Dragan S. Djordjević,[*] Milica Z. Kolundžija,[*] Sonja Radosavljević,[**] Dijana Mosić,[*]

A SURVEY ON 2×2 OPERATOR MATRICES AND THEIR APPLICATIONS

Abstract. This paper presents the survey of results related to the bounded linear operator matrices on Banach spaces (specially on Hilbert spaces) and the results in Banach algebras, which are related to 2×2 operator matrices. We observe bounded linear operators represented in a matrix form, as well as the elements of Banach algebra in the similar form. Also, we study Fredholm properties and generalized invertibility of such operators. The limited space did not allow us to present all related results, so we believe that the enlarged paper will be the future project. The paper contains some new results on generalized and hypergeneralized projections.

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*University of Niš, Faculty of Sciences and Mathematics, Serbia
`dragandjordjevic70@gmail.com` & `dragan@pmf.ni.ac.rs`,
`milica.kolundzija@gmail.com`, `dijana@pmf.ni.ac.rs`

**Stockholm Resilience Centre, Stockholm University, Stockholm & Department of Mathematics
KTH Royal Institute of Technology, Stockholm
`sonja.radosavljevic@su.se`

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1. Introduction

In this section we introduce the terms and notations that will be used later.

1.1. Banach algebras. Let $\mathcal{A}$ be a complex unital Banach algebra with the unit 1 and let $a \in \mathcal{A}$. Denote the spectrum, the spectral radius and the resolvent set of a by $\sigma(a)$, $r(a)$ and $\rho(a)$, respectively. The sets of all invertible, nilpotent, quasinilpotent and idempotent elements in $\mathcal{A}$ will be denoted by $\mathcal{A}^{-1}$, $\mathcal{A}^{\mathrm{nil}}$, $\mathcal{A}^{\mathrm{qnil}}$ and $\mathcal{A}^{\bullet}$, respectively.

In Banach algebras, we can use idempotents to represent elements in a matrix form. Let $p, q \in \mathcal{A}^{\bullet}$ be arbitrary idempotents. Then an element $a \in \mathcal{A}$ can be written as $a = paq + pa(1-q) + (1-p)aq + (1-p)a(1-q)$. We can use the following notation: $a_{11} = paq$, $a_{12} = pa(1-q)$, $a_{21} = (1-p)aq$, $a_{22} = (1-p)a(1-q)$.

Thus, idempotents $p, q \in \mathcal{A}^{\bullet}$ determine representation of the element $a \in \mathcal{A}$ as the sum such that $a_{11} \in p\mathcal{A}q, a_{12} \in p\mathcal{A}(1-q), a_{21} \in (1-p)\mathcal{A}q, a_{22} \in (1-p)\mathcal{A}(1-q)$, which can be written in the following matrix form

$$a = \begin{bmatrix} paq & pa(1-q) \\ (1-p)aq & (1-p)a(1-q) \end{bmatrix}_{p,q} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{p,q}.$$

We will also use the representation of an element in the matrix form in the case $p = q$. Thus, we represent the element $a \in \mathcal{A}$ as $a = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]_p$, where $a_{11} = pap$, $a_{12} = pa(1-p)$, $a_{21} = (1-p)ap$, $a_{22} = (1-p)a(1-p)$. We will frequently avoid the index p in $[\cdots]_p$ whenever there is no confusion.

Banach algebras $p\mathcal{A}p$ and $(1-p)\mathcal{A}(1-p)$ have the units. The unit in $p\mathcal{A}p$ is p, and the unit in $(1-p)\mathcal{A}(1-p)$ is $1-p$. Notice that $p = \left[\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right]_p$, $1 = \left[\begin{smallmatrix} p & 0 \\ 0 & 1-p \end{smallmatrix}\right]_p$.

1.2. Generalized inverses in Banach algebras. An element $a \in \mathcal{A}$ is inner generalized invertible (generalized invertible, inner invertible, relatively regular, regular), if there exists some $b \in \mathcal{A}$ such that $aba = a$ holds. In this case b is an inner (generalized) inverse of a. The set of all such inverses is denoted by $a\{1\}$, and the set of all inner invertible elements in $\mathcal{A}$ is denoted by $\widehat{\mathcal{A}} \equiv \mathcal{A}^{(1)}$. If $a \in \mathcal{A}^{-1}$, then a^{-1} is the only inner inverse of a.

An element $a \in \mathcal{A}$ is outer generalized invertible, if there exists some $b \in \mathcal{A}$ satisfying $b \neq 0$ and $b = bab$. Such b is called the outer generalized inverse of a. In this case ba and $1 - ab$ are idempotents corresponding to a and b. The set of all outer generalized invertible elements of $\mathcal{A}$ will be denoted with $\mathcal{A}^{(2)}$.

If b is both inner and outer inverse of a, then b is a reflexive inverse of a.

If $aba = a$ and $c = bab$, then $aca = a$ and $cac = c$. Thus, inner invertibility implies outer invertibility, and inner invertibility carries more invertibility properties than the outer invertibility.

Djordjević and Wei introduced outer generalized inverses with prescribed idempotents in [24]

Definition 1.1. [24] Let $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^{\bullet}$. An element $b \in \mathcal{A}$ satisfying

$$bab = b, \quad ba = p, \quad 1 - ab = q,$$

will be called a (p, q)-outer generalized inverse of a, written $a^{(2)}_{p,q} = b$.

The uniqueness of $a_{p,q}^{(2)}$ is provided in the following theorem.

Theorem 1.1. [24] *Let $a \in \mathcal{A}$ and $p, q \in \mathcal{A}^\bullet$. Then the following statements are equivalent*

1) $a_{p,q}^{(2)}$ *exists;*
2) $(1-q)a = (1-q)ap$, *and there exists some $b \in \mathcal{A}$ such that $pb = b$, $bq = 0$ and $ab = 1-q$.*

Moreover, if $a_{p,q}^{(2)}$ exists, then it is unique.

The set of all outer generalized invertible elements of $\mathcal{A}$ with prescribed idempotents $p, q \in \mathcal{A}^\bullet$ will be denoted with $\mathcal{A}_{p,q}^{(2)}$.

The generalized Drazin inverse of $a \in \mathcal{A}$ is the element $b \in \mathcal{A}$ which satisfies

$$bab = b, \quad ab = ba, \quad a - a^2 b \in \mathcal{A}^{\text{qnil}}.$$

If b exists, it is unique and will be denoted by a^d. The set $\mathcal{A}^d$ consists of all $a \in \mathcal{A}$ such that a^d exists. Koliha [35] studied the generalized Drazin inverse in Banach algebras. Harte gave an alternative definition of a generalized Drazin inverse in a ring [32].

The Drazin inverse is a special case of the generalized Drazin inverse for which $a - a^2 b \in \mathcal{A}^{\text{nil}}$. The group inverse is a special case of the Drazin inverse for which $a - a^2 b \in \mathcal{A}^{\text{nil}}$ is replaced with $a = aba$. By $a^\#$ will be denoted the group inverse of a.

An element $p = p^2 \in \mathcal{A}$ is a *spectral idempotent* of a if

$$ap = pa \in \mathcal{A}^{\text{qnil}}, \qquad a + p \in \mathcal{A}^{-1}.$$

Such an element is unique if it exists and it will be denoted by a^π [30, 33, 35, 36]. Recall that $a^\pi = 1 - aa^d$. For the theory of generalized inverses and its applications, we refer the reader to [3, 5, 22].

Theorem 1.2. 1) *An element $a \in \mathcal{A}$ is generalized Drazin invertible, if and only if $0 \notin \operatorname{acc} \sigma(a)$.*

2) *An element $a \in \mathcal{A}$ is Drazin invertible, if and only if $0 \notin \operatorname{acc} \sigma(a)$ and 0 is the pole of the resolvent $\lambda \mapsto (\lambda - a)^{-1}$.*

If a is Drazin invertible and $(a - a^2 a^d)^n = 0$, then the smallest such n is the Drazin index of a, denoted by $\operatorname{ixd}(a)$. If such n does not exist and a is generalized Drazin invertible, then $\operatorname{ixd}(a) = \infty$.

If $\mathcal{A}$ is a the algebra of all bounded linear operators on a Banach space X, then $A \in \mathcal{A}$ is Drazin invertible if and only if $\operatorname{asc}(A) < \infty$ and $\operatorname{dsc}(A) < \infty$. Here $\operatorname{asc}(A)$, the ascent of A, is the minimal n such that $\mathcal{N}(A^{n+1}) = \mathcal{N}(A^n)$ (if such n exists), and $\operatorname{dsc}(A)$, the descent of A, is the minimal n such that $\mathcal{R}(A^{n+1}) = \mathcal{R}(A^n)$. In this case, $\operatorname{ixd}(A) = \operatorname{asc}(A) = \operatorname{dsc}(A) = n$.

The Drazin inverse is very important in various applied mathematical fields such as iterative methods, singular differential equations, singular difference equations, Markov chains and so on. Under specific conditions many authors have studied representations for the Drazin inverse [11, 13, 23, 34, 46, 57].

2. Axiomatic spectrum

In this section we recall the axiomatic spectrum, which is introduced in [43] and widely investigated in [49, pages 51–58].

Definition 2.1. A non-empty set R ($R \subset \mathcal{A}$) is a regularity, provided that the following holds

1) If $a \in \mathcal{A}$ and $n \in \mathbb{N}$, then $a \in R$ if and only if $a^n \in R$;
2) If $a, b, c, d \in \mathcal{A}$ are mutually commuting elements and $ac + bd = 1$, then

$$ab \in R \iff a \in R \text{ and } b \in R.$$

Lemma 2.1. *If R is a regularity in $\mathcal{A}$, then $\mathcal{A}^{-1} \subset R$.*

Definition 2.2. If R is a regularity in $\mathcal{A}$ and $a \in \mathcal{A}$, then the R-spectrum of a is defined as $\sigma_R(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin R\}$.

We collect some properties of the regularity that we will use later.

Corollary 2.1. *If R is a regularity in $\mathcal{A}$, then*
(1) *$\sigma_R(a) \subset \sigma(a)$ for every $a \in \mathcal{A}$;*
(2) *$\sigma_R(a - \lambda) = \sigma_R(a) - \lambda$ for every $a \in \mathcal{A}$ and every $\lambda \in \mathbb{C}$.*

Definition 2.3. The mapping $a \mapsto \sigma_R(a)$ is upper semicontinuous, if

$$a_n, a \in \mathcal{A}, \ \lim_{n\to\infty} a_n = a, \ \lambda_n \in \sigma_R(a_n), \ \lim_{n\to\infty} \lambda_n = \lambda \implies \lambda \in \sigma_R(a).$$

The following result holds [49, page 55].

Theorem 2.1. *Let R be a regularity in $\mathcal{A}$. The the following statements are equivalent*

1) *The mapping $a \mapsto \sigma_R(a)$ is upper semicontinuous;*
2) *The mapping $a \mapsto \sigma_R(a)$ is upper semicontinuous and $\sigma_R(a)$ is closed for every $a \in \mathcal{A}$;*
3) *R is open in $\mathcal{A}$.*

Lemma 2.2. *Let $p \in \mathcal{A}^\bullet$ and let R be a regularity in $\mathcal{A}$. Then*

$$R_p = \left\{ a \in p\mathcal{A}p : a_1 = \begin{bmatrix} a & 0 \\ 0 & 1-p \end{bmatrix}_p \in R \right\}$$

is the regularity in $p\mathcal{A}p$ induced by p.

Proof. Let $a = pap \in p\mathcal{A}p$ and $n \in \mathbb{N}$. Then

$$a_1 = \begin{bmatrix} a & 0 \\ 0 & 1-p \end{bmatrix}_p, \quad a_1^n = \begin{bmatrix} a^n & 0 \\ 0 & 1-p \end{bmatrix}_p.$$

Since $a_1 \in R$ if and only if $a_1^n \in R$, it is obvious that $a^n \in R_p$ if and only if $a_1^n \in R_p$.

Now, let $a, b, c, d \in p\mathcal{A}p$ mutually commute and $ac + bd = p$. Again, let

$$a_1 = \begin{bmatrix} a & 0 \\ 0 & 1-p \end{bmatrix}_p, \ b_1 = \begin{bmatrix} b & 0 \\ 0 & 1-p \end{bmatrix}_p, \ c_1 = \begin{bmatrix} c & 0 \\ 0 & 1-p \end{bmatrix}_p, \ d_1 = \begin{bmatrix} d & 0 \\ 0 & 1-p \end{bmatrix}_p.$$

We obviously have

$$a_1c_1 + b_1d_1 = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}_p = 1.$$

Hence, $a_1b_1 \in R$ if and only if $a_1 \in R$ and $b_1 \in R$. The last is trivially equivalent to $ab \in R_p$ if and only if $a \in R_p$ and $b \in R_p$. □

Definition 2.4. An open regularity R in $\mathcal{A}$ is strong, if the following holds

$$a = c^{-1}bc \implies (a \in R \iff b \in R).$$

It seems that the notion of strong regularity is not investigated in [43] and [49]. As a corollary, we get the result.

Corollary 2.2. *If R is a strong regularity and $a = c^{-1}bc$, then $\sigma_R(a) = \sigma_R(b)$.*

2.1. Schur complement. Let M be a 2×2 block matrix $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$ and $D \in \mathbb{C}^{l\times k}$. If A is invertible, then the Schur complement of A in M is defined as

$$S = D - CA^{-1}B.$$

If M is invertible, then S is invertible, too, and M can be decomposed as

$$M = \begin{bmatrix} I_m & 0 \\ CA^{-1} & I_l \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I_m & A^{-1}B \\ 0 & I_l \end{bmatrix},$$

where I_t is the identity matrix of order t. In this case, the inverse of M can be written as

$$\begin{aligned} M^{-1} &= \begin{bmatrix} I_m & -A^{-1}B \\ 0 & I_l \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -CA^{-1} & I_l \end{bmatrix} \\ &= \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}. \end{aligned} \tag{2.1}$$

The result (2.1) is well known as the Banachiewicz–Schur form of M, and it has been used in dealing with inverses of block matrices.

Let $x = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]_p \in \mathcal{A}$ relative to the idempotent $p \in \mathcal{A}$. If $a \in (p\mathcal{A}p)^{-1}$ and the Schur complement $s = d - ca^{-1}b \in ((1-p)\mathcal{A}(1-p))^{-1}$, then the inverse of x has the Banachiewicz–Schur form

$$x^{-1} = \begin{bmatrix} a^{-1} + a^{-1}bs^{-1}ca^{-1} & -a^{-1}bs^{-1} \\ -s^{-1}ca^{-1} & s^{-1} \end{bmatrix}.$$

3. Bounded operators on Banach and Hilbert spaces

Let $\mathcal{L}(X,Y)$ denote the set of all bounded linear operators from X to Y. We abbreviate $\mathcal{L}(X) = \mathcal{L}(X,X)$. For $A \in \mathcal{L}(X,Y)$ we use $\mathcal{R}(A)$ and $\mathcal{N}(A)$ to denote the range and the null-space of A, respectively. The set of all finite rank operators from X to Y is denoted by $\mathcal{F}(X,Y)$. We abbreviate $\mathcal{F}(X) = \mathcal{F}(X,X)$.

We use $\mathcal{G}_l(X,Y)$, $\mathcal{G}_r(X,Y)$ and $\mathcal{G}(X,Y)$, respectively, to denote the set of all left, the set of all right and the set of all invertible operators from $\mathcal{L}(X,Y)$. The abbreviations $\mathcal{G}_l(X)$, $\mathcal{G}_r(X)$ and $\mathcal{G}(X)$ are clear. Recall that $A \in \mathcal{G}_l(X,Y)$ if and only if $\mathcal{N}(A) = \{0\}$ and $\mathcal{R}(A)$ is closed and complemented in Y. Also, $A \in \mathcal{G}_r(X)$ if and only if $\mathcal{N}(A)$ is complemented in X and $\mathcal{R}(A) = Y$.

Two Hilbert spaces, among other things, can be compared by their orthogonal dimensions. In the case of Banach spaces it seems that the existence of left or right invertible operators is a useful substitution.

Definition 3.1. If X and Y are Banach spaces, then X can be embedded in Y, if there exists a left invertible operator $W \in \mathcal{L}(X,Y)$. The notation is $X \preceq Y$.

Also, $X \preceq Y$ if and only if there exists right invertible operator $J_1 \colon Y \to X$.

If X and Y are Hilbert spaces, then $X \preceq Y$ if and only if $\dim X \leqslant \dim Y$.

We use X' to denote the dual space of X. If $A \in \mathcal{L}(X,Y)$, then $A' \in \mathcal{L}(Y',X')$ is the dual operator of A.

3.1. Operator matrices. Let Z be a Banach space, such that $Z = X \oplus Y$ for some closed and complementary subspaces X and Y. This sum will be also denoted by $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$. If Z is a Hilbert space, then we always assume that X and Y are closed and mutually orthogonal subspaces of Z, so in this case $Z = X \oplus Y$ denotes the orthogonal sum.

If W is a finite dimensional subspace of a Banach space, then $\dim W$ denotes the dimension of W. If W is infinite dimensional, then we simply write $\dim W = \infty$. However, if X is a Hilbert space and W is a closed subspace of X, then $\dim W$ is the orthogonal dimension of W.

If $Z = X \oplus Y$, then every bounded linear operator $M \in \mathcal{L}(Z)$ can be represented in the following matrix representation

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} X \\ Y \end{bmatrix},$$

for some $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y,X)$, $C \in \mathcal{L}(X,Y)$ and $D \in \mathcal{L}(Y)$. On the other hand, arbitrary operators A, B, C, D (bounded and linear on corresponding subspaces) give a bounded linear operator M on the space Z.

We will study operators in the following matrix representations

- $M_{(T,S)} = \left[\begin{smallmatrix} A & C \\ T & S \end{smallmatrix}\right]$, where the operators A and C are given, and operators T and S arbitrary. The notation $M_{(T,S)}$ is taken to be clear that the operator $M_{(T,S)}$ depends of operators T and S;
- $M_C = \left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right]$, where the operators A and B are given, and the operator C is arbitrary. Also, the notation M_C is clear.

Specially, if $C = 0$ in the operator M_C, we have diagonal operator matrix and we denote it as M_0. So, $M_0 = \left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$.

3.2. Applications to generalized inverses. Let $B \in \mathcal{L}(X,Y)$ be given. B is relatively regular (inner invertible) if there exists some $D \in \mathcal{L}(Y,X)$ such that $BDB = B$ holds. In this case D is an inner inverse of B. It is well-known that B is relatively regular, if and only if $\mathcal{R}(B)$ and $\mathcal{N}(B)$ are closed and complemented in Y and X, respectively. If $DBD = D$ holds and $D \neq 0$, then B is outer invertible, and D is an outer inverse of B. If $B \neq 0$, then it is a corollary of the Hahn–Banach theorem that there exists some non-zero outer inverse D of B. If D is both inner and outer inverse of B, then D is a reflexive inverse of B. Moreover, if D is an inner inverse of B, then DBD is a reflexive inverse of B.

If $D \in \mathcal{L}(Y,X)$ is a reflexive inverse of $B \in \mathcal{L}(X,Y)$, then BD is the projection from Y onto $\mathcal{R}(B)$ parallel to $\mathcal{N}(D)$, and DB is the projection from X onto $\mathcal{R}(D)$ parallel to $\mathcal{N}(B)$. On the other hand, if $X = U \oplus \mathcal{N}(B)$ and $Y = \mathcal{R}(B) \oplus V$ for closed subspaces: U of X and V of Y, then B have the matrix form

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} U \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ V \end{bmatrix},$$

and B_1 is invertible. It is easy to see that

$$D = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ V \end{bmatrix} \to \begin{bmatrix} U \\ \mathcal{N}(B) \end{bmatrix}$$

is the reflexive inverse of B satisfying $\mathcal{R}(B) = U$ and $\mathcal{N}(D) = V$.

It is also obvious that every inner generalized inverse of B has the form

$$E = \begin{bmatrix} B_1^{-1} & K \\ L & M \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ V \end{bmatrix} \to \begin{bmatrix} U \\ \mathcal{N}(B) \end{bmatrix},$$

where K, L, M are arbitrary bounded linear operators on appropriate spaces.

If H, K are Hilbert spaces, and $A \in \mathcal{L}(H,K)$, then the Moore–Penrose inverse of A is the unique operator $A^\dagger \in \mathcal{L}(K,H)$ (in the case when it exists) which satisfies

$$AA^\dagger A = A,\ A^\dagger AA^\dagger = A^\dagger,\ (AA^\dagger)^* = AA^\dagger,\ (A^\dagger A)^* = A^\dagger A.$$

The Moore–Penrose inverse of $A \in \mathcal{L}(H,K)$ exists if and only if $\mathcal{R}(A)$ is closed. If $A \in \mathcal{L}(H,K)$ is left (right) invertible, then $A^\dagger$ is a left (right) inverse of A. It is obvious that A and $A^\dagger$ have the following matrix forms with respect to the orthogonal decompositions of subspaces

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}, \quad A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix},$$

and A_1 is invertible.

3.3. Diagonal and triangular operator matrices. If R is a regularity in $\mathcal{L}(Z)$, then corresponding regularities in $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ are, accoridng to previous section, defined as follows

$$R_1 = \left\{ A \in \mathcal{L}(X) : \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \in R \right\}, \quad R_2 = \left\{ B \in \mathcal{L}(Y) : \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix} \in R \right\}.$$

Let $Z = X \oplus Y$, $A \in \mathcal{L}(X), B \in \mathcal{L}(Y), C \in \mathcal{L}(Y,X)$, and

$$M_0 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \quad M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}.$$

Lemma 3.1. [49, pages 53–54] $\sigma_R(M_0) = \sigma_{R_1}(A) \cup \sigma_{R_2}(B)$.

Lemma 3.2. *If R, R_1, R_2 are strong regularities, then $\sigma_R(M_C) \subset \sigma_{R_1}(A) \cup \sigma_{R_2}(B)$ for every $C \in \mathcal{L}(Y,X)$.*

Proof. Notice that

$$\begin{bmatrix} I & 0 \\ 0 & nI \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \frac{1}{n}I \end{bmatrix} = \begin{bmatrix} A & \frac{1}{n}C \\ 0 & B \end{bmatrix} = M_n.$$

Then $\sigma_R(M_C) = \sigma_R(M_n)$. Obviously, $\lim_{n\to\infty} M_n = M$. By the continuity of the R-spectrum, we conclude that $\sigma_R(M_C) \subset \sigma_{R_1}(A) \cup \sigma_{R_2}(B)$. □

4. Semi-Fredholm operators

An operator $A \in \mathcal{L}(X,Y)$ is upper semi-Fredholm, if $\mathcal{N}(A)$ is finite dimensional, and $\mathcal{R}(A)$ is closed. The set of all such operators is denoted by $\Phi_+(X,Y)$.

$A \in \mathcal{L}(X,Y)$ is lower semi-Fredholm, if $\mathcal{R}(A)$ is finite codimensional. It imediatelly follows that $\mathcal{R}(A)$ is closed. The set of all such operators is denoted by $\Phi_-(X,Y)$.

The set of Fredholm operators is $\Phi(X,Y) := \Phi_+(X,Y) \cap \Phi_-(X,Y)$.

Obviously, Fredholm operators are relatively regular, but semi-Fredholm operators are not necessarily relative regular. If we require relative regularity of semi-Fredholm operators, then we obtain left and right semi-Fredholm operators

$$\Phi_l(X,Y) := \Phi_+(X,Y) \cap \widehat{\mathcal{L}(X,Y)},$$
$$\Phi_r(X,Y) := \Phi_-(X,Y) \cap \widehat{\mathcal{L}(X,Y)}.$$

If $\mathcal{N}(A)$ is finite dimensional, then $\mathrm{nul}(A) = \dim \mathcal{N}(A)$. If $\mathcal{N}(A)$ is infinite dimensional, then $\mathrm{nul}(A) = \infty$. Similarly, if $\mathcal{R}(A)$ is finite codimensional, then $\mathrm{def}(A) = \dim Y/R(A)$. If $\mathcal{R}(A)$ is infinite codimensional, then $\mathrm{def}(A) = \infty$.

We restore the proof of main results in Fredholm theory using operator matrices.

Theorem 4.1. *Let $A \in \Phi_-(X,Y)$ and $B \in \Phi_-(Y,Z)$. Then $BA \in \Phi_-(X,Y)$.*

Proof. Since $\mathrm{def}(A) < \infty$ and $\mathrm{def}(B) < \infty$, we conclude that there exist: a finite dimensional subspace Y_1 of Y such that $Y = \mathcal{R}(A) \oplus Y_1$, and a finite dimensional subspace Z_1 of Z such that $Z = \mathcal{R}(B) \oplus Z_1$. It follows that we have matrix form of A

$$A = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} : X \to \begin{bmatrix} \mathcal{R}(A) \\ Y_1 \end{bmatrix},$$

where $A_1 \colon X \to \mathcal{R}(A)$ is onto. The matrix form of B is

$$B = \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ Y_1 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ Z_1 \end{bmatrix},$$

where $B_1 \colon \mathcal{R}(A) \to \mathcal{R}(B)$ and $B_2 \colon Y_1 \to \mathcal{R}(B)$. We have

$$\begin{aligned} \mathcal{R}(B) &= \left\{ \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} : x \in \mathcal{R}(A), y \in Y_1 \right\} \\ &= \left\{ \begin{bmatrix} B_1 x + B_2 y \\ 0 \end{bmatrix} : x \in \mathcal{R}(A), y \in Y_1 \right\} \\ &= \mathcal{R}(B_1) + \mathcal{R}(B_2). \end{aligned}$$

Since Y_1 is finite dimensional, we conclude that $\mathcal{R}(B_2) = B_2(Y_1)$ is finite dimensional. Thus, $\operatorname{def}(B_1) \leqslant \dim \mathcal{R}(B_2) < \infty$.

We see that

$$BA = \begin{bmatrix} B_1 A_1 \\ 0 \end{bmatrix} : X \to \begin{bmatrix} \mathcal{R}(B) \\ Z_1 \end{bmatrix}.$$

Using the fact that $A_1 \colon X \to \mathcal{R}(A)$ is onto, we get

$$\operatorname{def}(BA) = \operatorname{def}(B_1 A_1) + \dim Z_1 = \operatorname{def}(B_1) + \operatorname{def}(B) < \infty.$$

It follows that $\mathcal{R}(BA)$ is closed. The result is proved. □

Now we state the well-know results.

Lemma 4.1. *Let X be a normed space, let M be a closed subspace of X, and let N be a finite dimensional subspace of X. Then $M + N$ is a closed subspace of X.*

Proof. Without loss of generality, we can assume that $M \cap N = \{0\}$. Consider the quotient normed space $Y = X/M$, and the natural continuous epimorphism $\pi \colon X \to Y$. Since N is finite dimensional, we get that $\pi(N)$ is finite dimensional in Y, so $\pi(N)$ is closed in Y. From the continuity of π we get that $M+N = \pi^{-1}(\pi(N))$ is closed in X. □

Theorem 4.2. *Let $A \in \Phi_+(X, Y)$ and $B \in \Phi_+(Y, Z)$. Then $BA \in \Phi_+(X, Z)$.*

Proof. Since $\mathcal{N}(A)$ and $\mathcal{N}(B)$ are finite dimensional, there exist: a closed subspace X_1 of X such that $X = X_1 \oplus \mathcal{N}(A)$, and a closed subspace Y_1 of Y such that $Y = Y_1 \oplus \mathcal{N}(B)$. Thus, A has the matrix form

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} Y_1 \\ \mathcal{N}(B) \end{bmatrix},$$

where $A_1 \colon X_1 \to Y_1$ and $A_2 \colon X_1 \to \mathcal{N}(B)$. Also, B has the matrix form

$$B = \begin{bmatrix} B_1 & 0 \end{bmatrix} : \begin{bmatrix} Y_1 \\ \mathcal{N}(B) \end{bmatrix} \to Z,$$

where $B_1 \colon Y_1 \to Z$ is one-to-one with closed range.

We see that

$$BA = \begin{bmatrix} B_1 A_1 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ \mathcal{N}(A) \end{bmatrix} \to Z.$$

Let $x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X_1 \\ \mathcal{N}(A) \end{smallmatrix}\right]$. We have that $x \in \mathcal{N}(BA)$ if and only if $B_1 A_1 x_1 = 0$. Since B_1 is one-to-one, we conclude that $A_1 x_1 = 0$. Hence, $\mathcal{N}(BA) = \mathcal{N}(A_1) \oplus \mathcal{N}(A)$.

If we take $x_1 \in \mathcal{N}(A_1)$ and $x_1 \neq 0$, then it is not possible $A_2 x_1 = 0$. On the opposite, if $A_2 x_1 = 0$ then $x_1 \in X_1 \cap \mathcal{N}(A) = \{0\}$. We conclude that the restriction $A_2|_{\mathcal{N}(A_1)} \colon \mathcal{N}(A_1) \to \mathcal{N}(B)$ is one-to-one. This means that $\dim \mathcal{N}(A_1) \leqslant \dim \mathcal{N}(B) < \infty$. Finally,

$$\dim \mathcal{N}(BA) = \dim \mathcal{N}(A_1) + \dim \mathcal{N}(A) \leqslant \dim \mathcal{N}(B) + \dim \mathcal{N}(A) < \infty.$$

We have to prove that $\mathcal{R}(BA)$ is closed. We already mentioned that $\mathcal{N}(A_1)$ is finite dimensional in X_1. Hence, there exsits a closed subspace X_2 of X_1 such that $X_1 = X_2 \oplus \mathcal{N}(A_1)$. Thus, A_1 has the matrix form

$$A_1 = \begin{bmatrix} A_{11} & 0 \end{bmatrix} : \begin{bmatrix} X_2 \\ \mathcal{N}(A_1) \end{bmatrix} \to Y_1,$$

where $A_{11} \colon X_2 \to Y_1$ is one-to-one. Then, A_2 has the following matrix form

$$A_2 = \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} : \begin{bmatrix} X_2 \\ \mathcal{N}(A_1) \end{bmatrix} \to \mathcal{N}(B),$$

where $A_{21} \colon X_2 \to \mathcal{N}(B)$ and $A_{22} \colon \mathcal{N}(A_1) \to \mathcal{N}(B)$. Since $\mathcal{N}(B)$ is finite dimensional, we get that A_{21} and A_{22} are both finite rank operators. Moreover, A_{22} is one-to-one, but this is not important right now.

We get the matrix form of A as follows

$$A = \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \end{bmatrix} : \begin{bmatrix} X_2 \\ \mathcal{N}(A_1) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} Y_1 \\ \mathcal{N}(B) \end{bmatrix}.$$

It follows that

$$\mathcal{R}(A) = \left\{ \begin{bmatrix} A_{11}x \\ A_{21}x + A_{22}y \end{bmatrix} : x \in X_2, y \in \mathcal{N}(A_1) \right\} = \mathcal{R}\left(\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \right) + \mathcal{R}(A_{22}).$$

The subspace $\mathcal{R}(A_{22})$ is finite dimensional and $\mathcal{R}(A)$ is closed. From Lemma 4.1 it follows that the subspace $\mathcal{R}\left(\left[\begin{smallmatrix} A_{11} \\ A_{21} \end{smallmatrix}\right]\right)$ is closed.

It is easy to see that for dual spaces we have $(Y_1 \oplus \mathcal{N}(B))' = Y_1' \oplus \mathcal{N}(B)'$. Thus, similar holds for the dual operator

$$\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}' = \begin{bmatrix} A_{11}' & A_{21}' \end{bmatrix} : \begin{bmatrix} Y_1' \\ \mathcal{N}(B)' \end{bmatrix} \to X_2',$$

which has a closed range. Now, $\mathcal{R}\left(\begin{bmatrix} A_{11}' & A_{21}' \end{bmatrix}\right) = \mathcal{R}(A_{11}') + \mathcal{R}(A_{21}')$. The subspace $\mathcal{R}(A_{21}')$ is finite dimensional. Again, by Theorem 4.1 it follows that $\mathcal{R}(A_{11}')$ is closed, and consequently, $\mathcal{R}(A_{11})$ is closed.

Finaly, $\mathcal{R}(BA) = B_1(\mathcal{R}(A_{11}))$ is closed. □

The index of a semi-Fredholm operator A is defined as

$$\operatorname{ind}(A) = \operatorname{nul}(A) - \operatorname{def}(A).$$

5. Moore–Penrose and Drazin inverse of two projections on Hilbert space

Throughout this section, H, K will stand for Hilbert spaces. For $A \in \mathcal{L}(H, K)$ we use A^* to denote the adjoint operator of A.

An operator $P \in \mathcal{L}(H)$ is an idempotent if $P = P^2$, and P is an orthogonal projection if $P = P^2 = P^*$.

Generalized and hypergeneralized projections are inroduced in [27].

Definition 5.1. An operator $G \in \mathcal{L}(H)$ is

1) a generalized projection if $G^2 = G^*$,
2) a hypergeneralized projection if $G^2 = G^\dagger$.

The set of all generalized projecton on H is denoted by $\mathcal{GP}(H)$ and set of all hypergeneralized projecton is denoted by $\mathcal{HGP}(H)$.

Notice that if $A \in \mathcal{L}(H,K)$ has a closed range and $A^\dagger$ is the Moore–Penrose inverse of A, then $AA^\dagger$ is orthogonal projection from K onto $\mathcal{R}(A)$, and $A^\dagger A$ is the orthogonal projection from H onto $\mathcal{R}(A^*)$.

An essential property of any $P \in \mathcal{L}(H)$ is that P is an orthogonal projection if and only if it is expressible as $P = AA^\dagger$, for some $A \in \mathcal{L}(H)$.

Operator $A \in \mathcal{L}(H)$ is EP if $AA^\dagger = A^\dagger A$, or, in the other words, if $A^\dagger = A^D = A^\#$. There are many characterization of EP operators. In this paper, we use results from [20].

In what follows, $\bar{A}$ will stand for $I - A$ and P_A will stand for $AA^\dagger$.

