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ULTRAFILTERS AND THE KATĚTOV ORDER

Abstract. We review the use of the Katětov order in the classification of ultrafilters.

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1. Prologue: A brief and selective history of ultrafilters

The notions of a *filter* and *ultrafilter* were formally introduced by Cartan in 1937 [40, 41], allegedly after a memorable reunion of the *Bourbaki group* (see [126] for an entertaining narrative), in search of a suitable generalization of the notion of convergence. However, the story does not really start there. One can trace the development of the notion independently as part of three, originally quite separate, endeavors: *topological*-the aforementioned study of convergence, *algebraic*-the study of (principal) *ideals* in rings, and *measure theoretic*- questions related to the *measure extension problem*. We shall go through them one by one next.

Convergence in terms of nets had been developed by Moore and Smith [145], Birkhoff [10] and Tukey [181] in parallel to the filter-based treatment of convergence. Some authors (e.g. Engelking [66]) attribute the first occurrence of the notion of an ultrafilter to Riesz's 1908 paper [153] where on page 23 a family of subsets of a space is defined which if the word *verkettet* is interpreted as "intersect", indeed, defines an ultrafilter, however, in the context of the paper it becomes clear that what the author means is the family of sets which *accumulate* to a given point of the space, hence, the family of the positive sets with respect to the neighborhood filter of the point. A *filter base*, as a basis of the neighborhood filter of a point in a space is explicitly defined in Vietoris' [187] in 1921. The *space of ultrafilters* $\beta\mathbb{N}$ was first considered by Čech [42] in 1937 (no mention of ultrafilters there), though he himself attributes the definition and proof of existence of the so-called *Stone–Čech compactification* of completely regular spaces to Tychonoff [183]. Pospíšil [150] answering a question posed in [42] then proves that the size of $\beta\mathbb{N}$ is 2^{2^ω}.

The *measure extension problem* of Lebesgue [127] from 1904 was swiftly solved by Vitali [188] in 1905 in the negative. The *Polish school* of mathematics started looking at variants of the problem, in particular, into the existence of a *diffuse* finite or countably additive measure which measures all subsets of a given set. First, Banach and Kuratowski [2] in 1929 showed that assuming the Continuum Hypothesis there cannot be a countably additive such measure on the reals and then Ulam [185] (1930) showed that the minimal cardinality of a set which admits such a measure has to be inaccessible, thus initiating the study of measurable cardinals. For the finitely additive case Tarski [179] in 1930 published his celebrated *prime ideal theorem*, proving the existence of two-valued diffuse finitely additive measures on an arbitrary infinite set, i.e. the characteristic function of a free ultrafilter on the set. Curiously, (as acknowledged by Tarski at the end of his paper) the result is equivalent to a result of Ulam [184] published in the previous issue of Fundamenta Mathematicae.

Whereas the name filter is descriptively fitting, the dual notion of an *ideal* is not quite as self-explanatory. The reason for this is that the term ideal pre-dates the term filter by almost a full century and has its origin in the *ideal numbers* Kummer[1] [116] used in the proof of the failure of unique factorization for cyclotomic fields. Dedekind [55] then formalized and studied *ideals* as subsets of arbitrary rings. Stone's *representation* [177] (1936) and *duality* [178] (1937) theorems identified Boolean rings with Boolean algebras and the algebraic ideals with complements of filters. From today's perspective it is astounding that Stone's work precedes the notion of an ultrafilter.

It was Samuel [161] who made the natural switch from maximal ideals to ultrafilters in the proof of Stone duality and coined the term *space of ultrafilters* for $\beta\mathbb{N}$ which quickly became an important object of study. Rudin [160] in 1956 showed that the space $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ of *non-principal* ultrafilters on $\mathbb{N}$ is not homogeneous (assuming the Continuum Hypothesis) by noting that under CH there are *P-points* while non-P-points exist in ZFC, hence realizing that not all ultrafilters are the same.

At roughly the same time Łoś [128], building on previous work of Skolem [170] and Hewitt [89] introduced the *ultraproduct* and proved his famous theorem. The use of ultraproducts spread quickly through mathematics most notably in model theory, non-standard analysis and the theory of large cardinals (see e.g. [146] and [79,80]). We refer the interested reader to Keisler's survey [110] and the recent book by Goldbring [81] for further reading on (the history of) ultraproducts. We only mention the remarkable theorem of Keisler [108] that (assuming the Generalized Continuum Hypothesis) two models are elementarily equivalent if and only if they have isomorphic ultrapowers and Scott's [163] incompatibility of measurable cardinals with the Axiom of constructibility.

The study of ultrafilters as combinatorial object s in their own right started in the 1960's through the introduction of several interesting partial orders comparing ultrafilters. Isbell [105] in 1965 briefly mentions ultrafilters in his study of the

[1] Together with K. Weierstrass co-advisor of Georg Cantor.

Tukey order [181] of cofinal types of directed partial orders (the problem whether consistently all free ultrafilters on $\mathbb{N}$ are Tukey-equivalent became known as *Isbell's problem* and will be further discussed in this article). Rudin [158,159] studying the topological and similarity types of ultrafilters as points in $\beta\mathbb{N}$, Keisler [109] comparing ultrapowers, and Katětov [106,107] in the study of descriptive complexity of functions, independently introduced a pre-order on ultrafilters; Rudin and Keisler considered only ultrafilters, while Katětov, in fact, introduced two orders on filters in general which coincide when restricted to ultrafilters. Now, the more restrictive of the two is called *Rudin–Keisler order* and the other *Katětov order*. Yet another order was considered by Frolík [77] in order to give a ZFC proof of Rudin's non-homogeneity of $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$. This ordering is refreed to as the *Rudin–Frolík order*. The curious feature of Frolík's proof is that unlike Rudin' CH argument, his does not produce an explicit topological property satisfied by some but not all points of $\mathbb{N}^*$. This has been rectified only much later by Kunen [119] in his construction of *weak P-points* (i.e. ultrafilters which are not accumulation points of any countable subsets of $\mathbb{N}^*$). Choquet [47,53,129] and his school [53,54,129], in particular, Mokobodzki [144] identified important classes of ultrafilters: *selective* ultrafilters (denoted as *absolu*), *Q-points* (as *rare*), *Hausdorff* ultrafilters (denoted as *property C*) and *rapid* ones, and showed that these notions all differ assuming CH. Blass [14] and others later showed how these combinatorial properties correspond to model-theoretic properties of the associated ultrapowers.

At the beginning of the 1970's the study of not only topological but also combinatorial properties of ultrafilters became a very active area of research. Whole PhD theses were dedicated exclusively to the study of ultrafilters, most notably Booth's [23], Blass's [11], Ketonen's [111], Pitt's [148], Solomon's [173], Daguenet's [51], Devlin's [56], Rosen's [56], Laflamme's [121], García-Ferreira's [78], Benedikt's [8] and, more recently, Flašková's [73], Verner's [189], Medini's [134] and Cancino's [34].

Much work has been done on the Rudin–Keisler and Tukey orders and the topology of $\beta\mathbb{N}$. We refer the reader to the survey [138] by van Mill and its update [88] and Dobrinen's survey [58]. One has to highlight the work done by K. Kunen here. First [117] he showed how independent families can be used to give ZFC proofs of results about the Rudin–Keisler order originally proved using CH, then [119] he constructed, in ZFC, weak-P-points, giving the ultimate proof of non-homogeneity of $\mathbb{N}^*$. Much thanks to Kunen's insight it became clear that selective ultrafilters are special. They are Rudin–Keisler minimal, being selective is equivalent to being both a P-point and a Q-point, and to being Ramsey. A very useful charactarization of selective ultrafilters was given by Mathias [132] by proving that an ultrafilter is selective if and only if it intersects every tall analytic ideal. It was only fitting that Kunen [118] proved that selective ultrafilters consistently do not exist. This result was soon followed by Miller's proof [140] of the consistency of non-existence of Q-points and Shelah's celebrated construction of the model of ZFC without P-points (see [191] and [44]).

One also has to mention the first monograph dedicated to the theory of ultrafilters by Comfort and Negrepontis [49], the role ultrafilters played in Ramsey theory of semigroups (see [91, 93]), in particular, Hindman's *finite sums theorem* and the corresponding notion of a *union* ultrafilter, the work of Blass [13–15] and his students [39,65,122–125,155–157], and the work of Baumgartner [5–7] which ends this pre-introduction and leads us to the proper text.

2. Introduction

It is impossible to overstate the importance of ultrafilters in infinite combinatorics, general topology, and model theory. From constructing ultraproducts, finding limits on topological spaces, building compactifications of topological spaces and semigroups, and proving Ramsey-type theorems, applications of ultrafilters can be found across all of mathematics.

The study of ultrafilters is such a big area that it is impossible to survey all of it in a single paper. We will restrict ourselves to ultrafilters over countable sets. Even so, the topic is very extensive, so we will focus on the interaction between ultrafilters and definable ideals. Unfortunately, we will not have the opportunity to talk about the importance of ultrafilters in topology or model theory. Many of the topics that could not be covered in this survey can be consulted in the excellent recent book [81].

The general outline of the paper is as follows: In the third section, we review the preliminaries and notation. In the fourth chapter, we review the Katětov, Rudin–Keisler, Katětov–Blass and Rudin–Blass orderings on ideals. We prove some basic facts regarding definable ideals that will be needed in later chapters. In the fifth chapter, we introduce Baumgartner's notion of $\mathcal{I}$-ultrafilters, which allows us to classify ultrafilters using analytic (even Borel) ideals. We provide characterizations of some classes of ultrafilters that resemble the characterization of Mathias of selective ultrafilter. Now, our task is to find ways to construct $\mathcal{I}$-ultrafilters. We do this in chapters sixth where we study generic existence of classes of ultrafilters, and seven, where we use the parametrized diamonds introduced in [135] to build ultrafilters with special properties. In the last chapter, we tackle the question of: *How different can ultrafilters be?* We review consistency results regarding the non-existence of ultrafilters, we study the Tukey order and Isbell's problem. We also present an axiomatization of a model due to the first author where there are no $\mathcal{I}$-ultrafilters for any F_σ ideal $\mathcal{I}$.

3. Preliminaries and some notation

For completeness, we recall the definition of filters and ultrafilters.

Definition 3.1. Let X be a set and $\mathcal{F}$ a family of subsets of X. We say $\mathcal{F}$ *is a filter* if the following conditions hold:

(1) $X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.
(2) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
(3) If $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.

A filter $\mathcal{U}$ is called an *ultrafilter* if it is a maximal filter with respect to inclusion. Unless otherwise specified, we assume that ultrafilters are *not principal*, which means that no singleton belongs to $\mathcal{U}$.

Ideals are dual to filters. While a filter is a measure or notion of "largeness", an ideal is an abstraction of "smallness".

Definition 3.2. Let X be a set and $\mathcal{I}$ a family of subsets of X. We say $\mathcal{I}$ is an *ideal* if:

(1) $\emptyset \in \mathcal{I}$ and $X \notin \mathcal{I}$.
(2) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.
(3) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.

Unless explicitly stated otherwise we shall assume our ideals to contain all finite subsets of X and, dually, all filters to contain all co-finite subsets of X.

Given a set X and $A \subseteq X$, the complement of A relative to X is defined as $A^{\mathsf{c}} = X \setminus A$. If $\mathcal{F}$ is a filter on X, the *dual ideal of* $\mathcal{F}$ is defined as $\mathcal{F}^* = \{A^{\mathsf{c}} \mid A \in \mathcal{F}\}$. Similarly, for an ideal $\mathcal{I}$, define *dual filter* of $\mathcal{I}$ as $\mathcal{I}^* = \{A^{\mathsf{c}} \mid A \in \mathcal{F}\}$. It is straightforward to see that $\mathcal{F}^*$ is an ideal and $\mathcal{I}^*$ a filter. If $\mathcal{I}$ is an ideal on X, we let $\mathcal{I}^+ = \mathcal{P}(X) \setminus \mathcal{I}$ be the family of $\mathcal{I}$-*positive* sets, where $\mathcal{P}(X)$ denotes the powerset of X. If $\mathcal{F}$ is a filter, we define $\mathcal{F}^+ = (\mathcal{F}^*)^+$; it is easy to see that $\mathcal{F}^+$ is the family of all sets that have non-empty (infinite) intersection with every element of $\mathcal{F}$. If $A \in \mathcal{I}^+$ then *the restriction of* $\mathcal{I}$ to A, defined as $\mathcal{I} \restriction A = \mathcal{P}(A) \cap \mathcal{I}$, is an ideal on A. An ideal is called *tall* if every infinite set has an infinite subset belonging to the ideal. A stronger notion of tallness is that of being ω-*hitting;* an ideal $\mathcal{I}$ is ω-hitting if for every $\{X_n \mid n \in \omega\} \subseteq [\omega]^{\omega}$, there is a single $A \in \mathcal{I}$ such that $A \cap X_n$ is infinite for every $n \in \omega$. Let $A, B \subseteq \omega$. By $A \subseteq^* B$ (A is almost contained in B) we mean that $A \setminus B$ is finite. We say that A is a *pseudo-union (pseudo-intersection)* of a family $\mathcal{H} \subseteq [\omega]^{\omega}$ if A almost contains (is almost contained) in every $B \in \mathcal{H}$. We say that $\mathcal{I}$ is a P-*ideal* if every countable subfamily of $\mathcal{I}$ has a pseudounion in $\mathcal{I}$. Topology turns out to be extremely useful when studying ideals and filters on the natural numbers. We endow $\mathcal{P}(\omega)$ with the natural topology that makes it homeomorphic to 2^{ω}, the Cantor space. In this way, the topology of $\mathcal{P}(\omega)$ has as a subbase the sets of the form $\langle n \rangle_0 = \{A \subseteq \omega \mid n \notin A\}$ and $\langle n \rangle_1 = \{A \subseteq \omega \mid n \in A\}$, for $n \in \omega$. We view filters as subspaces of $\mathcal{P}(\omega)$. All notions of Borel, analytic meager are referred to this topology. A major topic in descriptive set theory is that nicely definable subspaces have very nice and desirable properties, this also applies to filters and ideals.

We now present the Borel ideals that will be critical in the present paper. The reader may read [96] and [97] to learn more about them. For the next definitions, given $n \in \omega$, denote the column $C_n = \{(n, m) \mid m \in \omega\}$ and $\mathcal{C} = \{C_n \mid n \in \omega\}$. Given $f \in \omega^{\omega}$, denote $D(f) = \{(n, m) \in \omega \times \omega \mid m \leqslant f(n)\}$.

- The ideal fin is the ideal of finite subsets of ω.
- The *eventually different* ideal $\mathcal{ED}$ is the ideal on ω^2 generated by $\mathcal{C}$ and the graphs of functions from ω to ω.
- The ideal $\mathcal{ED}_{\mathsf{fin}}$ is the restriction of $\mathcal{ED}$ to $\triangle = \{(n, m) \mid m \leqslant n\}$.
- The ideal $\mathsf{fin}\times\mathsf{fin}$ is the ideal on ω^2 generated by $\mathcal{C} \cup \{D(f) \mid f \in \omega^{\omega}\}$.
- The ideal conv is the ideal on $\mathbb{Q} \cap [0, 1]$ generated by all convergent sequences.

- The *nowhere dense ideal,* nwd is the ideal of nowhere dense subsets of the rational numbers.
- The *summable ideal* is the ideal $\mathcal{I}_{\frac{1}{n}} = \Big\{A \subseteq \omega \mid \sum_{n\in A} \frac{1}{n+1} < \infty\Big\}$.
- The *density zero ideal* is the ideal $\mathcal{Z} = \Big\{A \subseteq \omega \mid \lim_{n\longrightarrow\infty} \dfrac{|A \cap [2^n, 2^{n+1})|}{2^n} = 0\Big\}$.
- The ideal $\mathcal{R}$ is the ideal on ω generated by the cliques and free sets in the random graph.
- The ideal $\mathcal{G}_{\mathsf{FC}}$ is the ideal on $[\omega]^2$ generated by (the edges) of all finitely chromatic graphs.
- The non-flat ideal nflat is the ideal on $[\omega]^{<\omega}$ generated by $\{[\omega]^n : n \in \omega\} \cup \{X_f : f \in (\omega \smallsetminus \{0\})^\omega \text{ is increasing}\}$, where $X_f = \{s \in [\omega]^{<\omega} : (\exists k \in s)(s \cap (k, f(k)] \neq \emptyset)\}$.

