Dragan Latinčić

SPECTRUM PROJECTIONS IN MUSICAL-MATHEMATICAL ANALYSIS

Abstract. The text describes the method of rhythmic projections of the spectrum of harmonics, as well as the application of one of the most important isometric transformations to the projected metro-rhythmic entities of individual harmonics of the spectrum. It is a direct isometry called central rotation. The central rotation conditions the hemiol structuring of the meter. Hemioles are identified with regular and irregular geometric figures (primarily triangles) through the partition and composition (index) number of a specific spectral harmonic. The partition and composition of numbers dealt with by discrete mathematics, on the one hand, and the technique of horizontal hemioles, characteristic of the polyphony of the sub-Saharan region, on the other hand, served as a means of realizing the method by which the isometric transformation of the central rotation would be realized in (musical) time.

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1. Introduction

My compositional creativity is accompanied by knowledge from the natural sciences, primarily mathematics and physics. The desire to establish a transparent methodology in the system of my musical language and style arose during the preparation of the doctoral artistic project *Batal - preludes for string orchestra* with a theoretical study of *The application of microtonality* (*a*) *in instrumental Middle Eastern and Balkan music of folklore provenance, and* (*b*) *in instrumental, chamber and orchestral music of the acoustic type in contemporary Western art music* (*an attempt to establish an autonomous creative-composer concept*).

Many years of research in the field of microintervals in the music theory and practice of Middle Eastern and North African folkloric provenance, and the approximation of the language of mathematics to elementary musical means (rhythm, meter, modes, aliquot series) has resulted in several published works on this topic: the aforementioned theoretical study, and, two books: (1) *Microintervals in spectral geometry*, and, (2) *Spectral trigonometry* (*Establishing the Universal Musical-Mathematical Analysis*). The topic of all the mentioned papers is the basis of musical-mathematical analysis. This analysis would be based on the assumption of the justification of the connection between mathematics, specifically - geometry on the one hand, and music, on the other.

All the mentioned studies are based on the method of projection of individual harmonics of the spectrum. It is about an abstract representation of a projector that would create superpositioned harmonics (partial segments) of an arbitrary fundamental tone in the metro-rhythmic plane.

Projecting intervals or chords is a procedure in which the same elementary structure is retained, but its function is replaced. The vertical position of two different tonal pitches (harmonic interval) or the vertical position of three or more different pitches (harmonic chord) is "laid" by the projection process into the horizontal position of two different tonal lengths (projected interval) or the horizontal position of three or more different lengths (projected chord). By replacing the function, we mean that one harmonic relationship (which we call the height measured by two points in space) can be moved to a new position by means of the projector - a position measured by points in time that makes up the projected length, and from which the speed of propagation can be observed, defined and measured. The harmonic interval, therefore, now forms a unique relationship: original height - projected length. At the next step, a causal relationship of three basic functions could be established here: height-length-speed, - a relationship that, from the perspective of the harmonic vertical, has already been realized between any harmonics of the aliquot sequence. We say this also for the reason that a single metric set isolated in a given musical time should be subjected to the process of projection "uprighting" into a vertical position, whereby the cause-and-effect relationship of the mentioned three functions would be preserved. With the newly obtained projected intervals and chords, a new order in the system of proportions is established by replacing the function.

The super-individual relationships of higher harmonics to lower harmonics can practically be explained on the example of the three sides of the geometric figure of a triangle. We are talking about a chordal triangle that would consist of three independently isolated frequencies whose origin would be in an arbitrary fundamental tone of the spectrum. These frequencies would constitute the intersections of metrical lengths, which can be proved by means of the projector with mathematical formulations of metro-rhythmic distances.

The analytical approach of viewing the spectral series (through combinatorics and geometry) with all its characteristics of projections seems new and original to me, and in this connection I would like to mention that I have not come across such an approach (and similar ones) in the current professional literature. In the rest of the text, I tried to cover all the relevant areas based on the research work on the spectrum projection method with special reference to the isometric transformations of the projected geometric entities, in order to see a clear and transparent methodological framework for a wide compositional application.

2. Combinatorics of interval and chord funds of the spectrum

Interval and chord funds of the spectrum consist of numerical schemes according to which harmonics are superposed, and their projections are realized by real metric platforms on which the frequency of occurrence of distributed projected rhythmic units (change of the same rhythmic units) as well as their change (change of different rhythmic units) could be monitored. Each harmonic starting from the second index number, and in relation to its neighboring harmonics, contains sets of intervals and chords. Interval and chord funds can be classified according to each harmonic individually. Harmonics are, therefore, elements of an n-membered set[1], and, as such, they may or may not necessarily be repeated. In order to make the classification of interval and chordal funds as transparent as possible, we will use general equations from combinatorics in order to identify the types of combinations of sets with and without repetition of one of the elements.

$$C_n^k = \binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{2.1}$$

A mathematical formulaequation was used to review numerical *combinations without repetition* within the reference system[2] of a particular harmonic. The letter n denotes the harmonic, and the letter k denotes the number of members from the set. The value of k depends on the choice of the number of participating elements. On the examples of multi-member sets, we will notice the difference in the number of *combinations without repetition* and in the number of *combinations with repetitions* for interval and chord funds of the fourth harmonic of the spectrum.

If we apply eq. (2.1) for the two-membered set of elements of the fourth harmonic of the spectrum, we would get six (6) interval combinations without repetition:

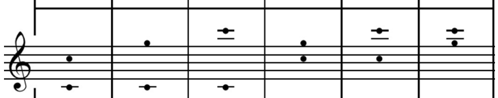
Formulation 2.1. Two-member set of elements of the fourth harmonic of the spectrum (combinations without repetition)

$$\binom{4}{2} = \frac{4!}{2! \cdot (4-2)!} = \frac{24}{4} = 6$$

The combinations are: 1 2; 1 3; 1 4; 2 3; 2 4; 3 4.

Table 1. Combinations without repetition in the two-member set of elements of the fourth harmonic of the spectrum.

1	2	3	4	5	6
2	3	4	3	4	4
1	1	1	2	2	3



However, in a two-member set of elements of the fourth harmonic of the spectrum, it is possible to form ten (10) interval combinations with repetition.

$$\bar{C}_n^k = \binom{n+k-1}{k} \tag{2.2}$$

[1] Two-membered sets are called intervals, while three-membered or multi-membered sets are called chordal sets.
[2] The reference system of the n-th harmonic of the spectrum is a set of intervals and chords formed by that harmonic in conjunction with its lower harmonics.

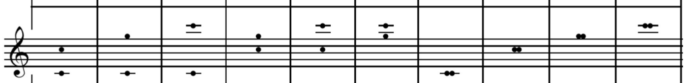
According to formulaeq. (2.2), which is used to review numerical combinations with repetition, it follows that:

Formulation 2.2. Two-member set of elements of the fourth harmonic of the spectrum (combinations with repetition)

$$\binom{5}{2} = \frac{5!}{2! \cdot (5-2)!} = \frac{120}{12} = 10$$

The combinations are: 1 1; 1 2; 1 3; 1 4; 2 2; 2 3; 2 4; 3 3; 3 4; 4 4.

TABLE 2. Combinations with repetition in a two-member set of elements of the fourth harmonic of the spectrum.

1	2	3	4	5	6	7	8	9	10
									
2	3	4	3	4	4	1	2	3	4
1	1	1	2	2	3	1	2	3	4

We notice that the final number of combinations of the two-membered set of elements of the fourth harmonic of the spectrum is different for both types of combinations.

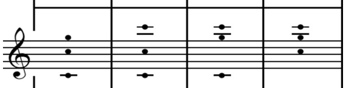
If we apply the first equation again, but now for a three-member set of elements of the fourth harmonic of the spectrum, we would get four chord combinations without repetition:

Formulation 2.3. Three-membered set of elements of the fourth harmonic of the spectrum (combinations without repetition)

$$\binom{4}{3} = \frac{4!}{3! \cdot (4-3)!} = \frac{24}{6} = 4$$

The combinations are: 1 2 3; 1 2 4; 1 3 4; 2 3 4.

TABLE 3. Combinations without repetition in the three-member set of elements of the fourth harmonic of the spectrum.

1	2	3	4
			
3	4	4	4
2	2	3	3
1	1	1	2

In the three-member set of elements of the fourth harmonic of the spectrum, the combination with repetition has 20.

Formulation 2.4. Three-member set of elements of the fourth harmonic of the spectrum (combinations with repetition)

$$\left(\frac{6!}{3! \cdot (6-3)!}\right) = \frac{720}{36} = 20$$

Table 4. Combinations with repetition in the three-member set of elements of the fourth harmonic of the spectrum (1-10).

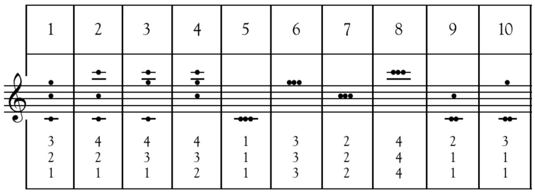
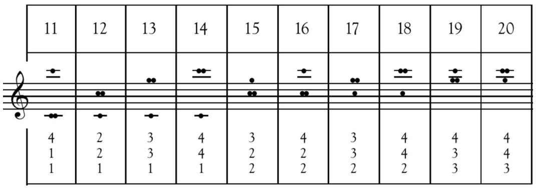


Table 5. Combinations with repetition in the three-member set of elements of the fourth harmonic of the spectrum (11-20).



The combinations are: 1 1 1; 1 1 2; 1 1 3; 1 1 4; 1 2 2; 1 2 3; 1 2 4; 1 3 3; 1 3 4; 1 4 4; 2 2 2; 2 2 3; 2 2 4; 2 3 3; 2 3 4; 2 4 4; 3 3 3; 3 3 4; 3 4 4; 4 4 4.

According to this procedure, in a four-member set of elements of the fourth harmonic of the spectrum, we would get one (1) chord combination without repetition:

Formulation 2.5. Four-member set of elements of the fourth harmonic of the spectrum (combinations without repetition)

$$\binom{4}{4} = \frac{4!}{4! \cdot (4-4)!} = \frac{25}{24} = 1$$

The combination is: 1 2 3 4.

In the four-member set of elements of the fourth harmonic, the combination with repetition has 35.

Formulation 2.6. Four-member set of elements of the fourth harmonic of the spectrum (combinations with repetition)

$$\binom{7}{4} = \frac{7!}{4! \cdot (7-4)!} = \frac{5040}{144} = 35$$

Going further, in the five-membered set of elements of the fourth harmonic of the spectrum, combinations without repetition do not exist because one member of the set is missing. The combination with repetitions has 56. We notice that the number of combinations without repetition decreases as we approach the number that defines the given reference system (4), while the number of combinations with repetition increases. Accordingly, the number of members of the set will depend on the index number of harmonics. If the final number of the set is greater than the index number of harmonics, combinations without repetition are not possible. Unlike the combinations without repetition, in the six-member set of elements of the fourth harmonic of the spectrum, the combination with repetition has eighty-four (84), and in the seven-member set - one hundred and twenty (120). The application of general equationss from combinatorics is valid for every reference system of interval and chord funds of any harmonic spectrum individually.

Unlike combinations, a permutation consists of all the elements of a set, including the order among them. The combination, on the other hand, consists of the elements of a set, among which not necessarily all of them must be included,[3] nor is the order according to which the selected elements are sorted important. The difference between permutation and combination will be made here only according to the importance of the order of how the elements in the set are classified, i.e. how the harmonics are juxtaposed. Considering that it is a question of multi-membered sets created by multiplying (often) the same harmonic, the mentioned permutations and combinations will always be observed with the repetition of the elements of the set. The order of the alphabetic letters follows the order of the harmonics of the spectrum ($a = 1$, $b = 2$, $c = 3$, and so on).

The number of combinations in the n-th member set of the second harmonic of the spectrum is developed by a series of ordinal numbers: (2), 3, 4, 5, 6, 7, 8, etc. The number of permutations in the n-th member set of the second harmonic of the spectrum is developed by a series of ordinal numbers consisting of the exponents of the ordinal number of harmonics in question (in this case the number 2), so: 2^1, 2^2, 2^3, 2^4, 2^5, 2^6, $\ldots$ that is: 2, 4, 8, 16, 32, 64 etc.

Thus, in the two-member set of the second harmonic of the spectrum, it is possible to realize three (3) combinations with repetition and four (the square of the number two) permutations, also with repetition. The combinations would be: a^2, b^2 and ab (these combinations can also be read as: aa, bb and ab). The permutations would be: a^2, b^2, ab and ba.[4]

Going further, in the three-membered set of the second harmonic of the spectrum, it is possible to realize four (4) combinations with repetition and eight (the cube of two) permutations with repetition. The combinations would be: a^3, b^3, a^2b and ab^2 (these combinations can also be read as: aaa, bbb, aab and abb). The permutations would be: a^3, b^3, a^2b, ba^2, aba, ab^2, b^2a, bab.

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[3] In the aforementioned combinations (with and without repetition) of the elements of the set of the fourth harmonic of the spectrum, all elements are still included. This principle will also apply in the following text.

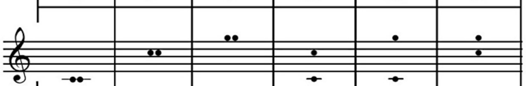
[4] A superscript next to an alphabetic letter is not read as an exponent, but as a multiplication of that letter, which is used to mark an individual harmonic from the set.

