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VARIOUS TYPES OF THE REVERSE ORDER LAWS FOR THE GROUP INVERSES IN RINGS

Abstract. We characterize different kinds of the reverse order laws for the group inverses in rings and we present the equivalences between some of them, extending the recent results.

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1. Introduction

Let $\mathcal{R}$ be an associative ring with the unit 1 and let $a \in \mathcal{R}$. An element $x \in \mathcal{R}$ is an *inner inverse* of a if $axa = a$, and we write $x = a^{(1)}$. In this case a is called *inner regular* (or *relatively regular*). If there exists $x \in \mathcal{R}$ such that $0 \neq x = xax$, then we say that x is an *outer inverse* of a, and write $x = a^{(2)}$. For such an a we say that it is *outer regular*. If x is both inner and outer inverse of a, then it is a *reflexive generalized inverse* (or *reflexive inverse*) of a. If x is an inner inverse of a, then xax is a reflexive generalized inverse of a. So, inner regularity implies outer regularity of a.

Recall that inner or outer inverses of a given $a \in \mathcal{R}$ do not necessarily exist. Also, neither inner, outer nor reflexive inverses are unique in general.

An element $a \in \mathcal{R}$ is *group invertible* if there exists $x \in \mathcal{R}$ satisfying

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (5)\ ax = xa;$$

such an x is a group inverse of a and it is uniquely determined by these equations, written $x = a^{\#}$. The group inverse $a^{\#}$ double commutes with a, that is, $ax = xa$ implies $a^{\#}x = xa^{\#}$ [1]. Denote by $\mathcal{R}^{\#}$ the set of all group invertible elements of $\mathcal{R}$.

An involution $a \mapsto a^*$ in a ring $\mathcal{R}$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^*a^*.$$

An element $a \in \mathcal{R}$ is *self-adjoint* (or *Hermitian*) if $a^* = a$.

An element $a \in \mathcal{R}$ is *Moore–Penrose invertible* if there exists $x \in \mathcal{R}$ satisfying the Penrose conditions [34]:

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (3)\ (ax)^* = ax, \quad (4)\ (xa)^* = xa;$$

such an x is the uniquely determined *Moore–Penrose inverse* of a denoted by $x = a^\dagger$. The set of all Moore–Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\dagger$.

If $\delta \subset \{1,2,3,4,5\}$ and b satisfies the equations (i) for all $i \in \delta$, then b is a δ–inverse of a. The set of all δ–inverses of a is denote by $a\{\delta\}$. Notice that $a\{1,2,5\} = \{a^{\#}\}$ and $a\{1,2,3,4\} = \{a^{\dagger}\}$. If a is invertible, then $a^{\#}$ and $a^{\dagger}$ each coincide with the ordinary inverse of a. The set of all invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^{-1}$.

For $a \in \mathcal{R}$, we define the following kernel ideals

$$a^{\circ} = \{x \in \mathcal{R} : ax = 0\}, \quad {}^{\circ}a = \{x \in \mathcal{R} : xa = 0\},$$

and image ideals

$$a\mathcal{R} = \{ax : x \in \mathcal{R}\}, \quad \mathcal{R}a = \{xa : x \in \mathcal{R}\}.$$

In the theory of generalized inverses, one of fundamental procedures is to find generalized inverses of products. If $a, b \in \mathcal{R}$ are invertible, then ab is also invertible, and the inverse of the product ab satisfies $(ab)^{-1} = b^{-1}a^{-1}$. This equality is called the reverse order law, and it can be used to simplify various expressions that involve inverses of products. Since this formula cannot trivially be extended to various generalized inverses of the product ab, the reverse order law for various generalized inverses yields a class of interesting problems that are fundamental in the theory of generalized inverses. Many authors studied these problems [1–3, 5–11, 13–15, 17, 20–23, 29, 33, 35, 41, 42].

Greville [14] proved that $(ab)^{\dagger} = b^{\dagger}a^{\dagger}$ holds for complex matrices, if and only if: $a^{\dagger}a$ commutes with bb^* and $bb^{\dagger}$ commutes with a^*a. In the case of linear bounded operators on Hilbert spaces, the analogous result was proved by Izumino [15]. The corresponding result in rings with involution was proved in [16].

Deng [5] presented some necessary and sufficient conditions concerning the reverse order law $(ab)^{\#} = b^{\#}a^{\#}$ for the group invertible linear bounded operators a and b on Hilbert space. He used the matrix form of operators induced by some natural decomposition of Hilbert spaces. In [28], the results from [5] were extended to more general setting of rings, giving some new conditions and providing simpler and more transparent proofs to already existing conditions. Mary [19] considered the reverse order law for the group inverse in semigroups and rings. Using the Drazin inverse, some equivalences of the reverse order law for the group inverse in rings were proved in [18]. These papers consider the group inverse because it is useful in several applications such as in the analysis of Markov chains.

Some necessary and sufficient conditions for the hybrid reverse order law $(ab)^{\#} = b^{\dagger}a^{\dagger}$ in rings with involution were studied in [31]. Also, several conditions equivalent to $(ab)^{\#} = b^{\dagger}a^{\dagger} = (ab)^{\dagger}$ (that is, the product of elements a and b be EP) were given in [31]. The rules $(ab)^{\#} = b^{\dagger}a^{\#}$, $(ab)^{\#} = b^{\#}a^{\dagger}$, $(ab)^{\#} = b^{\#}a^*$ and $(ab)^{\#} = b^*a^{\#}$ were characterized in [27].

If a and b are any matrices such that the product ab is defined, Cline [4] has developed a representation for the Moore–Penrose inverse of the product of ab, as follows: $(ab)^{\dagger} = (a^{\dagger}ab)^{\dagger}(abb^{\dagger})^{\dagger}$. In [40], Xiong and Qin generalized this result to the case of the weighted Moore–Penrose inverse.

The mixed-type reverse-order laws for matrix product ab like $(ab)^{\dagger} = (a^{\dagger}ab)^{\dagger}a^{\dagger}$, $(ab)^{\dagger} = b^{\dagger}(abb^{\dagger})^{\dagger}$, $(ab)^{\dagger} = (a^*ab)^{\dagger}a^*$, $(ab)^{\dagger} = b^*(abb^*)^{\dagger}$ have also been considered, see [36–38]. Using the setting of rings with involution these rules were investigated in [26].

The equations $(ab)^\# = (a^\dagger ab)^\dagger a^\dagger$, $(ab)^\# = (a^* ab)^\dagger a^*$, $(ab)^\# = b^\dagger (abb^\dagger)^\dagger$ and $(ab)^\# = b^*(abb^*)^\dagger$ were characterized in [31] for elements of a ring with involution. Assuming that a is Moore-Penrose invertible, and that b is group invertible, the reverse order laws $(ab)^\# = (a^\dagger ab)^\# a^\dagger$, $(ab)^\# = (a^* ab)^\# a^*$, $(a^\dagger ab)^\# = b^\# a^\dagger a$, $(a^* ab)^\# = b^\# a^\dagger a$, $(a^\dagger ab)^\# a^\dagger = b^\# a^\dagger$ and $(a^* ab)^\# a^* = b^\# a^*$ were studied in [27].

The reverse-order law $(ab)^\dagger = b^\dagger (a^\dagger abb^\dagger)^\dagger a^\dagger$ was first studied by Galperin and Waksman [12]. The results concerning the reverse order law $(ab)^\dagger = b^\dagger (a^\dagger abb^\dagger)^\dagger a^\dagger$ for complex matrices appeared in Tian's paper [39] and for elements of a ring with involution in [24]. A natural consideration is to see what will be obtained if we replace the Moore-Penrose inverse by the group inverses.

In [25], the equivalent conditions for $(ab)^\# = b^\# (a^\# abb^\#)^\# a^\#$ to hold in rings were investigated. Necessary and sufficient conditions for $(ab)^\# = b^\# (a^\dagger abb^\#)^\# a^\dagger$ and $(ab)^\# = b^\dagger (a^\# abb^\dagger)^\# a^\#$ to be satisfied in rings with involution are presented in [30]. Also, some equivalent conditions concerning the reverse order laws $(ab)^\# = b^\dagger a^\#$ and $(ab)^\# = b^\# a^\dagger$ are studied.

In [32], some necessary and sufficient conditions for the reverse order law $(ab)^\# = b^\dagger (a^\dagger abb^\dagger)^\dagger a^\dagger$ in rings with involution were presented. Also the equivalent conditions involving $a^\dagger abb^\dagger \in \mathcal{R}^\dagger$ to ensure that $(ab)^\# = b^\dagger a^\dagger$ is satisfied were studied.

In this paper we characterize various types of the reverse order laws for the group inverses in rings and we obtain the equivalences between some of them. Thus, we generalize some recent results replacing the assumptions a and b are group invertible or Moore-Penrose invertible with weaker conditions that $a^{(\circ)} \in a\{1\} \cup a\{2\}$ or $b^{(\circ)} \in b\{1\} \cup b\{2\}$ exist.

This work is organized as follows. In Section 2, firstly we consider the reverse order laws $(a^{(\circ)}ab)^\# = b^\# a^{(\circ)} a$ (or $(abb^{(\circ)})^\# = bb^{(\circ)} a^\#$) in the cases that $a^{(\circ)} \in a\{1\} \cup a\{2\}$ and $b \in \mathcal{R}^\#$ (or $a \in \mathcal{R}^\#$ and $b^{(\circ)} \in b\{1\} \cup b\{2\}$, respectively); and then we study equivalent conditions for $(ab)^\# = (a^{(1,2)}ab)^\# a^{(1,2)}$ and $(ab)^\# = b^{(1,2)}(abb^{(1,2)})^\#$. When a reflexive generalized inverse of a (b) exists, we present a necessary and sufficient condition for the reverse order law $(ab)^\# = ca^{(1,2)}$ ($(ab)^\# = b^{(1,2)}c$) to hold for arbitrary b (a) and c.

Section 3 contains some characterizations of the rules $(a^{(\circ)}abb^{(\circ)})^\# = bb^{(\circ)}a^{(\circ)}a$ (for $a^{(\circ)} \in a\{1\} \cup a\{2\}$ and $b^{(\circ)} \in b\{1\} \cup b\{2\}$), $(ab)^\# = b^{(1)}(a^{(1,2)}abb^{(1)})^\# a^{(1,2)}$ and $(ab)^\# = b^{(1,2)}(a^{(1)}abb^{(1,2)})^\# a^{(1)}$. We show that the inclusions $(ab)\{5\} \subseteq b^{(1)} \cdot (a^{(1,2)}abb^{(1)})\{1,5\} \cdot a^{(1,2)}$ and $(ab)\{5\} \subseteq b^{(1,2)} \cdot (a^{(1)}abb^{(1,2)})\{1,5\} \cdot a^{(1)}$ automatically imply equality. The equivalent and sufficient conditions which involve $a^{(1,2)}abb^{(1,2)} \in \mathcal{R}^\#$ to ensure that the reverse order law $(ab)^\# = b^{(1,2)}a^{(1,2)}$ is satisfied are given too. We prove that the reverse order laws $(cab)^\# = b^\# (cabb^\#)^\#$ and $(cabb^\#)^\# = b(cab)^\#$ (or $(abc)^\# = (a^\# abc)^\# a^\#$ and $(a^\# abc)^\# = (abc)^\# a$, respectively) are equivalent.

In Section 4 and Section 5, the different kinds of the reverse order laws for group inverses are investigated in a ring with involution. Precisely, in Section 4, first we present equivalent conditions for $(a^* ab)^\# = b^\# a^{(1,4)} a$, $(ab)^\# = (a^* ab)^\# a^*$ and

$(ab)^{\#} = (a^*ab)^{\dagger}a^*$ and then for $(ab)^{\#} = ca^{(1,2,3)}$, $(ab)^{\#} = b^{(1,2,4)}c$, $(ab)^{\#} = ca^*$ and $(ab)^{\#} = b^*c$. Sufficient conditions for the rules $(ab)^{\#} = ca^*$, $(ab)^{\#} = b^*c$ and $(ab)^{\#} = b^*a^* = (ab)^* = (ab)^{\dagger}$ are given too.

In Section 5, the reverse order laws $(a^*abb^{(\circ)})^{\#} = bb^{(\circ)}a^{(1,4)}a$, $(a^{(\circ)}abb^*)^{\#} = bb^{(1,3)}a^{(\circ)}a$, $(ab)^{\#} = b^{(1)}(a^*abb^{(1)})^{\#}a^*$ and $(ab)^{\#} = b^*(a^{(1)}abb^*)^{\#}a^{(1)}$ are characterized. Also, we obtain the conditions under which the rule $(ab)^{\#} = b^{(1,2)}a^*$ and $(ab)^{\#} = b^*a^{(1,2)}$ hold.

2. Reverse order laws in rings

In the first theorem of this section, the characterization for the reverse order law $(a^{(2)}ab)^{\#} = b^{\#}a^{(2)}a$ is presented in a ring.

Theorem 2.1. *If $a \in \mathcal{R}$ has an outer inverse $a^{(2)}$, and if $b, a^{(2)}ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(a^{(2)}ab)^{\#} = b^{\#}a^{(2)}a$,
(ii) $a^{(2)}ab = ba^{(2)}a$.

Proof. (i) $\Rightarrow$ (ii): By the assumption $(a^{(2)}ab)^{\#} = b^{\#}a^{(2)}a$, we get

$$a^{(2)}abb^{\#}a^{(2)}a = b^{\#}a^{(2)}aa^{(2)}ab = b^{\#}a^{(2)}ab,$$

$$b^{\#}a^{(2)}a = b^{\#}a^{(2)}a(a^{(2)}ab)b^{\#}a^{(2)}a = b^{\#}(a^{(2)}abb^{\#}a^{(2)}a) = b^{\#}b^{\#}a^{(2)}ab,$$

which imply

$$ba^{(2)}a = b^2(b^{\#}a^{(2)}a) = b^2b^{\#}b^{\#}a^{(2)}ab = bb^{\#}a^{(2)}ab, \tag{2.1}$$

$$(a^{(2)}abb^{\#}a^{(2)}a)b = b^{\#}a^{(2)}abb = b(b^{\#}b^{\#}a^{(2)}ab)b = bb^{\#}a^{(2)}ab. \tag{2.2}$$

Therefore, the equalities

$$a^{(2)}ab = a^{(2)}ab(a^{(2)}ab)^{\#}a^{(2)}ab = a^{(2)}abb^{\#}a^{(2)}aa^{(2)}ab = a^{(2)}abb^{\#}a^{(2)}ab,$$

(2.2) and (2.1) give that the condition (ii) is satisfied:

$$a^{(2)}ab = bb^{\#}a^{(2)}ab = ba^{(2)}a.$$

(ii) $\Rightarrow$ (i): Since the group inverse $b^{\#}$ double commutes with b, then $a^{(2)}ab = ba^{(2)}a$ yields $a^{(2)}ab^{\#} = b^{\#}a^{(2)}a$. Consequently, we get $b^{\#}a^{(2)}a \in (a^{(2)}ab)\{1, 2, 5\}$. □

The following results can be verified similarly as Theorem 2.1.

