Samuel M. Corson

WHY THE CONE GROUPS CAN BE ISOMORPHIC

Abstract. In this survey we give some motivation and also an outline of the argument for why the fundamental group of the Griffiths double cone space is isomorphic to that of the triple. Indications are also given on how to solve a couple of similar problems.

Mathematics Subject Classification (2010): NN-02, 03E75, 20A15, 55Q52; 20F10, 20F34

Keywords: Griffiths space, fundamental group, infinite word

Universidad Politécnica de Madrid
sammyc973@gmail.com
ORCID: 0000-0003-0050-2724

DOI: https://doi.org/10.64191/zr24040410105

Contents

1. Introduction

Given an abstract group homomorphism f between topological groups G and H, one wonders whether f is continuous. In other words, does respect for the algebraic structure imply respect for the topological structure? It is fairly easy to give examples where this fails. For example, if we consider the group of real numbers $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$, we pick (by the axiom of choice) a basis for $\mathbb{R}$, and then project to one of the coordinates. This gives a homomorphism from $\mathbb{R}$ to $\mathbb{Q}$, both having the natural topologies, which is not continuous (since $\mathbb{R}$ is connected and $\mathbb{Q}$ is not).

For another example, we take G to be the group of order two to the power of the natural numbers. We give G the product topology, with each coordinate group being discrete. Let U be a nonprincipal ultrafilter on the natural numbers, and take H to be the group of order two (under the discrete topology). Define a homomorphism from G to H by taking a sequence to the number 1 if and only if the sequence represents an element in the ultrafilter. This homomorphism is not continuous since each open neighborhood of identity in G includes elements which are not in the kernel.

Despite these examples, there are hypotheses on the groups G and H which force a homomorphism to be continuous. For example, it was shown by Dudley in 1961 [9] that if the topology on G is completely metrizable or locally compact and H is a free (abelian) group then a homomorphism is necessarily continuous. With similar hypotheses on G one can have H be torsion-free hyperbolic, Baumslag-Solitar, the Thompson group F, or a great many other groups and the conclusion still holds [6], and much can be said even when H is acylindrically hyperbolic [2] or has small torsion subgroups [15]. The beautiful theory surrounding ample generics, developed by Kechris and Rosendal [13], allows one to have G be the symmetric group on the natural numbers and H to be any Polish group and again an arbitrary homomorphism is continuous.

Automatic continuity questions arise also in algebraic topology. If X and Y are path connected spaces and $f\colon \pi_1(X, x) \to \pi_1(Y, y)$ is an abstract homomorphism, then one can ask whether there is a continuous function $c\colon X \to Y$ such that f is the induced homomorphism c^*. Typically one restricts attention to X and Y which are *Peano continua*—path connected, locally path connected compact metric spaces. A striking result in this regard is due to Kent, who showed that if X and Y are planar (i.e. $\subseteq \mathbb{R}^2$) or one-dimensional Peano continua then their fundamental groups are abstractly isomorphic if and only if the spaces are homotopy equivalent [14, Theorem 1.2]. In other words, isomorphism of fundamental groups holds in this situation precisely when there are continuous functions $c_1\colon X \to Y$ and $c_2\colon Y \to X$ with $c_2 \circ c_1$ and $c_1 \circ c_2$ being homotopic to the identity map on the appropriate space.

In this survey, we'll sketch out why this result is essentially the strongest possible, presenting a weakened version of Theorem A found in the paper [5]. Particularly we'll show that the Griffiths space $\mathbb{GS}_2$ (Figure 2) and the analogue $\mathbb{GS}_3$, having three cones instead of two, have isomorphic fundamental group. These are Peano continua, subspaces of $\mathbb{R}^3$, are two dimensional and almost look like manifolds with boundary. It is clear that no continuous function c from one space to the other can induce an isomorphism. For example, if $c\colon \mathbb{GS}_3 \to \mathbb{GS}_2$ is a continuous function then the loops in at least two of the cones in $\mathbb{GS}_3$ will have to eventually map into one of the cones in $\mathbb{GS}_2$, so c^* will have uncountable kernel. On the other hand, if $c\colon \mathbb{GS}_2 \to \mathbb{GS}_3$ is continuous then one of the cones of $\mathbb{GS}_3$ will have small loopss which are not mapped onto by c, and thus the index of the image of c^* will be uncountable. The construction of our abstract isomorphism is highly nonconstructive, and in fact what is truly shown in [5] is that $\pi_1(\mathbb{GS}_2)$ is isomorphic to $\pi_1(\mathbb{GS}_\kappa)$ where κ is a cardinal $2 \leqslant \kappa \leqslant 2^{\aleph_0}$.

In Sections 2 and 3 we present background and the main idea for the proof. In Section 4 we give some of technical lemmas leading to the isomorphism (Theorem 4.5). In Section 5 we indicate two additional results, whose proofs follow the same ideas, which have appeared in preprints.

2. The earring group and cone groups

We require a combinatorial characterization of the fundamental group of the Griffiths space. This uses a description of the fundamental group of the infinite earring, pictured in Figure 2. The infinite earring $\mathcal{E}$, also known as the Hawaiian earring, is the union $\mathcal{E} = \bigcup_{n\in\omega} C_n$ where C_n is the circle centered at $(\frac{1}{n+1}, 0) \in \mathbb{R}^2$ of radius $\frac{1}{n+1}$, with topology inherited from $\mathbb{R}^2$. The space $\mathcal{E}$ is compact, path connected, locally path connected. The fundamental group of $\mathcal{E}$ has a combinatorial description which resembles that of a free group (naturally, given the resemblance which $\mathcal{E}$ has to a bouquet of circles). However, this group is famously not free [12]. Our review of this group will be brief; the interested reader can find more information in [3].