The number of combinations in the n-th member set of the third harmonic of the spectrum is developed by a series of triangular numbers: 3, 6, 10, 15, 21, 28, 36, etc. The number of permutations in the n-th membered set of the third harmonic of the spectrum is developed by a series of ordinal numbers, which are still exponents of the ordinal number of harmonics in question (in this case the number 3), so: 3^1, 3^2, 3^3, 3^4, 3^5, 3^6, ie: 3, 9, 27, 81, 243, 729, etc.

Thus, in the two-member set of the third harmonic of the spectrum, it is possible to develop six (6) combinations with repetition and nine (the square of the number 3) permutations, also with repetition. The combinations would be: a^2, b^2, c^2, ab, ac and bc. The permutations would be: a^2, b^2, c^2, ab, ba, ac, ca, bc, and cb.

Table 6. Combinations with repetition in a two-member set of elements of the third harmonic of the spectrum.

1	2	3	4	5	6
a^2	b^2	c^2	ab	ac	bc
1 1	2 2	3 3	2 1	3 1	3 2



In the three-member set of the third harmonic of the spectrum, it is possible to realize ten (10) combinations with repetition and twenty-seven (cube of the number 3) permutations with repetition. The combinations would be: a^3, b^3, c^3, a^2b, ab^2, a^2c, ac^2, b^2c, bc^2 and abc. The permutations would be: a^3, b^3, c^3, a^2b, aba, ba^2, ab^2, bab, b^2a, a^2c, aca, ca^2, b^2c, bcb, cb^2, bc^2, cbc, c^2b, ac^2, cac, c^2a, abc, acb, bac, bca, cab, and cba.

As we can notice, the development of combinations or permutations for the reference system of each individual harmonic is uniform, followed by two clearly ordered number progressions.

Table 7. Combinations with repetition in the three-member set of elements of the third harmonic of the spectrum.

1	2	3	4	5	6	7	8	9	10
a^3	b^3	c^3	a^2b	ab^2	a^2c	ac^2	b^2c	bc^2	abc
1 1 1	2 2 2	3 3 3	2 1 1	2 2 1	3 1 1	3 3 1	3 2 2	3 3 2	3 2 1

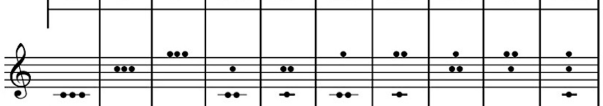


TABLE 8. Permutations with repetition in the three-membered set of elements of the third harmonic of the spectrum.

1	2	3	4	5	6	7	8	9
b a c	c a b	c b a	b c c	c b c	c c b	b b c	b c b	c b b
2 1 3	3 1 2	3 2 1	2 3 3	3 2 3	3 3 2	2 2 3	2 3 2	3 2 2

10	11	12	13	14	15	16	17	18
a a a	b b b	c c c	a a b	a b a	b a a	a a c	a c a	c a a
1 1 1	2 2 2	3 3 3	1 1 2	1 2 1	2 1 1	1 1 3	1 3 1	3 1 1

19	20	21	22	23	24	25	26	27
a b b	b b a	b a b	a c c	c c a	c a c	a b c	a c b	b c a
1 2 2	2 2 1	2 1 2	1 3 3	3 3 1	3 1 3	1 2 3	1 3 2	2 3 1

The number of combinations in the n-th member set of the fourth harmonic of the spectrum is developed by a series of pyramidal numbers with three dimensions. The Pythagoreans studied spatial figurative numbers and the law of gnomon formation. Tetrahedral (or pyramidal) numbers are: 1, 4, 10, 20, 35, 56, 84, $\dots$, $[n(n+1)(n+2)]/6, \dots$. In general, the n-th pyramidal number with three dimensions is: $[n(n+1)(n+2)]/3!$.

Professor Zoran Lučić also writes about the sequence of gnomons that can be represented by numerical sequences, referring to Theon: "Theon simply shows the sequence of gnomons of individual polygon numbers as numerous sequences. Thus, gnomons of triangular numbers are arranged in a series of natural numbers, square in a series of odd numbers, pentagonal in a series of 1, 4, 7, 10, 13, 16, 19, $\dots$ hexagonal in a series of 1, 5, 9, 13, 17, 21, 25, $\dots$ etc. Therefore, the gnomons of a p-togonal number whose sides are n in length (i.e. they consist of n units) are, respectively, the numbers 1, $1+(p-2)$, $1+2(p-2)$, $\dots$, $1+(n-1)(p-2)$. They are members of arithmetic progressions whose first members are always the number 1, and the differences are, respectively, 1, 2, 3, $\dots$. Iamblichus will use the numbers of units in the gnomons of a p-togonal number in order to calculate the number of units from which that p-togonal number consists" [5, p. 11].

TABLE 9. Development of the number of combinations with repetition for each reference system of a single harmonic of the spectrum.

	2−члан	3−члан	4−члан	5−члан	6−члан	7−члан	8−члан	9−члан	10−члан	11−члан	12−члан	13−члан	14−члан	15−члан	16−члан
1^0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2^0	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
3^0	6	10	15	21	28	36	45	55	66	78	91	105	120	136	153
4^0	10	20	35	56	84	120	165	220	286	364	455	560	680	816	969
5^0	15	35	70	126	210	330	495	715	1001	1365	1820	2380	3060	3876	4845
6^0	21	56	126	252	462	792	1287	2002	3003	4368	6188	8568	11628	15504	20349
7^0	28	84	210	462	924	1716	3003	5005	8008	12376	18564	27132	38760	54264	74613
8^0	36	120	330	792	1716	3432	6435	11440	19448	31824	50388	77520	116280	170544	245157
9^0	45	165	495	1287	3003	6435	12870	24310	43758	75582	125970	203490	319770	490314	735471
10^0	55	220	715	2002	5005	11440	24310	48620	92378	167960	293930	497420	817190	1307504	2042975
11^0	66	286	1001	3003	8008	19448	43758	92378	184756	352716	646646	1144066	1961256	3268760	5311735
12^0	78	364	1365	4368	12376	31824	75582	167960	352716	705432	1352078	2496144	4457400	7726160	13037895
13^0	91	455	1820	6188	18564	50388	125970	293930	646646	1352078	2704156	5200300	9657700	17383860	30421755
14^0	105	560	2380	8568	27132	77520	203490	497420	1144066	2496144	5200300	10400600	20058300	37442160	67863915
15^0	120	680	3060	11628	38760	116280	319770	817190	1961256	4457400	9657700	20058300	40116600	77558760	145422675
16^0	136	816	3876	15504	54264	170544	490314	1307504	3268760	7726160	17383860	37442160	77558760	155117520	300540195

TABLE 10. Development of the number of permutations with repetition for each reference system of a single harmonic of the spectrum.

	2−члан	3−члан	4−члан	5−члан	6−члан	7−члан	8−члан	9−члан	10−члан	11−члан	12−члан	13−члан	14−члан	15−члан	16−члан
1^0	1^2	1^3	1^4	1^5	1^6	1^7	1^8	1^9	1^{10}	1^{11}	1^{12}	1^{13}	1^{14}	1^{15}	1^{16}
2^0	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}	2^{15}	2^{16}
3^0	3^2	3^3	3^4	3^5	3^6	3^7	3^8	3^9	3^{10}	3^{11}	3^{12}	3^{13}	3^{14}	3^{15}	3^{16}
4^0	4^2	4^3	4^4	4^5	4^6	4^7	4^8	4^9	4^{10}	4^{11}	4^{12}	4^{13}	4^{14}	4^{15}	4^{16}
5^0	5^2	5^3	5^4	5^5	5^6	5^7	5^8	5^9	5^{10}	5^{11}	5^{12}	5^{13}	5^{14}	5^{15}	5^{16}
6^0	6^2	6^3	6^4	6^5	6^6	6^7	6^8	6^9	6^{10}	6^{11}	6^{12}	6^{13}	6^{14}	6^{15}	6^{16}
7^0	7^2	7^3	7^4	7^5	7^6	7^7	7^8	7^9	7^{10}	7^{11}	7^{12}	7^{13}	7^{14}	7^{15}	7^{16}
8^0	8^2	8^3	8^4	8^5	8^6	8^7	8^8	8^9	8^{10}	8^{11}	8^{12}	8^{13}	8^{14}	8^{15}	8^{16}
9^0	9^2	9^3	9^4	9^5	9^6	9^7	9^8	9^9	9^{10}	9^{11}	9^{12}	9^{13}	9^{14}	9^{15}	9^{16}
10^0	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9	10^{10}	10^{11}	10^{12}	10^{13}	10^{14}	10^{15}	10^{16}
11^0	11^2	11^3	11^4	11^5	11^6	11^7	11^8	11^9	11^{10}	11^{11}	11^{12}	11^{13}	11^{14}	11^{15}	11^{16}
12^0	12^2	12^3	12^4	12^5	12^6	12^7	12^8	12^9	12^{10}	12^{11}	12^{12}	12^{13}	12^{14}	12^{15}	12^{16}
13^0	13^2	13^3	13^4	13^5	13^6	13^7	13^8	13^9	13^{10}	13^{11}	13^{12}	13^{13}	13^{14}	13^{15}	13^{16}
14^0	14^2	14^3	14^4	14^5	14^6	14^7	14^8	14^9	14^{10}	14^{11}	14^{12}	14^{13}	14^{14}	14^{15}	14^{16}
15^0	15^2	15^3	15^4	15^5	15^6	15^7	15^8	15^9	15^{10}	15^{11}	15^{12}	15^{13}	15^{14}	15^{15}	15^{16}
16^0	16^2	16^3	16^4	16^5	16^6	16^7	16^8	16^9	16^{10}	16^{11}	16^{12}	16^{13}	16^{14}	16^{15}	16^{16}

The number of permutations in the n-th member set of the fourth harmonic of the spectrum is developed by a series of ordinal numbers which, again, form the exponent of the ordinal number of harmonics in question (in this case the number 4), so: 4^1, 4^2, 4^3, 4^4, 4^5, 4^6, ... i.e. 4, 16, 64, 256, 1024, 4096 etc.

The number of combinations in the n-th member set of the fifth harmonic of the spectrum is developed by a series of pentahedroid or hyper-pyramidal numbers with four dimensions[5]. It would be the following sequence of numbers: 1, 5, 15, 35, 70, 126, 210, 330, etc. The number of permutations in the n-th set of the fifth harmonic of the spectrum is developed by a series of ordinal numbers that also form the exponent of the ordinal number of harmonics in question (in this case the number 5), so: 5^1, 5^2, 5^3, 5^4, 5^5, 5^6, ... i.e. 5, 25, 125, 625, 3125, 15625 etc.

The number of combinations in the n-th member set of the sixth harmonic of the spectrum is developed by a series of hyper-pyramidal numbers with five dimensions. It would be the following sequence of numbers: 1, 6, 21, 56, 126, 252, 462, 792, etc. The number of permutations in the projected polyphony of the sixth harmonic of the spectrum is developed by a series of ordinal numbers that also form the exponent of the ordinal number of harmonics in question (in this case the number 6), so: 6^1, 6^2, 6^3, 6^4, 6^5, 6^6, ... i.e. 6, 36, 216, 1296, 7776, 46656 etc.

3. Classification of metrical and rhythmic projections of interval and chord funds of the spectrum. Rhythmic projections

There are two basic types of projections of spectral intervals and chords. The first type is rhythmic projection, while the second type is metric projection. Both types contain three subtypes:

(1) horizontal projection,
(2) vertical projection, and
(3) combined (horizontal-vertical) projection.

Vertical projections would serve to distribute each harmonic separately from the given set of harmonics being projected. Horizontal projections would serve the frequency (and periodicity) of each individual harmonic. The combined projection conditions both the distributiveness and the frequency of occurrence of given harmonics from the set being projected. According to this procedure, metric platforms do not necessarily have to be of equal length for both types of projections as well as for their subtypes.

Vertical rhythmic projection would be realized by distributing the reference rhythmic unit specific to a given harmonic from the set into its equal parts. For vertical projection, the order according to which the harmonics are classified as elements of the set is not important, so the permutation form of the set is not important. Before the vertical projection procedure, it is important to define the metric platform on which the projected rhythmic units will be distributed in time. The value of the projected rhythmic unit to be distributed in the vertical projection is reciprocal to the ordinal number of harmonics subjected to the projection.

[5] The n-th pyramid number with four dimensions is: $[n(n+1)(n+2)(n+3)]/4!$ that is, in general, the n-th pyramidal number with k dimensions is: $[n(n+1)(n+2)(n+k-1)]/k!$.

For example, for the index number of the fourth harmonic of the spectrum (4), the value of its projected rhythmic unit will be one quarter (1/4). If we think of a rhythmic quarter as a given metrical platform, and represent it with the number one (1), the projected set of harmonics (5:4:3) will be simultaneously distributed in three ways: (1) the reference rhythmic unit for the fifth harmonic would be distributed five times ($5 \cdot 1/5 = 1$); (2) the reference rhythmic unit for the fourth harmonic would be distributed four times ($4 \cdot 1/4 = 1$); and (3) the reference rhythmic unit for the third harmonic would be distributed three times ($3 \cdot 1/3 = 1$). The choice of metric length determines the expansion or contraction of the reference rhythmic unit for the projected harmonic. If we imagine the rhythmic half as a given metric platform, and represent it with the number two (2), the projected set of harmonics (5:4:3) will be distributed again in three ways: (1) the reference rhythmic unit (conditioned by the metric extent) for the fifth harmonic distributed repeat five times ($5 \cdot 2/5 = 2$); (2) the reference rhythmic unit for the fourth harmonic would be distributed four times ($4 \cdot 2/4 = 2$); and (3) the reference rhythmic unit for the third harmonic would be distributed three times ($3 \cdot 2/3 = 2$).

