \colon \beta \to \omega$) is coherent if for any $\alpha < \beta < \omega_1$, $\{\xi < \alpha : a(\xi, \alpha) \neq a(\xi, \beta)\}$ is finite.

Fact 2.8 ([27])**.** *ϱ_1 is coherent and $\rho_{1\beta}$ is finite-to-one for all β.*

Fact 2.9 ([12])**.** *(1) For limit ordinal $\beta > 0$, $\lim_{\alpha \to \beta} \min L(\alpha, \beta) = \beta$.*

(2) For $\alpha < \beta < \gamma$, if $L(\beta, \gamma) < L(\alpha, \beta)$, then $L(\alpha, \gamma) = L(\alpha, \beta) \cup L(\beta, \gamma)$.

The oscillation of two finite functions is well-known. Suppose that s and t are two functions defined on a common finite set of ordinals F. $\mathrm{Osc}(s, t; F)$ is the set of all ξ in $F \setminus \{\min F\}$ such that $s(\xi^-) \leqslant t(\xi^-)$ and $s(\xi) > t(\xi)$ where ξ^- is the greatest element of F less than ξ.

The osc map is then induced from the functions ϱ_1 and L.

Definition 2.10 ([12])**.** For $\alpha < \beta < \omega_1$, $\mathrm{Osc}(\alpha, \beta)$ denotes $\mathrm{Osc}(\varrho_{1\alpha}, \varrho_{1\beta}; L(\alpha, \beta))$ and $\mathrm{osc}(\alpha, \beta) = |\mathrm{Osc}(\alpha, \beta)|$ denotes the cardinality of $\mathrm{Osc}(\alpha, \beta)$.

We will also need Kronecker's Theorem.

Kronecker's Theorem ([10])**.** *Let A be a real $m \times n$ matrix and assume that $\{\boldsymbol{z} \in \mathbb{Q}^m : A^T\boldsymbol{z} \in \mathbb{Q}^n\} = \{\boldsymbol{0}\}$. Then for any $\epsilon > 0$, for any $b_0, \ldots, b_{m-1} \in \mathbb{R}$, there exist $p_0, \ldots, p_{m-1} \in \mathbb{Z}$, $\boldsymbol{q} \in \mathbb{Z}^n$ such that $|A_i\boldsymbol{q} - p_i - b_i| < \epsilon$ for all $i < m$ where A_i is the ith row of A.*

3. Earlier results and Moore's L space

3.1. Earlier results.

There are many consistent earlier results before Moore's first ZFC construction of an L space (see, e.g., [7, 21]). In this section, we review part of them.

Kunen proved that the existence of strong L spaces does not follow from ZFC.

Theorem 3.1 ([11])**.** *Assume MA_{ω_1}. There are no strong S or L spaces.*

In [6], Hajnal and Juhász constructed strong S and L spaces from CH. They then constructed strong S and L groups from CH in [6]. In [30], Zenor proved that there is a strong S space iff there is a strong L space.

Szentmiklóssy proved the consistency of no first countable L spaces.

Theorem 3.2 ([25])**.** *Assume MA_{ω_1}. There are no first countable L spaces.*

Above theorem generalizes an earlier result in [4] that there are no compact L spaces since HL spaces are point G_δ and character coincides with pseudo-character for compact spaces. Of course, the first consistent example of an L space—a Suslin line—is first countable. And its compactification is compact L.

Hajnal and Juhász consistently constructed an L space with large weight.

Theorem 3.3 ([5])**.** *It is consistent that there is an L space whose weight is greater than 2^ω.*

Since HL spaces have size $\leqslant 2^\omega$, an L space with large weight must have large character.

3.2. Moore's L space.

In this subsection, we recall Moore's L space construction and several properties of the L space.

The following combinatorial property of osc (see Definition 2.10) is the key to construct an L space.

Proposition 3.4 ([12])**.** *For every $\mathscr{A} \subset [\omega_1]^k$ and $\mathscr{B} \subset [\omega_1]^l$ which are uncountable families of pairwise disjoint sets and every $n < \omega$, there are a in $\mathscr{A}$ and b_m $(m < n)$ in $\mathscr{B}$ such that for all $i < k, j < l$, and $m < n$: $a < b_m$ and*
$\mathrm{osc}(a(i), b_m(j)) = \mathrm{osc}(a(i), b_0(j)) + m.$

Fix a rationally independent set

$$\{z_\alpha \in \mathbb{T} : \alpha < \omega_1\} \text{ where } \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}.$$

Definition 3.5 ([12])**.** $\mathscr{L} = \{w_\beta : \beta < \omega_1\} \subset \mathbb{T}^{\omega_1}$ where

$$w_\beta(\alpha) = \begin{cases} z_\alpha^{\mathrm{osc}(\alpha,\beta)+1} & : \alpha < \beta \\ 1 & : \alpha \geqslant \beta. \end{cases}$$

For $X \in [\omega_1]^{\omega_1}$, $\mathscr{L}_X = \{w_\beta \restriction_X : \beta \in X\}$.