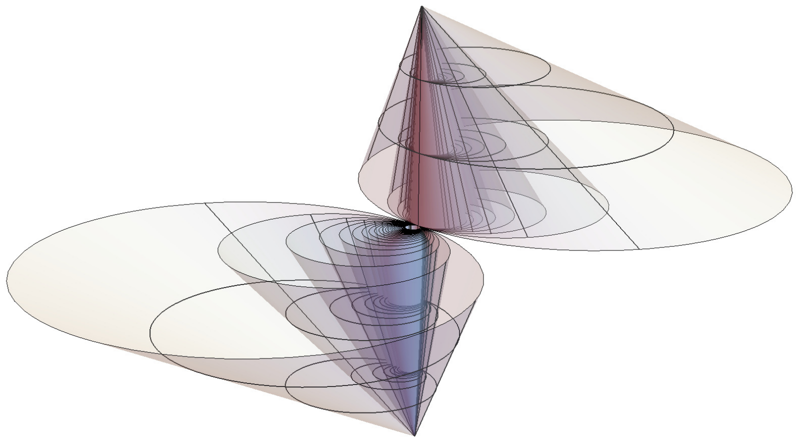

FIGURE 1. The Griffith space $\mathbb{GS}_2$

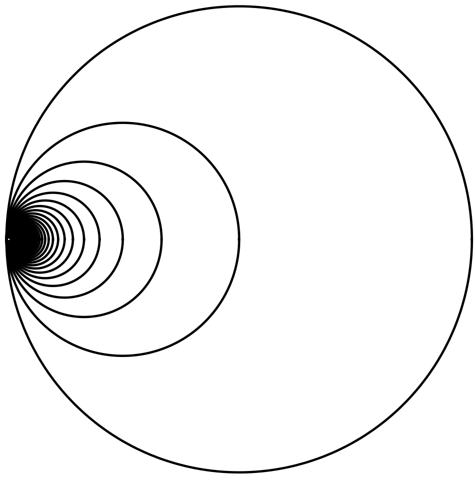

FIGURE 2. The infinite earring $\mathcal{E}$

Let $A = \{a_n^{\pm 1}\}_{n\in\omega}$ be a countably infinite set with formal inverses (this is the set of *letters*). A *word* is a function from a countable totally ordered set $\bar{W}$ to the set A such that for each $a \in A$ the set $\{i \in \bar{W} \mid W(i) = a\}$ is finite. For example one has the word $W\colon \omega \to A$ given by $n \mapsto a_n$, which one can "write" as $a_0a_1a_2\ldots$. More exotically one can have a word with domain $\mathbb{Q}$. We consider two words W and U to be the same, and write $W \equiv U$, if there exists an order isomorphism $\iota\colon \bar{W} \to \bar{U}$ such that for all $i \in \bar{W}$, $W(i) = U(\iota(i))$. When words are finite, the $\equiv$ equivalence agrees with the usual syntactic notion that the two words read the same, letter-for-letter. Let E denote the empty word; that is, the word with empty domain.

Each word W has an inverse which we will denote W^{-1}, which is given by taking $\overline{W^{-1}}$ to be the set $\bar{W}$ under the reverse order and letting $W^{-1}(i) = (W(i))^{-1}$. Given two words W_0 and W_1 we form their concatenation W_0W_1 by taking the domain $\overline{W_0W_1}$ to be the disjoint union $\overline{W_0} \sqcup \overline{W_1}$ which is ordered to extend the orders of $\overline{W_0}$ and $\overline{W_1}$ and places elements of $\overline{W_0}$ below those of $\overline{W_1}$. The function W_0W_1 is the following

$$W_0W_1(i) = \begin{cases} W_0(i) & \text{if } i \in \overline{W_0}, \\ W_1(i) & \text{if } i \in \overline{W_1}. \end{cases}$$

Importantly there is also a notion of *infinite concatenation*. Given a nonempty word W we let $\|W\| = \frac{1}{1+k}$ where k is the minimal subscript among the letters in the image of W; if $W \equiv E$ is the empty word then $\|W\| = 0$. Suppose that $\{W_\lambda\}_{\lambda\in\Lambda}$ is a collection of words indexed by a totally ordered set Λ and that for any real $\epsilon > 0$ the set $\{\lambda \in \Lambda \mid \|W_\lambda\| \geqslant \epsilon\}$ is finite. One forms the concatenation $W \equiv \prod_{\lambda\in\Lambda} W_\lambda$ by giving it the domain $\bigsqcup_{\lambda\in\Lambda} \overline{W_\lambda}$ ordered in the natural way and letting $W(i) = W_\lambda(i)$ where $i \in \overline{W_\lambda}$. It is easy to check that this function is indeed a word, and if each word W_λ is nonempty we know the index Λ is countable.

We are nearly ready to form the words into a group. For $n \in \omega$ and word W let $p_n(W)$ be the word given by the finite restriction $W \restriction \{i \in \bar{W} \mid W(i) \in \{a_m^{\pm 1}\}_{0\leqslant m\leqslant n}\}$. Write $W \sim U$ if for all $n \in \omega$ the words $p_n(W)$ and $p_n(U)$ are equal as words in the free group $F(a_0, \ldots, a_n)$. For example,

$$a_0a_1a_2 \ldots \ldots a_2^{-1}a_1^{-1}a_0^{-1} \sim E \tag{2.1}$$

Writing $[W]$ for the $\sim$ equivalence class of the word W, we obtain a group operation on the $\sim$ equivalence classes of words by defining $[W_0][W_1] := [W_0W_1]$. The identity element of the group is the equivalence class $[E]$ and the inverse is natural: $[W]^{-1} = [W^{-1}]$.

The group obtained is of cardinality $2^{\aleph_0}$. One can imagine the isomorphism between this group and the fundamental group of the infinite earring by taking for $i \in \omega$ a loop $l_i\colon [0,1] \to \mathcal{E}$ going counter-clockwise around the i-th circle C_i in $\mathcal{E}$. For a word W we realize a loop L_W by reading off the word W, and where the output is a_i (respectively a_i^{-1}) we then go around the loop l_i (respectively around the reverse loop l_i^{-1}). The map which takes an equivalence class $[W]$ and assigns to it the homotopy class of the loop L_W gives an isomorphism. Of course we are leaving out many details regarding parametrization and what is truly meant by "reading off" the word; also, the proof that this map is well-defined and an isomorphism requires work. The interested reader can find details in [3, §2.1].

As in a free group, we would prefer that the group elements be words instead of equivalence classes of words. We say a word W is *reduced* if whenever we write a concatenation $W \equiv W_0W_1W_2$ with $W_1 \sim E$ we have $W_1 \equiv E$. This definition is clearly an extension of that in free groups, and many of the same results can be obtained.

Lemma 2.1. [10, Theorem 1.4, Corollary 1.7] *For each word W there exists a unique up to $\equiv$ reduced word W_0 with $W \sim W_0$. Moreover if W and U are reduced there exist unique words W_0, W_1, U_0, U_1 such that*

(1) $W \equiv W_0 W_1$;
(2) $U \equiv U_0 U_1$;
(3) $W_1 \equiv U_0^{-1}$;
(4) $W_0 U_1$ *is reduced.*

One can obtain the reduced form of a word W via a (possibly infinite, complicated) process of pairing up elements of $\bar{W}$ whose outputs are inverse letters, and each point between elements in a pair also must be paired with a point between that pair. The reduced form of W is obtained by taking a maximal such *cancellation scheme* and restricting W to the set of points in $\bar{W}$ which do not appear in the scheme. It is a nontrivial fact that, up to $\equiv$, the word obtained does not depend on the maximal cancellation scheme which was used. In the benign example in (2.1) we take W to be the word on the left and pair the $\min(\bar{W})$ with $\max(\bar{W})$, pair $\min(\bar{W} \setminus \{\min(\bar{W}), \max(\bar{W})\})$ with $\max(\bar{W} \setminus \{\min(\bar{W}), \max(\bar{W})\})$, etc. As seen in Lemma 2.1, obtaining the reduced form of the product of two reduced words utilizes a similar uncomplicated cancellation scheme.

