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PYTHAGOREAN MUSIC THEORY

Abstract. A need for the use of arithmetic in explanations of musical phenomena has arisen as a consequence of the discovery of early Pythagoreans that some harmonic musical intervals can be "explained" by ratios of small positive integers. Following their steps, Euclid in his *Sectio Canonis* examines numerical ratios of concords. Using the language of music theory, at the beginning of the treatise he proved a few general arithmetical propositions that are proved in the *Elements* in a purely arithmetical manner. He continues with the propositions that are not of the general character since they are related to concrete intervals. The influence of arithmetic to the musical theory is obvious and dominant in the *Sectio Canonis*, but the influence of one of these two theories to the other was not only one-way. It is not possible to understand clearly the definitions of some arithmetical notions without understanding of Pythagorean music theory.

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Contents

The earliest source that gives us information about the relation between mathematics and music theory in the early period is attributed to Euclid. That is a short treatise entitled *Sectio Canonis*, which consists of the prologue and 20 propositions. [1] It contains the basics of the Pythagorean doctrine of musical harmony.

The first nine propositions of the *Sectio Canonis* are arithmetical theorems. Judging by them, this treatise is based on subtle number theory. In their proofs, the author uses propositions from the arithmetical books of the *Elements*. For instance, the proof of SC.2 relies on VIII.7, while SC.3 is an immediate consequence of VIII.8. However, the language that he uses in this treatise is not the same as the language of arithmetical books but suited to the musical theory. This is probably the reason why the analysts of the early music theory missed the fact that in the *Sectio Canonis* the author proves some propositions that are already proved in the *Elements*. Such are SC.1, SC.2, SC.4, and SC.5.

By SC.1 and SC.2 the author of the *Sectio Canonis* proves the assertions that are proved in proposition VIII.14 of the *Elements*. In SC.4 and SC.5, he proves assertions that are proved in VIII.16. Moreover, in the proofs of the propositions on

[1] Commenting *Ptolemy Harmonics*, Porphyry quotes 16 propositions of the *Sectio*, attributing them to "Euclid the Geometer". In *A Commentary on the First Book of Euclid's Elements* Proclus ascribes to Euclid writing on the *Elements of Music* (69.3). Boethius translated in Latin several propositions of the *Sectio* without naming the author. He ascribed the proof of SC.3 to Archytas (Diels, 47.A19). Ptolemy makes an explicit reference to SC.4 and SC.10, quoting them as examples of the Pythagorean approach to musical intervals [13, p. 19]. See also [1, p. 190] and [5, vol. I, p. 444].

music theory, he uses the same arguments as in the arithmetical books of the *Elements*. Thus, as VIII.14 and VIII.16 are Theaetetus's theorems, so are SC.1, SC.2, a part of SC.4, and the whole of SC.5. The second part of SC.4 is a consequence of Archytas's proposition SC.3.

1. Monochord

A need for the use of arithmetical notions and propositions in explanations of particular musical phenomena has arisen as a consequence of the discovery of early Pythagoreans that some harmonic musical intervals can be "explained" by ratios of numbers. As Theon of Smyrna claims, in the earliest period of Pythagoreanism, Lasus of Hermione at Peloponnesus[2], and, later, disciples of Hippasus of Metapontum found out that particular musical accords can be explained by using small numbers — 1, 2, 3, 4. They did it by performing acoustical experiments with vessels of equal sizes and made of the same material. In Theon's words (II.12 a, [9, p. 39]):

> *Taking several similar vessels of the same capacity, one was left empty and the other filled halfway with a liquid, then they were both struck, thus obtaining the consonance of the octave. Again leaving one vessel empty and filling the other up to one quarter, then striking them, the consonance of the fourth was obtained. For the accord of the fifth, a third of a vase was filled; the relationship of the empty spaces was 2 to 1 for the octave, 3 to 2 for the fifth, and 4 to 3 for the fourth.*

A similar experiment is ascribed to Hippasus but, according to Aristoxenus, instead of vessels he had been using bronze discs of the same diameters but of different thicknesses that were in the ratios of 2 : 1, 3 : 2 and 4 : 3. In Aristoxenus's words (Diels, 18 A 12),

> *Hippasus made four bronze discs in such a way that, while their diameters were equal, the first disc was one-third as thick as the second (4:3), a half as thick as the third (3:2), and twice as thick as the fourth (2:1). When struck, the discs sounded in a certain consonance (fr. 90).*

However, because of the use of discs of the precise ratios of thicknesses, Hippasus's experiment seems to be too complicated to be performed as the first attempt in which the ratios of the octave, fifth, and fourth were found. It looks more like an attempt to confirm an already-known fact.