Theorem 3.6 ([12]). *$\mathscr{L}$ is an L space.*

In fact, Moore's L space has additional structure.

Proposition 3.7 ([12]). *$\{w_\beta \restriction_\alpha : \alpha \leqslant \beta < \omega_1\}$ is an Aronszajn tree.*

The following result is a consequence of above proposition.

- ([12]) $\mathscr{L}^2$ contains an uncountable discrete subset.

The first author generalized above result to the following.

- ([15]) $\mathscr{L}^2$ contains an uncountable closed discrete subset.

Repovš and Zdomskyy proved that the semigroup generated by $\mathscr{L}$ is an L space.

- ([19]) The semigroup sgrp($\mathscr{L}$) generated by $\mathscr{L}$ is an L space.

Above result can not be generalized to group since by Proposition 3.7, $\{w_{a_\alpha(1)} - w_{a_\alpha(0)} : \alpha < \omega_1\}$ is discrete for sufficiently separated a_α's in $[\omega_1]^2$ with $w_{a_\alpha(1)} \restriction_\alpha = w_{a_\alpha(0)} \restriction_\alpha$.

We list several additional properties of $\mathscr{L}$ obtained in [12].

Theorem 3.8 ([12]).
1. *If $\mathscr{A} \subset [\omega_1]^k$ is uncountable pairwise disjoint, $B \in [\omega_1]^{\omega_1}$ and $\langle U_i : i < k\rangle$ is a sequence of open neighborhoods in $\mathbb{T}$, then there are $a \in \mathscr{A}$ and $\beta \in B$ such that $\max a < \beta$ and for all $i < k$, $w_\beta(a(i)) \in U_i$.*
2. *Every $X \in [\mathscr{L}]^{\omega_1}$ is dense above some $\alpha < \omega_1$, i.e., for every $a \in [\omega_1 \setminus \alpha]^{<\omega}$ and every sequence $\langle U_i : i < k\rangle$ of open neighborhoods in $\mathbb{T}$, there exists $x \in X$ such that $x(a(i)) \in U_i$ for all $i < k$.*
3. *Every continuous image of a subspace of $\mathscr{L}$ is countable.*

With a further investigation of the osc function, Moore discovered the following property.

Lemma 3.9 ([13]). *For every $\beta < \gamma < \omega_1$, there is $M < \infty$ such that whenever $\alpha < \beta$, $|\operatorname{osc}(\alpha, \beta) - \operatorname{osc}(\alpha, \gamma)| < M$.*

Applying above property, Moore constructed an L space whose square has an additional structure.

Theorem 3.10 ([13]). *There is an L space whose square has a σ-discrete dense subset.*

Due to $\mathscr{L}$'s Aronszajn property by Proposition 3.7, Todorcevic and the first author connected $\mathscr{L}$ with OCA (introduced in [28]).

For a topological space X, OCA_X is the assertion that if $[X]^2 = K_0 \cup K_1$ is a partition such that K_0 is open, then either there is an uncountable 0-homogeneous set, or X is a countable union of 1-homogeneous sets.

OCA_X can be viewed as a simple test question of topological basis problem. We list several results on OCA_X.

- ([28]) Assume PFA. If X^2 is hereditarily Lindelöf, then OCA_X holds.
- ([16]) Assume MA_{ω_1}. Suppose X is submetrizable and has size ω_1. If OCA_X holds, then X contains no uncountable discrete subset.

A space X is *submetrizable* if X has a weaker metric topology. And it turns out that submetrizability is necessary in above result [16]. More precisely, $\mathscr{L}^n$ is not submetrizable, has size ω_1 and contains an uncountable discrete subset when $n \geqslant 2$ while $\mathrm{OCA}_{\mathscr{L}^n}$ holds.

Theorem 3.11 ([16])**.** *Assume* PFA*.* $\mathrm{OCA}_{\mathscr{L}^n}$ *holds for every* $n < \omega$*.*

The L space problem is closely related to strong negative partition relations on ω_1. Moore's construction gives a strong coloring on ω_1.

- ([26]) $\omega_1 \to (\omega_1, (\operatorname{fin} \omega_1; \omega_1))^2$ is the assertion that if $c\colon [\omega_1]^2 \to 2$, then either there is an uncountable 0-homogeneous subset or there are uncountable pairwise disjoint $\mathscr{A} \subset [\omega_1]^{<\omega}$ and uncountable $B \subset \omega_1$ such that for all $a \in \mathscr{A}$ and $\beta \in B$ with $\max a < \beta$, there is $\alpha \in a$ with $c(\alpha, \beta) = 1$.