Having understood the earring group, we are ready to give a clean combinatorial description of the cone groups. We define words, reduced words, and the function $\|\cdot\|$ on the alphabet $\{a_n^{\pm 1}\}_{n\in\omega} \cup \{b_n^{\pm 1}\}_{n\in\omega}$ in the same way as above (a word is a finite-to-one function from a countable totally ordered set to the alphabet, etc.). Let $\mathrm{Red}_{a,b}$ denote the group of reduced words under the alphabet $\{a_n^{\pm 1}\}_{n\in\omega} \cup \{b_n^{\pm 1}\}_{n\in\omega}$. Let Red_a denote the group of reduced words in the original alphabet $\{a_n^{\pm 1}\}_{n\in\omega}$ and Red_b denote the group of reduced words in the alphabet $\{b_n^{\pm 1}\}_{n\in\omega}$. Both of Red_a and Red_b are subgroups of $\mathrm{Red}_{a,b}$. We point out that Lemma 2.1 still holds in each of these new settings.

The group $\mathrm{Red}_{a,b}$ is isomorphic to the fundamental group of the space obtained by taking two copies of the infinite earring and identifying at the interesting points. Incidentally, this space is homeomorphic to the earring, but we want the two earrings for defining the double cone group. One can view the Griffiths space $\mathbb{GS}_2$ as the space obtained by taking these conjoined two earrings, putting a topological cone over one earring, and then putting a topological cone over the other. The conjoined earrings at the "base" of $\mathbb{GS}_2$ generate the fundamental group; more precisely, the inclusion map of the base into $\mathbb{GS}_2$ induces a surjective homomorphism on fundamental groups. Note that any loop which remains in one of the earrings is nulhomotopic, since that earring has a topological cone above it. In fact, that observation precisely characterizes the fundamental group of $\mathbb{GS}_2$: by an application of the Seifert-van Kampen Theorem it is easy to see that the fundamental group of $\mathbb{GS}_2$ is isomorphic to $\mathrm{Red}_{a,b} / \langle\langle \mathrm{Red}_a \cup \mathrm{Red}_b \rangle\rangle$.

For example, the word

$$a_0 a_1 a_2 \ldots b_0 b_1 b_2 \ldots$$

represents the identity element in $\mathrm{Red}_{a,b} / \langle\langle \mathrm{Red}_a \cup \mathrm{Red}_b \rangle\rangle$, but the word

$$a_0 b_0 a_1 b_1 \ldots$$

does not. Thus, elements of the group $\mathrm{Red}_{a,b} / \langle\langle \mathrm{Red}_a \cup \mathrm{Red}_b \rangle\rangle$ are rather slippery objects because we can delete (sometimes in-)finitely many letters without changing the group element.

We close this section by defining $\mathrm{Red}_{c,d,e}$ to be the group of reduced words in the alphabet $\{c_n^{\pm 1}\}_{n\in\omega} \cup \{d_n^{\pm 1}\}_{n\in\omega} \cup \{e_n^{\pm 1}\}_{n\in\omega}$. By the same reasoning as before, the quotient $\mathrm{Red}_{c,d,e} / \langle\langle \mathrm{Red}_c \cup \mathrm{Red}_d \cup \mathrm{Red}_e \rangle\rangle$ is isomorphic to the fundamental group of the space $\mathbb{GS}_3$ obtained by identifying three copies of the infinite earring at their interesting points and placing a cone over each earring.

3. COI Triples: The building blocks of an isomorphism

Now that we have seen the combinatorial descriptions of $\pi_1(\mathbb{GS}_2)$ and $\pi_1(\mathbb{GS}_3)$ in Section 2, we can forget about spaces and focus instead on the combinatorics of infinite words.

3.1. Decomposition into pure subwords. We'll start with a straightforward concept.

Definition 3.1. We say that a word $W \in \mathrm{Red}_{a,b}$ is *a-pure* (respectively *b-pure*) if $W \in \mathrm{Red}_a$ (resp. $W \in \mathrm{Red}_b$). Generally a word in $\mathrm{Red}_{a,b}$ is *pure* if it is a-pure or b-pure. For $U \in \mathrm{Red}_{c,d,e}$ we say it is *c-pure*, *d-pure*, *e-pure* and *pure* in the comparable way.

For a word $W \in \mathrm{Red}_{a,b}$ we give a canonical way in which to write W as a concatenation of nonempty pure words. This is done by selecting (if W is not empty) an element $i \in \bar{W}$ and taking $I_i \subseteq \bar{W}$ to be the maximal interval such that $i \in I_i$ and $W \upharpoonright I_i$ is pure. Clearly for $j \in I_i$ we obtain $I_j = I_i$ under this process. This decomposition of $\bar{W}$ into nonempty, pairwise disjoint intervals is ordered in the natural way. Write $\mathrm{ind}(W)$ for this ordered set, which can be considered the index of the concatenation. Now we can write our word as a concatenation $W \equiv \prod_{I\in \mathrm{ind}(W)} W \upharpoonright I$, where each subword $W \upharpoonright I$ is a nonempty pure subword of W which has been made "as large as possible." For brevity, we will generally write instead $W \equiv_p \prod_{\lambda\in\mathrm{ind}(W)} W_\lambda$ to express that this decomposition of W is in the canonical way. The decomposition of $U \in \mathrm{Red}_{c,d,e}$ is given and denoted similarly: $U \equiv_p \prod_{\lambda\in\mathrm{ind}(U)} U_\lambda$. For the empty word we consider that $\mathrm{ind}(E) = \emptyset$.

We next give a special definition for those subwords of a word which respect this decomposition.

Definition 3.2. We say W_0 is a *p-chunk* of $W \equiv_p \prod_{\lambda\in\mathrm{ind}(W)} W_\lambda$ if there exists an interval $I \subseteq \mathrm{ind}(W)$ such that $W_0 \equiv \prod_{\lambda\in I} W_\lambda$. We shall also denote this word by $W \upharpoonright_p I$ (note that we may also write $W_0 \equiv_p \prod_{\lambda\in I} W_\lambda$). Write p-chunk$(W)$ for the set of all p-chunks of W.

One can imagine that a p-chunk as a subword which respects the pure subwords. Note that it can easily be the case that $\mathrm{ind}(W)$ is order isomorphic to the rationals $\mathbb{Q}$, so p-chunk(W) can be of size $2^{\aleph_0}$.

Definition 3.3. A subgroup G of $\mathrm{Red}_{a,b}$, or of $\mathrm{Red}_{c,d,e}$, is *p-fine* if for each $W \in G$ we have p-chunk$(W) \subseteq G$.