Theorem 2.2. *If $a \in \mathcal{R}$ has an inner inverse $a^{(1)}$, and if $b, a^{(1)}ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(a^{(1)}ab)^{\#} = b^{\#}a^{(1)}a$,
(ii) $a^{(1)}ab = ba^{(1)}a$.

Theorem 2.3. *If $b \in \mathcal{R}$ has an outer inverse $b^{(2)}$, and if $a, abb^{(2)} \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(abb^{(2)})^{\#} = bb^{(2)}a^{\#}$,
(ii) $abb^{(2)} = bb^{(2)}a$.

Theorem 2.4. *If $b \in \mathcal{R}$ has an inner inverse $b^{(1)}$, and if $a, abb^{(1)} \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(abb^{(1)})^{\#} = bb^{(1)}a^{\#}$,

(ii) $abb^{(1)} = bb^{(1)}a$.

In the next theorem, we characterize the reverse order law $(a^{(2)}ab)^{\#}a^{(2)} = b^{\#}a^{(2)}$.

Theorem 2.5. *If $a \in \mathcal{R}$ has an outer inverse $a^{(2)}$, and if $b, a^{(2)}ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(a^{(2)}ab)^{\#}a^{(2)} = b^{\#}a^{(2)}$,

(ii) $ba^{(2)}a = a^{(2)}aba^{(2)}a$,

(iii) $ba^{(2)}a\mathcal{R} \subset a^{(2)}\mathcal{R}$ (*or* ${}^{\circ}(a^{(2)}) \subset {}^{\circ}(ba^{(2)}a)$).

Proof. (i) ⇒ (ii): The hypothesis $(a^{(2)}ab)^{\#}a^{(2)} = b^{\#}a^{(2)}$ implies

$$a^{(2)}ab = a^{(2)}ab(a^{(2)}ab)^{\#}a^{(2)}ab = ((a^{(2)}ab)^{\#}a^{(2)})aba^{(2)}ab = b^{\#}a^{(2)}aba^{(2)}ab$$

and

$$b^{\#}a^{(2)} = (a^{(2)}ab)^{\#}a^{(2)} = (a^{(2)}ab)^{\#}a^{(2)}ab(a^{(2)}ab)^{\#}a^{(2)} = b^{\#}a^{(2)}abb^{\#}a^{(2)}.$$

Now, we obtain

$$\begin{aligned}(a^{(2)}ab)a^{(2)}a &= b^{\#}a^{(2)}aba^{(2)}aba^{(2)}a = b^{\#}b(b^{\#}a^{(2)}aba^{(2)}ab)a^{(2)}a \\ &= b^{\#}ba^{(2)}aba^{(2)}a = b(b^{\#}a^{(2)})aba^{(2)}a \\ &= b((a^{(2)}ab)^{\#}a^{(2)}ab)a^{(2)}a = ba^{(2)}ab((a^{(2)}ab)^{\#}a^{(2)})a \\ &= ba^{(2)}abb^{\#}a^{(2)}a,\end{aligned}$$

$$a^{(2)}aba^{(2)}a = ba^{(2)}abb^{\#}a^{(2)}a = b^2(b^{\#}a^{(2)}abb^{\#}a^{(2)})a = b^2b^{\#}a^{(2)}a = ba^{(2)}a.$$

So, the statement (ii) holds.

(ii) ⇒ (i): Assume that $ba^{(2)}a = a^{(2)}aba^{(2)}a$. Then, by

$$(a^{(2)}ab)^{\#} = a^{(2)}ab[(a^{(2)}ab)^{\#}]^2 = a^{(2)}a(a^{(2)}ab[(a^{(2)}ab)^{\#}]^2) = a^{(2)}a(a^{(2)}ab)^{\#},$$

we have

$$\begin{aligned}(a^{(2)}ab)^{\#}a^{(2)} &= (a^{(2)}aba^{(2)}a)b[(a^{(2)}ab)^{\#}]^3a^{(2)} \\ &= ba^{(2)}ab[(a^{(2)}ab)^{\#}]^3a^{(2)} \\ &= bb^{\#}(ba^{(2)}a)b[(a^{(2)}ab)^{\#}]^3a^{(2)} \\ &= bb^{\#}a^{(2)}aba^{(2)}ab[(a^{(2)}ab)^{\#}]^3a^{(2)} \\ &= bb^{\#}a^{(2)}a(a^{(2)}aba^{(2)}ab[(a^{(2)}ab)^{\#}]^3)a^{(2)} \\ &= b^{\#}(ba^{(2)}a)(a^{(2)}ab)^{\#}a^{(2)} \\ &= b^{\#}a^{(2)}ab(a^{(2)}a(a^{(2)}ab)^{\#})a^{(2)} \\ &= b^{\#}a^{(2)}ab(a^{(2)}ab)^{\#}a^{(2)}.\end{aligned} \tag{2.3}$$

From the equalities

$$\begin{aligned} b^{\#}a^{(2)} &= b^{\#}b^{\#}(ba^{(2)}a)a^{(2)} = b^{\#}b^{\#}(a^{(2)}ab)a^{(2)}aa^{(2)} \\ &= b^{\#}b^{\#}a^{(2)}ab((a^{(2)}ab)^{\#}a^{(2)}ab)a^{(2)} \\ &= b^{\#}b^{\#}(a^{(2)}aba^{(2)}a)b(a^{(2)}ab)^{\#}a^{(2)} \\ &= b^{\#}b^{\#}ba^{(2)}ab(a^{(2)}ab)^{\#}a^{(2)} \\ &= b^{\#}a^{(2)}ab(a^{(2)}ab)^{\#}a^{(2)} \end{aligned}$$

and (2.3), we deduce that $(a^{(2)}ab)^{\#}a^{(2)} = b^{\#}a^{(2)}$.

(ii) ⇔ (iii): Clearly, $ba^{(2)}a = a^{(2)}aba^{(2)}a$ gives $ba^{(2)}a\mathcal{R} \subset a^{(2)}\mathcal{R}$. On the other hand, if $ba^{(2)}a\mathcal{R} \subset a^{(2)}\mathcal{R}$, then $ba^{(2)}a = a^{(2)}z$ for some $z \in \mathcal{R}$. Thus, $ba^{(2)}a = a^{(2)}z = a^{(2)}a(a^{(2)}z) = a^{(2)}aba^{(2)}a$. □

In the same way as in Theorem 2.5, we can prove the following theorem.

Theorem 2.6. *If $b \in \mathcal{R}$ has an outer inverse $b^{(2)}$, and if $a, abb^{(2)} \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $b^{(2)}(abb^{(2)})^{\#} = b^{(2)}a^{\#}$,

(ii) $bb^{(2)}a = bb^{(2)}abb^{(2)}$,

(iii) $\mathcal{R}bb^{(2)}a \subset \mathcal{R}b^{(2)}$ (*or* $(b^{(2)})^{\circ} \subset (bb^{(2)}a)^{\circ}$).

Several equivalent conditions for the reverse order law $(ab)^{\#} = (a^{(1,2)}ab)^{\#}a^{(2)}$ are presented in the next result.

Theorem 2.7. *Let $a \in \mathcal{R}$ have a reflexive inverse $a^{(1,2)}$ and let $b \in \mathcal{R}$. If $a^{(1,2)}ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $ab \in \mathcal{R}^{\#}$ *and* $(ab)^{\#} = (a^{(1,2)}ab)^{\#}a^{(1,2)}$,

(ii) $(a^{(1,2)}ab)^{\#}a^{(1,2)} \in (ab)\{5\}$,

(iii) $abaa^{(1,2)} = ab$ *and* $(a^{(1,2)}ab)^{\#}a^{(1,2)}abaa^{(1,2)} = ab(a^{(1,2)}ab)^{\#}a^{(1,2)}$,

(iv) $(a^{(1,2)}ab)\{1,5\} \cdot a^{(1,2)} \subseteq (ab)\{5\}$.

Proof. (i) ⇒ (ii): Clearly.

(ii) ⇒ (iii): Observe that

$$ab(a^{(1,2)}ab)^{\#}a^{(1,2)}ab = a(a^{(1,2)}ab(a^{(1,2)}ab)^{\#}a^{(1,2)}ab) = aa^{(1,2)}ab = ab. \tag{2.4}$$

Now, the condition $(a^{(1,2)}ab)^{\#}a^{(1,2)} \in (ab)\{5\}$ implies

$$\begin{aligned} [abaa^{(1,2)} &= abab(a^{(2)}ab)^{\#}a^{(1,2)}aa^{(1,2)} = abab(a^{(1,2)}ab)^{\#}a^{(1,2)} = ab, \\ (a^{(1,2)}ab)^{\#}a^{(1,2)}(abaa^{(1,2)}) &= (a^{(1,2)}ab)^{\#}a^{(1,2)}ab = ab(a^{(1,2)}ab)^{\#}a^{(1,2)}. \end{aligned}$$

(iii) ⇒ (iv): If $(a^{(1,2)}ab)^{(1,5)} \in (a^{(1,2)}ab)\{1,5\}$, then

$$\begin{aligned} a^{(1,2)}ab(a^{(1,2)}ab)^{(1,5)} &= (a^{(1,2)}ab)^{\#}((a^{(1,2)}ab)^2(a^{(1,2)}ab)^{(1,5)}) \\ &= (a^{(1,2)}ab)^{\#}a^{(1,2)}ab. \end{aligned} \tag{2.5}$$

Using the equalities (2.5) and (iii), we get

$$(a^{(1,2)}ab)^{(1,5)}a^{(1,2)}ab = (a^{(1,2)}ab)^{\#}a^{(1,2)}abaa^{(1,2)} = ab(a^{(1,2)}ab)^{\#}a^{(1,2)}$$

$$= a(a^{(1,2)}ab(a^{(1,2)}ab)^{\#})a^{(1,2)} = ab(a^{(1,2)}ab)^{(1,5)}a^{(1,2)}.$$

So, $(a^{(1,2)}ab)^{(1,5)}a^{(1,2)} \in (ab)\{5\}$ and $(a^{(1,2)}ab)\{1,5\} \cdot a^{(1,2)} \in (ab)\{5\}$.

(iv) $\Rightarrow$ (i): Because $(a^{(1,2)}ab)^{\#} \in (a^{(1,2)}ab)\{1,5\}$ and the statement (iv) holds, we conclude that $(a^{(1,2)}ab)^{\#}a^{(1,2)} \in (ab)\{5\}$. Since the equalities (2.4) and

$$((a^{(1,2)}ab)^{\#}a^{(1,2)}ab(a^{(1,2)}ab)^{\#})a^{(1,2)} = (a^{(1,2)}ab)^{\#}a^{(1,2)}$$

hold, we have $(a^{(1,2)}ab)^{\#}a^{(1,2)} \in (ab)\{1,2\}$. Thus, the item (i) is satisfied. □

As Theorem 2.7, we get the following theorem related to $(ab)^{\#} = b^{(1,2)}(abb^{(1,2)})^{\#}$.

Theorem 2.8. *Let $b \in \mathcal{R}$ have a reflexive inverse $b^{(1,2)}$ and let $a \in \mathcal{R}$. If $abb^{(1,2)} \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) *$ab \in \mathcal{R}^{\#}$ and $(ab)^{\#} = b^{(1,2)}(abb^{(1,2)})^{\#}$,*
(ii) *$b^{(1,2)}(abb^{(1,2)})^{\#} \in (ab)\{5\}$,*
(iii) *$b^{(1,2)}bab = ab$ and $b^{(1,2)}babb^{(1,2)}(abb^{(1,2)})^{\#} = b^{(1,2)}(abb^{(1,2)})^{\#}ab$,*
(iv) *$b^{(1,2)} \cdot (abb^{(1,2)})\{1,5\} \subseteq (ab)\{5\}$.*

If we suppose that a has a reflexive inverse $a^{(1,2)}$, combining the conditions of Theorem 2.5 and Theorem 2.7, we get sufficient conditions for the reverse order law $(ab)^{\#} = b^{\#}a^{(1,2)}$ to hold. Similarly, Theorem 2.6 and Theorem 2.8 give sufficient conditions for $(ab)^{\#} = b^{(1,2)}a^{\#}$.

We check that $(ab)\{5\} \subseteq (a^{(1,2)}ab)\{1,5\} \cdot a^{(1,2)}$ is equivalent to $(ab)\{5\} = (a^{(1,2)}ab)\{1,5\} \cdot a^{(1,2)}$.

Theorem 2.9. *Let $a \in \mathcal{R}$ have a reflexive inverse $a^{(1,2)}$ and let $b \in \mathcal{R}$. If $ab, a^{(1,2)}ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) *$(ab)\{5\} \subseteq (a^{(1,2)}ab)\{1,5\} \cdot a^{(1,2)}$,*
(ii) *$(ab)\{5\} = (a^{(1,2)}ab)\{1,5\} \cdot a^{(1,2)}$.*

Proof. (i) $\Rightarrow$ (ii): Since $(ab)\{5\} \subseteq (a^{(1,2)}ab)\{1,5\} \cdot a^{(1,2)}$ and $(ab)^{\#} \in (ab)\{5\}$, there exists $(a^{(1,2)}ab)^{(1,5)} \in (a^{(1,2)}ab)\{1,5\}$ such that $(ab)^{\#} = (a^{(1,2)}ab)^{(1,5)}a^{(1,2)}$. The equalities (2.5) hold again and give

$$\begin{aligned}(a^{(1,2)}ab)^{\#}a^{(1,2)} &= (a^{(1,2)}ab)^{\#}a^{(1,2)}ab(a^{(1,2)}ab)^{\#}a^{(1,2)} \\ &= (a^{(1,2)}ab)^{(1,5)}a^{(1,2)}ab(a^{(1,2)}ab)^{(1,5)}a^{(1,2)} \\ &= (ab)^{\#}ab(ab)^{\#} = (ab)^{\#}.\end{aligned}$$

Using Theorem 2.7, notice that $(a^{(1,2)}ab)\{1,5\} \cdot a^{(1,2)} \subseteq (ab)\{5\}$. Therefore, the condition (ii) holds.