It also seems that before Lasus and Hippasus, Pythagoras came to the same conclusions on concordant accords. According to Diogenes Laertius (VIII. 11–12), "*he discovered the division of the monochord*." Xenocrates of Chalcedon [12, p. 268] also informs us that:

[2] Lasus was a musician and theoretician of music, the teacher of Pindar, and, according to the *Suda*, he was the first to write a book about music. Herodotus (7.6) places him in the time of the Pisistratidae. Thus, he was Pythagoras's contemporary, although not a Pythagorean.

> *Pythagoras discovered that musical intervals do not come to be apart from number; for they are a comparison of quantity with quantity.*

He also says that:

> *he therefore investigated under what conditions there result concordant or discordant intervals and everything harmonious or inharmonious. And turning to the generation of sound, he said that if from an equality a concordance is to be heard, it is necessary that there be some motion; but motion does not occur without number, and neither does number without quantity.*

Thus, it was Pythagoras who discovered the numerical ratios of the basic concords. But, although the sources are silent about that, these ratios could only have been the octave $(2:1)$, the fifth $(3:2)$, and the fourth $(4:3)$.

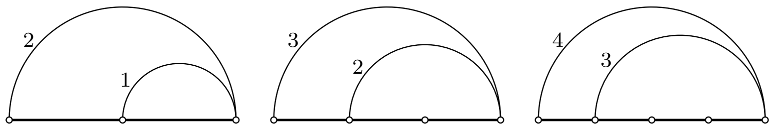

FIGURE 1. Octave, fifth and fourth

Lasus's and Hippasus's experiments were conducted to confirm what Pythagoras had already discovered by observation and experiments with a string instrument, though hardly with a monochord, which was probably not attested before the fourth century [4, pp. 374–375].

2. Intervals

Following the steps of Pythagoras, Lasus, and Hippasus, the author of the *Sectio Canonis* sets forth in a quest for numerical ratios of concords. Conveyed by the examples of the octave, of the fifth and the fourth, he investigates only the ratios of numbers, assuming that only such ratios could produce concordant musical intervals. He calls them shortly — *intervals*.

Of course, intervals $p:q$ and $r:s$ are considered to be equal if p,q and r,s are proportional pairs of numbers i.e., if $p:q=r:s$. However, if p and q are the least among proportional numbers,[3] they are relatively prime (VII.22), thus, each interval could be considered as a numerical ratio of two relatively prime numbers.

In the *Sectio Canonis* particular intervals have particular names. Some of them are named as *multiples*, some as *epimoric*, some as *epimeric*. However, the author of the *Sectio Canonis* is not defining them explicitly, but it is clear from the context what are these intervals. An interval is a *multiple* if it is determined by the ratio $n:1$. In particular, interval $2:1$ is called *double*, while $3:1$ is a *triple*. Obviously, if $p:q$ is a multiple interval i.e., if $p:q=n:1$, then $p=nq$ (VII.19), thus, the

[3] The Pythagoreans in the time of Archytas call such pairs of numbers — *pythmenes* (Diels, 47 A17).

notion of a *multiple interval* is not different from the notion of a *multiple* from the *Elements* (VII. def. 5).

Intervals that are of the form $(n+1):n$, are called *epimoric*.[4] Some of the epimoric intervals, such as $3:2$ and $4:3$, which produce the concords of the fifth and the fourth, have special names. Former is called *hemiolic*, later, *epitritic*. Further, the interval $9:8$ is called *epogdoic*, while intervals of the form $(n+k):k$, $k \neq 1,\ k \neq jn,\ j = 1, 2, \ldots,$ are named as *epimeric*.

However, $p:q$ and $q:p$ represent the same interval. Thus, any interval can be represented as a decreasing, or an increasing pair of numbers. The author of the *Sectio Canonis* uses both ways. In propositions SC.1 and SC.2 he uses increasing pairs of numbers, while in SC.3, decreasing.

3. Addition and subtraction of intervals

Although the operations of the addition and subtraction of intervals are frequently used in the *Sectio Canonis*, the definitions of these notions are missing. However, from the proofs of particular propositions, it is possible to understand what Euclid means by them. Explained in modern terms, these two operations are, in effect, multiplication, and division of ratios. Since ratios are not considered numbers, ancient Greeks could not define a product of ratios, but they could demonstrate the calculus of ratios following their musical theory.

To find what is the sum of two intervals $p:q$ and $r:s$, the author of the *Sectio Canonis* looks for three numbers i, j, k such that $p:q = i:j$ and $r:s = j:k$.[5] Then, interval $i:k$ serves as the *sum* of intervals $p:q$ and $r:s$.