Other than the non-flat ideal these ideals are Borel of low complexity:

Ideal	Borel Complexity
fin	F_σ
$\mathcal{ED}$	F_σ
$\mathcal{ED}_{\mathsf{fin}}$	F_σ
$\mathcal{R}$	F_σ
$\mathcal{I}_{\frac{1}{n}}$	F_σ
$\mathcal{G}_{\mathsf{FC}}$	F_σ
nwd	$F_{\sigma\delta}$
$\mathcal{Z}$	$F_{\sigma\delta}$
conv	$F_{\sigma\delta\sigma}$
$\mathsf{fin}\times\mathsf{fin}$	$F_{\sigma\delta\sigma}$

We actually do not know whether nflat is Borel (it is clearly analytic).

We will frequently refer to the *cardinal invariants of the continuum.* An excellent reference for this topic is [12]. The simplest cardinal invariant is $\mathfrak{c}$, which is the size of the set of the real numbers. Given $f, g \in \omega^\omega$, define $f \leqslant^* g$ if there are only finitely many n for which $g(n) < f(n)$. We say a family $\mathcal{B} \subseteq \omega^\omega$ is *unbounded* if there is no $g \in \omega^\omega$ such that $f \leqslant^* g$ for every $f \in \mathcal{B}$. On the other hand, a family $\mathcal{D} \subseteq \omega^\omega$ is *dominating* if for every $f \in \omega^\omega$, there is $g \in \mathcal{D}$ such that $f \leqslant^* g$. The *unboundedness number* $\mathfrak{b}$ is the least size of an unbounded family, while the *dominating number* $\mathfrak{d}$ is the smallest size of a dominating family. It is straightforward to see that $\omega_1 \leqslant \mathfrak{b} \leqslant \mathfrak{d} \leqslant \mathfrak{c}$. We say that $\mathcal{P} = \{P_n \mid n \in \omega\}$ is an *interval partition* if it is a partition of ω in (finite) intervals. There is an equivalent reformulation of $\mathfrak{b}$ and $\mathfrak{d}$ using interval partitions. Given $\mathcal{P}$ and $\mathcal{R}$ interval partitions, define $\mathcal{P} \leqslant^* \mathcal{R}$ if almost all intervals from $\mathcal{R}$ contain (at least) one interval from $\mathcal{P}$. In this way, $\mathfrak{b}$ is the smallest size of an unbounded family of interval partitions, while $\mathfrak{d}$ is the least size of a dominating family of interval partitions. Let $A, B \subseteq \omega$. We say that A *decides* B if $A \subseteq^* B$ or $A \subseteq^* B^{\mathsf{c}}$. A reaping family $\mathcal{R} \subseteq [\omega]^\omega$ is a family deciding each subset of ω. The *reaping number* $\mathfrak{r}$ is the least size of a reaping family. Now, a family $\mathcal{R}$ is σ*-reaping* if for every

$\{X_n \mid n \in \omega\} \subseteq [\omega]^\omega$, there is a single $R \in \mathcal{R}$ that decides each X_n. The *σ-reaping number* $\mathfrak{r}_\sigma$ is the least size of a σ-reaping family. It is currently unknown if $\mathfrak{r}$ and $\mathfrak{r}_\sigma$ are equal. It is easy to see that $\mathfrak{b} \leqslant \mathfrak{r} \leqslant \mathfrak{r}_\sigma \leqslant \mathfrak{c}$. The ultrafilter number $\mathfrak{u}$ is the smallest size of a base of an ultrafilter. Clearly we have that $\mathfrak{r} \leqslant \mathfrak{u} \leqslant \mathfrak{c}$.

Definition 3.3. Let X be a set and $\mathcal{I}$ an ideal on X.

(1) $\mathsf{add}(\mathcal{I})$ is the smallest size of an unbounded family in $(\mathcal{I}, \subseteq)$.
(2) $\mathsf{cov}(\mathcal{I})$ is the smallest size of a family $\mathcal{D} \subseteq \mathcal{I}$ such that $\cup\mathcal{D} = X$.
(3) $\mathsf{non}(\mathcal{I})$ is the smallest size of a subset of X that is not in $\mathcal{I}$.
(4) $\mathsf{cof}(\mathcal{I})$ is the smallest size of a cofinal family in $(\mathcal{I}, \subseteq)$.

It is easy to see that $\mathsf{add}(\mathcal{I}) \leqslant \mathsf{cov}(\mathcal{I})$, $\mathsf{non}(\mathcal{I}) \leqslant \mathsf{cof}(\mathcal{I}) \leqslant 2^{|X|}$. In general, there is no relation between $\mathsf{cov}(\mathcal{I})$ and $\mathsf{non}(\mathcal{I})$. In case $\mathcal{I}$ is an ideal on ω, the first three cardinals defined above are trivial. However, in [31] Brendle and Shelah found analogues of these invariants that are useful for ideals on countable sets, we recall their definitions. They defined these for filters, the notation used here follows [90].

Definition 3.4. Let $\mathcal{I}$ be a tall ideal on ω. Define:

(1) $\mathsf{add}^*(\mathcal{I})$ is the smallest size of an unbounded family in $(\mathcal{I}, \subseteq^*)$.
(2) $\mathsf{cov}^*(\mathcal{I})$ is the smallest size of a family $\mathcal{D} \subseteq \mathcal{I}$ such that for every $X \in [\omega]^\omega$, there is $A \in \mathcal{D}$ such that $A \cap X$ is infinite.
(3) $\mathsf{non}^*(\mathcal{I})$ is the smallest size of a family $\mathcal{H} \subseteq [\omega]^\omega$ such that for every $A \in \mathcal{I}$, there is $H \in \mathcal{H}$ such that $A \cap H$ is finite.

Once again, it is easy to see that $\mathsf{add^*}(\mathcal{I}) \leqslant \mathsf{cov^*}(\mathcal{I})$, $\mathsf{non^*}(\mathcal{I}) \leqslant \mathsf{cof}(\mathcal{I})$ and in general there is no relationship between $\mathsf{cov^*}(\mathcal{I})$ and $\mathsf{non^*}(\mathcal{I})$. By $\mathcal{M}$ we denote the ideal of meager sets on 2^ω. In this way, by $\mathsf{cov}(\mathcal{M})$ we denote the smallest size of a family of meager sets that are needed to cover 2^ω (equivalently, any perfect Polish space). It is well-known that $\omega_1 \leqslant \mathsf{cov}(\mathcal{M}) \leqslant \mathfrak{d}$, while there is no ZFC provable relation between $\mathfrak{b}$ and $\mathsf{cov}(\mathcal{M})$. Meagerness of filters can be reformulated in a very useful combinatorial way, as we will now review.

Definition 3.5. Let $\mathcal{I}$ be an ideal on ω and $\mathcal{P} = \{P_n \mid n \in \omega\}$ an interval partition. We say that $\mathcal{I}$ *is a Talagrand partition of* $\mathcal{I}$ if for every $X \in [\omega]^\omega$ we have that $\bigcup_{n \in X} P_n \in \mathcal{I}^+$.

The following is an important theorem, the reader may consult the book [3] for a proof.

Theorem 3.6 (Jalali–Naini–Talagrand [180])**.** *Let $\mathcal{I}$ be an ideal. The following are equivalent:*

(1) *$\mathcal{I}$ has the Baire property.*
(2) *$\mathcal{I}$ is meager.*
(3) *$\mathcal{I}$ has a Talagrand partition.*
(4) *The increasing enumerations of the elements of $\mathcal{I}^*$ is bounded.*

In this paper, a tree $p \subseteq \omega^{<\omega}$ is closed under taking initial segments. We denote by $[p]$ the set of cofinal branches of p.

Finally, we recall the combinatorial properties of ultrafilters on countable sets that are most used in the literature.

Definition 3.7. Let $\mathcal{U}$ be an ultrafilter in ω.

(1) $\mathcal{U}$ is *selective* (or *Ramsey)* if for every partition $\{P_n \mid n \in \omega\}$ of ω, either there is $n \in \omega$ such that $P_n \in \mathcal{U}$ or there is $X \in \mathcal{U}$ such that $|X \cap P_n| \leqslant 1$ for every $n \in \omega$.
(2) $\mathcal{U}$ is a *P-point* if for every decreasing $\{X_n \mid n \in \omega\} \subseteq \mathcal{U}$ there is $X \in \mathcal{U}$ such that $X \subseteq^* X_n$ for every $n \in \omega$.
(3) $\mathcal{U}$ is a *Q-point* if for every partition $\{P_n \mid n \in \omega\}$ of ω into finite sets, there is $X \in \mathcal{U}$ such that $|X \cap P_n| \leqslant 1$ for every $n \in \omega$.
(4) $\mathcal{U}$ is *rapid* if the family of increasing enumerations of the sets in $\mathcal{U}$ is a dominating family in $(\omega^\omega, \leqslant^*)$.
(5) $\mathcal{U}$ is *Hausdorff* if for every $f, g \in \omega^\omega$, either $\{n \mid f(n) = g(n)\} \in \mathcal{U}$ or there is $U \in \mathcal{U}$ such that $f[U] \cap g[U] = \emptyset$.

It is well-known that an ultrafilter is selective if and only if it is both a P-point and a Q-point, all Q-points are rapid and all selective ultrafilters are Hausdorff.

4. Orderings on ideals and ultrafilters

Since filters, ideals and ultrafilters play a fundamental role in infinite combinatorics, we need ways to classify and study them. As mentioned in the introduction, the Katětov, Rudin–Keisler and Tukey orderings have proven to be invaluable tools for their study.

Definition 4.1. Let X, Y be two sets, $\mathcal{I}$ an ideal on X, $\mathcal{J}$ an ideal on Y and $f\colon X \longrightarrow Y$.

(1) f is a *Katětov function from* $\mathcal{I}$ to $\mathcal{J}$ if for every $A \subseteq Y$, the following holds: If $A \in \mathcal{J}$, then $f^{-1}(A) \in \mathcal{I}$.
(2) f is a *Rudin–Keisler function from* $\mathcal{I}$ to $\mathcal{J}$ if for every $A \subseteq Y$, the following holds: $A \in \mathcal{J}$ if and only if $f^{-1}(A) \in \mathcal{I}$.
(3) f is a *Katětov-Blass function from* $\mathcal{I}$ to $\mathcal{J}$ if it is a Katětov function and it is finite.
(4) f is a *Rudin–Blass function from* $\mathcal{I}$ to $\mathcal{J}$ if it is a Rudin–Keisler function and it is finite-to-one.
(5) $\mathcal{J} \leqslant_{\mathsf{K}} \mathcal{I}$ if there is a Katětov function from $\mathcal{I}$ to $\mathcal{J}$. The orders $\leqslant_{\mathsf{KB}}$, $\leqslant_{\mathsf{RK}}$ and $\leqslant_{\mathsf{RB}}$ are defined analogously.
(6) $\mathcal{I}$ and $\mathcal{J}$ are *Katětov equivalent*(denoted as $\mathcal{I} =_{\mathsf{K}} \mathcal{J}$) in case $\mathcal{I} \leqslant_{\mathsf{K}} \mathcal{J}$ and $\mathcal{J} \leqslant_{\mathsf{K}} \mathcal{I}$. *Rudin–Keisler, Katětov–Blass* and *Rudin–Blass equivalences* are defined analogously.

The following is a list of easy facts regarding the orderings of ideals. We will be using these properties implicitly throughout the text.

Lemma 4.2. *Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω and $X \subseteq \omega$.*

(1) *If $\mathcal{I} \subseteq \mathcal{J}$, then $\mathcal{I} \leqslant_{\mathsf{KB}} \mathcal{J}$.*
(2) *If $\mathcal{I} \leqslant_{\mathsf{RK}} \mathcal{J}$, then $\mathcal{I} \leqslant_{\mathsf{K}} \mathcal{J}$.*

(3) *$\mathsf{fin} \leqslant_{\mathsf{KB}} \mathcal{I}$.*
(4) *$\mathcal{I}$ is Katětov equivalent to fin if and only if $\mathcal{I}$ is not tall.*
(5) *If $X \in \mathcal{I}^+$, then $\mathcal{I} \leqslant_{\mathsf{KB}} \mathcal{I} \upharpoonright X$.*
(6) *If $X \in \mathcal{I}^*$, then $\mathcal{I} =_{\mathsf{RB}} \mathcal{I} \upharpoonright X$.*
(7) *If $\mathcal{I} \leqslant_{\mathsf{K}} \mathcal{J}$, then $\mathsf{cov}^*(\mathcal{J}) \leqslant \mathsf{cov}^*(\mathcal{I})$.*
(8) *If $\mathcal{I} \leqslant_{\mathsf{KB}} \mathcal{J}$, then $\mathsf{non}^*(\mathcal{I}) \leqslant \mathsf{non}^*(\mathcal{J})$.*
(9) *$\mathcal{I}$ is meager if and only if $\mathsf{fin} \leqslant_{\mathsf{RB}} \mathcal{I}$.*

The following equivalence of the Katětov order is often useful:

Lemma 4.3. *Let X, Y be two sets, $\mathcal{I}$ an ideal on X, $\mathcal{J}$ an ideal on Y and $f\colon X \longrightarrow Y$. The following are equivalent:*

(1) *f is a Katětov function from $\mathcal{I}$ to $\mathcal{J}$.*
(2) *For every $A \subseteq X$, if $A \in \mathcal{I}^+$, then $f[A] \in \mathcal{J}^+$.*

Evidently, if $\mathcal{I} \leqslant_{\mathsf{RK}} \mathcal{J}$, then $\mathcal{I} \leqslant_{\mathsf{K}} \mathcal{J}$. However, in general the orderings are very different. For example, the ideal fin is Katětov-below any ideal, yet if $\mathcal{U}$ is an ultrafilter, it is easy to see that $\mathsf{fin} \nleqslant_{\mathsf{RK}} \mathcal{U}^*$. Nevertheless, these two orderings coincide when restricted to maximal ideals:

Lemma 4.4. *Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on ω. The following are equivalent:*

(1) *$\mathcal{U}^* \leqslant_{\mathsf{K}} \mathcal{V}^*$.*
(2) *$\mathcal{U}^* \leqslant_{\mathsf{RK}} \mathcal{V}^*$.*

The following is an important theorem of Mary Ellen Rudin and Saharon Shelah.

Theorem 4.5 (Shelah-Rudin [168])**.** *There are $2^{\mathfrak{c}}$ many $\leqslant_{\mathsf{RK}}$-incomparable ultrafilters.*

Although the Rudin–Keisler order has been studied extensively, many questions remain unsolved (see [88]). For example, the answer to the following problem is unknown.

Problem 4.6 (van Mill [138])**.** *Is there an ultrafilter that is $\leqslant_{\mathsf{RK}}$-comparable to any other ultrafilter?*

It is not hard to prove that there is no such ultrafilter in case that $\mathfrak{u} = \mathfrak{c}$. In [92] Hindman obtained some partial results in case $\mathfrak{c}$ is singular and Butkovičová [33] in case there is a $\kappa < \mathfrak{c}$ such that $2^{\kappa} > \mathfrak{c}$. It should be mentioned here that it is a theorem of ZFC that there is no least or largest ultrafilters the Rudin–Blass orders.

Another simple question about the Rudin–Keisler and Rudin–Blass orders is the following:

Problem 4.7 (H.-Sanchis-Tamariz–Mascarúa [102])**.** *Is there an ultrafilter that has the same RK-and RB-predecessors?*

Of course, every P-point is such [102], but it seems unknown whether such an ultrafilter exist in ZFC.

We will now prove that the Katětov order is both $\mathfrak{c}^+$-directed downwards an upwards when restricted to tall ideals. This means that every family of ideals of size $\mathfrak{c}$ has a lower and an upper bound in the Katětov order. This is an unpublished result of A. Blass, which appeared in [25]. We first need the following:

Definition 4.8. Let $\mathcal{P} \subseteq \omega^\omega$. We say that $\mathcal{P}$ is an *independent family of functions* if for every $f_0, \dots, f_n \in \mathcal{P}$ distinct and $m_0, \dots, m_n \in \omega$, there are infinitely many $a \in \omega$ such that $f_i(a) = m_i$ for every $i \leqslant n$.

It is easy to construct large independent families:

Lemma 4.9. *There is a perfect $\mathcal{P} \subseteq \omega^\omega$ independent family of functions.*

Proof. Let $\mathbb{P}$ be the set of all $p \subseteq \omega^{<\omega}$ with the following properties:

(1) p is a finite tree.
(2) Every node in p has an extension to the last level of p.