(ii) $\Rightarrow$ (i): This is obvious. □

Analogously to Theorem 2.9, we obtain the following theorem.

Theorem 2.10. *Let $b \in \mathcal{R}$ have a reflexive inverse $b^{(1,2)}$ and let $a \in \mathcal{R}$. If $ab, abb^{(1,2)} \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) *$(ab)\{5\} \subseteq b^{(1,2)} \cdot (abb^{(1,2)})\{1,5\}$,*
(ii) *$(ab)\{5\} = b^{(1,2)} \cdot (abb^{(1,2)})\{1,5\}$.*

Now, we give a necessary and sufficient condition which ensures that $(ab)^{\#} = ca^{(1,2)}$ holds.

Theorem 2.11. *Let $a \in \mathcal{R}$ have a reflexive inverse $a^{(1,2)}$ and let $b, c \in \mathcal{R}$. If $ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(ab)^{\#} = ca^{(1,2)}$,

(ii) $(ab)^{\#}a = ca^{(1,2)}a$ *and* $a^{(1,2)}ab = a^{(1,2)}abaa^{(1,2)}$.

Proof. (i) $\Rightarrow$ (ii): The assumption $(ab)^{\#} = ca^{(1,2)}$ gives $(ab)^{\#}a = ca^{(1,2)}a$ and $ab = abab(ab)^{\#} = ababca^{(1,2)}$. Thus,

$$a^{(1,2)}ab = a^{(1,2)}ababca^{(1,2)} = a^{(1,2)}(ababca^{(1,2)})aa^{(1,2)} = a^{(1,2)}abaa^{(1,2)}.$$

(ii) $\Rightarrow$ (i): Suppose that $(ab)^{\#}a = ca^{(1,2)}a$ and $a^{(1,2)}ab = a^{(1,2)}abaa^{(1,2)}$. Observe that $ca^{(1,2)} \in (ab)\{1,2\}$, by $ab(ca^{(1,2)}a)b = ab(ab)^{\#}ab = ab$ and

$$\begin{aligned} ca^{(1,2)}abca^{(1,2)} &= (ca^{(1,2)}a)b(ca^{(1,2)}a)a^{(1,2)} = (ab)^{\#}ab(ab)^{\#}aa^{(1,2)} \\ &= ((ab)^{\#}a)a^{(1,2)} = ca^{(1,2)}aa^{(1,2)} = ca^{(1,2)}. \end{aligned}$$

Further, from

$$\begin{aligned} (ca^{(1,2)}a)b &= (ab)^{\#}ab = (ab)^{\#}a(a^{(1,2)}ab) \\ &= (ab)^{\#}aa^{(1,2)}abaa^{(1,2)} = ((ab)^{\#}ab)aa^{(1,2)} \\ &= ab((ab)^{\#}a)a^{(1,2)} = abca^{(1,2)}aa^{(1,2)} \\ &= abca^{(1,2)}, \end{aligned}$$

we conclude that $ca^{(1,2)} \in (ab)\{5\}$ and (i) is satisfied. □

The next result related to the reverse order law $(ab)^{\#} = b^{(1,2)}c$ can be verified in the same manner as Theorem 2.11.

Theorem 2.12. *Let $b \in \mathcal{R}$ have a reflexive inverse $b^{(1,2)}$ and let $a, c \in \mathcal{R}$. If $ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(ab)^{\#} = b^{(1,2)}c$,

(ii) $b(ab)^{\#} = bb^{(1,2)}c$ *and* $abb^{(1,2)} = b^{(1,2)}babb^{(1,2)}$.

In the following theorem, sufficient conditions for the reverse order law $(ab)^{\#} = ca^{(\circ)}$ are presented in the cases that $a^{(\circ)} \in \{a^{(1,2)}, a^{(2,5)}, a^{(2)}\}$.

Theorem 2.13. *Let $a, b, c \in \mathcal{R}$ and $ab \in \mathcal{R}^{\#}$. Then the following statements hold:*

(i) *If a has a reflexive inverse $a^{(1,2)}$, $(ab)^{\#}a = ca^{(1,2)}a$ and $a^{(1,2)}ab = baa^{(1,2)}$, then $(ab)^{\#} = ca^{(1,2)}$.*

(ii) *If $a^{(2,5)} \in a\{2,5\}$, $a^{(2,5)}ab \in \mathcal{R}^{\#}$ and $(ab)^{\#}a = ca^{(2,5)}a = (a^{(2,5)}ab)^{\#}$, then $(ab)^{\#} = ca^{(2,5)}$.*

(iii) *If a has an outer inverse $a^{(2)}$, $a^{(2)}ab \in \mathcal{R}^{\#}$, $(ab)^{\#} = (a^{(2)}ab)^{\#}a^{(2)}$ and $(a^{(2)}ab)^{\#} = ca^{(2)}a$, then $(ab)^{\#} = ca^{(2)}$.*

Proof. (i) The assumptions $(ab)^\# a = ca^{(1,2)}a$ and $a^{(1,2)}ab = baa^{(1,2)}$ imply

$$\begin{aligned}abca^{(1,2)} &= (ab(ab)^\#)abca^{(1,2)} = ((ab)^\# a)babca^{(1,2)}\\ &= ca^{(1,2)}abab(ca^{(1,2)}a)a^{(1,2)}\\ &= ca^{(1,2)}(abab(ab)^\#)aa^{(1,2)}\\ &= ca^{(1,2)}a(baa^{(1,2)}) = ca^{(1,2)}aa^{(1,2)}ab\\ &= ca^{(1,2)}ab,\end{aligned}$$

i.e. $ca^{(1,2)} \in (ab)\{5\}$. Also, by $ab(ca^{(1,2)}a)b = ab(ab)^\# ab = ab$ and

$$\begin{aligned}ca^{(1,2)}abca^{(1,2)} &= (ca^{(1,2)}a)a^{(1,2)}ab(ca^{(1,2)}a)a^{(1,2)}\\ &= (ab)^\# aa^{(1,2)}ab(ab)^\# aa^{(1,2)}\\ &= ((ab)^\# a)a^{(1,2)} = ca^{(1,2)}aa^{(1,2)} = ca^{(1,2)},\end{aligned}$$

we deduce that $ca^{(1,2)} \in (ab)\{1,2\}$. So, the reverse order law $(ab)^\# = ca^{(1,2)}$ is satisfied.

(ii) First, from the equalities $(ab)^\# a = ca^{(2,5)}a = (a^{(2,5)}ab)^\#$, we have $ca^{(2,5)} \in (ab)\{1,2\}$:

$$ab(ca^{(2,5)}a)b = ab(ab)^\# ab = ab$$

and

$$\begin{aligned}ca^{(2,5)}abca^{(2,5)} &= (ca^{(2,5)}a)b(ca^{(2,5)}a)a^{(2,5)} = (ab)^\# ab(ab)^\# aa^{(2,5)}\\ &= ((ab)^\# a)a^{(2,5)} = ca^{(2,5)}aa^{(2,5)} = ca^{(2,5)}.\end{aligned}$$

Further, by $ca^{(2,5)}a = (a^{(2,5)}ab)^\#$, we conclude that

$$a^{(2,5)}abca^{(2,5)}a = ca^{(2,5)}aa^{(2,5)}ab = ca^{(2,5)}ab$$

which yields

$$\begin{aligned}abca^{(2,5)} &= (ab(ab)^\#)abca^{(2,5)} = ((ab)^\# a)babca^{(2,5)}\\ &= ca^{(2,5)}abab(ca^{(2,5)}a)a^{(2,5)} = ca^{(2,5)}abab(ab)^\# aa^{(2,5)}\\ &= (ca^{(2,5)}ab)aa^{(2,5)} = a^{(2,5)}abca^{(2,5)}aa^{(2,5)}a\\ &= a^{(2,5)}abca^{(2,5)}a = ca^{(2,5)}ab.\end{aligned}$$

Therefore, $ca^{(2,5)} \in (ab)\{5\}$ and $(ab)^\# = ca^{(2,5)}$ is satisfied.

(iii) Applying $(ab)^\# = (a^{(2)}ab)^\# a^{(2)}$ and $(a^{(2)}ab)^\# = ca^{(2)}a$, we obtain

$$(ab)^\# = (a^{(2)}ab)^\# a^{(2)} = ca^{(2)}aa^{(2)} = ca^{(2)}. \qquad \square$$

The next theorem can be proved in the similar way as Theorem 2.13.

Theorem 2.14. *Let $a, b, c \in \mathcal{R}$ and $ab \in \mathcal{R}^\#$. Then the following statements hold:*

(i) *If b has a reflexive inverse $b^{(1,2)}$, $b(ab)^\# = bb^{(1,2)}c$ and $b^{(1,2)}ba = abb^{(1,2)}$, then $(ab)^\# = b^{(1,2)}c$.*

(ii) *If $b^{(2,5)} \in b\{2,5\}$, $abb^{(2,5)} \in \mathcal{R}^\#$ and $b(ab)^\# = bb^{(2,5)}c = (abb^{(2,5)})^\#$, then $(ab)^\# = b^{(2,5)}c$.*

(iii) *If b has an outer inverse $b^{(2)}$, $abb^{(2)} \in \mathcal{R}^{\#}$, $(ab)^{\#} = b^{(2)}(abb^{(2)})^{\#}$ and $(abb^{(2)})^{\#} = bb^{(2)}c$, then $(ab)^{\#} = b^{(2)}c$.*

3. Reverse order laws involving $(a^{(\circ)}abb^{(\circ)})^{\#}$

Under the assumptions $a^{(\circ)} \in a\{1\} \cup a\{2\}$ and $b^{(\circ)} \in b\{1\} \cup b\{2\}$, the reverse order law $(a^{(\circ)}abb^{(\circ)})^{\#} = bb^{(\circ)}a^{(\circ)}a$ is studied in the following result.

Theorem 3.1. *Let $a, b \in \mathcal{R}$. If $a^{(\circ)} \in a\{1\} \cup a\{2\}$, $b^{(\circ)} \in b\{1\} \cup b\{2\}$ and $a^{(\circ)}abb^{(\circ)} \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(a^{(\circ)}abb^{(\circ)})^{\#} = bb^{(\circ)}a^{(\circ)}a$,

(ii) $a^{(\circ)}abb^{(\circ)} = bb^{(\circ)}a^{(\circ)}a$.

Proof. (i) ⇒ (ii): Since $(a^{(\circ)}abb^{(\circ)})^{\#} = bb^{(\circ)}a^{(\circ)}a$, we get

$$\begin{aligned} a^{(\circ)}abb^{(\circ)}a^{(\circ)}a &= a^{(\circ)}abb^{(\circ)}bb^{(\circ)}a^{(\circ)}a = bb^{(\circ)}a^{(\circ)}aa^{(\circ)}abb^{(\circ)} \\ &= bb^{(\circ)}a^{(\circ)}abb^{(\circ)}, \end{aligned}$$

which implies

$$\begin{aligned} a^{(\circ)}abb^{(\circ)} &= a^{(\circ)}abb^{(\circ)}(bb^{(\circ)}a^{(\circ)}a)a^{(\circ)}abb^{(\circ)} = a^{(\circ)}a(bb^{(\circ)}a^{(\circ)}abb^{(\circ)}) \\ &= a^{(\circ)}aa^{(\circ)}abb^{(\circ)}a^{(\circ)}a = a^{(\circ)}abb^{(\circ)}a^{(\circ)}a \\ bb^{(\circ)}a^{(\circ)}a &= bb^{(\circ)}a^{(\circ)}a(a^{(\circ)}abb^{(\circ)})bb^{(\circ)}a^{(\circ)}a = (bb^{(\circ)}a^{(\circ)}abb^{(\circ)})a^{(\circ)}a \\ &= a^{(\circ)}abb^{(\circ)}a^{(\circ)}aa^{(\circ)}a = a^{(\circ)}abb^{(\circ)}a^{(\circ)}a. \end{aligned}$$

Thus, $a^{(\circ)}abb^{(\circ)} = bb^{(\circ)}a^{(\circ)}a$ and the statement (ii) is satisfied.

(ii) ⇒ (i): By the hypothesis $a^{(\circ)}abb^{(\circ)} = bb^{(\circ)}a^{(\circ)}a$, observe that

$$(a^{(\circ)}abb^{(\circ)})(bb^{(\circ)}a^{(\circ)}a) = bb^{(\circ)}a^{(\circ)}aa^{(\circ)}abb^{(\circ)}.$$

Hence, $bb^{(\circ)}a^{(\circ)}a \in (a^{(\circ)}abb^{(\circ)})\{5\}$. We verify that $bb^{(\circ)}a^{(\circ)}a \in (a^{(\circ)}abb^{(\circ)})\{1,2\}$. So, the condition (i) holds. □

In the next theorem, we investigate the characterizations for the reverse order law $(ab)^{\#} = b^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)}$ in a ring.

Theorem 3.2. *Let $a \in \mathcal{R}$ have a reflexive inverse $a^{(1,2)}$ and let $b \in \mathcal{R}$ have an inner inverse $b^{(1)}$. If $a^{(1,2)}abb^{(1)} \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $ab \in \mathcal{R}^{\#}$ *and* $(ab)^{\#} = b^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)}$,

(ii) $b^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)} \in (ab)\{5\}$,

(iii) $abaa^{(1,2)} = ab$ *and*
$b^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)}abaa^{(1,2)} = abb^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)}$,

(iv) $b^{(1)} \cdot (a^{(1,2)}abb^{(1)})\{1,5\} \cdot a^{(1,2)} \subseteq (ab)\{5\}$.

Proof. (i) ⇒ (ii): This is trivial.

(ii) ⇒ (iii): The assumption $b^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)} \in (ab)\{5\}$ gives

$$abb^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)} = b^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)}ab.$$

From

$$abb^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)}ab = a(a^{(1,2)}abb^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)}abb^{(1)})b = aa^{(1,2)}abb^{(1)}b = ab, \tag{3.1}$$

we conclude that $b^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)} \in (ab)\{1\}$. Now, we obtain

$$abaa^{(1,2)} = ababb^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)}aa^{(1,2)} = ababb^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)} = ab$$

$$b^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)}(abaa^{(1,2)}) = b^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)}ab = abb^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)},$$

i.e. the statements (iii) is satisfed.

