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SMOOTH NUMBERS IN MUSIC AND ARCHITECTURE

Abstract. Besides their natural setting within mathematics, *smooth numbers* also appear in art, particularly in music and architecture.

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1. Introduction

"...thou hast ordered all things in measure and number and weight."

— Solomon

A natural number $n \in \mathbb{N}$ is *B-smooth* if none of its prime factors is greater than B. For example, 72 has prime factorization $2^3 3^2$ and is hence 3-smooth. Study of smooth number has long history. In particular, 5-smooth numbers, also called *regular numbers*, have a prominent role in Babylonian mathematics. In modern mathematics, smooth numbers feature prominently in cryptography and various fast Fourier transform algorithms such as Cooley-Tukey FFT algorithm.

Smooth numbers also appear in art, most notably, in music. In this note, we cover their connection to musical scales and examine how these scales, in a different guise, likewise emerge in architecture.

2. Pythagorean and Ptolemy's intense diatonic scales

"Music is the pleasure the human mind experiences from counting without being aware that it is counting."

— Leibniz

Pythagoras and his school have laid the foundations not only of European mathematics but of music theory too. In particular, their experimentation with the monochord led to the construction of a diatonic major scale generated by the progression of *perfect fifths*. A perfect fifth is a musical interval (i.e., difference in pitch between two sounds) corresponding to a pair of pitches with a frequency ratio of $\frac{3}{2}$.

Using the progression of perfect fifths we can obtain the following seven frequencies:

$$1 \to \frac{3}{2} \to \frac{9}{4} \to \frac{27}{8} \to \frac{81}{16} \to \frac{243}{32} \to \frac{729}{64}.$$

FIGURE 1. Woodcut from Theorica musicae by Franchino Gaffurio's *Theorica musicae* (1492) showing Pythagoras and his successor Philolaus doing musical experiments.

Choosing the corresponding frequencies within the base octave (i.e., dividing with the appropriate power of 2) and assuming, without loss of generality, that the base pitch corresponds to be C, we obtain the C diatonic major scale. Pythagorean

C	D	E	F	G	A	B	C
1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2

TABLE 1. Pythagorean scale.

scale is an example of *just tuning*, i.e., tuning in which the frequencies of notes are rational numbers. A B-limit tuning is a tuning that uses rational numbers whose numerators and denominators are B-smooth. Pythagorean tuning is, therefore, an example of 3-limit tuning. While exceptionally elegant in its generation, Pythagorean scale has several drawbacks. For example, the third, sixth, and seventh (notes of the scale) are difficult to tune. To overcame this, Ptolemy proposed a variation, called Ptolemy's intense diatonic scale, by lowering the third, sixth, and seventh (E, A, and B in our example) by $\frac{81}{80}$ (called syntonic comma). Ptolemy's

C	D	E	F	G	A	B	C
1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2

TABLE 2. Ptolemy's intense diatonic scale.

scale is an example of 5-limit tuning. Notice that a lot of the ratios appearing in it take the form $\frac{n+1}{n}$. Such ratios are called *superparticular ratios*. Moreover, the superparticular ratios in Ptolemy's scale are such that both the numerator and denominator are 5-smooth. Interestingly, only finitely many such pairs $(n, n+1)$ exist, a consequence of the famous Størmer's theorem [4].

Theorem 2.1 (Størmer's theorem)**.** *Let $P = \{p_1, \ldots, p_k\}$ be a set of distinct primes and let $N_P = \{p_1^{m_1}, \ldots, p_k^{m_k} \mid m_1, \ldots, m_k \in \mathbb{N}_0\}$ be a set of all P-smooth numbers. Then, N_P contains finitely consecutive pairs, i.e., the set*

$$\{n \in \mathbb{N} \mid n, n+1 \in N_P\}$$

is finite.

Moreover, Størmer proof is constructive and allows to obtain all pairs of consecutive B-smooth numbers. In particular, 5-smooth pairs $(n, n+1)$ with $n < 100$ are:

$$(1,2),(2,3),(3,4),(4,5),(5,6),(8,9),(9,10),(14,15),$$
$$(15,16),(20,21),(24,25),(27,28),(35,36),(63,64),(80,81).$$

Represented as superparticular rations, many of them musically meaningful.