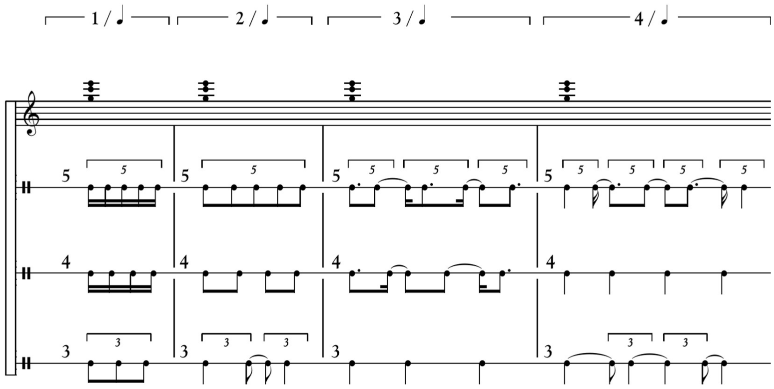


FIGURE 1. Vertical rhythmic projection of spectral ratio (5:4:3) with metric platform extension.

The horizontal rhythmic projection would be realized by projecting the complete set of harmonics (which now form sums of the form) according to the sum of the members of the form ($5 + 4 + 3$) and then distributing them. The value of the projected rhythmic unit that makes up the summation of the projected set is directly proportional to the ordinal number of harmonics with the rhythmic unit being selected (it is not necessarily reciprocal to the ordinal number of harmonics). Given that in the spectral chord (5:4:3) all three elements of the set are different, no rhythmic member will be of the same duration during horizontal projection.

For horizontal projection (as opposed to vertical projection), the order according to which the harmonics are sorted as elements of the set is important, so the permutational form of the set is important. If we imagine a rhythmic fourth as a given metric platform, and represent it with the number one (1), the set of harmonics (5:4:3) will be projected as follows: $[a = 5 \cdot 1/5 = 5/5 = 1]$; $[b = 4 \cdot 1/5 = 4/5]$; $[c = 3 \cdot 1/5 = 3/5]$. The horizontally projected form would read as follows: $[(5 + 4 + 3)/x] \cdot x$.

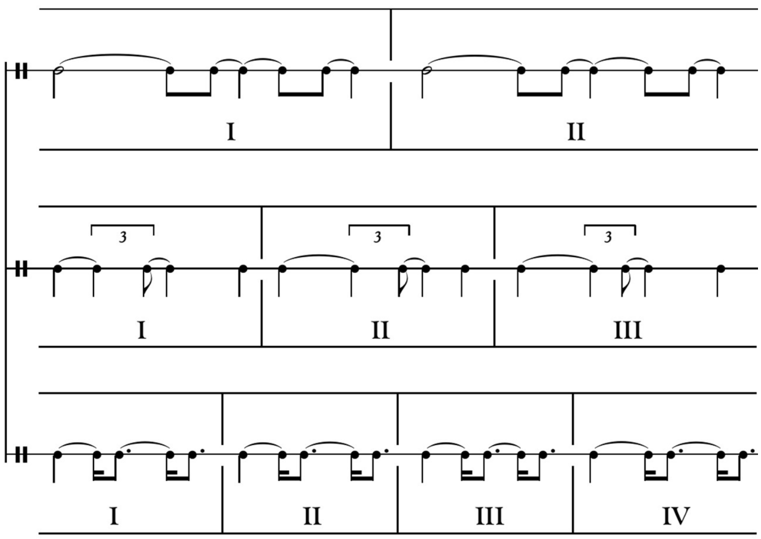


FIGURE 2. Horizontal rhythmic projection of the chord (5:4:3) towards half (♩/2) of the metric base (first line); horizontal rhythmic projection of the chord (5:4:3) according to the third (♩/3) of the metric base (second line); horizontal rhythmic projection of the chord (5:4:3) according to the quarter (♩/4) of the metric base (third row).

The distribution of horizontally projected harmonics from the set would depend on the choice of rhythmic unit. In the mathematical scheme, the choice of rhythmic unit is described by the denominator in the fraction. In horizontal projection, the length of the metric platform is the sum of all harmonics from the set. Considering that the sum of the three harmonics that are projected is twelve (12) and considering that, in the third line of example number two, a rhythmic unit consisting of a quarter (1/4) of the basic metrical unit (1) was selected for projection, projected the suit will therefore be distributed four times: $4 \cdot 12/4 = 48/4 = 12$.

It is obvious that the metric length depends on the sum of harmonics from the given set. The choice of rhythmic unit determines the expansion or contraction of the projected rhythmic form, while the length of the metrical platform does not change. The sum of the three harmonics (5:4:3) that are projected is twelve (12), and with the selection of a new rhythmic unit for projection that would be a third (1/3) of the basic metrical unit (1), the projected form will, now, distribute three times: $3 \cdot 12/3 = 36/3 = 12$ see Figure 2 - second row). By choosing a rhythmic unit that would be half (1/2) of the basic metrical unit (1), the projected form would be distributed twice: $2 \cdot 12/2 = 24/2 = 12$ (see Figure 2 - first line). A similar phase periodicity is also observed if the metric base is divided into smaller partial segments: fifths, sixths, sevenths and so on (see example number 3).

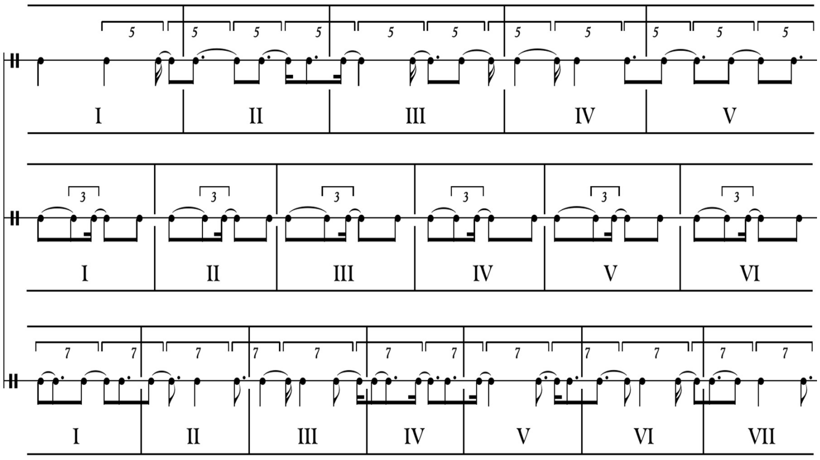


FIGURE 3. Horizontal rhythmic projection of the chord (5:4:3) according to the fifth (♩/5) of the metric base (first line); horizontal rhythmic projection of the chord (5:4:3) according to the sixth (♩/6) of the metric base (second line); horizontal rhythmic chord projection (5:4:3) according to the seventh (♩/7) of the metric base (third row).

Vertical metric projection would satisfy the condition that the choice of rhythmic unit is equal to the basic metric unit ($1 = 1$). However, the application of one or another type of projection has a different purpose.

It is possible to apply natural number partition and composition to rhythmic and metric projections. A partition of a natural number n is a representation of n in the form of a sum of several natural numbers, where the order of additions is irrelevant and that number itself represents one partition.

The partitions of the number 2 are: (2) and $(1 + 1)$; of the number 3 are: (3), $(2+1)$, $(1+1+1)$; of the number 4 are: (4), $(3+1)$, $(2+2)$, $(2+1+1)$, $(1+1+1+1)$. The number of partitions of a natural number is not always a prime number. For example, the number seven (7) contains fifteen (15) partitions, and the number fifteen (15) is divisible by both three (3) and five (5). Euler's theorem states that the number of ways in which a given natural number can be written as a sum of mutually different natural numbers $R(n)$ is equal to the number of ways in which the same number can be written as a sum of odd numbers $N(n)$.

For example, for the number five (5) it follows that: $5 = 4 + 1 = 3 + 2$, and on the other hand: $5 = 3 + 1 + 1 = 1 + 1 + 1 + 1 + 1$. It is observed that the equality $R(5) = N(5) = 3$ holds.

The partition is graphically represented by the Ferrer diagram:

The Figure 4 represents the partition of the number twelve (12) as $(5+4+3)$, where the number of dots in the vertical columns corresponds to one addition. The same example can be read in horizontal columns: $(3+3+3+2+1)$. Two partitions connected in this way are said to be conjugate. On the other hand, there are also such partitions in which the order of additions is the same if they are read along both horizontal and vertical columns.

Partitions connected in this way are said to be self-conjugate.

	5	4	3
3	•	•	•
3	•	•	•
3	•	•	•
2	•	•	
1	•		

FIGURE 4. Conjugate partitions of the number twelve (12)

In the rest of the text, the self-conjugate and conjugate-connected partitions in different reference systems of the individual harmonics of the spectrum are listed. In Tables 11 and 12 we see diagrams of self-conjugate partitions from the first to the fifth harmonic of the spectrum. Reference systems of higher harmonics, starting from the second index number, can contain more self-conjugate partitions, which is described in Table 12.

Euler's theorem is also applicable to any reference system starting from the second harmonic index number of the spectrum.

The composition of a natural number is an ordered partition or an ordered decomposition of the natural number n. The order of the additions is important in the composition of a natural number, unlike the partition.

The total number of compositions of the number n is equal to $c(n) = 2n - 1$. If we apply this theorem to the reference system of the fifth harmonic of the spectrum, we will get sixteen compositions for this harmonic: $c(5) = 25 - 1 = 24 = 16$. They would be in order:

5 (one composition to one addition);

$1+4$; $2+3$; $3+2$; $4+1$ (four compositions on two compilations);

$1+1+3$; $1+2+2$; $1+3+1$; $2+1+2$; $2+2+1$; $3+1+1$ (six compositions on three collections);

$1+1+1+2$; $1+1+2+1$; $1+2+1+1$; $2+1+1+1$ (four compositions on four collections);

$1+1+1+1+1$ (one composition of five additions).

TABLE 11. Self-conjugate partitions of low harmonics of the spectrum (with mathematical formulation of the number of compositions).

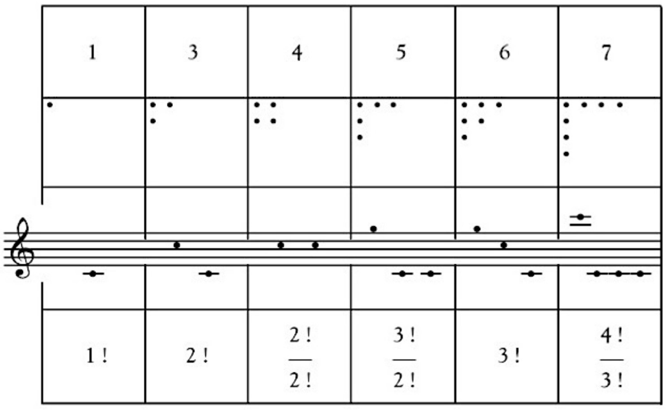
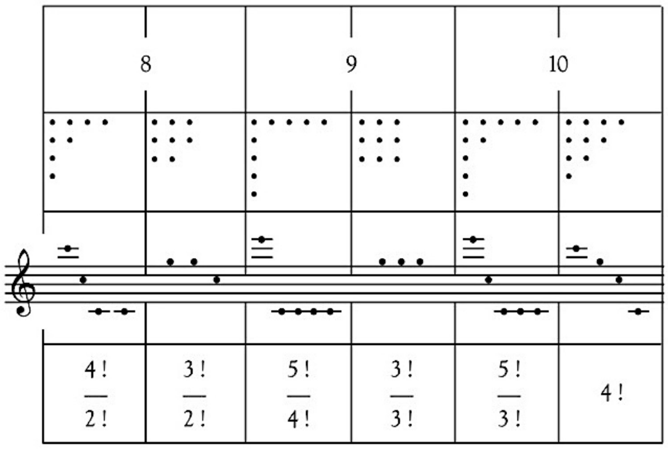


TABLE 12. Self-conjugate partitions of low harmonics of the spectrum (with mathematical formulation of the number of compositions).



4. Projections of number sequences

In the horizontal rhythmic projection, rhythmic structures that form sequences of ordinal, triangular, pyramidal numbers (i.e. structures that form sequences of spatial figurative numbers) or structures that form additive sequences in two or more steps (Fibonacci or Tribonacci sequence and similar) are particularly important in the horizontal rhythmic projection. On the (epitrite) platform, which would count seven (7) metric units, it is possible to synchronize two sequences starting from the first rhythmic member up to the seventh rhythmic member, with the fact that one sequence will be shorter than the other by one rhythmic member (it will have six rhythmic members). The rhythmic unit that is chosen for projection is a quarter (1/4) of the metric unit for the first voice (a rhythmic sixteenth if our meter is a whole rhythmic quarter) and a third (1/3) of a metric unit for the second voice (a triple eighth if our meter is a whole rhythmic quarter). The projected units are reciprocal to the index numbers of the third and fourth harmonics of the spectrum. It is the interval of a fourth (4:3). I state this because the epitrite rate, consisting of seven metric units in the writings of Nikola Mesarites (Mesarites) on the work of students in a higher Byzantine school, is closely related to the interval of a fourth (diatessaron): "Bearing in mind that folk music of the Balkans is, like cante jondo, a direct continuation of the Ancient Greeks' modes and genera through its Byzantine (and pre-Byzantine) heritage, one can hardly understand what led Wellesz, a most outstanding authority on Byzantine music but also - a fact which is perhaps of some interest - a Viennese pupil of Schoenberg, to dismiss as 'almost entirely nonsensical' the discussion, reported by Byzantine scholar Nicolas Mesarites, on the relationship between the interval diatessaron (3:4) and diapente (2:3) and respectively the meters epitritos $(3+4)$ and hemiolos $(2+3)$, the feet common in the Ancient Greeks' music and characteristic not only of the folk music of Andalusia or of the Basque region but also of the Balkans. On the other hand, ideas about the tonal systems of the ancient Greek and Byzantine music that are similar to those of De Falla are to be found in a brilliant analysis by Iannis Xenakis Vers une métamusique (La Nef, No. 29, Paris, 1967). Xenakis criticizes Wellesz regarding several times" [6, p. 37–38].