(iii) ⇒ (i): Suppose that $abaa^{(1,2)} = ab$ and $b^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)}abaa^{(1,2)} = abb^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)}$. Then we get $b^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)} \in (ab)\{1,5\}$, by (3.1) and

$$b^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)}(ab) = b^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)}abaa^{(1,2)} = abb^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)}.$$

Since

$$b^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)}abb^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)} = b^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)},$$

we deduce that $b^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)} \in (ab)\{2\}$. Hence, the condition (i) holds.

(ii) ⇒ (iv): Let $b^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)} \in (ab)\{5\}$ and let $(a^{(1,2)}abb^{(1)})^{(1,5)} \in (a^{(1,2)}abb^{(1)})\{1,5\}$. Then

$$\begin{aligned} a^{(1,2)}abb^{(1)}(a^{(1,2)}abb^{(1)})^{(1,5)} &= (a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)}abb^{(1)} \\ &\quad \times (a^{(1,2)}abb^{(1)}(a^{(1,2)}abb^{(1)})^{(1,5)}) \\ &= (a^{(1,2)}abb^{(1)})^{\#} \\ &\quad \times (a^{(1,2)}abb^{(1)}(a^{(1,2)}abb^{(1)})^{(1,5)}a^{(1,2)}abb^{(1)}) \\ &= (a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)}abb^{(1)}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} abb^{(1)}(a^{(1,2)}abb^{(1)})^{(1,5)}a^{(1,2)} &= a(a^{(1,2)}abb^{(1)}(a^{(1,2)}abb^{(1)})^{(1,5)})a^{(1,2)} \\ &= aa^{(1,2)}abb^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)} \\ &= abb^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)} \\ &= b^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)}ab \\ &= b^{(1)}((a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)}abb^{(1)})b \\ &= b^{(1)}(a^{(1,2)}abb^{(1)})^{(1,5)}a^{(1,2)}abb^{(1)}b \\ &= b^{(1)}(a^{(1,2)}abb^{(1)})^{(1,5)}a^{(1,2)}ab. \end{aligned}$$

Hence, we conclude that $b^{(1)}(a^{(1,2)}abb^{(1)})^{(1,5)}a^{(1,2)} \in (ab)\{5\}$. Since we prove that $b^{(1)}(a^{(1,2)}abb^{(1)})^{(1,5)}a^{(1,2)} \in (ab)\{5\}$, for any $(a^{(1,2)}abb^{(1)})^{(1,5)} \in (a^{(1,2)}abb^{(1)})\{1,5\}$, we deduce that the item (iv) holds.

(iv) ⇒ (ii): Consequently, because $(a^{(1,2)}abb^{(1)})^{\#} \in (a^{(1,2)}abb^{(1)})\{1,5\}$. □

In the similar manner as Theorem 3.2, we can check the following theorem considering the rule $(ab)^{\#} = b^{(1,2)}(a^{(1)}abb^{(1,2)})^{\#}a^{(1)}$.

Theorem 3.3. *Let $a \in \mathcal{R}$ have an inner inverse $a^{(1)}$ and let $b \in \mathcal{R}$ have a reflexive inverse $b^{(1,2)}$. If $a^{(1)}abb^{(1,2)} \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) *$ab \in \mathcal{R}^{\#}$ and $(ab)^{\#} = b^{(1,2)}(a^{(1)}abb^{(1,2)})^{\#}a^{(1)}$,*
(ii) *$b^{(1,2)}(a^{(1)}abb^{(1,2)})^{\#}a^{(1)} \in (ab)\{5\}$,*
(iii) *$b^{(1,2)}bab = ab$ and*
$b^{(1,2)}babb^{(1,2)}(a^{(1)}abb^{(1,2)})^{\#}a^{(1)} = b^{(1,2)}(a^{(1)}abb^{(1,2)})^{\#}a^{(1)}ab$,
(iv) *$b^{(1,2)} \cdot (a^{(1)}abb^{(1,2)})\{1,5\} \cdot a^{(1)} \subseteq (ab)\{5\}$.*

We will show that the inclusion $(ab)\{5\} \subseteq b^{(1)} \cdot (a^{(1,2)}abb^{(1)})\{1,5\} \cdot a^{(1,2)}$ automatically implies equality in the next result.

Theorem 3.4. *Let $a \in \mathcal{R}$ have a reflexive inverse $a^{(1,2)}$ and let $b \in \mathcal{R}$ have an inner inverse $b^{(1)}$. If $ab, a^{(1,2)}abb^{(1)} \in \mathcal{R}^{\#}$, then the inclusion*

$$(ab)\{5\} \subseteq b^{(1)} \cdot (a^{(1,2)}abb^{(1)})\{1,5\} \cdot a^{(1,2)}$$

is always an equality.

Proof. Suppose that $(ab)\{5\} \subseteq b^{(1)} \cdot (a^{(1,2)}abb^{(1)})\{1,5\} \cdot a^{(1,2)}$. Because $(ab)^{\#} \in (ab)\{5\}$, then there exists $(a^{(1,2)}abb^{(1)})^{(1,5)} \in (a^{(1,2)}abb^{(1)})\{1,5\}$ satisfying $(ab)^{\#} = b^{(1)}(a^{(1,2)}abb^{(1)})^{(1,5)}a^{(1,2)}$. Applying the equalities (3.2), we obtain

$$\begin{aligned} b^{(1)}(a^{(1,2)}abb^{(1)})^{\#}a^{(1,2)} &= b^{(1)}(a^{(1,2)}abb^{(1)})^{(1,5)}a^{(1,2)}abb^{(1)}(a^{(1,2)}abb^{(1)})^{(1,5)}a^{(1,2)} \\ &= (b^{(1)}(a^{(1,2)}abb^{(1)})^{(1,5)}a^{(1,2)})ab \\ &\quad \times (b^{(1)}(a^{(1,2)}abb^{(1)})^{(1,5)}a^{(1,2)}) \\ &= (ab)^{\#}ab(ab)^{\#} \\ &= (ab)^{\#}. \end{aligned}$$

Using Theorem 3.2, we have that $b^{(1)} \cdot (a^{(1,2)}abb^{(1)})\{1,5\} \cdot a^{(1,2)} \subseteq (ab)\{5\}$ which implies $(ab)\{5\} = b^{(1)} \cdot (a^{(1,2)}abb^{(1)})\{1,5\} \cdot a^{(1,2)}$. □

Similarly as Theorem 3.4, we can prove the following result.

Theorem 3.5. *Let $a \in \mathcal{R}$ have an inner inverse $a^{(1)}$ and let $b \in \mathcal{R}$ have a reflexive inverse $b^{(1,2)}$. If $ab, a^{(1)}abb^{(1,2)} \in \mathcal{R}^{\#}$, then the inclusion*

$$(ab)\{5\} \subseteq b^{(1,2)} \cdot (a^{(1)}abb^{(1,2)})\{1,5\} \cdot a^{(1)}$$

is always an equality.

Some equivalent condition for $b^{(2)}(a^{(2)}abb^{(2)})^{\#}a^{(2)} = b^{(2)}a^{(2)}$ to be satisfied are presented in the following theorem.

Theorem 3.6. *Let $a \in \mathcal{R}$ have an outer inverse $a^{(2)}$ and let $b \in \mathcal{R}$ have an outer inverse $b^{(2)}$. If $a^{(2)}abb^{(2)} \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $b^{(2)}(a^{(2)}abb^{(2)})^{\#}a^{(2)} = b^{(2)}a^{(2)}$,

(ii) $b^{(2)}a^{(2)}abb^{(2)}a^{(2)} = b^{(2)}a^{(2)}$,

(iii) $bb^{(2)}a^{(2)}a$ *is an idempotent.*

Proof. (i) ⇒ (ii): If we assume that $b^{(2)}(a^{(2)}abb^{(2)})^{\#}a^{(2)} = b^{(2)}a^{(2)}$, then we get

$$\begin{aligned}(b^{(2)}a^{(2)})ab(b^{(2)}a^{(2)}) &= b^{(2)}(a^{(2)}abb^{(2)})^{\#}a^{(2)}abb^{(2)}(a^{(2)}abb^{(2)})^{\#}a^{(2)} \\ &= b^{(2)}(a^{(2)}abb^{(2)})^{\#}a^{(2)} = b^{(2)}a^{(2)}.\end{aligned}$$

(ii) ⇒ (iii): The hypothesis $b^{(2)}a^{(2)}abb^{(2)}a^{(2)} = b^{(2)}a^{(2)}$ gives

$$bb^{(2)}a^{(2)}abb^{(2)}a^{(2)}a = bb^{(2)}a^{(2)}a,$$

that is, the condition (iii) is satisfied.

(iii) ⇒ (i): Let $bb^{(2)}a^{(2)}a$ be an idempotent. Now, using

$$\begin{aligned}a^{(2)}a(a^{(2)}abb^{(2)})^{\#} &= a^{(2)}aa^{(2)}abb^{(2)}[(a^{(2)}abb^{(2)})^{\#}]^2 \\ &= a^{(2)}abb^{(2)}[(a^{(2)}abb^{(2)})^{\#}]^2 \\ &= (a^{(2)}abb^{(2)})^{\#},\end{aligned}$$

we have that

$$\begin{aligned}b^{(2)}a^{(2)} &= b^{(2)}(bb^{(2)}a^{(2)}a)a^{(2)} = b^{(2)}(bb^{(2)}a^{(2)}a)^2a^{(2)} \\ &= b^{(2)}(a^{(2)}abb^{(2)})a^{(2)} \\ &= b^{(2)}(bb^{(2)}a^{(2)}abb^{(2)}a^{(2)}a)bb^{(2)}(a^{(2)}abb^{(2)})^{\#}a^{(2)} \\ &= b^{(2)}(bb^{(2)}a^{(2)}abb^{(2)}a^{(2)}a)(a^{(2)}abb^{(2)})^{\#}a^{(2)} \\ &= b^{(2)}(a^{(2)}a(a^{(2)}abb^{(2)})^{\#})a^{(2)} \\ &= b^{(2)}(a^{(2)}abb^{(2)})^{\#}a^{(2)}.\end{aligned}$$ □

In the analogy way as in the proof of Theorem 3.6, we obtain necessary and sufficient conditions for $a(bb^{(1)}a^{(1)}a)^{\#}b = ab$ to hold.

Theorem 3.7. *Let $a \in \mathcal{R}$ have an inner inverse $a^{(1)}$ and let $b \in \mathcal{R}$ have an inner inverse $b^{(1)}$. If $bb^{(1)}a^{(1)}a \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $a(bb^{(1)}a^{(1)}a)^{\#}b = ab$,

(ii) $abb^{(1)}a^{(1)}ab = ab$,

(iii) $a^{(1)}abb^{(1)}$ *is an idempotent.*

Using Theorem 3.6 and Theorem 3.7, we get the next consequence.

Corollary 3.1. *Let $a \in \mathcal{R}$ have a reflexive inverse $a^{(1,2)}$ and let $b \in \mathcal{R}$ have a reflexive inverse $b^{(1,2)}$. If $a^{(1,2)}abb^{(1,2)}, bb^{(1,2)}a^{(1,2)}a \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $b^{(1,2)}(a^{(1,2)}abb^{(1,2)})^{\#}a^{(1,2)} = b^{(1,2)}a^{(1,2)}$,

(ii) $b^{(1,2)}a^{(1,2)}abb^{(1,2)}a^{(1,2)} = b^{(1,2)}a^{(1,2)}$,

(iii) $bb^{(1,2)}a^{(1,2)}a$ *is an idempotent.*
(iv) $a(bb^{(1,2)}a^{(1,2)}a)^{\#}b = ab$,
(v) $abb^{(1,2)}a^{(1,2)}ab = ab$,
(vi) $a^{(1,2)}abb^{(1,2)}$ *is an idempotent.*

Proof. The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) follow by Theorem 3.6, and the equivalences (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) follow by Theorem 3.7.

(i) $\Rightarrow$ (vi): The condition $b^{(1,2)}(a^{(1,2)}abb^{(1,2)})^{\#}a^{(1,2)} = b^{(1,2)}a^{(1,2)}$ gives

$$\begin{aligned} a^{(1,2)}ab(b^{(1,2)}a^{(1,2)})abb^{(1,2)} &= a^{(1,2)}abb^{(1,2)}(a^{(1,2)}abb^{(1,2)})^{\#}a^{(1,2)}abb^{(1,2)} \\ &= a^{(1,2)}abb^{(2)}. \end{aligned}$$

Hence, the statement (vi) is satisfied.

(iv) $\Rightarrow$ (iii): Similarly as (i) $\Rightarrow$ (vi). □

When the corresponding generalized inverses of a and b exist, notice that the conditions of Theorem 3.1 imply the conditions of Theorem 3.6 (Theorem 3.7 and Corollary 3.1). The reverse implication fails.

In the case that a and b have the reflexive inverses $a^{(1,2)}$ and $b^{(1,2)}$, respectively, since the conditions of Theorem 3.1 give the conditions of Corollary 3.1, combining the conditions of Theorem 3.2 (Theorem 3.3) and Theorem 3.1, we get the sufficient conditions for the reverse order law $(ab)^{\#} = b^{(1,2)}a^{(1,2)}$ to be satisfied.

The equivalent condition which involve $a^{(1,2)}abb^{(1,2)} \in \mathcal{R}^{\#}$ to ensure that the reverse order law $(ab)^{\#} = b^{(1,2)}a^{(1,2)}$ holds is presented now.

Theorem 3.8. *Let $a \in \mathcal{R}$ have a reflexive inverse $a^{(1,2)}$ and let $b \in \mathcal{R}$ have a reflexive inverse $b^{(1,2)}$. If $ab, a^{(1,2)}abb^{(1,2)} \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(ab)^{\#} = b^{(1,2)}a^{(1,2)}$,
(ii) $(ab)^{\#} = b^{(1,2)}(a^{(1,2)}abb^{(1,2)})^{\#}a^{(1,2)}$ *and* $b^{(1,2)}(a^{(1,2)}abb^{(1,2)})^{\#}a^{(1,2)} = b^{(1,2)}a^{(1,2)}$.

Proof. (i) $\Rightarrow$ (ii): Since the reverse order law $(ab)^{\#} = b^{(1,2)}a^{(1,2)}$ holds, then $b^{(1,2)}a^{(1,2)}abb^{(1,2)}a^{(1,2)} = b^{(1,2)}a^{(1,2)}$. Applying Corollary 3.1, we conclude that $b^{(1,2)}(a^{(1,2)}abb^{(1,2)})^{\#}a^{(1,2)} = b^{(1,2)}a^{(1,2)}$. Therefore,

$$(ab)^{\#} = b^{(1,2)}a^{(1,2)} = b^{(1,2)}(a^{(1,2)}abb^{(1,2)})^{\#}a^{(1,2)}.$$

(ii) $\Rightarrow$ (i): This implication is obvious. □

When a and b have the reflexive inverses $a^{(1,2)}$ and $b^{(1,2)}$, respectively, if we combine the conditions of Theorem 3.2 (Theorem 3.3) and Corollary 3.1, we get a set of necessary and sufficient conditions for the reverse order law $(ab)^{\#} = b^{(1,2)}a^{(1,2)}$ to hold.