$\omega_1 \to (\omega_1, (\omega_1; \operatorname{fin} \omega_1))^2$ is similarly defined by replacing $\mathscr{A}$ by $A \subset \omega_1$ and B by pairwise disjoint $\mathscr{B} \subset [\omega_1]^{<\omega}$.

It is proved in [26] that PFA implies $\omega_1 \to (\omega_1, (\omega_1; \operatorname{fin} \omega_1))^2$ which in turn implies the non-existence of S spaces. Then Moore's construction induces a strong coloring witnessing the failure of $\omega_1 \to (\omega_1, (\operatorname{fin} \omega_1; \omega_1))^2$.

Theorem 3.12 ([12])**.** $\omega_1 \not\to (\omega_1, (\operatorname{fin} \omega_1; \omega_1))^2$*.*

4. L spaces with algebraic structure

In this section, we introduce L spaces with algebraic structure. The constructions rely on further investigation of Moore's osc function.

The semigroup generated by Moore's $\mathscr{L}$ is an L space by [19]. But the group generated by $\mathscr{L}$ is not HL by Proposition 3.7.

Then we constructed an L group in [17]. A topological group is an *L group* if the underlying space is an L space.

First we fix a rationally independent set of reals $\{\theta_\alpha : \alpha < \omega_1\}$ with the following additional property,

- for every $n < \omega$, every $\langle x_i \in span_{\mathbb{Q}}(\{\theta_\xi : \xi < \omega_1\}) : i < n\rangle$ with increasing heights[1] and every $\{q_i \in \mathbb{Q} \setminus \{0\} : i < n\}$, $\sum_{i<n} \frac{q_i}{x_i} \sin \frac{1}{x_i} \neq 0$.

Then the L group is defined as follows where $\operatorname{frac}(x)$ is the fraction part of x.

Definition 4.1. Denote $f(x) = \frac{1}{x} \sin \frac{1}{x}$ for $x \in \mathbb{R} \setminus \{0\}$.

(1) $\mathscr{L}' = \{w'_\beta : \beta < \omega_1\}$ where

$$w'_\beta(\alpha) = \begin{cases} f(\operatorname{frac}(\theta_\alpha \operatorname{osc}(\alpha, \beta) + \theta_\beta)) & : \alpha < \beta \\ 0 & : \alpha \geqslant \beta. \end{cases}$$

We view $\mathscr{L}'$ as a subspace of $\mathbb{R}^{\omega_1}$, equipped with the product topology.

(2) $\operatorname{grp}(\mathscr{L}')$ is the topological subgroup of $(\mathbb{R}^{\omega_1}, +)$ generated by $\mathscr{L}'$, where $+$ is the coordinatewise addition operation.

Theorem 4.2 ([17])**.** $\operatorname{grp}(\mathscr{L}')$ *is an L group.*

[1] The height of $x \in span_{\mathbb{Q}}(\{\theta_\xi : \xi < \omega_1\})$ is the least $\alpha < \omega_1$ such that $x \in span_{\mathbb{Q}}(\{\theta_\xi : \xi < \alpha\})$.

Since $\mathrm{grp}(\mathscr{L}')$ is HL, $\mathscr{L}'$ can not have an Aronszajn structure. In fact, $\mathscr{L}'$ is submetrizable. To see this, note that by our additional requirement on θ_α's, the projection of $\mathscr{L}'$ to the first coordinate, $w'_\beta \mapsto w'_\beta(0)$, is one-to-one.

Similar as Moore's L space, the L group $\mathrm{grp}(\mathscr{L}')$ has the following properties.

Theorem 4.3 ([17]). (1) *If $\mathscr{A} \subset [\omega_1]^k$ and $\mathscr{B} \subset [\omega_1]^l$ are uncountable families of pairwise disjoint sets, $\langle n_j : j < l\rangle$ is a sequence of non-zero integers and $\langle U_i \subset \mathbb{R} : i < k\rangle$ is a sequence of non-empty open sets, then there are $a \in \mathscr{A}$ and $b \in \mathscr{B}$ such that $\sum_{j<l} n_j w'_{b(j)}(a(i)) \in U_i$ for all $i < k$.*

(2) *Every $X \in [\mathrm{grp}(\mathscr{L}')]^{\omega_1}$ is dense above some $\alpha < \omega_1$.*

Then by [16, Theorem 2], under PFA (or MA_{ω_1}), $\mathrm{OCA}_{\mathrm{grp}(\mathscr{L}')^2}$ fails. But by [16, Corollary 3], the first assertion of $\mathrm{OCA}_{\mathrm{grp}(\mathscr{L}')^2}$ holds for some kinds of open sets under PFA.