The product of the metric platform (7) with the choice of reference rhythmic unit for the fourth harmonic is twenty-eight (28), and this number is the sum of the number seven: $(1+2+3+4+5+6+7=28)$. The product of the metric platform (7) with the choice of reference rhythmic unit for the third harmonic is twenty-one (21), and this number is the sum of the number six $(1+2+3+4+5+6=21)$. It is obvious that the sums of the rhythmic terms (28 and 21) divided by the index number of the fourth and third harmonics of the spectrum (4 and 3) result in the number seven (7) which constitutes the ancient epitritic foot. Look at Figure 5 and compare with Table 13 (third row).

If we apply the projection of the series of harmonics to other metric platforms, we will notice an analogy, first of all, in terms of odd lengths. On a (peon) platform that would count five (5) metric units, it is possible to synchronize two sequences starting from the first rhythmic member up to the fifth member, with the fact that one sequence will be shorter by one rhythmic member compared to the other (will have four rhythmic members). The rhythmic unit that is chosen for projection is a third (1/3) of the metric unit for the first voice (a triple eighth if our measure is a whole rhythmic quarter) and a half (1/2) of a metric unit for the second voice (a rhythmic eighth if our measure is a whole rhythmic quarter). The projected units are reciprocal to the index numbers of the second and third harmonics of the spectrum. It is an interval of a fifth (3:2). In the aforementioned Mesarite writing, it is also mentioned that Byzantine students connected the interval of a fifth (diapente) with the peonic foot: "7. And there [you will see] the people who are concerned

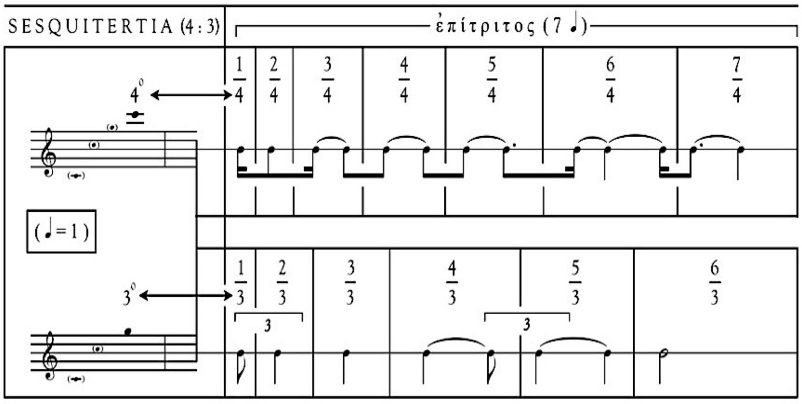


FIGURE 5. Projection of a sequence of ordinal numbers in the sesquitertia proportion on a metric platform of seven rhythmic quarters.

with geometrical lines and surfaces and three-dimensional bodies and plane and solid figures, triangles, I mean, and tetragons and hexahedrons and octahedrons, dodecahedrons and eicosahedrons and with pyramidal figures and the circular forms which comprise semicircles and circles and with the figure of the halo which is visible in the cloudes. 8. Near these [you will see] people who are concerned with tones and harmony, since this branch of learning took its beginnings from arithmetic; though it did not take these beginnings to itself immediately, but the mediator between it and the fundamentals of arithmetic, and the transmitter, was geometry; and this again [geometry], I think, costituted the most suitable intermediary of the subject with the highest of the sciences, to wit mathematics. 9. You can hear them indeed, disputing with each other, with words strange to most people, and never heard, talking to one another of *nêtê* and *hypatê* and *parhypatê* instead of strings, and of *mesê* and *paramesê*, and of how the interval which they call the *diatessarôn* is correspondingly called the *epitritos* by the mathematicians, while that which is called *diapente* seems to them [the musicians] to be the *hêmiolios*, corresponding to the *diapente* of the mathematicians; and of why the octave is called *diapasôn* and of how the first mode in it is found to be the principal, and of why the fifteenth string is called *disdiapasôn*, and why the whole instrument is called fifteen-stringed when it has sixteen strings" [2, 895–896].

The product of the metric platform (5) with the choice of reference rhythmic unit for the third harmonic is fifteen (15), and this number is the sum of the number five: $(1 + 2 + 3 + 4 + 5 = 15)$. The product of the metric platform (5) with the choice of reference rhythmic unit for the second harmonic is ten (10), and this number is the sum of the number four $(1 + 2 + 3 + 4 = 10)$. Note that the numerators of fractions are represented by triangular numbers. Dragutin Gostuški is right when he claims that the ratios of adjacent harmonics are synchronized during the projection, and that their sum (in the case of spectral fifths and fourths) constitutes peon and

Table 13. Projections of sequences of ordinal numbers according to adjacent pairs of spectrum harmonics.

2 : 1	$3 =$	$\left[\sum_{n=1}^{3} n\right]/2 = \left[\sum_{n=1}^{2} n\right]/1 =$	$6/2 = 3/1$
3 : 2	$5 =$	$\left[\sum_{n=1}^{5} n\right]/3 = \left[\sum_{n=1}^{4} n\right]/2 =$	$15/3 = 10/2$
4 : 3	$7 =$	$\left[\sum_{n=1}^{7} n\right]/4 = \left[\sum_{n=1}^{6} n\right]/3 =$	$28/4 = 21/3$
5 : 4	$9 =$	$\left[\sum_{n=1}^{9} n\right]/5 = \left[\sum_{n=1}^{8} n\right]/4 =$	$45/5 = 36/4$
6 : 5	$11 =$	$\left[\sum_{n=1}^{11} n\right]/6 = \left[\sum_{n=1}^{10} n\right]/5 =$	$66/6 = 55/5$
7 : 6	$13 =$	$\left[\sum_{n=1}^{13} n\right]/7 = \left[\sum_{n=1}^{12} n\right]/6 =$	$91/7 = 78/6$

epitrite rates: “However, it is clear that Byzantine students identified the interval of the fourth $(3 : 4)$ with the epitritic rhythm $(3 + 4)$, and the fifth $(2 : 3)$ with the hemiolic $(2 + 3)$. This had to be shown to them both by the geometric images of the relationship of the wires, and by the arithmetic of Euclid’s canon, which solves similar problems. The most interesting thing in this matter is that in the former territory of Byzantium, even today, hemiolic and epitritic rhythms are most consistently applied in music” [3, 205]. (Gostuški, 1968: 205).

So, here too, it becomes obvious that the sums of the rhythmic terms (15 and 10) divided by the index number of the third and second harmonics of the spectrum (3 and 2) result in the number five (5) which makes up the hemiolic or ancient peonic rate. Look at example 6 and compare with Table 13 (second row).

This principle is valid for all pairs of harmonics that are adjacent to each other (thus forming an even and an odd member), provided that the projections are made with a series of ordinal numbers. This is just a hint of a purposeful musical arithmetic (in the form of *metro-rhythmic* or *horizontal harmony*) which is naturally incorporated into mathematical formulations. Therefore, the whole story about the projection of adjacent harmonics of the spectrum can also be extended to non-adjacent harmonics whose gnomon[6] would be greater than one, which would certainly condition the dimensioning of the metric platform.

The goal of such dimensioning of the metrical number, that is, the rhythmic fraction, would be justified here through the analytical observation of metrorhythmic-structural synchronicity, primarily in polyphony.

[6] Gnomons were known in the Pythagorean teaching of figurative numbers. By the term gnomon we mean “first differences” in the relationship between two numbers.

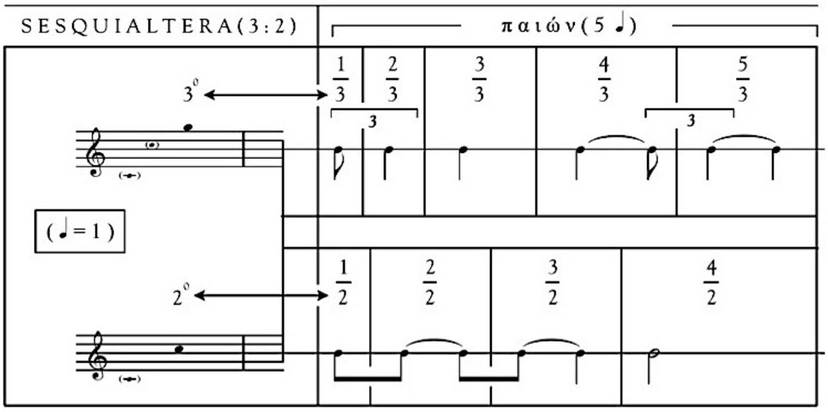


FIGURE 6. Projection of a sequence of ordinal numbers in the sesquialtera proportion on a metrical platform of five rhythmic quarters

5. Application of projection in the compositional process

Dragutin Gostuški succinctly explained the polymetric relationships in the book *Time of Art - Contribution of the Foundation of a General Science of Form* by the method of projection of vertical ratios of partial tones of the spectrum: "According to our principles, no reduction by addition should be made when comparing, but a pure horizontal projection of vertical ratios that, for example, appear in the series of the above tones: one time dimension should be converted into another. In this way, from the fifth ratio 3:2 (*proportio sesquialtera*) or fourth ratio 4:3 (*proportio sesquitertia*) we arrive at the succession $3 + 2$ or $4 + 3$. Therefore, in addition to the union of duole and triole, as a legitimate representative of vertical relationships in the rhythmic system, the so-called asymmetrical, 'axak' rhythm, of prehistoric, Asian origin, known in ancient music theory under the name of peonic and epitritic feet, and still alive today on the same, Macedonian ground" [3, p. 204].

I admit that it took me years to rationalize the postulate to the limit point where the results in the domain of the compositional process would begin to appear. On the way to new horizons of knowledge about rhythmic phenomena, I found myself faced with numerous trials. And only when I accepted the properties of elementary mathematical operations by incorporating them into the method of projection of partial harmonics of the spectrum - not only did Gostuski's postulate develop into a valuable arithmetic game during composition work, but it found its place where it originated - in deep history, in 12^{th} century, in Byzantium. I am deeply convinced that his reflections, based on and built upon the views of Lipps, Lalo and Helmholtz, which concern the methods of projecting spectrum harmonics, are extremely new and original. Therefore, for this occasion, I have selected a short section from my composition for piano *Inflections*, dedicated to my fellow composer and pianist Stanko Simić, who commissioned the composition, and with which I will spontaneously cover both the analytical and compositional-practical aspects of the phenomenon of rhythmic projection.

As we can see from examples number (7), (8) and (9) - rhythmic structuring, based on spectrum projection methods, I define according to three basic groups, on: (1) regular (hemiolic) groupings within which the value of the rhythmic unit is constant variable; to (2) regular (hemiolic) groupings within which the value of the rhythmic unit is constantly invariable; and, to (3) irregular (hemiolic) groupings within which the value of the rhythmic unit is variably variable.

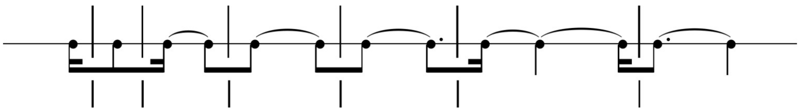


FIGURE 7. Regular (hemiolic) grouping within which the value of the rhythmic unit is constantly variable; harmonic projection: 1-2-3-4-5-6-7; (meter: 7/♩)

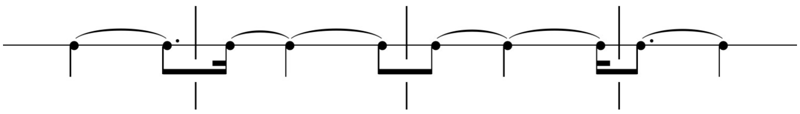


FIGURE 8. Regular (hemiolic) grouping within which the value of the rhythmic unit is constantly invariable; harmonic projection: 7-7-7-7; (meter: 7/♩)

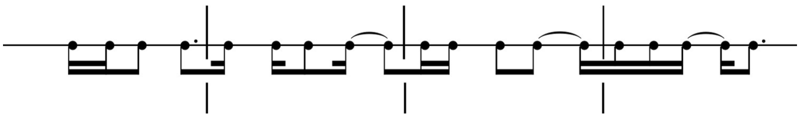


FIGURE 9. Irregular (hemiolic) grouping within which the value of the rhythmic unit is variably variable; harmonic projection: 1-1-2-3; (meter: 7/♩)

In the section from the composition *Inflections*[7], which we examine in example ten (10), I will describe the technique of crossing different projected rhythmic formations. Crossing was achieved through several layers (in the form of voices).