Let $P, Q \in \mathcal{L}(H)$ be orthogonal projectons and $\mathcal{R}(P) = L$. Since $H = R(P) \oplus R(P)^\perp = L \oplus L^\perp$, we have the following representaton of projections $P, \bar{P}, Q, \bar{Q} \in \mathcal{L}(H)$ with respect to the decomposition of space

$$
\begin{aligned}
P &= \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_L & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \to \begin{bmatrix} L \\ L^\perp \end{bmatrix}, \\
\bar{P} &= \begin{bmatrix} 0 & 0 \\ 0 & I_{L^\perp} \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \to \begin{bmatrix} L \\ L^\perp \end{bmatrix}, \\
Q &= \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \to \begin{bmatrix} L \\ L^\perp \end{bmatrix}, \\
\bar{Q} &= \begin{bmatrix} I_L - A & -B \\ -B^* & I_{L^\perp} - D \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \to \begin{bmatrix} L \\ L^\perp \end{bmatrix},
\end{aligned} \tag{5.1}
$$

with $A \in \mathcal{L}(L)$ and $D \in \mathcal{L}(L^\perp)$ being Hermitian and non-negative.

We prove the following two theorems, which are known for operators on $\mathbb{C}^n$ (see [2]).

Theorem 5.1. *Let $Q \in \mathcal{L}(H)$ be represented as in* (5.1)*, and suppose that $\mathcal{R}(A)$, $\mathcal{R}(\overline{A})$, $\mathcal{R}(D)$, $\mathcal{R}(\overline{D})$ and $\mathcal{R}(AA^* + BB^*)$ are closed. Then the following holds*

1) *$A = A^2 + BB^*$, or, equivalently, $A\bar{A} = BB^*$,*
2) *$B = AB + BD$, or, equivalently, $B^* = B^*A + DB^*$,*
3) *$D = D^2 + B^*B$, or, equivalently, $D\bar{D} = B^*B$.*

Proof. Since $Q = Q^2$, we obtain

$$\begin{bmatrix} A & B \\ B^* & D \end{bmatrix} \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} = \begin{bmatrix} A^2 + BB^* & AB + BD \\ B^*A + DB^* & B^*B + D^2 \end{bmatrix} = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$$

implying that $A = A^2 + BB^*$, $B = AB + BD$ and $D = D^2 + B^*B$. □

Theorem 5.2. *Let $Q \in \mathcal{L}(H)$ be represented as in* (3)*. Then*

1) $\mathcal{R}(B) \subseteq \mathcal{R}(A)$,	4) $\mathcal{R}(B^*) \subseteq \mathcal{R}(\bar{D})$,	7) *A is a contraction,*
2) $\mathcal{R}(B) \subseteq \mathcal{R}(\bar{A})$,	5) $A^\dagger B = B\bar{D}^\dagger$,	8) *D is a contraction,*
3) $\mathcal{R}(B^*) \subseteq \mathcal{R}(D)$,	6) $\bar{A}^\dagger B = BD^\dagger$,	9) $A - BD^\dagger B^* = I_L - \bar{A}\bar{A}^\dagger$.

Proof. 1) Since $A = A^2 + BB^*$, we have $\mathcal{R}(A) = \mathcal{R}(A^2 + BB^*) = \mathcal{R}(AA^* + BB^*)$. To prove that $\mathcal{R}(AA^*+BB^*) = \mathcal{R}(A)+\mathcal{R}(B)$, observe operator matrix $M = \left[\begin{smallmatrix} A & B \\ 0 & 0 \end{smallmatrix}\right]$.

For any $x \in \mathcal{R}(MM^*)$, there exisit $y \in H$ such that $x = MM^*y = M(M^*y)$ and $x \in \mathcal{R}(M)$. On the other hand, for $x \in \mathcal{R}(M)$, there is $y \in H$ and $x = My$. Besides, $MM^\dagger x = MM^\dagger My = My = x$ and $MM^\dagger = MM^*(MM^*)^\dagger = P_{\mathcal{R}(MM^*)}$ implying $x \in \mathcal{R}(MM^*)$. Hence, $\mathcal{R}(M) = \mathcal{R}(MM^*)$ and

$$\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(M) = \mathcal{R}(MM^*) = \mathcal{R}(AA^* + BB^*)$$

and we have $\mathcal{R}(A) = \mathcal{R}(A) + \mathcal{R}(B)$ implying $\mathcal{R}(B) \subseteq \mathcal{R}(A)$.

2) Since $A = I - \bar{A}$, from Theorem 5.1 (1), we get $\bar{A} = \bar{A}^2 + BB^*$. The rest of the proof is analogous to the point (1) of this theorem.

3), 4) Similarly.

5) Since $B = AB + BD$, we have $A^\dagger B = A^\dagger(AB + BD) = A^\dagger AB + A^\dagger BD$ and using the facts that $A^\dagger A = P_{\mathcal{R}(A^*)}$ and $\mathcal{R}(B) \subseteq \mathcal{R}(A^*)$, we get $A^\dagger AB = B$ and $A^\dagger B = B + A^\dagger BD$, or, equivalently $B = A^\dagger B\bar{D}$. Postmultiplying this equation by $\bar{D}^\dagger$ and using item (4) of this Theorem, in its equivalent form $B\bar{D}\,\bar{D}^\dagger = B$, we obtain (5).

6) Analogously to the previous proof.

7) Since $A = A^*$, from Theorem 5.1 (1), we have that

$$I_L - AA^* = I_L - (A - BB^*) = \bar{A} + BB^*,$$

and the right-hand side is nonnegative as a sum of two nonnegative operators implying that A is a contraction.

8) This part of the proof is dual to the part (7).

9) From Theorem 5.1 (1), item (6) of this Theorem and the fact that hermitian operator A commutes with its MP-inverse, it follows that

$$BD^\dagger B^* = \bar{A}^\dagger BB^* = \bar{A}^\dagger A\bar{A} = \bar{A}^\dagger(I - \bar{A})\bar{A} = \bar{A}^\dagger\bar{A} - \bar{A}^\dagger\bar{A}\,\bar{A} = \bar{A}^\dagger\bar{A} - \bar{A}$$

by taking into account that $\bar{A}\,\bar{A}^\dagger = \bar{A}^\dagger\bar{A}$. Now, we get $A - BD^\dagger B^* = I - \bar{A}^\dagger\bar{A}$, establishing the condition. □

Following the results of Gross and Trenkler for matrices, we will formulate a few theorems for generalized and hypergeneralized projections on arbitrary Hilbert space. We start with the result which is very similar to Theorem (1) in [27].

Theorem 5.3. *Let $G \in \mathcal{L}(H)$ be a generalized projection. Then G is a closed range operator and G^3 is an orthogonal projection on $\mathcal{R}(G)$. Moreover, H has decomposition $H = \mathcal{R}(G) \oplus \mathcal{N}(G)$ and G has the following matrix representaton*

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix},$$

where restriction $G_1 = G|_{\mathcal{R}(G)}$ is unitary on $\mathcal{R}(G)$.

Proof. If G is a generalized projection, then $G^4 = (G^2)^2 = (G^*)^2 = (G^2)^* = (G^*)^* = G$. From $GG^*G = G^4 = G$ follows that G is a partial isometry implying that

$$G^3 = GG^* = P_{\mathcal{R}(G)}, \quad G^3 = G^*G = P_{\mathcal{N}(G)^\perp}.$$

Thus, G^3 is an orthogonal projection onto $\mathcal{R}(G) = \mathcal{N}(G)^\perp = \mathcal{R}(G^*)$. Consequently, $\mathcal{R}(G)$ is a closed subset in H as a range of an orthogonal projection on a Hilbert space. From Lemma (1.2) in [20] we get the following decomposition of the space

$$H = \mathcal{R}(G^*) \oplus \mathcal{N}(G) = \mathcal{R}(G) \oplus \mathcal{N}(G).$$

Now, G has the following matrix representation in accordance with this decomposition

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix},$$

where $G_1^2 = G_1^*$, $G_1^4 = G_1$ and $G_1 G_1^* = G_1^* G_1 = G_1^3 = I_{\mathcal{R}(G)}$. □

Theorem 5.4. *Let $G, H \in \mathcal{GP}(H)$ and $H = \mathcal{R}(G) \oplus \mathcal{N}(G)$. Then G and H has the following representation with respect to decomposition of the space*

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix},$$

$$H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix},$$

where

$$H_1^* = H_1^2 + H_2 H_3, \quad H_2^* = H_3 H_1 + H_4 H_3, \quad H_3^* = H_1 H_2 + H_2 H_4, \quad H_4^* = H_3 H_2 + H_4^2.$$

Furthermore, $H_2 = 0$ if and only if $H_3 = 0$.

Proof. Let $H = \mathcal{R}(G) \oplus \mathcal{N}(G)$. Then representation of G follows from Theorem (1) in [27] and let H has representation

$$H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}.$$

Then, from

$$H^2 = \begin{bmatrix} H_1^2 + H_2 H_3 & H_1 H_2 + H_2 H_4 \\ H_3 H_1 + H_4 H_3 & H_3 H_2 + H_4^2 \end{bmatrix} = \begin{bmatrix} H_1^* & H_3^* \\ H_2^* & H_4^* \end{bmatrix} = H^*,$$

conclusion follows directly.

If $H_2 = 0$, then $H_3^* = H_1 H_2 + H_2 H_4 = 0$ and $H_3 = 0$. Analogously, $H_3 = 0$ implies $H_2 = 0$. □

Theorem 5.5. *Let $G \in \mathcal{L}(H)$ be a hypergeneralized projection. Then G is a closed range operator and H has decomposition $H = \mathcal{R}(G) \oplus \mathcal{N}(G)$. Also, G has the following matrix representaton with the respect to decomposition of the space*

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(G) \\ \mathcal{N}(G) \end{bmatrix},$$

where restriction $G_1 = G|_{\mathcal{R}(G)}$ satisfies $G_1^3 = I_{\mathcal{R}(G)}$.

Proof. If G is a hypergeneralized projecton, then G and $G^\dagger$ commute and G is EP. Using Lemma (1.2) in [20], we get the following decomposition of the space $H = \mathcal{R}(G) \oplus \mathcal{N}(G)$ and G has the required representation. □

6. Moore–Penrose and Drazin inverse of two orthogonal projections

We start this secton with theorem which gives matrix representation of the Moore–Penrose inverse of product, difference and sum of orthogonal projections.

Theorem 6.1. *Let orthogonal projections* $P, Q \in \mathcal{L}(H)$ *be represented as in* (1) *and* (2)*. If the Moore–Penrose inverses of* PQ*,* $P-Q$ *and* $P+Q$ *exist, then the following holds*

1) $(PQ)^\dagger = \begin{bmatrix} AA^\dagger & 0 \\ B^*A^\dagger & 0 \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \to \begin{bmatrix} L \\ L^\perp \end{bmatrix}$ *and* $\mathcal{R}(PQ) = \mathcal{R}(A)$,
2) $(P-Q)^\dagger = \begin{bmatrix} \bar{A}\bar{A}^\dagger & -BD^\dagger \\ -B^*\bar{A}^\dagger & -DD^\dagger \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \to \begin{bmatrix} L \\ L^\perp \end{bmatrix}$ *and* $\mathcal{R}(P-Q) = \mathcal{R}(\bar{A}) \oplus \mathcal{R}(D)$,
3) $(P+Q)^\dagger = \begin{bmatrix} \frac{1}{2}(I+\bar{A}\bar{A}^\dagger) & -BD^\dagger \\ -D^\dagger B^* & 2D^\dagger - DD^\dagger \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \to \begin{bmatrix} L \\ L^\perp \end{bmatrix}$ *and* $\mathcal{R}(P+Q) = L \oplus \mathcal{R}(D)$.

Proof. 1) Using representatons for orthogonal projections $P, Q \in \mathcal{L}(H)$, the well known Harte-Mbekhta formula $(PQ)^\dagger = (PQ)^*(PQ(PQ)^*)^\dagger$ and Theorem 5.1 1), we obtain

$$(PQ)^\dagger = \begin{bmatrix} A & 0 \\ B^* & 0 \end{bmatrix} \begin{bmatrix} A^2 + BB^* & 0 \\ 0 & 0 \end{bmatrix}^\dagger = \begin{bmatrix} AA^\dagger & 0 \\ B^*A^\dagger & 0 \end{bmatrix}.$$

From $PQ(PQ)^\dagger = P_{\mathcal{R}(PQ)}$, we obtain

$$(PQ)(PQ)^\dagger = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} AA^\dagger & 0 \\ B^*A^\dagger & 0 \end{bmatrix} = \begin{bmatrix} AA^\dagger & 0 \\ 0 & 0 \end{bmatrix},$$

or, in the other words, $\mathcal{R}(PQ) = \mathcal{R}(A)$.

2) Similarly to part 1), we can calculate the Moore–Penrose inverse of $P-Q$ as follows

$$\begin{aligned} (P-Q)^\dagger &= (P-Q)^*((P-Q)(P-Q)^*)^\dagger \\ &= \begin{bmatrix} \bar{A} & -B \\ -B^* & -D \end{bmatrix} \begin{bmatrix} \bar{A}^2 + BB^* & -\bar{A}B + BD \\ -B^*\bar{A} + DB^* & B^*B + D^2 \end{bmatrix}^\dagger \\ &= \begin{bmatrix} \bar{A} & -B \\ -B^* & -D \end{bmatrix} \begin{bmatrix} \bar{A}^\dagger & 0 \\ 0 & D^\dagger \end{bmatrix} = \begin{bmatrix} \bar{A}\bar{A}^\dagger & -BD^\dagger \\ -B^*\bar{A}^\dagger & -DD^\dagger \end{bmatrix}. \end{aligned}$$

For the range of $P-Q$ we have

$$\begin{aligned} P_{\mathcal{R}(P-Q)} &= (P-Q)(P-Q)^\dagger \\ &= \begin{bmatrix} \bar{A}\bar{A}\bar{A}^\dagger + BD^\dagger B^* & -\bar{A}BD^\dagger + BDD^\dagger \\ -B^*\bar{A}\bar{A}^\dagger + DD^\dagger B^* & B^*BD^\dagger + DDD^\dagger \end{bmatrix} = \begin{bmatrix} \bar{A}\bar{A}^\dagger & 0 \\ 0 & DD^\dagger \end{bmatrix}, \end{aligned}$$

implying $\mathcal{R}(P-Q) = \mathcal{R}(\bar{A}) \oplus \mathcal{R}(D)$.

3) The Moore–Penrose inverse of $P+Q$ has the following representation with the respect to decomposition of the space

$$(P+Q)^\dagger = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \to \begin{bmatrix} L \\ L^\perp \end{bmatrix}.$$

In order to calculate $(P+Q)^\dagger$, we will use Moore–Penrose equations. From the first Moore–Penrose equation, $(P+Q)(P+Q)^\dagger(P+Q) = P+Q$, we have

$$((I+A)X_1 + BX_3)(I+A) + ((I+A)X_2 + BX_4)B^* = I + A,$$

$$((I+A)X_1+BX_3)B+((I+A)X_2+BX_4)D=B,$$
$$(B^*X_1+DX_3)(I+A)+(B^*X_2+DX_4)B^*=B^*,$$
$$(B^*X_1+DX_3)B+(B^*X_2+DX_4)D=D.$$

The second Moore–Penrose equation, $(P+Q)^\dagger(P+Q)(P+Q)^\dagger=(P+Q)^\dagger$, implies

$$(X_1(I+A)+X_2B^*)X_1+(X_1B+X_2D)X_3=X_1,$$
$$(X_1(I+A)+X_2B^*)X_2+(X_1B+X_2D)X_4=X_2,$$
$$(X_3(I+A)+X_4B^*)X_1+(X_3B+X_4D)X_3=X_3,$$
$$(X_3(I+A)+X_4B^*)X_2+(X_3B+X_4D)X_4=X_4,$$

while the third and fourth Moore–Penrose equations, $((P+Q)(P+Q)^\dagger)^*=(P+Q)(P+Q)^\dagger$ and $((P+Q)^\dagger(P+Q))^*=(P+Q)^\dagger(P+Q)$, give $X_3=X_2^*$. Further calculations show that

$$(I+A)X_1+BX_2^*=I_L,\quad B^*X_1+DX_2^*=0,$$
$$(I+A)X_2+BX_4=0,\qquad B^*X_2+DX_4=DD^\dagger.$$

According to Theorem 5.2 2), 3), from $B^*X_1+DX_2^*=0$ we get $D^\dagger B^*X_1+X_2=0$, or equivalently, $X_2^*=-D^\dagger B^*X_1$.

From $(I+A)X_1+BX_2^*=I_L$ and Theorem 5.2 1), we get $(2I-\bar{A}\bar{A}^\dagger)X_1=I_L$ i.e. $X_1=(2I-\bar{A}\bar{A}^\dagger)^{-1}=\frac{1}{2}(I+\bar{A}\bar{A}^\dagger)$. Theorem 5.1 3) and $B^*X_2+DX_4=DD^\dagger$ imply $-B^*BD^\dagger+DX_4=DD^\dagger$. Finally, we have $X_2=-BD^\dagger$, $X_3=-D^\dagger B^*$, $X_4=2D^\dagger-DD^\dagger$ and

$$(P+Q)^\dagger=\begin{bmatrix}\frac{1}{2}(I+\bar{A}\bar{A}^\dagger) & -BD^\dagger\\ -D^\dagger B^* & 2D^\dagger-DD^\dagger\end{bmatrix}.$$

In the same way as in the part 2),

$$\begin{aligned}P_{\mathcal{R}(P+Q)}&=(P+Q)(P+Q)^\dagger\\&=\begin{bmatrix}\frac{1}{2}(I+A)(I+\bar{A}\bar{A}^\dagger)-BD^\dagger B^* & -(I+A)BD^\dagger+2BD^\dagger-BDD^\dagger\\ \frac{1}{2}B^*(I+\bar{A}\bar{A}^\dagger)-DD^\dagger B^* & -B^*BD^\dagger+2DD^\dagger-DDD^\dagger\end{bmatrix}\\&=\begin{bmatrix}I_L & 0\\ 0 & DD^\dagger\end{bmatrix},\end{aligned}$$

which proves that $\mathcal{R}(P+Q)=L\oplus\mathcal{R}(D)$.

To prove the existence of the Moore–Penrose inverse of PQ, $P-Q$ and $P+Q$, it is sufficient to prove that these operators have closed range. Since Q is an orthogonal projection, $\mathcal{R}(Q)$ is closed subset of H. Also,

$$\mathcal{R}(Q)=Q(H)=\begin{bmatrix}A & B\\ B^* & D\end{bmatrix}\begin{bmatrix}L\\ L^\perp\end{bmatrix}=\begin{bmatrix}\mathcal{R}(A)+\mathcal{R}(B)\\ \mathcal{R}(B^*)+\mathcal{R}(D)\end{bmatrix}=\mathcal{R}(A)+\mathcal{R}(D),$$

because items 1, 3 of Theorem 5.2 state that $\mathcal{R}(B)\subseteq\mathcal{R}(A)$ and $\mathcal{R}(B^*)\subseteq\mathcal{R}(D)$. This implies that $\mathcal{R}(A)$ and $\mathcal{R}(D)$ are closed subsets of L and $L^\perp$ respectively. If $\mathcal{R}(A)$ is closed, then for every sequence $(x_n)\subseteq L$, $x_n\to x$ and $Ax_n\to y$ imply $x\in L$ and $Ax=y$. Now, $(I-A)x_n\to x-y$ and $x-y\in L$, $(I-A)x=x-y$ which proves that $\mathcal{R}(I-A)$ is closed. Consequently, $\mathcal{R}(PQ)$, $\mathcal{R}(I-A)$ and $\mathcal{R}(I+A)$ are closed which completes the proof. □

Similar to Theorem 3.1 in [17], we have the following result.

Theorem 6.2. *Let orthogonal projections $P,Q\in\mathcal{L}(H)$ be represented as in a previous part. If the Drazin inverses of PQ, $P-Q$ and $P+Q$ exist, then $P-Q$ and $P+Q$ are EP operators and the following holds*

1) $(PQ)^D=\begin{bmatrix}A^D & (A^D)^2B\\ 0 & 0\end{bmatrix}$ *and* $\operatorname{ixd}(PQ)\leqslant\operatorname{ixd}(A)+1$,
2) $(P-Q)^D=(P-Q)^\dagger$ *and* $\operatorname{ixd}(P-Q)\leqslant 1$,
3) $(P+Q)^D=(P+Q)^\dagger$ *and* $\operatorname{ixd}(P+Q)\leqslant 1$.

Proof. (1) Theorem 6.1 proves that $\mathcal{R}(PQ)$ is closed subset of H. Thus, the Drazin inverse for this operators exists. According to representations of projections P,Q, their product PQ and the Drazin inverse $(PQ)^D$ can be written in the following way

$$PQ=\begin{bmatrix}A & B\\ 0 & 0\end{bmatrix},\quad (PQ)^D=\begin{bmatrix}X_1 & X_2\\ X_3 & X_4\end{bmatrix}:\begin{bmatrix}L\\ L^\perp\end{bmatrix}\to\begin{bmatrix}L\\ L^\perp\end{bmatrix}.$$

Equations that describe Drazin inverse are

$$(PQ)^DPQ(PQ)^D=\begin{bmatrix}X_1AX_1 & X_1AX_2\\ X_3AX_1 & X_3AX_2\end{bmatrix}=\begin{bmatrix}X_1 & X_2\\ X_3 & X_4\end{bmatrix}=(PQ)^D,$$

$$(PQ)^DPQ=\begin{bmatrix}X_1A & X_1B\\ X_3A & X_3B\end{bmatrix}=\begin{bmatrix}AX_1 & AX_2\\ 0 & 0\end{bmatrix}=PQ(PQ)^D,$$

$$(PQ)^{n+1}(PQ)^D=\begin{bmatrix}A^{n+1}X_1 & A^{n+1}X_2\\ 0 & 0\end{bmatrix}=\begin{bmatrix}A^n & A^{n-1}B\\ 0 & 0\end{bmatrix}=(PQ)^n.$$

Thus, from the first equation we have

$$X_1AX_1=X_1,\quad X_1AX_2=X_2,\quad X_3AX_1=X_3,\quad X_3AX_2=X_4,$$

from the second equation

$$X_1A=AX_1,\quad AX_2=X_1B,\quad X_3A=0,\quad X_3B=0,$$

and the third equation implies

$$A^{n+1}X_1=A^n,\quad A^{n+1}X_2=A^{n-1}B.$$

It is easy to conclude that $X_1=A^D$, $X_3=0$, $X_4=0$. Equations $X_1AX_2=X_2$ and $AX_2=X_1B$ give $X_1^2B=X_2$. Finally,

$$(PQ)^D=\begin{bmatrix}A^D & (A^D)^2B\\ 0 & 0\end{bmatrix}.$$

To estimate the Drazin index of PQ, suppose that $\operatorname{ixd}(A)=n$. Then

$$\begin{aligned}(PQ)^{n+2}(PQ)^D&=\begin{bmatrix}A^{n+2} & A^{n+1}B\\ 0 & 0\end{bmatrix}\begin{bmatrix}A^D & (A^D)^2B\\ 0 & 0\end{bmatrix}\\ &=\begin{bmatrix}A^{n+1} & A^{n+1}A^DB\\ 0 & 0\end{bmatrix}=\begin{bmatrix}A^{n+1} & A^nB\\ 0 & 0\end{bmatrix}=(PQ)^{n+1}\end{aligned}$$

implying that $\mathrm{ixd}(PQ) \leqslant \mathrm{ixd}(A) + 1$.

2) Since $(P-Q)(P-Q)^* = (P-Q)^*(P-Q)$ and $\mathcal{R}(P-Q) = \mathcal{R}(\bar{A}) \oplus \mathcal{R}(D)$ is closed, $P-Q$ is EP operator as normal operator with closed range and $(P-Q)^\dagger = (P-Q)^D$. Besides,

$$(P-Q)^2(P-Q)^D = (P-Q)(P-Q)^\dagger(P-Q) = P-Q$$

and $\mathrm{ixd}(P-Q) \leqslant 1$.

3) Similarly to 2), $P+Q$ is EP operator and $(P+Q)^\dagger = (P+Q)^D$, $\mathrm{ixd}(P+Q) \leqslant 1$. $\square$

Theorem 6.3. *Consider orthogonal projections $P, Q \in \mathcal{L}(H)$ as in a previous part. Then the following holds*

1) *If $PQ = QP$ or $PQP = PQ$, then*

$$(P+Q)^D = \begin{bmatrix} I_L - \frac{1}{2}A & 0 \\ 0 & D \end{bmatrix}, \quad (P-Q)^D = \begin{bmatrix} \bar{A} & 0 \\ 0 & -D \end{bmatrix}.$$

2) *If $PQP = P$, then*

$$(P+Q)^D = \begin{bmatrix} \frac{1}{2}I_L & 0 \\ 0 & D \end{bmatrix}, \quad (P-Q)^D = \begin{bmatrix} 0 & 0 \\ 0 & -D \end{bmatrix}.$$

3) *If $PQP = Q$, then*

$$(P+Q)^D = \begin{bmatrix} I_L - \frac{1}{2}A & 0 \\ 0 & 0 \end{bmatrix}, \quad (P-Q)^D = \begin{bmatrix} \bar{A} & 0 \\ 0 & 0 \end{bmatrix} = P-Q.$$

4) *If $PQP = 0$, then*

$$(P+Q)^D = \begin{bmatrix} I_L & 0 \\ 0 & D \end{bmatrix} = P+Q, \quad (P-Q)^D = \begin{bmatrix} I_L & 0 \\ 0 & -D \end{bmatrix} = P-Q.$$

Proof. Let

$$(P+Q)^D = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} L \\ L^\perp \end{bmatrix} \to \begin{bmatrix} L \\ L^\perp \end{bmatrix}.$$

1) If

$$PQ = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ B^* & 0 \end{bmatrix} = QP \quad \text{or} \quad PQP = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} = PQ,$$

then $B = B^* = 0$, $I_L + A$ is invertible and $(I_L + A)^{-1} = I_L - \frac{1}{2}A$ and according to Theorem 5.1 3), $D = D^2$. Thus, we can write

$$Q = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \quad P+Q = \begin{bmatrix} I_L + A & 0 \\ 0 & D \end{bmatrix}, \quad (P+Q)^n = \begin{bmatrix} (I_L + A)^n & 0 \\ 0 & D \end{bmatrix}.$$

Verifying the equation

$$\begin{aligned}(P+Q)^2(P+Q)^D &= \begin{bmatrix} (I_L + A)^2 X_1 & (I_L + A)^2 X_2 \\ DX_3 & DX_4 \end{bmatrix} \\ &= \begin{bmatrix} I_L + A & 0 \\ 0 & D \end{bmatrix} = P+Q\end{aligned}$$

we get $X_2 = X_3 = 0$, $DX_4 = D$. The other two equations,

$$(P+Q)^D(P+Q)(P+Q)^D = (P+Q)^D \quad \text{and} \quad (P+Q)^D(P+Q) = (P+Q)(P+Q)^D,$$

give $X_4DX_4 = X_4$, $X_4D = DX_4$ i.e., $X_4 = D$. Thus,

$$(P+Q)^D = \begin{bmatrix} I_L - \frac{1}{2}A & 0 \\ 0 & D \end{bmatrix}.$$

Formula

$$(P-Q)^D = \begin{bmatrix} \bar{A} & 0 \\ 0 & -D \end{bmatrix}$$

follows form Theorem 6.2 2) and the fact that $A = A^2$ implies $\bar{A}^D = \bar{A}^2 = \bar{A}$.

2) If $PQP = P$, then $A = I_L$ and Theorem 5.1 implies $B = B^* = 0$. Then,

$$Q = \begin{bmatrix} I_L & 0 \\ 0 & D \end{bmatrix}$$

and from part 1) of this Theorem we conclude

$$(P+Q)^D = \begin{bmatrix} \frac{1}{2}I_L & 0 \\ 0 & D \end{bmatrix}, \quad (P-Q)^D = \begin{bmatrix} 0 & 0 \\ 0 & -D \end{bmatrix}.$$

3) From $PQP = Q$ we get $B = B^* = D = 0$ and $A = A^2$. Now, $I_L + A$ is invertible and

$$(P+Q)^D = (P+Q)^{-1} = \begin{bmatrix} (I_L + A)^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_L - \frac{1}{2}A & 0 \\ 0 & 0 \end{bmatrix}, \quad (P-Q)^D = \begin{bmatrix} \bar{A} & 0 \\ 0 & 0 \end{bmatrix}.$$

4) If $PQP = 0$, then $A = 0$ and since $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, we conclude $B = B^* = 0$. In this case,

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}, \quad P+Q = \begin{bmatrix} I_L & 0 \\ 0 & D \end{bmatrix}$$

implying

$$(P+Q)^D = P+Q = \begin{bmatrix} I_L & 0 \\ 0 & D \end{bmatrix}, \quad (P-Q)^D = P-Q = \begin{bmatrix} I_L & 0 \\ 0 & -D \end{bmatrix}. \qquad \square$$

Theorem 6.4. *Consider orthogonal projections $P, Q \in \mathcal{L}(H)$. If $(PQ)^D$ exists, then $(PQ)^D = (QP)^\dagger(PQ)^\dagger(QP)^\dagger$. Moreover, if $PQ = QP$, then PQ is the EP operator and $(PQ)^D = (PQ)^\dagger$, $\operatorname{ixd}(PQ) \leqslant 1$.*

Proof. Corollary 5.2 in [37] states that $(PQ)^\dagger$ is idempotent for every orthogonal projections P and Q. Thus, we can write

$$(PQ)^\dagger = \begin{bmatrix} I & 0 \\ K & 0 \end{bmatrix}, \quad PQ = (PQ)^{\dagger\dagger} = \begin{bmatrix} (I + K^*K)^{-1} & (I + K^*K)^{-1}K^* \\ 0 & 0 \end{bmatrix}.$$

Denote by $A = (I+K^*K)^{-1}$ and $B = (I+K^*K)^{-1}K^* = AK^*$. Then $PQ = \left[\begin{smallmatrix} A & B \\ 0 & 0 \end{smallmatrix}\right]$ and according to Theorem 6.3 (1),

$$\begin{aligned}(PQ)^D &= \begin{bmatrix} A^D & (A^D)^2B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I+K^*K & (I+K^*K)^2(I+K^*K)^{-1}K^* \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} I+K^*K & (I+K^*K)K^* \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I+K^*K & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & K^* \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} I & K^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ K & 0 \end{bmatrix} \begin{bmatrix} I & K^* \\ 0 & 0 \end{bmatrix} = (QP)^\dagger (PQ)^\dagger (QP)^\dagger,\end{aligned}$$

where we used $((PQ)^\dagger)^* = ((PQ)^*)^\dagger = (QP)^\dagger$.

If P and Q commute, then PQ is normal operator with closed range which means it is EP operator $(PQ)^\dagger = (PQ)^D$. □

7. Moore–Penrose and Drazin inverse of generalized and hypergeneralized projections

Some of the results obtained in the previous section can be extended to generalized and hypergeneralized projections.

Theorem 7.1. *Let $G, H \in \mathcal{L}(H)$ be two generalized or hypergeneralized projections. If the Moore–Penrose inverse of GH exists, then it has the following matrix representation*

$$(GH)^\dagger = \begin{bmatrix} (G_1H_1)^*D^{-1} & 0 \\ (G_1H_2)^*D^{-1} & 0 \end{bmatrix},$$

where $D = G_1H_1(G_1H_1)^* + G_1H_2(G_1H_2)^* > 0$ is invertible.

Proof. From Theorems 5.3, 5.4 and 5.5, we see that $\mathcal{R}(G) = \mathcal{R}(G_1)$ is closed and pair of generalized or hypergeneralized projections has matrix form

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}.$$

Then

$$GH = \begin{bmatrix} G_1H_1 & G_1H_2 \\ 0 & 0 \end{bmatrix}$$

and analogously to the proof of Theorem 6.1 1), we obtain mentioned matrix form. Since $\mathcal{R}(GH) = \mathcal{R}(G_1)$ is closed, the Moore–Penrose $(GH)^\dagger$ exists. □

Theorem 7.2. *Let $G, H \in \mathcal{L}(H)$ be two generalized or hypergeneralized projections. If the Drazin inverse of GH exists, then it has the following matrix representation*

$$(GH)^D = \begin{bmatrix} (G_1H_1)^D & ((G_1H_1)^D)^2G_1H_2 \\ 0 & 0 \end{bmatrix}.$$

Proof. Similarly to the proof of Theorem 6.2 1) and using Theorem 5.5. □

Theorem 7.3. *Let $G, H \in \mathcal{L}(H)$ be two generalized projections. If the appropriate operators have the Drazin or the Moore–Penrose inverse, then*

1) *If* $GH = HG$, *then* GH *is EP and*

$$(GH)^\dagger = (GH)^D = (GH)^* = (GH)^2 = (GH)^{-1},$$

$$(GH)^\dagger = \begin{bmatrix} (G_1H_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

2) *If* $GH = HG = 0$, *then* $G + H$ *is EP and*

$$(G+H)^\dagger = (G+H)^D = (GH)^* = (G+H)^2 = (G+H)^{-1},$$

$$(G+H)^\dagger = \begin{bmatrix} G_1^{-1} & 0 \\ 0 & H_4^{-1} \end{bmatrix}.$$

3) *If* $GH = HG = H^*$, *then* $G - H$ *is EP and*

$$(G-H)^\dagger = (G-H)^D = (GH)^* = (G-H)^2 = (G-H)^{-1},$$

$$(G-H)^\dagger = \begin{bmatrix} (G_1 - H_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof. 1) If $G, H \in \mathcal{L}(H)$ are two commuting generalized projections, then from

$$(GH)^* = H^*G^* = H^2G^2 = (HG)^2 = (GH)^2$$

we conclude that GH is also generalized projection, and therefore EP operator. Checking the Moore–Penrose equations for $(GH)^2$, we see that they hold. From the uniqueness of the Moore–Penrose inverse follows $(GH)^2 = (GH)^\dagger$ and

$$(GH)^\dagger = (GH)^D = (GH)^2.$$

From $GH(GH)^\dagger = P_{\mathcal{R}(GH)}$, using matrix form of GH, we get $G_1H_1(G_1H_1)^\dagger = I$, or equivalently, $(G_1H_1)^\dagger = (G_1H_1)^{-1}$. Finally,

$$(GH)^\dagger = (GH)^D = (GH)^* = (GH)^2 = (GH)^{-1}.$$

2) If $GH = HG = 0$, then $(G+H)^2 = G^2 + H^2 = G^* + H^* = (G+H)^*$ and $G + H$ is a generalized projection. The rest of the proof is similar to part 1).