Order $p \leqslant q$ if p is an end-extention of q. Given a filter $G \subseteq \mathbb{P}$, define $T_G = \bigcup_{p \in G} p$. It is not hard to find countably many dense sets of $\mathbb{P}$ and a sufficiently generic filter G such that $\mathcal{P} = [T_G]$ is a perfect independent family of functions. □

We can now prove the following:

Proposition 4.10 (Blass, see [25])**.** *The Katětov order is $\mathfrak{c}^+$-directed upwards and downwards when restricted to tall ideals.*

Proof. Let $\mathcal{L} = \{\mathcal{I}_\alpha \mid \alpha < \mathfrak{c}\}$ be a family of tall ideals on ω. We want to find a lower and upper bound of $\mathcal{L}$ in the Katětov order.

We first find an upper bound. Let $\mathcal{P} = \{f_\alpha \mid \alpha < \mathfrak{c}\} \subseteq \omega^\omega$ be an independent family of functions. Define $\mathcal{J}$ as the ideal generated by $\bigcup_{\alpha < \mathfrak{c}} f_\alpha^{-1}(\mathcal{I}_\alpha)$. We claim that $\mathcal{J}$ is a proper ideal. To prove this, it is enough to show that for every $\alpha_0, \dots, \alpha_n \in \mathfrak{c}$ and $A_{\alpha_0}, \dots, A_{\alpha_n} \subseteq \omega$ with $A_{\alpha_i} \in \mathcal{I}_{\alpha_i}$, the set $B = \bigcup_{i \leqslant n} f_{\alpha_i}^{-1}(A_{\alpha_i})$ is co-infinite. Since each A_{α_i} is a proper subset of ω, we choose $m_i \notin A_{\alpha_i}$. Since $\mathcal{P}$ is independent, it follows that there are infinitely many $a \in \omega$ such that $f_i(a) = m_i$ for every $i \leqslant n$, so $a \notin B$. Finally, it is clear that f_α is a Katětov function from $\mathcal{J}$ to $\mathcal{I}_\alpha$, so $\mathcal{J}$ is a $\leqslant_{\mathsf{K}}$-upper bound of $\mathcal{L}$. In order to find a lower bound let $\mathcal{A} = \{A_\alpha \mid \alpha < \mathfrak{c}\}$ be a MAD family. Define an ideal $\mathcal{K}$ such that $\mathcal{A} \subseteq \mathcal{K}^+$ and $\mathcal{K} \restriction A_\alpha$ is Katětov equivalent to $\mathcal{I}_\alpha$. Since $\mathcal{A}$ is maximal and each $\mathcal{I}_\alpha$ is tall, it follows that $\mathcal{K}$ is tall. Since each $\mathcal{I}_\alpha$ is Katětov equivalent to a restriction of $\mathcal{K}$, it follows that $\mathcal{K}$ is a $\leqslant_{\mathsf{K}}$-lower bound of $\mathcal{L}$. □

The next lemma will be used in the proof of Proposition 5.8. It first appeared in [125]:

Lemma 4.11 (C. Laflamme, J. P. Zhu [125])**.** *There exists a family $\mathcal{D} \subseteq \omega^\omega$ of finite-to-one functions of size $\mathfrak{d}$ such that for any two ultrafilters $\mathcal{V} \leqslant_{RB} \mathcal{U}$, there is $h \in \mathcal{D}$ such that $h(\mathcal{U}) \leqslant_{RB} \mathcal{V}$. Moreover, for each $h \in \mathcal{D}$, there is a partition $\{I_n : n \in \omega\}$ of ω into intervals such that $h(k) = n$ if and only if $k \in I_n$.*

A simple Baire category argument shows that there are no G_δ ideals (see [3]). In this way, F_σ is the lowest possible complexity of an ideal. In [131] Mazur found a very simple representation of F_σ-ideals, which we will review now.

Definition 4.12. We say that $\varphi\colon \mathcal{P}(\omega) \longrightarrow [0, \infty]$ is a *lower semicontinuous submeasure* if the following hold:

(1) $\varphi(A) \leqslant \varphi(B)$ whenever $A \subseteq B$.
(2) $\varphi(A \cup B) \leqslant \varphi(A) + \varphi(B)$ for every $A, B \subseteq \omega$.
(3) $\varphi(A) < \infty$ for every finite $A \subseteq \omega$.
(4) If $A \subseteq \omega$, then $\varphi(A) = sup\{\varphi(A \cap n) \mid n \in \omega\}$.

Given a lower semi-continuous submeasure φ we define the following (possibly improper) ideals:

(1) $\mathsf{fin}(\varphi) = \{A \subseteq \omega \mid \varphi(A) < \infty\}$.
(2) $\mathsf{exh}(\varphi) = \{A \subseteq \omega \mid \lim_{n \longrightarrow \infty} \varphi(A \setminus n) = 0\}$.

The theorem of Mazur is as follows,

Theorem 4.13 (Mazur [131])**.** *Let $\mathcal{I}$ be an ideal on ω. The following are equivalent:*

(1) *$\mathcal{I}$ is F_σ.*
(2) *There is a lower semicontinuous submeasure φ with $\varphi(\omega) = \infty$ such that $\mathcal{I} = \mathit{fin}(\varphi)$.*

The following theorem by Grebík and Vidnyánszky is worth pointing out.

Theorem 4.14 (Grebík, Vidnyánszky [83])**.** *Every tall analytic ideal contains a tall F_σ-ideal.*

Similar to the theorem of Mazur, Solecki found a useful representation of analytic P-ideals.

Theorem 4.15 (Solecki [171])**.** *Let $\mathcal{I}$ be an ideal on ω.*

(1) *$\mathcal{I}$ is an analytic P-ideal if and only if there is a lower semicontinuous submeasure φ such that $\mathcal{I} = \mathit{exh}(\varphi)$.*
(2) *$\mathcal{I}$ is an F_σ P-ideal if and only if there is a lower semicontinuous submeasure φ such that $\mathcal{I} = \mathit{exh}(\varphi) = \mathit{fin}(\varphi)$.*

It follows from the theorem that all analytic P-ideals are actually $F_{\sigma\delta}$.

Using submeasures, we can define the following important class of ideals:

Definition 4.16. Let $g\colon \omega \longrightarrow [0, \infty)$ be a function such that $\sum_{n\in\omega} g(n)$ diverges to infinity. Define $\varphi_g\colon \mathcal{P}(\omega) \longrightarrow [0, \infty)$ as $\varphi_g(A) = \sum_{n\in A} g(n)$, for each $A \in \mathcal{P}(\omega)$. It turns out that φ_g is a lower semicontinuous submeasure on ω, so it defines an ideal which we denote by $\mathcal{J}_g = \mathsf{fin}(\varphi_g)$.

Ideals of this form are known as *summable ideals*. Obviously, the summable ideal is of this form.

Proposition 4.17. *Let $g\colon \omega \longrightarrow [0, \infty)$ be such that $\sum_{n\in\omega} g(n)$ diverges. Then:*

(1) *$\mathcal{J}_g$ is an F_σ P-ideal.*
(2) *$\mathcal{J}_g$ is a tall ideal if and only if $\langle g(n) : n \in \omega\rangle$ converges to 0.*

The following two operations are often useful, they are used to push and pull-back ultrafilters through functions:

Definition 4.18. Let X, Y be two sets, $\mathcal{I}$ an ideal on X, $\mathcal{J}$ an ideal on Y and $f\colon X \longrightarrow Y$.

(1) $f(\mathcal{I})$ is an ideal on Y defined as $f(\mathcal{I}) = \{A \subseteq Y \mid f^{-1}(A) \in \mathcal{I}\}$.
(2) $f^{-1}(\mathcal{J})$ is a (possibly improper) ideal on X generated by the collection $\{f^{-1}(B) \mid B \in \mathcal{J}\}$.

In the above situation, note that since $f^{-1}(B \cup C) = f^{-1}(B) \cup f^{-1}(C)$ for $B, C \subseteq Y$, it follows that a set $D \subseteq X$ is in $f^{-1}(\mathcal{J})$ if and only if there is $B \in \mathcal{J}$ such that $D \subseteq f^{-1}(B)$. In this way, $f^{-1}(\mathcal{J}) = \{A \subseteq X \mid f[A] \in \mathcal{J}\}$. Note that it is very possible that $f^{-1}(\mathcal{J})$ is a trivial ideal. In fact, we have the following:

Lemma 4.19. *Let X, Y be two sets, $\mathcal{J}$ an ideal on Y and $f\colon X \longrightarrow Y$. The following are equivalent:*
(1) *$f^{-1}(\mathcal{J})$ is a proper ideal.*
(2) *$\mathsf{im}(f) \in \mathcal{J}^+$.*

We have the following:

Proposition 4.20. *Let $\mathcal{I}$ be an ideal on ω, $f\colon \omega \longrightarrow \omega$ and $X = \mathsf{im}(f)$. If $f^{-1}(\mathcal{I})$ is a proper ideal, then the following holds:*
(1) *$\mathcal{I} \leqslant_{\mathsf{K}} f^{-1}(\mathcal{I})$ as witnessed by f.*
(2) *$\mathcal{I} \restriction X \leqslant_{\mathsf{RK}} f^{-1}(\mathcal{I})$ as witnessed by f.*
(3) *$\mathcal{I} \restriction X \leqslant_{\mathsf{RB}} f^{-1}(\mathcal{I})$ if f is finite-to-one.*
(4) *If $X \in \mathcal{I}^*$, then $\mathcal{I} \leqslant_{\mathsf{RK}} f^{-1}(\mathcal{I})$.*
(5) *If $\mathcal{J}$ is an ideal on ω such that f is a Katětov function from $\mathcal{J}$ to $\mathcal{I}$, then $f^{-1}(\mathcal{I}) \subseteq \mathcal{J}$.*

Lemma 4.21. *Let $\mathcal{I}$ be an ideal on ω and $f\colon \omega \longrightarrow \omega$.*
(1) *$f(\mathcal{I})$ is the ideal generated by $\{f[B^{\mathsf{c}}]^{\mathsf{c}} \mid B \in \mathcal{I}\}$.*
(2) *$\mathsf{cof}(f(\mathcal{I})) \leqslant \mathsf{cof}(\mathcal{I})$.*
(3) *$f(\mathcal{I}) \leqslant_{\mathsf{K}} \mathcal{I}$.*

We will now prove a result regarding the complexity of these ideals.

Proposition 4.22. *Let $\mathcal{I}$ be an ideal on ω and $f\colon \omega \longrightarrow \omega$.*
(1) *If $\mathcal{I}$ is Borel (analytic), then $f(\mathcal{I})$ is Borel (analytic).*
(2) *If $\mathcal{I}$ is analytic, then $f^{-1}(\mathcal{I})$ is analytic.*
(3) *If $\mathcal{I}$ is Borel and f is finite-to-one, then $f^{-1}(\mathcal{I})$ is Borel. Moreover, the Borel complexity of $f^{-1}(\mathcal{I})$ is at most the Borel complexity of $\mathcal{I}$.*

Proof. Before starting the proof, we make some (well-known) remarks. Define the functions $F, G\colon \mathcal{P}(\omega) \longrightarrow \mathcal{P}(\omega)$ where $F(A) = f^{-1}(A)$ and $G(A) = f[A]$.

Claim 4.22.1. (1) F is continuous.
(2) If f is finite-to-one, then G is continuous.

In both cases, it is enough to see that the preimage of every element of the subbase $\{\langle n\rangle_i \mid n \in \omega \wedge i \in 2\}$ is open. In the case of F:

$$\begin{aligned} F^{-1}(\langle n\rangle_1) &= \{A \mid n \in F(A)\}, \\ &= \{A \mid n \in f^{-1}(A)\}, \end{aligned}$$

$$
\begin{aligned}
&= \{A \mid f(n) \in A\},\\
&= \langle f(n)\rangle_1.\\
F^{-1}(\langle n\rangle_0) &= \{A \mid n \notin F(A)\},\\
&= \{A \mid n \notin f^{-1}(A)\},\\
&= \{A \mid f(n) \notin A\},\\
&= \langle f(n)\rangle_0.
\end{aligned}
$$

Now, assuming f is finite-to-one:

$$
\begin{aligned}
G^{-1}(\langle n\rangle_1) &= \{A \mid n \in G(A)\},\\
&= \{A \mid n \in f[A]\},\\
&= \textstyle\bigcup_{f(m)=n}\langle m\rangle_1.\\
G^{-1}(\langle n\rangle_0) &= \{A \mid n \notin G(A)\},\\
&= \{A \mid n \notin f[A]\},\\
&= \mathcal{P}(\omega) \smallsetminus \textstyle\bigcup_{f(m)=n}\langle m\rangle_1.
\end{aligned}
$$

Since f is finite-to-one, it follows that $\bigcup_{f(m)=n}\langle m\rangle_1$ is clopen, so $G^{-1}(\langle n\rangle_0)$ is open. Now, we have that $F^{-1}(\mathcal{I}) = f(\mathcal{I})$, so the first point of the proposition follows. We now prove the second point, so assume that $\mathcal{I}$ is an analytic ideal. It then follows that $F[\mathcal{I}] = \{f^{-1}(A) \mid A \in \mathcal{I}\}$ is an analytic set. Since $f^{-1}(\mathcal{I}) = \{B \mid \exists A \in F[\mathcal{I}](A \subseteq B)\}$, we get that $f^{-1}(\mathcal{I})$ is analytic as well.

Finally, assume that $\mathcal{I}$ is Borel and f is finite-to-one. We now have that $G^{-1}(\mathcal{I}) = f^{-1}(\mathcal{I})$, so $f^{-1}(\mathcal{I})$ is Borel. □

Lemma 4.23. *Let $\mathcal{I}$ be a tall analytic P-ideal.*

(1) *If $X \in \mathcal{I}^+$, then $\mathcal{I} \restriction X$ is an analytic P-ideal.*
(2) *There is a tall summable ideal $\mathcal{J}$ such that $\mathcal{J} \subseteq \mathcal{I}$.*
(3) *If $f\colon \omega \longrightarrow \omega$ is finite-to-one, then $f^{-1}(\mathcal{I})$ is an analytic P-ideal.*

Proof. The first point is trivial. We prove the second point. Let φ be a lower semi-continuous submeasure such that $\mathcal{I} = \mathsf{exh}(\varphi)$. Define the function $g\colon \omega \longrightarrow [0,\infty)$ given by $g(n) = \varphi(\{n\})$. It is not difficult to verify that $\mathcal{J}_g \subseteq \mathcal{I}$. We now prove the third point. Assume $f \in \omega^\omega$ is finite-to-one. By Proposition 4.22, it is enough to prove that $f^{-1}(\mathcal{I})$ is a P-ideal. It is enough to prove that for every $\{B_n \mid n \in \omega\} \subseteq \mathcal{I}$, the family $\{f^{-1}(B_n) \mid n \in \omega\}$ has a pseudounion. Since $\mathcal{I}$ is a P-ideal, we know there is $A \in \mathcal{I}$ such that $B_n \subseteq^* A$ for every $n \in \omega$. Since f is finite-to-one, it follows that $f^{-1}(B_n) \subseteq^* f^{-1}(A)$ for every $n \in \omega$. □

In the next section will need the following theorem, which was proved in [99]:

Theorem 4.24 (H.-Meza-Minami, [99])**.** *Let $\mathcal{I}$ be a Borel ideal on ω. The following are equivalent:*

(1) *$\mathcal{I}$ is ω-hitting.*
(2) *$\mathcal{ED}_{\mathsf{fin}} \leqslant_{\mathsf{KB}} \mathcal{I}$.*

5. $\mathcal{I}$-ultrafilters

The following notion was formally introduced by J. Baumgartner in [5].

Definition 5.1. Let $\mathcal{U}$ be an ultrafilter on ω, X a set and $\mathcal{I} \subseteq \mathcal{P}(X)$ closed under subsets and containing all singletons.

(1) $\mathcal{U}$ is an *$\mathcal{I}$-ultrafilter* if for every function $g\colon \omega \longrightarrow X$, there is $A \in \mathcal{U}$ such that $g[A] \in \mathcal{I}$.

(2) $\mathcal{U}$ is a *weak $\mathcal{I}$-ultrafilter* if for every finite-to-one function $g\colon \omega \longrightarrow X$, there is $A \in \mathcal{U}$ such that $g[A] \in \mathcal{I}$.