[7] Three inflections for piano were written in the spring of 2019. They were created as an order from a dear and respected colleague, composer and pianist Stanko Simić. The title of the composition was chosen for two reasons: (1) in the writings of the Spanish composer Manuel de Falla, it is mentioned as a stylistic reference to the cante jondo; (2) in mathematical analysis, the term inflection refers to an odd function which, with the change of x to -x, does not change its absolute values, but changes its sign (at the point of inflection, the convex part of the function changes to concave, or vice versa, and the inflection tangent touches and intersects the graph functions).
In the analysis of the musical elements of cante jondo, de Falla states: “We should highlight three facts from Spanish history that left very different consequences on our overall cultural life and left

Primo rivolto section. Poco più mosso ♩ = 126 makes fourteen bars, that is, considering the two-quarter time, it makes twenty-eight rhythmic quarters. Thus, the section is properly divided into four sentences of equal length, and each sentence is seven rhythmic quarters long ($28/4 = 7$). In each sentence, the projection procedure was applied with the intersection of projected rhythmic groupings, through three main layers. The rhythmic unit chosen for the projection is a third (1/3) metric unit for the first, highest layer (which would be a triple eighth if our measure is a whole rhythmic quarter) and a quarter (1/4) metric unit for the second and third layer, that is, the middle and lowest layer (which would be a rhythmic sixteenth if our measure is a whole rhythmic quarter). The projected units are reciprocal to the index numbers of the third and fourth harmonics of the spectrum. Projection formulations are written below the musical text in the examples that follow. The first layer is described in the highest voice by the pitch cis^4, on the first metrical length of seven rhythmic quarters; the same layer is described in the highest voice by pitch e^4, on the second metrical length of seven rhythmic quarters; then it is described by the pitch h^3, on the third metrical length of seven rhythmic quarters; to finally be described by the pitch fis^3, on the fourth metrical length of seven rhythmic quarters. This layer is structured as a regular (hemiolic) grouping within which the value of the rhythmic unit is constantly variable.

In the first sentence the structure is: $6/3+5/3+4/3+3/3+2/3+1/3 = 21/3 = 7$♩ (follow the pitch cis^4 in note example 11). In the second sentence, the structure is rhythmically inverted: $1/3 + 2/3 + 3/3 + 4/3 + 5/3 + 6/3 = 21/3 = 7$♩, with the fact that within the last projected rhythmic member (6/3) inserted substructure, in the function of decoration, and in the form of a sub-sequence: $1/3 + 2/3 + 3/3$ (follow the pitch e^4 in note example 11). In the third sentence, the structure is the same as in the second: $1/3 + 2/3 + 3/3 + 4/3 + 5/3 + 6/3 = 21/3 = 7$♩ (follow the pitch h^3 in note example 11) . In the fourth sentence, the structure is the same as in the previous two: $1/3 + 2/3 + 3/3 + 4/3 + 5/3 + 6/3 = 21/3 = 7$♩ (follow the pitch fis^3 in note example 11).

The next lower layer is in the adjacent lower (middle) voice, and is more difficult to detect as the pitch moves from the upper system to the lower system. This layer is described by the pitch cis^3, on the first metrical length of seven rhythmic quarters; the same layer is described in the middle voice by the pitch e^3, on the second metrical length of seven rhythmic quarters; then it is described by pitch h^2, on the third metrical length of seven rhythmic quarters; to finally be described by

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a more than clear mark in the history of music. And they are: (a) the Byzantine chant accepted by the Spanish church; (b) the Arab conquest of the Iberian Peninsula; (c) the settlement of Gypsies in Spain and thus the arrival of their musical groups. [...]
We would add to this that in one of the numerous Andalusian chants, in the one in which, in our opinion, this ancient spirit has been preserved in the most vivid form, in Sigiriya, we find the following elements of Byzantine liturgical chanting: the presence of tonal modes of the original systems (which is not should not be confused with the modes that we now call Greek, although they also sometimes participate in the structure of these systems); the presence of an enharmonic genus that is inseparable from the original modes, that is, the division of critical tones into degrees and semi-degrees in the function of beautifying the tonality; and, finally, the absence of metrical rhythm in the melodic line and its richness in modulating inflections" [1, 119].

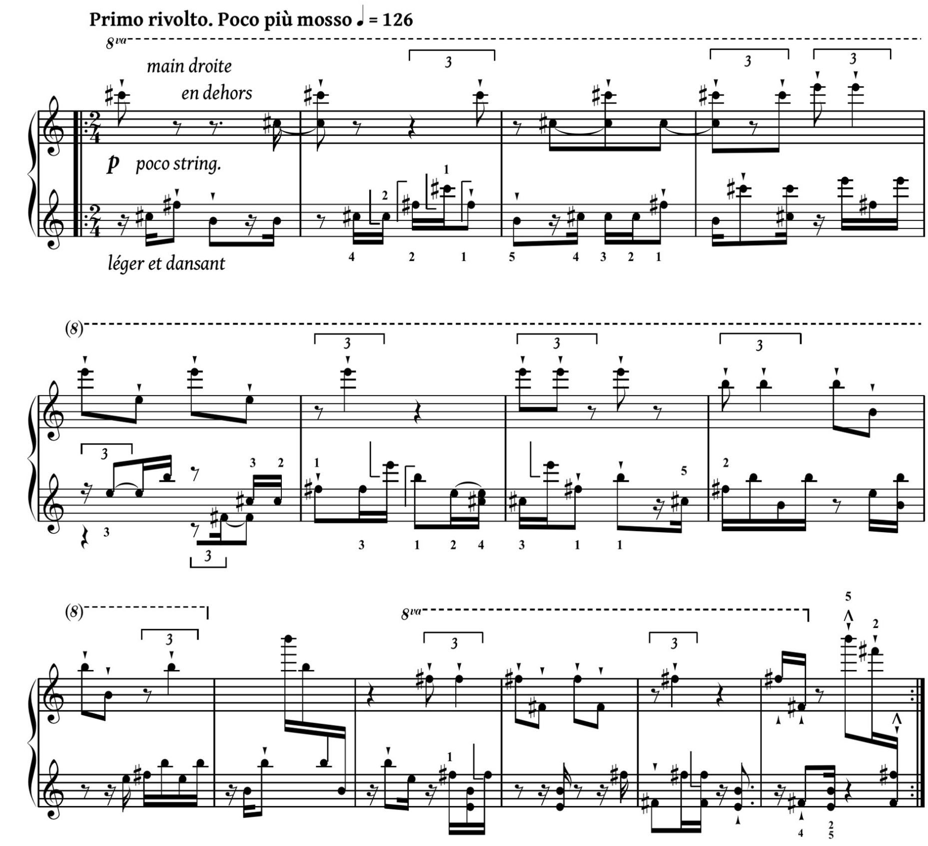


FIGURE 10. IrrInflections for piano (First movement); Section: *Primo rivolto. Poco più mosso* ♩ = 126)

the pitch fis^2, on the fourth metrical length of seven rhythmic quarters. This layer is structured, like the first layer, as a regular (hemiolic) grouping within which the value of the rhythmic unit is constantly variable.

In the first sentence the structure is: $7/4 + 6/4 + 5/4 + 4/4 + 3/4 + 2/4 + 1/4 = 28/4 = 7$♩ (follow the pitch cis^3 in note example 12). In the second sentence, the structure is rhythmically inverted: $1/4+2/4+3/4+4/4+5/4+6/4+7/4 = 28/4 = 7$♩ (follow the pitch e^3 in the musical example 12). In the third sentence, the structure is the same as in the second: $1/4 + 2/4 + 3/4 + 4/4 + 5/4 + 6/4 + 7/4 = 28/4 = 7$♩ (follow the pitch h^2 in musical example 12). In the fourth sentence, the structure is the same as in the previous two: $1/4+2/4+3/4+4/4+5/4+6/4+7/4 = 28/4 = 7$♩ (follow the pitch fis^2 in the musical example 12).

The textual complexity of this section is also reflected in the superposition of the new (third or lowest layer) which forms an irregular grouping, and within which the value of the rhythmic unit is variably variable. The length of the rhythmic model

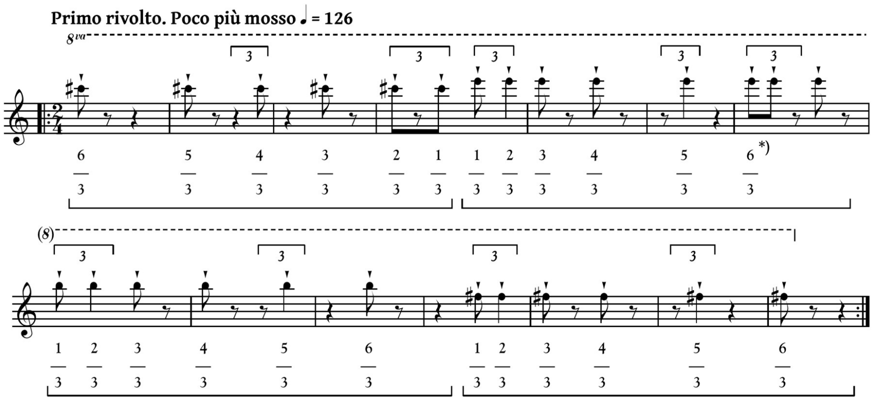
Table 14. Conjugate connected partitions of the number seven (7) used for the third layer in the *Primo rivolto section. Poco più mosso* ♩ = 126 (according to Ferrer's diagram)

	4	2	1
3	•	•	•
2	•	•	
1	•		
1	•		

(which is seven rhythmic sixteenths) is determined by the metrical length of the sentence (which is seven rhythmic quarters). According to this choice, the model will be repeated exactly four times within one sentence unit. The repetition of the model here should be accepted conditionally, because it is a rhythmic turning of the model.

The structure of the model was chosen according to the partition of the number seven (7) into four rhythmic members $(1 + 1 + 2 + 3)$ in the first two sentences; that is, on three rhythmic members $(2 + 4 + 1)$ in the other two sentences. These are two conjugately connected partitions.

The original form of the model is not easily discernible because the term is omitted at the very beginning (follow the numbers in parentheses below the description of the omitted pitches in example 12). The omission of certain pitches could already be noticed in the previous layers. However, when turning the model over, it is easy to guess which pitches should have been written. The reason why I gave up noting



*) $\frac{1}{3} + \frac{2}{3} + \frac{3}{3} = \frac{6}{3}$

Figure 11. Inflections for piano (First movement); The structure of the first layer in the *Primo rivolto section. Poco più mosso* ♩ = 126)

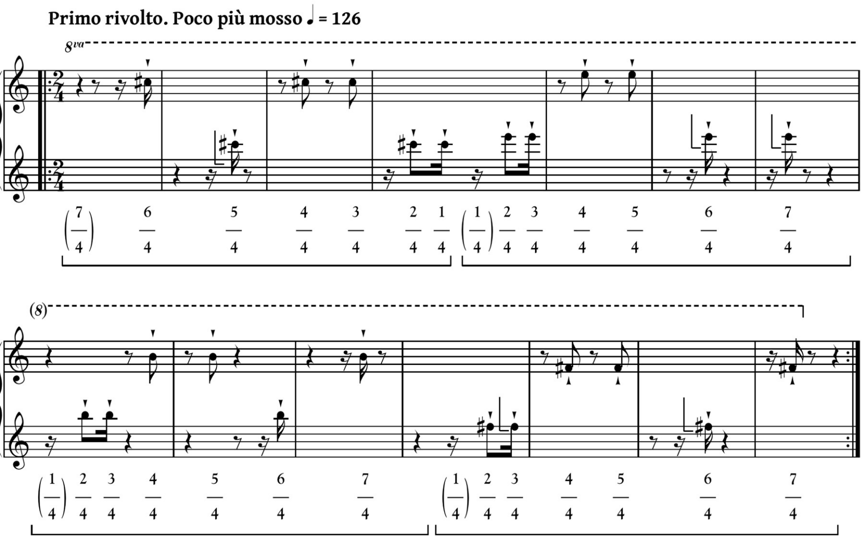


FIGURE 12. Inflections for piano (First movement); The structure of the second layer in the *Primo rivolto section. Poco più mosso* ♩ = 126) (octave punctuation only applies to the above system).

individual members of the model is of an interpretative nature. The expression and gesture that are born in the pianist's consciousness are rightly prioritized over the arithmetic game of numbers. The arithmetical game of numbers is now transposed into a virtuoso game of fingers and strums, which makes the reason for renouncing certain notes obvious. Reversal of the model is achieved by moving each rhythmic member to the place of the member next to it during repetition. If moving (or shifting) a member is done to the right, each move is defined as an increment; and, if moving (or shifting) the member is to the left, each shift is defined as a decrement. The original models were rotated for the first partition in the form of increments (1123 - 3112 - 2311 - 1231). The original models were rotated for the second partition in the form of decrements (241 - 412 - 124 - 241). See table 15 and 16.

TABLE 15. Increments of the rhythmic model from the first partition

I				II				III				IV			
1	1	2	3	3	1	1	2	2	3	1	1	1	2	3	1

What can be noticed is that the pitch that describes the rhythmic member does not change within one sentence structure. The first rhythmic model (1-1-2-3) from the first partition is described by pitches: cis^2 - cis^2 - fis^2 - h^1, while the second rhythmic model (1-1-2-3) from the first partition is the same with the fact that in the fourth rhythmic member, the pitch is displaced an octave upward (the pitches are:

TABLE 16. Decrements of the rhythmic model from the second partition

I			II			III			IV		
2	4	1	4	1	2	1	2	4	2	4	1

cis^2 - cis^2 - fis^2 - h^2). The third and fourth rhythmic models (2-4-1) from the second partition differ in the choice of pitches. The third model is described by pitches: (fis^2 or h^2) - h^1 - e^2, while the fourth model is described by pitches: fis^1 - (e^1-h^1 as a fifth) - fis^2. The first term from the third model is variable because, within the third sentence, it is first described by the height fis^2, and then, within the same sentence, it is described by the height h^2. The pitch h^2 is repeated, i.e. embellished, in the tenth measure, although, according to the scheme of the rhythmic model, it should be described only by a rhythmic eighth, and not by two rhythmic sixteenths.