3) If $GH = HG = H^*$, then $(G-H)^2 = G^2 - H^2 = G^* - H^* = (G-H)^*$ and the rest of the proof is similar to part 1).

Matrix representations are easily obtained by using canonical forms of G and H given in Theorem 5.4. □

Theorem 7.4. *Let* $G, H \in \mathcal{L}(H)$ *be two hypergeneralized projections. If the appropriate operators have Drazin or Moore–Penrose inverse, then*

1) *If* $GH = HG$, *then* GH *is EP and*

$$(GH)^\dagger = (GH)^D = (GH)^2 = (GH)^{-1},$$

$$(GH)^\dagger = \begin{bmatrix} (G_1H_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

2) *If* $GH = HG = 0$, *then* $G + H$ *is EP and*

$$(G+H)^\dagger = (G+H)^D = (G+H)^2 = (G+H)^{-1},$$

$$(G+H)^\dagger = \begin{bmatrix} G_1^{-1} & 0 \\ 0 & H_4^{-1} \end{bmatrix}.$$

3) *If $GH = HG = H^*$, then $G - H$ is EP and*

$$(G-H)^\dagger = (G-H)^D = (G-H)^2 = (G-H)^{-1},$$
$$(G-H)^\dagger = \begin{bmatrix} (G_1 - H_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof. 1) GH is EP operator and $(GH)^4 = GH$, so it is a hypergeneralized projection. Since $(GH)^2 = (GH)^\dagger$, operator GH commutes with its Moore–Penrose inverse and $(GH)^\dagger = (GH)^D$. From $GH(GH)^\dagger = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$ follows $(GH)^\dagger = (GH)^{-1}$. Thus,

$$(GH)^\dagger = (GH)^D = (GH)^2 = (GH)^{-1}.$$

2) If $GH = HG = 0$, then $(G+H)^2 = (G+H)^\dagger$ and $H_1 = H_2 = H_3 = 0$ implies

$$G + H = \begin{bmatrix} G_1 & 0 \\ 0 & H_4 \end{bmatrix}, \quad (G+H)^\dagger = \begin{bmatrix} G_1^\dagger & 0 \\ 0 & H_4^\dagger \end{bmatrix}.$$

From $(G+H)(G+H)^\dagger = P_{\mathcal{R}(G+H)} = P_{\mathcal{R}(G)} + P_{\mathcal{R}(H)}$ and

$$(G+H)(G+H)^\dagger = \begin{bmatrix} G_1 G_1^\dagger & 0 \\ 0 & H_4 H_4^\dagger \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

we conclude that $G_1^\dagger = G_1^{-1}$, $H_4^\dagger = H_4^{-1}$ and $(G+H)^\dagger = (G+H)^{-1}$.

3) Similarly to 2). □

8. Invertibility of operator matrices

8.1. Right invertibility of operator matrices $M_{(T,S)}$. In the first part of this section we investigate the right invertibility of the operator $M_{(T,S)}$.

Theorem 8.1. *Let $A \in \mathcal{L}(X)$ and $C \in \mathcal{L}(Y, X)$ be given operators. Then the following statements are equivalent*

1) *There exist some $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y)$ such that $M_{(T,S)}$ is right invertible;*

2) *$[A\ C] \in \mathcal{L}(X \oplus Y, Y)$ is right invertible and $Y \preceq \mathcal{N}([A\ C])$.*

Proof. 1) $\Longrightarrow$ 2): Suppose that $M_{(T,S)}$ is right invertible for some $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y)$. Then there exists a bounded linear operator

$$\begin{bmatrix} E & G \\ H & F \end{bmatrix} : \begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} X \\ Y \end{bmatrix} \quad \text{such that} \quad \begin{bmatrix} A & C \\ T & S \end{bmatrix} \begin{bmatrix} E & G \\ H & F \end{bmatrix} = \begin{bmatrix} I_X & 0 \\ 0 & I_Y \end{bmatrix}.$$

It follows that $\left[\begin{smallmatrix} A & C \end{smallmatrix}\right] \left[\begin{smallmatrix} E \\ H \end{smallmatrix}\right] = I_X$, so $[A \quad C]$ is right invertible. On the other hand, we have $[T \quad S] \left[\begin{smallmatrix} G \\ F \end{smallmatrix}\right] = I_Y$ and $[A \quad C] \left[\begin{smallmatrix} G \\ F \end{smallmatrix}\right] = 0$, so there exists a left invertible operator $\left[\begin{smallmatrix} G \\ F \end{smallmatrix}\right] : Y \to \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ such that $\mathcal{R}\left(\left[\begin{smallmatrix} G \\ F \end{smallmatrix}\right]\right) \subseteq \mathcal{N}([A \quad C])$. Hence $Y \preceq \mathcal{N}([A \quad C])$.

2) $\Longrightarrow$ 1): Suppose that $[A \quad C] \colon \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right] \to X$ is right invertible, and suppose that $Y \preceq \mathcal{N}([A \quad C])$ holds. Let $K = \left[\begin{smallmatrix} E \\ H \end{smallmatrix}\right] : X \to \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ be a bounded right inverse of $[A \quad C]$. Then $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right] = \mathcal{R}(K) \oplus (\mathcal{N}([A \quad C]))$ and

$$[A \quad C] \begin{bmatrix} E \\ H \end{bmatrix} = I_X. \tag{8.1}$$

Let $L \colon Y \to \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ be a left invertible operator such that $\mathcal{R}(L) \subset \mathcal{N}([A \quad C])$. Then $L = \left[\begin{smallmatrix} G \\ F \end{smallmatrix}\right] \colon Y \to \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$. Since $\mathcal{R}(L) \subset \mathcal{N}([A \quad C])$, we get that

$$[A \quad C] \begin{bmatrix} G \\ F \end{bmatrix} = 0. \tag{8.2}$$

$\mathcal{N}([A \quad C])$ is complemented in $X \oplus Y$ and $\mathcal{R}(L)$ is complemented in $X \oplus Y$. From $\mathcal{R}(L) \subset \mathcal{N}([A \quad C])$ it follows that $\mathcal{R}(L)$ is complemented in $\mathcal{N}([A \quad C])$. It follows that there exists a closed subspace W such that $\mathcal{N}([A \quad C]) = \mathcal{R}(L) \oplus W$. Now we have $X \oplus Y = \mathcal{R}(K) \oplus \mathcal{N}([A \quad C]) = \mathcal{R}(K) \oplus W \oplus \mathcal{R}(L)$. There exists the bounded left inverse N of L, such that $\mathcal{N}(N) = \mathcal{R}(K) \oplus W$. Such N has the matrix form $N = [T \quad S] \colon \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right] \to Y$. Then

$$[T \quad S] \begin{bmatrix} G \\ F \end{bmatrix} = I_Y. \tag{8.3}$$

From $\mathcal{R}(K) \subset \mathcal{N}(N)$ we have

$$[T \quad S] \begin{bmatrix} E \\ H \end{bmatrix} = 0. \tag{8.4}$$

Finally, from (8.1), (8.2), (8.3) and (8.4) it follows that

$$\begin{bmatrix} A & C \\ T & S \end{bmatrix} \begin{bmatrix} E & G \\ H & F \end{bmatrix} = \begin{bmatrix} I_X & 0 \\ 0 & I_Y \end{bmatrix}.$$

Thus, the proof is completed. □

As a corollary, we obtain the following result for Hilbert space operators, which is proved in [12, Theorem 1.1].

Corollary 8.1. *Let $Z = X \oplus Y$ be a Hilbert space, where X, Y are closed and mutually orthogonal. Let $A \in \mathcal{L}(X)$ and $C \in \mathcal{L}(Y, X)$ be given operators. Then the following statements are equivalent*

1) *There exist some $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y)$ such that $M_{(T,S)}$ is right invertible;*
2) *$\mathcal{R}(A) + \mathcal{R}(C) = Y$ and $\dim Y \leqslant \mathcal{N}([A \quad C])$;*
3) *$[A \quad C]$ is right invertible operator and $\dim Y \leqslant \mathcal{N}([A \quad C])$.*

Notice that 2) is equivalent to 3) from the following reason: $[A \quad C]$ is right invertible if and only if $\mathcal{R}([A \quad C]) = Y$; on the other hand, it is easy to see that $\mathcal{R}([A \quad C]) = \mathcal{R}(A) + \mathcal{R}(C)$.

Finally, we mention that it seems more difficult to prove the analogous result considering the left invertibility of $M_{(T,S)}$ on the Banach space $X \oplus Y$.

8.2. Invertibility of operator matrices M_C. In this part, we will investigate the invertibility of operator matrices $M_C = \left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right]$.

Actually, the invertibility of operators A and B implies the invertibility of operator M_C for arbitrary operator C. The following lemma holds.

Lemma 8.1. *Let X and Y be Banach spaces. If the operators $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ are invertible, then the operator $M_C \in \mathcal{L}(X \oplus Y)$ is invertible for every operator $C \in \mathcal{L}(Y, X)$.*

Proof. Let us prove that operator $N = \left[\begin{smallmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{smallmatrix}\right]$ is the inverse of operator M_C. Indeed, it holds $NM_C = I_{X\oplus Y}, M_C N = I_{X\oplus Y}$. So, for every operator $C \in \mathcal{L}(Y, X)$ it is $M_C^{-1} = \left[\begin{smallmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{smallmatrix}\right]$. □

Specially, the inverse of operator $M_0 = \left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$ is the diagonal matrix $M_0^{-1} = \left[\begin{smallmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{smallmatrix}\right]$.

It is natural to ask ourselves if the opposite direction holds. More precisely, does the invertibility of M_0 implies the invertibility of any of operators A and B. In the case of invertibility of operator M_0, it holds that operator A is invertible if and only if the operator B is invertible. This property is proved in the next well known result.

Lemma 8.2. *Let X and Y be Banach spaces and let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $M_0 = \left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right] : \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right] \to \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ be operators. If two out of three given operators are invertible, then the third one is also invertible.*

Proof. The proof implies directly from the decompositions $\mathcal{N}(M_0) = \mathcal{N}(A) \oplus \mathcal{N}(B)$ and $\mathcal{R}(M_0) = \mathcal{R}(A) \oplus \mathcal{R}(B)$. □

The main result in the paper [29] gives us necessary and sufficient conditions for operator matrix M_C to be invertible.

Theorem 8.2. *Operator matrix M_C is invertible for some operator $C \in \mathcal{L}(Y, X)$ if and only if operators $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ satisfy the following conditions:*

1) *A is left invertible,* 2) *B is right invertible,* 3) *$X/\mathcal{R}(A) \cong \mathcal{N}(B)$.*

Based on this theorem, we will examine equivalent conditions under which the M_C operator is injective or surjective.

As the part of the paper [21], we proved next theorem concerning sufficient conditions for operator M_C to be injective.

Theorem 8.3. *Suppose that $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ satisfy the following: A is left invertible, $\mathcal{N}(B)$ is complemented, and $\mathcal{N}(B) \preceq X/\mathcal{R}(A)$. Then there exists some $C \in \mathcal{L}(Y, X)$ such that M_C is injective.*

Proof. There exist closed subspaces V of Y and W of X, such that $Y = \mathcal{N}(B) \oplus V$ and $X = W \oplus \mathcal{R}(A)$. Since $\mathcal{N}(B) \preceq X/\mathcal{R}(A)$, there exists a left invertible operator $C_0 \in \mathcal{L}(\mathcal{N}(B), W)$. Define $C \in \mathcal{L}(Y, X)$ as follows

$$C = \begin{bmatrix} C_0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ V \end{bmatrix} \to \begin{bmatrix} W \\ \mathcal{R}(A) \end{bmatrix}.$$

We prove that M_C is injective. Let $z = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in (X \oplus Y)$. From $M_C z = 0$, we have

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then, $Ax + Cy = 0$ and $By = 0$. From the first equation we have $Ax = -Cy \in \mathcal{R}(A) \cap \mathcal{R}(C) \subseteq \mathcal{R}(A) \cap W = \{0\}$. Now, we have $Ax = Cy = 0$. Since A is injective, we get $x = 0$. From $By = 0$, it follows that $y \in \mathcal{N}(B)$. Now, we have $Cy = C_0 y = 0$. Since C_0 is left invertible, it is also injective. From $C_0 y = 0$ we conclude that $y = 0$. Thus, $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ and this proves that M_C is injective. □

As it can be seen from the proof of the previous theorem, the condition $\mathcal{N}(B) \preceq X/\mathcal{R}(A)$ can be replaced by a weaker condition. More precisely, assumption that there exists left invertible operator from $\mathcal{N}(B)$ to $X/\mathcal{R}(A)$ can be replaced by the assumption about existence of injective operator.

So, the theorem is then as follows.

Theorem 8.4. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ be operators such that following conditions are satisfied*

1) *A is left invertible,*
2) *$\mathcal{N}(B)$ is compemented in Y,*
3) *There exists an injective operator from $\mathcal{N}(B)$ to $X/\mathcal{R}(A)$.*

Then, the operator M_C is injective for some operator $C \in \mathcal{L}(Y, X)$.

Now, it is a question if the opposite direction holds in the previous theorems. It is not proved the opposite direction, but it holds that two of the conditions in Theorem 8.3 are actually equivalent to the conditions for left invertibility of operator M_C. The following theorem from [40] proves this.

Theorem 8.5. *Operator M_C is left invertible for some operator $C \in \mathcal{L}(Y, X)$ if and only if operators $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ are such that satisfy the following conditions:*

1) *A is left invertible,* 2) *$\mathcal{N}(B) \preceq X/\mathcal{R}(A)$.*

Proof. ($\Longleftarrow$:) The same as it is in the proof of Theorem 8.3, let operator $C_0 \in \mathcal{L}(\mathcal{N}(B), W)$ be left invertible and let the decomposition $X = W \oplus \mathcal{R}(A)$ holds. The operator $C \in \mathcal{L}(Y, X)$ is defined as

$$Cy = \begin{cases} C_0 y, & y \in \mathcal{N}(B) \\ 0, & y \notin \mathcal{N}(B) \end{cases}.$$

Denote a left inverse of A with A_1, and left inverse of C_0 with D_0. Then, the operator $D_0 \colon W \to \mathcal{N}(B)$ is surjective and $D_0 C_0 = I_{\mathcal{N}(B)}$. Define the operator $D \in \mathcal{L}(X, Y)$ such that

$$D = \begin{bmatrix} D_0 & 0 \end{bmatrix} : \begin{bmatrix} W \\ \mathcal{R}(A) \end{bmatrix} \to Y.$$

Let $V = Y \backslash \mathcal{N}(B)$. Notice that the restriction operator $B \colon V \to \mathcal{R}(B)$ is invertible. Denote with $B_0 \colon \mathcal{R}(B) \to V$ the inverse of this restriction. It holds $B_0 B = I_V$.

Let the operator $B_1 \in \mathcal{L}(Y)$ be defined as follows

$$B_1 y = \begin{cases} B_0 y, & y \in \mathcal{R}(B) \\ 0, & y \notin \mathcal{R}(B) \end{cases}.$$

Let

$$N = \begin{bmatrix} A_1 & 0 \\ D & B_1 \end{bmatrix} : \begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} X \\ Y \end{bmatrix}.$$

Then,

$$NM_C = \begin{bmatrix} A_1 & 0 \\ D_1 & B_1 \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \begin{bmatrix} A_1 A & A_1 C \\ DA & DC + B_1 B \end{bmatrix}.$$

Since A_1 is a left inverse of A, then $A_1 A = I_X$. Further, we have $\mathcal{R}(C) \subset W \subset \mathcal{N}(A_1)$, so $A_1 C = 0$. From the definition of D, we have $\mathcal{R}(A) \subset \mathcal{N}(D_1)$, which implies $D_1 A = 0$.

Let $y \in Y$ be arbitrary. Then

$$(DC + B_1 B)y = \begin{cases} D_0 C_0 y, & y \in \mathcal{N}(B) \\ B_0 B y, & y \notin \mathcal{N}(B) \end{cases} = \begin{cases} y, & y \in \mathcal{N}(B) \\ y, & y \notin \mathcal{N}(B) \end{cases} = y,$$

which implies $D_1 C + B_1 B = I_Y$. Now, it holds

$$NM_C = \begin{bmatrix} A_1 A & A_1 C \\ D_1 A & D_1 C + B_1 B \end{bmatrix} = \begin{bmatrix} I_X & 0 \\ 0 & I_Y \end{bmatrix} = I_{X \oplus Y}.$$

So, the operator M_C is left invertible and N is its left inverse.

($\Longrightarrow$:) Let M_C be left invertible for some $C \in \mathcal{L}(Y, X)$. If $C = 0$, the statement trivially holds. Assume that C is not equal to zero.

Let

$$N = \begin{bmatrix} A_1 & C_1 \\ D_1 & B_1 \end{bmatrix} : \begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} X \\ Y \end{bmatrix}$$

be a left inverse of M_C. Then

$$NM_C = \begin{bmatrix} A_1 & C_1 \\ D_1 & B_1 \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \begin{bmatrix} A_1 A & A_1 C + C_1 B \\ D_1 A & D_1 C + B_1 B \end{bmatrix} = \begin{bmatrix} I_X & 0 \\ 0 & I_Y \end{bmatrix} = I_{X \oplus Y}.$$

The equality $A_1 A = I_X$ proves the left invertibility of A, so we have that there exists a subspace $W \subset X$ such that $X = W \oplus \mathcal{R}(A)$. Then, the condition 1) is proved.

Let $C_0 \colon \mathcal{N}(B) \to X$ be the restriction of operator C on $\mathcal{N}(B)$. We will prove that $\mathcal{R}(C_0) \subset W$. Let $x \in \mathcal{R}(A) \cap \mathcal{R}(C_0)$ be arbitrary. Then, there exist $x_0 \in X$ and $y_0 \in \mathcal{N}(B)$ such that $x = Ax_0 = C_0 y_0 = Cy_0$. It holds

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} -x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} -Ax_0 + Cy_0 \\ By_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since the operator M_C is left invertible, then it is also injective, so the previous equality implies $\left[\begin{smallmatrix} -x_0 \\ y_0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, and then $x = 0$. So, we have $\mathcal{R}(A) \cap \mathcal{R}(C_0) = \{0\}$, which implies that $\mathcal{R}(C_0) \subset W$.

Now, we look at the equality $D_1 C + B_1 B = I_Y$. Let $y \in \mathcal{N}(B)$ be arbitrary. Then we have $y = I_{\mathcal{N}(B)} y = D_1 C y + B_1 B y = D_1 C y = D_1 C_0 y$. So, $D_1 C_0 = I_{\mathcal{N}(B)}$.

It is already proved that $\mathcal{R}(C_0) \subset W$. Because of that, there exists left invertible operator from the subspace $\mathcal{N}(B)$ to $X/\mathcal{R}(A)$, so the condition 2) is also satisfied.

So, the proof is completed. □

Now, we study when operator M_C is surjective. The result is presented in [40].

Theorem 8.6. *Let operators $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ be such that the following conditions are satisfied*

1) *B is right invertible,*
2) *$\mathcal{R}(A)$ is complemented in X,*
3) *There exists the surjective operator from $\mathcal{N}(B)$ to $X/\mathcal{R}(A)$.*

Then, the operator M_C is surjective for some operator $C \in \mathcal{L}(Y, X)$.

Proof. Since B is right invertible and $\mathcal{R}(A)$ is complemented, there exist closed subspaces $V \subset Y$ and $W \subset X$ such that $Y = \mathcal{N}(B) \oplus V$ and $X = W \oplus \mathcal{R}(A)$.

Let $C_0 \in \mathcal{L}(\mathcal{N}(B), W)$ be the surjective operator from $\mathcal{N}(B)$ to $X/\mathcal{R}(A)$. It holds $C_0(\mathcal{N}(B)) = W$. Define the operator operator $C \in \mathcal{L}(Y, X)$ in the following way

$$C = \begin{bmatrix} C_0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ V \end{bmatrix} \to \begin{bmatrix} W \\ \mathcal{R}(A) \end{bmatrix}.$$

Then, we have

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} A(X) + C(Y) \\ B(Y) \end{bmatrix}.$$

Since $\mathcal{R}(C) = C_0(\mathcal{N}(B)) = W$, it holds $A(X) + C(Y) = \mathcal{R}(A) \oplus W = X$. On the other hand, the operator B is right invertible, so B is surjective and $\mathcal{R}(B) = Y$. Thus, we have $\left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right]\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right] = \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$, which proves that operator M_C is surjective. □

9. Generalized invertibility of operator matrices

9.1. Regularity of operator matrices M_C.

In this part we investigate relative regularity of M_C.

Theorem 9.1. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ be relatively regular. If $\mathcal{N}(B) \preceq X/\mathcal{R}(A)$, then there exists some $C \in \mathcal{L}(Y, X)$ such that M_C is relative regular.*

Proof. Let $A_1 \in \mathcal{L}(X)$ and $B_1 \in \mathcal{L}(Y)$ denote reflexive inverses of A and B, respectively. Then $Y = \mathcal{R}(B_1) \oplus \mathcal{N}(B)$ and $X = \mathcal{N}(A_1) \oplus \mathcal{R}(A)$. Let $J : \mathcal{N}(B) \to \mathcal{N}(A_1)$ be a left invertible mapping and let $J_1 : \mathcal{N}(A_1) \to \mathcal{N}(B)$ be a left inverse of J. Define $C \in \mathcal{L}(Y, X)$ and $C_1 \in \mathcal{L}(X, Y)$ in the following way

$$C = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{R}(B_1) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(A_1) \\ \mathcal{R}(A) \end{bmatrix},$$

$$C_1 = \begin{bmatrix} J_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(A_1) \\ \mathcal{R}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{R}(B_1) \end{bmatrix}.$$

Consider the operator $N = \left[\begin{smallmatrix} A_1 & 0 \\ C_1 & B_1 \end{smallmatrix}\right] \in \mathcal{L}(X \oplus Y)$. Then we find

$$NM_C = \begin{bmatrix} A_1 A & A_1 C \\ C_1 A & C_1 C + B_1 B \end{bmatrix}.$$

Since $\mathcal{R}(C) \subset \mathcal{N}(A_1)$ and $\mathcal{R}(A) \subset \mathcal{N}(C_1)$, we have $A_1C = 0$ and $C_1A = 0$, respectively. Also, B_1B is the projection from Y onto $\mathcal{R}(B_1)$ parallel to $\mathcal{N}(B)$, and C_1C is the projection from Y onto $\mathcal{N}(B)$ parallel to $\mathcal{R}(B_1)$. Hence $C_1C+B_1B = I$, and

$$NM_C = \begin{bmatrix} A_1A & 0 \\ 0 & I \end{bmatrix}.$$

Since $AA_1A = A$ and $A_1AA_1 = A_1$, we have

$$M_CNM_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} A_1A & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} AA_1A & C \\ 0 & B \end{bmatrix} = M_C,$$

and M_C is relatively regular. □

We state the following result concerning the Moore–Penrose inverse of M_C.

Theorem 9.2. *Let H, K be mutually orthogonal Hilbert spaces and $Z = H \oplus K$. If $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(K)$ both have closed ranges, and if $\mathrm{nul}(B) = \mathrm{def}(A)$, then there exists some $C \in \mathcal{L}(K, H)$ such that M_C has a closed range, and*

$$M_C^\dagger = \begin{bmatrix} A^\dagger & 0 \\ C^\dagger & B^\dagger \end{bmatrix}.$$

Proof. Recall the notations from the proof of Theorem 9.1, with one assumption: J is invertible. We have the following

$$NM_CN = \begin{bmatrix} A_1A & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ C_1 & B_1 \end{bmatrix} = \begin{bmatrix} A_1AA_1 & 0 \\ C_1 & B_1 \end{bmatrix} = N,$$

$$M_CN = \begin{bmatrix} AA_1 + CC_1 & CB_1 \\ BC_1 & BB_1 \end{bmatrix}.$$

Since $\mathcal{R}(B_1) = \mathcal{N}(C)$ and $\mathcal{R}(C_1) = \mathcal{N}(B)$, it follows that $CB_1 = 0$ and $C_1B = 0$. Also, AA_1 is the projection on $\mathcal{R}(A)$ parallel to $\mathcal{N}(A_1)$. Since J is invertible, we have that CC_1 is the projection on $\mathcal{N}(A_1)$ parallel to $\mathcal{R}(A)$. Hence, $AA_1+CC_1 = I$. Thus, N is a reflexive inverse of M_C.

Now, we take $A_1 = A^\dagger$ and $B_1 = B^\dagger$. Then all previous results hold, with one more nice property: we have orthogonal decompositions. Precisely, $X = \mathcal{N}(A_1) \oplus \mathcal{R}(A) = \mathcal{N}(A^*) \oplus \mathcal{R}(A)$ and $Y = \mathcal{N}(B) \oplus \mathcal{R}(B_1) = \mathcal{N}(B) \oplus \mathcal{R}(B^*)$. Since J is invertible, we have $J_1 = J^{-1}$ and consequently $C_1 = C^\dagger$. The operator N_C is still a reflexive inverse of M_C. Furthermore, we have

$$NM_C = \begin{bmatrix} A^\dagger A & 0 \\ 0 & I \end{bmatrix} : \begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} X \\ Y \end{bmatrix}, \quad \text{and} \quad M_CN = \begin{bmatrix} I & 0 \\ 0 & BB^\dagger \end{bmatrix} : \begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} X \\ Y \end{bmatrix}.$$

Projections NM_C and M_CN are obviously selfadjoint, so $N = M_C^\dagger$. □

9.2. Generalized invertibility in Banach algebras. Let $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_u \in \mathcal{A}$ relative to the idempotent $u \in \mathcal{A}$. If $a \in (u\mathcal{A}u)$ is not invertible but has the outer generalized inverse with prescribed idempotents $p_1, q_1 \in (u\mathcal{A}u)^\bullet$, we can observe the generalized Schur complement $s = d - ca^{(2)}_{p_1,q_1}b$. Accordingly, we investigate equivalent conditions under which $x^{(2)}_{p,q}$ has the generalized Banachiewicz–Schur form in a Banach algebra.

We use the following auxiliary results.

Lemma 9.1. *Let p, q be idempotents in a Banach algebra $\mathcal{A}$. The following stataments are equivalent:*

$$1)\ p+q \in \mathcal{A}^\bullet, \quad 2)\ pq = qp = 0.$$

Proof. 1) ⇒ 2): Suppose $p+q \in \mathcal{A}^\bullet$. We have

$$(p+q)^2 = p+q \Rightarrow pq+qp = 0 \Rightarrow pq = -qp.$$

Since the following holds

$$pq = p^2q^2 = p(pq)q = p(-qp)q = -pq(pq) = pqqp = pqp = -ppq = -pq,$$

we obtain $pq = 0$. The analogous proof holds for $qp = 0$.

2) ⇒ 1): Let $p, q \in \mathcal{A}^\bullet$ such that $pq = qp = 0$. Then

$$(p+q)^2 = p^2 + pq + qp + q^2 = p + q,$$

so $p+q \in \mathcal{A}^\bullet$. □

If $u \in \mathcal{A}^\bullet$, then the product of arbitrary elements from algebra $u\mathcal{A}u$ and $(1-u)$ $\mathcal{A}(1-u)$ is equal to 0, i.e. for all $a \in u\mathcal{A}u$ and for all $b \in (1-u)\mathcal{A}(1-u)$, we have $ab = 0$.
Now, as a corollary of Lemma 9.1, we state the following result.

Lemma 9.2. *Let $u \in \mathcal{A}^\bullet$. If $p_1 \in (u\mathcal{A}u)^\bullet$ and $p_2 \in ((1-u)\mathcal{A}(1-u))^\bullet$, then $p = p_1 + p_2 \in \mathcal{A}$ is an idempotent.*

We also need the following known results in Banach algebra.

Lemma 9.3. [52] [53, Theorem 1.6.15] *Let $\mathcal{A}$ be a complex unital Banach algebra with unit 1, and let p be an idempotent of $\mathcal{A}$. If $x \in p\mathcal{A}p$, then $\sigma_{p\mathcal{A}p}(x) \cup \{0\} = \sigma_{\mathcal{A}}(x)$, where $\sigma_{\mathcal{A}}(x)$ denotes the spectrum of x in the algebra $\mathcal{A}$, and $\sigma_{p\mathcal{A}p}(x)$ denotes the spectrum of x in the algebra $p\mathcal{A}p$.*

Lemma 9.4. [10, Lemma 2.4] *Let $b, q \in \mathcal{A}^{\text{qnil}}$ and let $qb = 0$. Then $q + b \in \mathcal{A}^{\text{qnil}}$.*

Lemma 9.5. *Let $b \in \mathcal{A}^d$ and $a \in \mathcal{A}^{\text{qnil}}$.*

1) [10, Corollary 3.4] *If $ab = 0$, then $a+b \in \mathcal{A}^d$ and $(a+b)^d = \sum_{n=0}^{+\infty}(b^d)^{n+1}a^n$.*
2) *If $ba = 0$, then $a+b \in \mathcal{A}^d$ and $(a+b)^d = \sum_{n=0}^{+\infty} a^n(b^d)^{n+1}$.*

9.3. (p, q)-outher generalized inverse.

The first result gives the additive properties of the (p, q)-outer generalized inverse.

Theorem 9.3. *Let $p, q \in \mathcal{A}^\bullet$ and $a, b \in \mathcal{A}^{(2)}_{p,q}$. If*

$$a^{(2)}_{p,q}b + b^{(2)}_{p,q}a + 1 = 0, \qquad ab^{(2)}_{p,q} + ba^{(2)}_{p,q} + 1 = 0, \tag{9.1}$$

then $a + b \in \mathcal{A}^{(2)}_{p,q}$ and $(a+b)^{(2)}_{p,q} = a^{(2)}_{p,q} + b^{(2)}_{p,q}$.

Proof. Using the fact that $a, b \in \mathcal{A}^{(2)}_{p,q}$, Theorem 1.1 and conditions (9.1), we have

$$\begin{aligned}&(a^{(2)}_{p,q} + b^{(2)}_{p,q})(a+b)(a^{(2)}_{p,q} + b^{(2)}_{p,q})\\ &\quad = a^{(2)}_{p,q} + pb^{(2)}_{p,q} + a^{(2)}_{p,q}ba^{(2)}_{p,q} + a^{(2)}_{p,q}(1-q) + b^{(2)}_{p,q}(1-q) + b^{(2)}_{p,q}ab^{(2)}_{p,q} + pa^{(2)}_{p,q} + b^{(2)}_{p,q}\end{aligned}$$

$$
\begin{aligned}
&= a^{(2)}_{p,q} + b^{(2)}_{p,q} + a^{(2)}_{p,q} b a^{(2)}_{p,q} + a^{(2)}_{p,q} + b^{(2)}_{p,q} + b^{(2)}_{p,q} a b^{(2)}_{p,q} + a^{(2)}_{p,q} + b^{(2)}_{p,q} \\
&= a^{(2)}_{p,q} + b^{(2)}_{p,q} + a^{(2)}_{p,q}(b a^{(2)}_{p,q} + 1) + b^{(2)}_{p,q}(1 + a b^{(2)}_{p,q}) + a^{(2)}_{p,q} + b^{(2)}_{p,q} \\
&= a^{(2)}_{p,q} + b^{(2)}_{p,q} + a^{(2)}_{p,q}(-a b^{(2)}_{p,q}) + b^{(2)}_{p,q}(-b a^{(2)}_{p,q}) + a^{(2)}_{p,q} + b^{(2)}_{p,q} \\
&= a^{(2)}_{p,q} + b^{(2)}_{p,q} - p b^{(2)}_{p,q} - p a^{(2)}_{p,q} + a^{(2)}_{p,q} + b^{(2)}_{p,q} = a^{(2)}_{p,q} + b^{(2)}_{p,q},
\end{aligned}
$$

$$
\begin{aligned}
(a^{(2)}_{p,q} + b^{(2)}_{p,q})(a+b) &= a^{(2)}_{p,q} a + a^{(2)}_{p,q} b + b^{(2)}_{p,q} a + b^{(2)}_{p,q} b \\
&= p + p a^{(2)}_{p,q} b + p b^{(2)}_{p,q} a + p \\
&= p + p(a^{(2)}_{p,q} b + b^{(2)}_{p,q} a + 1) \\
&= p,
\end{aligned}
$$

and also

$$
\begin{aligned}
(a+b)(a^{(2)}_{p,q} + b^{(2)}_{p,q}) &= a a^{(2)}_{p,q} + b a^{(2)}_{p,q} + a b^{(2)}_{p,q} + b b^{(2)}_{p,q} \\
&= (1-q) + b a^{(2)}_{p,q} + a b^{(2)}_{p,q} + (1-q) \\
&= (1-q) + b a^{(2)}_{p,q}(1-q) + a b^{(2)}_{p,q}(1-q) + (1-q) \\
&= (1-q) + (b a^{(2)}_{p,q} + a b^{(2)}_{p,q} + 1)(1-q) \\
&= (1-q).
\end{aligned}
$$

Thus, we proved $(a+b)^{(2)}_{p,q} = a^{(2)}_{p,q} + b^{(2)}_{p,q}$. □

The following theorem gives us equivalent conditions under which $x^{(2)}_{p,q}$ has the generalized Banachiewicz–Schur form in a Banach algebra.