Necessary and sufficient conditions for reverse order law $(cab)^{\#} = b^{\#}(cabb^{\#})^{\#}$ to be satisfied are investigated in the following theorem. Notice that we will prove that the reverse order laws $(cab)^{\#} = b^{\#}(cabb^{\#})^{\#}$ and $(cabb^{\#})^{\#} = b(cab)^{\#}$ are equivalent.

Theorem 3.9. *Let $a, c \in \mathcal{R}$. If $b, cab, cabb^\# \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(cab)^\# = b^\#(cabb^\#)^\#$,
(ii) $b^\#(cabb^\#)^\# \in (cab)\{5\}$,
(iii) $bb^\#cab = cab$ *and* $bcabb^\#(cabb^\#)^\# = (cabb^\#)^\#cab$,
(iv) $b\{1,5\} \cdot (cabb^\#)\{1,5\} \subseteq (cab)\{5\}$,
(v) $(cabb^\#)^\# = b(cab)^\#$,
(vi) $b(cab)^\# \in (cabb^\#)\{5\}$,
(vii) $b \cdot (cab)\{1,5\} \subseteq (cabb^\#)\{5\}$.

Proof. (i) ⇒ (ii): This is clear.

(ii) ⇒ (iii): Suppose that $b^\#(cabb^\#)^\# \in (cab)\{5\}$. We firstly observe that $b^\#(cabb^\#)^\# \in (cab)\{1\}$, by

$$cabb^\#(cabb^\#)^\#cab = (cabb^\#(cabb^\#)^\#cabb^\#)b = cabb^\#b = cab. \tag{3.3}$$

Now, we obtain

$$bb^\#(cab) = bb^\#b^\#(cabb^\#)^\#(cab)^2 = b^\#(cabb^\#)^\#(cab)^2 = cab, \tag{3.4}$$

$$\begin{aligned} bcabb^\#(cabb^\#)^\# &= bb^\#(cabb^\#)^\#cab = bb^\#((cabb^\#)^\#cabb^\#)b \\ &= (bb^\#cab)b^\#(cabb^\#)^\#b = cabb^\#(cabb^\#)^\#b \\ &= (cabb^\#)^\#cabb^\#b = (cabb^\#)^\#cab. \end{aligned}$$

Hence, the condition (iii) holds.

(iii) ⇒ (i): If $bb^\#cab = cab$ and $bcabb^\#(cabb^\#)^\# = (cabb^\#)^\#cab$, then we get

$$b^\#((cabb^\#)^\#cab) = (b^\#bcab)b^\#(cabb^\#)^\# = cabb^\#(cabb^\#)^\#,$$

which implies that $b^\#(cabb^\#)^\# \in (cab)\{5\}$. Notice that $b^\#(cabb^\#)^\# \in (cab)\{1,2\}$, by the equalities (3.3) and $b^\#((cabb^\#)^\#cabb^\#(cabb^\#)^\#) = b^\#(cabb^\#)^\#$. Thus, the statement (i) is satisfied.

(ii) ⇒ (iv): Let $b^{(1,5)} \in b\{1,5\}$ and $(cabb^\#)^{(1,5)} \in (cabb^\#)\{1,5\}$. Then the equalities $b^{(1,5)}b = (b^{(1,5)}bb)b^\# = bb^\#$, $cabb^\#(cabb^\#)^{(1,5)} = (cabb^\#)^\#cabb^\#$ and (3.4) are satisfied. Since $b^\#(cabb^\#)^\# \in (cab)\{5\}$, we have that

$$\begin{aligned} b^{(1,5)}(cabb^\#)^{(1,5)}cab &= b^{(1,5)}((cabb^\#)^{(1,5)}cabb^\#)b \\ &= b^{(1,5)}((cabb^\#)^\#cabb^\#)b \\ &= b^{(1,5)}(cab)b^\#(cabb^\#)^\#b \\ &= (b^{(1,5)}b)b^\#cabb^\#(cabb^\#)^\#b \\ &= b^\#bb^\#(cabb^\#(cabb^\#)^\#)b \\ &= b^\#(cabb^\#)^\#cabb^\#b = cabb^\#(cabb^\#)^\# \\ &= ca(bb^\#)(cabb^\#)^{(1,5)} \\ &= cabb^{(1,5)}(cabb^\#)^{(1,5)}. \end{aligned}$$

Therefore, we deduce that $b^{(1,5)}(cabb^{\#})^{(1,5)} \in (cab)\{5\}$, for any $b^{(1,5)} \in b\{1,5\}$ and $(cabb^{\#})^{(1,5)} \in (cabb^{\#})\{1,5\}$. So, the statement (iv) is satisfied.

(iv) $\Rightarrow$ (ii): Because $b^{\#} \in b\{1,5\}$ and $(cabb^{\#})^{\#} \in (cabb^{\#})\{1,5\}$, this implication follows.

(i) $\Rightarrow$ (v): Assume that $(cab)^{\#} = b^{\#}(cabb^{\#})^{\#}$. From the part (i) $\Rightarrow$ (iii), we have $bb^{\#}cab = cab$. Hence, by

$$\begin{aligned}(cabb^{\#})^{\#} &= (cab)b^{\#}[(cabb^{\#})^{\#}]^2 \\ &= bb^{\#}(cabb^{\#}[(cabb^{\#})^{\#}]^2) \\ &= b(b^{\#}(cabb^{\#})^{\#}) = b(cab)^{\#},\end{aligned}$$

the reverse order law (v) holds.

(v) $\Rightarrow$ (i): This can be verified as the implication part (i) $\Rightarrow$ (v).

(v) $\Leftrightarrow$ (vi): Since $b(cab)^{\#} \in (cabb^{\#})\{1,2\}$, note that $b(cab)^{\#} \in (cabb^{\#})\{5\}$ is equivalent to $(cabb^{\#})^{\#} = b(cab)^{\#}$.

(vi) $\Leftrightarrow$ (vii): In the same way as (ii) $\Leftrightarrow$ (iv), we check this part. □

Analogously to Theorem 3.9, we can show that the reverse order laws $(abc)^{\#} = (a^{\#}abc)^{\#}a^{\#}$ and $(a^{\#}abc)^{\#} = (abc)^{\#}a$ are equivalent.

Theorem 3.10. *Let $b, c \in \mathcal{R}$. If $a, abc, a^{\#}abc \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(abc)^{\#} = (a^{\#}abc)^{\#}a^{\#}$,
(ii) $(a^{\#}abc)^{\#}a^{\#} \in (abc)\{5\}$,
(iii) $abcaa^{\#} = abc$ *and* $(a^{\#}abc)^{\#}a^{\#}abca = abc(a^{\#}abc)^{\#}$,
(iv) $(a^{\#}abc)\{1,5\} \cdot a\{1,5\} \subseteq (abc)\{5\}$,
(v) $(a^{\#}abc)^{\#} = (abc)^{\#}a$,
(vi) $(abc)^{\#}a \in (a^{\#}abc)\{5\}$,
(vii) $(abc)\{1,5\} \cdot a \subseteq (a^{\#}abc)\{5\}$.

Observe that we check the following characterizations for the reverse order laws $(cab)^{\#} = b^{(1,2)}(cabb^{(1,2)})^{\#}$ and $(abc)^{\#} = (a^{(1,2)}abc)^{\#}a^{(1,2)}$ in the same way as in the proof of Theorem 3.9.

Theorem 3.11. (a) *Let $a, c \in \mathcal{R}$ and let $b \in \mathcal{R}$ have a reflexive inverse $b^{(1,2)}$. If $cab, cabb^{(1,2)} \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(cab)^{\#} = b^{(1,2)}(cabb^{(1,2)})^{\#}$,
(ii) $b^{(1,2)}(cabb^{(1,2)})^{\#} \in (cab)\{5\}$,
(iii) $b^{(1,2)}bcab = cab$ *and* $bcabb^{(1,2)}(cabb^{(1,2)})^{\#} = bb^{(1,2)}(cabb^{(1,2)})^{\#}cab$.

(b) *Let $b, c \in \mathcal{R}$ and let $a \in \mathcal{R}$ have a reflexive inverse $a^{(1,2)}$. If $abc, a^{(1,2)}abc \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(abc)^{\#} = (a^{(1,2)}abc)^{\#}a^{(1,2)}$,
(ii) $(a^{(1,2)}abc)^{\#}a^{(1,2)} \in (abc)\{5\}$,
(iii) $abcaa^{(1,2)} = abc$ *and* $(a^{(1,2)}abc)^{\#}a^{(1,2)}abca = abc(a^{(1,2)}abc)^{\#}a^{(1,2)}a$.

The relation between $(cab)\{5\} \subseteq b\{1,5\} \cdot (cabb^{\#})\{1,5\}$ and $(cab)\{5\} = b\{1,5\} \cdot (cabb^{\#})\{1,5\}$ is presented in the following theorem.

Theorem 3.12. *If $a, c \in \mathcal{R}$ and $b, cab, cabb^{\#} \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(cab)\{5\} \subseteq b\{1,5\} \cdot (cabb^{\#})\{1,5\}$ *and* $bb^{\#}cab = cab$,

(ii) $(cab)\{5\} = b\{1,5\} \cdot (cabb^{\#})\{1,5\}$.

Proof. (i) ⇒ (ii): If we suppose that $(cab)\{5\} \subseteq b\{1,5\} \cdot (cabb^{\#})\{1,5\}$ and $bb^{\#}cab = cab$, then there exist $b^{(1,5)} \in b\{1,5\}$ and $(cabb^{\#})^{(1,5)} \in (cabb^{\#})\{1,5\}$ satisfying $(cab)^{\#} = b^{(1,5)}(cabb^{\#})^{(1,5)}$. Using equalities $cabb^{\#}(cabb^{\#})^{(1,5)} = (cabb^{\#})^{\#}cabb^{\#}$ and $bb^{\#} = bb^{(1,5)}$, we obtain

$$\begin{aligned}
(cabb^{\#})^{\#} &= (cabb^{\#})^{(1,5)}cabb^{\#}(cabb^{\#})^{(1,5)},\\
b^{\#}(cabb^{\#})^{\#} &= b^{\#}(bb^{\#})(cabb^{\#})^{(1,5)}ca(bb^{\#})(cabb^{\#})^{(1,5)}\\
&= b^{\#}b(b^{(1,5)}(cabb^{\#})^{(1,5)})cab(b^{(1,5)}(cabb^{\#})^{(1,5)})\\
&= b^{\#}b(cab)^{\#}cab(cab)^{\#} = (b^{\#}bcab)[(cab)^{\#}]^2\\
&= cab[(cab)^{\#}]^2 = (cab)^{\#}.
\end{aligned}$$

Applying Theorem 3.9, we conclude that $b\{1,5\} \cdot (cabb^{\#})\{1,5\} \subseteq (cab)\{5\}$. So, the condition (ii) holds.

(ii) ⇒ (i): From Theorem 3.9, this part is obvious. □

In the analogy way as in Theorem 3.12, we can consider the conditions $(abc)\{5\} \subseteq (a^{\#}abc)\{1,5\} \cdot a\{1,5\}$ and $(abc)\{5\} = (a^{\#}abc)\{1,5\} \cdot a\{1,5\}$.

Theorem 3.13. *If $b, c \in \mathcal{R}$ and $a, abc, a^{\#}abc \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(abc)\{5\} \subseteq (a^{\#}abc)\{1,5\} \cdot a\{1,5\}$ *and* $abca^{\#}a = abc$,

(ii) $(abc)\{5\} = (a^{\#}abc)\{1,5\} \cdot a\{1,5\}$.

Remark that, using the similar argument as in proof of Theorem 3.12, we get that the inclusions $(cab)\{5\} \subseteq b^{\#} \cdot (cabb^{\#})\{1,5\}$ and $(abc)\{5\} \subseteq (a^{\#}abc)\{1,5\} \cdot a^{\#}$ are always the equalities.

The equality $(cabb^{\#})\{5\} = b \cdot (cab)\{1,5\}$ is studied in the following theorem.

Theorem 3.14. *If $a, c \in \mathcal{R}$ and $b, cab, cabb^{\#} \in \mathcal{R}^{\#}$, the inclusion $(cabb^{\#})\{5\} \subseteq b \cdot (cab)\{1,5\}$ is always an equality.*

Proof. Assume that $(cabb^{\#})\{5\} \subseteq b \cdot (cab)\{1,5\}$. Then there exists $(cab)^{(1,5)} \in (cab)\{1,5\}$ such that $(cabb^{\#})^{\#} = b(cab)^{(1,5)}$. Since

$$\begin{aligned}
b(cab)^{\#} &= b(cab)^{(1,5)}cab(cab)^{(1,5)}\\
&= (b(cab)^{(1,5)})cabb^{\#}(b(cab)^{(1,5)})\\
&= (cabb^{\#})^{\#}cabb^{\#}(cabb^{\#})^{\#} = (cabb^{\#})^{\#},
\end{aligned}$$

by Theorem 3.9, we have that $b \cdot (cab)\{1,5\} \subseteq (cabb^{\#})\{5\}$. Thus, the equality $(cabb^{\#})\{5\} = b \cdot (cab)\{1,5\}$ holds. □

As Theorem 3.14, we prove the next theorem related to equality $(a^{\#}abc)\{5\} = (abc)\{1,5\} \cdot a$.

Theorem 3.15. *If $b, c \in \mathcal{R}$ and $a, abc, a^{\#}abc \in \mathcal{R}^{\#}$, the inclusion $(a^{\#}abc)\{5\} \subseteq (abc)\{1,5\} \cdot a$ is always an equality.*

In the following result, we give some sufficient conditions for the reverse order law $(ab)^{\#} = b^{\#}(cabb^{\#})^{\#}c$.