Baumgartner was obviously unaware of the work of M. Daguenet [53] who considered exactly the same notion almost 20 years earlier and, in particular, showed that P-points are exactly the $\mathsf{fin}\times\mathsf{fin}$-ultrafilters, in the form of an instance of the following simple lemma.

Lemma 5.2. *Let $\mathcal{I}$ be an ideal on ω and $\mathcal{U}$ an ultrafilter on ω.*

(1) *$\mathcal{U}$ is an $\mathcal{I}$-ultrafilter if and only if $\mathcal{I} \not\leq_{\mathsf{K}} \mathcal{U}^*$.*

(2) *$\mathcal{U}$ is a weak $\mathcal{I}$-ultrafilter if and only if $\mathcal{I} \not\leq_{\mathsf{KB}} \mathcal{U}^*$.*

It is easy to find, for each ideal $\mathcal{I}$, ultrafilters that are not $\mathcal{I}$-ultrafilters, just extend $\mathcal{I}^*$ to any ultrafilter. The notions of (weak) $\mathcal{I}$-ultrafilters allows us to classify ultrafilters using Borel ideals as parameters. In practice, it turns out that most of the more interesting combinatorial properties of ultrafilters can be reformulated in terms of $\mathcal{I}$-ultrafilters.

Remark 5.3. The following table summarizes the relatioships between combinatorial properties of ultrafilters and the ideals (or classes of ideals) which charactarize these:

Type of Ultraf.	$\mathcal{I}$-ultrafilter	weak $\mathcal{I}$-ultrafilter	Ref.
Selective	$\mathcal{ED}$		[96]
Selective	$\mathcal{R}$		[96]
Selective	all tall analytic ideals		[132]
P-point	$\mathsf{fin}\times\mathsf{fin}$		[53]
P-point	conv		[5]
Q-point		$\mathcal{ED}_{\mathsf{fin}}$	[96]
Rapid		all tall analytic P-ideals	[186]
Hausdorff	$\mathcal{G}_{\mathsf{FC}}$		[98]
Non-flat	nflat		[68, 114]

We make one comment on non-flat ultrafilters (which we will not define). *Flat ultrafilters* were introduced in [68], in connection with the space of bounded operators of a separable, infinite dimensional complex Hilbert space. Later it was found that flat ultrafilters are those whose dual ideal is Katětov above the ideal nflat, i.e., ultrafilters such that $\mathsf{nflat} \leqslant_K \mathcal{U}^*$, giving a purely combinatorial characterization [114, 115]. It is known that P-points are non-flat, however, the existence of nflat-ultrafilters in ZFC remains unknown.

Problem 5.4 (Farah-Phillip-Steprans [68]). *Does* ZFC *prove the existence of a* nflat*-ultrafilter?*

There are other natural classes of combinatorial properties of ultrafilters which can be characterized as $\mathcal{I}$-ultrafilters (see [7, 9, 27, 28, 43, 54, 68, 70–72, 76, 84, 86, 94, 95, 113, 124, 130, 155, 156, 174, 175]).

The reader may note from the table above that there may be non-Katětov equivalent ideals $\mathcal{I}$ and $\mathcal{J}$ for which the classes of $\mathcal{I}$-ultrafilters and $\mathcal{J}$-ultrafilters coincide. At the moment, it is not clear if there is a combinatorial characterization of when two different ideals induce the same class of ultrafilters. This topic was studied by R. Filipów, K. Kowitz and A. Kwela in [70] where (among other results) they proved the following:

Theorem 5.5 (Filipów–Kowitz–Kwela [70]). *Assume* CH. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω such that $\mathcal{I}$ is tall, $\mathcal{J}$ is Katětov uniform and $\mathcal{P}(\omega)/\mathcal{J}$ is σ-closed. There is an $\mathcal{I}$-ultrafilter that is not a $\mathcal{J}$-ultrafilter if and only if $\mathcal{I} \not\leqslant_{\mathsf{K}} \mathcal{J}$.*

Here, $\mathcal{J}$ is *Katětov uniform* if it is Katětov equivalent to all its restrictions. An extreme case where many different ideals induce same class of ultrafilters is the model constructed by the first author in [35], where all tall F_σ ideals induce the same class of ultrafilters.

We now introduce the following notion, which can be viewed as the notion of an ultrafilter that "has no interesting combinatorial properties".

Definition 5.6. Let $\mathcal{U}$ be an ultrafilter on ω. We say that $\mathcal{U}$ *is unremarkable* if $\mathcal{I} \leqslant_{\mathsf{K}} \mathcal{U}^*$ for every analytic ideal $\mathcal{I}$.

Since there are only $\mathfrak{c}$-many analytic ideals, Proposition 4.10 yields the following:

Proposition 5.7. *There is an unremarkable ultrafilter.*

On the other hand, assuming additional hypothesis to ZFC we can obtain stronger versions of this proposition, which we shall prove in this section.

Proposition 5.8. *Assume $\mathfrak{b} = \mathfrak{c}$. Then there is an ultrafilter all of whose Rudin–Blass predecessors are unremarkable.*

An ultrafilter like the one in the previous proposition cannot be constructed just on the basis of ZFC, since in the Miller's model P-points are dense in the Rudin–Blass ordering. Moreover, in Miller's model, $\mathcal{I}$-ultrafilters are dense in the Rudin–Blass ordering whenever $\mathcal{I}$ is an analytic P-ideal (see [36]). Hausdorff utrafilters are also dense in the Rudin–Blass ordering in the Miller's model, as was proved in [4]. The best known ZFC approximation to an ultrafilter like in Proposition 5.8 was given in [125] by presenting an ultrafilter all of whose Rudin–Blass precessors are Katětov–Blass above $\mathcal{ED}_{fin}$, that is, no RB-predecessor is a Q-point.

Theorem 5.9 (C. Laflamme, J. P. Zhu [125]). *There is an ultrafilter $\mathcal{U}$ such that for all $\mathcal{V} \leqslant_{\mathsf{RB}} \mathcal{U}$, $\mathcal{ED}_{fin} \leqslant_{KB} \mathcal{V}$. Therefore, no Rudin–Blass predecessor of $\mathcal{U}$ is a Q-point.*

Note that the previous theorem implies that for any ideal $\mathcal{I}$ which is KB-below the ideal $\mathcal{ED}_{fin}$, there is an ultrafilter for which all RB-predecessors are KB-above $\mathcal{I}$. We don't know of any other ideal having this property.

Definition 5.10. Given two ideals $\mathcal{I}$ and $\mathcal{J}$ on ω, we say they are *compatible* if $\mathcal{I} \cup \mathcal{J}$ generates a proper ideal, in which case we denote by $\langle \mathcal{I} \cup \mathcal{J} \rangle$ the ideal that $\mathcal{I} \cup \mathcal{J}$ generates.

Lemma 5.11. *If $\mathcal{I}$ and $\mathcal{J}$ are analytic, then $\langle \mathcal{I} \cup \mathcal{J} \rangle$ is analytic.*

Proof. Note that $\bigcup \colon \mathcal{P}(\omega) \times \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is continuous, $\mathcal{I} \times \mathcal{J}$ is analytic and $\langle \mathcal{I} \cup \mathcal{J} \rangle = \bigcup(\mathcal{I} \times \mathcal{J})$ □

Lemma 5.12. *Let $\mathcal{I}$ be a meager ideal on ω and $f \colon \omega \to \omega$ a finite-to-one function. Then $f(\mathcal{I})$ is meager.*

Proof. Let $\{I_n : n \in \omega\}$ be a partition of ω into intervals witnessing that $\mathcal{I}$ is meager. Define a sequence of natural numbers $\langle k_n : n \in \omega \rangle$ as follows:

(1) $k_0 = 0$.
(2) Once k_n is defined, let k_{n+1} be big enough so there is $m \in \omega$ such that $I_m \subseteq f^{-1}([k_n, k_{n+1}))$.

Now define $J_n = [k_n, k_{n+1})$ for each $n \in \omega$. We claim that $\{J_n : n \in \omega\}$ witnesses $f(\mathcal{I})$ is meager. Assume otherwise there is $A \in [\omega]^\omega$ such that $\bigcup_{n \in A} J_n \in f(\mathcal{I})$. Then $f^{-1}(\bigcup_{n \in A} J_n) \in \mathcal{I}$. However, by the construction of $\{k_j : j \in \omega\}$, for each $n \in A$, there is $m_n \in \omega$ such that $I_{m_n} \subseteq f^{-1}[J_n]$, So, for all $n \in A$, $I_{m_n} \subseteq f^{-1}(\bigcup_{n \in A} J_n)$, which implies $f^{-1}(\bigcup_{n \in A} J_n) \notin \mathcal{I}$, a contradiction. □

Lemma 5.13. *Let $\{\mathcal{I}_\alpha : \alpha < \lambda\}$ be a $\subseteq$-increasing family of meager ideals on ω where $\lambda < \mathfrak{b}$. Then $\mathcal{I} = \bigcup_{\alpha < \lambda} I_\alpha$ is a meager ideal.*

Proof. It is clear that $\mathcal{I}$ is an ideal, so all we have to prove is that $\mathcal{I}$ is indeed a meager ideal. For each $\alpha < \lambda$, let $\mathcal{P}_\alpha = \{I_n^\alpha : n \in \omega\}$ be a partition witnessing that $\mathcal{I}_\alpha$ is a meager ideal. Now, since $\lambda < \mathfrak{d}$, there is an interval partition $\mathcal{P} = \{I_n : n \in \omega\}$ such that for each $\alpha < \lambda$, there is $k \in \omega$ such that for all $n \geqslant k$, I_n contains some interval from $\mathcal{P}_\alpha$. This implies that $\mathcal{P}$ witnesses that $\mathcal{I}$ is a meager ideal. □

Lemma 5.14. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω and $f \colon \omega \to \omega$ such that $f(\mathcal{I}) = \mathsf{fin}$. Then:*

(1) *If $\mathsf{im}(f) \in \mathcal{J}^+$, then $f^{-1}(\mathcal{J})$ and $\mathcal{I}$ are compatible.*
(2) *If $\mathcal{J}$ is meager, $\mathsf{im}(f) \in \mathcal{J}^+$ and f is finite-to-one, then $\langle f^{-1}(\mathcal{J}) \cup \mathcal{I} \rangle$ is meager.*
(3) *If $\mathcal{I}$ and $\mathcal{J}$ are analytic, then $\langle f^{-1}(\mathcal{J}) \cup \mathcal{I} \rangle$ is analytic.*

Proof. (1) Let us assume that $f^{-1}(\mathcal{J})$ and $\mathcal{I}$ are not compatible, so there are $A \in \mathcal{I}$ and $B \in \mathcal{J}$ such that $\omega = A \cup f^{-1}[B]$. Then, for each $n \notin B$, we have $f^{-1}(\{n\}) \subseteq A$. Since $\mathsf{im}(f) \in \mathcal{J}^+$, we have $\mathsf{im}(f) \setminus B$ is infinite, so for each $n \in \mathsf{im}(f) \setminus B$, $f^{-1}(\{n\}) \subseteq A$, that is $f^{-1}(\omega \setminus B) \subseteq A$, which contradicts that $A \in \mathcal{I}$.

(2) Let $\langle I_n : n \in \omega \rangle$ be a Talagrand partition for $\mathcal{J}$. For each $n \in \omega$, define $D_n = f^{-1}(I_n)$. Clearly $\{D_n : n \in \omega\}$ is a partition of ω into finite sets. We claim that for any $X \in \langle f^{-1}(\mathcal{J}) \cup \mathcal{I} \rangle$, X contains at most finitely many sets D_n. Suppose otherwise there is $X \in \langle f^{-1}(\mathcal{J}) \cup \mathcal{I} \rangle$ such that for infinitely many $n \in \omega$, $D_n \subseteq X$. We can assume $X = A \cup f^{-1}[B]$ where $A \in \mathcal{I}$ and $B \in \mathcal{J}$. Let $Z = \bigcup\{D_n : D_n \subseteq X\}$, so $Z \subseteq X$ and it is the union of infinitely many sets D_n.

Therefore, there are infinitely many $n \in \omega$ such that $I_n \subseteq f(Z)$, so $f(Z) \in \mathcal{J}^+$, and we can find $Y \in [\omega]^\omega$ and $\{k_n : n \in Y\}$ such that $k_n \in I_n \setminus B$ and $I_n \subseteq f(Z)$ for all $n \in Y$. Then, for each $n \in Y$, $f^{-1}(\{k_n\}) \subseteq A$, which implies $A \in \mathcal{I}^+$, a contradiction.

(3) Follows directly from Lemma 5.11 and Proposition 4.22. □

Proof of Proposition 5.8. Let $\mathcal{D} \subseteq \omega^\omega$ be the family given by Lemma 4.11, let $\mathcal{A}$ be the family of all analytic tall ideals and fix $\{(h_\alpha, \mathcal{J}_\alpha) : \alpha < 2^\omega\}$ an enumeration of $\mathcal{D} \times \mathcal{A}$. By recursion we construct a family of ideals $\{\mathcal{I}_\alpha : \alpha \in \mathfrak{d}\}$ such that for each $\alpha < 2^\omega$:

(1) $\mathcal{I}_\alpha$ is meager.
(2) $\mathcal{I}_\alpha \subseteq \mathcal{I}_{\alpha+1}$.
(3) If α is a limit ordinal, $\bigcup_{\beta<\alpha} \mathcal{I}_\beta \subseteq \mathcal{I}_\alpha$.
(4) There is $g_\alpha \colon \omega \to \omega$ such that $g_\alpha \circ h_\alpha$ is Katětov from $\mathcal{I}_\alpha$ to $\mathcal{J}_\alpha$.

We start with $\mathcal{I}_0 = h_0^{-1}(\mathcal{J}_0)$. Note that by Proposition 4.22(2), $\mathcal{I}_0$ is a meager ideal. Suppose $\mathcal{I}_\alpha$ has been defined. By Lemma 5.12, we have that $h_{\alpha+1}(\mathcal{I}_\alpha)$ is a meager ideal, so by Talagrand's theorem there is a finite-to-one function $g_{\alpha+1}$ such that $\mathsf{fin} = g_{\alpha+1}(h_{\alpha+1}(\mathcal{I}_\alpha))$. Let $f_{\alpha+1} = g_{\alpha+1} \circ h_{\alpha+1}$. Note that this implies that $f_{\alpha+1}(\mathcal{I}_\alpha) = \mathsf{fin}$: assume there is an infinite set $A \in f_{\alpha+1}(\mathcal{I}_\alpha)$, then $h^{-1}_{\alpha+1}[g^{-1}_{\alpha+1}[A]] = (g_{\alpha+1} \circ h_{\alpha+1})^{-1}[A] = f^{-1}_{\alpha+1}[A] \in \mathcal{I}_\alpha$, which implies that $g^{-1}_{\alpha+1}[A] \in h_{\alpha+1}(\mathcal{I}_\alpha)$, so $A \in g_{\alpha+1}(h_{\alpha+1}(\mathcal{I}_\alpha)) = \mathsf{fin}$, a contradiction.

Let us see first that $\mathsf{im}(f_{\alpha+1})$ is co-finite. Assume otherwise and let $A \subseteq \omega$ be an infinite set such that $A \cap \mathsf{im}(f_{\alpha+1}) = \emptyset$. Note that $\mathsf{im}(g_{\alpha+1})$ is cofinite (this is because $g_{\alpha+1}$ is a Rudin–Blass function from $h_{\alpha+1}(\mathcal{I}_\alpha)$ to fin), so we can assume $A \subseteq \mathsf{im}(g_{\alpha+1})$. Then, since $A \cap \mathsf{im}(f_{\alpha+1}) = \emptyset$, we should have $g^{-1}_{\alpha+1}[A] \cap \mathsf{im}(h_{\alpha+1}) = \emptyset$, which in turn implies that $h^{-1}_{\alpha+1}((g^{-1}_{\alpha+1}[A])) = \emptyset$, from which it follows that $h^{-1}_{\alpha+1}((g^{-1}_{\alpha+1}[A])) \in \mathcal{I}_\alpha$; thus, $g^{-1}_{\alpha+1}[A] \in h_{\alpha+1}(\mathcal{I}_\alpha)$, which in turn implies $A \in g_{\alpha+1}(h_{\alpha+1}(\mathcal{I}_\alpha)) = \mathsf{fin}$, a contradiction, since A was assumed to be infinite.