Theorem 9.4. *Let $x = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]_u \in \mathcal{A}$ relative to the idempotent $u \in \mathcal{A}$, $p_1, q_1 \in (u\mathcal{A}u)^\bullet$ and $p_2, q_2 \in ((1-u)\mathcal{A}(1-u))^\bullet$ and let $p = p_1 + p_2 \in \mathcal{A}$ and $q = q_1 + q_2 \in \mathcal{A}$. Let $a \in (u\mathcal{A}u)^{(2)}_{p_1,q_1}$ and let $s = d - c a^{(2)}_{p_1,q_1} b \in ((1-u)\mathcal{A}(1-u))^{(2)}_{p_2,q_2}$ be the generalized Schur complement of a in x. Then the following statements are equivalent*

1) *$x \in \mathcal{A}^{(2)}_{p,q}$ and $x^{(2)}_{p,q} = r$, where*

$$
r = \begin{bmatrix} a^{(2)}_{p_1,q_1} + a^{(2)}_{p_1,q_1} b s^{(2)}_{p_2,q_2} c a^{(2)}_{p_1,q_1} & -a^{(2)}_{p_1,q_1} b s^{(2)}_{p_2,q_2} \\ -s^{(2)}_{p_2,q_2} c a^{(2)}_{p_1,q_1} & s^{(2)}_{p_2,q_2} \end{bmatrix} \tag{9.2}
$$

2) *$c a^{(2)}_{p_1,q_1} a = s s^{(2)}_{p_2,q_2} c$ and $a a^{(2)}_{p_1,q_1} b = b s^{(2)}_{p_2,q_2} s$.*

Proof. By Lemma 9.2 we obtain that p and q are idempotents.
Using the assumptions $a \in (u\mathcal{A}u)^{(2)}_{p_1,q_1}$ and $s \in ((1-u)\mathcal{A}(1-u))^{(2)}_{p_2,q_2}$, we verify $rxr = r$.
The equation $rx = p$ is equivalent to the equations

$$
s^{(2)}_{p_2,q_2} c = s^{(2)}_{p_2,q_2} c a^{(2)}_{p_1,q_1} a \quad \text{and} \quad a^{(2)}_{p_1,q_1} b = a^{(2)}_{p_1,q_1} b s^{(2)}_{p_2,q_2} s.
$$

On the other hand, $1 - xr = q$ is equivalent to

$$
b s^{(2)}_{p_2,q_2} = a a^{(2)}_{p_1,q_1} b s^{(2)}_{p_2,q_2} \quad \text{and} \quad c a^{(2)}_{p_1,q_1} = s s^{(2)}_{p_2,q_2} c a^{(2)}_{p_1,q_1}.
$$

Therefore, x has (p,q)-outer generalized inverse if and only if

$$s^{(2)}_{p_2,q_2}c = s^{(2)}_{p_2,q_2}ca^{(2)}_{p_1,q_1}a, \quad a^{(2)}_{p_1,q_1}b = a^{(2)}_{p_1,q_1}bs^{(2)}_{p_2,q_2}s,$$
$$bs^{(2)}_{p_2,q_2} = aa^{(2)}_{p_1,q_1}bs^{(2)}_{p_2,q_2}, \quad ca^{(2)}_{p_1,q_1} = ss^{(2)}_{p_2,q_2}ca^{(2)}_{p_1,q_1},$$

which are equivalent to

$$ca^{(2)}_{p_1,q_1}a = ss^{(2)}_{p_2,q_2}c, \quad bs^{(2)}_{p_2,q_2}s = aa^{(2)}_{p_1,q_1}b. \qquad \square$$

As a corollary, we formulate the following result.

Corollary 9.1. *Let* $x = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathcal{A}$ *relative to the idempotent* $u \in \mathcal{A}$, $p_1, q_1 \in (u\mathcal{A}u)^\bullet$ *and* $p_2, q_2 \in ((1-u)\mathcal{A}(1-u))^\bullet$ *and let* $p = p_1 + p_2 \in \mathcal{A}$ *and* $q = q_1 + q_2 \in \mathcal{A}$. *Let* $a \in (u\mathcal{A}u)^{(2)}_{p_1,q_1}$ *and let* $s = d - ca^{(2)}_{p_1,q_1}b \in ((1-u)\mathcal{A}(1-u))^{(2)}_{p_2,q_2}$. *The following stataments are equivalent*

1) $ca^{(2)}_{p_1,q_1} = a^{(2)}_{p_1,q_1}b = bs^{(2)}_{p_2,q_2} = s^{(2)}_{p_2,q_2}c = 0$,
2) $ca^{(2)}_{p_1,q_1}a = ss^{(2)}_{p_2,q_2}c, \quad aa^{(2)}_{p_1,q_1}b = bs^{(2)}_{p_2,q_2}s, \quad a^{(2)}_{p_1,q_1}bs^{(2)}_{p_2,q_2} = s^{(2)}_{p_2,q_2}ca^{(2)}_{p_1,q_1} = 0$.

If one of these conditions is satisfied, then $x \in \mathcal{A}^{(2)}_{p,q}$ *and*

$$x^{(2)}_{p,q} = \begin{bmatrix} a^{(2)}_{p_1,q_1} + a^{(2)}_{p_1,q_1}bs^{(2)}_{p_2,q_2}ca^{(2)}_{p_1,q_1} & -a^{(2)}_{p_1,q_1}bs^{(2)}_{p_2,q_2} \\ -s^{(2)}_{p_2,q_2}ca^{(2)}_{p_1,q_1} & s^{(2)}_{p_2,q_2} \end{bmatrix}.$$

10. Drazin inverse of block matrices

The Drazin inverse plays an important role in Markov chains, singular differential and difference equations, iterative methods in numerical linear algebra, etc.

Representations for the Drazin inverse of block matrices under certain conditions where given in the literature [6, 7, 9, 17, 18, 23, 34, 46, 57].

In [15], a representation for the Drazin inverse of an anti-triangular block matrix under some conditions was obtained, generalizing in different ways results from [8, 34].

Block anti-triangular matrices arise in numerous applications, ranging form constrained optimization problems to solution of differential equations, etc. Deng [16] presented some formulas for the generalized Drazin inverse of an anti-triangular operator matrix $M = \left[\begin{smallmatrix} A & B \\ C & 0 \end{smallmatrix}\right]$, acting on a Banach space, with the assumption that CA^dB is invertible.

In this part of the paper, we present the results from [42] where were studied the equivalent conditions under which the generalized Drazin inverse has the generalized Banahievich-Shur form in Banach algebra. Also, several representations were obtained under different conditions for the generalized Drazin inverse of the anti-triangular block matrix $x = \left[\begin{smallmatrix} a & b \\ c & 0 \end{smallmatrix}\right]_p$ in Banach algebra $\mathcal{A}$ relative to the idempotent p. Thus, we get the particular cases of results from [14–16].

Hartwig et al. [34] gave expressions for the Drazin inverse of a 2×2 block matrix in the cases when the generalized Schur complement is nonsingular and it is equal to zero. These results are generalized in [47] under different conditions and the hypothesis the Schur complement is either nonsingular or zero.

In [11], Castro-González and Martínez-Serrano developed conditions under which the Drazin inverse of a block matrix having generalized Schur complement group invertible, can be expressed in terms of a matrix in the Banachiewicz–Schur form and its powers.

Deng and Wei [17] introduced several explicit representations for the Drazin inverse of a block–operator matrix with Drazin invertible Schur complement under different conditions.

Let

$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{A} \tag{10.1}$$

relative to the idempotent $p \in \mathcal{A}$, $a \in (p\mathcal{A}p)^d$ and let $s = d - ca^d b \in ((1-p)\mathcal{A}(1-p))^d$ be the generalized Schur complement of a in x.

In this section, when we say that x is defined as in (10.1), we assume that x has a representation as in (10.1) relative to the idempotent $p \in \mathcal{A}$, $a \in (p\mathcal{A}p)^d$ and $s = d - ca^d b \in ((1-p)\mathcal{A}(1-p))^d$.

In the following lemma, we present necessary and sufficient conditions for an element $x = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ of Banach algebra to have the generalized Drazin inverse with the generalized Banachiewicz–Schur form. We recover result concerning the Drazin inverse of Hilbert space operators (see [14, Corollary 3]).

Lemma 10.1. *Let x be defined as in* (10.1)*, $a \in (p\mathcal{A}p)^\#$, and let $s = d - ca^\# b \in ((1-p)\mathcal{A}(1-p))^\#$ be the generalized Schur complement of a in x. Then the following statements are equivalent*

1) $x \in \mathcal{A}^d$ *and*

$$x^d = \begin{bmatrix} a^\# + a^\# b s^\# c a^\# & -a^\# b s^\# \\ -s^\# c a^\# & s^\# \end{bmatrix}; \tag{10.2}$$

2) $a^\pi b s^\# = a^\# b s^\pi$, $\quad s^\pi c a^\# = s^\# c a^\pi \quad$ *and* $\quad z = \left[\begin{smallmatrix} 0 & bs^\pi \\ ca^\pi & 0 \end{smallmatrix}\right] \in \mathcal{A}^{\mathrm{qnil}}$;
3) $a^\pi b = b s^\pi$, $\quad s^\pi c = c a^\pi \quad$ *and* $\quad z = \left[\begin{smallmatrix} 0 & a^\pi b \\ s^\pi c & 0 \end{smallmatrix}\right] \in \mathcal{A}^{\mathrm{qnil}}$.

Proof. 1) $\Leftrightarrow$ 2): If the right side of (10.2) is denoted by y, then we obtain

$$xy = \begin{bmatrix} aa^\# - a^\pi b s^\# c a^\# & a^\pi b s^\# \\ s^\pi c a^\# & s s^\# \end{bmatrix},$$

$$yx = \begin{bmatrix} a^\# a - a^\# b s^\# c a^\pi & a^\# b s^\pi \\ s^\# c a^\pi & s^\# s \end{bmatrix}.$$

So, $xy = yx$ if and only if $a^\pi b s^\# = a^\# b s^\pi$ and $s^\pi c a^\# = s^\# c a^\pi$, because these equalities imply $(a^\pi b s^\#) c a^\# = a^\# b (s^\pi c a^\#) = a^\# b s^\# c a^\pi$. Further, we can verify that $yxy = y$. Using $s = d - ca^\# b$, $a^\pi b s^\# = a^\# b s^\pi$ and $s^\pi c a^\# = s^\# c a^\pi$, we have

$$x - x^2 y = \begin{bmatrix} -b s^\# c a^\pi & b s^\pi \\ c a^\pi & 0 \end{bmatrix}.$$

From $a^\# bs^\pi = a^\pi bs^\# = (p - aa^\#)bs^\# = bs^\# - aa^\# bs^\#$, we obtain $bs^\# = a^\# bs^\pi + aa^\# bs^\#$ which gives $ca^\pi bs^\# = 0 = bs^\# ca^\pi bs^\#$ and

$$x - x^2 y = \begin{bmatrix} p & -bs^\# \\ 0 & 1-p \end{bmatrix} z \begin{bmatrix} p & bs^\# \\ 0 & 1-p \end{bmatrix}.$$

Since $r(x - x^2 y) = r\big(\left[\begin{smallmatrix} p & bs^\# \\ 0 & 1-p \end{smallmatrix}\right] \left[\begin{smallmatrix} p & -bs^\# \\ 0 & 1-p \end{smallmatrix}\right] z\big) = r(z)$, we deduce that $x - x^2 y \in \mathcal{A}^{\text{qnil}}$ is equivalent to $z \in \mathcal{A}^{\text{qnil}}$. 2) $\Leftrightarrow$ 3): We prove that $a^\pi bs^\# = a^\# bs^\pi$ is equivalent to $a^\pi b = bs^\pi$. Indeed, multiplying $a^\pi bs^\# = a^\# bs^\pi$ from the right side by s and from the left side by a, respectively, we obtain $a^\pi bs^\# s = 0$ and $aa^\# bs^\pi = 0$. Therefore, $bs^\# s = aa^\# bs^\# s = aa^\# b$ and

$$a^\pi b = b - aa^\# b = b - bs^\# s = bs^\pi.$$

On the other hand, if $a^\pi b = bs^\pi$, then $(a^\pi b)s^\# = bs^\pi s^\# = 0$ and $a^\#(bs^\pi) = a^\# a^\pi b = 0$, i.e. $a^\pi bs^\# = a^\# bs^\pi$.

In the same manner, we can verify that $s^\pi ca^\# = s^\# ca^\pi$ is equivalent to $s^\pi c = ca^\pi$. Hence, the equivalence 2) $\Leftrightarrow$ 3) holds. □

By Lemma 10.1, the following corollary recovers [4, Theorem 2].

Corollary 10.1. *Let x be defined as in* (10.1)*, $a \in (p\mathcal{A}p)^\#$, and let $s = d - ca^\# b \in ((1-p)\mathcal{A}(1-p))^\#$ be the generalized Schur complement of a in x. Then $x \in \mathcal{A}^\#$ and*

$$x^\# = \begin{bmatrix} a^\# + a^\# bs^\# ca^\# & -a^\# bs^\# \\ -s^\# ca^\# & s^\# \end{bmatrix}$$

if and only if $a^\pi b = 0 = bs^\pi$, and $s^\pi c = 0 = ca^\pi$.

Now, we extend the well known result concerning the Drazin inverse of complex matrices to the generalized Drazin inverse of Banach algebra elements, see [15, Theorem 3.5].

Theorem 10.1. *Let*

$$x = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \in \mathcal{A} \tag{10.3}$$

relative to the idempotent $p \in \mathcal{A}$, $a \in (p\mathcal{A}p)^d$ and let $s = -ca^d b \in ((1-p)\mathcal{A}(1-p))^d$. If

$$ss^d ca^\pi b = 0, \quad ss^d ca^\pi a = 0, \quad aa^d bs^\pi c = 0, \quad bs^\pi ca^\pi = 0, \tag{10.4}$$

then $x \in \mathcal{A}^d$ and

$$x^d = \left(r + \sum_{n=0}^{+\infty} \begin{bmatrix} aa^\pi & a^\pi bs^\pi \\ s^\pi ca^\pi & 0 \end{bmatrix}^n \begin{bmatrix} 0 & a^\pi bss^d \\ s^\pi caa^d & 0 \end{bmatrix} r^{n+2} \right) \times \left(1 + r \begin{bmatrix} 0 & aa^d bs^\pi \\ ss^d ca^\pi & 0 \end{bmatrix} \right), \tag{10.5}$$

where

$$r = \begin{bmatrix} a^d + a^d bs^d ca^d & -a^d bs^d \\ -s^d ca^d & s^d \end{bmatrix}. \tag{10.6}$$

Proof. Applying $aa^d + a^\pi = p$ and $ss^d + s^\pi = 1 - p$, we have

$$x = \begin{bmatrix} a^2a^d & aa^db \\ ss^dc & 0 \end{bmatrix} + \begin{bmatrix} aa^\pi & a^\pi b \\ s^\pi c & 0 \end{bmatrix} := u + v.$$

The equalities $a^da^\pi = 0$ and (10.4) give $uv = 0$.

First, we show that $u \in \mathcal{A}^d$. If we write

$$u = \begin{bmatrix} a^2a^d & aa^dbss^d \\ ss^dcaa^d & 0 \end{bmatrix} + \begin{bmatrix} 0 & aa^dbs^\pi \\ ss^dca^\pi & 0 \end{bmatrix} := u_1 + u_2,$$

we can get $u_2u_1 = 0$ and $u_2^2 = 0$. Let $A_{u_1} \equiv a^2a^d$, $B_{u_1} \equiv aa^dbss^d$, $C_{u_1} \equiv ss^dcaa^d$ and $D_{u_1} \equiv 0$. Then $u_1 = \left[\begin{smallmatrix} A_{u_1} & B_{u_1} \\ C_{u_1} & D_{u_1} \end{smallmatrix}\right]$ and, by $(a^2a^d)^\# = a^d$, $A_{u_1} \in (p\mathcal{A}p)^\#$. Also, from $s = -ca^db$, $S_{u_1} \equiv D_{u_1} - C_{u_1}A_{u_1}^\# B_{u_1} = s^2s^d \in ((1-p)\mathcal{A}(1-p))^\#$ and $(s^2s^d)^\# = s^d$. Consequently,

$$A_{u_1}^\pi B_{u_1} S_{u_1}^\# = 0 = A_{u_1}^\# B_{u_1} S_{u_1}^\pi, \quad S_{u_1}^\pi C_{u_1} A_{u_1}^\# = 0 = S_{u_1}^\# C_{u_1} A_{u_1}^\pi$$

and $\left[\begin{smallmatrix} 0 & B_{u_1}S_{u_1}^\pi \\ C_{u_1}A_{u_1}^\pi & 0 \end{smallmatrix}\right] = 0 \in \mathcal{A}^{\mathrm{qnil}}$. By Lemma 10.1, notice that $u_1 \in \mathcal{A}^d$ and

$$u_1^d = \begin{bmatrix} A_{u_1}^\# + A_{u_1}^\# B_{u_1} S_{u_1}^\# C_{u_1} A_{u_1}^\# & -A_{u_1}^\# B_{u_1} S_{u_1}^\# \\ -S_{u_1}^\# C_{u_1} A_{u_1}^\# & S_{u_1}^\# \end{bmatrix} = r.$$

Using Lemma 9.51), $u \in \mathcal{A}^d$ and $u^d = u_1^d + (u_1^d)^2u_2 = r + r^2u_2$.

To prove that $v \in \mathcal{A}^{\mathrm{qnil}}$, observe that

$$v = \begin{bmatrix} aa^\pi & a^\pi bs^\pi \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ s^\pi ca^\pi & 0 \end{bmatrix} + \begin{bmatrix} 0 & a^\pi bss^d \\ s^\pi caa^d & 0 \end{bmatrix} := v_1 + v_2 + v_3.$$

If $z = \left[\begin{smallmatrix} m & t \\ 0 & n \end{smallmatrix}\right]$, then $\lambda 1 - z = \left[\begin{smallmatrix} \lambda p - m & -t \\ 0 & \lambda(1-p) - n \end{smallmatrix}\right]$. Therefore

$$\lambda \in \rho_{p\mathcal{A}p}(m) \cap \rho_{(1-p)\mathcal{A}(1-p)}(n) \ \Rightarrow \ \lambda \in \rho(z),$$

i.e.

$$\sigma(z) \subseteq \sigma_{p\mathcal{A}p}(m) \cup \sigma_{(1-p)\mathcal{A}(1-p)}(n).$$

Notice that, by $aa^\pi \in (p\mathcal{A}p)^{\mathrm{qnil}}$, $v_1 \in \mathcal{A}^{\mathrm{qnil}}$. It can be verified that $v_1v_2 = 0$ and $v_2^2 = 0$, i.e. $v_2 \in \mathcal{A}^{\mathrm{nil}}$. Now, by Lemma 9.4, $v_1 + v_2 \in \mathcal{A}^{\mathrm{qnil}}$. Using Lemma 9.4 again, from $v_3^2 = 0$ and $v_3(v_1 + v_2) = 0$, we conclude that $v \in \mathcal{A}^{\mathrm{qnil}}$.

Applying Lemma 9.52), we deduce that $x \in \mathcal{A}^d$ and

$$x^d = \left(1 + \sum_{n=0}^{+\infty} v^{n+1}(u^d)^{n+2}\right)u^d = \left(1 + \sum_{n=0}^{+\infty} v^{n+1}(u^d)^{n+2}\right)r(1 + ru_2).$$

Since $u_2r = u_2u_1^d = (u_2u_1)(u_1^d)^2 = 0$, then $(u^d)^{n+2} = (r + r^2u_2)^{n+2} = r^{n+2}(1 + ru_2)$. From $r = \left[\begin{smallmatrix} aa^d & 0 \\ 0 & ss^d \end{smallmatrix}\right]r$, we obtain $vr = v\left[\begin{smallmatrix} aa^d & 0 \\ 0 & ss^d \end{smallmatrix}\right]r = \left[\begin{smallmatrix} 0 & a^\pi bss^d \\ s^\pi caa^d & 0 \end{smallmatrix}\right]$. By $v^{n+1} = (v_1 + v_2)^n v$, we have $v^{n+1}(u^d)^{n+2} = (v_1 + v_2)^n \left[\begin{smallmatrix} 0 & a^\pi bss^d \\ s^\pi caa^d & 0 \end{smallmatrix}\right] r^{n+1}(1 + ru_2)$. Applying $u_2r = 0$ again, we get (10.5). □

From Theorem 10.1, we get the following consequence.

Corollary 10.2. *Let x be defined as in* (10.3)*, $a \in (p\mathcal{A}p)^d$ and let r be defined as in* (10.6)*.*

1) *If* $ca^\pi = 0$ *and the generalized Schur complement* $s = -ca^d b$ *is invertible, then* $x \in \mathcal{A}^d$ *and*

$$x^d = r + \sum_{n=0}^{+\infty} \begin{bmatrix} aa^\pi & 0 \\ 0 & 0 \end{bmatrix}^n \begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix} r^{n+2}.$$

2) *If* $ca^\pi = 0$, $a^\pi b = 0$ *and the generalized Schur complement* $s = -ca^d b$ *is invertible, then* $x \in \mathcal{A}^d$ *and*

$$x^d = \begin{bmatrix} a^d + a^d b s^{-1} c a^d & -a^d b s^{-1} \\ -s^{-1} c a^d & s^{-1} \end{bmatrix}.$$

3) *If* $ca^\pi b = 0$, $ca^\pi a = 0$ *and the generalized Schur complement* $s = -ca^d b$ *is invertible, then* $x \in \mathcal{A}^d$ *and*

$$x^d = \left(r + \sum_{n=0}^{+\infty} \begin{bmatrix} 0 & a^n a^\pi b \\ 0 & 0 \end{bmatrix} r^{n+2} \right) \left(1 + r \begin{bmatrix} 0 & 0 \\ ca^\pi & 0 \end{bmatrix} \right).$$

In the following theorems, we assume that $s = -ca^d b$ is the generalized Drazin invertible, and we prove representations of the generalized Drazin inverse of anti-triangular block matrices. Several results from [16] are extended.

Theorem 10.2. *Let* x *be defined as in* (10.3), $a \in (p\mathcal{A}p)^d$ *and let* $s = -ca^d b \in ((1-p)\mathcal{A}(1-p))^d$. *If* $bca^\pi = 0$ *and* $aa^d bs^\pi = 0$, *then* $x \in \mathcal{A}^d$ *and*

$$x^d = \sum_{n=0}^{+\infty} \begin{bmatrix} aa^\pi & a^\pi b \\ ca^\pi & 0 \end{bmatrix}^n \left(1 + \begin{bmatrix} 0 & 0 \\ s^\pi c & 0 \end{bmatrix} r \right) r^{n+1}, \tag{10.7}$$

where r *be defined as in* (10.6).

Proof. We can write

$$x = \begin{bmatrix} a^2 a^d & aa^d b \\ caa^d & 0 \end{bmatrix} + \begin{bmatrix} aa^\pi & a^\pi b \\ ca^\pi & 0 \end{bmatrix} := y + q.$$

Now, we get $yq = 0$, by the assumption $bca^\pi = 0$.

In order to prove that $y \in \mathcal{A}^d$, note that

$$\begin{aligned} y &= \begin{bmatrix} a^2 a^d & aa^d bss^d \\ ss^d caa^d & 0 \end{bmatrix} + \begin{bmatrix} 0 & aa^d bs^\pi \\ s^\pi caa^d & 0 \end{bmatrix} \\ &= \begin{bmatrix} a^2 a^d & aa^d bss^d \\ ss^d caa^d & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ s^\pi caa^d & 0 \end{bmatrix} := y_1 + y_2, \end{aligned}$$

$y_1 y_2 = 0$ and $y_2^2 = 0$. Using Lemma 10.1, we have $y_1 \in \mathcal{A}^d$ and $y_1^d = r$. By Lemma 9.52), $y \in \mathcal{A}^d$ and $y^d = y_1^d + y_2 (y_1^d)^2 = r + y_2 r^2$.

Further, we verify that $q \in \mathcal{A}^{\mathrm{qnil}}$. Let

$$q = \begin{bmatrix} aa^\pi & a^\pi b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ ca^\pi & 0 \end{bmatrix} := q_1 + q_2.$$

Thus, we deduce that $q_1 \in \mathcal{A}^{\mathrm{qnil}}$ and $q_2 \in \mathcal{A}^{\mathrm{nil}}$, because $aa^\pi \in (p\mathcal{A}p)^{\mathrm{qnil}}$ and $q_2^2 = 0$. Since $q_1 q_2 = 0$, by Lemma 9.4, $q \in \mathcal{A}^{\mathrm{qnil}}$.

By Lemma 9.52), $x \in \mathcal{A}^d$ and

$$x^d = \sum_{n=0}^{+\infty} q^n (y^d)^{n+1} = \sum_{n=0}^{+\infty} q^n (1 + y_2 r) r^{n+1}.$$

The equality $r = \left[\begin{smallmatrix} aa^d & 0 \\ 0 & ss^d \end{smallmatrix}\right] r$ give $y_2 r = \left[\begin{smallmatrix} 0 & 0 \\ s^\pi c & 0 \end{smallmatrix}\right] r$ implying (10.7). □

Replacing the hypothesis $aa^d bs^\pi = 0$ with $s^\pi caa^d = 0$ in Theorem 10.2, we get the following theorem.

Theorem 10.3. *Let x be defined as in* (10.3)*, $a \in (p\mathcal{A}p)^d$ and let $s = -ca^d b \in ((1-p)\mathcal{A}(1-p))^d$. If $bca^\pi = 0$ and $s^\pi caa^d = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \sum_{n=0}^{+\infty} \begin{bmatrix} aa^\pi & a^\pi b \\ ca^\pi & 0 \end{bmatrix}^n r^{n+1} \left(1 + r \begin{bmatrix} 0 & bs^\pi \\ 0 & 0 \end{bmatrix}\right), \tag{10.8}$$

where r is defined in the same way as in (10.6)*.*

Proof. In the similar way as in the proof of Theorem 10.2, using

$$y = \begin{bmatrix} a^2 a^d & aa^d bss^d \\ ss^d caa^d & 0 \end{bmatrix} + \begin{bmatrix} 0 & aa^d bs^\pi \\ 0 & 0 \end{bmatrix} := y_1 + y_2$$

and $y_2 y_1 = 0$, we check this theorem. □

If $s = -ca^d b \in ((1-p)\mathcal{A}(1-p))^{-1}$ and $s' = -s$, then $s^\pi = 0$ and $(s')^{-1} = -s^{-1}$. As a special case of Theorem 10.2 (or Theorem 10.3), we obtain the following result which recovers [16, Theorem 3.1] for bounded linear operators on a Banach space.

Corollary 10.3. *Let x be defined as in* (10.3)*, $a \in (p\mathcal{A}p)^d$ and let $s' = ca^d b \in ((1-p)\mathcal{A}(1-p))^{-1}$. If $bca^\pi = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \sum_{n=0}^{+\infty} \begin{bmatrix} aa^\pi & a^\pi b \\ ca^\pi & 0 \end{bmatrix}^n t^{n+1}, \quad \textit{where } t = \begin{bmatrix} a^d - a^d b(s')^{-1} ca^d & a^d b(s')^{-1} \\ (s')^{-1} ca^d & -(s')^{-1} \end{bmatrix}.$$

Sufficient conditions under which the generalized Drazin inverse x^d is represented by (10.7) or (10.8) are investigated in the following result.

Theorem 10.4. *Let x be defined as in* (10.3)*, $a \in (p\mathcal{A}p)^d$ and let $s = -ca^d b \in ((1-p)\mathcal{A}(1-p))^d$. Suppose that $aa^d bca^\pi = 0$ and $ca^\pi b = 0$.*

1) *If $aa^d bs^\pi = 0$ and ($aa^\pi b = 0$ or $caa^\pi = 0$), then $x \in \mathcal{A}^d$ and* (10.7) *is satisfied.*
2) *If $s^\pi caa^d = 0$ and ($aa^\pi b = 0$ or $caa^\pi = 0$), then $x \in \mathcal{A}^d$ and* (10.8) *is satisfied.*

Proof. This result can be proved similarly as Theorem 10.2 and Theorem 10.3, applying $q_2 q_1 = 0$ when $caa^\pi = 0$, and the decomposition

$$q = \begin{bmatrix} aa^\pi & 0 \\ ca^\pi & 0 \end{bmatrix} + \begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix}$$

when $aa^\pi b = 0$. □

Remark 10.1. In the preceding theorem, if $ca^d b \in ((1-p)\mathcal{A}(1-p))^{-1}$, then we obtain as a particular case [16, Theorem 3.2] for Banach space operator.

The following result is well-known for complex matrices.

Lemma 10.2. *Let x be defined as in* (10.1), *$a \in (p\mathcal{A}p)^d$ and let $w = aa^d + a^d bca^d$ be such that $aw \in (p\mathcal{A}p)^d$. If $ca^\pi = 0$, $a^\pi b = 0$ and the generalized Schur complement $s = d - ca^d b$ is equal to 0, then*

$$x^d = \begin{bmatrix} p & 0 \\ ca^d & 0 \end{bmatrix} \begin{bmatrix} [(a\omega)^d]^2 a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p & a^d b \\ 0 & 0 \end{bmatrix}. \tag{10.9}$$

Proof. Denote by y the right side of (10.9). Then we obtain

$$xy = \begin{bmatrix} (a+bca^d)[(a\omega)^d]^2 a & (a+bca^d)[(a\omega)^d]^2 b \\ (c+dca^d)[(a\omega)^d]^2 a & (c+dca^d)[(a\omega)^d]^2 b \end{bmatrix},$$

$$yx = \begin{bmatrix} [(a\omega)^d]^2(a^2+bc) & [(a\omega)^d]^2(ab+bd) \\ ca^d[(a\omega)^d]^2(a^2+bc) & ca^d[(a\omega)^d]^2(ab+bd) \end{bmatrix}.$$

By $ca^\pi = 0$ and $a^\pi b = 0$, we can conclude that $a + bca^d$ commutes with $a\omega$. Indeed,

$$\begin{aligned} (a+bca^d)(a\omega) &= (a^2 + bca^d a)(aa^d + a^d bca^d) \\ &= (a^2 + aa^d bc)a^d(a+bca^d) = (a\omega)(a+bca^d). \end{aligned}$$

Since $a + bca^d$ commutes with $a\omega$, it also commutes with $(a\omega)^d$ and we have

$$(a+bca^d)[(a\omega)^d]^2 a = [(a\omega)^d]^2(a+bca^d)a = [(a\omega)^d]^2(a^2+bc).$$

From $s = 0$, we get $c + dca^d = ca^d a + ca^d bca^d = ca^d(a+bca^d)$. Thus,

$$(c+dca^d)[(a\omega)^d]^2 a = ca^d(a+bca^d)[(a\omega)^d]^2 a = ca^d[(a\omega)^d]^2(a^2+bc).$$

Also, $ab + bd = ab + bca^d b = (a+bca^d)b$ and we obtain

$$\begin{aligned} (a+bca^d)[(a\omega)^d]^2 b &= [(a\omega)^d]^2(ab+bd) \\ (c+dca^d)[(a\omega)^d]^2 b &= ca^d[(a\omega)^d]^2(ab+bd). \end{aligned}$$

So, we proved that

$$xy = yx = \begin{bmatrix} [(a\omega)^d]^2(a+bca^d)a & [(a\omega)^d]^2(a+bca^d)b \\ ca^d[(a\omega)^d]^2(a+bca^d)a & ca^d[(a\omega)^d]^2(a+bca^d)b \end{bmatrix}.$$

Further, we can verify that $yxy = y$. Indeed, we have

$$\begin{aligned} yxy = &\begin{bmatrix} [(a\omega)^d]^2 a & [(a\omega)^d]^2 b \\ ca^d[(a\omega)^d]^2 a & ca^d[(a\omega)^d]^2 b \end{bmatrix} \\ &\times \begin{bmatrix} [(a\omega)^d]^2(a+bca^d)a & [(a\omega)^d]^2(a+bca^d)b \\ ca^d[(a\omega)^d]^2(a+bca^d)a & ca^d[(a\omega)^d]^2(a+bca^d)b \end{bmatrix} \\ &= \begin{bmatrix} [(a\omega)^d]^4(a+bca^d)^2 a & [(a\omega)^d]^4(a+bca^d)^2 b \\ ca^d[(a\omega)^d]^4(a+bca^d)^2 a & ca^d[(a\omega)^d]^4(a+bca^d)^2 b \end{bmatrix}. \end{aligned}$$

The equalities $a + bca^d = a - a^2a^d + a^2a^d + bca^d = aa^\pi + a\omega$ and $a^\pi\omega = 0 = \omega a^\pi$ give $(a + bca^d)^2 = a^2a^\pi + (a\omega)^2$. Therefore,

$$(a\omega)^d(a + bca^d)^2 = (a\omega)^d(a^2a^\pi + (a\omega)^2)$$
$$= [(a\omega)^d]^2(a\omega)a^\pi a^2 + (a\omega)^d(a\omega)^2 = (a\omega)^d(a\omega)^2$$

and $[(a\omega)^d]^4(a + bca^d)^2 = [(a\omega)^d]^4(a\omega)^2 = [(a\omega)^d]^2$ implying

$$yxy = \begin{bmatrix} [(a\omega)^d]^2a & [(a\omega)^d]^2b \\ ca^d[(a\omega)^d]^2a & ca^d[(a\omega)^d]^2b \end{bmatrix} = y.$$

We obtain

$$x - x^2y = \begin{bmatrix} (a\omega)^\pi a & (a\omega)^\pi b \\ ca^d(a\omega)^\pi a & ca^d(a\omega)^\pi b \end{bmatrix} = \begin{bmatrix} p & 0 \\ ca^d & 0 \end{bmatrix} \begin{bmatrix} (a\omega)^\pi a & (a\omega)^\pi b \\ 0 & 0 \end{bmatrix}.$$

Notice that, by $a + bca^d = aa^\pi + a\omega$, $(a\omega)^\pi(a + bca^d) = aa^\pi + (a\omega)(a\omega)^\pi$. Since $aa^\pi, (a\omega)(a\omega)^\pi \in (p\mathcal{A}p)^{\text{qnil}}$ and $aa^\pi(a\omega)(a\omega)^\pi = 0$, by Lemma 9.4, we have that $aa^\pi + (a\omega)(a\omega)^\pi \in (p\mathcal{A}p)^{\text{qnil}}$ and $r_{p\mathcal{A}p}((a\omega)^\pi(a + bca^d)) = 0$. From

$$r(x - x^2y) = r\left(\begin{bmatrix} (a\omega)^\pi a & (a\omega)^\pi b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p & 0 \\ ca^d & 0 \end{bmatrix}\right)$$
$$= r\left(\begin{bmatrix} (a\omega)^\pi(a + bca^d) & 0 \\ 0 & 0 \end{bmatrix}\right) = r_{p\mathcal{A}p}((a\omega)^\pi(a + bca^d)) = 0,$$

we deduce that $x - x^2y \in \mathcal{A}^{\text{qnil}}$ and prove that $x^d = y$. □

In the following theorem, we extend [16, Theorem 3.3 and Theorem 3.4] for Banach space operators to elements of a Banach algebra.