Theorem 3.16. *Suppose that $a, c \in \mathcal{R}$ and $b, ab, cab, abb^{\#}, cabb^{\#} \in \mathcal{R}^{\#}$. Then each of the following conditions is sufficient for $(ab)^{\#} = b^{\#}(cabb^{\#})^{\#}c$ to hold:*

(i) $(cab)^{\#} = b^{\#}(cabb^{\#})^{\#}$ *and* $(abb^{\#})^{\#} = (cabb^{\#})^{\#}c$,
(ii) $(ab)^{\#} = (cab)^{\#}c$ *and* $(cab)^{\#} = b^{\#}(cabb^{\#})^{\#}$,
(iii) $(ab)^{\#} = b^{\#}(abb^{\#})^{\#}$ *and* $(abb^{\#})^{\#} = (cabb^{\#})^{\#}c$.

Proof. (i) The assumptions $(cab)^{\#} = b^{\#}(cabb^{\#})^{\#}$ and $(abb^{\#})^{\#} = (cabb^{\#})^{\#}c$ imply

$$abb^{\#}(cabb^{\#})^{\#}c = (cabb^{\#})^{\#}cabb^{\#} = cabb^{\#}(cabb^{\#})^{\#} = b^{\#}(cabb^{\#})^{\#}cab.$$

So, we conclude that $b^{\#}(cabb^{\#})^{\#}c \in (ab)\{5\}$. Furthermore, by

$$abb^{\#}(cabb^{\#})^{\#}cab = abb^{\#}(abb^{\#})^{\#}abb^{\#}b = abb^{\#}b = ab,$$
$$b^{\#}(cabb^{\#})^{\#}cabb^{\#}(cabb^{\#})^{\#}c = b^{\#}(cabb^{\#})^{\#}c,$$

we have that $b^{\#}(cabb^{\#})^{\#}c \in (ab)\{1,2\}$. Hence, we get that the reverse order law $(ab)^{\#} = b^{\#}(cabb^{\#})^{\#}c$ holds.

(ii) Since $(ab)^{\#} = (cab)^{\#}c$ and $(cab)^{\#} = b^{\#}(cabb^{\#})^{\#}$, then we obtain

$$(ab)^{\#} = (cab)^{\#}c = b^{\#}(cabb^{\#})^{\#}c.$$

(iii) This part follows similarly as part (ii). □

The condition $(cab)^{\#} = b^{\#}(cabb^{\#})^{\#}$ of Theorem 3.16 can be replaced by some equivalent conditions from Theorem 3.9.

Sufficient conditions for the reverse order law $(ab)^{\#} = c(a^{\#}abc)^{\#}a^{\#}$ given in the next theorem can be verified in the same manner as in the proof of Theorem 3.16.

Theorem 3.17. *Suppose that $b, c \in \mathcal{R}$ and $a, ab, abc, a^{\#}ab, a^{\#}abc \in \mathcal{R}^{\#}$. Then each of the following conditions is sufficient for $(ab)^{\#} = c(a^{\#}abc)^{\#}a^{\#}$ to hold:*

(i) $(a^{\#}ab)^{\#} = c(a^{\#}abc)^{\#}$ *and* $(abc)^{\#} = (a^{\#}abc)^{\#}a^{\#}$,
(ii) $(ab)^{\#} = (a^{\#}ab)^{\#}a^{\#}$ *and* $(a^{\#}ab)^{\#} = c(a^{\#}abc)^{\#}$,
(iii) $(ab)^{\#} = c(abc)^{\#}$ *and* $(abc)^{\#} = (a^{\#}abc)^{\#}a^{\#}$.

The relation between the next two reverse order laws $(ab)^{\#} = (a^{(1,2)}ab)^{\#}a^{(1,2)}$ and $(a^{(1,2)}ab)^{\#} = (ab)^{\#}a$ is investigated now.

Theorem 3.18. *Let $b \in \mathcal{R}$ and let $a \in \mathcal{R}$ have a reflexive inverse $a^{(1,2)}$. If $ab, a^{(1,2)}ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(ab)^{\#} = (a^{(1,2)}ab)^{\#}a^{(1,2)}$ *and* $a^{(1,2)}aba^{(1,2)}a = a^{(1,2)}ab$,
(ii) $(a^{(1,2)}ab)^{\#} = (ab)^{\#}a$ *and* $abaa^{(1,2)} = ab$.

Proof. (i) ⇒ (ii): Assume that $(ab)^{\#} = (a^{(1,2)}ab)^{\#}a^{(1,2)}$ and $a^{(1,2)}aba^{(1,2)}a = a^{(1,2)}ab$. Then, by the equalities

$$abaa^{(1,2)} = (ab)^2(ab)^{\#}aa^{(1,2)} = (ab)^2(a^{(1,2)}ab)^{\#}a^{(1,2)}aa^{(1,2)}$$

$$= (ab)^2(a^{(1,2)}ab)^{\#}a^{(1,2)} = ab,$$

$$(ab)^{\#}a = (a^{(1,2)}ab)^{\#}a^{(1,2)}a = [(a^{(1,2)}ab)^{\#}]^2(a^{(1,2)}aba^{(1,2)}a)$$
$$= [(a^{(1,2)}ab)^{\#}]^2a^{(1,2)}ab = (a^{(1,2)}ab)^{\#},$$

we deduce that (ii) holds.

(ii) $\Rightarrow$ (i): Applying the equalities $(a^{(1,2)}ab)^{\#} = (ab)^{\#}a$ and $abaa^{(1,2)} = ab$, we get

$$a^{(1,2)}aba^{(1,2)}a = (a^{(1,2)}ab)^2(a^{(1,2)}ab)^{\#}a^{(1,2)}a = (a^{(1,2)}ab)^2(ab)^{\#}aa^{(1,2)}a$$
$$= (a^{(1,2)}ab)^2(ab)^{\#}a = a^{(1,2)}ab,$$
$$(a^{(1,2)}ab)^{\#}a^{(1,2)} = (ab)^{\#}aa^{(1,2)} = [(ab)^{\#}]^2(abaa^{(1,2)}) = [(ab)^{\#}]^2ab = ab.$$

Hence, the item (i) is satisfied. □

Analogously to Theorem 3.18, we can verify the following.

Theorem 3.19. *Let $a \in \mathcal{R}$ and let $b \in \mathcal{R}$ have a reflexive inverse $b^{(1,2)}$. If $ab, abb^{(1,2)} \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(ab)^{\#} = b^{(1,2)}(abb^{(1,2)})^{\#}$ *and* $bb^{(1,2)}abb^{(1,2)} = abb^{(1,2)}$,

(ii) $(abb^{(1,2)})^{\#} = b(ab)^{\#}$ *and* $b^{(1,2)}bab = ab$.

4. Reverse order laws in rings with involution

Throughout Section 4 and Section 5, we will denote by $\mathcal{R}$ a ring with involution.

The reverse order laws $(a^*ab)^{\#} = b^{\#}a^{(1,4)}a$ and $(abb^*)^{\#} = bb^{(1,3)}a^{\#}$ are firstly investigated in a ring with involution.

Theorem 4.1. *Let $a \in \mathcal{R}$ have an $\{1,4\}$–inverse $a^{(1,4)}$ and let $b, a^*ab \in \mathcal{R}^{\#}$. Then the following statements are equivalent:*

(i) $(a^*ab)^{\#} = b^{\#}a^{(1,4)}a$,

(ii) $a^*ab = ba^{(1,4)}a$.

Proof. (i) $\Rightarrow$ (ii): Using $(a^*ab)^{\#} = b^{\#}a^{(1,4)}a$, we have

$$b^{\#}a^{(1,4)}a = b^{\#}a^{(1,4)}a(a^*ab)b^{\#}a^{(1,4)}a = b^{\#}(a^*abb^{\#}a^{(1,4)}a)$$
$$= b^{\#}b^{\#}a^{(1,4)}aa^*ab = b^{\#}b^{\#}a^*ab.$$

So,

$$a^*ab = (a^*ab)^{\#}a^*aba^*ab = b^{\#}a^{(1,4)}aa^*aba^*ab = b(b^{\#}b^{\#}a^*ab)a^*ab$$
$$= bb^{\#}a^{(1,4)}aa^*ab = bb^{\#}a^*ab = bb(b^{\#}b^{\#}a^*ab) = bbb^{\#}a^{(1,4)}a$$
$$= ba^{(1,4)}a.$$

(ii) $\Rightarrow$ (i): The hypothesis $a^*ab = ba^{(1,4)}a$ implies

$$a^*ab = ba^{(1,4)}a = bb^{\#}(ba^{(1,4)}a) = bb^{\#}a^*ab,$$
$$(a^*ab)^{\#} = (a^*ab)[(a^*ab)^{\#}]^2 = bb^{\#}a^*ab[(a^*ab)^{\#}]^2$$

$$= bb^{\#}a^{(1,4)}a(a^*ab[(a^*ab)^{\#}]^2) = bb^{\#}a^{(1,4)}a(a^*ab)^{\#}.$$

Therefore,

$$\begin{aligned} b^{\#}a^{(1,4)}a &= b^{\#}b^{\#}(ba^{(1,4)}a) = b^{\#}b^{\#}(a^*ab) = b^{\#}b^{\#}(a^*ab)a^*ab(a^*ab)^{\#} \\ &= b^{\#}b^{\#}ba^{(1,4)}aa^*ab(a^*ab)^{\#} = b^{\#}(a^*ab)(a^*ab)^{\#} \\ &= b^{\#}ba^{(1,4)}a(a^*ab)^{\#} = (a^*ab)^{\#}. \end{aligned}$$

□

In the same way as Theorem 4.1, we prove the following result.

Theorem 4.2. *Let $b \in \mathcal{R}$ have an $\{1,3\}$–inverse $b^{(1,3)}$ and let $a, abb^* \in \mathcal{R}^{\#}$. Then the following statements are equivalent:*

(i) $(abb^*)^{\#} = bb^{(1,3)}a^{\#}$,
(ii) $abb^* = bb^{(1,3)}a$.

The next theorems related to the rules $(a^*ab)^{\#}a^* = b^{\#}a^*$ and $b^*(abb^*)^{\#} = b^*a^{\#}$ can be checked in the similar way as Theorem 2.5.

Theorem 4.3. *Let $a \in \mathcal{R}$ have an $\{1,4\}$–inverse $a^{(1,4)}$ and let $b, a^*ab \in \mathcal{R}^{\#}$. Then the following statements are equivalent:*

(i) $(a^*ab)^{\#}a^* = b^{\#}a^*$,
(ii) $ba^{(1,4)}a = a^*aba^{(1,4)}a$.

Theorem 4.4. *Let $b \in \mathcal{R}$ have an $\{1,3\}$–inverse $b^{(1,3)}$ and let $a, abb^* \in \mathcal{R}^{\#}$. Then the following statements are equivalent:*

(i) $b^*(abb^*)^{\#} = b^*a^{\#}$,
(ii) $bb^{(1,3)}a = bb^{(1,3)}abb^*$.

Remark that if conditions of Theorem 4.1 and Theorem 4.2 are satisfied, then the conditions of Theorem 4.3 and Theorem 4.4 holds, respectively.

We study now some equivalent conditions for $(ab)^{\#} = (a^*ab)^{\#}a^*$ and $(ab)^{\#} = b^*(abb^*)^{\#}$ to be satisfied.

Theorem 4.5. *Let $b \in \mathcal{R}$ and let $a \in \mathcal{R}$ have an $\{1,3\}$–inverse $a^{(1,3)}$. If $a^*ab \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $ab \in \mathcal{R}^{\#}$ *and* $(ab)^{\#} = (a^*ab)^{\#}a^*$,
(ii) $(a^*ab)^{\#}a^* \in (ab)\{5\}$,
(iii) $abaa^{(1,3)} = ab$ *and* $(a^*ab)^{\#}a^*aba = ab(a^*ab)^{\#}a^*a$,
(iv) $(a^*ab)\{1,5\} \cdot a^* \subseteq (ab)\{5\}$.

Proof. We prove this result similarly as Theorem 2.7, by $a = (a^{(1,3)})^*a^*a$ and $a^* = a^*aa^{(1,3)}$. □

Theorem 4.6. *Let $a \in \mathcal{R}$ and let $b \in \mathcal{R}$ have an $\{1,4\}$–inverse $b^{(1,4)}$. If $abb^* \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $ab \in \mathcal{R}^{\#}$ *and* $(ab)^{\#} = b^*(abb^*)^{\#}$,
(ii) $b^*(abb^*)^{\#} \in (ab)\{5\}$,
(iii) $b^{(1,4)}bab = ab$ *and* $babb^*(abb^*)^{\#} = bb^*(abb^*)^{\#}ab$,
(iv) $b^* \cdot (abb^*)\{1,5\} \subseteq (ab)\{5\}$.

If we consider the Moore–Penrose inverses of a^*ab and abb^* instead of the group inverses of these elements in Theorem 4.5 and Theorem 4.6, respectively, we get the following theorems.

Theorem 4.7. *Let $b \in \mathcal{R}$ and let $a \in \mathcal{R}$ have an $\{1,3\}$–inverse $a^{(1,3)}$. If $a^*ab \in \mathcal{R}^\dagger$, then the following statements are equivalent:*

(i) $ab \in \mathcal{R}^\#$ *and* $(ab)^\# = (a^*ab)^\dagger a^*$,

(ii) $(a^*ab)^\dagger a^* \in (ab)\{5\}$,

(iii) $abaa^{(1,3)} = ab$ *and* $(a^*ab)^\dagger a^*aba = ab(a^*ab)^\dagger a^*a$,

(iv) $(a^*ab)\{1,3,4\} \cdot a^* \subseteq (ab)\{5\}$.

Theorem 4.8. *Let $a \in \mathcal{R}$ and let $b \in \mathcal{R}$ have an $\{1,4\}$–inverse $b^{(1,4)}$. If $abb^* \in \mathcal{R}^\dagger$, then the following statements are equivalent:*

(i) $ab \in \mathcal{R}^\#$ *and* $(ab)^\# = b^*(abb^*)^\dagger$,

(ii) $b^*(abb^*)^\dagger \in (ab)\{5\}$,

(iii) $b^{(1,4)}bab = ab$ *and* $babb^*(abb^*)^\dagger = bb^*(abb^*)^\dagger ab$,

(iv) $b^* \cdot (abb^*)\{1,3,4\} \subseteq (ab)\{5\}$.

As Theorem 2.9, we verify that inclusions $(ab)\{5\} \subseteq (a^*ab)\{1,5\} \cdot a^*$, $(ab)\{5\} \subseteq b^* \cdot (abb^*)\{1,5\}$, $(ab)\{5\} \subseteq (a^*ab)\{1,3,4\} \cdot a^*$ and $(ab)\{5\} \subseteq b^* \cdot (abb^*)\{1,3,4\}$ are always equalities.