It follows that $\mathsf{im}(f_{\alpha+1}) \in \mathcal{J}^+_{\alpha+1}$, so we can apply clause (1) of Lemma 5.14 to get that $\mathcal{I}_\alpha$ and $f^{-1}_{\alpha+1}(\mathcal{J}_{\alpha+1})$ are compatible, which means that $\langle \mathcal{I}_\alpha \cup f^{-1}_{\alpha+1}(\mathcal{J}_{\alpha+1})\rangle$ is a proper ideal, and by clause (2) of Lemma 5.14, we also get that $\langle \mathcal{I}_\alpha \cup f^{-1}_{\alpha+1}(\mathcal{J}_{\alpha+1})\rangle$ is a meager ideal.

Define $\mathcal{I}_{\alpha+1} = \langle \mathcal{I}_\alpha \cup f^{-1}_{\alpha+1}(\mathcal{J}_{\alpha+1})\rangle$. This finishes the successor step. Now assume α is a limit ordinal and $\mathcal{I}_\beta$ has been defined for all $\beta < \alpha$. Let $\mathcal{I}'_\alpha = \bigcup_{\beta<\alpha} \mathcal{I}_\beta$. By Lemma 5.13, $\mathcal{I}'_\alpha$ is a meager ideal, Lemma 5.12 implies $h_\alpha(\mathcal{I}'_\alpha)$ is meager. Let g_α be a finite-to-one function such that $\mathsf{fin} = g_\alpha(h_\alpha(\mathcal{I}'_\alpha))$, let $f_\alpha = g_\alpha \circ h_\alpha$. By clause (2) from Lemma 5.14, $\langle \mathcal{I}'_\alpha \cup f^{-1}_\alpha(\mathcal{J}_\alpha)\rangle$ is a meager ideal. Then define $\mathcal{I}_\alpha = \langle \mathcal{I}'_\alpha \cup f^{-1}_\alpha(\mathcal{J}_\alpha)\rangle$. This finishes the construction.

Now define $\mathcal{I} = \bigcup_{\alpha<2^\omega} \mathcal{I}_\alpha$ and let $\mathcal{U}$ be any ultrafilter such that $\mathcal{I}^* \subseteq \mathcal{U}$. We claim that no RB-predecessor of $\mathcal{U}$ is a $\mathcal{J}$-ultrafilter for any analytic ideal $\mathcal{J}$. Let $\mathcal{V} \leqslant_{\mathsf{RB}} \mathcal{U}$ and an analytic ideal $\mathcal{J}$ be arbitrary. Then, by the choice of the family $\mathcal{D}$ (see Lemma 4.11), there is $h \in \mathcal{D}$ such that $h(\mathcal{U}) \leqslant_{\mathsf{RB}} \mathcal{V}$. There is $\alpha < 2^\omega$ such that $(h_\alpha, \mathcal{J}_\alpha) = (h, \mathcal{J})$. Thus, $g_\alpha \circ h_\alpha$ is a Katětov function from $\mathcal{I}_\alpha$ to $\mathcal{J}$, so for any $X \in \mathcal{J}$, $(g_\alpha \circ h_\alpha)^{-1}[X] \in \mathcal{I}_\alpha$, that is, $h_\alpha^{-1}[g_\alpha^{-1}[X]] \in \mathcal{I}_\alpha$, so $h_\alpha^{-1}[g_\alpha^{-1}[X]] \in \mathcal{U}^*$, which in turn implies $g_\alpha^{-1}[X] \in h_\alpha(\mathcal{U})^*$. Since $h = h_\alpha$, we have proved that $\mathcal{J} \leqslant_{\mathsf{KB}} h(\mathcal{U})^*$. Since $h(\mathcal{U}) \leqslant_{\mathsf{RB}} \mathcal{V}$, this implies that $\mathcal{J} \leqslant_{\mathsf{KB}} \mathcal{V}^*$, so $\mathcal{U}$ is not a $\mathcal{J}$-ultrafilter. Since $\mathcal{V} \leqslant_{\mathsf{RB}} \mathcal{U}$ and $\mathcal{J}$ were arbitrary, we have that no RB-predecessor of $\mathcal{U}$ is a $\mathcal{J}$-ultrafilter, for any analytic ideal $\mathcal{J}$. □

Under CH, the Proposition 5.8 can be strengthened to Rudin–Keisler ordering.

Proposition 5.15. *Asume* CH. *There is an ultrafilter all of whose Rudin–Keisler predecessors are unremarkable.*

Proof. Let $\{(f_\alpha, \mathcal{I}_\alpha) : \alpha \in \omega_1\}$ be an enumeration of all ordered pairs $(f, \mathcal{I})$ such that $f \in \omega^\omega$ and $\mathcal{I}$ is an analytic ideal. By recursion we construct a sequence $\{\mathcal{J}_\alpha : \alpha \in \omega_1\}$ such that:

(1) $\mathcal{J}_0 = [\omega]^{<\omega}$.
(2) $\mathcal{J}_\alpha$ is an analytic ideal.
(3) If $\alpha < \beta$, then $\mathcal{J}_\alpha \subseteq \mathcal{J}_\beta$.
(4) $\mathcal{I}_\alpha \leqslant_K \mathcal{J}_{\alpha+1}$.

At limit steps α we define $\mathcal{J}_\alpha = \bigcup_{\beta<\alpha} \mathcal{J}_\beta$. Note that $\mathcal{J}_\alpha$ is analytic since the countable union of analytic sets if analytic. Now suppose $\mathcal{J}_\alpha$ has been defined and we have to define $\mathcal{J}_{\alpha+1}$. By clause (1) from Lemma 4.22, $f_\alpha(\mathcal{J}_\alpha)$ is analytic, so by Talagrand's theorem there is a finite-to-one function $h_\alpha \colon \omega \to \omega$ such that $h_\alpha(f_\alpha(\mathcal{J}_\alpha)) = \mathsf{fin}$. A similar argument to that of Proposition 5.8 shows that $\mathsf{im}(h_\alpha \circ f_\alpha)$ is a cofinite set, so we can apply clause (1) of Lemma 5.14 to get that $\mathcal{J}_\alpha$ and $(h_\alpha \circ f_\alpha)^{-1}(\mathcal{I}_\alpha)$ are compatible, so $\langle \mathcal{J}_\alpha \cup (h_\alpha \circ f_\alpha)^{-1}(\mathcal{I}_\alpha)\rangle$ is a proper ideal. Define $\mathcal{J}_{\alpha+1} = \langle \mathcal{J}_\alpha \cup (h_\alpha \circ f_\alpha)^{-1}(\mathcal{I}_\alpha)\rangle$. By clause (3) from 5.14, $\mathcal{J}_{\alpha+1}$ is also an analytic ideal. It is clear from the construction that $\mathcal{I}_\alpha \leqslant_K \mathcal{J}_{\alpha+1}$. This finishes the construction. Now define $\mathcal{H} = \bigcup_{\alpha<\omega_1} \mathcal{J}_\alpha$, which is a proper ideal, and let $\mathcal{U}$ be an ultrafilter extending $\mathcal{H}^*$. Let us see that $\mathcal{U}$ satisfies the lemma. Let $\mathcal{I}$ be an analytic ideal and $f \in \omega^\omega$. By construction, there is $\alpha \in \omega_1$ such that $(\mathcal{I}, f) = (\mathcal{I}_\alpha, f_\alpha)$. Then, for any $A \in \mathcal{I}$, $f_\alpha^{-1}[h_\alpha^{-1}[A]] = (h_\alpha \circ f_\alpha)^{-1}[A] \in \mathcal{U}^*$, which means that $h_\alpha^{-1}[A] \in f_\alpha(\mathcal{U})^*$. Thus, h_α witnesses $\mathcal{I} \leqslant_K f(\mathcal{U})^*$. □

In [132] Mathias obtained a very nice characterization of Ramsey ultrafilters: *An ultrafilter is Ramsey if and only if it has non-empty intersection with every tall analytic ideal* (with the aid of large cardinals, this can be extended to a larger class of ideals, see [192] for more details). We aim to obtain analogous characterizations for other classes of ultrafilters.

Proposition 5.16. *Let* $\mathcal{I}$ *be an ideal and* $\mathcal{U}$ *an ultrafilter. The following are equivalent:*

(1) $\mathcal{U}$ *is an (weak)* $\mathcal{I}$*-ultrafilter.*

(2) *$\mathcal{U} \cap f^{-1}(\mathcal{I}) \neq \emptyset$ for every $f \in \omega^\omega$ (that is finite-to-one).*
(3) *If $\mathcal{J}$ is an ideal on ω such that $\mathcal{I} \leqslant_{K} \mathcal{J}$ ($\mathcal{I} \leqslant_{KB} \mathcal{J}$), then $\mathcal{J} \cap \mathcal{U} \neq \emptyset$.*
(4) *If $\mathcal{J}$ is an ideal on ω for which there is $X \in \mathcal{I}^+$ such that $\mathcal{I} \restriction X \leqslant_{RK} \mathcal{J}$, ($\mathcal{I} \restriction X \leqslant_{RB} \mathcal{J}$), then $\mathcal{J} \cap \mathcal{U} \neq \emptyset$.*

Proof. We only prove the version corresponding to $\mathcal{I}$-ultrafilters, the one for weak $\mathcal{I}$-ultrafilters is analogous.

Clearly 1) and 2) are equivalent. We prove that 1) implies 3) by contrapositive. Assume there is an ideal $\mathcal{J}$ on ω such that $\mathcal{I} \leqslant_{\mathsf{K}} \mathcal{J}$ but $\mathcal{J} \cap \mathcal{U} = \emptyset$. This means that $\mathcal{J} \subseteq \mathcal{U}^*$, so $\mathcal{I} \leqslant_{\mathsf{K}} \mathcal{U}^*$. We will now prove that 3) implies 4). Let $\mathcal{J}$ be an ideal on ω for which there is $X \in \mathcal{I}^+$ such that $\mathcal{I} \restriction X \leqslant_{\mathsf{RK}} \mathcal{J}$. It follows that $\mathcal{I} \leqslant_{\mathsf{K}} \mathcal{J}$, so $\mathcal{J} \cap \mathcal{U} \neq \emptyset$. We now prove that 4) implies 2). Let $f \colon \omega \longrightarrow \omega$, we need to find $A \in \mathcal{U}$ such that $f[A] \in \mathcal{I}$. If $f^{-1}(\mathcal{I})$ is not a proper ideal, there is nothing to do. Assume that it is a proper ideal and let $X = \mathsf{im}(f)$. We know that $X \in \mathcal{I}^+$ and $\mathcal{I} \restriction X \leqslant_{\mathsf{RK}} f^{-1}(\mathcal{I})$ (see Proposition 4.20), so we are done. □

In the case that $\mathcal{I}$ is definable, by applying Proposition 4.22, we have the following:

Proposition 5.17. *Let $\mathcal{I}$ be an analytic ideal and $\mathcal{U}$ an ultrafilter. The following are equivalent:*

(1) *$\mathcal{U}$ is a $\mathcal{I}$-ultrafilter.*
(2) *If $\mathcal{J}$ is an analytic ideal on ω such that $\mathcal{I} \leqslant_{K} \mathcal{J}$, then $\mathcal{J} \cap \mathcal{U} \neq \emptyset$.*
(3) *If $\mathcal{J}$ is an analytic ideal on ω for which there is $X \in \mathcal{I}^+$ such that $\mathcal{I} \restriction X \leqslant_{RB} \mathcal{J}$, then $\mathcal{J} \cap \mathcal{U} \neq \emptyset$.*

In case for weak $\mathcal{I}$-ultrafilters and $\mathcal{I}$ Borel, we only need to check the intersection for Borel ideals.

Proposition 5.18. *Let $\mathcal{I}$ be a Borel (analytic) ideal and $\mathcal{U}$ an ultrafilter. The following are equivalent:*

(1) *$\mathcal{U}$ is a weak $\mathcal{I}$-ultrafilter.*
(2) *If $\mathcal{J}$ is a Borel (analytic) ideal on ω such that $\mathcal{I} \leqslant_{KB} \mathcal{J}$, then $\mathcal{J} \cap \mathcal{U} \neq \emptyset$.*
(3) *If $\mathcal{J}$ is a Borel (analytic) ideal on ω for which there is $X \in \mathcal{I}^+$ such that $\mathcal{I} \restriction X \leqslant_{RB} \mathcal{J}$, then $\mathcal{J} \cap \mathcal{U} \neq \emptyset$.*

Now we will give the following theorem, which encompasses all the characterizations that we know of ultrafilters in the style of Mathias's characterization of Ramsey ultrafilters.

Theorem 5.19. *Let $\mathcal{U}$ be an ultrafilter on ω. Then:*

(1) (*Mathias* [132]) *$\mathcal{U}$ is selective if and only if $\mathcal{U} \cap \mathcal{I} \neq \emptyset$ for every tall analytic ideal $\mathcal{I}$.*
(2) *$\mathcal{U}$ is selective if and only if $\mathcal{U} \cap \mathcal{I} \neq \emptyset$ for every tall F_σ (or Borel) ideal $\mathcal{I}$.*
(3) (*H-Reyes Saenz* [101]) *$\mathcal{U}$ is a Q-point if and only if $\mathcal{U} \cap \mathcal{I} \neq \emptyset$ for every ω-hitting Borel ideal $\mathcal{I}$.*
(4) (*Vojtáš* [186]) *$\mathcal{U}$ is rapid if and only if $\mathcal{U} \cap \mathcal{I} \neq \emptyset$ for all tall analytic P-ideals $\mathcal{I}$.*

Proof. The first point is a direct application of Remark 5.3 and Proposition 5.17. The second point follows inmediately from the first and the theorem of Grebík and Vidnyánszky 4.14. Alternatively, it is enough to note that in order for an ultrafilter to satisfy the partition relation $\mathcal{U} \longrightarrow (\mathcal{U})^2_2$, it is only needed to intersect F_σ ideals, since the ideal of monochromatic sets of a coloring has this complexity. Point 3 follows from Remark 5.3, Theorem 4.24 and Proposition 5.18. Point 4 follows by Remark 5.3 and Lemma 4.23. □

Regarding Vojtáš' characterization of a rapid ultrafilter, we would like to mention the following refinement due to J. Flašková and the first author (see [75] and [36]).

Theorem 5.20 (Flašková [75], Cancino-Manríquez [36])**.**
(1) *There is a family $\mathcal{D}$ of summable ideals of size $\mathfrak{d}$ such that an ultrafilter $\mathcal{U}$ is rapid if and only if $\mathcal{U} \cap \mathcal{I} \neq \emptyset$ for every $\mathcal{I} \in \mathcal{D}$.*
(2) *If $\mathcal{D}$ is a family of summable ideals of size less than $\mathfrak{d}$, then there is a non-rapid ultrafilter $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{I} \neq \emptyset$ for every $\mathcal{I} \in \mathcal{D}$.*

We have established the importance of $\mathcal{I}$-ultrafilters. Naturally, now the question is how to build them. There are several methods, but there are more questions than answers. For some ideals $\mathcal{I}$ it is possible to find $\mathcal{I}$-ultrafilters in ZFC, while for others it is consistent that they do not exist. Moreover, there are ideals, such as the zero density ideal $\mathcal{Z}$ for which it is unknown if it is consistent that $\mathcal{Z}$-ultrafilters do not exist. A characterization of the ideals for which their respective ultrafilters exist in ZFC is unknown. Two canonical constructions to build $\mathcal{I}$-ultrafilters are the "generic existence method" and with the aid of parameterized diamonds, two methods we will explore below. Of course, there are other constructions, such as Todorcevic's ingenious construction of a selective ultrafilter under $\mathfrak{m} > \omega_1$ (see [182]).

6. Generic existence of $\mathcal{I}$-ultrafilters

The "generic existence method" is probably the most direct of straightforward approach to building a combinatorial object. In our context, we simply have to list all the requirements that we need to satisfy and try to solve them recursively, in at most c steps. Of course, there is a risk that the recursion cannot continue, so often a cardinal invariant hypothesis is needed to ensure that the recursion does not get stuck before we can satisfy all the requirements. Although it is not always possible to perform a generic existence construction, it is very important to understand when it is possible and what are the obstructions for achieving such construction.

Definition 6.1. Let $\mathcal{P}$ be a property which an ultrafilter may or may not have. We say that *ultrafilters with property $\mathcal{P}$ exist generically* if every filter $\mathcal{F}$ generated by less than $\mathfrak{c}$ many sets, can be extended to an ultrafilter with property $\mathcal{P}$.