Theorem 10.5. *Let x be defined as in* (10.3), *$a \in (p\mathcal{A}p)^d$ and let $k = a^2a^d + aa^dbca^d \in (p\mathcal{A}p)^d$. If $ca^db = 0$ and if one of the following conditions holds*

1) *$bca^\pi = 0$;*
2) *$aa^dbca^\pi = 0$, $aa^\pi b = 0$ and $ca^\pi b = 0$;*
3) *$aa^dbca^\pi = 0$, $caa^\pi = 0$ and $ca^\pi b = 0$;*

then $x \in \mathcal{A}^d$ and

$$x^d = \sum_{n=0}^{+\infty} \begin{bmatrix} aa^\pi & a^\pi b \\ ca^\pi & 0 \end{bmatrix}^n \begin{bmatrix} (k^d)^2a & (k^d)^2b \\ ca^d(k^d)^2a & ca^d(k^d)^2b \end{bmatrix}^{n+1}. \tag{10.10}$$

Proof. To prove the part (1) suppose that $x = y + q$, where s and y are defined as in the proof of Theorem 10.2. It follows that $yq = 0$ and $q \in \mathcal{A}^{\text{qnil}}$. Applying Lemma 10.2, we conclude that $y \in \mathcal{A}^d$ and

$$y^d = \begin{bmatrix} p & 0 \\ ca^d & 0 \end{bmatrix} \begin{bmatrix} (k^d)^2a^2a^d & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p & a^db \\ 0 & 0 \end{bmatrix}.$$

Since $kaa^d = k$, then $k^daa^d = k^d$ and

$$y^d = \begin{bmatrix} (k^d)^2a & (k^d)^2b \\ ca^d(k^d)^2a & ca^d(k^d)^2b \end{bmatrix}.$$

Using Lemma 9.52), we conclude that $x \in \mathcal{A}^d$ and $x^d = \sum_{n=0}^{+\infty} q^n (y^d)^{n+1}$. Thus, (10.10) holds.

The parts (2) and (3) can be checked in the similar manner as in the part (1) and in the proof of Theorem 10.4. $\square$

If $c = 0$ or $b = 0$ in Theorem 10.5, we have $k = a^2 a^d \in (p\mathcal{A}p)^d$ and $k^d = a^d$. As a consequence of Theorem 10.5, we obtain the following result.

Corollary 10.4. *Let x be defined as in* (10.3) *and let $a \in (p\mathcal{A}p)^d$.*

1) *If $c = 0$, then $x \in \mathcal{A}^d$ and* $x^d = \begin{bmatrix} a^d & (a^d)^2 b \\ 0 & 0 \end{bmatrix}$.
2) *If $b = 0$, then $x \in \mathcal{A}^d$ and* $x^d = \begin{bmatrix} a^d & 0 \\ c(a^d)^2 & 0 \end{bmatrix}$.

In the following part of this section, we present the results from the paper [41].

The following auxiliary results will be used in the rest of the section.

Lemma 10.3. *Let x be defined as in* (10.1) *and assume that $w_0 = p + a^d b s^\pi c a^d$ is invertible. Then $w_0 a^2 a^d$ is group invertible, $(w_0 a^2 a^d)^\# = a^d w_0^{-1}$ and $(wa^2a^d)^\pi = a^\pi$.*

Proof. Let us prove that $a^d w_0^{-1}$ is group inverse of $w_0^{-1} a^2 a^d$. Indeed,

$$\begin{aligned}(w_0 a^2 a^d)(a^d w_0^{-1}) &= w_0 a a^d w_0^{-1} = (p + a^d b s^\pi c a^d) a a^d w_0^{-1} \\ &= (a^d a + a^d a a^d b s^\pi c a^d) w_0^{-1} = a^d a w_0 w_0^{-1} = a^d a \\ &= a^d a^2 a^d = (a^d w_0^{-1})(w_0 a^2 a^d),\end{aligned}$$

$$(w_0 a^2 a^d)(a^d w_0^{-1})(w_0 a^2 a^d) = w_0 a^2 a^d a^d a^2 a^d = w_0 a^2 a^d,$$

$$(a^d w_0^{-1})(w_0 a^2 a^d)(a^d w_0^{-1}) = a^d a^2 a^d a^d w_0^{-1} = a^d w_0^{-1}$$

implies that $(w_0 a^2 a^d)^\# = a^d w_0^{-1}$. Spectral idempotent of $w_0 a^2 a^d$ is equal to $(w_0 a^2 a^d)^\pi = p - (w_0 a^2 a^d)(a^d w_0^{-1}) = p - a a^d = a^\pi$. $\square$

Lemma 10.4. *Let $x \in \mathcal{A}^d$ and $u \in \mathcal{A}$ be an invertible element. Then $u^{-1} x u \in \mathcal{A}^d$ and $(u^{-1} x u)^d = u^{-1} x^d u$.*

The following lemma will extend to the generalized Drazin inverse of Banach algebra elements a well known result concerning the Drazin inverse of Hilbert space operators.

Lemma 10.5. *Let x be defined as in* (10.1). *Then the following statements are equivalent*

1) *$x \in \mathcal{A}^d$ and $x^d = r$, where*

$$r = \begin{bmatrix} a^d + a^d b s^d c a^d & -a^d b s^d \\ -s^d c a^d & s^d \end{bmatrix}; \tag{10.11}$$

2) *$a^\pi b s^d = a^d b s^\pi$, $s^\pi c a^d = s^d c a^\pi$ and* $y = \left[\begin{smallmatrix} a a^\pi & a^\pi b \\ s^\pi c a^\pi & s s^\pi \end{smallmatrix}\right] \in \mathcal{A}^{\mathrm{qnil}}$;
3) *$a^\pi b = b s^\pi$, $s^\pi c = c a^\pi$ and* $y = \left[\begin{smallmatrix} a a^\pi & b s^\pi \\ c a^\pi & s s^\pi \end{smallmatrix}\right] \in \mathcal{A}^{\mathrm{qnil}}$.

Proof. 1)⇔2): We can verify that $rxr = r$. Since $a^\pi bs^d = a^d bs^\pi$ and $s^\pi ca^d = s^d ca^\pi$ imply $a^\pi bs^d ca^d = a^d bs^d ca^\pi$, by elementary computations, we observe that $xr = rx$ if and only if $a^\pi bs^d = a^d bs^\pi$ and $s^\pi ca^d = s^d ca^\pi$. Now, we can obtain

$$x - x^2 r = \begin{bmatrix} p & -a^d b \\ 0 & 1-p \end{bmatrix} y \begin{bmatrix} p & a^d b \\ 0 & 1-p \end{bmatrix},$$

$$r(x - x^2 r) = r\left(\begin{bmatrix} p & a^d b \\ 0 & 1-p \end{bmatrix} \begin{bmatrix} p & -a^d b \\ 0 & 1-p \end{bmatrix} y \right) = r(y).$$

Hence, $x - x^2 r \in \mathcal{A}^{\mathrm{qnil}}$ is equivalent to $y \in \mathcal{A}^{\mathrm{qnil}}$.

2)⇔3): First, we check that $a^\pi bs^d = a^d bs^\pi$ is equivalent to $a^\pi b = bs^\pi$. If we multiply the equality $a^\pi bs^d = a^d bs^\pi$ from the right side by s and from the left side by a, respectively, we get $a^\pi bs^d s = 0$ and $aa^d bs^\pi = 0$. So, $bs^d s = aa^d bs^d s = aa^d b$ and

$$a^\pi b = b - aa^d b = b - bs^d s = bs^\pi.$$

On the other hand, $a^\pi b = bs^\pi$ gives $(a^\pi b)s^d = bs^\pi s^d = 0$ and $a^d(bs^\pi) = a^d a^\pi b = 0$. Hence, $a^\pi bs^d = a^d bs^\pi$.

Similarly, we can prove that $s^\pi ca^d = s^d ca^\pi$ is equivalent to $s^\pi c = ca^\pi$. Thus, we deduce that 2) ⇔ 3). □

Remark 10.2. Using Lemma 10.5, if x is defined as in (10.1) and r is defined as in (10.6), then $x \in \mathcal{A}^\#$ and $x^\# = r$ if and only if $a \in (p\mathcal{A}p)^\#$, $s \in ((1-p)\mathcal{A}(1-p))^\#$, $a^\pi b = 0 = bs^\pi$ and $s^\pi c = 0 = ca^\pi$. This results is well-known for a complex matrix [4, Theorem 2] (see also [11, Corollary 2.3]). The expression (10.6) is called the generalized Banachiewicz–Schur form of x. For more details see [1, 4, 11, 34].

Now we present a formula for the generalized Drazin inverse of block matrix x in (10.1) in terms of the generalized Drazin invertible Schur complement s. We extend [17, Theorem 7] concerning the Drazin inverse of 2×2 block-operator matrix to more general setting.

Theorem 10.6. *Let x be defined as in* (10.1). *If*

$$ca^\pi bss^d = 0, \qquad aa^\pi bss^d = 0, \qquad ss^\pi c = 0, \qquad a^\pi bs^\pi c = bs^\pi caa^d = 0, \tag{10.12}$$

then $x \in \mathcal{A}^d$ and

$$x^d = \left(\begin{bmatrix} 0 & a^\pi b \\ s^\pi c & s^\pi d \end{bmatrix} r + 1 \right) r \left(1 + \sum_{n=0}^{\infty} r^{n+1} \begin{bmatrix} 0 & bs^\pi \\ ca^\pi & ds^\pi \end{bmatrix} \begin{bmatrix} aa^\pi & bs^\pi \\ ca^\pi & ds^\pi \end{bmatrix}^n \right), \tag{10.13}$$

where r is defined as in (10.6).

Proof. Notice that, by $a^\pi + aa^d = p$ and $s^\pi + ss^d = 1 - p$,

$$x = \begin{bmatrix} aa^\pi & bs^\pi \\ ca^\pi & ds^\pi \end{bmatrix} + \begin{bmatrix} a^2 a^d & bss^d \\ caa^d & dss^d \end{bmatrix} := y + z.$$

From $a^d a^\pi = 0 = s^\pi s^d$, $d = s + ca^d b$, $bs^\pi ca^d = (bs^\pi caa^d)a^d = 0$ and (10.12), we get $yz = 0$.

To prove that $y \in \mathcal{A}^{\text{qnil}}$, we observe that

$$y = \begin{bmatrix} aa^\pi & a^\pi bs^\pi \\ 0 & ss^\pi \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ s^\pi ca^\pi & s^\pi ca^d bs^\pi \end{bmatrix} + \begin{bmatrix} 0 & aa^d bs^\pi \\ ss^d ca^\pi & ss^d ds^\pi \end{bmatrix} := y_1 + y_2 + y_3.$$

Recall that if $u = \left[\begin{smallmatrix} a_1 & 0 \\ c_1 & b_1 \end{smallmatrix}\right]$, then $\lambda 1 - u = \left[\begin{smallmatrix} \lambda p - a_1 & 0 \\ -c_1 & \lambda(1-p) - b_1 \end{smallmatrix}\right]$ and

$$\lambda \in \rho_{p\mathcal{A}p}(a_1) \cap \rho_{(1-p)\mathcal{A}(1-p)}(b_1) \ \Rightarrow \ \lambda \in \rho(u),$$

i.e., $\sigma(u) \subseteq \sigma_{p\mathcal{A}p}(a_1) \cup \sigma_{(1-p)\mathcal{A}(1-p)}(b_1)$.

Since $aa^\pi \in (p\mathcal{A}p)^{\text{qnil}}$ and $ss^\pi \in ((1-p)\mathcal{A}(1-p))^{\text{qnil}}$, we deduce that $y_1 \in \mathcal{A}^{\text{qnil}}$. By $r(s^\pi ca^d bs^\pi) = r(bs^\pi ca^d) = r(0) = 0$ and

$$\sigma_{\mathcal{A}}(s^\pi ca^d bs^\pi) = \sigma_{(1-p)\mathcal{A}(1-p)}(s^\pi ca^d bs^\pi) \cup \{0\}$$

(Lemma 9.3), $y_2 \in \mathcal{A}^{\text{qnil}}$. We can check that $y_1 y_2 = 0$ which gives that $y_1 + y_2 \in \mathcal{A}^{\text{qnil}}$, by Lemma 9.4. Also, by Lemma 9.4, $y_3^2 = 0$ (i.e. $y_3 \in \mathcal{A}^{\text{nil}}$) and $(y_1 + y_2)y_3 = 0$ imply $y \in \mathcal{A}^{\text{qnil}}$.

In order to show that $z \in \mathcal{A}^d$, we write

$$z = \begin{bmatrix} a^2 a^d & aa^d bss^d \\ ss^d caa^d & ss^d dss^d \end{bmatrix} + \begin{bmatrix} 0 & a^\pi bss^d \\ s^\pi caa^d & s^\pi dss^d \end{bmatrix} := z_1 + z_2.$$

We can verify that $z_1 z_2 = 0$ and $z_2^2 = 0$. If $z_1 = \left[\begin{smallmatrix} A_{z_1} & B_{z_1} \\ C_{z_1} & D_{z_1} \end{smallmatrix}\right]$, we note that $A_{z_1} \equiv a^2 a^d \in (p\mathcal{A}p)^\#$, $(a^2 a^d)^\# = a^d$, $S_{z_1} \equiv D_{z_1} - C_{z_1} A_{z_1}^\# B_{z_1} = s^2 s^d \in ((1-p)\mathcal{A}(1-p))^\#$ and $(s^2 s^d)^\# = s^d$. Using Lemma 10.5, we have $z_1 \in \mathcal{A}^d$ and $z_1^d = r$. Further, by Lemma 9.5, $z \in \mathcal{A}^d$ and $z^d = z_1^d + z_2 (z_1^d)^2$.

Applying again Lemma 9.5, we conclude that $x \in \mathcal{A}^d$ and

$$x^d = \sum_{n=0}^{\infty} (z^d)^{n+1} y^n = (1 + z_2 z_1^d) z_1^d \left(1 + \sum_{n=0}^{\infty} (z_1^d)^{n+1} y^{n+1}\right).$$

Then, observe that $z_1^d = r = r \begin{bmatrix} aa^d & 0 \\ 0 & ss^d \end{bmatrix} = \begin{bmatrix} aa^d & 0 \\ 0 & ss^d \end{bmatrix} r$,

$$z_2 z_1^d = \begin{bmatrix} 0 & a^\pi b \\ s^\pi c & s^\pi d \end{bmatrix} \begin{bmatrix} aa^d & 0 \\ 0 & ss^d \end{bmatrix} r = \begin{bmatrix} 0 & a^\pi b \\ s^\pi c & s^\pi d \end{bmatrix} r,$$

$$ry = r \begin{bmatrix} aa^d & 0 \\ 0 & ss^d \end{bmatrix} y = r \begin{bmatrix} aa^d & 0 \\ 0 & ss^d \end{bmatrix} \begin{bmatrix} 0 & bs^\pi \\ ca^\pi & ds^\pi \end{bmatrix} = r \begin{bmatrix} 0 & bs^\pi \\ ca^\pi & ds^\pi \end{bmatrix}$$

yield (10.13). □

The condition of Theorem 10.6 are cumbersome and complicated, but the theorem itself have a number of useful consequences.

By Theorem 10.6, we obtain the following corollary which recovers [11, Theorem 2.5] for the Drazin inverse of complex matrices.

Corollary 10.5. *Let x be defined as in* (10.1)*, $a \in (p\mathcal{A}p)^\#$ and let $s \in ((1-p)\mathcal{A}(1-p))^\#$. If $ca^\pi = 0$ and $bs^\pi = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \begin{bmatrix} p - a^\pi bs^\# ca^\# & a^\pi bs^\# \\ s^\pi ca^\# & 1-p \end{bmatrix} \begin{bmatrix} a^\# + a^\# bs^\# ca^\# & -a^\# bs^\# \\ -s^\# ca^\# & s^\# \end{bmatrix}.$$

If we assume that the generalized Schur complement s is invertible in Theorem 10.6, then $s^\pi = 0$ and the next corollary which covers [34, Theorem 3.1] follows.

Corollary 10.6. *Let x be defined as in* (10.1), *and let $s \in ((1-p)\mathcal{A}(1-p))^{-1}$. If $ca^\pi b = 0$ and $aa^\pi b = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \left(\begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix} r_1 + 1 \right) r_1 \left(1 + \sum_{n=0}^{\infty} r_1^{n+1} \begin{bmatrix} 0 & 0 \\ ca^n a^\pi & 0 \end{bmatrix} \right),$$

where $r_1 = \begin{bmatrix} a^d + a^d bs^{-1} ca^d & -a^d bs^{-1} \\ -s^{-1} ca^d & s^{-1} \end{bmatrix}$.

In the following result we introduce the other expression for the generalized Drazin inverse of x which include an invertible element $w_0 = p + a^d bs^\pi ca^d$.

Theorem 10.7. *Let x be defined as in* (10.1). *If*

$$(10.14)\quad aa^\pi - a^\pi bs^d ca^\pi = 0, \quad s^\pi ca^\pi = 0, \quad ca^\pi b = 0, a^\pi bs^\pi = 0, \quad ss^\pi c = 0 = bss^\pi$$

and $w_0 = p + a^d bs^\pi ca^d$ is invertible, then $x \in \mathcal{A}^d$ and

$$(10.15)\qquad x^d = \left(\begin{bmatrix} 0 & a^\pi b \\ s^\pi c & s^\pi d \end{bmatrix} r + 1 \right) wrw \left(1 + r \begin{bmatrix} 0 & bs^\pi \\ ca^\pi & ds^\pi \end{bmatrix} \right),$$

where r is defined as in (10.6) *and* $w = \begin{bmatrix} w_0^{-1} & 0 \\ 0 & 1-p \end{bmatrix}$.

Proof. First, we observe that $u = \begin{bmatrix} p & a^d b \\ s^d c & (1-p) + s^d ca^d b \end{bmatrix}$ is invertible in $\mathcal{A}$ and its inverse is $u^{-1} = \begin{bmatrix} p + a^d bs^d c & -a^d b \\ -s^d c & 1-p \end{bmatrix}$.

Let us denote $X = uxu^{-1}$, so we have

$$\begin{aligned} X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = uxu^{-1} &= \begin{bmatrix} p & a^d b \\ s^d c & (1-p) + s^d ca^d b \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p + a^d bs^d c & -a^d b \\ -s^d c & (1-p) \end{bmatrix} \\ &= \begin{bmatrix} a - a^\pi bs^d c + a^d bs^\pi c & a^\pi b + a^d bs \\ s^\pi c + s^d c(a - a^\pi bs^d c + a^d bs^\pi c) & s + s^d c(a^\pi b + a^d bs) \end{bmatrix}. \end{aligned}$$

The first and the third conditions from (10.14) give us equations $caa^\pi = 0$ and $aa^\pi b = 0$. The second condition implies $s^\pi caa^d = s^\pi c$.

Applying these equations along with $a = aa^\pi + a^2 a^d$, we have

$$\begin{aligned} A &= a - a^\pi bs^d c + a^d bs^\pi c = w_0 a^2 a^d + aa^\pi - a^\pi bs^d c, \\ B &= a^\pi b + a^d bs, \\ C &= s^\pi c + s^d c(a - a^\pi bs^d c + a^d bs^\pi c) = s^\pi c + s^d cw_0 a^2 a^d, \\ D &= s + s^d c(a^\pi b + a^d bs) = s + s^d ca^d bs. \end{aligned}$$

From Lemma 10.3, we have $w_0 a^2 a^d \in (p\mathcal{A}p)^\#$, $(w_0 a^2 a^d)^\# = a^d w_0^{-1}$ and $(wa^2 a^d)^\pi = a^\pi$. Further,

$$(aa^\pi - a^\pi bs^d c)^2 = (aa^\pi - a^\pi bs^d ca^\pi)a - aa^\pi bs^d c + a^\pi bs^d ca^\pi bs^d c = 0$$

implies $(aa^\pi - a^\pi bs^d c) \in (p\mathcal{A}p)^{\text{nil}} \subseteq (p\mathcal{A}p)^{\text{qnil}}$ and it holds

$$w_0 a^2 a^d (aa^\pi - a^\pi bs^d c) = 0.$$

Applying Lemma 9.52), we conclude that $A \in (p\mathcal{A}p)^d$ and

$$\begin{aligned} A^d &= (w_0a^2a^d)^{\#} + (aa^\pi - a^\pi bs^dc)((w_0a^2a^d)^{\#})^2 \\ &= (p - a^\pi bs^dca^dw_0^{-1})a^dw_0^{-1} \end{aligned}$$

Since $w_0aa^d = aa^dw_0$ implies $(w_0a^2a^d)(a^dw_0^{-1}) = aa^d$ and it holds $a^dw_0^{-1}a^\pi = (w_0a^2a^d)^{\#}(w_0a^2a^d)^\pi = 0$, we have

$$\begin{aligned} A^\pi = p - AA^d &= p - (w_0a^2a^d + aa^\pi - a^\pi bs^dc)(p - a^\pi bs^dca^dw_0^{-1})a^dw_0^{-1} \\ &= p - w_0a^2a^da^dw_0^{-1} - aa^\pi a^dw_0^{-1} + a^\pi bs^dca^dw_0^{-1} \\ &\quad + w_0a^2a^da^\pi bs^dca^dw_0^{-1}a^dw_0^{-1} + aa^\pi bs^dca^dw_0^{-1}a^dw_0^{-1} \\ &\quad - a^\pi bs^dca^\pi bs^dca^dw_0^{-1}a^dw_0^{-1} \\ &= a^\pi + a^\pi bs^dca^dw_0^{-1}. \end{aligned}$$

Notice that $AA^\pi = 0$. Therefore, $A^d = A^{\#}$.

Now,

$$\begin{aligned} S = D - CA^{\#}B &= s + s^dca^dbs \\ &\quad - (s^\pi c + s^dcw_0a^2a^d)(p - a^\pi bs^dca^dw_0^{-1})a^dw_0^{-1}(a^\pi b + a^dbs) \\ &= s + s^dca^dbs - (s^\pi c + s^dcw_0a^2a^d)a^dw_0^{-1}a^dbs \\ &= s + s^dca^dbs - s^\pi ca^dw_0^{-1}a^dbs - s^dcw_0a^2a^da^dw_0^{-1}a^dbs \\ &= s - s^\pi ca^dw_0^{-1}a^dbs. \end{aligned}$$

Since

$$s \in ((1-p)\mathcal{A}(1-p))^d, \quad (s^\pi ca^dw_0^{-1}a^dbs)^2 = 0, \quad s(s^\pi ca^dw_0^{-1}a^dbs) = 0,$$

applying Lemma 9.52), we have that $S \in ((1-p)\mathcal{A}(1-p))^d$ and

$$S^d = s^d - s^\pi ca^dw_0^{-1}a^dbs^d.$$

Then,

$$S^\pi = (1-p) - SS^d = s^\pi + s^\pi ca^dw_0^{-1}a^dbss^d.$$

The following equations hold

$$CA^\pi = 0, \quad BS^\pi = 0, \quad AA^\pi = 0, \quad SS^\pi C = 0,$$

which implies that X satisfies the conditions (10.12) from Theorem 10.6. Using this Theorem, we conclude $X \in \mathcal{A}^d$ and

$$X^d = \left(\begin{bmatrix} 0 & A^\pi B \\ S^\pi C & S^\pi D \end{bmatrix} R + 1\right) R, \text{ where } R = \begin{bmatrix} A^{\#} + A^{\#}BS^dCA^{\#} & -A^{\#}BS^d \\ -S^dCA^{\#} & S^d \end{bmatrix}$$

Then, applying Lemma 10.4 on $x = u^{-1}Xu$ we have

$$x^d = u^{-1}X^du = u^{-1}\left(\begin{bmatrix} 0 & A^\pi B \\ S^\pi C & S^\pi D \end{bmatrix} R + 1\right) Ru.$$

Observe that

$$Ru = \begin{bmatrix} A^\# & 0 \\ 0 & S^d \end{bmatrix} \begin{bmatrix} p + BS^dCA^\# & -BS^d \\ -CA^\# & (1-p) \end{bmatrix} \begin{bmatrix} p & a^db \\ s^dc & (1-p) + s^dca^db \end{bmatrix}.$$

Since

$$\begin{aligned}
&\begin{bmatrix} p + BS^dCA^\# & -BS^d \\ -CA^\# & (1-p) \end{bmatrix} \begin{bmatrix} p & a^db \\ s^dc & (1-p) + s^dca^db \end{bmatrix} \\
&\quad = \begin{bmatrix} p + a^\pi b(s^d)^2caa^d + a^dbs^dcaa^d & -a^\pi bs^d - a^dbss^d \\ -s^\pi ca^dw_0^{-1} - s^dcaa^d & (1-p) \end{bmatrix} \\
&\quad\quad \times \begin{bmatrix} p & a^db \\ s^dc & (1-p) + s^dca^db \end{bmatrix} \\
&\quad = \begin{bmatrix} p - a^\pi b(s^d)^2ca^\pi - a^dbs^dca^\pi & a^dbs^\pi - a^\pi bs^d \\ -s^\pi ca^dw_0^{-1} + s^dca^\pi & (1-p) - s^\pi ca^dw_0^{-1}a^db \end{bmatrix},
\end{aligned}$$

we have

$$\begin{aligned}
Ru &= \begin{bmatrix} A^\# & 0 \\ 0 & S^d \end{bmatrix} \begin{bmatrix} p - a^\pi b(s^d)^2ca^\pi - a^dbs^dca^\pi & a^dbs^\pi - a^\pi bs^d \\ -s^\pi ca^dw_0^{-1} + s^dca^\pi & (1-p) - s^\pi ca^dw_0^{-1}a^db \end{bmatrix} \\
&= \begin{bmatrix} A^\# & 0 \\ 0 & S^d \end{bmatrix} \begin{bmatrix} p - a^dbs^dca^\pi & a^dbs^\pi \\ s^dca^\pi & (1-p) \end{bmatrix} \\
&\quad + \begin{bmatrix} A^\# & 0 \\ 0 & S^d \end{bmatrix} \begin{bmatrix} -a^\pi b(s^d)^2ca^\pi & -a^\pi bs^d \\ -s^\pi ca^dw_0^{-1} & -s^\pi ca^dw_0^{-1}a^db \end{bmatrix} \\
&= \begin{bmatrix} A^\# & 0 \\ 0 & S^d \end{bmatrix} \left(1 + \begin{bmatrix} -a^dbs^dca^\pi & a^dbs^\pi \\ s^dca^\pi & 0 \end{bmatrix}\right) \\
&\quad + \begin{bmatrix} (p - a^\pi bs^dca^dw_0^{-1})a^dw_0^{-1} & 0 \\ 0 & ((1-p) - s^\pi ca^dw_0^{-1}a^db)s^d \end{bmatrix} \\
&\quad\quad \times \begin{bmatrix} -a^\pi b(s^d)^2ca^\pi & -a^\pi bs^d \\ -s^\pi ca^dw_0^{-1} & -s^\pi ca^dw_0^{-1}a^db \end{bmatrix} \\
&= \begin{bmatrix} A^\# & 0 \\ 0 & S^d \end{bmatrix} \left(1 + \begin{bmatrix} a^d + a^dbs^dca^d & -a^dbs^d \\ -s^dca^d & s^d \end{bmatrix} \begin{bmatrix} 0 & bs^\pi \\ ca^\pi & ds^\pi \end{bmatrix}\right) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} A^\# & 0 \\ 0 & S^d \end{bmatrix} \left(1 + r \begin{bmatrix} 0 & bs^\pi \\ ca^\pi & ds^\pi \end{bmatrix}\right)
\end{aligned}$$

We can write

$$\begin{aligned}
\begin{bmatrix} A^\# & 0 \\ 0 & S^d \end{bmatrix} &= \begin{bmatrix} p - a^\pi bs^dca^dw_0^{-1} & 0 \\ 0 & (1-p) - s^\pi ca^dw_0^{-1}a^dbss^d \end{bmatrix} \\
&\quad \times \begin{bmatrix} a^d & 0 \\ 0 & s^d \end{bmatrix} \begin{bmatrix} w_0^{-1} & 0 \\ 0 & (1-p) \end{bmatrix} \\
&= \begin{bmatrix} p - a^\pi bs^dca^dw_0^{-1} & 0 \\ 0 & (1-p) - s^\pi ca^dw_0^{-1}a^dbss^d \end{bmatrix} \begin{bmatrix} a^d & 0 \\ 0 & s^d \end{bmatrix} w.
\end{aligned}$$

Therefore,

$$Ru = \begin{bmatrix} p - a^\pi bs^d ca^d w_0^{-1} & 0 \\ 0 & (1-p) - s^\pi ca^d w_0^{-1} a^d bss^d \end{bmatrix} \begin{bmatrix} a^d & 0 \\ 0 & s^d \end{bmatrix} w \times \left(1 + r \begin{bmatrix} 0 & bs^\pi \\ ca^\pi & ds^\pi \end{bmatrix}\right)$$

Denote $M = w\left(1 + r\left[\begin{smallmatrix} 0 & bs^\pi \\ ca^\pi & ds^\pi \end{smallmatrix}\right]\right)$. Notice $r\left[\begin{smallmatrix} a & 0 \\ 0 & s \end{smallmatrix}\right]\left[\begin{smallmatrix} a^d & 0 \\ 0 & s^d \end{smallmatrix}\right] = r$. Using the equation $a^d w_0^{-1}(p + a^d bs^d ca^d a) = a^d w_0^{-1}(a^d + a^d bs^d ca^d)a$, we have

$$\begin{aligned}
x^d &= u^{-1}\left(\begin{bmatrix} 0 & A^\pi B \\ S^\pi C & S^\pi D \end{bmatrix} R + 1\right) \\
&\quad \times \begin{bmatrix} p - a^\pi bs^d ca^d w_0^{-1} & 0 \\ 0 & (1-p) - s^\pi ca^d w_0^{-1} a^d bss^d \end{bmatrix} \begin{bmatrix} a^d & 0 \\ 0 & s^d \end{bmatrix} M \\
&= u^{-1} \begin{bmatrix} p - A^\pi BS^d CA^\# & A^\pi BS^d \\ S^\pi CA^\# & (1-p) \end{bmatrix} \\
&\quad \times \begin{bmatrix} p - a^\pi bs^d ca^d w_0^{-1} & 0 \\ 0 & (1-p) - s^\pi ca^d w_0^{-1} a^d bss^d \end{bmatrix} \begin{bmatrix} a^d & 0 \\ 0 & s^d \end{bmatrix} M \\
&= u^{-1} \times \begin{bmatrix} p - a^\pi b(s^d)^2 caa^d - a^\pi bs^d ca^d w_0^{-1} a^d bs^d caa^d & a^\pi bs^d + a^\pi bs^d ca^d w_0^{-1} a^d bss^d \\ s^\pi ca^d w_0^{-1}(a^d + a^d bs^d ca^d)a & (1-p) \end{bmatrix} \\
&\quad \times \begin{bmatrix} p - a^\pi bs^d ca^d w_0^{-1} & 0 \\ 0 & (1-p) - s^\pi ca^d w_0^{-1} a^d bss^d \end{bmatrix} \begin{bmatrix} a^d & 0 \\ 0 & s^d \end{bmatrix} M \\
&= u^{-1} \times \begin{bmatrix} p - a^\pi b(s^d)^2 caa^d - a^\pi bs^d ca^d w_0^{-1}(a^d + a^d bs^d ca^d)a & a^\pi bs^d + a^\pi bs^d ca^d w_0^{-1} a^d bss^d \\ s^\pi ca^d w_0^{-1}(a^d + a^d bs^d ca^d)a & (1-p) - s^\pi ca^d w_0^{-1} a^d bss^d \end{bmatrix} \\
&\quad \times \begin{bmatrix} a^d & 0 \\ 0 & s^d \end{bmatrix} M \\
&= u^{-1}\left(1 + \begin{bmatrix} -a^\pi bs^d ca^d w_0^{-1} & a^\pi bs^d \\ s^\pi ca^d w_0^{-1} & 0 \end{bmatrix} \begin{bmatrix} (a^d + a^d bs^d ca^d)a & -a^d bs^d s \\ -s^d ca^d a & s^d s \end{bmatrix}\right) \begin{bmatrix} a^d & 0 \\ 0 & s^d \end{bmatrix} M \\
&= u^{-1}\left(1 + \begin{bmatrix} -a^\pi bs^d ca^d & a^\pi bs^d \\ s^\pi ca^d & 0 \end{bmatrix} \begin{bmatrix} w_0^{-1} & 0 \\ 0 & (1-p) \end{bmatrix} r \begin{bmatrix} a & 0 \\ 0 & s \end{bmatrix}\right) \begin{bmatrix} a^d & 0 \\ 0 & s^d \end{bmatrix} M \\
&= \left(\begin{bmatrix} p + a^d bs^d c & -a^d b \\ -s^d c & (1-p) \end{bmatrix} \begin{bmatrix} a^d & 0 \\ 0 & s^d \end{bmatrix}\right. \\
&\quad \left. + u^{-1} \begin{bmatrix} 0 & a^\pi b \\ s^\pi c & s^\pi d \end{bmatrix} r \begin{bmatrix} w_0^{-1} & 0 \\ 0 & (1-p) \end{bmatrix} r \begin{bmatrix} a & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} a^d & 0 \\ 0 & s^d \end{bmatrix}\right) M \\
&= \left(r + \begin{bmatrix} p + a^d bs^d c & -a^d b \\ -s^d c & (1-p) \end{bmatrix} \begin{bmatrix} 0 & a^\pi b \\ s^\pi c & s^\pi d \end{bmatrix} rwr\right) M \\
&= \left(\begin{bmatrix} w_0 & 0 \\ 0 & (1-p) \end{bmatrix} + \left(\begin{bmatrix} a^d bs^d c & -a^d b \\ -s^d c & 0 \end{bmatrix} + 1\right) \begin{bmatrix} 0 & a^\pi b \\ s^\pi c & s^\pi d \end{bmatrix} r\right) wrM \\
&= \left(\begin{bmatrix} w_0 & 0 \\ 0 & (1-p) \end{bmatrix} + \begin{bmatrix} -a^d bs^\pi c & -a^d bs^\pi d \\ 0 & 0 \end{bmatrix} r + \begin{bmatrix} 0 & a^\pi b \\ s^\pi c & s^\pi d \end{bmatrix} r\right) wrM \\
&= \left(\begin{bmatrix} p + a^d bs^\pi ca^d & 0 \\ 0 & (1-p) \end{bmatrix} + \begin{bmatrix} -a^d bs^\pi ca^d & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a^\pi b \\ s^\pi c & s^\pi d \end{bmatrix} r\right) wrM \\
&= \left(1 + \begin{bmatrix} 0 & a^\pi b \\ s^\pi c & s^\pi d \end{bmatrix} r\right) wrM.
\end{aligned}$$

Replacing M, we get (10.15). □

If $s \in ((1-p)\mathcal{A}(1-p))^{\#}$ in Theorem 10.7, then $ss^\pi = 0$ and we recover as a special case [17, Theorem 9]. If $s \in ((1-p)\mathcal{A}(1-p))^{-1}$ in Theorem 10.7, we get the following consequence.