The proof that such numbers i, j, and k exist is simple. If j is the least common multiple for q and r, then i and k are determined by $p:q = i:j$ and $r:s = j:k$. This is the idea that Euclid uses in the proof of VIII.4. However, instead of the least common multiple j, the product qr may serve as a middle proportional for $i = pr$ and $k = qs$. This is so because $p:q = pr:qr$ (VII.18) and $r:s = qr:qs$ (VII.17). Then, the interval $pr:qs$ serves as the sum of intervals $p:q$ and $r:s$[6]. Thus, in modern writing, $(p:q)(r:s) = pr:qs$, which proves Euclid's proposition VIII.5.

This method of addition (or a combination) of two intervals has arisen from the musical theory. Indeed, if a tone is produced at a musical instrument by a vibrating string of integer length i, the tone that produces a vibrating string of integer length j $(j < i)$ responds to the interval $p:q$, while the tone that produces a string of integer length k $(k < j)$ responds to the interval $r:s$, then the sum (or combination) of intervals $p:q$ $(= i:j)$ and $r:s$ $(= j:k)$ is an interval $i:k$ (Fig. 2).

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[4] They are also known by Latin name — *superparticular*. In SC.3, numbers in an epimoric interval are characterized as those that the difference between the grater and the less is such as to be a factor of the less.

[5] This problem Euclid solves by proposition VIII.4. However, his solution is about "continued proportions" with arbitrary many terms, not only about three terms proportions.

[6] In SC.7 Euclid finds that the sum of intervals $2:1 = 6:3$ and $3:2$ is the interval $6:2$, i.e., $3:1$.

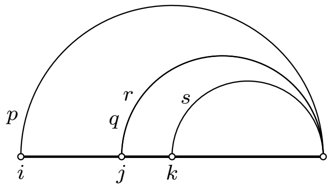

FIGURE 2. Interval $i : k$ is the sum of intervals $p : q$ and $r : s$

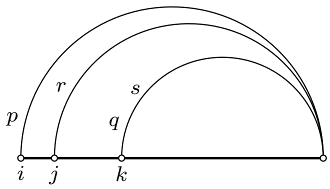

FIGURE 3. Interval $i : j$ is the difference of intervals $p : q$ and $r : s$

In order to subtract $r : s$ from $p : q$, the author of the *Sectio Canonis* looks for numbers i, j and k such that $p : q = i : k$ and $r : s = j : k$. Then, the interval $i : j$ is the difference of $p : q$ and $r : s$ (Fig. 3). If k is the least common multiple of q and s, then i and j are determined by the proportions $p : q = i : k$ and $r : s = j : k$.[7] But, as $p : q = ps : qs$ (VII.18) and $r : s = qr : qs$ (VII.17), the product qs can take the role of k. Therefore, for any two intervals $p : q$ and $r : s$ there exists their difference $ps : qr$. In modern writing, $(p : q) : (r : s) = ps : qr$.

4. Doubling and halving intervals

Since doubling of an interval is the addition of that interval to itself, to double $p : q$ the author of the *Sectio Canonis* is finding numbers i, j, and k such that $p : q = i : j = j : k$ to obtain an interval $i : k$ which is a result of *putting together twice* of the interval $p : q$.[8] Since $p : q = p^2 : pq = pq : q^2$ (VII.17–18), it could be taken that $i = p^2$, $j = pq$, and $k = q^2$, thus, the doubling of an interval $p : q$ results in the interval $p^2 : q^2$. In other words, $(p : q)^2 = p^2 : q^2$. But, the same relation holds if p and q are any two magnitudes, not only numbers. Moreover, the arguments that could be used in the proof of this assertion are almost the same as in the case of numbers, but with the difference that makes the usage of proposition V.15 instead of VII.17. This is what makes Euclid confident to claim in the definition V. def. 9 that, *when three magnitudes are proportional* $(i : j = j : k)$, *the first is said to have to the third* $(i : k)$ *the duplicate ratio of that which it has to the second* $(i : j)$. This also makes clear that the definition V. def. 9 has its roots in the Pythagorean musical theory.

[7] In SC.8 Euclid subtracts 4 : 3 from 3 : 2 to obtain 9 : 8.

[8] For instance, the result of doubling of a multiple interval $(n : 1)$ is a multiple interval $(n^2 : 1)$, as proved in SC.1.