The relevant cardinal invariant in our context is the "exterior cofinality" of an ideal. This cardinal was studied by Brendle and Flašková in [28] and by Hong and Zhang in [94].

Definition 6.2. Let $\mathcal{I}$ be an ideal on ω. The *exterior cofinality of $\mathcal{I}$* (also called the *generic existence number of $\mathcal{I}$*) is defined as $\mathfrak{ge}(\mathcal{I}) = \min\{\mathsf{cof}(\mathcal{J}) \mid \mathcal{I} \subseteq \mathcal{J}\}$.

The next lemma is useful to work with the generic existence numbers.

Lemma 6.3. *Let $\mathcal{I}$ be an ideal on ω. The following cardinal invariants are equal.*

(1) $\mathfrak{ge}(\mathcal{I})$.
(2) $\min\{cof(\mathcal{J}) \mid \mathcal{I} \leqslant_K \mathcal{J}\}$.
(3) $\min\{cof(\mathcal{J}) \mid \mathcal{I} \leqslant_{KB} \mathcal{J}\}$.

Proof. Denote $\mu = min\{\mathsf{cof}(\mathcal{J}) \mid \mathcal{I} \leqslant_{\mathsf{K}} \mathcal{J}\}$ and $\mu_0 = min\{\mathsf{cof}(\mathcal{J}) \mid \mathcal{I} \leqslant_{\mathsf{KB}} \mathcal{J}\}$. It is clear that $\mu \leqslant \mu_0 \leqslant \mathfrak{ge}(\mathcal{I})$, so it is enough to prove that $\mathfrak{ge}(\mathcal{I}) \leqslant \mu$. Let $\mathcal{J}$ be an ideal such that $\mathsf{cof}(\mathcal{J}) = \mu$ and $\mathcal{I} \leqslant_{\mathsf{K}} \mathcal{J}$. Find $f \in \omega^\omega$ a Katětov function from $\mathcal{J}$ to $\mathcal{I}$. It follows that $\mathcal{I} \subseteq f(\mathcal{J})$. In this way, $\mathfrak{ge}(\mathcal{I}) \leqslant \mathsf{cof}(f(\mathcal{J})) \leqslant \mu$ (see Lemma 4.21). □

We now write some basic remarks regarding this cardinal invariant.

Lemma 6.4. *Let $\mathcal{I}$, $\mathcal{J}$ be tall ideals on ω.*

(1) $\omega_1 \leqslant \mathfrak{ge}(\mathcal{I}) \leqslant \mathfrak{c}$.
(2) $non^*(\mathcal{I}) \leqslant \mathfrak{ge}(\mathcal{I}) \leqslant cof(\mathcal{I})$.
(3) *If $\mathcal{I} \leqslant_K \mathcal{J}$, then $\mathfrak{ge}(\mathcal{I}) \leqslant \mathfrak{ge}(\mathcal{J})$.*

The following is a very important theorem, which enables us to perform generic existence constructions of ultrafilters.

Theorem 6.5 (Brendle–Flašková [28], Hong–Zhang [94])**.** *Let $\mathcal{I}$ be an ideal on ω. The following are equivalent:*

(1) *$\mathcal{I}$-ultrafilters exist generically.*
(2) *Weak $\mathcal{I}$-ultrafilters exist generically.*
(3) $\mathfrak{ge}(\mathcal{I}) = \mathfrak{c}$.

Proof. Obviously the first point implies the second one. We now argue that point 2 implies point 3 by proving that if $\mathfrak{ge}(\mathcal{I}) < \mathfrak{c}$, then weak $\mathcal{I}$-ultrafilters do not exist generically. Let $\mathcal{J}$ be an ideal extending $\mathcal{I}$ such that $\mathsf{cof}(\mathcal{J}) < \mathfrak{c}$. It is clear that every ultrafilter extending $\mathcal{J}^*$ is not a weak $\mathcal{I}$-ultrafilter.

We now prove that if $\mathfrak{ge}(\mathcal{I}) = \mathfrak{c}$, then $\mathcal{I}$-ultrafilters exist generically. Let $\mathcal{F}$ be a filter such that $\kappa = \mathsf{cof}(\mathcal{F}) < \mathfrak{c}$. Fix an enumeration $\omega^\omega = \{f_\alpha \mid \alpha < \mathfrak{c}\}$. We recursively define $\{\mathcal{F}_\alpha \mid \alpha < \mathfrak{c}\}$ a family of filters such that for every $\alpha < \mathfrak{c}$, the following holds:

(1) $\mathcal{F}_0 = \mathcal{F}$.
(2) If $\xi < \alpha$, then $\mathcal{F}_\xi \subseteq \mathcal{F}_\alpha$.
(3) If α is limit, then $\mathcal{F}_\alpha = \bigcup_{\xi<\alpha}\mathcal{F}_\xi$.
(4) $\mathsf{cof}(\mathcal{F}_\alpha) \leqslant \kappa + |\alpha|$.
(5) There is $A \in \mathcal{F}_{\alpha+1}$ such that $f_\alpha[A] \in \mathcal{I}$.

Assume $\mathcal{F}_\alpha$ is already defined, we will find $\mathcal{F}_{\alpha+1}$. Since $\mathsf{cof}(\mathcal{F}_\alpha) < \mathfrak{ge}(\mathcal{I})$, it follows that $\mathcal{I} \nleq_{\mathsf{K}} \mathcal{F}_\alpha^*$. In particular, f_α is not a Katětov function from $\mathcal{F}_\alpha^*$ to $\mathcal{I}$. This means that there is $A \in \mathcal{F}_\alpha^+$ such that $f_\alpha[A] \in \mathcal{I}$. Let $\mathcal{F}_{\alpha+1}$ be the filter generated by $\mathcal{F}_\alpha \cup \{A\}$. This finishes the construction.

Any ultrafilter extending $\bigcup_{\alpha<\mathfrak{c}}\mathcal{F}_\alpha$ is an $\mathcal{I}$-ultrafilter. □

We now present the following chart, containing some of the ideals and their exterior cofinalities.

Ideal	$\mathfrak{ge}$	Ultrafilter	Reference
fin×fin	$\mathfrak{d}$	P-point	[94]
conv	$\mathfrak{d}$	P-point	[28]
$\mathcal{ED}$	$\mathsf{cov}(\mathcal{M})$	Selective	[94]
$\mathcal{R}$	$\mathsf{cov}(\mathcal{M})$	Selective	[28]
nwd	$\mathsf{cof}(\mathcal{M})$	Nowhere dense	[27]
mz	$\max\{\mathsf{non}(\mathcal{E}), \mathfrak{d}\}$	Measure zero	[27]

For more computations of the exterior cofinality, see [28].

7. Simple existence under parametrized diamonds

Generic existence of $\mathcal{I}$-ultrafilters is a rather powerful kind of existence when compared to the simple existence of $\mathcal{I}$-ultrafilters. In this section we turn our attention to two cardinal invariants on ideals which, in the presence of parametrized diamond principles, imply the simple existence of $\mathcal{I}$-ultrafilters. The parametrized diamonds are guessing principles introduced by Džamonja, Moore and the third author in [135]. These principles are weakenings of Jensen's diamond, but they have the advantage that they are valid in a large number of models, even in models where the Continuum Hypothesis fails.

Definition 7.1. Let $\mathcal{I}$ be an ideal on ω. The cardinal invariant $\mathfrak{z}(\mathcal{I})$ is defined as follows:

$$\mathfrak{z}(\mathcal{I}) = \min\{|\mathcal{D}| : (\forall f \in \omega^\omega)(\exists A \in \mathcal{D})(f[A] \in \mathcal{I})\}$$

Let FtO denote the family of finite-to-one functions from ω to ω. Then $\mathfrak{z}_{fin}(\mathcal{I})$ is defined as follows:

$$\mathfrak{z}_{fin}(\mathcal{I}) = \min\{|\mathcal{D}| : (\forall f \in \mathsf{FtO})(\exists A \in \mathcal{D})(f[A] \in \mathcal{I})\}$$

We will now introduce the principles $\diamondsuit(\mathfrak{z}(\mathcal{I}))$ and $\diamondsuit(\mathfrak{z}_{fin}(\mathcal{I}))$ that we will use to build $\mathcal{I}$-ultrafilters. These are instances of the diamond principles from [135]. We will say that a function $F\colon 2^{<\omega_1} \to X$ is a Borel function if for all $\alpha < \omega$, $F \upharpoonright 2^\alpha$ is Borel.

Definition 7.2. Let $\mathcal{I}$ a Borel ideal. The parametrized diamond principle $\diamondsuit(\mathfrak{z}(\mathcal{I}))$ is the following assertion:

> For any Borel function $F\colon 2^{<\omega_1} \to \omega^\omega$, there is a function $g\colon \omega_1 \to [\omega]^\omega$ such that for all $f \in 2^{\omega_1}$ the set $\{\alpha \in \omega_1 : F(f \upharpoonright \alpha)[g(\alpha)] \in \mathcal{I}\}$ is stationary.

The function g is called a $\diamondsuit(\mathfrak{z}(\mathcal{I}))$-guessing sequence. The principle $\diamondsuit(\mathfrak{z}_{fin}(\mathcal{I}))$ is defined in a similar way (the function F takes values in FtO instead of ω^ω).

Proposition 7.3 (C. [34])**.** *Let $\mathcal{I}$ be a Borel ideal. Then:*

1. $\diamondsuit(\mathfrak{z}(\mathcal{I}))$ *implies the existence of $\mathcal{I}$-ultrafilters.*
2. $\diamondsuit(\mathfrak{z}_{fin}(\mathcal{I}))$ *implies the existence of weak $\mathcal{I}$-ultrafilters.*

Proof. The two proofs are similar, so we only prove the first one. We can assume that the domain of the function F consists of ordered pairs $(f, \vec{A})$, where $f \in \omega^\omega$ and $\vec{A} = \langle A_\beta : \beta < \alpha \rangle$ is a sequence of countable length of infinite subsets of ω. Define $F(f, \vec{A})$ as follows:

(1) If $\vec{A}$ is a centered family, let $p(\vec{A})$ be a pseudointersection of $\langle A_\beta : \beta < \alpha \rangle$ (defined in a recursive or in a Borel way), and $\varphi_{\vec{A}} \colon \omega \to p(\vec{A})$ be its increasing enumeration. Then make $F(f, \vec{A}) = f \circ \varphi_{\vec{A}}$.
(2) If $\vec{A}$ is not a centered family, then $F(f, \vec{A}) = id$.

Let g be a $\diamondsuit(\mathfrak{z}(\mathcal{I}))$-guessing sequence for F. Then construct a sequence $\vec{B} = \langle B_\alpha : \alpha \in \omega_1 \rangle$ as follows. Start with $B_n = \omega \setminus n$ and suppose $\langle B_\beta : \beta < \alpha \rangle$ has been defined. Then define $B_\alpha = \varphi_{\langle B_\beta : \beta < \alpha \rangle}[g(\alpha)]$. Clearly, $\vec{B}$ is a $\subseteq^*$-decreasing sequence of sets, so it is centered.

Let us see that $\vec{B}$ is a witness for $\mathfrak{z}(\mathcal{I})$. Pick any $f \in \omega^\omega$ and consider $(f, \vec{B})$. Then the set $\{\alpha \in \omega_1 \colon F(f, \vec{B} \restriction \alpha)[g(\alpha)] \in \mathcal{I}\}$ is stationary. Let $\alpha \in \omega_1$ be such that $F(f, \vec{B} \restriction \alpha)[g(\alpha)] \in \mathcal{I}$. Since $\vec{B} \restriction \alpha$ is a centered family, it follows from the definition of F that $F(f, \vec{B} \restriction \alpha) = f \circ \varphi_{\vec{B} \restriction \alpha}$, so $f \circ \varphi_{\langle B_\beta : \beta < \alpha \rangle}[g(\alpha)] = f[\varphi_{\langle B_\beta : \beta < \alpha \rangle}[g(\alpha)]] = f[B_\alpha]$, and due to the choice of α, $f[B_\alpha] \in \mathcal{I}$.

Let $\mathcal{U}$ be any ultrafilter extending $\vec{B}$. It is clear that $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter. □

It is well known that the principle $\diamondsuit(\mathfrak{j})$ implies that $\mathfrak{j} \leqslant \aleph_1$ for every Borel cardinal invariant $\mathfrak{j}$, while $\diamondsuit(\mathfrak{j})$ holds in many models where $\mathfrak{j} \leqslant \aleph_1$ (see [135]). In particular, by virtue of Proposition 7.3 in order to obtain information about the existence of $\mathcal{I}$-ultrafilters it is useful to evaluate the cardinal invariants $\mathfrak{z}(\mathcal{I})$ and $\mathfrak{z}_{fin}(\mathcal{I})$ for definable ideals $\mathcal{I}$.

Proposition 7.4 (C. [34]). *Let $\mathcal{I}, \mathcal{J}$ be ideals on ω. Then:*

(1) *If $\mathcal{I} \leqslant_K \mathcal{J}$, then $\mathfrak{z}(\mathcal{J}) \leqslant \mathfrak{z}(\mathcal{I})$.*
(2) *If $\mathcal{I} \leqslant_{KB} \mathcal{J}$, then $\mathfrak{z}_{fin}(\mathcal{J}) \leqslant \mathfrak{z}_{fin}(\mathcal{I})$*

Proposition 7.5 (C. [34]). *If $\mathcal{I}$ is an ideal on ω such that for some $n, k \in \omega$ there exists a coloring $\varphi \colon [\omega]^n \to k$ whose homogeneous sets are in the ideal $\mathcal{I}$, then $\mathfrak{z}_{fin}(\mathcal{I}) \leqslant \max\{\mathfrak{d}, \mathfrak{r}_\sigma\}$.*

Proposition 7.6 (C. [34]). *For any meager ideal $\mathcal{I}$, $\min\{\mathfrak{r}, \mathfrak{d}\} \leqslant \mathfrak{z}_{fin}(\mathcal{I})$.*

For the next proposition, $\mathfrak{r}_{part}$ is the minimal cardinality of a family $\mathcal{R} \subseteq [\omega]^\omega$ such that for any partition $\{P_n : n \in \omega\}$ into infinite sets, there is $A \in \mathcal{R}$ such that either, there is $n \in \omega$ such that $A \subseteq P_n$, or for all $n \in \omega$, $A \cap P_n$ is finite.

Proposition 7.7 (C. [34]). (1) *$\mathfrak{z}(\mathsf{conv}) = \mathfrak{r}_\sigma$.*
(2) *$\mathfrak{r} \leqslant \mathfrak{z}(\mathsf{nwd}) \leqslant \mathfrak{r}_\sigma$.*
(3) *$\mathfrak{z}_{fin}(\mathcal{ED}_{fin}) = \mathfrak{d}$.*
(4) *$\mathfrak{z}(\mathsf{Fin} \times \mathsf{Fin}) = \mathfrak{z}_{fin}(\mathsf{Fin} \times \mathsf{Fin}) = \mathfrak{r}_{part}$.*
(5) *$\mathfrak{z}(\mathcal{ED}) = \mathfrak{z}_{fin}(\mathcal{ED}) = \max\{\mathfrak{r}_{part}, \mathfrak{d}\}$.*
(6) *$\max\{\mathfrak{r}_{part}, \mathfrak{d}\} \leqslant \mathfrak{z}_{fin}(\mathcal{R}) \leqslant \mathfrak{z}(\mathcal{R}) \leqslant \max\{\mathfrak{r}_\sigma, \mathfrak{d}\}$.*
(7) *If $\mathcal{I}$ is an analytic P-ideal, $\min\{\mathfrak{r}_\sigma, \mathfrak{d}\} \leqslant \mathfrak{z}_{fin}(\mathcal{I}) \leqslant \mathfrak{d}$.*
(8) *$\mathfrak{z}_{fin}(\mathcal{Z}) = \mathfrak{z}_{fin}(\mathcal{SC}) = \min\{\mathfrak{r}_\sigma, \mathfrak{d}\}$.*

8. How different can ultrafilters be?

At first glance, all ultrafilters might look the same. We may ask, how can we build two really different ultrafilters? From a topological point of view, this question translates to whether ω^* is homogeneous. As mentioned in the introduction, this led Walter Rudin to introduce P-points and thus finding really different ultrafilters under the Continuum Hypothesis. Some time later, Frolik proved from ZFC that ω^* is not homogeneous. Finally, it was Kunen who explicitly found ultrafilters with different topological properties. Van Mill continued with this task by finding many points on ω^* with different topological type (see [138]). Although the question in the topological sense is completely resolved, there are still several criteria for which there is still no complete answer. Naturally, assuming the Continuum Hypothesis, it is possible to build all kinds of ultrafilters. So the question really is if we can find a model where all ultrafilters are as similar as possible.