Corollary 10.7. *Let x be defined as in (10.1), and let $s \in ((1-p)\mathcal{A}(1-p))^{-1}$. If $aa^\pi - a^\pi bs^{-1}ca^\pi = 0$ and $ca^\pi b = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \left(\begin{bmatrix} 0 & a^\pi b \\ 0 & 0 \end{bmatrix} r_1 + 1\right) r_1 \left(1 + r_1 \begin{bmatrix} 0 & 0 \\ ca^\pi & 0 \end{bmatrix}\right),$$

where r_1 is defined as in Corollary 2.2.

We give a representation of x^d in the next theorem under conditions $a^\pi b = 0$ and $s^\pi caa^d = 0$.

Theorem 10.8. *Let x be defined as in (10.1). If $a^\pi b = 0$ and $s^\pi caa^d = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \sum_{n=0}^{\infty} r^{n+1} \left(1 + \begin{bmatrix} -a^d bs^d ca^\pi & a^d bs^\pi \\ s^d ca^\pi & 0 \end{bmatrix}\right) \begin{bmatrix} aa^\pi & 0 \\ s^\pi c & s^\pi s \end{bmatrix}^n, \tag{10.16}$$

where r is defined as in (10.6).

Proof. By the assumption $a^\pi b = 0$ and $s^\pi caa^d = 0$, $s^\pi ca^d = 0$ and we can write

$$x = \begin{bmatrix} aa^\pi & a^\pi b \\ s^\pi c & s^\pi d \end{bmatrix} + \begin{bmatrix} a^2 a^d & aa^d b \\ ss^d c & ss^d d \end{bmatrix} = \begin{bmatrix} aa^\pi & 0 \\ s^\pi c & s^\pi s \end{bmatrix} + \begin{bmatrix} a^2 a^d & aa^d b \\ ss^d c & ss^d d \end{bmatrix} := y + z.$$

Now, we obtain that $yz = 0$ and $y \in \mathcal{A}^{\text{qnil}}$, because $aa^\pi \in (p\mathcal{A}p)^{\text{qnil}}$ and $ss^\pi \in ((1-p)\mathcal{A}(1-p))^{\text{qnil}}$.

To prove that $z \in \mathcal{A}^d$, we observe that

$$z = \begin{bmatrix} a^2 a^d & aa^d bss^d \\ ss^d caa^d & ss^d dss^d \end{bmatrix} + \begin{bmatrix} 0 & aa^d bs^\pi \\ ss^d ca^\pi & ss^d ds^\pi \end{bmatrix} := z_1 + z_2.$$

From Lemma 10.5, we have $z_1 \in \mathcal{A}^d$ and $z_1^d = r$. Since $z_2 z_1 = 0$ and $z_2^2 = 0$, by Lemma 9.51), $z \in \mathcal{A}^d$ and $z^d = z_1^d + (z_1^d)^2 z_2 = r + r^2 z_2$.

Therefore, using Lemma 9.51), $x \in \mathcal{A}^d$ and $x^d = \sum_{n=0}^{\infty} r^{n+1}(1 + rz_2)y^n$ which gives (10.16). □

Also we can obtain the following expression for the generalized Drazin inverse of block matrix x.

Theorem 10.9. *Let x be defined as in (10.1). If $a^\pi b = 0 = bs^\pi$ and $s^\pi scaa^d = 0$, then $x \in \mathcal{A}^d$ and*

$$x^d = \left(\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r + 1\right) r \left(1 + \sum_{n=0}^{\infty} r^{n+1} \begin{bmatrix} 0 & 0 \\ ca^n a^\pi & 0 \end{bmatrix}\right), \tag{10.17}$$

where r is defined as in (10.6).

Proof. In the similar way as in the proof of Theorem 10.6, using the following decomposition

$$x = \begin{bmatrix} aa^\pi & 0 \\ ca^\pi & ss^\pi \end{bmatrix} + \begin{bmatrix} a^2a^d & bss^d \\ caa^d & dss^d \end{bmatrix} := y + z,$$

we verify this result. □

Using Theorem 10.8, we get necessary and sufficient conditions for the existence and the expression of the group inverse of x. The following result recovers [17, Theorem 12] and [11, Theorem 2.2].

Theorem 10.10. *Let x be defined as in* (10.1). *Suppose that $a^\pi b = 0$ and $s^\pi caa^d = 0$. Then*

$$x \in \mathcal{A}^\# \text{ if and only if } a \in (p\mathcal{A}p)^\#, \ s \in ((1-p)\mathcal{A}(1-p))^\# \text{ and } s^\pi ca^\pi = 0.$$

Furthermore, if $a \in (p\mathcal{A}p)^\#$, $s \in ((1-p)\mathcal{A}(1-p))^\#$, $a^\pi b = 0$ and $s^\pi c = 0$, then

$$x^\# = \begin{bmatrix} a^\# + a^\# bs^\# ca^\# & -a^\# bs^\# \\ -s^\# ca^\# & s^\# \end{bmatrix} \begin{bmatrix} p - a^\# bs^\# ca^\pi & a^\# bs^\pi \\ s^\# ca^\pi & 1-p \end{bmatrix}. \tag{10.18}$$

Proof. If $x \in \mathcal{A}^\#$, by Theorem 10.8, $x^\#$ is equal to the right hand side of (10.16). Since $xr^2 = (xr)r = \left[\begin{smallmatrix} aa^d & 0 \\ 0 & ss^d \end{smallmatrix}\right] r = r$, then

$$\begin{aligned} x^2x^\# &= x^2 r \left(1 + \begin{bmatrix} -a^d bs^d ca^\pi & a^d bs^\pi \\ s^d ca^\pi & 0 \end{bmatrix}\right) \\ &\quad + \begin{bmatrix} aa^d & 0 \\ 0 & ss^d \end{bmatrix} \left(1 + \begin{bmatrix} -a^d bs^d ca^\pi & a^d bs^\pi \\ s^d ca^\pi & 0 \end{bmatrix}\right) \begin{bmatrix} aa^\pi & 0 \\ s^\pi c & s^\pi s \end{bmatrix} \\ &\quad + \sum_{n=2}^{\infty} r^{n-1} \left(1 + \begin{bmatrix} -a^d bs^d ca^\pi & a^d bs^\pi \\ s^d ca^\pi & 0 \end{bmatrix}\right) \begin{bmatrix} aa^\pi & 0 \\ s^\pi c & s^\pi s \end{bmatrix}^n := I_1 + I_2 + I_3. \end{aligned}$$

By the equality $x - x^2x^\# = 0$, we obtain $I_3 = x - I_1 - I_2$. Now, notice that

$$\begin{aligned} x^\# &= r \left(1 + \begin{bmatrix} -a^d bs^d ca^\pi & a^d bs^\pi \\ s^d ca^\pi & 0 \end{bmatrix}\right) \\ &\quad + r^2 \left(1 + \begin{bmatrix} -a^d bs^d ca^\pi & a^d bs^\pi \\ s^d ca^\pi & 0 \end{bmatrix}\right) \begin{bmatrix} aa^\pi & 0 \\ s^\pi c & s^\pi s \end{bmatrix} + r^2 I_3 \\ &= r \left(1 + \begin{bmatrix} -a^d bs^d ca^\pi & a^d bs^\pi \\ s^d ca^\pi & 0 \end{bmatrix}\right) + r^2(x - I_1) \\ &= r \left(1 + \begin{bmatrix} -a^d bs^d ca^\pi & a^d bs^\pi \\ s^d ca^\pi & 0 \end{bmatrix}\right). \end{aligned}$$

Hence,

$$\begin{aligned} x^2x^\# &= \begin{bmatrix} a^2a^d + a^\pi bs^d ca^\pi & aa^d bs^\pi + bss^d \\ caa^d + ss^d ca^\pi & ca^d b + s^2 s^d \end{bmatrix} \\ &= \begin{bmatrix} a^2a^d & bs^\pi + bss^d \\ ss^d caa^d + ss^d ca^\pi & d - s + s^2 s^d \end{bmatrix} = \begin{bmatrix} a^2a^d & b \\ ss^d c & d - s + s^2 s^d \end{bmatrix} \end{aligned}$$

and $x^2x^\# = x$ imply $a^2a^d = a$, $s^2s^d = s$ and $ss^dc = c$. So, $a \in (p\mathcal{A}p)^\#$, $s \in ((1-p)\mathcal{A}(1-p))^\#$ and $s^\pi ca^\pi = ca^\pi - ca^\pi = 0$.

Assume that $a \in (p\mathcal{A}p)^\#$, $s \in ((1-p)\mathcal{A}(1-p))^\#$ and $s^\pi ca^\pi = 0$. Then $s^\pi c = s^\pi ca^\pi + s^\pi caa^\# = 0$. Denote by u the right hand side of (10.18). Using Theorem 10.8, we get that $x \in \mathcal{A}^d$ and $x^d = u$. We can show that $xx^dx = xux = x$ which implies that $x \in \mathcal{A}^\#$ and $x^\# = u$. $\square$

Applying Theorem 10.9, we prove the next result related to the group inverse $x^\#$ which is an extension of [17, Theorem 13].

Theorem 10.11. *Let x be defined as in* (10.1)*. If $a^\pi b = 0 = bs^\pi$ and $s^\pi scaa^d = 0$. Then*

$$x \in \mathcal{A}^\# \text{ if and only if } a \in (p\mathcal{A}p)^\#,\ s \in ((1-p)\mathcal{A}(1-p))^\# \text{ and } s^\pi ca^\pi = 0.$$

Furthermore, if $a \in (p\mathcal{A}p)^\#$, $s \in ((1-p)\mathcal{A}(1-p))^\#$, $a^\pi b = 0 = bs^\pi$ and $s^\pi ca^\pi = 0$, then

$$x^\# = \left(\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r + 1\right) r \left(1 + r \begin{bmatrix} 0 & 0 \\ ca^\pi & 0 \end{bmatrix}\right), \tag{10.19}$$

where r is defined as in (10.6)*.*

Proof. Let $x \in \mathcal{A}^\#$. Using Theorem 10.9, $x^\#$ is equal to the right-hand side of (10.17). From $\left[\begin{smallmatrix} 0 & 0 \\ ca^na^\pi & 0 \end{smallmatrix}\right] x = \left[\begin{smallmatrix} 0 & 0 \\ ca^{n+1}a^\pi & 0 \end{smallmatrix}\right]$, we get

$$\begin{aligned} x^\# x &= \left(\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r + 1\right) r \left(x + \sum_{n=0}^{\infty} r^{n+1} \begin{bmatrix} 0 & 0 \\ ca^{n+1}a^\pi & 0 \end{bmatrix}\right) \\ &= \left(\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r + 1\right) \left(rx + \sum_{n=1}^{\infty} r^{n+1} \begin{bmatrix} 0 & 0 \\ ca^na^\pi & 0 \end{bmatrix}\right) \\ &= \left(\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r + 1\right) \left(rx - r\begin{bmatrix} 0 & 0 \\ ca^\pi & 0 \end{bmatrix} + \sum_{n=0}^{\infty} r^{n+1} \begin{bmatrix} 0 & 0 \\ ca^na^\pi & 0 \end{bmatrix}\right) \\ &= \left(\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r + 1\right) \left(r\begin{bmatrix} a & b \\ caa^d & d \end{bmatrix} + \sum_{n=0}^{\infty} r^{n+1} \begin{bmatrix} 0 & 0 \\ ca^na^\pi & 0 \end{bmatrix}\right) \\ &= \left(\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r + 1\right) \left(\begin{bmatrix} aa^d & 0 \\ 0 & ss^d \end{bmatrix} + \sum_{n=0}^{\infty} r^{n+1} \begin{bmatrix} 0 & 0 \\ ca^na^\pi & 0 \end{bmatrix}\right). \end{aligned}$$

Observe that $x\left[\begin{smallmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{smallmatrix}\right] r = x\left[\begin{smallmatrix} 0 & 0 \\ s^\pi ca^d & 0 \end{smallmatrix}\right] = 0$ gives

$$x = x^2x^\# = xx^\# x = x\begin{bmatrix} aa^d & 0 \\ 0 & ss^d \end{bmatrix} + x\sum_{n=0}^{\infty} r^{n+1} \begin{bmatrix} 0 & 0 \\ ca^na^\pi & 0 \end{bmatrix}.$$

So $x\sum_{n=0}^{\infty} r^{n+1}\left[\begin{smallmatrix} 0 & 0 \\ ca^na^\pi & 0 \end{smallmatrix}\right] = x - x\left[\begin{smallmatrix} aa^d & 0 \\ 0 & ss^d \end{smallmatrix}\right] = x\left[\begin{smallmatrix} a^\pi & 0 \\ 0 & s^\pi \end{smallmatrix}\right]$. By this equality and the equation $rxr = r$, we obtain

$$x^\# = \left(\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r + 1\right) r \left(1 + \sum_{n=0}^{\infty} r^{n+1} \begin{bmatrix} 0 & 0 \\ ca^na^\pi & 0 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r + 1\right) r \left(1 + rx \sum_{n=0}^{\infty} r^{n+1} \begin{bmatrix} 0 & 0 \\ ca^n a^\pi & 0 \end{bmatrix}\right)$$
$$= \left(\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r + 1\right) r \left(1 + rx \begin{bmatrix} a^\pi & 0 \\ 0 & s^\pi \end{bmatrix}\right)$$
$$= \left(\begin{bmatrix} 0 & 0 \\ s^\pi c & s^\pi d \end{bmatrix} r + 1\right) r \left(1 + r \begin{bmatrix} aa^\pi & 0 \\ ca^\pi & ss^\pi \end{bmatrix}\right)$$

implying

$$\begin{aligned} x^2 x^\# = x^2 r \left(1 + r \begin{bmatrix} aa^\pi & 0 \\ ca^\pi & ss^\pi \end{bmatrix}\right) &= x \begin{bmatrix} aa^d & 0 \\ s^\pi ca^d & ss^d \end{bmatrix} \left(1 + \begin{bmatrix} -a^d bs^d ca^\pi & 0 \\ s^d ca^\pi & 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} a^2 a^d & bss^d \\ caa^d & dss^d \end{bmatrix} \left(1 + \begin{bmatrix} -a^d bs^d ca^\pi & 0 \\ s^d ca^\pi & 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} a^2 a^d & bss^d \\ caa^d & dss^d \end{bmatrix} + \begin{bmatrix} -aa^d bs^d ca^\pi + bs^d ca^\pi & 0 \\ -ca^d bs^d ca^\pi + ds^d ca^\pi & 0 \end{bmatrix} \\ &= \begin{bmatrix} a^2 a^d & bss^d \\ caa^d & dss^d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ ss^d ca^\pi & 0 \end{bmatrix} = \begin{bmatrix} a^2 a^d & b \\ c - s^\pi ca^\pi & d - ss^\pi \end{bmatrix}. \end{aligned}$$

Because $x^2 x^\# = x$, we deduce that $a^2 a^d = a$, $ss^\pi = 0$ and $s^\pi ca^\pi = 0$ which yield $a \in (p\mathcal{A}p)^\#$, $s \in ((1-p)\mathcal{A}(1-p))^\#$ and $s^\pi ca^\pi = 0$.

Suppose that $a \in (p\mathcal{A}p)^\#$, $s \in ((1-p)\mathcal{A}(1-p))^\#$ and $s^\pi ca^\pi = 0$. Thus $a^n a^\pi = 0$ for all $n \geqslant 1$. If we denote by v the right-hand side of (10.19), by Theorem 10.9, $x \in \mathcal{A}^d$ and $x^d = v$. Since $xx^d x = xrx = x$, then $x \in \mathcal{A}^\#$ and $x^\# = v$. □

11. Right and left Fredholm operator $M_{(T,S)}$

In this part of the paper, we are interested in the properties of the right and left Fredholm operator of type $M_{(T,S)}$. For given A and C, we are interested to find T and S, such that $M_{(T,S)}$ is right or left Fredholm operator.

For this purpose we need to review some properties of right and left Fredholm operators. An operator $A \in \mathcal{L}(X,Y)$ is right Fredholm, if $\operatorname{def}(A) = \dim Y/\mathcal{R}(A) < \infty$, and $\mathcal{N}(A)$ is complemented in X. Notice that if A is right Fredholm, then it follows that $\mathcal{R}(A)$ has to be a closed and complemented subspace of Y. The set of all right Fredholm operators from X to Y is denoted by $\Phi_r(X,Y)$. It is well-known that $A \in \Phi_r(X,Y)$ if and only if there exist $B \in \mathcal{L}(Y,X)$ and $F \in \mathcal{F}(Y)$ such that $AB = I_Y + F$ holds.

An operator $A \in \mathcal{L}(X,Y)$ is left Fredholm, if $\operatorname{nul}(A) = \dim \mathcal{N}(A) < \infty$, and $\mathcal{R}(A)$ is closed and complemented in Y. The set of all left Fredholm operators from X to Y is denoted by $\Phi_l(X,Y)$. It is well-known that $A \in \Phi_l(X,Y)$ if and only if there exist $B \in \mathcal{L}(Y,X)$ and $F \in \mathcal{F}(X)$ such that $BA = I_X + F$ holds.

If $A \in \Phi_r(X,Y)$ and $B \in \Phi_r(Y,Z)$, then $BA \in \Phi_r(X,Z)$. The similar result holds for the class Φ_l. The set of Fredholm operators is defined as $\Phi(X,Y) = \Phi_r(X,Y) \cap \Phi_l(X,Y)$.

We formulate the following well-known results.

Lemma 11.1. *Let X,Y,Z be Banach spaces and let $A \in \mathcal{L}(X,Y)$, $B \in \mathcal{L}(Y,Z)$. If $BA \in \Phi(X,Z)$, then the following holds: $A \in \Phi(X,Y)$ if and only if $B \in \Phi(Y,Z)$.*

Lemma 11.2. *Let X,Y be Banach spaces, and let $A \in \Phi_r(X,Y)$, $P \in \mathcal{F}(X,Y)$. Then $A+P \in \Phi_r(X,Y)$. The analogous result holds for classes Φ_l and Φ.*

Lemma 11.3. *Let M_1, M_2 and N be the vector subspaces of the vector space X. If $M_1 \subseteq M_2$, then $\dim M_1/(M_1 \cap N) \leqslant \dim M_2/(M_2 \cap N)$.*

Properties of right (left) Fredholm and related operators can be found in [30] and [49]. For the importance and applications of operator matrices we refer to [12,19,25,26,29,38,45,56]. Particularly, this paper is related to the research in [12] and [38], where the left and right invertibility of $M_{(T,S)}$ is considered.

11.1. Right Fredholm operator.

Now, we consider right Fredholm properties of $M_{(T,S)}$.

Theorem 11.1. *Let $A \in \mathcal{L}(X)$ and $C \in \mathcal{L}(Y,X)$ be given. The following statements are equivalent*

1) $[A \quad C] \in \Phi_r(X \oplus Y, X) \smallsetminus \Phi(X \oplus Y, X)$, *and there exists an operator* $J \in \Phi_l(Y, \mathcal{N}([A \quad C]) \smallsetminus \Phi(Y, \mathcal{N}([A \quad C]))$.
2) $M_{(T,S)} \in \Phi_r(X \oplus Y) \smallsetminus \Phi(X \oplus Y)$ *for some* $T \in \mathcal{L}(X,Y)$ *and* $S \in \mathcal{L}(Y)$.

Proof. 1) $\Longrightarrow$ 2): Suppose that $[A \quad C] \in \Phi_r(X \oplus Y, X) \smallsetminus \Phi(X \oplus Y, X)$. It follows that $\mathcal{N}([A \quad C])$ is infinite dimensional. By the assumption, there exists an operator $J \in \Phi_l(Y, \mathcal{N}([A \quad C]) \smallsetminus \Phi(Y, \mathcal{N}([A \quad C]))$, so $\mathcal{N}(J)$ is finite dimensional and $\mathcal{N}([A \quad C])/R(J)$ is infinite dimensional. The operator J has the form

$$J = \begin{bmatrix} E \\ G \end{bmatrix} : Y \to \begin{bmatrix} X \\ Y \end{bmatrix}.$$

Since $\mathcal{R}(J)$ is closed and complemented in $\mathcal{N}([A \quad C])$, and $\mathcal{N}([A \quad C])$ is closed and complemented in $X \oplus Y$, we obtain that there exist closed subspaces V and W such that $\mathcal{N}[A \quad C]) = R(J) \oplus V$ and $X \oplus Y = \mathcal{N}([A \quad C]) \oplus W = R(J) \oplus V \oplus W$. Notice that V is infinite dimensional.

There exists a closed subspace Y_1 such that $Y = \mathcal{N}(J) \oplus Y_1$. Now, the reduction operator $J \colon Y_1 \to \mathcal{R}(J)$ is invertible, so let $K_1 \colon \mathcal{R}(J) \to Y_1$ denote its inverse. Define the operator $K \in \mathcal{L}(X \oplus Y, Y)$ in the following way

$$Kx = \begin{cases} K_1 x, & x \in \mathcal{R}(J), \\ 0, & x \in V \oplus W. \end{cases}$$

Then $K \in \mathcal{L}(X \oplus Y, Y)$ is a right Fredholm operator, such that $\mathcal{N}(K) = V \oplus W$. The operator K has the matrix form

$$K = [T \quad S] \colon \begin{bmatrix} X \\ Y \end{bmatrix} \to Y.$$

We also have

$$KJ = [T \quad S]\begin{bmatrix} E \\ G \end{bmatrix} = I_Y - P_1, \tag{11.1}$$

where P_1 is the projection from Y onto the finite dimensional subspace $\mathcal{N}(J)$, parallel to Y_1.

From $\mathcal{R}(J) \subset \mathcal{N}([A \quad C])$ we get that

$$[A \quad C]\begin{bmatrix} E \\ G \end{bmatrix} = 0. \tag{11.2}$$

Since $[A \quad C] \in \Phi_r(X \oplus Y, X)$, we have the following decompositions of spaces: $X \oplus Y = \mathcal{N}([A \quad C]) \oplus W$ and $X = \mathcal{R}([A \quad C]) \oplus U$, where U is finite dimensional. Since the reduction $[A \quad C]\colon W \to \mathcal{R}([A \quad C])$ is invertible, define $L_1\colon \mathcal{R}([A \quad C]) \to W$ to be its inverse. Then consider the operator $L \in \mathcal{L}(X, X \oplus Y)$, which is defined as follows

$$Lx = \begin{cases} L_1 x, & x \in \mathcal{R}([A \quad C]) \\ 0, & x \in U. \end{cases}$$

The operator L has the matrix form

$$L = \begin{bmatrix} D \\ F \end{bmatrix} : X \to \begin{bmatrix} X \\ Y \end{bmatrix}.$$

Then $L \in \Phi_l(X, X \oplus Y)$, $\mathcal{R}(L) = W$, and

$$[A \quad C]L = [A \quad C]\begin{bmatrix} D \\ F \end{bmatrix} = I_X - P_2, \tag{11.3}$$

where P_2 is the projection from X onto the finite dimensional subspace U, parallel to $\mathcal{R}([A \quad C])$. Since $\mathcal{N}([T \quad S]) = V \oplus W$, we conclude that

$$[T \quad S]\begin{bmatrix} D \\ F \end{bmatrix} = 0. \tag{11.4}$$

Finally, from (11.1), (11.2), (11.3) and (11.4), we get that for $M = \left[\begin{smallmatrix} A & C \\ T & S \end{smallmatrix}\right]$ i $N = \left[\begin{smallmatrix} D & E \\ F & G \end{smallmatrix}\right]$ the following holds

$$MN = \begin{bmatrix} A & C \\ T & S \end{bmatrix}\begin{bmatrix} D & E \\ F & G \end{bmatrix} = \begin{bmatrix} I_X & 0 \\ 0 & I_Y \end{bmatrix} + \begin{bmatrix} -P_2 & 0 \\ 0 & -P_1 \end{bmatrix}.$$

Since $\left[\begin{smallmatrix} -P_2 & 0 \\ 0 & -P_1 \end{smallmatrix}\right]$ is finite rank, we conclude that M is right Fredholm. Moreover, we notice that

$$\mathcal{N}(M) = \mathcal{N}([A \quad C]) \cap \mathcal{N}([T \quad S]) = V,$$
$$\mathcal{R}(N) = \mathcal{R}\left(\begin{bmatrix} D \\ F \end{bmatrix}\right) + \mathcal{R}\left(\begin{bmatrix} E \\ G \end{bmatrix}\right) = W \oplus \mathcal{R}(J),$$
$$X \oplus Y = \mathcal{R}(J) \oplus V \oplus W.$$

Since V is infinite dimensional, we obtain that both M and N are not Fredholm operators.

2) $\Longrightarrow$ 1): Suppose that there exist some $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y)$ such that $M_{(T,S)} \in \Phi_r(X \oplus Y) \setminus \Phi(X, Y)$. Then there exist operators $N \in \mathcal{L}(X \oplus Y)$ and

$P \in \mathcal{F}(X \oplus Y)$ such that $MN = I + P$. The last equality holds in the matrix form as follows

$$\begin{bmatrix} A & C \\ T & S \end{bmatrix} \begin{bmatrix} D & E \\ F & G \end{bmatrix} = \begin{bmatrix} I_X & 0 \\ 0 & I_Y \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},$$

where all P_{ij} are finite rank operators. It also follows that $N = \left[\begin{smallmatrix} D & E \\ F & G \end{smallmatrix}\right] \in \Phi_l(X \oplus Y)$.

In particular, we obtain

$$[A \quad C] \begin{bmatrix} D \\ F \end{bmatrix} = I_X + P_{11},$$

so $[A \quad C]$ is right Fredholm. The operator $I_X + P_{11}$ is Fredholm. If we suppose that $[A \quad C]$ is Fredholm, by Lemma 11.1 it follows that $\left[\begin{smallmatrix} D \\ F \end{smallmatrix}\right]$ is also Fredholm. Since

$$\mathcal{R}\left(\begin{bmatrix} D & E \\ F & G \end{bmatrix}\right) = \mathcal{R}\left(\begin{bmatrix} D \\ F \end{bmatrix}\right) + \mathcal{R}\left(\begin{bmatrix} E \\ G \end{bmatrix}\right) \supset \mathcal{R}\left(\begin{bmatrix} D \\ F \end{bmatrix}\right),$$

it follows that $\left[\begin{smallmatrix} D & E \\ F & G \end{smallmatrix}\right]$ belongs to $\Phi_r(X \oplus Y)$, so $\left[\begin{smallmatrix} D & E \\ F & G \end{smallmatrix}\right]$ is Fredholm. By Lemma 11.1 again, we obtain that $\left[\begin{smallmatrix} A & C \\ T & S \end{smallmatrix}\right]$ is Fredholm (since $I + P$ is Fredholm from Lemma 11.2). The last statement is not possible, so we obtain that $[A \quad C] \in \Phi_r(X \oplus Y, X) \setminus \Phi(X \oplus Y, X)$.

Denote with $L = \left[\begin{smallmatrix} E \\ G \end{smallmatrix}\right] \in \mathcal{L}(Y, X \oplus Y)$. We have $[T \quad S]L = I_Y + P_{22}$, so $L \in \Phi_l(Y, X \oplus Y) \setminus \Phi(Y, X \oplus Y)$. Otherwise, if L is Fredholm, then also $\left[\begin{smallmatrix} D & E \\ F & G \end{smallmatrix}\right]$ is Fredholm, so $\left[\begin{smallmatrix} A & C \\ T & S \end{smallmatrix}\right]$ is Fredholm.

Since we have the following decomposition of space $X \oplus Y = \mathcal{N}([A \quad C]) \oplus W$, the operator L has the matrix form

$$L = \begin{bmatrix} J \\ K \end{bmatrix} : Y \to \begin{bmatrix} \mathcal{N}([A \quad C]) \\ W \end{bmatrix}.$$

From the fact that $\mathcal{R}(P_{12}) = \mathcal{R}([A \quad C]L) = \mathcal{R}\big([A \quad C]\left[\begin{smallmatrix} J \\ K \end{smallmatrix}\right]\big) = [A \quad C](\mathcal{R}(K))$ is a finite dimensional space and the reduction $[A \quad C] \colon W \to \mathcal{R}([A \quad C])$ is a bijection, we obtain that $\mathcal{R}(K)$ is a finite dimensional subspace of W.

Since $L \in \Phi_l(Y, X \oplus Y) \setminus \Phi(Y, X \oplus Y)$, we have the following decompositions of spaces $Y = \mathcal{N}(L) \oplus U$ and $X \oplus Y = \mathcal{R}(L) \oplus U_1$, where $\dim \mathcal{N}(L) < \infty$ and $\dim U_1 = \infty$. The reduction operator $L \colon U \to \mathcal{R}(L)$ is invertible, so let $L_1 \colon \mathcal{R}(L) \to U$ be its inverse.

As it was shown, $\mathcal{R}(K)$ is a finite dimensional subspace, so $Y_1 = L_1(\mathcal{R}(K))$ have to be finite dimensional subspace of U and there exists a closed subspace Y_2 such that $U = Y_1 \oplus Y_2$.

Now, the operator L has the following matrix form

$$L = \begin{bmatrix} J & 0 & 0 \\ 0 & K & 0 \end{bmatrix} : \begin{bmatrix} Y_2 \\ Y_1 \\ \mathcal{N}(L) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}([A \quad C]) \\ W \end{bmatrix},$$

where Y_1 is finite dimensional. We obtain that $\mathcal{N}(J) = Y_1 \oplus \mathcal{N}(L)$, so $\dim \mathcal{N}(J) < \infty$.

From the fact that $[T \quad S]L = I_Y + P_{22}$ follows that

$$L_1(\mathcal{N}([T \quad S]) \cap \mathcal{R}(L)) \subseteq \mathcal{N}(I_Y + P_{22}).$$

Since $I_Y + P_{22}$ is Fredholm operator, we have that $L_1(\mathcal{N}([T \quad S]) \cap \mathcal{R}(L))$ is finite dimensional, so $\mathcal{N}([T \quad S]) \cap \mathcal{R}(L)$ is also finite dimensional subspace.

Denote with $V = \mathcal{N}([A \quad C]) \cap \mathcal{N}([T \quad S]) \cap \mathcal{R}(J)$. Further,

$$V \subseteq \mathcal{N}([T \quad S]) \cap \mathcal{R}(J) \subseteq \mathcal{N}([T \quad S]) \cap \mathcal{R}(L),$$

so it follows that $\dim V < \infty$. Then, there exists a closed subspace V_1 such that $\mathcal{N}(M_{(T,S)}) = \mathcal{N}([A \quad C]) \cap \mathcal{N}([T \quad S]) = V \oplus V_1$. Since $\mathcal{N}(M_{(T,S)})$ is infinite dimensional, then V_1 is also infinite dimensional subspace.

Now, applying the Lemma 11.3 on the spaces $\mathcal{N}([A \ \ C]) \cap \mathcal{N}([T \ \ S])$, $\mathcal{N}([A \ \ C])$ and $\mathcal{R}(J)$, we obtain

$$\dim V_1 = \dim(\mathcal{N}([A \quad C]) \cap \mathcal{N}([T \quad S]))/V \leqslant \dim \mathcal{N}([A \quad C])/\mathcal{R}(J).$$

We conclude that $\dim \mathcal{N}([A \quad C])/\mathcal{R}(J) = \infty$.

Lastly, we proved for the operator $J \colon Y \to \mathcal{N}([A \quad C])$ that $\dim \mathcal{N}(J) < \infty$ and $\dim \mathcal{N}([A \quad C])/\mathcal{R}(J) = \infty$.

So, there exists the operator $J \in \Phi_l(Y, \mathcal{N}([A \quad C]) \setminus \Phi(Y, \mathcal{N}([A \quad C]))$. □

11.2. Left Fredholm operators.

Now, we investigate the left Fredholm properties of $M_{(T,S)}$. We consider two separate cases according to the dimension of Y.

Theorem 11.2. *Let X be infinite dimensional, and let Y be finite dimensional. For given $A \in \mathcal{L}(X)$ and $C \in \mathcal{L}(Y, X)$, the following statements are equivalent*

1) $M_{(T,S)} \in \Phi_l(X \oplus Y) \setminus \Phi(X \oplus Y)$ *for every $T \in \mathcal{L}(X, Y)$ and every operator $S \in \mathcal{L}(Y)$;*
2) $A \in \Phi_l(X) \setminus \Phi(X)$.

Proof. Before the proof of the equivalence, note that

$$\mathcal{N}\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = \mathcal{N}(A) \oplus Y, \quad \mathcal{R}\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = \mathcal{R}(A) \oplus \{0\}.$$

Since Y is finite dimensional, we have that $A \in \Phi_l(X) \setminus \Phi(X)$ if and only if $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \Phi_l(X \oplus Y) \setminus \Phi(X \oplus Y)$.