Further, as $p : q = p^3 : p^2q = p^2q : pq^2 = pq^2 : q^3$ (VII.17–18), tripling of an interval $p : q$ results in an interval $p^3 : q^3$, its quadrupling in an interval $p^4 : q^4$ etc. Thus, $(p : q)^3 = p^3 : q^3, (p : q)^4 = p^4 : q^4$, etc. This is what Euclid claims by V. def. 10. Therefore, this definition has its roots in the Pythagorean music theory. Furthermore, by proving in Book VIII the propositions on continued proportions with arbitrary many terms, Euclid enabled the finding of an interval that is a result of arbitrary many times repeated addition of an interval to itself.[9]

The author of the *Sectio Canonis* is also solving the reverse problem that is about dividing a given interval into n equal intervals. When he attempts to halve an interval $p : q$, he looks for numbers i, j, and k such that $p : q = i : k$ and $i : j = j : k$, to find an interval $i : j$, which is a half of $p : q$.[10] Is it possible to halve an interval $p : q$ depends on the existence of a middle proportional of numbers p and q. If p and q are not similar plane numbers, then their middle proportional does not exist (VIII.20).[11] Since two similar plane numbers are one to another as a square number to a square number (VIII.26), numbers p and q must be proportional to two square numbers to exist their middle term. Conversely, if p and q are proportional to square numbers r^2 and s^2, from the existence of their middle proportional rs (VIII.11) follows that there exists a middle proportional of p and q (VIII.8). Thus, it is possible to halve an interval $p : q$ if and only if p and q are proportional to two square numbers. This is not proved explicitly in the *Sectio Canonis*, but the fact is a simple consequence of the propositions on continued proportions from Book VIII of the *Elements*. In the same vein, an interval $p : q$ can be divided into three equal intervals if and only if p and q are proportional to two cube numbers. Further, it can be divided into four equal intervals if and only if p and q are proportional to two fourth powers, etc.

Now it is clear why the theory on continued proportions from Book VIII had been so important in the early period. The reason was a need for the development of the calculus of intervals that was essential for the progress of Pythagorean music theory. In this theory proposition VIII.8 is unavoidable. Its immediate consequence is SC.3, which Boethius attributes to Archytas. Therefore, VIII.8 should also be attributed to him as the division of an interval into equal intervals is based on the use of this proposition.

However, the application of the theory of continued proportions had not been limited only to musical theory. Some problems that troubled mathematics of the fifth century had been solved as a result of the early development of the theory of continued proportions. By finding two middle proportionals for two magnitudes, Archytas solved the geometric problem of doubling the cube (Diels, 47.A14), whereas by using the arithmetic theory of continued proportions Theaetetus proved that the cube root of a given number is rational if and only if that number is a perfect cube.

[9] In the proof of SC.9 Euclid finds an interval that is a result of *putting together* six times the interval $9 : 8$.

[10] This is what he does in the proof of SC.3.

[11] By Boethius's claim, Archytas proved that it is not possible to halve superparticular (epimoric) interval (Diels, 47.A19). Euclid proves this in SC.3.

Of course, before that, he proved that the same holds for square roots and perfect squares.

5. Propositions SC.1–5

The propositions at the beginning of the *Sectio Canonis* are about addition and subtraction of multiples and epimoric intervals [1, p. 194–196]:

SC.1: *If a multiple interval* $(1 : n)$ *put together twice makes some interval, this interval too will be multiple* $(1 : k)$.

SC.2: *If an interval put together twice makes a whole that is multiple* $(1 : k)$, *then that interval will also be multiple* $(1 : n)$.

The proof of SC.1 begins with the assumptions that an interval $p : q$ is a multiple, i.e., that there is a number n such that $q = np$, and that r is a number such that $p : q = q : r$.[12] Then p divides q but, as $p : q = q : r$ (VII. def. 20), q divides r, thus, p divides r.[13] Hence, there is a number k such that $r = kp$. Therefore, $p : r$ is a multiple.

The proposition SC.2 is the converse of SC.1. In the proof Euclid assumes that $p : q$ is an interval, and that there is a number $r = kp$ such that $p : q = q : r$. Since p divides r, it also divides q (VIII.7), hence, there is a number n such that $q = np$, which proves the proposition.

In effect, in SC.1 is proved that, if $p : q$ is a multiple, then $p^2 : q^2$ is also a multiple. Indeed, if $p : q = 1 : n$, then there is a number k such that $p^2 : q^2 = 1 : k$. In other words, if $q = np$, then $q^2 = kp^2$. Therefore, if p divides q, then p^2 divides q^2. By SC.2, the converse also holds: if p^2 divides q^2, then p divides q. Thus, by SC.1 and SC.2 it is proved that p divides q if and only if p^2 divides q^2. This is what Euclid proves in the *Elements* as proposition VIII.14 [10]. SC.1 is equivalent to the simpler implication in this proposition. It could be proved by pebble arithmetic means, so it might originate from the early Pythagorean period. The major implication in VIII.14 is proved as SC.2. It is proved by the use of the same arguments as in VIII.14, including the reference to VIII.7. The only difference makes the language, as in the *Sectio Canonis* it is suited to the musical theory.