Now we compare above L space $\mathscr{L}'$ with Moore's L space $\mathscr{L}$. First, we view Moore's L space in the following way.

- $\mathscr{L} = \{w_\beta : \beta < \omega_1\}$ where

$$w_\beta(\alpha) = \begin{cases} \mathrm{frac}(\theta_\alpha(\mathrm{osc}(\alpha,\beta)+1)) & : \alpha < \beta \\ 0 & : \alpha \geqslant \beta. \end{cases}$$

The $+1$ after $\mathrm{osc}(\alpha,\beta)$ in above definition is not important (see [13] for a modification). The f is not important for $\mathscr{L}'$. The role of f is to guarantee that the generated group is HL. So if we ignore $+1$ in definition of $\mathscr{L}$ and f in definition of $\mathscr{L}'$, w'_β is a modification of w_β by adding an interruption θ_β at each coordinate $< \beta$ and $\mathscr{L}'$ can be viewed as refining the topology of $\mathscr{L}$ by adding the topology of reals. More precisely,

- $X = \{x_\beta : \beta < \omega_1\}$ is homeomorphic to $X' = \{x'_\beta : \beta < \omega_1\}$ where $x_\beta(0) = \theta_\beta$, $x_\beta(\alpha) = \mathrm{frac}(\theta_\alpha(\mathrm{osc}(\alpha,\beta)+1))$ for $0 < \alpha < \beta$, $x_\beta(\alpha) = x'_\beta(\alpha) = 0$ for $\alpha \geqslant \beta$ and $x'_\beta(\alpha) = \mathrm{frac}(\theta_\alpha \,\mathrm{osc}(\alpha,\beta) + \theta_\beta)$ for $\alpha < \beta$.

This also indicates that f is necessary to guarantee HL of $\mathrm{grp}(\mathscr{L}')$ since $\mathrm{grp}(X)$ and hence $\mathrm{grp}(X')$ is not HL. In fact, f is used to determine a major element of $\sum_{j<m} n_j w'_{\beta_j}(\alpha)$—the j^* such that $\mathrm{frac}(\theta_\alpha \,\mathrm{osc}(\alpha,\beta_{j^*}) + \theta_{\beta_{j^*}})$ is close to 0. This also is the reason to have the interruption $+\theta_\beta$, e.g., in the case $w'_\beta(\alpha) - w'_{\beta'}(\alpha)$ with $\mathrm{osc}(\alpha,\beta) = \mathrm{osc}(\alpha,\beta')$, $f(\mathrm{frac}(\theta_\alpha \,\mathrm{osc}(\alpha,\beta))) - f(\mathrm{frac}(\theta_\alpha \,\mathrm{osc}(\alpha,\beta'))) = 0$ while $f(\mathrm{frac}(\theta_\alpha \,\mathrm{osc}(\alpha,\beta) + \theta_\beta)) - f(\mathrm{frac}(\theta_\alpha \,\mathrm{osc}(\alpha,\beta') + \theta_{\beta'}))$ still has a major part.

Several topological properties of $\mathrm{grp}(\mathscr{L}')^2$ are also investigated in [17].

- ([17]) $\mathrm{grp}(\mathscr{L}')^2$ contains an uncountable closed discrete subset. $\mathrm{grp}(\mathscr{L}')^2$ is neither normal nor weakly paracompact.

It turns out that the above idea can be generalized to control nth power of an L space/group for fixed $n < \omega$.

Theorem 4.4 ([17]). *For every positive integer n, there is a topological group G such that G^n is an L group while G^{n+1} is neither normal nor weakly paracompact. Moreover, G^{n+1} contains an uncountable closed discrete subset.*

Note that we can not expect an L group whose all finite powers are HL since such group does not exist under MA_{ω_1} by [11].

The proof also induces strong colorings on ω_1. We first recall some definition from [24] and its generalization.

- ([24]) $\mathrm{Pr}_0(\omega_1, \theta, \sigma)$ asserts that there is a function $c\colon [\omega_1]^2 \to \theta$ such that whenever we are given $n < \sigma$, an uncountable pairwise disjoint $\mathscr{A} \subset [\omega_1]^n$ and a function $h\colon n \times n \to \theta$, then there are $a < b$ in $\mathscr{A}$ such that $c(a(i), b(j)) = h(i, j)$ for every $i, j < n$.
- $\mathrm{Pr}_0(\omega_1, \theta, (\sigma; \eta))$ asserts that there is a function $c\colon [\omega_1]^2 \to \theta$ such that whenever we are given $n < \sigma$, $m < \eta$, two uncountable pairwise disjoint $\mathscr{A} \subset [\omega_1]^n$, $\mathscr{B} \subset [\omega_1]^m$ and a function $h\colon n \times m \to \theta$, then there are $a \in \mathscr{A}$ and $b \in \mathscr{B}$ such that $a < b$ and $c(a(i), b(j)) = h(i, j)$ for every $i < n, j < m$.