Lemma 3.4. *If $\mathcal{X} \subseteq \mathrm{Red}_{a,b}$ then the subgroup generated by $\bigcup_{W \in \mathcal{X}}$ p-chunk(W) in $\mathrm{Red}_{a,b}$ is p-fine, and is the smallest p-fine subgroup which includes the set $\mathcal{X}$. Similarly for $\mathcal{X} \subseteq \mathrm{Red}_{c,d,e}$.*

This lemma requires a bit of work to prove. One can give a full description of the elements of this subgroup, using Lemma 2.1 . The elements are precisely those words $W \equiv_p \prod_{\lambda \in \mathrm{ind}(W)} W_\lambda$ for which there is a finite collection of intervals $I_0, \ldots, I_j$ in $\mathrm{ind}(W)$ with $\mathrm{ind}(W) = \bigsqcup_{i=0}^{j} I_i$ and for $0 \leqslant i \leqslant j$ we have

(1) $I_i = \{\lambda\}$ is a singleton and W_λ is a product of pure elements (and inverses) in $\bigcup_{W \in \mathcal{X}}$ p-chunk(W); or
(2) $\prod_{\lambda \in I_i} W_\lambda$ is a p-chunk, or the inverse of a p-chunk, of an element in $\mathcal{X}$.

Definition 3.5. For $\mathcal{X} \subseteq \mathrm{Red}_{a,b}$, or $\mathcal{X} \subseteq \mathrm{Red}_{c,d,e}$, we write $\mathrm{Pfine}(\mathcal{X})$ for the minimal p-fine subgroup including $\mathcal{X}$.

3.2. Close subsets, coi triples.

Definition 3.6. If Λ is a totally ordered set we say that a subset $\Lambda_0 \subseteq \Lambda$ *is close in* Λ if for each infinite interval $I \subseteq \Lambda$ we have $\Lambda_0 \cap I \neq \emptyset$.

For example, a subset of $\mathbb{Q}$ is close provided it is dense. A subset of ω is close if it is unbounded.

Definition 3.7. A *close order isomorphism* (abbreviated *coi*) between two totally ordered sets Λ and Θ is an order isomorphism $\iota\colon \Lambda_0 \to \Theta_0$ with Λ_0 close in Λ and Θ_0 close in Θ.

From a coi $\iota\colon \Lambda_0 \to \Theta_0$ between Λ and Θ we get a (not necessarily one-to-one) correspondence between the intervals in Λ and those in Θ. More specifically if $I \subseteq \Lambda$ is an interval then we let $\underline{\iota}(I)$ denote the smallest interval in Θ which includes the set $\iota(I \cap \Lambda_0)$. Similarly from an interval $J \subseteq \Theta$ we get an interval $\underline{\iota^{-1}}(J) \subseteq \Lambda$. The correspondence allows some forgetfulness: given an interval $I \subseteq \Lambda$ we have that $\underline{\iota^{-1}}(\underline{\iota}(I))$ is a subinterval in I (it is precisely the smallest interval in I which includes the set $I \cap \Lambda_0$), and $I \setminus \underline{\iota^{-1}}(\underline{\iota}(I))$ is finite.

Definition 3.8. A *coi triple* is an ordered triple (W, ι, U) such that

(1) $W \in \mathrm{Red}_{a,b}$;
(2) $U \in \mathrm{Red}_{c,d,e}$; and
(3) ι is a close order isomorphism between $\mathrm{ind}(W)$ and $\mathrm{ind}(U)$.

We let

$$\beth_{a,b}\colon\ \mathrm{Red}_{a,b} \to \mathrm{Red}_{a,b} /\langle\langle \mathrm{Red}_a \cup \mathrm{Red}_b\rangle\rangle$$

and

$$\beth_{c,d,e} : \mathrm{Red}_{c,d,e} /\langle\langle \mathrm{Red}_c \cup \mathrm{Red}_d \cup \mathrm{Red}_e\rangle\rangle$$

denote the quotient maps. The following mouthful of a definition delineates the tool that we use to construct our isomorphism.

Definition 3.9. A collection $\{(W_x, \iota_x, U_x)\}_{x \in X}$ of coi triples is *coherent* if

(1) for any $x, x' \in X$, intervals $I \subseteq \mathrm{ind}(W_x)$ and $I' \subseteq \mathrm{ind}(W_{x'})$, and $i \in \{-1, 1\}$ if
$$W_x \restriction_p I \equiv (W_{x'} \restriction I')^i$$
then
$$\sqsupset_{c,d,e}(U_x \restriction_p \underline{\iota_x}(I)) = \sqsupset_{c,d,e}((U_{x'} \restriction_p \underline{\iota_{x'}}(I'))^i)$$
and

(2) for any $x, x' \in X$, intervals $J \subseteq \mathrm{ind}(U_x)$ and $J' \subseteq \mathrm{ind}(U_{x'})$, and $j \in \{-1, 1\}$ if
$$U_x \restriction_p J \equiv (U_{x'} \restriction J')^j$$
then
$$\sqsupset_{a,b}(W_x \restriction_p \underline{\iota_x^{-1}}(J)) = \sqsupset_{a,b}((W_{x'} \restriction_p \underline{\iota_{x'}^{-1}}(J'))^j).$$

The definition essentially says that, up to deletion of finitely many p-chunks, the collection tells a compatible algebraic "story" on the p-chunks of the words appearing in the coi triples. Conditions (1) and (2) give the definition a very symmetric flavor, and this gets used to prove Theorem 3.10 below.

We point out that given a collection of coi triples it is a very serious matter to check that it is coherent. This is even the case if the collection has only one coi triple in it. For example suppose we have a collection of cardinality one, $\{(W, \iota, U)\}$, and $\mathrm{ind}(W)$ is of order type $\mathbb{Q}$. There are $2^{\aleph_0}$ intervals in $\mathrm{ind}(W)$, and to see if the collection is coherent we must check that if $W \restriction_p I$ is equivalent to $W \restriction_p I'$, or to its inverse, then the appropriate words determined in $\mathrm{Red}_{c,d,e}$ are equal under the quotient $\sqsupset_{c,d,e}$. We must also check this among all of the intervals $J, J' \subseteq \mathrm{ind}(U)$. The reader can imagine how tedious this becomes when the collection contains more than one element.

Nevertheless, insofar as one can manage to produce a large coherent collection they are rewarded with an isomorphism (see [5, Proposition 3.16]).

Theorem 3.10. *Suppose $\{(W_x, \iota_x, U_x)\}_{x\in X}$ is a coherent collection of coi triples. Then there exist homomorphisms $\phi_0\colon \mathrm{Pfine}(\{W_x\}_{x\in X}) \to \sqsupset_{c,d,e}(\mathrm{Pfine}(\{U_x\}_{x\in X}))$ and $\phi_1\colon \mathrm{Pfine}(\{U_x\}_{x\in X}) \to \sqsupset_{a,b}(\mathrm{Pfine}(\{W_x\}_{x\in X}))$ such that for all $x \in X$, $\phi_0(W_x) = \sqsupset_{c,d,e}(U_x)$ and $\phi_1(U_x) = \sqsupset_{a,b}(W_x)$. If in addition $\mathrm{Pfine}(\{W_x\}_{x\in X}) = \mathrm{Red}_{a,b}$ and $\mathrm{Pfine}(\{U_x\}_{x\in X}) = \mathrm{Red}_{c,d,e}$, the homomorphism ϕ_0 descends to an isomorphism*
$$\Phi\colon \mathrm{Red}_{a,b} / \langle\langle \mathrm{Red}_a \cup \mathrm{Red}_b \rangle\rangle \to \mathrm{Red}_{c,d,e} / \langle\langle \mathrm{Red}_c \cup \mathrm{Red}_d \cup \mathrm{Red}_e \rangle\rangle$$
and ϕ_1 descends to Φ^{-1}.