The increments and decrements from the tables can also be compared with the example (13) that follows.[8]

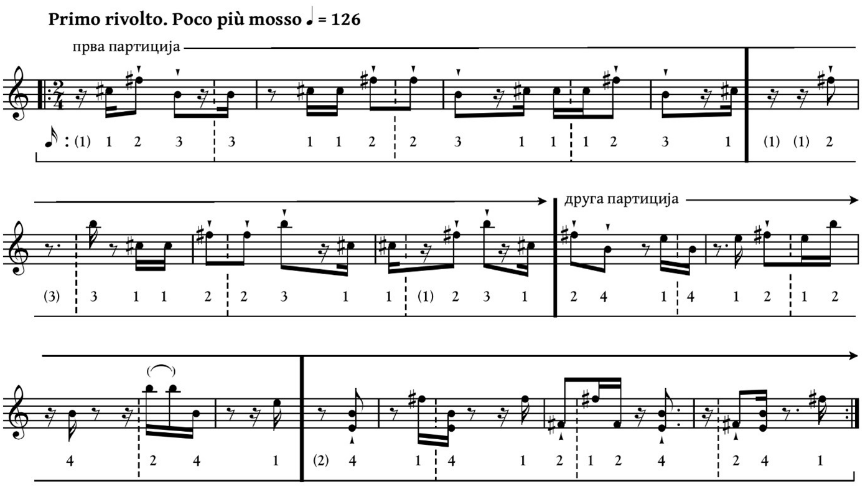


FIGURE 13. Inflections for piano (First movement); The structure of the third layer in the *Primo rivolto section. Poco più mosso* ♩ = 126).

From the example of the *First Inflection* described above, it can be noted that I literally introduced into my compositional style those metro-rhythmic formulations that our composer and musicologist Dragutin Gostuški recognized and defined as harmonic projection in his study *Time of Art.* Contribution to the foundation of a

[8] In the second sentence, a fourth layer is discreetly introduced, in which the outlines of a large rhythmic sextole are taken on a meter of seven rhythmic quarters (6:7). That voice is not consistently implemented in the second sentence.

general science of forms, and especially in the second part of the book related to the horizontalization of vertical interval entities of the spectrum. I upgraded his thesis with my own methods of spectrum projection, specifically in the above-mentioned example - the method of horizontal projection of number sequences on an ancient epitrite platform.

On the other hand, I explained the principle of rhythmic rotation in my paper *Central Rotation of Regular (and Irregular) Musical Polygons* [4]: "With enumerated regular polygons, which are the reference for each harmonic individually, the principle of rhythmic rotation can be realized by means of isometric transformation, specifically - central rotation. This principle is valuable because we find its etymological roots in the folkloric rhythms of Central African and West African music. It is achieved by "moving" the accents that make up a geometric figure (regular polygon) to the adjacent reference rhythmic unit in the direction: left or right. The temporal length from one accent to another forms the side of a regular polygon. This is how the impression of the temporal kinematics of an isolated sound entity is created. If the shift of accents is in the "right" direction, each shift is called an increment. If the shift of accents is in the direction "to the left", each shift is called a decrement [4, p. 216].

The projection of sequences of ordinal numbers of adjacent harmonics of the spectrum (octaves 2:1 or fifths 3:2 or fourths 4:3 and so on) is also applicable to a selection of other numerical sequences; let us say, on the selection of sentences of triangular or pyramidal numbers. The triangle number counts the objects that can form an equilateral triangle. Triangular numbers are given by the following explicit equation: Triangular numbers

$$T_n = \sum_{k-1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} = \binom{n+1}{2} \tag{5.1}$$

The coefficient that we read as "n plus one over two" represents the number of different pairs that can be selected from $n + 1$ objects.

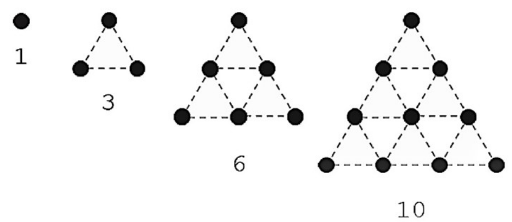


FIGURE 14. Triangular numbers.

On a platform that would count ten (10) metric units, it is possible to synchronize two series of triangular numbers starting from the first rhythmic member up to the fourth member, with the fact that one series will be shorter than the other by one rhythmic member (it will have three rhythmic members). The rhythmic unit that is selected for projection is half (1/2) of the metric unit for the first voice (rhythmic eighth if our measure is a whole rhythmic quarter) and the value of the metric unit (1/1) which is a unit for the second voice (rhythmic quarter if our measure is a whole rhythmic quarter). The projected units are reciprocal to the index numbers of the second and first harmonics of the spectrum. It is an octave interval. The product of the metric platform (10) with the choice of the reference rhythmic unit for the second harmonic is twenty halves (20/2) or twenty rhythmic eighths, and the numerator in the fraction is the sum of the fourth term in the series of triangular numbers: $(1 + 3 + 6 + 10 = 20)$. The product of the metric platform (10) with the choice of the reference rhythmic unit for the first harmonic is ten units (10/1) or ten rhythmic quarters, and this number is the sum of the third term in the series of triangular numbers: $(1 + 3 + 6 = 10)$. See the first row in Table 17.

TABLE 17. Projections of strings of triangular numbers according to adjacent pairs of spectrum harmonics.

2 : 1	$10 =$	$\left[\sum_{n=1}^{4}\frac{n(n+1)}{2}\right]/2 = \left[\sum_{n=1}^{3}\frac{n(n+1)}{2}\right]/1 =$	$20/2 = 10/1$
3 : 2	$28 =$	$\left[\sum_{n=1}^{7}\frac{n(n+1)}{2}\right]/3 = \left[\sum_{n=1}^{6}\frac{n(n+1)}{2}\right]/2 =$	$84/3 = 56/2$
4 : 3	$55 =$	$\left[\sum_{n=1}^{10}\frac{n(n+1)}{2}\right]/4 = \left[\sum_{n=1}^{9}\frac{n(n+1)}{2}\right]/3 =$	$220/4 = 165/3$
5 : 4	$91 =$	$\left[\sum_{n=1}^{13}\frac{n(n+1)}{2}\right]/5 = \left[\sum_{n=1}^{12}\frac{n(n+1)}{2}\right]/4 =$	$455/5 = 364/4$
6 : 5	$136 =$	$\left[\sum_{n=1}^{16}\frac{n(n+1)}{2}\right]/6 = \left[\sum_{n=1}^{15}\frac{n(n+1)}{2}\right]/5 =$	$816/6 = 680/5$
7 : 6	$190 =$	$\left[\sum_{n=1}^{19}\frac{n(n+1)}{2}\right]/7 = \left[\sum_{n=1}^{18}\frac{n(n+1)}{2}\right]/6 =$	$1330/7 = 1140/6$

By correctly arranging the points in the plane, different polygonal shapes can be obtained (triangle, square, pentagon and so on). With this procedure, as we could see, the aforementioned two-dimensional polygonal numbers are created. However, the points can also be properly arranged in space, in three-dimensional bodies (pyramid, cube, octahedron, dodecahedron or icosahedron). This process produces pyramidal, cubical, octahedral numbers and so on. A pyramidal number can be represented in space by means of points arranged in a pyramid whose base is a regular polygon. Considering the base of the pyramid, we can divide the pyramid numbers into triangular, square, pentagonal, hexagonal and so on. We will dwell on the first mentioned type (triangular pyramidal numbers) because it is the only one where the development of the projection of sequences according to the arrangement of adjacent harmonics continues.

On a platform that would count thirty-five (35) metric units, it is possible to synchronize two series of pyramidal numbers starting from the first member up to the fifth member, with the fact that one series will be shorter than the other by one rhythmic member (it will have four rhythmic member). The rhythmic unit chosen for projection is half (1/2) of the metric unit for the first voice (rhythmic eighth if our measure is a whole rhythmic quarter) and one (1/1) of the metric unit (rhythmic quarter if our measure is a whole rhythmic quarter) for another voice. The projected units are reciprocal to the index numbers of the second and first harmonics of the spectrum. It is an octave interval (2:1). The product of the metric platform (35) with the choice of the reference rhythmic unit for the second harmonic is seventy halves or seventy rhythmic eighths (70/2), and the numerator in the fraction is the sum of the fifth term in the series of pyramidal numbers: $(1+4+10+20+35=70)$. The product of the metric platform (35) with the choice of the reference rhythmic unit for the first harmonic is thirty-five units or thirty-five rhythmic quarters (35/1), and the numerator in the fraction is the sum of the fourth term in the series of pyramidal numbers:

$(1+4+10+20=35)$. See table 18 (first row).

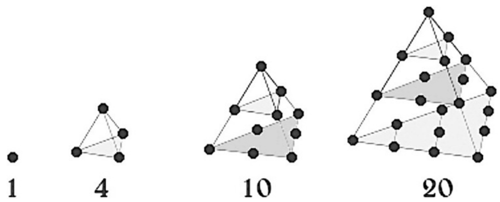


Figure 15. Pyramid numbers.

Table 18. Projections of series of pyramidal numbers according to adjacent pairs of spectrum harmonics.

2 : 1	35 =	$\left[\sum_{n=1}^{5}\frac{n(n+1)(n+2)}{3!}\right]/2$	$=\left[\sum_{n=1}^{4}\frac{n(n+1)(n+2)}{3!}\right]/1 =$	70/2 = 35/1
3 : 2	165 =	$\left[\sum_{n=1}^{9}\frac{n(n+1)(n+2)}{3!}\right]/3$	$=\left[\sum_{n=1}^{8}\frac{n(n+1)(n+2)}{3!}\right]/2 =$	495/3 = 330/2
4 : 3	455 =	$\left[\sum_{n=1}^{13}\frac{n(n+1)(n+2)}{3!}\right]/4$	$=\left[\sum_{n=1}^{12}\frac{n(n+1)(n+2)}{3!}\right]/3 =$	1820/4 = 1365/3
5 : 4	969 =	$\left[\sum_{n=1}^{17}\frac{n(n+1)(n+2)}{3!}\right]/5$	$=\left[\sum_{n=1}^{16}\frac{n(n+1)(n+2)}{3!}\right]/4 =$	4845/5 = 3876/4
6 : 5	1771 =	$\left[\sum_{n=1}^{21}\frac{n(n+1)(n+2)}{3!}\right]/6$	$=\left[\sum_{n=1}^{20}\frac{n(n+1)(n+2)}{3!}\right]/5 =$	10626/6 = 8855/5
7 : 6	2925 =	$\left[\sum_{n=1}^{25}\frac{n(n+1)(n+2)}{3!}\right]/7$	$=\left[\sum_{n=1}^{24}\frac{n(n+1)(n+2)}{3!}\right]/6 =$	20475/7 = 17550/6

A further sequence of projections of sequences of figurative numbers with n-dimensions is possible. The next higher projection would refer to sequences of pentahedroid or hyperpyramidal numbers with four dimensions. However, here we will refer only to the branching of harmonics of the spectrum that we subject to projection.

On the example of the peon rate (5), we saw that through the reciprocal values of the projected second and projected third harmonics, a series of ordinal numbers was structured. On the example of the epitritic rate (7), we saw that through the reciprocal values of the projected third and projected fourth harmonics, a series of ordinal numbers was also structured.

The question arises whether there is an order in the projection of each individual harmonic and what it looks like. My answer is that there is and that the reciprocities that make up the projected rhythmic units are in agreement with the harmonic relationships even when the harmonics are not adjacent to each other. Moreover, the row to be discussed precisely shows the expansion of the intervallic ambitus of two harmonics, with the metrical platform remaining unchanged. The octave interval (2:1) in the projection of a series of ordinal numbers forms a metric platform of three metric units. The next interval related to it (but related only by the projection of a series of ordinal numbers, i.e. by the projection that would form a meter of the same length) would be (7:5) and this interval in the projection of a series of ordinal numbers forms a metric platform of three metric units. The gnomon, in the case of an octave (2:1), is one (1) because it is about adjacent harmonics. However, the gnomon in the case of the spectral interval (7:5) is two (2) because it is one (sixth) skipped harmonic. The next interval related to it (but related again only by the projection of a series of ordinal numbers) would be the major third (15:12). And this interval in the projection of a series of ordinal numbers forms a metric platform of three metric units. The gnomon, in the case of the spectral interval (15:12), is three (3) because it is about two skipped harmonics (thirteenth and fourteenth). This order can be easily proved mathematically by the ratio of the sums formed by the harmonics with the square of the gnomon: In the case of an octave interval (2:1), the formulation reads: $(1 + 2)/1^2 = 3$. In the case of an interval (7:5), we apply the same procedure: $(7 + 5)/2^2 = 3$. In the case of the interval (15:12) we apply the same procedure again: $(15 + 12)/3^2 = 3$. This kind of branching of harmonics that are not adjacent to each other, and which in the projection form series of ordinal numbers, can be traced further on the example of fifths (3:2), and on the example of fourths (4:3), major thirds (5:4) , minor thirds (6:5), that is, on each similar (adjacent) interval (spectral) step. In the case of the fifth interval (3:2) we have the scheme: $(3 + 2)/1^2 = 5$. In the case of the interval (11:9) we apply the same procedure: $(11 + 9)/2^2 = 5$. In the case of the interval (24: 21) again apply the same procedure: $(24 + 21)/3^2 = 5$.