$\mathrm{Pr}_0(\omega_1, \omega, \sigma)$ ($\mathrm{Pr}_0(\omega_1, \omega, (\sigma; \eta))$) is equivalent to $\mathrm{Pr}_0(\omega_1, \omega_1, \sigma)$ ($\mathrm{Pr}_0(\omega_1, \omega_1, (\sigma; \eta))$) (see the beginning of Section 4 in [27]). But we do not know if $\mathrm{Pr}_0(\omega_1, \omega, \omega)$ is equivalent to $\mathrm{Pr}_0(\omega_1, 2, \omega)$. MA_{ω_1} implies $\neg\,\mathrm{Pr}_0(\omega_1, 2, \omega)$. But we do not know if $\neg\,\mathrm{Pr}_0(\omega_1, 2, \omega)$ implies, e.g., $\mathfrak{t} > \omega_1$ (see [28, 1.5]).

Question 4.5. Does $\mathfrak{t} = \omega_1$ imply $\mathrm{Pr}_0(\omega_1, 2, \omega)$?

$\mathrm{Pr}_0(\omega_1, 2, (\omega; 2))$ implies $\omega_1 \not\to (\omega_1, (\mathrm{fin}\,\omega_1; \omega_1))^2$. Moore actually proved $\mathrm{Pr}_0(\omega_1, \omega, (\omega; 2))$ in [12].

Our construction for higher powers shows the following.

Theorem 4.6 ([17])**.** $\mathrm{Pr}_0(\omega_1, \omega, (\omega; n))$ *holds for every* $2 < n < \omega$. *In particular,* $\mathrm{Pr}_0(\omega_1, \omega, n)$ *holds for every* $2 < n < \omega$.

This is optimal since MA_{ω_1} implies $\neg\,\mathrm{Pr}_0(\omega_1, 2, \omega)$.

One might expect that some strengthened algebraic property may reject L spaces. For example, stronger algebraic property makes differences for the metrization of pseudo-compact spaces. Shakhmatov [22] proved that every pesudo-compact subspace of a topological field is metrizable. But the situation for topological groups is different since every completely regular space can be embedded into a topological group—$\mathbb{R}^\kappa$.

However, we show that this is not the case for L spaces by constructing an L field. A *topological field* is a field endowed with a topology with respect to which all algebraic operations are continuous. A topological field is an *L field* if the underline space is an L space.

We first fix a function $g\colon (0, 1] \to \mathbb{C}$ such that

- $g(x) = x$ for $x \in [\frac{1}{3}, 1]$;
- for any integer $n \geqslant 3$, g is a linear function on $[\frac{1}{n+1}, \frac{1}{n}]$;
- for any positive integer n, $g(\frac{1}{n}) = p + qi$ for some rational number p, q;
- for any positive integer n, $g[(0, \frac{1}{n}]]$ is dense in $\mathbb{C}$.

Then we fix an algebraically independent set of reals $\{\theta_\alpha, \theta'_\alpha : \alpha < \omega_1\}$ with the following property,

- for every $l < \omega$, every $b \in [\omega_1]^l$ and every pair of polynomials $P(z_0, \dots z_{l-1})$, $Q(z_0, \dots z_{l-1})$ with rational coefficients,
$$\frac{P(\theta_{b(0)} + \theta'_{b(0)} i, \dots, \theta_{b(l-1)} + \theta'_{b(l-1)} i)}{Q(\theta_{b(0)} + \theta'_{b(0)} i, \dots, \theta_{b(l-1)} + \theta'_{b(l-1)} i)} \notin \mathbb{R}$$
provided that $\frac{P(z_0, \dots z_{l-1})}{Q(z_0, \dots z_{l-1})}$ is not a constant.

The L field is defined as follows.

Definition 4.7. (1) $\mathscr{L}'' = \{w''_\beta : \beta < \omega_1\}$ where
$$w''_\beta(\alpha) = \begin{cases} g(\mathrm{frac}(\theta_\alpha \operatorname{osc}(\alpha, \beta) + \theta_\beta)) & : \alpha < \beta \\ \theta_\beta & : \alpha = \beta \\ \theta_\beta + \theta'_\beta i & : \alpha > \beta. \end{cases}$$
We view $\mathscr{L}''$ as a subspace of $\mathbb{C}^{\omega_1}$, equipped with the product topology.

(2) field($\mathscr{L}''$) is the topological field generated by $\mathscr{L}''$, where the addition $+$ and multiplication $\cdot$ are coordinatewise addition and coordinatewise multiplication.

Theorem 4.8 ([18]). (1) field($\mathscr{L}''$) *is an L field.*

(2) *Every* $X \in [\mathrm{field}(\mathscr{L}'')]^{\omega_1}$ *is dense above some* $\alpha < \omega_1$.