Theorem 4.9. *Let $a, b \in \mathcal{R}$ and $ab \in \mathcal{R}^\#$.*

(i) *If a has an $\{1,3\}$–inverse $a^{(1,3)}$ and $a^*ab \in \mathcal{R}^\#$, then*
$$(ab)\{5\} \subseteq (a^*ab)\{1,5\} \cdot a^* \textit{ if and only if } (ab)\{5\} = (a^*ab)\{1,5\} \cdot a^*.$$

(ii) *If b has an $\{1,4\}$–inverse $b^{(1,4)}$ and $abb^* \in \mathcal{R}^\#$, then*
$$(ab)\{5\} \subseteq b^* \cdot (abb^*)\{1,5\} \textit{ if and only if } (ab)\{5\} = b^* \cdot (abb^*)\{1,5\}.$$

(iii) *If a has an $\{1,3\}$–inverse $a^{(1,3)}$ and $a^*ab \in \mathcal{R}^\dagger$, then*
$$(ab)\{5\} \subseteq (a^*ab)\{1,3,4\} \cdot a^* \textit{ if and only if } (ab)\{5\} = (a^*ab)\{1,3,4\} \cdot a^*.$$

(iv) *If b has an $\{1,4\}$–inverse $b^{(1,4)}$ and $abb^* \in \mathcal{R}^\dagger$, then*
$$(ab)\{5\} \subseteq b^* \cdot (abb^*)\{1,3,4\} \textit{ if and only if } (ab)\{5\} = b^* \cdot (abb^*)\{1,3,4\}.$$

The following characterizations of reverse order laws $(ab)^\# = ca^{(1,2,3)}$, $(ab)^\# = b^{(1,2,4)}c$, $(ab)^\# = ca^*$ and $(ab)^\# = b^*c$ in rings with involution, can be proved as Theorem 2.11.

Theorem 4.10. *Let $a, b, c \in \mathcal{R}$ and $ab \in \mathcal{R}^\#$.*

(i) *If a has an $\{1,2,3\}$–inverse $a^{(1,2,3)}$, then*
$$(ab)^\# = ca^{(1,2,3)} \textit{ if and only if } (ab)^\# a = ca^{(1,2,3)}a \textit{ and } a^*ab = a^*abaa^{(1,2,3)}.$$

(ii) *If b has an $\{1,2,4\}$–inverse $b^{(1,2,4)}$, then*
$$(ab)^\# = b^{(1,2,4)}c \textit{ if and only if } b(ab)^\# = bb^{(1,2,4)}c \textit{ and } abb^* = b^{(1,2,4)}babb^*.$$

(iii) *If a has an $\{1,3\}$–inverse $a^{(1,3)}$, then*
$$(ab)^\# = ca^* \textit{ if and only if } (ab)^\# a = ca^*a \textit{ and } a^*ab = a^*abaa^{(1,3)}.$$

(iv) *If* b *has an* $\{1,4\}$*–inverse* $b^{(1,4)}$*, then*

$$(ab)^{\#} = b^*c \text{ if and only if } b(ab)^{\#} = bb^*c \text{ and } abb^* = b^{(1,4)}babb^*.$$

Sufficient conditions for the reverse order laws $(ab)^{\#} = ca^*$ and $(ab)^{\#} = b^*c$ are given in the next results.

Theorem 4.11. *Suppose that* $a, b, c \in \mathcal{R}$ *and* $ab \in \mathcal{R}^{\#}$*. Then each of the following conditions is sufficient for* $(ab)^{\#} = ca^*$ *to hold:*

(i) a *has an* $\{1,3\}$*–inverse* $a^{(1,3)}$, $(ab)^{\#}a = ca^*a$ *and* $a^*ab = baa^*$;
(ii) a *has an* $\{1,3\}$*–inverse* $a^{(1,3)}$, $(ab)^{\#}a = ca^*a$ *and* $a^*ab = baa^{(1,3)}$;
(iii) a *has an* $\{1,4\}$*–inverse* $a^{(1,4)}$, $a^{(1,4)}ab \in \mathcal{R}^{\#}$, $(ab)^{\#} = (a^{(1,4)}ab)^{\#}a^*$ *and* $(a^{(1,4)}ab)^{\#} = ca^{(1,4)}a$;
(iv) a *has an* $\{1,4\}$*–inverse* $a^{(1,4)}$, $a^*ab \in \mathcal{R}^{\#}$, $(ab)^{\#} = (a^*ab)^{\#}a^*$ *and* $(a^*ab)^{\#} = ca^{(1,4)}a$.

Theorem 4.12. *Suppose that* $a, b, c \in \mathcal{R}$ *and* $ab \in \mathcal{R}^{\#}$*. Then each of the following conditions is sufficient for* $(ab)^{\#} = b^*c$ *to hold:*

(i) b *has an* $\{1,4\}$*–inverse* $b^{(1,4)}$, $b(ab)^{\#} = bb^*c$ *and* $b^*ba = abb^*$;
(ii) b *has an* $\{1,4\}$*–inverse* $b^{(1,4)}$, $b(ab)^{\#} = bb^*c$ *and* $b^{(1,4)}ba = abb^*$;
(iii) b *has an* $\{1,3\}$*–inverse* $b^{(1,3)}$, $abb^{(1,3)} \in \mathcal{R}^{\#}$, $(ab)^{\#} = b^*(abb^{(1,3)})^{\#}$ *and* $(abb^{(1,3)})^{\#} = bb^{(1,3)}c$;
(iv) b *has an* $\{1,3\}$*–inverse* $b^{(1,3)}$, $abb^* \in \mathcal{R}^{\#}$, $(ab)^{\#} = b^*(abb^*)^{\#}$ *and* $(abb^*)^{\#} = bb^{(1,3)}c$.

If we assume that $c = b^*$ or $c = a^*$ in Theorem 4.11 and Theorem 4.12, respectively, we get sufficient conditions for the rule $(ab)^{\#} = b^*a^* = (ab)^* = (ab)^{\dagger}$.

Corollary 4.1. *Suppose that* $a, b, c \in \mathcal{R}$ *and* $ab \in \mathcal{R}^{\#}$*. Then each of the following conditions is sufficient for* $(ab)^{\#} = b^*a^* = (ab)^{\dagger}$ *to hold:*

(i) a *has an* $\{1,3\}$*–inverse* $a^{(1,3)}$, $(ab)^{\#}a = b^*a^*a$ *and* $a^*ab = baa^*$;
(ii) a *has an* $\{1,3\}$*–inverse* $a^{(1,3)}$, $(ab)^{\#}a = b^*a^*a$ *and* $a^*ab = baa^{(1,3)}$;
(iii) a *has an* $\{1,4\}$*–inverse* $a^{(1,4)}$, $a^{(1,4)}ab \in \mathcal{R}^{\#}$, $(ab)^{\#} = (a^{(1,4)}ab)^{\#}a^*$ *and* $(a^{(1,4)}ab)^{\#} = b^*a^{(1,4)}a$;
(iv) a *has an* $\{1,4\}$*–inverse* $a^{(1,4)}$, $a^*ab \in \mathcal{R}^{\#}$, $(ab)^{\#} = (a^*ab)^{\#}a^*$ *and* $(a^*ab)^{\#} = b^*a^{(1,4)}a$;
(v) b *has an* $\{1,4\}$*–inverse* $b^{(1,4)}$, $b(ab)^{\#} = bb^*a^*$ *and* $b^*ba = abb^*$;
(vi) b *has an* $\{1,4\}$*–inverse* $b^{(1,4)}$, $b(ab)^{\#} = bb^*a^*$ *and* $b^{(1,4)}ba = abb^*$;
(vii) b *has an* $\{1,3\}$*–inverse* $b^{(1,3)}$, $abb^{(1,3)} \in \mathcal{R}^{\#}$, $(ab)^{\#} = b^*(abb^{(1,3)})^{\#}$ *and* $(abb^{(1,3)})^{\#} = bb^{(1,3)}a^*$;
(viii) b *has an* $\{1,3\}$*–inverse* $b^{(1,3)}$, $abb^* \in \mathcal{R}^{\#}$, $(ab)^{\#} = b^*(abb^*)^{\#}$ *and* $(abb^*)^{\#} = bb^{(1,3)}a^*$.

5. Reverse order laws involving $(a^*abb^{(\circ)})^{\#}$

In the following theorem, necessary and sufficient conditions for the reverse order law $(a^*abb^{(\circ)})^{\#} = bb^{(\circ)}a^{(1,4)}a$ are obtained in the case that $b^{(\circ)} \in b\{1\} \cup b\{2\}$ in a ring with involution.

Theorem 5.1. *Let $b \in \mathcal{R}$ and let $a \in \mathcal{R}$ have an $\{1,4\}$–inverse $a^{(1,4)}$. If $b^{(\circ)} \in b\{1\} \cup b\{2\}$ and $a^*abb^{(\circ)} \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(a^*abb^{(\circ)})^{\#} = bb^{(\circ)}a^{(1,4)}a$,

(ii) $a^*abb^{(\circ)} = bb^{(\circ)}a^{(1,4)}a$.

Proof. (i) ⇒ (ii): Since the reverse order law $(a^*abb^{(\circ)})^{\#} = bb^{(\circ)}a^{(1,4)}a$ holds, then

$$\begin{aligned} bb^{(\circ)}a^{(1,4)}a &= bb^{(\circ)}a^{(1,4)}a(a^*abb^{(\circ)})bb^{(\circ)}a^{(1,4)}a \\ &= bb^{(\circ)}(a^*abb^{(\circ)}bb^{(\circ)}a^{(1,4)}a) \\ &= bb^{(\circ)}bb^{(\circ)}a^{(1,4)}aa^*abb^{(\circ)} = bb^{(\circ)}a^*abb^{(\circ)}. \end{aligned}$$

Using the previous equality, we get

$$\begin{aligned} a^*abb^{(\circ)} &= (a^*abb^{(\circ)})^{\#}a^*abb^{(\circ)}a^*abb^{(\circ)} = bb^{(\circ)}a^{(1,4)}aa^*abb^{(\circ)}a^*abb^{(\circ)} \\ &= (bb^{(\circ)}a^*abb^{(\circ)})a^*abb^{(\circ)} = bb^{(\circ)}a^{(1,4)}aa^*abb^{(\circ)} \\ &= bb^{(\circ)}a^*abb^{(\circ)} = bb^{(\circ)}a^{(1,4)}a. \end{aligned}$$

(ii) ⇒ (i): Applying the hypothesis $a^*abb^{(\circ)} = bb^{(\circ)}a^{(1,4)}a$, we obtain

$$a^*abb^{(\circ)} = bb^{(\circ)}a^{(1,4)}a = bb^{(\circ)}(bb^{(\circ)}a^{(1,4)}a) = bb^{(\circ)}a^*abb^{(\circ)}.$$

By this equality, we have

$$\begin{aligned} (a^*abb^{(\circ)})^{\#} &= (a^*abb^{(\circ)})[(a^*abb^{(\circ)})^{\#}]^2 = bb^{(\circ)}a^*abb^{(\circ)}[(a^*abb^{(\circ)})^{\#}]^2 \\ &= bb^{(\circ)}a^{(1,4)}a(a^*abb^{(\circ)}[(a^*abb^{\#})^{\#}]^2) = bb^{(\circ)}a^{(1,4)}a(a^*abb^{(\circ)})^{\#} \end{aligned}$$

implying

$$\begin{aligned} bb^{(\circ)}a^{(1,4)}a &= bb^{(\circ)}(bb^{(\circ)}a^{(1,4)}a) = bb^{(\circ)}(a^*abb^{(\circ)}) \\ &= bb^{(\circ)}(a^*abb^{(\circ)})a^*abb^{(\circ)}(a^*abb^{(\circ)})^{\#} \\ &= bb^{(\circ)}bb^{(\circ)}a^{(1,4)}aa^*abb^{(\circ)}(a^*abb^{(\circ)})^{\#} \\ &= bb^{(\circ)}(a^*abb^{(\circ)})(a^*abb^{(\circ)})^{\#} \\ &= bb^{(\circ)}bb^{(\circ)}a^{(1,4)}a(a^*abb^{(\circ)})^{\#} \\ &= bb^{(\circ)}a^{(1,4)}a(a^*abb^{(\circ)})^{\#} = (a^*abb^{(\circ)})^{\#}. \end{aligned}$$ □

If we repeat the argument of the proof of Theorem 5.1, we get the next result.

Theorem 5.2. *Let $a \in \mathcal{R}$ and let $b \in \mathcal{R}$ have an $\{1,3\}$–inverse $b^{(1,3)}$. If $a^{(\circ)} \in a\{1\} \cup a\{2\}$ and $a^{(\circ)}abb^* \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(a^{(\circ)}abb^*)^{\#} = bb^{(1,3)}a^{(\circ)}a$,

(ii) $a^{(\circ)}abb^* = bb^{(1,3)}a^{(\circ)}a$.

The reverse order law $(ab)^{\#} = b^{(1)}(a^*abb^{(1)})^{\#}a^*$ is considered in a ring with involution in the following theorem.

Theorem 5.3. *Let $a \in \mathcal{R}$ have an $\{1,3\}$–inverse $a^{(1,3)}$ and let $b \in \mathcal{R}$ have an inner inverse $b^{(1)}$. If $a^*abb^{(1)} \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $ab \in \mathcal{R}^{\#}$ *and* $(ab)^{\#} = b^{(1)}(a^*abb^{(1)})^{\#}a^*$,
(ii) $b^{(1)}(a^*abb^{(1)})^{\#}a^* \in (ab)\{5\}$,
(iii) $abaa^{(1,3)} = ab$ *and* $b^{(1)}(a^*abb^{(1)})^{\#}a^*abaa^{(1,3)} = abb^{(1)}(a^*abb^{(1)})^{\#}a^*$,
(iv) $b^{(1)} \cdot (a^*abb^{(1)})\{1,5\} \cdot a^* \subseteq (ab)\{5\}$.

Proof. In the analogy way as in the proof of Theorem 3.2, applying the equalities $a = (a^{(1,3)})^*a^*a$ and $a^* = a^*aa^{(1,3)}$, we verify this result. □

The next result related to the rule $(ab)^{\#} = b^*(a^{(1)}abb^*)^{\#}a^{(1)}$ can be proved as Theorem 5.3.