Judging by Boethius's comment, the next proposition of the *Sectio Canonis* — SC.3, is not Theaetetus's but Archytas's discovery (Diels, 47.A19).[14] By this proposition [1, p. 195]:

SC.3: *In the case of an epimoric interval, no mean number, neither one nor more than one, will fall within it proportionally.*

This is an immediate consequence of Euclid's proposition VIII.8. Indeed, if between two numbers, which are in epimoric ratio, there is one or more numbers in continued proportion, then there is an equal number of middle proportionals between the least numbers in the same epimoric ratio. However, the least numbers are two consecutive numbers n and $n+1$, but there is no number between them. The proof

[12] Obviously, such number exists as $p : np = np : n^2p$.

[13] Euclid nowhere proves the property of transitivity of divisibility.

[14] For detailed analysis of different proofs of SC.3 see [8, pp. 212–225]. See also [5, vol. I, pp. 215–216].

of SC.3 is a little more detailed because it contains an explanation why there is no number between the two least numbers in epimoric ratio. This is so because of understanding of the author of the *Sectio Canonis* that two numbers are in epimoric ratio "*when the difference between the greater and the less is such as to be a factor of the less*" (Theon, 76.21). [15] By the arguments of the author of the *Sectio Canonis*, as the unit is the greatest common measure of two least numbers in epimoric ratio, if suppose that there is a middle proportional between them, then this number divides the unit, which is impossible.

Archytas's proof, which is quoted by Boethius (Diels, 47.A19), is similar. It also relies on VIII.8, but his proof that the least numbers that are in epimoric ratio differs by the unit, is by contradiction. He assumes that n and $n+l$ ($l \neq 1$) are the least numbers in epimoric ratio and concludes that l divides both n and $n+l$.[16] Then n and $n+l$ are not relatively prime and, consequently, they are not the least numbers in that proportion.[17] Therefore, $l = 1$.

Propositions SC.4 and SC.5 are consequences of the first three propositions of the *Sectio Canonis*. In the first among them the author claims:

SC.4: *If an interval which is not multiple is put together twice, the whole will be neither multiple nor epimoric.*

In the proof he supposes that an interval $p:q$ is not a multiple and concludes by SC.2 that its double $p^2:q^2$ is not a multiple neither. It is not an epimoric interval neither because, by SC.3, for a double interval $p^2:q^2$ there is a middle term pq (VIII.11).

But, if $p^2:q^2$ is not a multiple, then, by SC.1, $p:q$ is not a multiple neither. This proves the following proposition:

SC.5: *If an interval put together twice does not make a whole that is multiple, that interval itself will not be multiple either.*

Similarly to the proposition SC.1, which is equivalent to one direction in VIII.14, and to SC.2, which is equivalent to the other direction in the same proposition, the proposition SC.5 is equivalent to one direction in VIII.16. That the reverse of SC.5 is also true is proved in SC.4. Thus, the conjunction of the propositions SC.4 and SC.5 is equivalent to VIII.16. However, this is not the whole truth, as in SC.4 is also proved that the double of an interval that is not a multiple, is an interval that is not epimoric. In the proof the author of the *Sectio Canonis* relies on Archytas's theorem SC.3. Thus, SC.4 is a hybrid, composed of the consequence of Theaetetus's theorem SC.2 (or VIII.14), and of the consequence of Archytas theorem SC.3 (or VIII.8) [2, p. 382].

6. Discordance and irrationality

The proposition SC.2 implies that, if interpreted in wording of musical theory, the statement that the square root of 2 is irrational number reads: *the double*

[15] See [7, pp. 458–459].

[16] In the proof, Archytas uses the notion of a *part*, defined by Euclid in VII. def. 3. See [7, p. 457–66].

[17] This is by Euclid's proposition VII.22 which, obviously, has been known to Archytas.

interval (the octave) cannot be divided into two equal multiples. The proposition SC.3 is a generalisation of this statement as it claims that any interval that is of the form $(n+1):n$ cannot be divided into two, or more than two equal intervals.

Although the division of the octave into two equal multiples is not possible, it is possible to divide it into two equal ratios. In order to prove this, suppose that B is the midpoint of the segment OA and that Z is the point of intersection of the segment AB and the circle centered at O, with the radius that is equal to the diagonal of the square with the side OB. If d is the length of the diagonal and a is the length of the side of the square, then $d^2 = 2a^2$ by Pythagoras's theorem.[18] Consequently, $2a : d = d : a$. Thus, the interval $2a : a = 2 : 1$ is divided by the point Z into two equal ratios $2a : d$ and $d : a$. The figure that illustrates the fact is simple and it was reachable by early Pythagoreans (Fig. 4). It shows that the doubling of the ratio $2a : d$ results in the octave $2 : 1$.