2/1	the octave
3/2	the perfect fifth
4/3	the perfect fourth
5/4	the major third
6/5	the minor third
9/8	the whole step
10/9	the minor tone
16/15	the minor second
25/24	the minor semitone
81/80	the syntonic comma

TABLE 3. Some superparticular ratios and their musical interpretation.

3. Floor plans in Palladian architecture

"Music is liquid architecture;
architecture is frozen music."

— Goethe

Andrea Palladio, one of the foremost architects of the Italian Renaissance, left a large body of work both practical — large part of it protected as part of the UNESCO World Heritage Site, 'Palladian Villas of the Veneto', and theoretical - "The Four Books on Architecture" [3] being his most well-known treatise. The zeitgeist of his time emphasized rationalism and revival and reinterpretation of Greco-Roman antiquity, which naturally led to aesthetic objectivism and the quest to explain the world in therms of ideal proportions. This is evident in the architecture of the time and in Palladio's work in particular. Especially interesting are Palladio's thoughts regarding floor plans. In the aforementioned treatise, he lists seven ideal shapes for floor plans:

- circular;
- rectangular with a length l and width w that satisfy

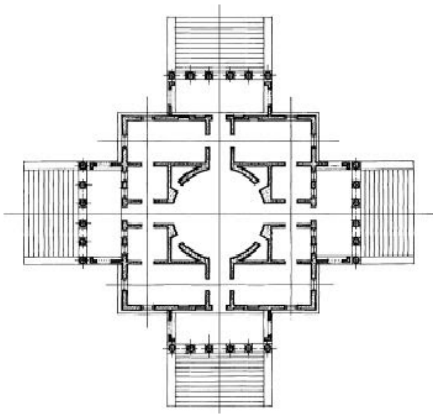
$$\frac{l}{w} \in \left\{\frac{1}{1}, \frac{4}{3}, \frac{\sqrt{2}}{1}, \frac{3}{2}, \frac{5}{3}, \frac{2}{1}\right\}$$

The preference for circular and square plans can be easily explained due to their pleasing symmetry. The choice of non-square rectangular plans, however, requires elaboration. Rudolf Wittkower was the first to propose that the ratios of Palladio's rectangular plans can be explained via musical scales ([5]; see [1],[2] for a more detailed study of this question).

Indeed, if we compare the ratios suggested by Palladio with those of Ptolemy's intense diatonic scale, we see that, with the exception of $\frac{\sqrt{2}}{1}$, they correspond to unison (1) C, subdominant $\left(\frac{4}{3}\right)$ F, dominant $\left(\frac{3}{2}\right)$ G, $\left(\frac{5}{3}\right)$ A, and octave (2) C. The connection with Ptolemy's scale is not unexpected as it was widely used during Renaissance. The famous musical theorist and composer Gioseffo Zarlino, Palladio's contemporary, even declared it the only reasonable scale for singing.

To interpret the remaining ratio $\left(\frac{\sqrt{2}}{1}\right)$, it is necessary to move from Ptolomy's scale to the *equal tempered scale*. By the end of the Baroque period, equal temperament tuning had become dominant in European music, overcoming the practical difficulties in tuning instruments to accommodate key changes. The beginning of this shift can be traced back to the Renaissance. For example, one of the early proponents of equal temperament was musical theorist and composers Vincenzo Galilei (father of Galileo Galilei). It equal temperament, the octave is divided into 12 equal semitones, each pitch therefore having frequency that is a power of $\sqrt[12]{2}$. In particular, $\sqrt{2} = \sqrt[12]{2}^6$ which means that the ratio $\frac{\sqrt{2}}{1}$ corresponds to a *tritone* (augmented fourth i.e. diminished fifth). It is interesting to note that while other ratios suggested by Palladio correspond to consonant intervals, the tritone is not only dissonant but possibly the most dissonant (it was historically referred

(a) Villa La Rotonda.



(b) Floor plan of Villa Rotonda.

Figure 2: Andrea Palladio's famous Villa Rotonda.

as *diabolus in musica* and there were even attempts to officially ban its use in composition).

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