Without a doubt, the best-known types of ultrafilters are the selective, P-points and Q-points. In each case, it is known that their existence cannot be proven from ZFC. In [118] Kunen proved that in the random model there are no selective ultrafilters. In [140] and [139], Miller proved that there are no Q points in the Laver and Miller models. Finally, Shelah proved (see [191] and [164]) that it is consistent that there are no P-points. In [44] the second author and Chodounský proved that there are no P-points in the Silver model. Regarding nwd-ultrafilters, Shelah in [166] built a model where they do not exist. The method developed by Shelah was later expanded by Brendle in [27]. Recently, the first author built a model where there are no $\mathcal{I}$-ultrafilters, for any ideal $\mathcal{I}$ that is F_σ. Regarding the existence of P-points and Q-points, the following is known:

Proposition 8.1. (1) $\mathfrak{d} = \omega_1$ *implies that there is a Q-point.*
(2) (*Ketonen* [112]) $\mathfrak{d} = \mathfrak{c}$ *implies that there is a P-point.*
(3) *In this way,* $\mathfrak{c} \leqslant \omega_2$ *implies that there is either a P-point or a Q-point.*

The following problem is one of the most important open question regarding ultrafilters on countable sets, see [142]:

Problem 8.2. *Is it consistent that there are no P-points and no Q-points?*

One of the first difficulties in building a model in which there are no P-points and no Q-points, is that the continuum must be larger than ω_2. In this way, we cannot use usual countable support iterations of proper forcings since they do not allow us to pass ω_2, while usual finite support iterations are also ruled out since they add Cohen reals. Although Proposition 8.1 points out to a deeper problem. In order to destroy Q-points, we need to add unbounded reals; however, adding unbounded reals cofinally often will create P-points. It is the general belief that it should be possible to build a model without a P-point or a Q-point, but the reality could be completely different.

As we have discussed before, ultrafilters can be classified using Borel ideals and the Katětov order. We may wonder if this classification can be trivial; *Is it possible that every ultrafilter is Katětov above every tall Borel ideal?* (in other words, if every ultrafilter is unremarkable). *Is it possible to find a Borel ideal $\mathcal{I}$ for which* *we can prove in ZFC that $\mathcal{I}$-ultrafilters exist?* According to Theorem 6.5, it is enough to find a Borel ideal for which exterior cofinality is provable to be $\mathfrak{c}$. An old construction of Pospíšil (see [150]) can be used to find an analytic ideal with that property.

Definition 8.3. Let $\mathcal{P}$ be an independent family. By $\mathsf{Pos}(\mathcal{P})$ denote the ideal generated by $\{A^{\mathsf{c}} \mid A \in \mathcal{P}\} \cup \{\bigcap C \mid C \in [\mathcal{P}]^{\omega}\}$.

We now have the following:

Proposition 8.4 ([150], [86])**.** *Let $\mathcal{P}$ be an independent family.*

(1) *$\mathsf{Pos}(\mathcal{P})$ is a proper ideal.*
(2) *If $|\mathcal{P}| = \mathfrak{c}$, then $\mathfrak{ge}(\mathsf{Pos}(\mathcal{P})) = \mathfrak{c}$.*
(3) *If $\mathcal{P}$ is perfect, then $\mathsf{Pos}(\mathcal{P})$ is an analytic ideal.*

Proof. For the first point, we need to prove that $\omega \notin \mathsf{Pos}(\mathcal{P})$. Let $A_0, \ldots, A_n \in \mathcal{P}$ and $C_0, \ldots, C_n \in [\mathcal{P}]^{\omega}$. We need to show that $X = (\bigcup A_i^{\mathsf{c}}) \cup (\bigcap C_0) \cup \cdots \cup (\bigcap C_n)$ is co-infinite. Since each C_i is infinite, we can choose $B_i \in C_i$ such that $B_i \notin \{A_0, \ldots, A_n\}$. It follows that $(\bigcap C_0) \cup \cdots \cup (\bigcap C_n) \subseteq B_0 \cup \cdots \cup B_n$. Since $\mathcal{P}$ is independent, $(\bigcup A_i^{\mathsf{c}}) \cup (\bigcup B_i)$ has infinite complement, so we are done.

We now prove the second point by contradiction. Assume that $|\mathcal{P}| = \mathfrak{c}$ and there is an ideal $\mathcal{I}$ with $\mathsf{cof}(\mathcal{I}) < \mathfrak{c}$ and $\mathsf{Pos}(\mathcal{P}) \subseteq \mathcal{I}$. Let $\mathcal{B} \subseteq \mathcal{I}$ be a base of size less than $\mathfrak{c}$. It follows that there must be $B \in \mathcal{B}$ and $\{A_n \mid n \in \omega\} \subseteq \mathcal{P}$ such that $A_n^{\mathsf{c}} \subseteq B$ for every $n \in \omega$. In this way, $\bigcup_{n\in\omega} A_n^{\mathsf{c}} \subseteq B$, so $\bigcup_{n\in\omega} A_n^{\mathsf{c}} \in \mathcal{I}$. However, we know that $\bigcap_{n\in\omega} A_n$ is also in $\mathcal{I}$, which entails that $\omega \in \mathcal{I}$. We leave the computation of the complexity of the ideal to the reader. □

As perfect independent families exist (see e.g. [86]) we conclude:

Corollary 8.5. *There is an analytic ideal $\mathcal{I}$ for which $\mathcal{I}$-ultrafilters exist generically.*

We can now obtain a Borel ideal as above appealing to the following theorem of H. Sakai:

Theorem 8.6 (Sakai [162])**.** *Every analytic ideal is contained in a Borel ideal.*

The theorem of Sakai uses the *Luzin separation Theorem* and does not provide an explicit complexity of the Borel ideal. In [86] the second and third author found a variation $\mathsf{Pos}(\mathcal{P})$ (named $\mathsf{Pos}_B(\mathcal{P})$) that is $F_{\sigma\delta\sigma}$ and extends $\mathsf{Pos}(\mathcal{P})$. In particular, it follows that $\mathfrak{ge}(\mathsf{Pos}_B(\mathcal{P})) = \mathfrak{c}$. Hence we conclude:

Theorem 8.7 (G.-H. [86])**.** *There is an $F_{\sigma\delta\sigma}$-ideal $\mathcal{I}$ for which $\mathcal{I}$-ultrafilters exist generically.*

Moreover, using ideas from [28], it is proved in [86] that the complexity $F_{\sigma\delta\sigma}$ is optimal.

Theorem 8.8 (G.-H. [86])**.** *It is consistent that $\mathcal{I}$-ultrafilters do not exist generically for every $F_{\sigma\delta}$-ideal $\mathcal{I}$.*

Of course, the theorem above does not rule out the existence of $\mathcal{I}$-ultrafilters for $F_{\sigma\delta}$ or even F_{σ} ideals. In fact, it is not even known if there are $\mathcal{Z}$-ultrafilters in ZFC.

Problem 8.9 (H. [97]). *Are there $\mathcal{Z}$-ultrafilters in* ZFC*?*

Problem 8.10 (G.-H. [86]). *Is it consistent that there are no $\mathcal{I}$-ultrafilters for any $F_{\sigma\delta}$-ideal $\mathcal{I}$?*

In general, we would like to understand for which Borel ideals $\mathcal{I}$, does ZFC imply the existence of $\mathcal{I}$-ultrafilters (or weak $\mathcal{I}$-ultrafilters). The case of F_σ ideals was solved by the first author in [35], where it was proved that consistently there are no weak $\mathcal{I}$-ultrafilters for any F_σ ideal $\mathcal{I}$. There is a way to present the results from [35] in an axiomatic framework employing the principle of Near Coherence of Filters.

Definition 8.11 (Blass [16]). Let $\mathcal{U}$ and $\mathcal{V}$ be two non-principal ultrafilters on ω. We say that $\mathcal{U}$ and $\mathcal{V}$ are *nearly coherent* if there is a finite-to-one function $f\colon \omega \to \omega$ such that $f(\mathcal{U}) = f(\mathcal{V})$.

The *Near Coherence of Filters*, denoted by NCF, is the assertion that any two different ultrafilters on ω are nearly coherent.

The NCF principle has some quite interesting consequences, among them we can find the following:

Theorem 8.12 (Blass [16]). *The Near Coherence of Filters principle implies the following:*

(1) *There are no Q-points.*
(2) $\mathfrak{u} < \mathfrak{d}$.
(3) *The P-points are dense in the Rudin–Keisler ordering.*
(4) *For any family $\{\mathcal{U}_\alpha : \alpha < \lambda\}$ of ultrafilters where $\lambda < \mathfrak{d}$, there is a finite-to-one function $h\colon \omega \to \omega$ such that for all $\alpha, \beta \in \lambda$, $h(\mathcal{U}_\alpha) = h(\mathcal{U}_\beta)$.*

Theorem 8.13. *Assume* NCF *and let $\mathcal{I}$ be an ideal on ω.*

(1) *If $\mathfrak{u} < \mathfrak{z}(\mathcal{I})$, then there is no $\mathcal{I}$-ultrafilter.*
(2) *If $\mathfrak{u} < \mathfrak{z}_{fin}(\mathcal{I})$, then there is no weak $\mathcal{I}$-ultrafilter.*

Proof. Let $\mathcal{U}$ be an non-principal ultrafilter on ω, we need to find a function $f\colon \omega \to \omega$ such that $\mathcal{I} \cap f(\mathcal{U}) = \emptyset$. Let $\lambda = \mathfrak{z}(\mathcal{I})$. Let $\mathcal{V}_0$ be an ultrafilter generated by less than λ sets. By NCF, there is a finite-to-one function $h\colon \omega \to \omega$ such that $h(\mathcal{U}) = h(\mathcal{V}_0)$. Since $\mathcal{V}_0$ is generated by less than λ sets, $h(\mathcal{U})$ is also generated by less than λ sets. Let $\mathcal{B}$ be an ultrafilter base for $h(\mathcal{U})$ of cardinality smaller than λ. Then there is a function $f\colon \omega \to \omega$ such that $f(\mathcal{B}) \cap \mathcal{I} = \emptyset$. Therefore, we also get $f(h(\mathcal{U})) \cap \mathcal{I} = \emptyset$, so $f \circ h$ witnesses $\mathcal{U}$ is not a $\mathcal{I}$-ultrafilter.

Note that if $\mathfrak{u} < \mathfrak{z}_{fin}(\mathcal{I})$ we can take the function $f\colon \omega \to \omega$ to be finite-to-one, so $f \circ h$ is finite-to-one; thus, in this case there are not even weak $\mathcal{I}$-ultrafilters. □

Note that from Proposition 7.7 we have that $\mathfrak{z}_{fin}(\mathcal{ED}_{fin}) = \mathfrak{d}$. It is well known that in the Miller's model $\mathfrak{d} = \mathfrak{c}$ and the Near Coherence of Filters holds, so the previous theorem gives us a proof that in the Miller's model there is no Q-point. It turns out that this can be generalized to the case of ideals of Borel complexity F_σ.

Theorem 8.14 (C. [35]). *The conjunction of the next assertions is relatively consistent:*

(1) *The Near Coherence of Filters.*
(2) *For any F_σ ideal $\mathcal{I}$, $\mathfrak{z}_{fin}(\mathcal{I}) = \mathfrak{c}$.*

Therefore, it is relatively consistent that for any F_σ ideal $\mathcal{I}$, there is no weak $\mathcal{I}$-ultrafilter.

Note that the previous theorem implies that no $F_{\sigma\delta\sigma}$-ideal from Proposition 8.7 can be extended to an F_σ-ideal.

Another way to classify ultrafilters is with the *Tukey order,* which is used to study directed sets (that is, partial orders in which any two elements have a common upper bound).

Definition 8.15. Let $\mathbb{D} = (D, \leqslant_D)$ and $\mathbb{E} = (E, \leqslant_E)$ be two directed partial orders.

(1) Let $f: D \longrightarrow E$. We say that f is a *cofinal function from* $\mathbb{D}$ *to* $\mathbb{E}$ if it maps cofinal subsets of $\mathbb{D}$ to cofinal subsets of $\mathbb{E}$.
(2) We say that $\mathbb{E} \leqslant_\mathsf{T} \mathbb{D}$ ($\mathbb{E}$ is *Tukey below* $\mathbb{D}$) if there is a cofinal function from $\mathbb{D}$ to $\mathbb{E}$.
(3) We say $\mathbb{E}$ and $\mathbb{D}$ are *Tukey equivalent* (denoted $\mathbb{D} =_\mathsf{T} \mathbb{E}$) if $\mathbb{E} \leqslant_\mathsf{T} \mathbb{D}$ and $\mathbb{D} \leqslant_\mathsf{T} \mathbb{E}$.

The Tukey order can be formulated in terms of the Katětov order as follows: for every directed set $\mathbb{D}$, define $\mathsf{ncf}(\mathbb{D})$ as the ideal on $\mathbb{D}$ of all non cofinal subsets of $\mathbb{D}$. It follows that $\mathbb{E} \leqslant_\mathsf{T} \mathbb{D}$ if and only if $\mathsf{ncf}(\mathbb{E}) \leqslant_\mathsf{K} \mathsf{ncf}(\mathbb{D})$. Moreover, if we define $\mathsf{bnd}(\mathbb{D})$ as the ideal of bounded subsets of $\mathbb{D}$, it is also possible to prove that $\mathbb{E} \leqslant_\mathsf{T} \mathbb{D}$ if and only if $\mathsf{bnd}(\mathbb{D}) \leqslant_\mathsf{K} \mathsf{bnd}(\mathbb{E})$. In this way, we get the following:

Lemma 8.16. *Let $\mathbb{D} = (D, \leqslant_D)$ and $\mathbb{E} = (E, \leqslant_E)$ be two directed partial orders. The following are equivalent:*

(1) $\mathbb{E} \leqslant_T \mathbb{D}$.
(2) *There is $f: D \longrightarrow E$ that maps cofinal subsets of $\mathbb{D}$ to cofinal subsets of $\mathbb{E}$.*
(3) *There is $g: E \longrightarrow D$ that maps unbounded subsets of $\mathbb{E}$ to unbounded subsets of $\mathbb{D}$ (a function with this property is called a* Tukey function*).*

An example of a directed set is $([\kappa]^{<\omega}, \subseteq)$ (where κ is an infinite cardinal). These directed sets play a crucial role in the study of the Tukey order:

Lemma 8.17. *Let $\mathbb{D} = (D, \leqslant_D)$ be a directed set. If $|D| \leqslant \kappa$, then $\mathbb{D} \leqslant_T [\kappa]^{<\omega}$.*

Proof. Take an enumeration (maybe with repetitions) $D = \{d_\alpha \mid \alpha \in \kappa\}$. Define $f: [\kappa]^{<\omega} \longrightarrow D$ such that if $s = \{\alpha_1, \ldots, \alpha_n\}$, then $d_{\alpha_1}, \ldots, d_{\alpha_n} \leqslant_D f(s)$. It is easy to see that f is a cofinal function. □

The following types of sets are useful:

Definition 8.18. Let $\mathbb{D} = (D, \leqslant_D)$ be a directed set. We say that $\mathcal{B} \subseteq \mathcal{D}$ is *strongly unbounded* if no infinite subset of $\mathcal{B}$ has an upper bound.

The following is a simple description of the Tukey class of $[\kappa]^{<\omega}$. We sketch the argument for completeness.

Lemma 8.19. *Let $\mathbb{D} = (D, \leqslant_D)$ be a directed set.*

(1) $[\kappa]^{<\omega} \leqslant_T \mathbb{D}$ *if and only if* $\mathbb{D}$ *has a strongly unbounded subset of size* κ,
(2) *Assume* $|D| = \kappa$. $[\kappa]^{<\omega} =_T \mathbb{D}$ *if and only if* $\mathbb{D}$ *has a strongly unbounded subset of size* κ.