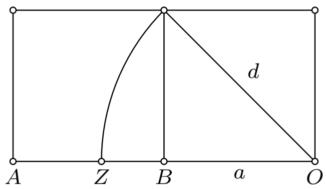

FIGURE 4. Halving the octave

After the discovery that some harmonic musical intervals like octave, quint and quart, can be explained by the ratios of numbers, but also that there are many intervals that produce discordant accords, it seems that the explanation of what is the cause of accordance or discordance of musical accords must had been in the focus of interest of early Pythagoreans and of Pythagoras himself. Otherwise, we can not explain Xenocrates's claim that it was Pythagoras who "*investigated under what conditions there result concordant or discordant intervals.*"

In a quest for concordant and discordant accords, the question about halving octave or, equivalently, about the ratio of the diagonal and the side of a square, seems unavoidable. But, by making experiments on a string instrument, early Pythagoreans could only conclude that this ratio results in a discordant accord.[19] In the time when every ratio was considered as an interval, the natural question arises — what interval is the ratio of the diagonal and the side of a square?

Early Pythagoreans solved the problem proving that there is no such interval and consequently that the square root of 2 is irrational. It seems that the problem was born "from the spirit of music" []Szabo, Borzacchini2007, but its solution was arithmetical. Indeed, from the fact that the square whose side is a diagonal of a

[18] In this form Pythagoras's theorem is not proved in the *Elements*. Judging by Hippocrates' solution to the problem of the quadrature of the lune, the pre-Eudoxian proof of the Pythagorean theorem could only have been grounded on the notions of length and area. See [11].

[19] They could possibly come to the same conclusion about the ratio of the diagonal and the side of a regular pentagon. See [15].

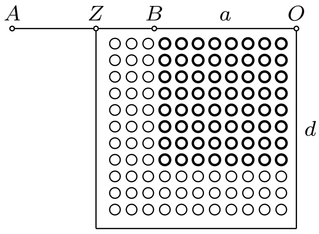

FIGURE 5. Halving the octave and doubling a square number

given square is two times larger than that square,[20] it follows that the ratio $2a : d$ is an interval if and only if $d : a$ is the ratio of two numbers. Obviously, this is possible only if there is a square number of an integer side a that is two times larger than a square number of an integer side d (Fig. 5). The use of pebble representation of numbers makes evident that such two numbers do not exist [10]. Consequently, the half of the double interval is not an interval. Thus, the half of the octave is not an interval.

7. Propositions SC.6–9

The propositions SC.6–9 are about concrete epimoric intervals: hemiolic (3 : 2), epitritic (4 : 3) and epogdoic (9 : 8), and about multiples 2 : 1 and 3 : 1. That the result of the addition of 4 : 3 to 3 : 2 is 2 : 1, is proved by SC.6. By SC.7 is proved that the addition of 3 : 2 to 2 : 1 results in 3 : 1, while by the SC.8 that the subtraction of 4 : 3 from 3 : 2 results in the interval 9 : 8. In the first among these propositions the author of the *Sectio Canonis* claims:

SC.6: *The double interval is composed of the two greatest epimoric intervals, the hemiolic and the epitritic.*

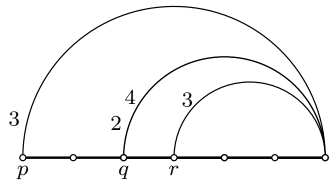

FIGURE 6. SC.6

In the proof, he assumes that the ratio of the lengths of line segments p and q is hemiolic, i.e., that $p : q = 3 : 2$, and concludes that $p - q$ is the third of p, and the half of q. Similarly, if the ratio of the lengths of q and r is epitritic, i.e., if $q : r = 4 : 3$, then $q - r$ is a quarter of q, and the third of r. Since $q - r$ is a quarter of q, and $p - q$ is its half, $q - r$ is a half of $p - q$. Thus, $p - q$ is a third of p, while

[20] This fact has been known even to Meno's slave boy.

$q - r$ is the sixth of p, so r is a half of p. Therefore, the addition of hemiolic $(3:2)$ and the epitritic interval $(4:3)$ results in a double interval $(2:1)$.

The author uses similar arguments in the proof of the next proposition:

SC.7: *From the double interval and the hemiolic, a triple interval is generated.*

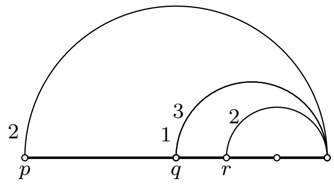

FIGURE 7. SC.7

He assumes that $p : q = 2 : 1$, i.e., that $p = 2q$, and that $q : r = 3 : 2$ i.e., that $2q = 3r$, and concludes that $p = 3r$, i.e., that $p : r = 3 : 1$. Thus, the sum of the double and the hemiolic interval is the triple interval.