Theorem 5.4. *Let $a \in \mathcal{R}$ have an inner inverse $a^{(1)}$ and let $b \in \mathcal{R}$ have an $\{1,4\}$–inverse $b^{(1,4)}$. If $a^{(1)}abb^* \in \mathcal{R}^{\#}$, then the following statements are equivalent:*
(i) $ab \in \mathcal{R}^{\#}$ *and* $(ab)^{\#} = b^*(a^{(1)}abb^*)^{\#}a^{(1)}$,
(ii) $b^*(a^{(1)}abb^*)^{\#}a^{(1)} \in (ab)\{5\}$,
(iii) $b^{(1,4)}bab = ab$ *and* $b^{(1,4)}babb^*(a^{(1)}abb^*)^{\#}a^{(1)} = b^*(a^{(1)}abb^*)^{\#}a^{(1)}ab$,
(iv) $b^* \cdot (a^{(1)}abb^*)\{1,5\} \cdot a^{(1)} \subseteq (ab)\{5\}$.

Exactly as Theorem 3.6, we can show the following theorems.

Theorem 5.5. *Let $a \in \mathcal{R}$ have an $\{1,3\}$–inverse $a^{(1,3)}$ and let $b \in \mathcal{R}$ have an outer inverse $b^{(2)}$. If $a^*abb^{(2)} \in \mathcal{R}^{\#}$, then the following statements are equivalent:*
(i) $b^{(2)}(a^*abb^{(2)})^{\#}a^* = b^{(2)}a^*$,
(ii) $b^{(2)}a^*abb^{(2)}a^* = b^{(2)}a^*$,
(iii) $bb^{(2)}a^*a$ *is an idempotent.*

Theorem 5.6. *Let $a \in \mathcal{R}$ have an outer inverse $a^{(2)}$ and let $b \in \mathcal{R}$ have an $\{1,4\}$–inverse $b^{(1,4)}$. If $a^{(2)}abb^* \in \mathcal{R}^{\#}$, then the following statements are equivalent:*
(i) $b^*(a^{(2)}abb^*)^{\#}a^{(2)} = b^*a^{(2)}$,
(ii) $b^*a^{(2)}abb^*a^{(2)} = b^*a^{(2)}$,
(iii) $bb^*a^{(2)}a$ *is an idempotent.*

In the same manner as Theorem 3.7, we can show the next results.

Theorem 5.7. *Let $a \in \mathcal{R}$ have an $\{1,3\}$–inverse $a^{(1,3)}$ and let $b \in \mathcal{R}$ have an inner inverse $b^{(1)}$. If $bb^{(1)}a^*a \in \mathcal{R}^{\#}$, then the following statements are equivalent:*
(i) $a(bb^{(1)}a^*a)^{\#}b = ab$,
(ii) $abb^{(1)}a^*ab = ab$,
(iii) $a^*abb^{(1)}$ *is an idempotent.*

Theorem 5.8. *Let $a \in \mathcal{R}$ have an inner inverse $a^{(1)}$ and let $b \in \mathcal{R}$ have an $\{1,4\}$–inverse $b^{(1,4)}$. If $bb^*a^{(1)}a \in \mathcal{R}^{\#}$, then the following statements are equivalent:*
(i) $a(bb^*a^{(1)}a)^{\#}b = ab$,
(ii) $abb^*a^{(1)}ab = ab$,
(iii) $a^*abb^{(1)}$ *is an idempotent.*

Similarly as Corollary 3.1, we can prove the following results applying Theorem 5.5 and Theorem 5.7 (or Theorem 5.6 and Theorem 5.8).

Corollary 5.1. *Let $a \in \mathcal{R}$ have an $\{1,3\}$–inverse $a^{(1,3)}$ and let $b \in \mathcal{R}$ have a reflexive inverse $b^{(1,2)}$. If $a^*abb^{(1,2)}, bb^{(1,2)}a^*a \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $b^{(1,2)}(a^*abb^{(1,2)})^{\#}a^* = b^{(1,2)}a^*$,
(ii) $b^{(1,2)}a^*abb^{(1,2)}a^* = b^{(1,2)}a^*$,
(iii) $bb^{(1,2)}a^*a$ *is an idempotent,*
(iv) $a(bb^{(1,2)}a^*a)^{\#}b = ab$,
(v) $abb^{(1,2)}a^*ab = ab$,
(vi) $a^*abb^{(1,2)}$ *is an idempotent.*

Corollary 5.2. *Let $a \in \mathcal{R}$ have a reflexive inverse $a^{(1,2)}$ and let $b \in \mathcal{R}$ have an $\{1,4\}$–inverse $b^{(1,4)}$. If $a^{(1,2)}abb^*, bb^*a^{(1,2)}a \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $b^*(a^{(1,2)}abb^*)^{\#}a^{(1,2)} = b^*a^{(1,2)}$,
(ii) $b^*a^{(1,2)}abb^*a^{(1,2)} = b^*a^{(1,2)}$,
(iii) $bb^*a^{(1,2)}a$ *is an idempotent,*
(iv) $a(bb^*a^{(1,2)}a)^{\#}b = ab$,
(v) $abb^*a^{(1,2)}ab = ab$,
(vi) $a^{(1,2)}abb^*$ *is an idempotent.*

The next characterizations of the rules $(ab)^{\#} = b^{(1,2)}a^*$ and $(ab)^{\#} = b^*a^{(1,2)}$ can be verified in the same way as in Theorem 3.8.

Theorem 5.9. (a) *Let $a \in \mathcal{R}$ have an $\{1,3\}$–inverse $a^{(1,3)}$ and let $b \in \mathcal{R}$ have a reflexive inverse $b^{(1,2)}$. If $ab, a^*abb^{(1,2)} \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(ab)^{\#} = b^{(1,2)}a^*$,
(ii) $(ab)^{\#} = b^{(1,2)}(a^*abb^{(1,2)})^{\#}a^*$ *and* $b^{(1,2)}(a^*abb^{(1,2)})^{\#}a^* = b^{(1,2)}a^*$.

(b) *Let $a \in \mathcal{R}$ have a reflexive inverse $a^{(1,2)}$ and let $b \in \mathcal{R}$ have an $\{1,4\}$–inverse $b^{(1,4)}$. If $ab, a^{(1,2)}abb^* \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(ab)^{\#} = b^*a^{(1,2)}$,
(ii) $(ab)^{\#} = b^*(a^{(1,2)}abb^*)^{\#}a^{(1,2)}$ *and* $b^*(a^{(1,2)}abb^*)^{\#}a^{(1,2)} = b^*a^{(1,2)}$.

If $a \in \mathcal{R}$ has an $\{1,3\}$–inverse $a^{(1,3)}$ and $b \in \mathcal{R}$ has a reflexive inverse $b^{(1,2)}$ (or if a has a reflexive inverse $a^{(1,2)}$ and b has an $\{1,4\}$–inverse $b^{(1,4)}$) combining the conditions of Theorem 5.3 and Corollary 5.1 (Theorem 5.4 and Corollary 5.2, respectively), by Theorem 5.9, we get a list of equivalent conditions for the reverse order law $(ab)^{\#} = b^{(1,2)}a^*$ ($(ab)^{\#} = b^*a^{(1,2)}$, respectively) to be satisfied.

As a consequences of Theorem 3.9 and Theorem 3.10, we can check the following results.

Corollary 5.3. *Let $a \in \mathcal{R}$. If $b, a^*ab, a^*abb^{\#} \in \mathcal{R}^{\#}$, then the following statements are equivalent:*

(i) $(a^*ab)^{\#} = b^{\#}(a^*abb^{\#})^{\#}$,
(ii) $b^{\#}(a^*abb^{\#})^{\#} \in (a^*ab)\{5\}$,
(iii) $bb^{\#}a^*ab = a^*ab$ *and* $ba^*abb^{\#}(a^*abb^{\#})^{\#} = (a^*abb^{\#})^{\#}a^*ab$,

(iv) $b\{1,5\}\cdot(a^*abb^\#)\{1,5\}\subseteq(a^*ab)\{5\}$,
(v) $(a^*abb^\#)^\#=b(a^*ab)^\#$,
(vi) $b(a^*ab)^\#\in(a^*abb^\#)\{5\}$,
(vii) $b\cdot(a^*ab)\{1,5\}\subseteq(a^*abb^\#)\{5\}$.

Corollary 5.4. *Let $b\in\mathcal{R}$. If $a, abb^*, a^\#abb^*\in\mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(abb^*)^\#=(a^\#abb^*)^\#a^\#$,
(ii) $(a^\#abb^*)^\#a^\#\in(abb^*)\{5\}$,
(iii) $abb^*aa^\#=abb^*$ *and* $(a^\#abb^*)^\#a^\#abb^*a=abb^*(a^\#abb^*)^\#$,
(iv) $(a^\#abb^*)\{1,5\}\cdot a\{1,5\}\subseteq(abb^*)\{5\}$,
(v) $(a^\#abb^*)^\#=(abb^*)^\#a$,
(vi) $(abb^*)^\#a\in(a^\#abb^*)\{5\}$,
(vii) $(abb^*)\{1,5\}\cdot a\subseteq(a^\#abb^*)\{5\}$.

Some equivalent conditions for the rules $(cab)^\#=b^*(cabb^*)^\#$ and $(abc)^\#=(a^*abc)^\#a^*$ can be obtained as Theorem 3.11.

Corollary 5.5. (a) *Let $a,c\in\mathcal{R}$ and let $b\in\mathcal{R}$ have an $\{1,4\}$–inverse $b^{(1,4)}$. If $cab, cabb^*\in\mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(cab)^\#=b^*(cabb^*)^\#$,
(ii) $b^*(cabb^*)^\#\in(cab)\{5\}$,
(iii) $b^{(1,4)}bcab=cab$ *and* $bcabb^*(cabb^*)^\#=bb^*(cabb^*)^\#cab$.

(b) *Let $b,c\in\mathcal{R}$ and let $a\in\mathcal{R}$ have an $\{1,3\}$–inverse $a^{(1,3)}$. If $abc, a^*abc\in\mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(abc)^\#=(a^*abc)^\#a^*$,
(ii) $(a^*abc)^\#a^*\in(abc)\{5\}$,
(iii) $abcaa^{(1,3)}=abc$ *and* $(a^*abc)^\#a^*abca=abc(a^*abc)^\#a^*a$.

Using Theorem 3.12 and Theorem 3.13, we get the following corollary.

Corollary 5.6. (a) *If $a\in\mathcal{R}$ and $b, a^*ab, a^*abb^\#\in\mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(a^*ab)\{5\}\subseteq b\{1,5\}\cdot(a^*abb^\#)\{1,5\}$ *and* $bb^\#a^*ab=a^*ab$,
(ii) $(a^*ab)\{5\}=b\{1,5\}\cdot(a^*abb^\#)\{1,5\}$.

(b) *If $b\in\mathcal{R}$ and $a, abb^*, a^\#abb^*\in\mathcal{R}^\#$, then the following statements are equivalent:*

(i) $(abb^*)\{5\}\subseteq(a^\#abb^*)\{1,5\}\cdot a\{1,5\}$ *and* $abb^*a^\#a=abb^*$,
(ii) $(abb^*)\{5\}=(a^\#abb^*)\{1,5\}\cdot a\{1,5\}$.

Also, we can prove that the inclusions $(a^*ab)\{5\}\subseteq b^\#\cdot(a^*abb^\#)\{1,5\}$ and $(abb^*)\{5\}\subseteq(a^\#abb^*)\{1,5\}\cdot a^\#$ are always the equalities.

The next result is a consequence of Theorem 3.14 and Theorem 3.15.

Corollary 5.7. (a) *If $a\in\mathcal{R}$ and $b, a^*ab, a^*abb^\#\in\mathcal{R}^\#$, then the inclusion $(a^*abb^\#)\{5\}\subseteq b\cdot(a^*ab)\{1,5\}$ is always an equality.*

(b) *If $b\in\mathcal{R}$ and $a, abb^*, a^\#abb^*\in\mathcal{R}^\#$, then the inclusion $(a^\#abb^*)\{5\}\subseteq(abb^*)\{1,5\}\cdot a$ is always an equality.*

We obtain the following corollary applying Theorem 3.16 and Theorem 3.17.

Corollary 5.8. (a) *Suppose that* $a \in \mathcal{R}$ *and* $b, ab, a^*ab, abb^\#, a^*abb^\# \in \mathcal{R}^\#$. *Then each of the following conditions is sufficient for* $(ab)^\# = b^\#(a^*abb^\#)^\# a^*$ *to hold:*

(i) $(a^*ab)^\# = b^\#(a^*abb^\#)^\#$ *and* $(abb^\#)^\# = (a^*abb^\#)^\# a^*$,

(ii) $(ab)^\# = (a^*ab)^\# a^*$ *and* $(a^*ab)^\# = b^\#(a^*abb^\#)^\#$,

(iii) $(ab)^\# = b^\#(abb^\#)^\#$ *and* $(abb^\#)^\# = (a^*abb^\#)^\# a^*$.

(b) *Suppose that* $b \in \mathcal{R}$ *and* $a, ab, abb^*, a^\#ab, a^\#abb^* \in \mathcal{R}^\#$. *Then each of the following conditions is sufficient for* $(ab)^\# = b^*(a^\#abb^*)^\# a^\#$ *to hold:*

(i) $(a^\#ab^*)^\# = b^*(a^\#abb^*)^\#$ *and* $(abb^*)^\# = (a^\#abb^*)^\# a^\#$,

(ii) $(ab)^\# = (a^\#ab)^\# a^\#$ *and* $(a^\#ab)^\# = b^*(a^\#abb^*)^\#$,

(iii) $(ab)^\# = b^*(abb^*)^\#$ *and* $(abb^*)^\# = (a^\#abb^*)^\# a^\#$.

Similarly as Theorem 3.18, we prove the next results.

Corollary 5.9. (a) *Let* $b \in \mathcal{R}$ *and let* $a \in \mathcal{R}$ *have a reflexive inverse* $a^{(1,2)}$. *If* $ab, a^*ab \in \mathcal{R}^\#$, *then the following statements are equivalent:*

(i) $(ab)^\# = (a^*ab)^\# a^{(1,2)}$ *and* $a^*aba^{(1,2)}a = a^*ab$,

(ii) $(a^*ab)^\# = (ab)^\# a$ *and* $abaa^{(1,2)} = ab$.

(b) *Let* $a \in \mathcal{R}$ *and let* $b \in \mathcal{R}$ *have a reflexive inverse* $b^{(1,2)}$. *If* $ab, abb^* \in \mathcal{R}^\#$, *then the following statements are equivalent:*

(i) $(ab)^\# = b^{(1,2)}(abb^*)^\#$ *and* $bb^{(1,2)}abb^* = abb^*$,

(ii) $(abb^*)^\# = b(ab)^\#$ *and* $b^{(1,2)}bab = ab$.

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