(3) field$(\mathscr{L}'')^2$ *contains an uncountable closed discrete subset.*

(4) field$(\mathscr{L}'')^2$ *is neither normal nor weakly paracompact.*

A similar result about L vector space is also proved in [18]. A *topological vector space* is a vector space endowed with a topology with respect to which the vector operations are continuous. Here we consider vector spaces over $\mathbb{R}$. A topological vector space is an *L vector space* if the underline space is an L space.

Theorem 4.9 ([18]). *There is an L vector space whose square is neither normal nor weakly paracompact.*

5. Combinatorial properties of osc

Moore's method may have potential to construct more spaces with various properties. This probably relies on the discovery of combinatorial properties of osc. In this section, we recall its properties and indicate its connection to corresponding constructions.

The first combinatorial property is [12, Theorem 4.3] (Proposition 3.4 above). For n, a, b as in Proposition 3.4,
$$z_{a(i)}^{\operatorname{osc}(a(i), b_m(0))+1} = z_{a(i)}^{\operatorname{osc}(a(i), b_0(0))+1} * z_{a(i)}^m.$$
So in order to find m with $w_{b_m(0)}(a(i)) \in U_i$ for all $i < k$ for Theorem 3.8 (1), it suffices to require in advance that for appropriate $\epsilon > 0$,
$$\{(z_{a(0)}^m, \cdots, z_{a(k-1)}^m) : m < n\} \text{ is } \epsilon\text{-dense in } \mathbb{T}^k.$$
Say a set is ϵ-dense if it meets every ball of radius ϵ.

Now, for a given uncountable pairwise disjoint $\mathscr{A} \subset [\omega_1]^k$, apply Kronecker's Theorem to find $\mathscr{A}' \in [\mathscr{A}]^{\omega_1}$ and n such that every $a \in \mathscr{A}'$ satisfies the ϵ-dense condition.

The key combinatorial property of osc to construct an L group in [17] is the following.

Proposition 5.1 ([17])**.** *For uncountable families of pairwise disjoint sets $\mathscr{A} \subset [\omega_1]^k$ and $\mathscr{B} \subset [\omega_1]^l$, there are $\mathscr{A}' \in [\mathscr{A}]^{\omega_1}$, $\mathscr{B}' \in [\mathscr{B}]^{\omega_1}$ and $\langle c_{ij} : i < k, j < l\rangle \in \mathbb{Z}^{k\times l}$ such that for every $a \in \mathscr{A}'$ and every $b \in \mathscr{B}'$, if $a < b$, then*

$$\operatorname{osc}(a(i), b(j)) = \operatorname{osc}(a(i), b(0)) + c_{ij} \textit{ for all } i < k, j < l.$$

Note that $c_{i,0} = 0$ for every $i < k$ in above proposition. To simplify the notation, we use the following function defined on $\mathbb{R} \setminus \mathbb{Z}$:

$$f^*(x) = f(\operatorname{frac}(x)) \text{ where } f(x) = \frac{1}{x} sin \frac{1}{x}.$$

For $a \in \mathscr{A}', b \in \mathscr{B}'$ as in above proposition and a sequence of non-zero integers $\langle n_j : j < l\rangle$,

$$\begin{aligned}\sum_{j<l} n_j w'_{b(j)}(a(i)) &= \sum_{j<l} n_j f^*(\theta_{a(i)} \operatorname{osc}(a(i), b(j)) + \theta_{b(j)})\\ &= \sum_{j<l} n_j f^*(\theta_{a(i)} \operatorname{osc}(a(i), b(0)) + \theta_{a(i)}c_{ij} + \theta_{b(j)})\\ &= \sum_{j<l} n_j f^*(x + \theta_{a(i)}c_{ij} + \theta_{b(j)})\end{aligned}$$

where $x = \theta_{a(i)} \operatorname{osc}(a(i), b(0))$.

Now we view $\sum_{j<l} n_j f^*(x + \theta_{a(i)}c_{ij} + \theta_{b(j)})$ as a function with variable x. As x approaches $-\theta_{b(0)}$ from the right,

$$\sum_{0<j<l} n_j f^*(x + \theta_{a(i)}c_{ij} + \theta_{b(j)}) \to \sum_{0<j<l} n_j f^*(-\theta_{b(0)} + \theta_{a(i)}c_{ij} + \theta_{b(j)})$$

and $f^*(x + \theta_{b(0)})$ ranges over $\mathbb{R}$. In particular, for every non-empty open $U \subset \mathbb{R}$, there is $x \in (-\theta_{b(0)}, -\theta_{b(0)} + \epsilon)$ such that

$$\sum_{j<l} n_j f^*(x + \theta_{a(i)}c_{ij} + \theta_{b(j)}) \in U.$$

And the collection of such x is open.