In [5, Proposition 3.16] it was claimed that from a coherent $\{(W_x, \iota_x, U_x)\}_{x\in X}$ one obtains an isomorphism $\Phi\colon \sqsupset_{a,b}(\mathrm{Pfine}(\{W_x\}_{x\in X})) \to \sqsupset_{c,d,e}(\mathrm{Pfine}(\{U_x\}_{x\in X}))$ (without assuming $\mathrm{Pfine}(\{W_x\}_{x\in X}) = \mathrm{Red}_{a,b}$ and $\mathrm{Pfine}(\{U_x\}_{x\in X}) = \mathrm{Red}_{c,d,e}$). This requires an argument that, if N is the normal subgroup of $\mathrm{Pfine}(\{W_x\}_{x\in X})$ normally generated by the set $\mathrm{Pfine}(\{W_x\}_{x\in X}) \cap (\mathrm{Red}_a \cup \mathrm{Red}_b)$, and M is the normal subgroup of $\mathrm{Red}_{a,b}$ normally generated by $\mathrm{Red}_a \cup \mathrm{Red}_b$, the equality
$$N = M \cap \mathrm{Pfine}(\{W_x\}_{x\in X})$$

holds. In case X has cardinality less than $2^{\aleph_0}$, [5, Proposition 3.16] is correct simply because we show that the collection can be extended to a larger one whose p-fine subgroups are respectively $\mathrm{Red}_{a,b}$ and $\mathrm{Red}_{c,d,e}$. We simply make the more conservative Theorem 3.10 given above and this is sufficient for the task.

Theorem 3.10 is proved very carefully using the decomposition mentioned in Lemma 3.4. One defines a homomorphism in the most straighforward way imaginable, works out the details to show that it is well-defined and that it is a bijection. An essential fact used is that in the images of $\sqsupset_{a,b}$ and $\sqsupset_{c,d,e}$ one can add or delete pure p-chunks without changing the group element. Now the game is clear. We obtain the desired isomorphism if we succeed in constructing a coherent coi collection $\{(W_x, \iota_x, U_x)\}_{x\in X}$ such that $\mathrm{Pfine}(\{W_x\}_{x\in X}) = \mathrm{Red}_{a,b}$ and $\mathrm{Pfine}(\{U_x\}_{x\in X}) = \mathrm{Red}_{c,d,e}$.

It is well to return to the example when the collection, $\{(W, \iota, U)\}$, has only one element and $\mathrm{ind}(W)$ is of order type $\mathbb{Q}$. If this collection is coherent, then using the work done in Theorem 3.10 one obtains homomorphisms $\phi_0\colon \mathrm{Pfine}(\{W_x\}_{x\in X}) \to \sqsupset_{c,d,e}(\mathrm{Pfine}(\{U_x\}_{x\in X}))$ and $\phi_1\colon \mathrm{Pfine}(\{U_x\}_{x\in X}) \to \sqsupset_{a,b}(\mathrm{Pfine}(\{W_x\}_{x\in X}))$, each have domain of cardinality $2^{\aleph_0}$. This once again illustrates the amount of information that one coi triple can encode. It also illustrates a potential pitfall that one must overcome. It may be that in the course of constructing a coherent coi collection we accidentally make $\mathrm{Pfine}(\{U_x\}_{x\in X}) = \mathrm{Red}_{c,d,e}$ while $\mathrm{Pfine}(\{W_x\}_{x\in X})$ is still a proper subgroup of $\mathrm{Red}_{a,b}$, or vice versa. So, we might exhaust one side without having exhausted the other. Of course another potential difficulty could simply be that we produce a coherent coi collection $\{(W_x, \iota_x, U_x)\}_{x\in X}$ and we fail to make either $\mathrm{Pfine}(\{W_x\}_{x\in X})$ or $\mathrm{Pfine}(\{U_x\}_{x\in X})$ large enough because our ingenuity in extending a coherent collection runs out.

We surmount all challenges by ensuring that the collection has cardinality strictly smaller than $2^{\aleph_0}$ at each stage of the induction. Diagonalization arguments show that it is always possible to add one more coi triple to the collection.

4. Adding another coi triple to a small collection

We have seen that it may be difficult to enlarge a coherent collection of coi triples. We show how this is done, proving the main theorem. Lemmas 4.1, 4.2, 4.3, 4.4 coorrespond with Lemma 3.18 and Propositions 3.20, 3.21, 3.23 of [5], respectively.

A useful warmup result is the following.

Lemma 4.1. *Suppose that* $\{(W_x, \iota_x, U_x)\}_{x\in X}$ *is a coherent collection of coi triples and* $\epsilon > 0$.

(a) *If* $W \in \mathrm{Pfine}(\{W_x\}_{x\in X})$ *then there exists a* $U \in \mathit{Red}_{c,d,e}$ *with* $\|U\| < \epsilon$ *and a close order isomorphism* ι *such that* $\{(W_x, \iota_x, U_x)\}_{x\in X} \cup \{(W, \iota, U)\}$ *is coherent.*

(b) *If* $U \in \mathrm{Pfine}(\{U_x\}_{x\in X})$ *then there exists a* $W \in \mathrm{Red}_{a,b}$ *with* $\|W\| < \epsilon$ *and a close order isomorphism* ι *such that* $\{(W_x, \iota_x, U_x)\}_{x\in X} \cup \{(W, \iota, U)\}$ *is coherent.*

This lemma is proved using the description of elements in $\mathrm{Pfine}(\cdot)$ used in Lemma 3.4. If $W \in \mathrm{Pfine}(\{W_x\}_{x\in X})$ then we can write W as a finite concatenation of words $W \equiv W_0 \dots W_j$, where each W_i is (1) pure, or (2) an element of $(\bigcup_{x \in X} \text{p-chunk}(W_x)) \cup (\bigcup_{x \in X} \text{p-chunk}(W_x^{-1}))$. For a W_i of type (2) we can write $W_i \equiv_p (W_{x_i} \upharpoonright_p I_i)^{\alpha_i}$ with $I_i \subseteq \text{ind}(W_{x_i})$ an interval and $\alpha_i \in \{\pm 1\}$, and we carefully modify the word $U_{x_i} \upharpoonright_p \underline{\iota_{x_i}}(I_i)$ at finitely many elements in $\underline{\iota_{x_i}}(I_i)$ to obtain a word $U_i' \in \text{Red}_{c,d,e}$ with $|\underline{U_i'}| < \epsilon$. Now the coi ι_{x_i} can be restricted to give a coi ι_i from $\text{ind}(W_i)$ to $\text{ind}((U_i')^{\alpha_i})$. If W_i is of type (1) then we take U_i' to be the empty word. The desired word U is given by $U_1 \dots U_j$, where U_i is either U_i' or is $U_i'V_i$ (where V_i is pure and $|V_i| < \epsilon$) as needed in order to ensure that U is a reduced word. The coi ι is the union of the ι_i considered in type (2). The check that $\{(W_x, \iota_x, U_x)\}_{x \in X} \cup \{(W, \iota, U)\}$ is coherent is straightforward but tedious. The proof of (b) is essentially the same, but using the inverses of the coi.