Low neighboring harmonics, as we could notice, in the projections of sequences of ordinal numbers, form metric platforms of odd lengths. The projection of the first and second harmonics forms a meter of three rhythmic quarters; the projection of the second and third harmonics forms a meter of five rhythmic quarters; the third and fourth form a meter of seven rhythmic quarters and so on. However, low harmonics that would not be adjacent, in the projections of sequences of ordinal numbers, can also form metric platforms of even lengths. The first such interval is the major sixth (5:3). The next interval related to it (but related only by the

TABLE 19. Projections of sequences of ordinal numbers of adjacent and non-adjacent harmonics of the spectrum.

				2 ♩	5:3	18:14	39:33
3 ♩	2:1	7:5	15:12				
				4 ♩	9:7	34:30	75:69
5 ♩	3:2	11:9	24:21				
				6 ♩	13:11	50:46	111:105
7 ♩	4:3	15:13	33:30				
				8 ♩	17:15	66:62	147:141
9 ♩	5:4	19:17	42:39				
				10 ♩	21:19	82:78	183:177
11 ♩	6:5	23:21	51:48				
				12 ♩	25:23	98:94	219:213
13 ♩	7:6	27:25	60:57				
				14 ♩	29:27	114:110	255:249
15 ♩	8:7	31:29	69:66				
				16 ♩	33:31	130:126	291:285
17 ♩	9:8	35:33	78:75				

projection of a series of ordinal numbers, i.e. by the projection that would form a meter of the same length) would be (18:14). And this interval in the projection of a series of ordinal numbers forms a metric platform of two (2) metric units. The gnomon, in the case of the major sixth (5:3), is two (2) because it is one (fourth) skipped harmonic. However, the gnomon, in the case of the spectral interval (18:14), amounts to four (4) because it is about three skipped harmonics. The next interval related to it (but related again only by the projection of a series of ordinal numbers) would be (39:33). And this interval in the projection of a series of ordinal numbers forms a metric platform of two metric units. The gnomon, in the case of the spectral interval (39:33), is six (6) because it is about five skipped harmonics. Even now, this order can be easily proved mathematically by the ratio of the sums formed by the harmonics with the square of the gnomon: In the case of the interval of the major sixth (5:3) we have the scheme: $(5+3)/2^2 = 2$. In the case of the interval (18:14) we apply the same procedure: $(18+14)/4^2 = 2$. In the case of the interval (39:33) we apply the same procedure again: $(39+33)/6^2 = 2$. Table 19 lists the metric lengths up to the seventeenth regular of numbers formed by projecting adjacent and non-adjacent harmonics of the spectrum.

The consequence of replacing the functions of the rhythmic unit and the metrical basis is reflected in the orderly expansion of the metrical interval. In particular, the projection of a sequence of ordinal numbers can be represented by a metrical progression in the choice of a single rhythmic unit to be selected. Let us say, on the metric basis of one rhythmic quarter (1=3/3), the ordered form of rhythmic additions according to a series of ordinal numbers (in the projection of the third harmonic of the spectrum) contains a triple eighth (1/3) and a triple quarter (2/3), i.e. two beats (or two accents).

On the metric basis of two rhythmic quarters (2=6/3), the ordered form of rhythmic additions according to a sequence of ordinal numbers (in the projection of the third harmonic of the spectrum) contains a triple eighth (1/3), a triple quarter (2/3) and a rhythmic quarter (3/3), that is, three beats (or three accents). In the next step, on the metric basis of five rhythmic quarters (5=15/3), the ordered form of rhythmic additions according to a series of ordinal numbers (in the projection of the third harmonic of the spectrum) contains a triple eighth (1/3), a triple quarter (2/3), a rhythmic quarter (3/3), a rhythmic quarter linked by a triple eighth (4/3) and a triple quarter linked by a rhythmic quarter (5/3), i.e. five beats (or five accents). Tables (20, 21 and 22) list the progressions of the metric base for a series of ordinal numbers when projecting the third, fourth and fifth harmonics of the spectrum.

Table 20. Metric progression of a series of ordinal numbers in the projection of the third harmonic of the spectrum.

	1 ♩	2 ♩										
	+ 1		5 ♩	7 ♩								
			+ 2		12 ♩	15 ♩						
1/3					+ 3		22 ♩	26 ♩				
							+ 4		35 ♩	40 ♩		
									+ 5		51 ♩	57 ♩
											+ 6	

Table 21. Metric progression of a series of ordinal numbers in the projection of the fourth harmonic of the spectrum.

	7 ♩	9 ♩										
	+ 2		30 ♩	34 ♩								
			+ 4		69 ♩	75 ♩						
1/4					+ 6		124 ♩	132 ♩				
							+ 8		195 ♩	205 ♩		
									+ 10		282 ♩	294 ♩
											+ 12	

Table 22. Metric progression of a series of ordinal numbers in the projection of the fifth harmonic of the spectrum.

	2 ♩	3 ♩										
	+1		9 ♩	11 ♩								
			+2		21 ♩	24 ♩						
1/5					+3		38 ♩	42 ♩				
							+4		60 ♩	65 ♩		
									+5		87 ♩	93 ♩
											+6	

On the epitritic foot, which counts seven metrical units, we constructed a series of seven different rhythmic members. For this construction, the reciprocal value of the fourth harmonic of the spectrum was chosen as the building rhythmic unit (1/4). A sequence of seven different rhythmic members on the same metrical platform (which would count seven metrical units) is also achievable with a change in rhythmic resolution.

In relation to the basic sequence, the conditions for the construction of (derived) sequences on the same metrical platform would be the following: (1) the rhythmic unit must be shorter than the unit defining the terms for the basic sequence $[1/(4+n)]$, and (2) rhythmic members are moved to the place of the adjacent higher rhythmic member according to the choice of the rhythmic unit. The expression $[1/(4+n)]$ could conditionally be understood as a coefficient of rhythmic resolution that serves to build a similar homothetically growing form on the same metrical basis.

The sum of the sum of the rhythmic members, described by the numerator in the fraction, according to the choice of the rhythmic unit described by the denominator in the fraction, is followed by the following comparison: $28/4 = 35/5 = 42/6 = 49/7 = 56/8 = 63/9$ and so on. If we introduce the structure of rhythmic terms into the numerator of the fraction, we arrive at the following comparative solution: $(1+2+3+4+5+6+7)/4 = (2+3+4+5+6+7+8)/5 = (3+4+5+6+7+8+9)/6 = (4+5+6+7+8+9+10)/7$ and so on. If the rhythmic unit were longer than the unit defining the terms for the basic sequence $[1/(4-n)]$ with the condition: $n < 4$, the derived sequence would be shorter by (by) two rhythmic terms: $(1+2+3+4+5+6)/3 = (2+3+4+5)/2 = (3+4)/1$.

What should be mentioned in an aesthetic sense and is related to the projections of sequences of ordinal, triangular or pyramidal numbers according to the ratios of spectral harmonics is that the sound result is harmonic and has an effect; it is often in the service of contemplation, it is meditative, slightly hypnotizing, visual, it imitates the sounds of nature in a fresh realistic way, and when it is playful and lively, it is easy to create the impression of a harmonic-periodic alternation of beats like a kind of game of beads. This is supported by a complex compositional procedure that accepts the above-mentioned mathematical formulations only as a starting point for composing. When composing, only the dimensioning of the number, and in this case it is the counter of the rhythmic fraction, causes a lively correspondence of two or more voices.

Metric projections can also be vertical, horizontal and combined. A vertical metrical projection would satisfy the condition that the choice of rhythmic unit is equal to the basic metrical unit. The difference is reflected in the control of distribution elements. The functions of rhythm and meter are directly opposite if the types of projections are compared.

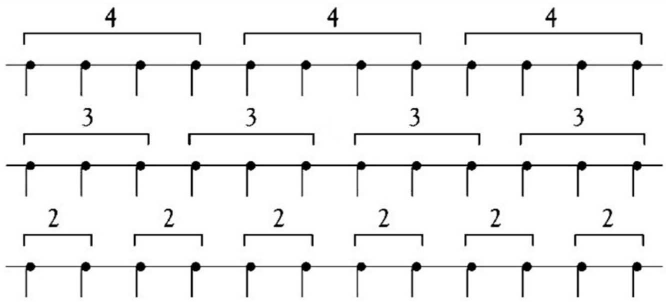


Figure 16. Vertical metric projection (4 : 3 : 2).

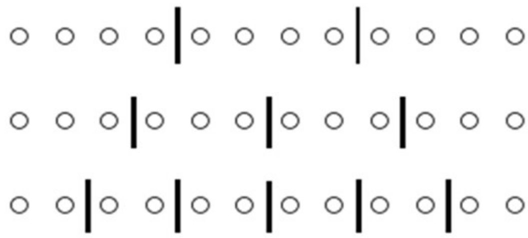


Figure 17. Vertical metric projection (4 : 3 : 2).

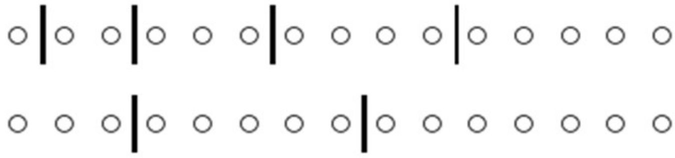


Figure 18. Horizontal polymetric projection (1 : 2 : 3 : 4 : 5) and (3 : 5 : 7).

6. Isometric transformations of musical polygons

Both regular and reproducible (hemiol) rhythmic groups can be viewed as regular polygons. A triplet would make an equilateral triangle on any metric basis. A quartola would make an equilateral quadrilateral (square), a quintola a pentagon, a sextola a hexagon, a septimola a heptagon, and so on. With the enumerated regular polygons, which are the reference for each harmonic individually, the principle of rhythmic rotation can be realized by means of direct and indirect isometric transformations.

In my study *Central Rotation of Regular* (*and Irregular*) *Musical Polygons*, I described direct isometry on different types of triangle figures (equilateral, isosceles obtuse, isosceles acute and right triangles): “With enumerated regular polygons, which are referential for each harmonic individually, one can to realize the principle of rhythmic turning by means of isometric transformation, specifically - central rotation. This principle is valuable because we find its etymological roots in the folkloric rhythms of Central African and West African music. It is achieved by ‘moving’ the accents that make up a geometric figure (regular polygon) to the adjacent reference rhythmic unit in the direction: left or right. The temporal length from one accent to another forms the side of a regular polygon. This is how the impression of the temporal kinematics of an isolated sound entity is created. If the shift of accents is in the ‘right’ direction, each shift is called an increment. If the shift of accents is in the direction ‘to the left’, each shift is called a ‘decrement” [4, 216].

Isometric transformation is direct if it “preserves” the orientation of the plane, i.e. if every triangle of that plane maps into a triangle of the same orientation. An isometric transformation is indirect if every triangle of that plane maps into a triangle of the opposite orientation. All isometric transformations, in the rest of the text, will be listed according to general definitions related to the type of isometry, that is, to the fact of how many fixed points one transformation can contain. Fixed points are made up of accents arranged on the appropriate metrical platform.

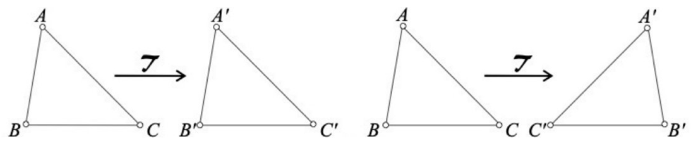


FIGURE 19. Direct and indirect isometry.

Direct isometry (central rotation and central symmetry) of the musical polygon. For the central rotation procedure, the center of the described circle around the regular polygon, that is, the hemiol group, is taken. It is a circle that passes through all the vertices of the polygon. The center of this circle is located at the intersection of the bisectors of the sides and its radius is the distance of the center from any vertex of the polygon. Central rotation is a direct isometry in which the movement of each accent occurs phase-periodically to the adjacent rhythmic unit in order to preserve the orientation of the musical figure.

For the central rotation procedure, the center of the described circle around the regular polygon, that is, the hemiol group, is taken. It is a circle that passes through all the vertices of the polygon. The center of this circle is located at the intersection of the bisectors of the sides and its radius is the distance of the center from any vertex of the polygon.

The angle of rotation (φ) is the angle of rotational displacement of the hemiola, i.e. the angle that is valid for the time in which one side of a regular polygon is mirrored on its adjacent side. The rotation angle of the quartola is equal to a quarter of the full angle of the circumscribed circle around the equilateral quadrilateral or square ($\varphi_1 = 360°/4 = 90°$)[9] for any scale. The angle of rotation of the triole, for any scale, is equal to a third of the full angle of the circumscribed circle around the equilateral triangle ($\varphi_2 = 360°/3 = 120°$).