So in order to have $\sum_{j<l} n_j w'_{b(j)}(a(i)) \in U_i$, it suffices to find x such that

- $\sum_{j<l} n_j f^*(x + \theta_{a(i)}c_{ij} + \theta_{b(j)}) \in U_i$ (as analyzed above);
- $\operatorname{frac}(\theta_{a(i)} \operatorname{osc}(a(i), b(0)))$ is close enough to $\operatorname{frac}(x)$ (as in Moore's construction).

Another property of osc which is dual to that of Proposition 5.1 is the following.

Proposition 5.2 ([17]). *For every $X \in [\omega_1]^{\omega_1}$ and every $\langle c_{ij} : i < k, j < l \rangle \in \mathbb{Z}^{k\times l}$ such that $c_{i0} = 0$ whenever $i < k$, there are uncountable pairwise disjoint families $\mathscr{A} \subset [X]^k$ and $\mathscr{B} \subset [X]^l$ such that for every $a \in \mathscr{A}$ and $b \in \mathscr{B}$, if $a < b$, then*

$$\operatorname{osc}(a(i), b(j)) = \operatorname{osc}(a(i), b(0)) + c_{ij} \text{ for all } i < k, j < l.$$

The behaviors of Proposition 5.1 and Proposition 5.2 are different when we take one pairwise disjoint family instead of two.

Proposition 5.3 ([17]). *For every uncountable family of pairwise disjoint sets $\mathscr{A} \subset [\omega_1]^k$, there are $\mathscr{A}' \in [\mathscr{A}]^{\omega_1}$ and $\langle c_{ij} : i, j < k \rangle \in \omega^{k\times k}$ such that for any a, b in $\mathscr{A}'$, if $a < b$, then*

$$\operatorname{osc}(a(i), b(j)) = \operatorname{osc}(a(i), b(i)) + c_{ij} \text{ for all } i, j < k.$$

Unlike Proposition 5.1, $c_{i,j}$'s in above proposition are non-negative. This means that for each $i < k$, $\operatorname{osc}(a(i), b(i))$ is the smallest among $\operatorname{osc}(a(i), b(j))$'s (see Observation after [18, Corollary 5.3] for an explanation).

Proposition 5.4 ([18]). *The following statement is independent of* ZFC.

- *For every $X \in [\omega_1]^{\omega_1}$, every $k < \omega$ and every $\langle c_{ij} : i, j < k \rangle \in \omega^{k\times k}$ such that $c_{ii} = 0$ for $i < k$, there is an uncountable pairwise disjoint family $\mathscr{A} \subset [X]^k$ such that for all $a, b \in \mathscr{A}$, if $a < b$, then*

$$\operatorname{osc}(a(i), b(j)) = \operatorname{osc}(a(i), b(i)) + c_{ij} \text{ for all } i, j < k.$$

There is an easy way to determine for what $\langle c_{ij} : i, j < k \rangle \in \omega^{k\times k}$ can we find $\mathscr{A}$ to satisfy the conclusion of above proposition.

Lemma 5.5 ([18]). *For a given $X \in [\omega_1]^{\omega_1}$ and $\langle c_{ij} : i, j < k \rangle \in \omega^{k\times k}$ such that $c_{ii} = 0$ whenever $i < k$, the following statements are equivalent.*

(1) *For every uncountable pairwise disjoint family $\mathscr{A} \subset [X]^k$, there are $a < b$ in $\mathscr{A}$ and $i, j < k$ such that*

$$\operatorname{osc}(a(i), b(j)) \neq \operatorname{osc}(a(i), b(i)) + c_{ij}.$$

(2) *There is a club E such that for every $\delta \in E$ and every $a \in [X \setminus \delta]^k$, there are $i, j < k$ such that*

$$|\operatorname{Osc}(\varrho_{1a(i)}, \varrho_{1a(j)}; L(\delta, a(j)))| \neq c_{ij}.$$

(3) *There is an uncountable set $A \subset \omega_1$ such that for every $\delta \in A$ and every $a \in [X \setminus \delta]^k$, there are $i, j < k$ such that*

$$|\operatorname{Osc}(\varrho_{1a(i)}, \varrho_{1a(j)}; L(\delta, a(j)))| \neq c_{ij}.$$

Consequently, for $X \in [\omega_1]^{\omega_1}$ and $k < \omega$, let $\mathcal{C}_k(X)$ be the collection of all patterns $\langle c_{ij} : i, j < k \rangle \in \omega^{k\times k}$ that can be realized inside X, in the sense that for some uncountable family $\mathscr{A} \subset [X]^k$ of pairwise disjoint sets, $\operatorname{osc}(a(i), b(j)) = \operatorname{osc}(a(i), b(i)) + c_{ij}$ whenever $a < b$ are in $\mathscr{A}$ and $i, j < k$. Fix a countable elementary submodel $\mathcal{M} \prec H(\omega_2)$ containing the C-sequence and X. Then

$$\mathcal{C}_k(X) = \{\langle |\operatorname{Osc}(\varrho_{1a(i)}, \varrho_{1a(j)}; L(\mathcal{M} \cap \omega_1, a(j)))| : i, j < k \rangle : a \in [X \setminus \mathcal{M}]^k\}.$$

6. L spaces with other structures

In this section, we review two longstanding open problems concerning L spaces.