There was no requirement in the previous lemma that the collection of coi triples was smaller than $2^{\aleph_0}$, but this becomes essential in what follows. The next lemma says that if $W \in \text{Red}_{a,b}$ is written as an ω type concatenation of words which are already in $\text{Pfine}(\{W_x\}_{x \in X})$, then we can coherently extend the coi collection so that W also is in the generated p-fine group.

Lemma 4.2. *Assume $\{(W_x, \iota_x, U_x)\}_{x \in X}$ is a coherent collection of coi triples with $|X| < 2^{\aleph_0}$.*

(a) *Suppose $W \in \text{Red}_{a,b}$ is such that $\text{ind}(W) = \bigsqcup_{n \in \omega} I_n$ with each $I_n \neq \emptyset$ an interval in $\text{ind}(W)$ and elements of I_n are below elements of I_m when $n < m$. Suppose further that $W \upharpoonright_p I_n \in \text{Pfine}(\{W_x\}_{x \in X})$ for all $n \in \omega$ but $W \notin \text{Pfine}(\{W_x\}_{x \in X})$. Then there exists $U \in \text{Red}_{c,d,e}$ and coi ι from $\text{ind}(W)$ to $\text{ind}(U)$ such that $\{(W_x, \iota_x, U_x)\}_{x \in X} \cup \{(W, \iota, U)\}$ is coherent.*

(b) *Suppose $U \in \text{Red}_{c,d,e}$ is such that $\text{ind}(U) = \bigsqcup_{n \in \omega} J_n$ with each $J_n \neq \emptyset$ an interval in $\text{ind}(U)$ and elements of J_n are below elements of J_m when $n < m$. Suppose further that $U \upharpoonright_p J_n \in \text{Pfine}(\{U_x\}_{x \in X})$ for all $n \in \omega$ but $U \notin \text{Pfine}(\{U_x\}_{x \in X})$. Then there exists $W \in \text{Red}_{a,b}$ and coi ι from $\text{ind}(W)$ to $\text{ind}(U)$ such that $\{(W_x, \iota_x, U_x)\}_{x \in X} \cup \{(W, \iota, U)\}$ is coherent.*

This lemma is proved by iterative use of Lemma 4.1. We extend to a coherent collection $\{(W_x, \iota_x, U_x)\}_{x \in X} \cup \{(W \upharpoonright_p I_0, \iota_0, U_0')\}$, then to a coherent collection $\{(W_x, \iota_x, U_x)\}_{x \in X} \cup \{(W \upharpoonright_p I_0, \iota_0, U_0'), (W \upharpoonright_p I_1, \iota_1, U_1')\}$, etc. so that $\|U_k'\| < \frac{1}{k}$ and each ι_k has nonempty domain (this latter condition can be easily added). The requirement that $\|U_k'\| < \frac{1}{k}$ makes it so that the concatenation which we define later is in fact a word. It is clear that the collection $\{(W_x, \iota_x, U_x)\}_{x \in X} \cup \{(W \upharpoonright I_k, \iota_k, U_k')\}_{k \in \omega}$ is coherent, as an increasing union of coherent collections. The word U is given as a concatenation $U \equiv U_0 U_1 U_2 \dots$ where $U_k \equiv U_k' V_k$ with $\|V_k\| < \frac{1}{k}$. The interstitial words V_k have $1 \leqslant |\text{ind}(V_k)| \leqslant 2$ and some p-chunk of V_k is not in $\text{Pfine}(\{U_x\}_{x \in x} \cup \{U_k\}_{k \in \omega})$ (the ability to do this is guaranteed by the fact that the collection is of size $< 2^{\aleph_0}$). Each V_k also ensures that U is a reduced word (i.e. it interrupts any cancellation that might otherwise occur between U_k' and U_{k+1}'). The coi ι is the union of the ι_k. The check that the extension is coherent is even more tedious than in Lemma 4.1; the fact that each V_k has a p-chunk which is not an element in $\text{Pfine}(\{U_x\}_{x \in x} \cup \{U_k\}_{k \in \omega})$ ensures that part (2) of Definition 3.9 holds.

Next we switch from ω type concatenation to $\mathbb{Q}$ type.

Lemma 4.3. *Assume* $\{(W_x, \iota_x, U_x)\}_{x\in X}$ *is a coherent collection of coi triples with* $|X| < 2^{\aleph_0}$.

(a) *Suppose* $W \in \mathrm{Red}_{a,b}$ *is such that* $\mathrm{ind}(W) = \bigsqcup_{q\in\mathbb{Q}} I_q$ *with each* $I_q \neq \emptyset$ *an interval in* $\mathrm{ind}(W)$ *and elements of* I_q *are below elements of* $I_{q'}$ *when* $q < q'$. *Suppose further that* $W \upharpoonright_p I_q \in \mathrm{Pfine}(\{W_x\}_{x\in X})$ *for all* $q \in \mathbb{Q}$ *and* $W \upharpoonright_p I \notin \mathrm{Pfine}(\{W_x\}_{x\in X})$ *implies that* $I \subseteq I_q$ *for some* $q \in \mathbb{Q}$. *Then there exists* $U \in \mathrm{Red}_{c,d,e}$ *and coi* ι *from* $\mathrm{ind}(W)$ *to* $\mathrm{ind}(U)$ *such that* $\{(W_x, \iota_x, U_x)\}_{x\in X} \cup \{(W, \iota, U)\}$ *is coherent.*

(b) *Suppose* $U \in \mathrm{Red}_{c,d,e}$ *is such that* $\mathrm{ind}(U) = \bigsqcup_{q\in\mathbb{Q}} J_q$ *with each* $J_q \neq \emptyset$ *an interval in* $\mathrm{ind}(U)$ *and elements of* J_q *are below elements of* $J_{q'}$ *when* $q < q'$. *Suppose further that* $U \upharpoonright_p I_q \in \mathrm{Pfine}(\{U_x\}_{x\in X})$ *for all* $q \in \mathbb{Q}$ *and* $U \upharpoonright_p J \in \mathrm{Pfine}(\{U_x\}_{x\in X})$ *implies that* $J \subseteq J_q$ *for some* $q \in \mathbb{Q}$. *Then there exists* $W \in \mathrm{Red}_{a,b}$ *and coi* ι *from* $\mathrm{ind}(W)$ *to* $\mathrm{ind}(U)$ *such that* $\{(W_x, \iota_x, U_x)\}_{x\in X} \cup \{(W, \iota, U)\}$ *is coherent.*