By the next proposition the author of the *Sectio Canonis* proves that:

SC.8: *If from a hemiolic interval an epitritic interval is subtracted, the remainder left is epogdoic.*

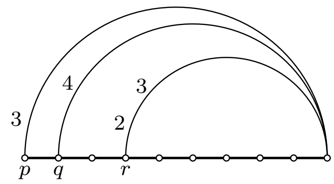

FIGURE 8. SC.8

In the proof, he assumes that the ratio of two line segments of lengths p and r is hemiolic, i.e., that $p : r = 3 : 2$, and concludes that $8p = 12r$, i.e., that $p : r = 12 : 8$. In the same way, if $q : r$ is the epitritic interval, i.e., if $q : r = 4 : 3$, then $9q = 12r$, i.e., $q : r = 12 : 9$. Therefore, $8p = 9q$, i.e., $p : q = 9 : 8$, thus, the ratio of line segments of lengths p and q is epogdoic.

By SC.9, according to which:

SC.9: *Six epogdoic intervals are greater than one double interval,*

the author proves, in effect, that $(9:8)^6 > 2$. For that purpose he looks for seven middle proportionals in the ratio $8 : 9$. Using VIII.2 he finds that

$$8^6 : 9 \cdot 8^5 = 9 \cdot 8^5 : 9^2 \cdot 8^4 = 9^2 \cdot 8^4 : 9^3 \cdot 8^3 = 9^3 \cdot 8^3 : 9^4 \cdot 8^2 = 9^4 \cdot 8^2 : 9^5 \cdot 8 = 9^5 \cdot 8 : 9^6.$$

Therefore, $(8:9)^6 = 8^6 : 9^6$. But, $8^6 = 262.144$ and $9^6 = 531.441$,[21] therefore, $2 \cdot 8^6 = 2 \cdot 262.144 = 524.288 < 531.441 = 9^6$ and, consequently, $9^6 : 8^6 > 2 : 1$.

[21] The other middle proportionals are $9 \cdot 8^5 = 294.912$, $9^2 \cdot 8^4 = 331.776$, $9^3 \cdot 8^3 = 373.248$, $9^4 \cdot 8^2 = 419.904$ and $9^5 \cdot 8 = 472.372$.

But, six intervals $9:8$, i.e., $(9:8)^6$, is the interval $9^6:8^6$, so $(9:8)^6 > 2:1$. Thus, six epogdoic intervals are greater than the double interval.

8. Calculus of intervals

The first five propositions of the *Sectio Canonis* are general theorems on intervals. They are about multiples, and epimoric intervals. The propositions that follow are not of a general character since they are related to concrete intervals $2:1$, $3:1$, $3:2$, $4:3$ and $9:8$. They are independent from the propositions that precede them and they do not demand the knowledge of sophisticated propositions from the theory of continued proportions such as VIII.7 and VIII.8. Their proofs are obvious as they could be illustrated by figures or by a demonstration on a string instrument. However, the proposition SC.9 stands out from them as it relies on VIII.2, which is about continued proportions with arbitrary many terms.

Thus, the propositions SC.6–8 are simpler for proving and certainly older than the propositions that precede them. This conclusion is confirmed by a preserved fragment from Philolaus that refers to the assertions proved in SC.6 and SC.8. By Philolaus's words (Diels, 44 B 6.):

> *The content of the Harmony (Octave) is the major fourth and the major fifth; the fifth is greater than the fourth by a whole tone; for from the highest string (lowest note) to the middle is a fourth, and from the middle to the lowest string (highest note) is a fifth. From the lowest to the third string is a fourth, from the third to the highest string is a fifth. Between the middle and third strings is a tone. The major fourth has the ratio 3:4, the fifth 2:3, and the octave 1:2. Thus the Harmony (Octave) consists of five whole tones and two semitones, the fifth consists of three tones and a semitone, and the fourth consists of two tones and a semitone.*

However, Philolaus was rather a musical theorist than an arithmetician. His conclusions are more empirical than theoretical. In contrast to the author of the *Sectio Canonis*, who makes the difference between a musical interval and the arithmetical notion of interval, Philolaus applies notions of the musical theory as if they are arithmetical. In this manner, instead of addition of hemiolic and epitritic interval in order to obtain the double interval like in the proof of SC.6, Philolaus adds the fourth to the fifth, which results in the octave. Moreover, he emphasises that the addition is commutative, letting us know that in the middle of the fifth century Pythagoreans already had developed the calculus of intervals.