1) $\Longrightarrow$ 2): Suppose that $M_{(T,S)}$ is left Fredholm but not Fredholm, for every $T \in \mathcal{L}(X, Y)$ and every $S \in \mathcal{L}(Y)$. We have that $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} A & C \\ T & S \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & -C \\ -T & -S \end{smallmatrix}\right]$ where $\left[\begin{smallmatrix} 0 & -C \\ -T & -S \end{smallmatrix}\right]$ is finite rank operator. Applying Lemma 11.2, we obtain that $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is left Fredholm operator.

Suppose that $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is Fredholm. Applying Lemma 11.2 to $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]$ we conclude that $M_{(T,S)}$ has to be Fredholm, which does not hold. Hence, $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is left Fredholm but not Fredholm operator, so we have that $A \in \Phi_l(X) \setminus \Phi(X)$.

2) $\Longrightarrow$ 1): Suppose that A is left Fredholm but not Fredholm, so the operator $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is also left Fredholm but not Fredholm.

Let $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y)$ be arbitrary operators. Then the operator $M_{(T,S)}$ is a finite-rank perturbation of $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Indeed, $\left[\begin{smallmatrix} A & C \\ T & S \end{smallmatrix}\right] = \left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & C \\ T & S \end{smallmatrix}\right]$, where $\left[\begin{smallmatrix} 0 & C \\ T & S \end{smallmatrix}\right]$ is a finite rank operator because Y is finite dimensional space. Applying Lemma 11.2 to $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]$ we get that $M_{(T,S)}$ is left Fredholm operator. If we suppose that $M_{(T,S)}$ is Fredholm, from Lemma 11.2, we conclude that $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]$ have to be

Fredholm, which does not hold. We obtain that $M_{(T,S)}$ is left Fredholm but not Fredholm operator. $\square$

Theorem 11.3. *Let X and Y be infinite dimensional, such that Y is isomorphic to $Z = X \oplus Y$. Let $A \in \mathcal{L}(X)$ and $C \in \mathcal{L}(Y, X)$ be arbitrary. Then $M_{(T,S)} \in \Phi_l(X \oplus Y) \setminus \Phi(X \oplus Y)$ for some $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y)$.*

Proof. Since Y is isomorphic with Z, then $Y = Y_1 \oplus Y_2$, where X is isomorphic to Y_1, and Y is isomorphic to Y_2. Let $T \in \mathcal{L}(X, Y_1)$ and $S \in \mathcal{L}(Y, Y_2)$ be those isomorphisms. Then $T \in \mathcal{L}(X, Y)$ is left invertible with a left inverse $K \in \mathcal{L}(Y, X)$ and $\mathcal{N}(K) = Y_2$. Also, $S \in \mathcal{L}(Y, Y_2)$ is left invertible with a left inverse L and $\mathcal{N}(L) = Y_1$. Then

$$\begin{bmatrix} 0 & K \\ 0 & L \end{bmatrix} \begin{bmatrix} A & C \\ T & S \end{bmatrix} = \begin{bmatrix} I_X & 0 \\ 0 & I_Y \end{bmatrix},$$

so $M_{(T,S)}$ is left invertible. It follows that $M_{(T,S)}$ is left Fredholm for chosen operators T and S. Suppose that $M_{(T,S)}$ is Fredholm. Since $\left[\begin{smallmatrix} I_X & 0 \\ 0 & I_Y \end{smallmatrix}\right]$ is Fredholm, from Lemma 11.1 it follows that N is also Fredholm. However, we notice $\mathcal{N}(N) = X$, which is infinite dimensional. Hence, N is not Fredholm. Then $M_{(T,S)}$ is not Fredholm also, i.e. $M_{(T,S)} \in \Phi_l(X \oplus Y) \setminus \Phi(X \oplus Y)$. $\square$

We formulate a corollary for Hilbert space operators.

Corollary 11.1. *Let X and Y be infinite dimensional and mutually orthogonal subspaces of a Hilbert space $Z = X \oplus Y$. Suppose that $\dim_H Y = \dim_H Z$. Let $A \in \mathcal{L}(X)$ and $C \in \mathcal{L}(Y, X)$ be arbitrary. Then $M_{(T,S)} \in \Phi_l(X \oplus Y) \setminus \Phi(X \oplus Y)$ for some $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y)$.*

11.3. Left Browder invertibility of M_C.

As part of Fredholm theory, the Browder operators are studied.

An operator $T \in B(X)$ is left Browder, if it is left Fredholm with finite ascent. Analogously, T is right Browder, if it is right Fredholm with finite descent. These classes of operators are denoted, respectively, by $\mathcal{B}_l(X)$ and $\mathcal{B}_r(X)$. The set of all Browder operators on X is defined as $\mathcal{B}(X) = \mathcal{B}_l(X) \cap \mathcal{B}_r(X)$.

Among left Browder operators, we distinguish one new class of operators as follows

$$\mathcal{B}_{lc}(X) = \{T \in B_l(X) : \overline{\mathcal{R}(T) + \mathcal{N}(T^{\operatorname{asc}(T)})} \text{ is complemented in } X\}.$$

Analogously, among right Browder operators we distinguish the following class of operators

$$\mathcal{B}_{rc}(X) = \{T \in B_r(X) : \overline{\mathcal{R}(T^{\operatorname{dsc}(T)}) + \mathcal{N}(T)} \text{ is complemented in } X\}.$$

Now, we prove the following result concerning the left Browder invertibility of M_C.

Theorem 11.4. *Suppose that the following hold: $A \in \mathcal{B}_{lc}(X)$, B is relatively regular, and $\mathcal{N}(B)$ is isomorphic to $X/\overline{(\mathcal{R}(A) + \mathcal{N}(A^{\operatorname{asc}(A)}))}$. Then there exists some $C \in \mathcal{L}(Y, X)$ such that $M_C \in B_l(Z)$.*

Proof. Let $A \in \mathcal{B}_{lc}(X)$, $\operatorname{asc}(A) = p$, and let W be a closed subspace of X such that $X = \overline{\mathcal{R}(A) + \mathcal{N}(A^p)} \oplus W$. Since $\mathcal{N}(B)$ is complemented, then $Y = \mathcal{N}(B) \oplus V$ for a closed subspace V. Since there exists a linear bounded and invertible operator $T\colon \mathcal{N}(B) \to W$, we can define operator $C\colon Y \to X$ by

$$C = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ V \end{bmatrix} \to \begin{bmatrix} W \\ \overline{\mathcal{R}(A) + \mathcal{N}(A^p)} \end{bmatrix}.$$

We prove that M_C is left Fredholm. Let $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathcal{N}(M_C)$, so it is $Ax + Cy = 0$ and $By = 0$. We have $Ax = -Cy = -Ty \in \mathcal{R}(A) \cap W \subseteq \overline{\mathcal{R}(A) + \mathcal{N}(A^p)} \cap W = \{0\}$. Since $y \in \mathcal{N}(B)$ we have $Cy = Ty$, so $x \in \mathcal{N}(A)$ and $Ty = 0$. Since T is invertible, we have $y = 0$. It means that $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathcal{N}(A) \oplus \{0\}$, so $\mathcal{N}(M_C) \subseteq \mathcal{N}(A) \oplus \{0\}$. It follows that $\operatorname{nul}(M_C) \leqslant \operatorname{nul}(A) < \infty$.

Notice that we have obviously $\mathcal{N}(A) \subset \mathcal{N}(M_C)$, so actually we have $\operatorname{nul}(M_C) = \operatorname{nul}(A)$.

Let S be a reflexive inverse of A, let K be a reflexive inverse of B, and let $L = \left[\begin{smallmatrix} T^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$. We prove that $N = \left[\begin{smallmatrix} S & 0 \\ L & K \end{smallmatrix}\right]$ is an inner inverse of M_C. We have

$$M_C N M_C = \begin{bmatrix} ASA + CLA & ASC + CLC + CKB \\ BLA & BLC + BKB \end{bmatrix}.$$

Since $\mathcal{R}(A) \subseteq \overline{\mathcal{R}(A) + \mathcal{N}(A^p)} = \mathcal{N}(L)$, we have $LA = 0$ which induces $BLA = 0$ and $CLA = 0$. From the fact that S is a reflexive inverse of A, we have $ASA = A$, and AS is a projection from X on $\mathcal{R}(A)$. Since $\mathcal{R}(C) = W$, $W \cap \mathcal{R}(A) = \{0\}$ and AS is a projection on $\mathcal{R}(A)$, it follows that $ASC = 0$. Analogously, from the fact that K is a reflexive inverse of B, we have $BKB = B$ and KB is a projection from Y on V. Since $V = \mathcal{N}(C)$ and $\mathcal{R}(KB) = V$, it holds $CKB = 0$. We have that $LC = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] \colon \left[\begin{smallmatrix} \mathcal{N}(B) \\ V \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{N}(B) \\ V \end{smallmatrix}\right]$, so $\mathcal{R}(LC) \subseteq \mathcal{N}(B)$ and then $BLC = 0$. Obviously, $CLC = C$ holds.

It follows that

$$\begin{bmatrix} ASA + CLA & ASC + CLC + CKB \\ BLA & BLC + BKB \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = M_C.$$

Thus M_C is relatively regular. This induces $M_C \in \Phi_l$.

Now, we prove that $\operatorname{asc}(M_C) < \infty$. It is enough to prove that $\mathcal{N}(M_C^{p+1}) \subseteq \mathcal{N}(M_C^p)$. Let $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathcal{N}(M_C^{p+1})$, then

$$A^{p+1}x + A^pCy + A^{p-1}CBy + \cdots + ACB^{p-1}y + CB^py = 0, \quad B^{p+1}y = 0.$$

Since $B^py \in \mathcal{N}(B)$, it follows that $A^{p+1}x + A^pCy + A^{p-1}CBy + \cdots + ACB^{p-1}y = -CB^py \in \mathcal{R}(A) \cap W \subseteq \overline{\mathcal{R}(A) + \mathcal{N}(A^p)} \cap W = \{0\}$. Thus

$$A^{p+1}x + A^pCy + A^{p-1}CBy + \cdots + ACB^{p-1}y = 0, \quad CB^py = 0.$$

From the definition of C and from $B^p \in \mathcal{N}(B)$, we know that $CB^py = TB^py = 0$. Since T is invertible, we conclude that $B^py = 0$.

From the fact that $A^{p+1}x + A^pCy + A^{p-1}CBy + \cdots + ACB^{p-1}y = 0$, we have that $x_1 = A^px + A^{p-1}Cy + A^{p-2}CBy + \cdots + ACB^{p-2}y + CB^{p-1}y \in \mathcal{N}(A)$. Then

$$A^px + A^{p-1}Cy + A^{p-2}CBy + \cdots + ACB^{p-2}y - x_1 + CB^{p-1}y = 0, \quad B^py = 0.$$

Thus $B^{p-1}y \in \mathcal{N}(B)$. It induces that

$$\begin{aligned} &A^p x + A^{p-1}Cy + A^{p-2}CBy + \cdots + ACB^{p-2}y - x_1 \\ &\qquad = -CB^{p-1}y \in (\mathcal{R}(A) + \mathcal{N}(A)) \cap W \subseteq \overline{\mathcal{R}(A) + \mathcal{N}(A^p)} \cap W = \{0\}, \end{aligned}$$

then $B^{p-1}y = 0$ and $A^p x + A^{p-1}Cy + A^{p-2}CBy + \cdots + ACB^{p-2}y = x_1$. Since $x_1 \in \mathcal{N}(A)$, it follows that $A^{p-1}x + A^{p-2}Cy + A^{p-3}CBy + \cdots + CB^{p-2}y \in \mathcal{N}(A^2)$. Let $x_2 = A^{p-1}x + A^{p-2}Cy + A^{p-3}CBy + \cdots + CB^{p-2}y$. Then

$$A^{p-1}x + A^{p-2}Cy + A^{p-3}CBy + \cdots + ACB^{p-3}y - x_2 + CB^{p-2}y = 0, \quad B^{p-1}y = 0.$$

If we continue this process, we get $A^2x + ACy - x_{p-1} + CBy = 0$, $B^2y = 0$, where $x_{p-1} \in \mathcal{N}(A^{p-1})$. Then there exists $x_p \in \mathcal{N}(A^p)$ such that $Ax + Cy - x_p = 0$, $By = 0$. Thus $Ax - x_p = -Cy \in \overline{\mathcal{R}(A) + \mathcal{N}(A^p)} \cap W = \{0\}$. It follows that $x \in \mathcal{N}(A^{p+1}) = \mathcal{N}(A^p)$ and $y = 0$, so $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathcal{N}(M_C^p)$. Since $\mathcal{N}(M_C^{p+1}) \subseteq \mathcal{N}(M_C^p)$, we get $\operatorname{asc}(M_C) \leqslant p$. □

12. Perturbations of spectra of operator matrices

The spectral theory is an essential part of functional analysis. It has great application in several branches of mathematics and physics such as complex analysis, function theory, matrix theory, differential and integral equations, quantum physics, control theory, etc. The book [49] is also an important contribution.

Various types of spectra have been studied throughout history. The spectrum for operator $A \in \mathcal{L}(X)$ is defined as follows: $\sigma(A) = \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not invertible}\}$.

The spectrum can also be defined in relation to another set of elements such as regular, Fredholm, left and right invertible elements, etc. Some of these spectra for operator $A \in \mathcal{L}(X)$ are defined as follows

Left spectrum:	$\sigma_l(A) = \{\lambda \in \mathbb{C} \mid A - \lambda I \notin \mathcal{G}_l(X)\}$
Right spectrum:	$\sigma_r(A) = \{\lambda \in \mathbb{C} \mid A - \lambda I \notin \mathcal{G}_r(X)\}$
Regular spectrum:	$\sigma_g(A) = \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not regular}\}$
Essential spectrum:	$\sigma_e(A) = \{\lambda \in \mathbb{C} \mid A - \lambda I \notin \Phi(X)\}$
Left Fredholm spectrum:	$\sigma_{le}(A) = \{\lambda \in \mathbb{C} \mid A - \lambda I \notin \Phi_l(X)\}$
Right Fredholm spectrum:	$\sigma_{re}(A) = \{\lambda \in \mathbb{C} \mid A - \lambda I \notin \Phi_r(X)\}$
Point spectrum:	$\sigma_p(A) = \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not "1–1"}\}$

The spectrum of operator matrices has been studied in the literature, see [12, 19, 25, 29, 39, 45]. When it comes to operator matrices of type M_C, it is interesting to observe how the spectrum of these matrices looks like for arbitrary C, and to find $\bigcap_{C \in \mathcal{L}(Y,X)} \sigma_\tau$, where σ_τ is one of the mentioned spectrum.

This is the subject of study in [19].

12.1. Perturbation of the essential spectrum. We start with the following result.

Lemma 12.1. [31] *If $T\in\mathcal{L}(X,Y)$, $S\in\mathcal{L}(Y,Z)$, $ST\in\mathcal{L}(X,Z)$ are relatively regular, then*

$$\mathcal{N}(T)\times\mathcal{N}(S)\times Z/\mathcal{R}(ST)\cong\mathcal{N}(ST)\times Y/\mathcal{R}(T)\times Z/\mathcal{R}(S).$$

Definition 12.1. Banach spaces U,V are isomorphic up to a finite dimensional subspace, if one of the following two statements hold:
1)There exists a bounded below operator $J_1\colon U\to V$ such that $\dim V/J_1(U)<\infty$, or
2) There exists a bounded below operator $J_2\colon V\to U$ such that $\dim U/J_2(V)<\infty$.

Lemma 12.2. *Let X,Y be Banach spaces and let M,N be finite dimensional spaces. If $M\oplus X\cong N\oplus Y$, then X and Y are isomorphic up to a finite dimensional subspace. Particualry, if $\dim M=\dim N$, then $X\cong Y$.*

Proof. If at least one of X,Y is finite dimensional, then the result is trivial. Hence, we suppose that both X and Y are infinite dimensional. Let $\dim M=m$, $\dim N=n$, and $J\colon M\oplus X\to N\oplus Y$ be a Banach space isomorphism. Let $x_1,\dots,x_k\in X$ be a system of linearly independent vectors in X, such that $Jx_1,\dots,Jx_k$ are linearly independent modulo Y. We conclude $0\leqslant k\leqslant n$. There exists a system of $n-k$ vectors $z_1,\dots,z_{n-k}$ in $N\oplus Y$, which are linearly independent modulo $\operatorname{span}\{Jx_1,\dots,Jx_k\}\oplus Y$. We get $0\leqslant n-k\leqslant n$. Denote by $y_i=J^{-1}z_i$, for all $i+1,\dots,n-k$, in the case when $n-k>0$. All vectors $y_1,\dots,y_{n-k}$ must be linearly independent modulo X. In general, we get $0\leqslant n-k\leqslant m$. There exists a system of exactly $l=m-(n-k)$ vectors $u_1,\dots,u_l$ which are linearly independent modulo $\operatorname{span}\{y_1,\dots,y_{n-k}\}\oplus X$. There exists a Banach space X_1 such that

$$\operatorname{span}\{x_1,\dots,x_k\}\oplus X_1=X,$$

$$M\oplus X=\operatorname{span}\{y_1,\dots,y_{n-k}\}\oplus\operatorname{span}\{u_1,\dots,u_l\}\oplus\operatorname{span}\{x_1,\dots,x_k\}\oplus X_1.$$

Let $v_i=Ju_i$, $i=1,\dots,l$. Vectors $v_1,\dots,v_l$ are linearly independent modulo

$$\operatorname{span}\{Jx_1,\dots,Jx_k\}\oplus\operatorname{span}\{z_1,\dots,z_{n-k}\}.$$

Let $Y_1=J(X_1)$. Then Y_1 is closed, $X_1\cong Y_1$ and

$$N\oplus Y=\operatorname{span}\{Jx_1,\dots,Jx_k\}\oplus\operatorname{span}\{z_1,\dots,z_{n-k}\}\oplus\operatorname{span}\{v_1,\dots,v_l\}\oplus Y_1.$$

Since $\operatorname{span}\{Jx_1,\dots,Jx_k\}\oplus\operatorname{span}\{z_1,\dots,z_{n-k}\}$ is linearly independent modulo Y, we conclude

$$N\oplus Y=\operatorname{span}\{Jx_1,\dots,Jx_k\}\oplus\operatorname{span}\{z_1,\dots,z_{n-k}\}\oplus Y.$$

Hence,

$$Y\cong\frac{N\oplus Y}{\operatorname{span}\{Jx_1,\dots,Jx_k\}\oplus\operatorname{span}\{z_1,\dots,z_{n-k}\}}\cong\operatorname{span}\{v_1,\dots,v_l\}\oplus Y_1.$$

We have to add a k-dimensional subspace to X_1, to get space which is isomorphic to X. We have to add an l-dimensional space to Y_1 to get a space which is isomorphic to Y. Since $X_1\cong Y_1$, we conclude that X and Y are isomorphic up to a finite dimensional subspace.

Particularly, if $m=n$, then $k=l$, so $X\cong Y$. □

Now, let $Z = X \oplus Y$ be a topological direct sum of closed subspaces X, Y of Z. Let

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} : \begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} X \\ Y \end{bmatrix}.$$

Since $\Phi(Z)$, $\Phi(X)$ and $\Phi(Y)$ are strong regularities, we have the result

$$(\forall C \in \mathcal{L}(Y, X)) \quad \sigma_e(M_C) \subset \sigma_e(A) \cup \sigma_e(B).$$

We prove the following result.

Theorem 12.1. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ be given and consider the following statements*

1) $M_C \in \Phi(Z)$ *for some* $C \in \mathcal{L}(Y, X)$.
2) 2.1) $A \in \Phi_l(X)$;
   2.2) $B \in \Phi_r(Y)$;
   2.3) $\mathcal{N}(B)$ *and* $X/\overline{\mathcal{R}(A)}$ *are isomorphic up to a finite dimensional subspace.*

Then 1) $\Longleftrightarrow$ 2).

Proof. 1) $\Longrightarrow$ 2): Let $M_C \in \Phi(Z)$ for some $C \in \mathcal{L}(Y, X)$, and denote $B_1 = \left[\begin{smallmatrix} I & 0 \\ 0 & B \end{smallmatrix}\right]$, $C_1 = \left[\begin{smallmatrix} I & C \\ 0 & I \end{smallmatrix}\right]$, $A_1 = \left[\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right]$. Obviously, C_1 is invertible in $\mathcal{L}(Z)$. From $M_C = B_1 C_1 A_1 \in \Phi(Z)$ it follows that $B_1, B_1 C_1 \in \Phi_r(Z)$, and $A_1, A_1 C_1 \in \Phi_l(Z)$. Thus 2.1) and 2.2) are proved.

Applying Lemma 12.1 to $M_C = (B_1 C_1) A_1$, we get

$$\mathcal{N}(A) \times \mathcal{N}(B_1 C_1) \times (X \oplus Y)/\mathcal{R}(M_C) \cong \mathcal{N}(M_C) \times X/\mathcal{R}(A) \times Y/\mathcal{R}(B).$$

Now we apply Lemma 12.1 to $B_1 C_1$ and get

$$\mathcal{N}(B) \times Y/\mathcal{R}(B) \cong \mathcal{N}(B_1 C_1) \times Y/\mathcal{R}(B).$$

Since $\operatorname{def}(B) < \infty$, from Lemma 12.2 we obtain $\mathcal{N}(B) \cong \mathcal{N}(B_1 C_1)$. Finally, the following hold

$$\mathcal{N}(A) \times \mathcal{N}(B) \times Z/\mathcal{R}(M_C) \cong \mathcal{N}(M_C) \times X/\mathcal{R}(A) \times Y/\mathcal{R}(B).$$

Since $\mathcal{N}(A)$, $Z/\mathcal{R}(M_C)$, $\mathcal{N}(M_C)$, $Y/\mathcal{R}(B)$ are finite dimensional, we conclude that $\mathcal{N}(B)$ and $X/\mathcal{R}(A)$ are isomorphic up to a finite dimensional subspace. Thus 2.3) is proved.

2) $\Longrightarrow$ 1) Suppose that $A \in \Phi_l(X)$, $B \in \Phi_r(Y)$ and $\mathcal{N}(B)$ and $X/\mathcal{R}(A)$ are isomorphic up to a finite dimensional subspace. There exists closed subspaces U and V of X and Y, repsectively, such that $\mathcal{R}(A) \oplus U = X$ and $\mathcal{N}(B) \oplus V = Y$. We consider two cases.

Case 1. Suppose that there exists a bounded below operator $J \colon \mathcal{N}(B) \to U$, such that $\dim U/J(\mathcal{N}(B)) < \infty$. There exists a finite dimensional subspace W of X such that $J(\mathcal{N}(B)) \oplus W = U$. We define $C \in \mathcal{L}(Y, X)$ as follows

$$C = \begin{bmatrix} J & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ V \end{bmatrix} \to \begin{bmatrix} J(\mathcal{N}(B)) \\ W \mathcal{R}(A) \end{bmatrix}.$$

Obviously, $\mathcal{R}(C) = J(\mathcal{N}(B))$. Now, $\mathcal{R}(M_C) = [\mathcal{R}(A) \oplus J(\mathcal{N}(B))] \oplus \mathcal{R}(B)$ and $\dim Z/\mathcal{R}(M_C) = \dim W + \operatorname{def}(B) < \infty$. It also follows that $\mathcal{R}(M_C)$ is closed. On the other hand, if $M_C \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = 0$, then $y \in \mathcal{N}(B)$ and $Ax = -Cy$, implying $x \in \mathcal{N}(A)$ and $y = 0$. We get $\mathcal{N}(M_C) = \mathcal{N}(A)$, so $M_C \in \Phi(Z)$.

Case 2. Assume that there exists a bounded below operator $J \colon U \to \mathcal{N}(B)$, such that $\dim \mathcal{N}(B)/J(U) < \infty$. There exists a finite dimensional subspace K of $\mathcal{N}(B)$ such that $\mathcal{N}(B) = J(U) \oplus K$. Let $J_1 \colon J(U) \to U$ denote the inverse of the redusction $J \colon U \to J(U)$. Define $C \in \mathcal{L}(Y, X)$ as follows

$$C = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} J(U) \\ K \\ V \end{bmatrix} \to \begin{bmatrix} U \\ \mathcal{R}(A) \end{bmatrix}.$$

Obviously, $\mathcal{R}(C) = U$. We conclude that $\mathcal{R}(M_C) = X \oplus \mathcal{R}(B)$, so $\dim Z/\mathcal{R}(M_C) = \operatorname{def}(B) < \infty$ and $\mathcal{R}(M_C)$ is closed. Also, $\mathcal{N}(M_C) = \mathcal{N}(A) \oplus Z$, so it follows that $M_C \in \Phi(Z)$. □

We get the following consequence.

Corollary 12.1. *For given $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ the following holds*

$$\bigcap_{C \in \mathcal{L}(X,Y)} \sigma_e(M_C) = \sigma_{le}(A) \cup \sigma_{re}(B) \cup \mathcal{W}(A, B),$$

where

$$\mathcal{W}(A, B) = \{\lambda \in \mathbb{C} : \ \mathcal{N}(B - \lambda) \text{ and } X/\overline{\mathcal{R}(A - \lambda)} \text{ are not isomorphic up to a finite dimensional subspace}\}.$$

We know which part of the set $\sigma_e(A) \cup \sigma_e(B)$ can be perturbed out by choosing a suitable operator $C \in \mathcal{L}(Y, X)$.

Theorem 12.2. *Assume that there exists an operator $A \in \mathcal{L}(Y, X)$ such that the inclusion $\sigma_e(M_C) \subset \sigma_e(A) \cup \sigma_e(B)$ is proper. Then*

$$[\sigma_e(A) \cup \sigma_e(B)] \setminus \sigma_e(M_C) \subset \sigma_e(A) \cup \sigma_e(B).$$

Proof. Suppose that $[\sigma_e(A) \setminus \sigma_e(B)] \setminus \sigma_e(M_C)$. Then $A - \lambda \notin \Phi(X)$ and $B - \lambda \in \Phi(Y)$. Since $\operatorname{nul}(B - \lambda) < \infty$, by Corollary 12.1 we conclude $\operatorname{def}(A - \lambda) < \infty$. It follows that $\lambda \notin \sigma_e(A)$, and it is in contradiction with the choice of λ. Thus

$$[\sigma_e(A) \setminus \sigma_e(B)] \setminus \sigma_e(C) = \emptyset.$$

In the same manner we can prove

$$[\sigma_e(B) \setminus \sigma_e(A)] \setminus \sigma_e(M_C) = \emptyset. \qquad \square$$

Consider the following classes of operators

$$\mathcal{S}_+(X) = \{T \in \mathcal{L}(X) : \operatorname{nul}(T - \lambda) \geqslant \operatorname{def}(T - \lambda) \text{ if at least one of these quantities is finite}\},$$

$$\mathcal{S}_+(X) = \{T \in \mathcal{L}(X) : \operatorname{nul}(T - \lambda) \leqslant \operatorname{def}(T - \lambda)$$

if at least one of these quantities is finite},

Theorem 12.3. *If $A \in \mathcal{S}_+(X)$ or $B \in \mathcal{S}_-(Y)$, then for every $C \in \mathcal{L}(Y,X)$ we have $\sigma_e(M_C) = \sigma_e(A) \cup \sigma_e(B)$.*

Proof. It is enough to prove the inclusion $\supset$. Suppose that $\lambda \in [\sigma_e(A) \cup \sigma_e(B)] \setminus \sigma_e(M_C)$. Then $A - \lambda \in \Phi_l(X)$, $B - \lambda \in \Phi_r(Y)$, and $\mathcal{N}(B-\lambda)$ and $X/\mathcal{R}(A-\lambda)$ are isomirpshic up to a finite dimensional subspace.

If $A \in \mathcal{S}_+(X)$, then $\operatorname{def}(A-\lambda) \leqslant \operatorname{nul}(A-\lambda) < \infty$ and $A - \lambda \in \Phi(X)$. Hence $\mathcal{N}(B-\lambda)$ must be finite dimensional subspace and $B - \lambda \in \Phi(Y)$.

If $B \in \mathcal{S}_-(Y)$, then $\operatorname{nul}(B-\lambda) \leqslant \operatorname{def}(B-\lambda) < \infty$ and $B - \lambda \in \Phi(Y)$. Then $X/\mathcal{R}(A-\lambda)$ must be finite dimensional and $A - \lambda \in \Phi(X)$.

In both cases we obtain $A-\lambda \in \Phi(X)$ and $B-\lambda \in \Phi(Y)$, which is in contradiction with our assumtion $\lambda \in \sigma_e(A) \cup \sigma_e(B)$. □

12.2. Perturbation of the Weyl and Browder spectrum.

We consider the Weyl spectrum of M_C.

Theorem 12.4. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ be given and consider the statements*

1) *$M_C \in \phi_0(Z)$ for some $C \in \mathcal{L}(Y,X)$.*
2) *$A \in \Phi_l(X)$, $B \in \Phi_r(Y)$, $\mathcal{N}(A) \oplus \mathcal{N}(B) \cong X/\overline{\mathcal{R}(A)} \oplus Y/\overline{\mathcal{R}(B)}$.*

Proof. 1) $\Longrightarrow$ 2): Follows from (3.1) and Lemma 2.3.

2) $\Longrightarrow$ 1): Let $A \in \Phi_l(X)$, $B \in \Phi_r(X)$ and

$$\mathcal{N}(A) \oplus \mathcal{N}(B) \cong X/\mathcal{R}(A) \oplus Y/R(B). \tag{12.1}$$

There exists closed subspaces U and V such that $X = \mathcal{R}(A) \oplus U$ and $Y = \mathcal{N}(B) \oplus V$. We consider three cases.

Case 1. Let $\operatorname{nul}(A) = \operatorname{def}(B) < \infty$. From (12.1) it follows that $\mathcal{N}(B) \cong X/\mathcal{R}(A)$. Let $J\colon \mathcal{N}(B) \to U$ be an arbitrary isomorphism. Define $C \in \mathcal{L}(Y,X)$ as follows

$$C = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ V \end{bmatrix} \to \begin{bmatrix} U \\ \mathcal{R}(A) \end{bmatrix}.$$

We get that $\mathcal{R}(M_C) = X \oplus \mathcal{R}(B)$,, $\mathcal{N}(M_C) = \mathcal{N}(A)$, so M_C is Weyl.

Case 2. Let $\operatorname{nul}(A) < \operatorname{def}(B) < \infty$. From (12.1) it follows that there exists a bounded below operator $J\colon U \to \mathcal{N}(B)$, such that $\dim \mathcal{N}(B)/J(U) = \operatorname{def}(B) - \operatorname{nul}(A)$. The reduction $J\colon U \to J(U)$ is invertible, so let $J_1\colon J(U) \to U$ denote its inverse. There exists a finite dimensional subspace U_1 such that $J(U) \oplus U_1 = \mathcal{N}(B)$ and $\dim U_1 = \operatorname{def}(B) - \operatorname{nul}(A)$. Define $C \in \mathcal{L}(Y,X)$ as

$$C = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} J(U) \\ U_1 \\ V \end{bmatrix} \to \begin{bmatrix} U \\ \mathcal{R}(A) \end{bmatrix}.$$

We get that $\mathcal{R}(M_C) = X \oplus \mathcal{R}(B)$ and $\mathcal{N}(M_C) = \mathcal{N}(A) \oplus U_1$, so we conclude that M_C is Weyl.

Case 3. Let $\operatorname{def}(B) < \operatorname{nul}(A) < \infty$. From (12.1) it follows that there exists a bounded below operator $J\colon \mathcal{N}(B) \to U$ such that $\dim Z/J(\mathcal{N}(B)) = \operatorname{nul}(A) - \operatorname{def}(B)$. There exists a finite dimensional space U_2 such that $J(\mathcal{N}(B)) \oplus U_2 = U$ and $\dim U_2 = \operatorname{nul}(A) - \operatorname{def}(B)$. We define $C \in \mathcal{L}(Y, X)$ as

$$C = \begin{bmatrix} J & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ V \end{bmatrix} \to \begin{bmatrix} J(\mathcal{N}(B)) \\ U_2 \\ \mathcal{R}(A) \end{bmatrix}.$$

It follows that $\mathcal{R}(M_C) = [\mathcal{R}(A) \oplus J(\mathcal{N}(B))] \oplus \mathcal{R}(B)$, $\mathcal{N}(M_C) = \mathcal{N}(A)$, and we conclude that M_C is Weyl. □

As a corollary we get the following result.

Corollary 12.2. *For given $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ the following holds*

$$\bigcap_{C \in \mathcal{L}(Y,X)} \sigma_w(M_C) = \sigma_{le}(A) \cup \sigma_{re}(B) \cup \mathcal{W}_0(A, B),$$

where

$$\mathcal{W}_0(A, B) = \big\{\lambda \in \mathbb{C} : \mathcal{N}(A - \lambda) \oplus \mathcal{N}(B) \text{ is not isomorphic to } X/\overline{R(A - \lambda)} \oplus Y/\overline{\mathcal{R}(B - \lambda)}\big\}.$$

We formulate the result for the Browder spectrum.

Corollary 12.3. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ be given. Consider the following statements*

1) *$A \in \Phi_l(X)$; $B \in \Phi_r(Y)$; $\mathcal{N}(B)$ and $X/\overline{\mathcal{R}(A)}$ are isomorphic up to a finite dimensional subspace; A and B are Drazin invertible.*
2) *$M_C \in \mathcal{B}(Z)$ for some $C \in \mathcal{L}(Y, X)$.*

Then 1) $\implies$ 2).
Moreover, if $0 \notin \operatorname{acc}(\sigma(A) \cup \sigma(B))$, then 1) $\iff$ 2).

Proof. Follows from Theorem 12.1. □

We have more details concerning the perturbation of the Browder spectrum.

Theorem 12.5. *If $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$, then*

$$\bigcap_{C \in \mathcal{L}(Y,X)} \sigma_b(M_C) \subset \sigma_{le}(A) \cup \sigma_{re}(B) \cup \mathcal{W}(A, B) \cup \mathcal{W}_1(A, B), \tag{12.2}$$

where $\mathcal{W}(A, B)$ is defined in Corollary 12.2 and

$$\mathcal{W}_1(A, B) = \{\lambda \in \mathbb{C} : \text{ one of } A - \lambda \text{ or } B - \lambda \text{ is not Drazin invertible}\}.$$

If $\operatorname{acc} \sigma(A) \cup \operatorname{acc} \sigma(B) = \emptyset$, then the equality holds in (12.2).
If $\sigma_a(A) = \sigma(A)$ and $\sigma_d(B) = \sigma(B)$, then the equality holds in (12.2).
If $\sigma(A) \cup \sigma(B)$ does not have interior points, then the equality holds in (12.2).