Proof. We start with the first point. Let $\mathbb{D} = (D, \leqslant_D)$ be a directed set and $S = \{d_\alpha \mid \alpha < \kappa\}$ a strongly unbounded subset of $\mathbb{D}$. Define $g\colon [\kappa]^{<\omega} \longrightarrow D$ such that for every $s \in [\kappa]^{<\omega}$, we have that $g(s)$ is an upper bound of $\{d_\alpha \mid \alpha \in s\}$. It is easy to see that g is a Tukey function. For the other implication, assume that $[\kappa]^{<\omega} \leqslant_T \mathbb{D}$ and let $g\colon [\kappa]^{<\omega} \longrightarrow D$ be a Tukey function. Define $S = \{g(\{\alpha\}) \mid \alpha \in \kappa\}$. It is easy to see that $|S| = \kappa$ (since every point in D can only have finite preimage) and is strongly unbounded in $\mathbb{D}$ (since $[\kappa]^1$ is strongly unbounded in $[\kappa]^{<\omega}$). The second point follows by the first and the previous lemma. □

Let $\mathcal{U}$ be an ultrafilter on ω. It follows that $\mathcal{U}$ is a directed set when ordered with the reverse inclusion (it is also a directed set when ordered with inclusion, but a trivial one). It follows that every ultrafilter is Tukey below $[\mathfrak{c}]^{<\omega}$. This is the motivation for the following definition:

Definition 8.20. Let $\mathcal{U}$ be an ultrafilter on ω. We say that $\mathcal{U}$ is *Tukey top* if $\mathcal{U} =_T [\mathfrak{c}]^{<\omega}$.

Equivalently, $\mathcal{U}$ is Tukey top if there is $\mathcal{W} \subseteq \mathcal{U}$ of size $\mathfrak{c}$ such that for every $\mathcal{B} \in [\mathcal{W}]^\omega$, we have that $\cap\mathcal{B} \notin \mathcal{U}$.

Proposition 8.21 (Isbell [105])**.** *There is a Tukey top ultrafilter.*

Proof. Let $\mathcal{P}$ be an independent family of size $\mathfrak{c}$ and $\mathcal{U}$ any ultrafilter extending $\mathsf{Pos}(\mathcal{P})^*$. It is easy to see that $\mathcal{P} \subseteq \mathcal{U}$ is a strongly unbounded set. □

However, the following is unknown:

Problem 8.22 (Isbell [105])**.** *Is there a non-Tukey top ultrafilter?*[2]

As far as we know, it might be consistent that all ultrafilters have the same Tukey type. It is known that consistently the problem has an affirmative answer. In fact, Dobrinen and Todorcevic proved in [60] that P-points are not Tukey top. They also introduced the following class of ultrafilters:

Definition 8.23. Let $\mathcal{U}$ be an ultrafilter on ω. We say that $\mathcal{U}$ is *basically generated* if there is a base $\mathcal{B} \subseteq \mathcal{U}$ closed under intersections such that every convergent sequence on $\mathcal{B}$ (when viewed as a subspace of $\mathcal{P}(\omega)$) has a bounded (in $\mathcal{U}$) subsequence.

This is a variation of the notion of *basic directed set* introduced earlier by Solecki and Todorcevic in [172]. The following is proved in [60]:

Proposition 8.24 (Dobrinen–Todorcevic [60])**.** (1) *Every P-point is basically generated.*

[2] Added in proof: recently, the first author and J. Zapletal proved that consistenly all ultrafilters are Tukey top (see [38]).

(2) *If* $\mathcal{U}$ *is basically generated, then* $[\omega_1]^{<\omega} \not\leqslant_T \mathcal{U}$*. In particular,* $\mathcal{U}$ *is not Tukey top.*
(3) *The class of basically generated ultrafilters is closed under taking limits.*

We would also like to highlight the following (particular case) of a Theorem of Dobrinen and Todorcevic, which says that Tukey functions between P-points may be assumed to be continuous:

Theorem 8.25 (Dobrinen–Todorcevic [60])**.** *Let* $\mathcal{U}$ *be a* P*-point and* $\mathcal{V}$ *an ultrafilter such that* $\mathcal{V} \leqslant_T \mathcal{U}$*. There is a continuous and monotone* $f \colon \mathcal{P}(\omega) \longrightarrow \mathcal{P}(\omega)$ *such that* $f \restriction \mathcal{U}$ *is a cofinal function from* $\mathcal{U}$ *to* $\mathcal{V}$*.*

Proposition 8.26. *Any non P-point ultrafilter is Tukey above* $(\omega^\omega, \leqslant^*)$*.*

Proof. Let $\{A_n : n \in \omega\}$ be a partition of ω witnessing $\mathcal{U}$ is not a P-point, and such that $\min(A_n) < \min(A_{n+1})$ for all $n \in \omega$. Then, for each $X \in \mathcal{U}$, there are infinitely many $n \in \omega$ such that $A_n \cap X$ is infinite. For each $X \in \mathcal{U}$, define $f_X \colon \omega \to \omega$ as $f_X(n) = \min A_{k_n} \cap X$ where $k_n \in \omega$ is the minimal $l \geqslant n$ such that $X \cap A_l$ is infinite. Let us see that for any cofinal $\mathcal{D} \subseteq \mathcal{U}$, the set $\{f_X : X \in \mathcal{D}\}$ is cofinal in $(\omega^\omega, \leqslant^*)$. Fix $h \in \omega^\omega$, define $X^h = \bigcup_{n\in\omega} A_n \setminus h(n)$. We can assume h is strictly increasing. Since $\{A_n : n \in \omega\}$ witnesses that $\mathcal{U}$ is not a P-point, we have $X^h \in \mathcal{U}$. Since $\mathcal{D}$ is cofinal in $\mathcal{U}$, there is $B \in \mathcal{U}$ such that $B \subseteq X^h$. It is easy to see that $h \leqslant f_B$. □

Proposition 8.27. *It is consistent that all ultrafilters are Tukey above* $(\omega^\omega, \leqslant^*)$*.*

To learn more about the Tukey order on ultrafilters, the reader may consult the paper [60], the survey [58] and the references in there.

It has been very fruitful to study the preservation of ultrafilters under forcing. Let $\mathcal{U}$ be an ultrafilter on ω and $\mathbb{P}$ a forcing notion. Unless $\mathbb{P}$ does not add new reals, $\mathcal{U}$ will no longer be an ultrafilter after forcing with $\mathbb{P}$. However, it might still generate an ultrafilter.

Definition 8.28. Let $\mathcal{U}$ be an ultrafilter on ω and $\mathbb{P}$ a partial order. We say that $\mathbb{P}$ *preserves* $\mathcal{U}$ (or $\mathcal{U}$ is $\mathbb{P}$*-indestructible*) if $\mathcal{U}$ generates an ultrafilter after forcing with $\mathbb{P}$. Equivalently, for every $p \in \mathbb{P}$ and $\dot{X}$ such that $p \Vdash$"$\dot{X} \subseteq \omega$", there are $q \leqslant p$ and $A \in \mathcal{U}$ such that either $q \Vdash$"$A \subseteq \dot{X}$" or $q \Vdash$"$A \cap \dot{X} = \emptyset$".

Evidently, if $\mathbb{P}$ does not add new reals, then $\mathbb{P}$ preserves all ultrafilters. On the other hand, any filter adding either a Cohen, random or dominating real will destroy all ultrafilters. As it is often the case, preservation under Sacks forcing is particularly interesting, as we will now see. The following theorem is basically a compilation of results obtained independently by Miller, Eisworth (unpublished) and Yiparaki.

Theorem 8.29 (Eisworth, Miller [141], Yiparaki [190])**.** *Let* $\mathcal{U}$ *be an ultrafilter on* ω*. The following are equivalent:*

(1) *Sacks forcing preserves* $\mathcal{U}$*.*
(2) *There is a forcing* $\mathbb{P}$ *that adds reals and preserves* $\mathcal{U}$*.*
(3) *For every* $p \in \mathbb{S}$ *and* $c \colon p \longrightarrow 2$*, there is* $q \leqslant p$ *and* $A \in \mathcal{U}$ *such that* c *is constant on* $q \restriction A$*.*

(4) *For every* $p \in \mathbb{S}$*, there is* $q \leqslant p$ *and* $A \in \mathcal{U}$ *such that either* $A \subseteq x$ *for every* $x \in [q]$ *or* $A \cap x = \emptyset$ *for every* $x \in [q]$*.*

It is interesting to compare $\mathbb{S}$ preservability with being Tukey top.

Corollary 8.30. *Let* $\mathcal{U}$ *be an ultrafilter on* ω*.*

(1) *The following are equivalent:*
  (a) $\mathcal{U}$ *is Tukey top.*
  (b) *There is* $\mathcal{B} \subseteq \mathcal{U}$ *of size* $\mathfrak{c}$ *such that if* $\mathcal{D} \subseteq \mathcal{B}$ *is infinite, then* $\bigcap \mathcal{D} \in \mathcal{U}^*$*.*
(2) *The following are equivalent:*
  (a) $\mathcal{U}$ *is not preserved by Sacks forcing.*
  (b) *There is a perfect* $\mathcal{P} \subseteq \mathcal{U}$ *such that if* $\mathcal{R} \subseteq \mathcal{P}$ *is perfect, then* $\bigcap \mathcal{R} \in \mathcal{U}^*$*.*

Although both notions are strikingly similar, it is not obvious if there is any implication betweem them.

Problem 8.31 (Blass)**.** *What is the relationship between being preserved by Sacks forcing and not being Tukey top?*

It was proved by Bartoszyński, Goldstern, Judah and Shelah [3, Theorem 6.2.2] that there is an analytic ideal $\mathcal{I}$ such that every ultrafilter $\mathcal{U}$ extending $\mathcal{I}^*$ is not preserved by Sacks forcing (equivalently, is not preserved by any forcing adding new reals). It then follows by Theorem 8.6 that there is a Borel ideal with this property. It was realized by Chodounský, the second and third authors that the density zero filter has this property. In fact, the same is true for being Tukey top.

Theorem 8.32 (Chodounský–G-H [45])**.** *If* $\mathcal{U}$ *is not a* $\mathcal{Z}$*-ultrafilter* (*in particular, if* $\mathcal{Z}^* \subseteq \mathcal{U}$)*, then* $\mathcal{U}$ *is both Tukey top and destroyed by Sacks forcing.*

Proof. For every $n \in \omega$, define $P_n = [2^n, 2^{n+1})$ and $\sigma_n \colon \mathcal{P}(\omega) \longrightarrow \mathbb{R}$ where $\sigma_n(A) = \frac{|A \cap P_n|}{2^n}$. For $A \subseteq \omega$, denote $\sigma(A) = lim_{n \longrightarrow \infty}\, \sigma_n(A \cap P_n)$ in case the limit exists. In this way, $\mathcal{Z}$ is the set of all $A \subseteq \omega$ such that $\sigma(A) = 0$. We now have the following:

Claim 8.32.1. There is $p \in \mathbb{S}$ such that:

(1) If $x \in [p]$, then $\sigma(x) = \frac{1}{2}$.
(2) If $\mathcal{B} \subseteq [p]$ is infinite, then $\sigma(\bigcup \mathcal{B}) = 1$ and $\sigma(\bigcap \mathcal{B}) = 0$.

The idea is to build p such that all any n of its branches are "independent" in almost all of the P_m. The precise construction can be consulted in [45].

Now, let $\mathcal{U}$ be an ultrafilter such that $\mathcal{Z} \leqslant_{\mathsf{K}} \mathcal{U}^*$. Pick $f \in \omega^\omega$ a Katětov function and fix p the Sacks tree constructed above. Given $x \in [p]$, denote $x^0 = x$ and $x^1 = \omega \setminus x$. Find $i_x \in 2$ such that $f^{-1}(x^{i_x}) \in \mathcal{U}$. In this way, we can find $i \in 2$ such that $\mathcal{B} = \{x \in [p] \mid i_x = i\}$ has size $\mathfrak{c}$. For concreteness, assume that $i = 1$ (the other case is similar). We claim that $\{f^{-1}(x^1) \mid x \in \mathcal{B}\} \subseteq \mathcal{U}$ is strongly unbounded. Let $\{x_n \mid n \in \omega\} \subseteq \mathcal{B}$ and $C = \bigcap_{n \in \omega} f^{-1}(x^1_n)$. We now have the following:

$$f[C] = f[\bigcap_{n \in \omega} f^{-1}(x^1_n)] \subseteq \bigcap_{n \in \omega} f[f^{-1}(x^1_n)] \subseteq$$

$$\subseteq \bigcap_{n \in \omega} x^1_n = \bigcap_{n \in \omega} x^{\mathsf{c}}_n = \Big(\bigcup_{n \in \omega} x_n\Big)^{\mathsf{c}}.$$

So $f[C] \in \mathcal{Z}$. Since f is a Katětov function, we conclude that $C \in \mathcal{U}^*$. The proof that $\mathcal{U}$ is Sacks destructible is similar (see [45] for more details). □

It was proved by Baumgartner and Laver in [6] that Ramsey ultrafilters are Sacks-indestructible. It was later noted that P-points are enough for Sacks preservation. In this way, it is consistent that there are Sacks-indestructible ultrafilters. However, the following is unknown:

Problem 8.33 (Miller [141])**.** *Is it consistent that there are no Sacks-indestructible ultrafilters?*

Although Sacks indestructible ultrafilters are very interesting, the indestructibility of P-points is the one that has been studied the most. Miller proved the following theorem:

Theorem 8.34 (Miller [139])**.**
(1) *Let $\mathbb{P}$ be a partial order. If $\mathbb{P}$ adds an unbounded real, then $\mathbb{P}$ destroys all non P-point.*
(2) *Miller forcing preserves an ultrafilter $\mathcal{U}$ if and only if $\mathcal{U}$ is a P-point.*

It follows by the previous Theorem that $\mathcal{U} \times \mathcal{U}$ is never preserved by Miller forcing, even if $\mathcal{U}$ is preserved. In contrast, if $\mathcal{U}$ and $\mathcal{V}$ are preserved by Sacks forcing, then $\mathcal{U} \times \mathcal{V}$ is also preserved. One of the reasons that the preservation of P-points is so important is that this property can be iterated, as was proved by Blass and Shelah.

Theorem 8.35 (Blass–Shelah, see [164])**.** *Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta \rangle$ be a countable support iteration of proper forcings and $\mathcal{U}$ a P-point. If for every $\alpha < \delta$, we have that $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha$ preserves $\mathcal{U}$", then $\mathbb{P}_\delta$ preserves $\mathcal{U}$.*

Since most of our tools for preserving ultrafilters use P-points, it is natural to ask the following:

Problem 8.36 (Nyikos)**.** *Is it consistent that there is an ultrafilter generated by ω_1 many sets, yet there are no P-points?*

A positive solution to this problem was announced in [167], but the result has not been published so far. Zapletal studied the preservation of P-points and Ramsey ultrafilters under definable forcing. In case where a forcing is suitable definable, preservation of this kind of ultrafilters is equivalent to simpler and easier to check conditions.

Theorem 8.37 ((CH+LC) (Zapletal, [193])**.** *Let $\mathbb{P}$ be a suitably definable proper forcing. The following are equivalent:*

(1) *$\mathbb{P}$ preserves all P-points.*
(2) *$\mathbb{P}$ does not add a splitting real and has the weak Laver property.*

Theorem 8.38 ((LC) (Zapletal, [192])**.** *Let $\mathbb{P}$ be a suitably definable proper forcing and $\mathcal{U}$ a Ramsey ultrafilter. The following are equivalent:*

(1) *$\mathbb{P}$ preserves $\mathcal{U}$ and it generates a Ramsey ultrafilter in the extension.*
(2) *$\mathbb{P}$ does not add a splitting real and is ω^ω-bounding.*

The LC above denotes a large cardinal hypothesis. In practice, it is often the case that no large cardinals are needed at all. The reader may find more information (as well as the definition of the undefined notions) in Zapletal's book [192].

We started this section wondering how different ultrafilters can be. It is worth pointing out that all selective ultrafilters are equal in some sense. This can be seen from the following theorem of Todorcevic:

Theorem 8.39 ((LC) Todorcevic, see [67])**.** *Let $\mathcal{U}$ be an ultrafilter. The following are equivalent:*

(1) *$\mathcal{U}$ is selective.*
(2) *$\mathcal{U}$ is $\mathcal{P}(\omega)/\mathsf{fin}$-generic over $L(\mathbb{R})$.*

A similar characterization of $\mathcal{P}(\omega)/\mathcal{I}$ generic ultrafilters for $\mathcal{I}$ an F_σ-ideal was found by Chodounský and Zapletal in [46].

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