By Philolaus's findings, if the longest string which is of length p produces the lowest tone, while the (middle) string which is of the length s produces the middle tone which is its fourth, and if the shortest string which is of the length r produces the highest tone which is the fifth of a tone that produces the string of the length s, then the tone that produces the shortest string is the octave of the tone that produces the longest string. Therefore, the sum of the fourth and the fifth is the octave. In other words, if $p:s = 4:3$ and $s:r = 3:2$, then $p:r = 2:1$ (Fig. 9). However, if the string of the length q produces a tone which is the fifth of tone

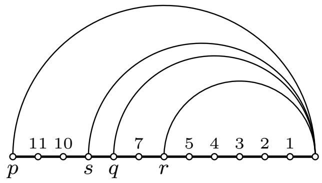

FIGURE 9. The sum of $p : s$ and $s : r$ equals the sum of $p : q$ and $q : r$

that produces a string of a length p, then the tone which produces the string of the length r is the fourth of the tone which produces the string of the length q. In other words, if $p : q = 3 : 2$ and $q : r = 4 : 3$, then $p : r = 2 : 1$,[22] so the sum of of a fifth and the fourth is again the octave. Thus, the addition of intervals is commutative.

9. Tones and semitones

Claiming that *the fifth is greater than the fourth by a whole tone*, Philolaus stating that the whole tone is produced by an interval $9 : 8$. A simple calculation shows that the subtraction of two intervals $9 : 8$ from $4 : 3$ results in an interval $256 : 243$, while the subtraction of three intervals $9 : 8$ from $3 : 2$, also results in $256 : 243$. Thus, an interval $2 : 1$ can be divided into five intervals $9 : 8$ and two intervals $256 : 243$. This is in accordance with Philolaus's claim that the octave consists of five whole tones and two semitones.

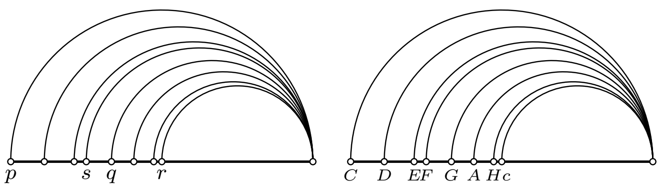

FIGURE 10. Scala

Philolaus does not explain what is the pattern of the whole tones and semitones in his scale. However, in the discussion about the world's soul in the *Timaeus* (36 b), Plato fills the gap describing the following pattern:[23]

$$9 : 8,\ \ 9 : 8,\ \ 256 : 243,\ \ 9 : 8,\ \ 9 : 8,\ \ 9 : 8,\ \ 256 : 243.$$

Thus, an interval of the fourth $(p : s)$ consists of two intervals of $9 : 8$, followed by an interval of $256 : 243$, while an interval of the fifth $(s : r)$, of three intervals of $9 : 8$ followed by an interval of $256 : 243$ (Fig. 10). This is in line with Philolaus's structure of a scale that he calls the *Harmony*. If C is a tone produced by a

[22] This is proved by SC.6.

[23] Plato explains that the interval of $4 : 3$ consists of two intervals of $9 : 8$ and the reminder $256 : 243$, but he is not giving any further information about the motives for this examination. This is discussed by [6, pp. 149–150].

vibrating string of the length p, then tones produced by strings in a pattern of tones described by Plato, are C, D, E, F, G, A, H, c.

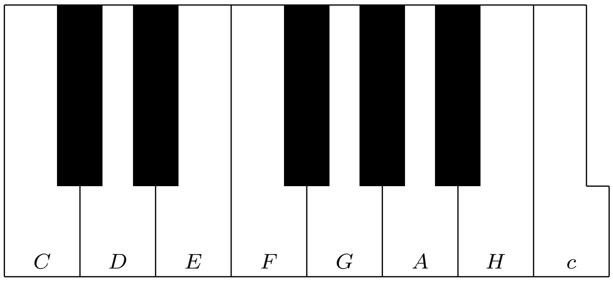

FIGURE 11. Keyboard

The interpolation of semitones between C and D, then between D and E, and also between F and G, G and A, A and H (black keys on a keyboard), results in a scale that consists of twelve semitones. However, this scale has one bad property. The sum of two semitones is not equal to a whole tone as $(256/243)^2 = 1.110 < 1.125 = 9/8$.[24]

Philolaus is not giving any explanation of this bad property. He just concludes that the octave consists of five whole tones and two semitones. However, by SC.9, the sum of six whole tones is greater than the octave, which proves the fact. Judging by the proof which relies on VIII.2, this proposition could not have been proved before the time of Philolaus's disciple Archytas.[25]

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