To prove this lemma, we write down a list $\{W_k\}_{k\in\omega}$ of words in $\mathrm{Pfine}(\{W_x\}_{x\in X})$ such that for each $q \in \mathbb{Q}$ there is a unique $k \in \omega$ such that $W \upharpoonright_p I_q \equiv W_k$ or $W \upharpoonright_p \equiv W_k^{-1}$. Now as in Lemma 4.2 we use Lemma 4.1 to extend to a coherent collection $\{(W_x, \iota_x, U_x)\}_{x\in X} \cup \{(W_k, \iota_k, U'_k)\}_{k\in\omega}$ such that $\|U'_k\| < \min\{\frac{1}{k}, \|U'_{k-1}\|\}$ and the domain of ι_k is nonempty. Again, we are requiring $\|U'_k\| < \min\{\frac{1}{k}, \|U'_{k-1}\|\}$ so that a concatenation defined later will indeed be a word. Next we select pure words $V_{k,0}$ and $V_{k,1}$ so that $\|V_{k,0}\| = \|V_{k,1}\| = \|W_k\|$ and $V_{k,j} \notin \mathrm{Pfine}(\{U_x\}_{x\in X} \cup \{U'_k\}_{k\in\omega})$ when $j = 0, 1$. We also require that $V_{k,0}U'_kV_{k,1}$ is a reduced word.

The word $U \in \mathrm{Red}_{c,d,e}$ is given by

$$U \equiv \prod_{q\in\mathbb{Q}} (V_{k_q,0}U'_{k_q}V_{k_q,1})^{\alpha_q}$$

where $k_q \in \omega$ and $\alpha_q \in \{\pm 1\}$ are such that $W \upharpoonright_p I_q \equiv W_{k_q}^{\alpha_q}$ (note that the only word which is $\equiv$ to its own inverse is the empty word, so α_q is well-defined). One must check that U is in fact a reduced word. To this end, one imagines a cancellation scheme on U and modifies it in such a way that it pulls back to a cancellation scheme on W, giving a contradiction.

The coi ι from $\mathrm{ind}(W)$ to $\mathrm{ind}(U)$ is produced by gluing together the coi ι_{k_q} which go from $W_{k_q}^{\alpha_q}$ to $(U'_{k_q})^{\alpha_q}$. Coherence of $\{(W_x, \iota_x, U_x)\}_{x\in X} \cup \{(W, \iota, U)\}$ is checked very carefully. The use of $V_{k,0}$ and $V_{k,1}$ for $k \in \omega$ aids us in checking part (2) of Definition 3.9.

It turns out that the extension lemmas proven so far are already sufficient to give us the general induction below.

Lemma 4.4. *Assume* $\{(W_x, \iota_x, U_x)\}_{x\in X}$ *is a coherent collection of coi triples with* $|X| < 2^{\aleph_0}$.

(1) *If* $W \in \mathrm{Red}_{a,b}$ *then there exists a* $U \in \mathrm{Red}_{c,d,e}$ *and coi* ι *such that*

$$\{(W_x, \iota_x, U_x)\}_{x\in X} \cup (W, \iota, U)$$

is coherent.

(2) *If* $U \in \mathrm{Red}_{c,d,e}$ *then there exists* $W \in \mathrm{Red}_{a,b}$ *and coi* ι *such that*

$$\{(W_x, \iota_x, U_x)\}_{x\in X} \cup (W, \iota, U)$$

is coherent.

Of course if $W \equiv E$ then there is little work to do, so we assume W is not empty. One begins by adding for each $i \in \mathrm{ind}(W)$ a rather trivial coi triple $(W \upharpoonright_p \{i\}, \iota_i, E)$, where ι_i is the empty function, to the collection $\{(W_x, \iota_x, U_x)\}_{x\in X}$. This enlarged collection $\{(W_x, \iota_x, U_x)\}_{x\in X_0}$ is coherent. Next we look at the collection $\mathcal{I}_0$ of intervals I in $\mathrm{ind}(W)$ such that $W \upharpoonright_p I \in \mathrm{Pfine}(\{W_x\}_{x\in X_0})$. Because of the initial enlargement we have each singleton $\{i\}$, $i \in \mathrm{ind}(W)$, is an element of $\mathcal{I}_0$. If for some I in this collection there is some ω sequence $I = I_0 \subseteq I_1 \subseteq \ldots$ for which each $I_k \in \mathcal{I}_0$ but $\bigcup_{k\in\omega} I_k \notin \mathcal{I}_0$ then we use Lemma 4.2, perhaps once in the forward and once in the backward direction, to enlarge to a collection $\{(W_x, \iota_x, U_x)\}_{x\in X_1}$ such that the now larger $\mathcal{I}_1$ has $\bigcup_{k\in\omega} I_k$ as an element. We keep doing this until it is not possible to do so anymore. By employing bookkeeping at each implementation of this, one sees that this can only be done countably often, so there is some $\alpha < \aleph_1$ such that the collection $\{(W_x, \iota_x, U_x)\}_{x\in X_\alpha}$ cannot be enlarged in this manner. Note that this collection is of size $< 2^{\aleph_0}$. This collection $\mathcal{I}_\alpha$ is such that each $i \in \mathrm{ind}(W)$ is an element of a unique maximal $I \in \mathcal{I}_\alpha$. Also, the collection $\mathcal{M}$ of maximal elements in $\mathcal{I}_\alpha$ is such that if $I, I' \in \mathcal{M}$ have $I \cap I' \neq \emptyset$ then $I = I'$ (otherwise $W \upharpoonright_p I \cup I'$ is a concatenation of two elements of $\mathrm{Pfine}(\{W_x\}_{x\in X_\alpha})$). Ordering the elements of $\mathcal{M}$ through comparing their elements, by the same reasoning if $I < I'$ are in $\mathcal{M}$ then there exists $I'' \in \mathcal{M}$ with $I < I'' < I'$. Thus $\mathcal{M}$ is a nonempty, countable, dense ordered set. If $\mathcal{M}$ is a singleton, then we can apply Lemma 4.1 to immediately obtain the desired ι and U. Else, $\mathcal{M}$ is order isomorphic to $\mathbb{Q} \cup L$ where $L \subseteq \{-\infty, \infty\}$. Use Lemma 4.3 to make a larger collection $\{(W_x, \iota_x, U_x)\}_{x\in X'}$ so that $W \upharpoonright_p \bigcup\{I_q\}_{q\in\mathbb{Q}} \in \mathrm{Pfine}(\{W_x\}_{x\in X'})$, and so W is a finite concatenation of (at most three) elements of $\mathrm{Pfine}(\{W_x\}_{x\in X'})$ and applying Lemma 4.1 we obtain U and ι.

Now we can easily prove the main theorem.

Theorem 4.5. $\pi_1(\mathbb{GS}_2) \simeq \pi_1(\mathbb{GS}_3)$.