6.1. Symmetrizable L spaces. The concept symmetric traces back to Nemytskii and Aleksandrov (see [1]). For a set X, a mapping $d\colon X\times X \to [0,\infty)$ is a *symmetric* if for any $x, y \in X$,

- $d(x, y) = 0$ iff $x = y$;
- $d(x, y) = d(y, x)$.

A symmetric is a weakening of metric by omitting triangle inequality. The topology generated by a symmetric d is as follows: U is open iff for every $x \in U$, there exists $\epsilon > 0$ with $B(x, \epsilon) \subset U$ where $B(x, \epsilon) = \{y : d(x, y) < \epsilon\}$. A topological space is *symmetrizable* if its topology can be generated by a symmetric.

It is well-known that for a metric space, the four properties are equivalent: separable, Lindelöf, HS, HL. Does any equivalence hold in the class of symmetrizable spaces?

First, separability does not imply any other property, e.g., $\mathcal{A}\cup\omega$ for an uncountable almost disjoint family $\mathcal{A} \subset [\omega]^\omega$ with topology generated by $\{\{n\}, \{a\}\cup a \setminus m : n, m < \omega, a \in \mathcal{A}\}$.

Then, Nedev [14] proved that for symmetrizable spaces, HS implies HL and Lindelöf is equivalent to HL.

In 1966, Arkhangel'skii posted the following question (see [23]): is every Lindelöf symmetrizable space separable? Note that a counterexample must be an L space by Nedev's result. Then Shakhmotov consistently constructed a counterexample.

It is still open if there is a ZFC counterexample.

Question 6.1. Is there a symmetrizable L space?

It is worth mentioning that Balogh, Burke and Davis proved the following results in [2].

- In ZFC, there is a Hausdorff non-regular symmetrizable space which is HL and not separable.
- There is no left separated Lindelöf symmetrizable space of uncountable cardinality.

Although every L space has a left separated L subspace, symmetrizability is not closed under taking subspaces. It seems likely that a new method of constructing L spaces is needed.

6.2. Cardinal function. Cardinal function is one of the most useful and important concepts in set-theoretic topology. We mainly consider hd and hL here. See [8] for more concepts and results on cardinal functions.

For a topological space X,

- $d(X) = \min\{|S| : S \subseteq X, \bar{S} = X\} + \omega$ is the density of X;
- $hd(X) = \sup\{d(Y) : Y \subseteq X\}$ is the hereditary density of X;
- $L(X) = \min\{\kappa$: every open cover of X has a subcover of cardinality $\leqslant \kappa\}+\omega$ is the Lindelöf degree;

- $hL(X) = \sup\{L(Y) : Y \subseteq X\}$ is the hereditary Lindelöf degree.
- $w(X) = \min\{|\mathscr{B}| : \mathscr{B}$ is a base for $X\} + \omega$ is the weight of X.

We recall several related results for a regular space X.

- (Pospíšil) $|X| \leqslant 2^{2^{d(X)}}$.
- $w(X) \leqslant 2^{d(X)}$.
- (De Groot) $|X| \leqslant 2^{hL(X)}$.
- $w(X) \leqslant 2^{|X|} \leqslant 2^{2^{hL(X)}}$.

So for HS space X, $|X| \leqslant 2^{2^\omega}$ and for HL space X, $w(X) \leqslant 2^{2^\omega}$.

Is the upper bound optimal? In [5] and [6], Hajnal and Juhász consistently constructed examples of S spaces of size 2^{2^ω} and L spaces of weight 2^{2^ω}. Then Juhász and Shelah [9] consistently constructed such examples with 2^ω and 2^{2^ω} arbitrarily large.

On the other hand, Todorcevic [26] proved that under PFA, $|X| \leqslant 2^\omega$ for HS space (or space without uncountable discrete subset). But we do not have any analogous result on HL spaces. Note that by results listed above, if an HL space has weight $> 2^\omega$, then it is not separable and has character $> 2^\omega$. The following question is still open.

Question 6.2. Is there an L space with character (or equivalently, weight) $> 2^\omega$?

Again, it seems likely that a new method of constructing L spaces is needed.

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