In contrast to the rotation angle, there is also a phase angle (ω) which is measured in relation to the choice of projection of a certain spectral proportion. If the quartola was rotated in the sesquitertial scale, and taking into account the product of the distribution ($3 \cdot 4$) of the reference rhythmic unit (rhythmic sixteenth), that product is now divided by a full angle: ($\omega_1 = 360°/12 = 30°$). Thus, we come to the conclusion that twelve phase shifts (increments) and four rotational shifts occurred during the quartola rotation. More precisely, we could say that the quartola rotation angle ($\varphi_1 = 90°$) is valid for three phase shifts, i.e. three increments ($\omega_1 = \varphi_1/3 = 90°/3 = 30°$) in the metro-rhythmic projection of the sesquitercial scale. On the other hand, again taking into account the product of the distribution ($3 \cdot 4$), but now for the triole (triole eighth), we come to the conclusion that the rotation angle of the triole ($\varphi_2 = 120°$) is valid for four phase shifts or four increments ($\omega_2 = \varphi_2/4 = 120°/4 = 30°$) in the metro-rhythmic projection of the sesquitertial ratio. During the rotation of the triole, and unlike the rotation of the quartole, there were also twelve phase shifts, i.e. twelve increments, but only three rotational shifts. A quarto's rotary shift is worth three increments, while a triplet's rotary shift is worth four increments.

Phase shifts ($\omega_{1,2}$) for the sesquitertial ratio (4:3) are the quotient of the reference angle of rotation of the third harmonic (120°) with the order number of the fourth harmonic (4), that is, the quotient of the reference angle of rotation of the fourth harmonic (90°) with the order number of the third harmonic (3).

$$\omega_{1,2} = 120°/4 = 90°/3 = 30° \ (\omega_1 = \omega_2) \quad \text{See table 23.}$$

The above-described principle of metro-rhythmic rotation, which was valid for sesquitertial projection, can be realized for any interval ratio of two harmonics. Let's look at the superbipartite projection of the spectral interval (5:3). The angle of rotation of the quintola is for any scale equal to a fifth of the full angle of the

[9] The polygonal (square) number 4 is a number that is polygonally constructed by partitioning its four equal sides ($1 + 1 + 1 + 1$), thus forming a special case of a rectangle, i.e. square. The spectral square (regular polygon) is a geometric figure that is a reference for the system of the fourth harmonic of the spectrum. On the other hand, the polygonal (triangular or hexagonal) number 6 would be a number that can also be constructed polygonally (square) by partitioning four of its sides ($1 + 1 + 2 + 2$). This would be the spectral rectangle or geometric figure that is the reference for the sixth harmonic system of the spectrum.

described circle around the equilateral pentagon: ($\varphi_1 = 360°/5 = 72°$). As we have already said, the angle of rotation of the triole for any scale (including the one in question) is equal to a third of the full angle of the described circle around the equilateral triangle ($\varphi_2 = 360°/3 = 120°$).

If the quintola was rotated in a superbipartite ratio (5:3), and taking into account the product of the distribution ($3 \cdot 5$) of the reference rhythmic unit (the quintola sixteenth), that product is now divided by a full angle: ($\omega_1 = 360°/15 = 24°$). Thus, we come to the conclusion that fifteen phase shifts (increments) and five rotational shifts occurred during the rotation of the quintola. More precisely, we could say that the rotation angle of the quintola ($\varphi_2 = 72°$) is valid for three phase shifts, i.e. three increments ($\omega_1 = \varphi_1/3 = 72°/3 = 24°$) in the metro-rhythmic projection of the superbipartite scale (5:3).

On the other hand, again taking into account the product of the distribution ($3 \cdot 5$), but now for the triole (triole eighth), we come to the conclusion that the rotation angle of the triole (120°) is valid for five phase shifts or five increments ($\omega_2 = \varphi_2/5 = 120°/5 = 24°$) in metro-rhythmic projection of superbipartite scale (5:3). During the rotation of the triplet, and in contrast to the rotation of the quintole, there were also fifteen phase shifts, i.e. fifteen increments, but only three rotational shifts. A quintola's rotational displacement is worth three increments, while a triplet's rotational displacement is worth five increments.

The phase shifts ($\omega_{1,2}$) for the superbipartite ratio (5:3) are the quotient of the reference angle of rotation of the fifth harmonic (72°) with the ordinal number of the third harmonic (3), i.e., the quotient of the reference angle of rotation of the third harmonic (120°) with the ordinal number of the fifth harmonic (5).

$$\omega_{1,2} = 120°/5 = 72°/3 = 24° \ (\omega_1 = \omega_2) \text{ See table 24.}$$

As already stated, the angle of rotation of the quartola is for any scale equal to a quarter of the full angle of the circumscribed circle around the square ($360°/4 = 90°$). The angle of rotation of the quintola is for any scale equal to one-fifth of the full angle of the circumscribed circle around the equilateral pentagon: ($360°/5 = 72°$). Thus, the phase shifts ($\varphi_{1,2}$) for the sesquiquart ratio (5:4) would be: (a) the quotient of the reference angle of rotation of the fifth harmonic (72°) with the ordinal number of the fourth harmonic (4), and, (b) the quotient of the reference angle of rotation of the fourth harmonic (90°) with the ordinal number of the fifth harmonic (5).

$$\omega_{1,2} = 72°/4 = 90°/5 = 18° \ (\omega_1 = \omega_2)$$

The angle of rotation is the angle in which one side of the triangle, i.e. the temporal line that divides the two accents (vertices) on the metric surface, is mirrored on the adjacent side. With an equilateral triangle, in the first phase of rotation, side a is mapped to side b, side b to side c, and side c to side a. In the next rotation phase, side b is copied to side c, side c to side a, and side a to side b; so that in the last (third) rotation phase, side c would be copied to side a, side a to side b, and side b to side c. Since it has three rotational movements, we divided the full angle (360°) into three equal parts (360°/3).

The angle of rotation of an equilateral triangle, which is 120°, is therefore constant. On the other hand, the phase angle consequently changes by sharpening the rhythmic

Table 23. Central rotation of square and equilateral triangle in sesquitertial scale (4:3); $0° = 360°$ (In the shaded fields, the angles of phase shifts (ω) coincide with the angles of rotation (φ). Parameters for the square: $a = 3/4$ ($a = b = c = d$); Parameters for the triangle: $a = 4/3$ ($a = b = c$).

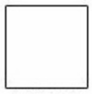
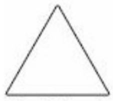
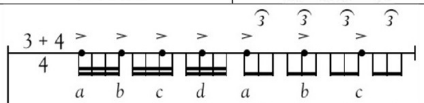
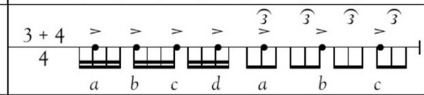
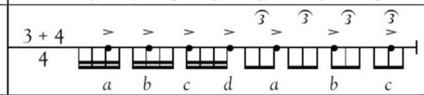
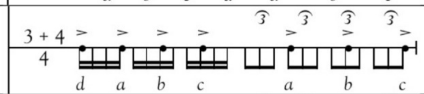
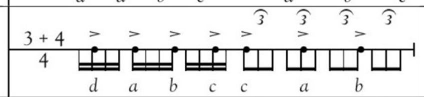
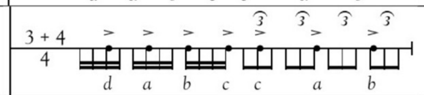
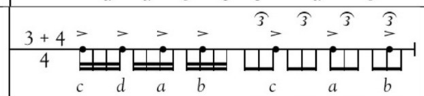
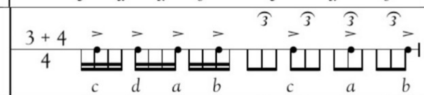
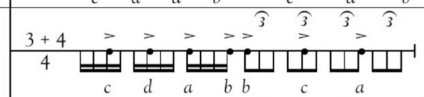
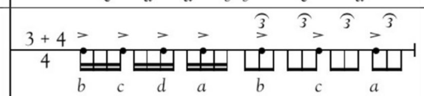
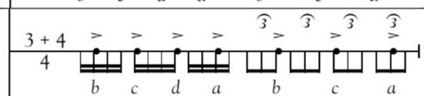
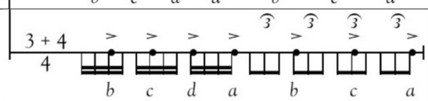
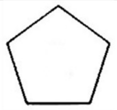
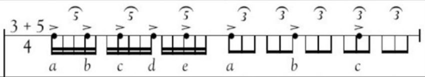
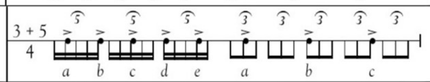
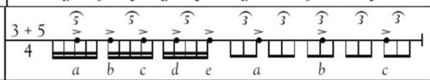
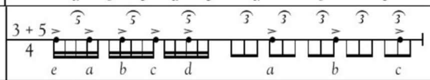
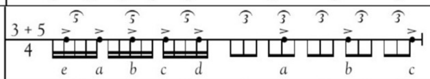
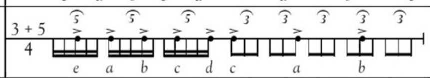
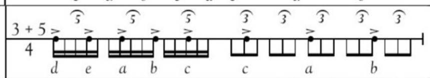
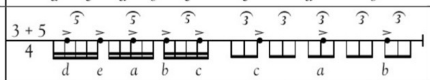
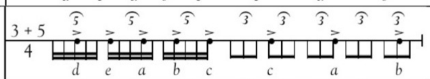
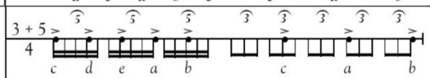
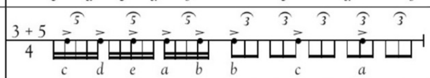
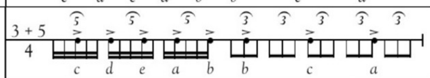
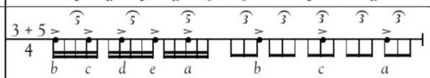
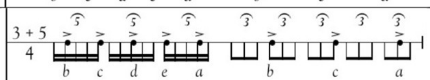
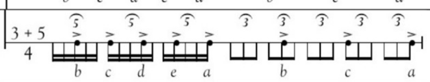
increments	ω_1	 	ω_2	increments
I	0°		0°	I
II	30°		30°	II
III	60°		60°	III
IV	90°		90°	IV
V	120°		120°	V
VI	150°		150°	VI
VII	180°		180°	VII
VIII	210°		210°	VIII
IX	240°		240°	IX
X	270°		270°	X
XI	300°		300°	XI
XII	330°		330°	XII

TABLE 24. Central rotation of a regular pentagon and an equilateral triangle in superbipartite scale (5:3); $0° = 360°$ (In the shaded fields, the angles of phase shifts (ω) coincide with the angles of rotation (φ). Parameters for the pentagon: $a = 3/5$ ($a = b = c = d = e$); Parameters for the triangle: $a = 5/3$ ($a = b = c$).

increments	ω_1			ω_2	increments
I	0°			0°	I
II	24°			24°	II
III	48°			48°	III
IV	72°			72°	IV
V	96°			96°	V
VI	120°			120°	VI
VII	144°			144°	VII
VIII	168°			168°	VIII
IX	192°			192°	IX
X	216°			216°	X
XI	240°			240°	XI
XII	264°			264°	XII
XIII	288°			288°	XIII
XIV	312°			312°	XIV
XV	336°			336°	XV

resolution, but it also changes by choosing a new temporal length that divides the two accents (vertices) on the metrical surface. Thus, in the sesquitertial scale (4:3), where the side $a = 4/3$ (and where the equality of the sides of an equilateral triangle holds: $a = b = c$), we identified a phase shift (increment) at an angle of 30°. Already in the superbipartite scale (5:3), where the side $a = 5/3$ (and where the sides are also equal) the phase shift is identified at an angle of 24°.

If we were to sharpen the rhythmic resolution already in the sesquitertial scale (4:3), and describe side a as $a = 8/6$, we would identify the increment at a sharper angle (15°), so the rotational displacement would be worth eight increments. And in the superbipartite scale (5:3) if we sharpened the rhythmic resolution, and described side a as $a = 10/6$, we would identify the increment at an even sharper angle (12°), and the rotational displacement would be worth, now, ten increments. We notice that the number of increments until the next rotational shift depends on the numerator in the fractional constellation.[10]

Hemiol groups are distributed into equal rhythmic units. A triple can be made up of three triple sixteenths, three triple eighths, and so on; a quintola can be made up of five sixteenths of a quintola, five eighths of a quintola, and so on. However, as the goal of this metro-rhythmic research is aimed at examining the identification of geometric figures in the plane, the fact cannot be ignored that apart from regular hemiol groups that would make up projected regular musical polygons, there are also irregular hemiol groups.

Examples of irregular hemiol groups would be attached to irregular polygons. Isosceles or many-sided triangles are irregular compared to an equilateral triangle. They can be constructed in the metro-rhythmic plane, and more importantly, all isometric transformations can be performed on them, including central rotation.

7. Conclusion

Projection of harmonics of the spectrum brings with it an innumerable number of possibilities through practical-compositional and analytical aspects. In this paper, I tried to cover only the essential characteristics that describe my style to the greatest extent. I consider the introduction of strict mathematical formulations relevant even though I try to avoid giving the impression of any predictability in the music I compose.

Also, in the text I included a description of the origin of the relevant algorithms through the projection of harmonics of the spectrum, which, in my opinion, are an inseparable part of deep philosophical reflection, as well as the analytical and compositional-practical experience of two composers and modern Pythagoreans: Dragutin Gostuški, and my professor, composer and academician Vlastimir Trajković. Therefore, I owe the greatest credit for profiling my style, which implies both expression, form and language.

[10] Not all metric lengths are suitable for examining the amount of phase shifts within a single phase of rotation. Those lengths at which the increase in increments (increment expressed in whole numbers) is best observed at the level of sharpening of the rhythmic resolution are the lengths that are the result of the whole number of the given fraction, for example: 3/3, 4/4, 5/5 and so on, or: 6/3, 8/4, 10/5 and the like.

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