Take $\prec_{a,b}$ to be a well-order on $\mathrm{Red}_{a,b}$ such that each W has fewer than $2^{\aleph_0}$ elements below it, and take $\prec_{c,d,e}$ similarly for $\mathrm{Red}_{c,d,e}$. By Lemma 4.4 inductively define a coherent collection $\{(W_\gamma, \iota_\gamma, U_\gamma)\}_{\gamma<2^{\aleph_0}}$ so that for an even ordinal γ (i.e. $\gamma = \beta + 2m$ with β limit and $m \in \omega$) W_γ is the $\prec_{a,b}$-minimal element of $\mathrm{Red}_{a,b} \setminus \{W_\delta\}_{\delta<\gamma}$, and similarly U_γ is the $\prec_{c,d,e}$-minimal element of $\mathrm{Red}_{c,d,e} \setminus \{U_\delta\}_{\delta<\gamma}$ for odd γ. Now apply Theorem 3.10.

5. Further applications

The techniques above yield more results. The *harmonic archipelago* $\mathbb{HA}$ [1] is a subspace of $\mathbb{R}^3$ which is formed by starting with a topological disc, selecting a point on the boundary, and raising hills of height 1 whose bases shrink in diameter and converge to the distinguished point on the boundary (see Figure 5). This space looks in many ways like a disc, and the space obtained by removing the distinguished

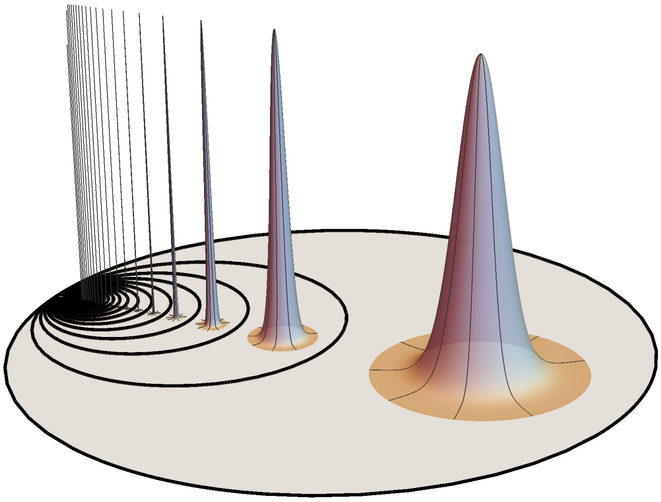

FIGURE 3. The harmonic archipelago $\mathbb{HA}$

point is homeomorphic to a disc minus a boundary point. To see that $\pi_1(\mathbb{HA})$ is nontrivial, take a loop which traverses the boundary and notice that any attempt to nulhomotopy this loop would require moving over infinitely many of the hills, and this violates the continuity of a homotopy. In fact, the fundamental group is uncountable.

Cannon and Conner conjectured that the group $\pi_1(\mathbb{HA})$ is isomorphic to $\pi_1(\mathbb{GS}_2)$. In the arXiv version of [5] it is shown that this is indeed true. The proof uses a combinatorial description of $\pi_1(\mathbb{HA})$ provided in [7]. One can make a comparable description of pure words and prove comparable extension results for ω and $\mathbb{Q}$ concatenations as given above.

As another application, suppose that $\{H_n\}_{n\in\omega}$ is a countable collection of groups. The elements of the inverse limit of the free products $\varprojlim *_{n=0}^m H_n$ can be considered as infinite countable words, and the subgroup consisting of those words which utilize elements in H_n only finitely often for each n we call $\circledast_{n\in\omega} H_n$. The free product $*_{n\in\omega}H_n$ is the subgroup of $\circledast_{n\in\omega} H_n$ consisting of finite words. If each H_n is isomorphic to $\mathbb{Z}$ the group $\circledast_{n\in\omega} H_n$ is isomorphic to the fundamental group of the earring. There is a natural notion of reduced word for elements in $\circledast_{n\in\omega} H_n$.

The group $\circledast_{n\in\omega} H_n/\langle\langle *_{n\in\omega}H_n\rangle\rangle$ has a similar feel to the groups that we have been discussing—elements are represented by infinite words and one can make finite deletions in these words without changing the group element. In [4] it is shown, by the appropriate modificaions to the techniques, that if each H_n has no involutions and $2 \leqslant |H_n| \leqslant 2^{\aleph_0}$ then $\circledast_{n\in\omega} H_n/\langle\langle *_{n\in\omega}H_n\rangle\rangle$ is also isomorphic to $\pi_1(\mathbb{GS}_2)$. This corrects and generalizes a result in [7]. It is not known whether the requirement that the groups be involution-free is essential. One of the main hurdles in this particular modification is ensuring that a constructed word in a ω or $\mathbb{Q}$ concatenation is reduced.

Acknowledgments. This survey sketches the principal ideas used in proving the main theorem of [5], first published in Pacific Journal of Mathematics in Vol. 327 (2023), No. 2, published by Mathematical Sciences Publishers. The beautiful pictures are provided by Jeremy Brazas. The work of the author in writing this survey was supported by the Basque Government Grant IT1483-22 and Spanish Government Grants PID2019-107444GA-I00 and PID2020-117281GB-I00.

References

[1] W. A. Bogley, A. J. Sieradski, *Weighted combinatorial group theory and wild metric complexes*, In Y. G Baik et. al. (eds), *Groups*, de Gruyter, Berlin, 2000, 53–80.
[2] O. Bogopolski, S. M. Corson, *Abstract homomorphisms from some topological groups to acylindrically hyperbolic groups*, Math. Ann. **384** (2022), 1017–1055.
[3] J. Cannon, G. R. Conner, *The combinatorial structure of the Hawaiian earring group*, Topology Appl. **106** (2000), 225-271.
[4] S. M. Corson, *The nonabelian product modulo sum*, arXiv 2206.02682.
[5] S. M. Corson, *The Griffiths double cone group is isomorphic to the triple*, Pac. J. Math. **327** (2023), 297–336.
[6] G. R. Conner, S. M. Corson, *A note on automatic continuity*, Proc. Am. Math. Soc. **147** (2019), 1255–1268.
[7] G. R. Conner, W. Hojka, M. Meilstrup, *Archipelago groups*, Proc. Am. Math. Soc. **143** (2015), 4973–4988.
[8] S. M. Corson, O. Varghese, *A Nunke type classification in the locally compact setting*, J. Algebra **563** (2020), 49–52.
[9] R. M. Dudley, *Continuity of homomorphisms*, Duke Math. J. **28** (1961), 587–594.
[10] K. Eda, *Free σ-products and noncommutatively slender groups*, J. Algebra **148** (1992), 243–263.
[11] L. Fuchs, *Abelian Groups*, Springer, 2015.
[12] G. Higman, *Unrestricted free products and varieties of topological groups*, J. Lond. Math. Soc. **27** (1952), 73-81.
[13] A. S. Kechris, C. Rosendal, *Turbulence, amalgamation, and generic automorphisms of homogeneous structures*, Proc. Lond. Math. Soc. **94** (2007), 349–371.
[14] C. Kent, *Homotopy type of planar Peano continua*, Adv. Math. **391** (2021), 107971.
[15] D. Keppeler, P. Möller, O. Varghese, *Automatic continuity for groups whose torsion subgroups are small*, J. Group Theory **25** (2022), 1017–1043.