Proof. The result of (12.2) follows immediately from Theorem 3.9. If the assumption $\operatorname{acc}\sigma(A) \cup \operatorname{acc}\sigma(B) = \emptyset$ is satisfied, then from Theorem 12.5 it follows that equaltiy holds in (12.2).

Suppose that $\sigma_a(A) = \sigma(A)$ and $\sigma_d(B) = \sigma(B)$, and

$$\lambda \notin \bigcap_{C \in \mathcal{L}(Y,X)} \sigma_b(M_C).$$

There exists some $C \in \mathcal{L}(Y,X)$ such that $M_C - \lambda \in \mathcal{B}(Z)$. From Theorem 12.1 it follows that $A - \lambda \in \Phi_l(X)$, $B - \lambda \in \Phi_r(Y)$ and $X/\mathcal{R}(A-\lambda)$ is isomorphic to $\mathcal{N}(B-\lambda)$ up to a finite dimensional subspace. Let $\operatorname{asc}(M_C - \lambda) = \operatorname{dsc}(M_C - \lambda) = p < \infty$. Also, $\lambda \notin \operatorname{acc}(M_C)$. Hence, there exists an $\epsilon > 0$ such that if $0 < |z-\lambda| < \epsilon$ then $z \notin \sigma(M_C - \lambda)$. For such z the operator $M_C - \lambda$ is invertible and it is eaasy to prove that $A - \lambda - z$ is left invertible and $B - \lambda - z$ is right invertible. It follows that $\lambda \notin \operatorname{acc}\sigma_a(A) \cup \operatorname{acc}\sigma_d(B) = \operatorname{acc}\sigma(A) \cup \operatorname{acc}\sigma(B)$. It follows that $A - \lambda$ and $B - \lambda$ are Drazin invertible.

Let $\operatorname{int}(\sigma(A) \cup \sigma(B)) = \emptyset$. If $\lambda \notin \bigcap_{C \in \mathcal{L}(Y,X)} \sigma_b(M_C)$, in the same way as above we can prove that $\lambda \notin \operatorname{acc}\sigma_a(A) \cup \operatorname{acc}\sigma_d(B)$. We will prove that $\lambda \notin \operatorname{acc}\sigma(A) \cup \operatorname{acc}\sigma(B)$. Since λ can not be an interior point of $\sigma(A) \cup \sigma(B)$, it follows that λ must be a boundary point of $\sigma(A) \cup \sigma(B)$. If $\lambda \in \operatorname{acc}\sigma(A)$, then there exists a sequence $(x_n)_n$, $x_n \in \partial\sigma(A) \subset \sigma_a(A)$, such that $\lim_{n\to\infty} x_n = \lambda$. It follows that $\lambda \in \operatorname{acc}\sigma_a(A)$ and this is in contradiction with our previous statement $\lambda \notin \operatorname{acc}\sigma_a(A) \cup \operatorname{acc}\sigma_d(B)$. We conclude that $\lambda \notin \operatorname{acc}\sigma(A)$. Similarly, since $\partial\sigma(B) \subset \sigma_d(B)$, we get $\lambda \notin \sigma(B)$. Now, it follows that $A - \lambda$ and $B - \lambda$ are Drazin invertible. □

12.3. Perturbation of the left and right essential spectra.

We formulate the following statement.

Lemma 12.3. *For given $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ and $C \in \mathcal{L}(Y,X)$, the following inclusion holds $\sigma_{re}(M_C) \subset \sigma_{re}(A) \sup \sigma_{re}(B)$. Particulary, if $A \in \Phi_r(X)$ and $B \in \Phi_r(Y)$, then $M_C \in \Phi(Z)$ for every $C \in \mathcal{L}(Y,X)$.*

The notion of embedded spaces is introduced.

Definition 12.2. Let X and Y be Banach spaces. The space X can be embedded in Y, and we write $X \preceq Y$, if there exists a left invertible operator $J\colon X \to Y$.

The space X can essentially be embedded in Y, denoted by $X \prec Y$, if $X \preceq Y$ and $Y/T(X)$ is infinite dimensional space for every $T \in \mathcal{L}(X,Y)$.

Remark 12.1. Obviously, $X \preceq Y$ if and only if there exists a right invertible operator $J_1\colon Y \to X$.

If H, K are Hilbert spaces, then $H \preceq K$ if and only if $\dim H \leqslant \dim K$ (and $\dim H$ is the orthogobal dimension of H). Moreover, $H \prec K$ if and only if $\dim H < \dim K$ and K is infinite dimensional.

The main result of this subsection follows.

Theorem 12.6. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ be given operators. Consider the following statements*

1) *$B \in \Phi_r(X)$ and $[A \in \Phi_r(X)$ or $(\mathcal{R}(A)$ is closed and complemented in X and $X/\overline{\mathcal{R}(A)} \preceq \mathcal{N}(B))]$.*
2) *$M_C \in \Phi_r(Z)$ for some $C \in \mathcal{L}(Y,X)$.*
3) *$B \in \Phi_r(Y)$ and $[A \in \Phi_r(X)$, or $\mathcal{R}(A)$ is not closed, or $\mathcal{N}(B) \prec X/\overline{\mathcal{R}(A)}$ does not hold].*

Then 1) $\Longrightarrow$ 2) $\Longrightarrow$ 3).

Proof. 1) $\Longrightarrow$ 2): Let $B \in \Phi_r(Y)$. If $A \in \Phi_r(X)$, we get that $M_C \in \Phi_r(Z)$ for every $C \in \mathcal{L}(Y,X)$.

Hence, assume $B \in \Phi_r(Y)$, $A \notin \Phi_r(X)$, $\mathcal{R}(A)$ is closed and complemented in X and $X/\mathcal{R}(A) \preceq \mathcal{N}(B)$. There exists a closed subspace U of X such that $\mathcal{R}(A) \oplus U = X$. Let $J\colon U \to \mathcal{N}(B)$ be a left invertible operator and $J_1\colon \mathcal{N}(B) \to U$ its left inverse. There exists a closed subspace V of Y such that $\mathcal{N}(B) \oplus V = Y$. Define the operator $C \in \mathcal{L}(Y,X)$ in the following way

$$C = \begin{bmatrix} J_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ V \end{bmatrix} \to \begin{bmatrix} U \\ \mathcal{R}(A) \end{bmatrix}.$$

Then $\mathcal{R}(M_C) = X \oplus \mathcal{R}(B)$ and $\operatorname{def}(M_C) = \operatorname{def}(B) < \infty$. Hence, $M_C \in \Phi_r(Z)$.

2) $\Longrightarrow$ 3): Let $M - C \in \Phi_r(Z)$ for some $C \in \mathcal{L}(Y,X)$. Then $\mathcal{R}(M_C) \subset [\mathcal{R}(A) + \mathcal{R}(C)] \oplus \mathcal{R}(B)$. If $x_1, \dots, x_n \in X$ are linearly independent modulo $\mathcal{R}(A) + \mathcal{R}(C)$, and if $y_1, \dots, y_m$ are linearly independent modulo $\mathcal{R}(B)$, then $n + m \leqslant \operatorname{def}(M_C) < \infty$. Hence, $\operatorname{def}(B) < \infty$ and $B \in \Phi_r(Y)$. Thus we have proved the first statement of 3).

Moreover, assume that the second statement in 3) does not hold. Then $A \notin \Phi_r(X)$, $\mathcal{R}(A)$ is closed and $\mathcal{N}(B) \prec X/\overline{\mathcal{R}(A)}$. It follows that $X/\mathcal{R}(A)$ is an infinite dimensional space, and hence $X/[\mathcal{R}(A) + C(\mathcal{N}(B))]$ is infinite dimensional. Let $z_1, \dots, z_n \in X$ be linearly independent modulo $(X \oplus Y)/\mathcal{R}(M_C)$. Suppose that there exists complex numbers $\alpha_1, \dots, \alpha_n$, such that $\alpha_1 z_1 + \cdots + \alpha_n z_n = z \in \mathcal{R}(M_C)$. Then there exists a vector $x \in Z$ such that $M_C z = x$. We can find $u \in X$ and $v \in Y$ such that $x = u + v$. Since $z = (Au + Cv) + Bv \in X$, $Au + Cv \in X$ and $Bv \in Y$, we get $Bv = 0$. Thus, $\alpha_1 z_1 + \cdots + \alpha_n z_n = z \in \mathcal{R}(A) + C(\mathcal{N}(B))$. This is in contradiction with the choice of $z_1, \dots, z_n$, so $z_1, \dots, z_n \in X$ must be linearly independent modulo $\mathcal{R}(M_C)$. It follows that $Z/\mathcal{R}(M_C)$ is an infinite dimensional space, so $M_C \notin \Phi_r(Z)$. This is in contradiction with our previous assumption $M_C \in \Phi_r(Z)$. Thus we have proved that the second statement in 3) holds.

3) $\Longrightarrow$ 1): This is obvious. □

As a corollary we get the following result.

Corollary 12.4. *Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ be given. Then*

$$\begin{aligned} \sigma_{re}(B) \cup \{\lambda \in \sigma_{re}(A) : R(A-\lambda) \text{ is closed and } \mathcal{N}(B-\lambda) \prec X/\mathcal{R}(A-\lambda)\} \\ \subset \bigcap_{C \in \mathcal{L}(Y,X)} \sigma_{re}(M_C) \\ \subset \sigma_{re}(B) = \cup\{\lambda \in \sigma_{re}(A) : \mathcal{R}(A-\lambda) \text{ is not closed and complemented}\} \\ \cup\{\lambda \in \sigma_{re}(A) : X/\overline{\mathcal{R}(A-\lambda)} \preceq \mathcal{N}(B-\lambda) \text{ does not hold}\}. \end{aligned}$$

Analogously we can prove similar results for the left Fredholm spectrum.

Theorem 12.7. *Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ be given operators and consider the following statements*

1) *$A \in \Phi_l(X)$ and [$B \in \Phi_l(Y)$, or ($\mathcal{R}(B)$ and $\mathcal{N}(B)$ are closed and complemented subspaces of Y and $\mathcal{N}(B) \preceq X/\overline{\mathcal{R}(A)}$)].*
2) *$M_C \in \Phi_l(Z)$ for some $C \in \mathcal{L}(Y,X)$.*
3) *$A \in \Phi_l(X)$ and [$B \in \Phi_l(Y)$, or $\mathcal{R}(B)$ is not closed, or $\mathcal{R}(A)^\circ \prec \mathcal{N}(B)'$ does not hold)].*

Then 1) $\Longrightarrow$ 2) $\Longrightarrow$ 3).

Proof. 1) $\Longrightarrow$ 2): If $A \in \Phi_l(X)$ and $B \in \Phi_l(Y)$, then we get that $M_C \in \Phi_l(Z)$ for every $C \in \mathcal{L}(Y,X)$. Otherwise, let 1) hold and $B \notin \Phi_l(Y)$. There exist closed subspace U of X and V, W of Y, such that $\mathcal{R}(A) \oplus U = X$ and $\mathcal{N}(B) \oplus V = \mathcal{R}(B) \oplus W = Y$. Let $J \colon \mathcal{N}(B) \to U$ be an arbitrary left invertible operator. There exists a closed subspace Z such that $\mathcal{R}(J) \oplus Z = U$. Define $C \in \mathcal{L}(Y,X)$ as follows

$$C = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ V \end{bmatrix} \to \begin{bmatrix} U \\ \mathcal{R}(A) \end{bmatrix}.$$

Then $\mathcal{R}(M_C) = \mathcal{R}(A) \oplus \mathcal{R}(J) \oplus \mathcal{R}(B)$. From the decomposition

$$X \oplus Y = \mathcal{R}(A) \oplus \mathcal{R}(J) \oplus U \oplus \mathcal{R}(B) \oplus W$$

it follows that $\mathcal{R}(M_C)$ is closed. Also, it is easy to verify $\mathcal{N}(M_C) = \mathcal{N}(A)$. Hence, $M_C \in \Phi(Z)$.

2) $\Longrightarrow$ 3): Suppose that $M_C \in \Phi(Z)$ for some $C \in \mathcal{L}(Y,X)$. If $f \in X'$, we can take $f|_Y = 0$. Hence, $X \oplus Y' = X \oplus Y'$. Notice that

$$M_C' = \begin{bmatrix} A' & 0 \\ C' & B' \end{bmatrix} \in \Phi_r(X' \oplus Y').$$

We can prove $A' \in \Phi_r(X')$, so $A \in \Phi_l(X)$. Thus, the first statement of 3) is proved.

Suppose that the second part of 3) does not hold. Then $B \notin \Phi_l(Y)$, $\mathcal{R}(B)$ is closed and $\mathcal{R}(A)^\circ \prec \mathcal{N}(B)'$ holds. Notice that $\mathcal{R}(A)^\circ = \mathcal{N}(A')$ and

$$\mathcal{N}(B)' \cong Y'/\mathcal{N}(B)^\circ = Y'/\mathcal{R}(B').$$

Since $B \notin \Phi_l(Y)$, we know that $Y'/\mathcal{R}(B')$ is infinite dimensional. We can prove that $(X' \oplus Y')/\mathcal{R}(M_C')$ is infinite dimensional. Hence, $M_C' \notin \Phi_r(X' \oplus Y')$ and $M_C \notin \Phi_r(X \oplus Y)$. Thus, the second statement of 3) is proved.

3) $\Longrightarrow$ 1): This is obvious. ☐

The following result concerning the perturbation of the left Fredholm spectrum holds.

Corollary 12.5. *Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ be given. Then*

$$\begin{aligned} &\sigma_{le}(A) \cup \{\lambda \in \sigma_{le}(B) : \mathcal{R}(B-\lambda) \textit{ is closed and } \mathcal{R}(A-\lambda)^\circ \prec \mathcal{N}(B-\lambda)'\} \\ &\quad \subset \bigcap_{C \in \mathcal{L}(Y,X)} \sigma_{le}(M_C) \subset \sigma_{le}(A) \cup \{\lambda \in \sigma_{le}(B) : \mathcal{R}(B-\lambda) \textit{ and } \mathcal{N}(B-\lambda) \\ &\qquad\qquad \textit{are not closed and complemented}\} \end{aligned}$$

$$\cup\{\lambda \in \sigma_{le}(B) : \mathcal{N}(B-\lambda) \preceq X/\overline{\mathcal{R}(A-\lambda)} \textit{ does not hold}\}.$$

Remark 12.2. Notice that the mapping $\mathcal{L}(Y,X) \ni T \mapsto T' \in \mathcal{L}(X',Y')$ is injective, but not necessarily surjective.

Finally, we get the result for perturbations of the Fredholm spectrum for Hilbert space operators. This result can also be obtained from Corollary 12.2.

Corollary 12.6. *Let $H \oplus K$ be the orthogonal sum of infinite dimensional Hilbert spaces. Then*

$$\bigcap_{C\in\mathcal{L}(K,H)} \sigma_e(M_C) = \sigma_{le}(A) \cup \sigma_{re}(B) \cup \mathcal{W}_2(A,B),$$

where

$$\mathcal{W}_2(A,B) = \{\lambda \in \mathbb{C} : \dim \mathcal{N}(B-\lambda) \neq \dim \mathcal{R}(A-\lambda)^{\perp} \\ \textit{and at least one of these spaces is infinite dimensional}\}.$$

12.4. Perturbation of the left and right spectra. We begin with the following statement.

Lemma 12.4. *Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ be given. Then the inclusion*

$$\sigma_l(M_C) \subset \sigma_l(A) \cup \sigma_l(B)$$

holds for every $C \in \mathcal{L}(Y,X)$. Particularly, if A, B are left invertible, then M_C is left invertible for every $C \in \mathcal{L}(Y,X)$.

For the left invertibility of an operator matrix we can prove the following result.

Theorem 12.8. *Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ be given. Consider the following statements*

1) *$A \in \mathcal{G}_l(X)$, $\mathcal{N}(B) \preceq X/\overline{\mathcal{R}(A)}$ and B is generalized invertible.*
2) *$M_C \in \mathcal{G}_l(Z)$ for some $C \in \mathcal{L}(Y,X)$.*
3) *$A \in \mathcal{G}_l(X)$ and $X/\overline{\mathcal{R}(A)}$ does not hold.*

Then 1) $\Longrightarrow$ 2).

Moreover, if H, K are infinite dimensional Hilbert spaces, and $Z = H \oplus H$ is the orthogonal sum, then 2) $\Longrightarrow$ 3).

Proof. 1) $\Longrightarrow$ 2): Assume that $A \in \mathcal{G}_l(X)$, $\mathcal{N}(B) \preceq X/\mathcal{R}(A)$ and B is generalized invertible. Let $B_1 \in \mathcal{L}(Y)$ denote a generalized inverse of B. Then $Y = \mathcal{R}(B_1) \oplus \mathcal{N}(B)$. Let $A_1 \in \mathcal{L}(X)$ be left inverse of A. Then $X = \mathcal{N}(A_1) \oplus \mathcal{R}(A)$. Let $J : \mathcal{N}(B) \to \mathcal{N}(A_1)$ be a left invertible mapping and let $J_1 : \mathcal{N}(A_1) \to \mathcal{N}(B)$ denote a left inverse of J. Hence, $\mathcal{N}(A_1) = \mathcal{R}(J) \oplus \mathcal{N}(J_1)$. Define $C \in \mathcal{L}(Y,X)$ as follows

$$C = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{R}(B_1) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(A_1) \\ \mathcal{R}(A) \end{bmatrix}.$$

We have the decomposition

$$Z = X \oplus Y = \mathcal{R}(A) \oplus \mathcal{R}(J) \oplus \mathcal{N}(J_1) \oplus \mathcal{R}(B) \oplus \mathcal{N}(B_1).$$

It follows that $\mathcal{R}(M_C) = \mathcal{R}(A) \oplus \mathcal{R}(J) \oplus \mathcal{R}(B)$ is closed.

Define $C_1 \in \mathcal{L}(Y, X)$ in the following way

$$C_1 = \begin{bmatrix} J_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(A_1) \\ \mathcal{R}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{R}(B_1) \end{bmatrix}.$$

Consider the operator

$$N = \begin{bmatrix} A_1 & 0 \\ C_1 & B_1 \end{bmatrix} \in \mathcal{L}(X \oplus Y).$$

Then we have

$$NM_C = \begin{bmatrix} A_1 A & A_1 C \\ C_1 A & C_1 C + B_1 B \end{bmatrix}.$$

Since $\mathcal{R}(C) \subset \mathcal{N}(A_1)$ we get $A_1 A = I$. From $\mathcal{R}(A) \subset \mathcal{N}(C_1)$ we conclude $C_1 A = 0$. Also, $B_1 B$ is the projection from Y onto $\mathcal{R}(B_1)$ parallel to $\mathcal{N}(B)$, and $C_1 C$ is the projection from Y onto $\mathcal{N}(B)$ parallel to $\mathcal{R}(B_1)$. Hence, $C_1 C + B_1 B = I$ and N is the left inverse of M_C. Thus, 2) is proved.

2) $\implies$ 3): Let H, K be Hilbert spaces and let M_C be left invertible. It immediately follows that A is left invertible. Hence, the first part of 3) is proved.

Suppose that $H/\mathcal{R}(A) \prec \mathcal{N}(B)$ holds, i.e. $\dim \mathcal{R}(A)\perp < \dim \mathcal{N}(B)$.

Assume that $\mathcal{N}(C) \cap \mathcal{N}(B) \neq \{0\}$. Then for all non-zero vectors $z \in \mathcal{N}(C) \cap \mathcal{N}(B)$ we have $M_C z = 0$. We conclude that M_C is not one-to-one and $M_C \notin \mathcal{G}_l(H \oplus K)$.

We conclude that $\mathcal{N}(C) \cap \mathcal{N}(B) = \{0\}$. Hence, $C|_{\mathcal{N}(B)}$ is one-to-one. By [28, Problem 42] it follows that $\dim \overline{C(\mathcal{N}(B))} = \dim \mathcal{N}(B)$. Hence,

$$\dim \overline{C(\mathcal{N}(B))} = \mathrm{nul}(B) > \mathrm{def}(A).$$

Since $\mathcal{R}(A)$ is closed, we get $\mathcal{R}(A) \cap \overline{C(\mathcal{N}(B))} \neq \{0\}$. We take a non-zero vector $y_1 \in \mathcal{R}(A) \cap \overline{C(\mathcal{N}(B))}$. There exist: some $y_2 \in H$ and a sequence $(z_n)_n$ in $\mathcal{N}(B)$ such that $Ay_2 = y_1 = \lim_{n\to\infty} Cz_n$. Obviously, $\lim_{n\to\infty} z_n \neq 0$, so we can assume that there exists an $\epsilon > 0$ such that for every n we have $\|z_n\| \geqslant \epsilon$. Notice that $\|y_2 - z_n\| \geqslant \sqrt{\|y_2\|^2 + \epsilon^2}$. Now,

$$\lim_{n\to\infty} \left\| M_C \frac{y_2 - z_n}{\|y_2 - z_n\|} \right\| \leqslant \frac{1}{\sqrt{\|y_2\|^2 + \epsilon^2}} \lim_{n\to\infty} \|Ay_2 - Cz_n - Bz_n\| = 0.$$

It follows that $M_C \notin \mathcal{G}_l(H \oplus K)$. Thus, the second part of 3) is proved. □

As a corollary we get the following result.

Corollary 12.7. *Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ be given. Then the following inclusion holds*

$$\bigcap_{C \in \mathcal{L}(Y,X)} \sigma_l(M_C) \subset \sigma_l(A) \cup \sigma_g(B) \cup \{\lambda \in \mathbb{C} : \mathcal{N}(B-\lambda) \preceq X/\overline{\mathcal{R}(A-\lambda)} \text{ does not hold.}$$

If $H \oplus K$ is the orthogonal sum of infinite dimensional Hilbert spaces H and K, then

$$\sigma_l(A) \cup \{\lambda \in \mathbb{C} : \dim \mathcal{R}(A)^\perp < \dim \mathcal{N}(B-\lambda)\} \subset \bigcap_{C \in \mathcal{L}(K,H)} \sigma_l(M_C).$$

Analogously, we can prove a similar result concerning the right spectrum and right invertibility of M_C.

Theorem 12.9. *Let $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ be given operators, and consider statements*

1) $B \in \mathcal{G}_r(Y)$, $X/\overline{\mathcal{R}(A)} \preceq \mathcal{N}(B)$, *$A$ is generalized invertible.*
2) $M_C \in \mathcal{G}_r(X \oplus Y)$ *for some* $C \in \mathcal{L}(Y, X)$.
3) $B \in \mathcal{G}_r(Y)$, *and* $\mathcal{N}(B) \prec X/\overline{\mathcal{R}(A)}$ *does not hold.*

Then 1) $\implies$ 2).

If $H \oplus K$ is the orthogonal sum of infinite dimensional Hilbert spaces, then 2) $\implies$ 3).

Proof. 1) $\implies$ 2) Let AB_1 be a right inverse of B, and let A_1 be a generalized inverse of A. Then $X = \mathcal{R}(A) \oplus \mathcal{N}(A_1)$ and $Y = \mathcal{N}(B) \oplus \mathcal{R}(B_1) = \mathcal{R}(B) \oplus \mathcal{N}(B_1)$. There exists a left invertible operator $J\colon \mathcal{N}(A_1) \to \mathcal{N}(B)$ and denote by $J_1\colon \mathcal{N}(B) \to \mathcal{N}(A_1)$ its left inverse. Define an operator $C \in \mathcal{L}(Y, X)$ in the following way

$$C = \begin{bmatrix} J_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ \mathcal{R}(B_1) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(A_1) \\ \mathcal{R}(A) \end{bmatrix}.$$

Then $\mathcal{R}(M_C) = X \oplus Y$. Since $\mathcal{N}(M_C) = \mathcal{N}(A)$, from the decomposition

$$X \oplus Y = \mathcal{N}(A) \oplus \mathcal{R}(A_1) \oplus Y$$

we conclude that $\mathcal{N}(M_C)$ is a complemented subspace of $X \oplus Y$. Hence, M_C is right invertible and 2) is proved.

2) $\implies$ 3): Let H, K be Hilbert spaces and let M_C be right invertible. It follows that $B \in \mathcal{G}_r(K)$, so the first part of 3) is proved.

Assume that the second part of 3) is not satisfied, i.e. $\dim \mathcal{N}(B) < \dim \mathcal{R}(A)^{\perp}$. Consider the conjugate operator M_C

$$M_C = \begin{bmatrix} A^* & 0 \\ C^* & B^* \end{bmatrix} \in \mathcal{L}(H \oplus K).$$

If $\mathcal{N}(C^*) \cap \mathcal{N}(A^*) \neq 0$, then there exists some $z \in \mathcal{N}(C^*) \cap \mathcal{N}(A^*)$ and $z \neq 0$. It follows that $M_C^* z = 0$, M_C^* is not left invertible and hence M_C is not right invertible.

We conlcude that $\mathcal{N}(C^*) \cap \mathcal{N}(A^*) = \{0\}$ holds. Then

$$\dim \overline{C(\mathcal{N}(A^*))} = \dim \mathcal{N}(A^*) > \dim \mathcal{N}(B) = \dim \mathcal{R}(B^*)^{\perp}.$$

Since $\mathcal{R}(B)$ is closed, we obtain

$$\overline{C(\mathcal{N}(A^*))} \cap \mathcal{R}(B) \neq \{0\}.$$

We can prove that $M_C^* \notin \mathcal{G}_l(H \oplus K)$ holds similalry as in the proof of Theorem 12.8. Thus the second part of 3) is proved. ☐

As a corollary we get the following result.

Corollary 12.8. *For given $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ the following inclusion holds*

$$\bigcap_{C \in \mathcal{L}(Y,X)} \sigma_r(M_C) \subset \sigma_r(B) \cup \sigma_g(A) \cup \big\{\lambda \in \mathbb{C} : X/\overline{\mathcal{R}(A-\lambda)} \preceq \mathcal{N}(B-\lambda) \text{ does not hold}\big\}.$$

Moreover, if $H \oplus K$ is the orthogonal sum of infinite dimensional Hilbert spaces, then

$$\sigma_r(B) \cup \{\lambda \in \mathbb{C} : \dim \mathcal{N}(B-\lambda) < \dim \mathcal{R}(A-\lambda)^{\perp}\} \subset \bigcap_{C \in \mathcal{L}(K,H)} \sigma_r(M_C).$$

As a corollary, we get the following main result.

Corollary 12.9. *Let $H \oplus K$ be the orthogonal sum of infinite dimensional Hilbert spaces. For given $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(K)$ the following equality holds*

$$\bigcap_{C \in \mathcal{L}(K,H)} \sigma(M_C) = \sigma_l(A) \cup \sigma_r(B) \cup \{\lambda \in \mathbb{C} : \dim \mathcal{N}(B-\lambda) \neq \dim \mathcal{R}(A-\lambda)^{\perp}\}.$$

12.5. Special classes of operators. In this subsection we will consider special classes of operators and related results.

Theorem 12.10. *Let H, K be infinite dimensional Hilbert spaces, $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(K)$.*

If $A \in \mathcal{S}_+(H)$ and $B \in \mathcal{S}_-(K)$, then for every $C \in \mathcal{L}(K,H)$ we have $\sigma_l(M_C) = \sigma_l(A) \cup \sigma_l(B)$.

If $A \in \mathcal{S}_+(H)$ or $B \in \mathcal{S}_-(K)$, then $\sigma(M_C) = \sigma(A) \cup \sigma(B)$.

Proof. Since $\sigma_l(A) \subset \sigma_l(M_C)$, by Proposition 12.4 it is enough to prove that $\sigma_l(B) \subset \sigma_l(M_C)$. Suppose that $\lambda \in \sigma_l(B) \setminus \sigma_l(M_C)$. From Corollary 12.7 we get that $A-\lambda$ is left invertible and $\dim \mathcal{N}(B-\lambda) \leqslant \dim \mathcal{R}(A-\lambda)^{\perp}$. Since $A \in \mathcal{S}_+(X)$, we conclude $\operatorname{def}(A-\lambda) \leqslant \operatorname{nul}(A-\lambda) = 0$. Now $\operatorname{nul}(N-\lambda) = 0$ and $\operatorname{def}(B-\lambda) = 0$. Hence, $A-\lambda$ and $B-\lambda$ are invertible and $M_C-\lambda$ must be invertible. Thus, the equality $\sigma_l(M_C) = \sigma_l(A) \cup \sigma_l(B)$ is proved.

To prove the second equality, notice that $\sigma(M_C) \subset \sigma(A) \cup \sigma(B)$. Let $\lambda \in (\sigma(A) \cup \sigma(B)) \setminus \sigma(M_C)$. From Corollary 12.9 we get that $A-\lambda$ is left invertible, $B-\lambda$ is right invertible and $\operatorname{nul}(B-\lambda) = \operatorname{def}(A-\lambda)$.

If $A \in \mathcal{S}_-(X)$, then we get

$$\operatorname{nul}(B-\lambda) = \operatorname{def}(A-\lambda) \leqslant \operatorname{nul}(A-\lambda) = 0.$$

Hence, $A-\lambda$ and $B-\lambda$ are invertible, which is in contradiction with the assumption $\lambda \in \sigma(A) \cup \sigma(B)$.

If $B \in \mathcal{S}_-(K)$, then

$$\operatorname{def}(B-\lambda) = \operatorname{nul}(B-\lambda) \leqslant \operatorname{def}(B-\lambda) = 0.$$

We also get that $A-\lambda$ and $B-\lambda$ are invertible.

Thus, $\sigma(M_C) = \sigma(A) \cup \sigma(B)$ for every $C \in \mathcal{L}(K,H)$. □

Finally, we consider four block operator matrices. For given $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(K)$ and $C \in \mathcal{L}(K,H)$, we take $T \in \mathcal{L}(H,K)$ and

$$G_T = \begin{bmatrix} A & C \\ T & B \end{bmatrix} \in \mathcal{L}(H \oplus K).$$

We prove the following result.

Theorem 12.11. *Let $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(K)$, $C \in \mathcal{L}(K,H)$ be given operators, and let $\lambda \in \mathbb{C} \setminus \sigma_l(A)$.*

1) *If $\mathcal{N}(C) \cap \mathcal{N}(B-\lambda) \neq \{0\}$, then $\lambda \in \sigma_p(G_T)$ for every $T \in \mathcal{L}(H,K)$.*
2) *If $\mathcal{R}(A-\lambda) \cap \mathcal{R}(C) \neq \{0\}$, then there exists a rank-one operator $T \in \mathcal{L}(H,K)$ such that $\lambda \in \sigma_p(G_T)$.*
3) *If neither* 1) *nor* 2) *is satisfied, then $\lambda \notin \sigma_p(G_T)$ for every $T \in \mathcal{L}(H,K)$.*

Proof. To prove 1), suppose that $\mathcal{N}(C) \cap \mathcal{N}(B-\lambda) \neq \{0\}$. There exists a non-zero vector $v \in \mathcal{N}(C) \cap \mathcal{N}(B-\lambda)$, so $(G_T - \lambda)v = 0$ for every $T \in \mathcal{L}(H,K)$, and consequently $\lambda \in \sigma_p(G_T)$.

To prove 2), suppose that $\mathcal{R}(A-\lambda) \cap \mathcal{R}(C) \neq \{0\}$. Let us take an arbitrary non-zero vector $z \in \mathcal{R}(A-\lambda) \cap \mathcal{R}(C)$. There exists an operator $A_1 \colon \mathcal{R}(A-\lambda) \to H$ such that $A_1(A-\lambda) = I_H$ and $(A-\lambda)A_1 = I_{\mathcal{R}(A-\lambda)}$. There exist vectors: $x_1 = A_1 z \in H$, and $x_2 \in K$, such that $Cx_2 = z$. We define a rank-one operator $T \in \mathcal{L}(H,K)$, such that for every $x \in H$ we have

$$T(x) = \frac{1}{\|x_1\|^2} \langle x, x_1 \rangle (B-\lambda) x_2.$$

Taking $x = -x_1 + x_2$, we get $(G_T - \lambda)x = 0$, so $\lambda \in \sigma_p(G_T)$.

To prove 3), suppose that neither 1) nor 2) is satisfied. Let $0 \neq x \in \mathcal{N}(G_T - \lambda)$ for some $T \in \mathcal{L}(H,K)$. Then $x = u + v$, $u \in H$, $v \in K$, and

$$(A-\lambda)u + Cv = Tu + (B-\lambda)v.$$

Since $\mathcal{R}(A-\lambda) \cap \mathcal{R}(C) = \{0\}$, we get $(A-\lambda)u = Cv = 0$. Also, $u = 0$, $v \in \mathcal{N}(C) \cap \mathcal{N}(B-\lambda)$ and $v = 0$. The obtained contradiction completes the proof. □

13. The pseudospectrum and the condition spectrum

The pseudospectrum and the condition spectrum were studied in [44, 54, 55].

Definition 13.1. [55] (Pseudospectrum) Let $\epsilon > 0$. The ϵ-pseudospectrum of an element $a \in \mathcal{A}$ is defined as

$$\Lambda_\epsilon(a) = \left\{ z \in \mathbb{C} \mid a - z \text{ is not invertible or } \|(a-z)^{-1}\| \geqslant \epsilon \right\}.$$

Definition 13.2. [44] (Condition spectrum) Let $0 < \epsilon < 1$. The ϵ-condition spectrum of an element $a \in \mathcal{A}$ is defined as

$$\sigma_\epsilon(a) = \left\{ z \in \mathbb{C} \mid a - z \text{ is not invertible or } \|(a-z)^{-1}\| \cdot \|a - z\| \geqslant 1/\epsilon \right\}.$$

The pseudospectrum is used in numerical calculations, while the conditional spectrum is useful in finding the numerical solution of operator equations. Let

$A \in \mathcal{L}(X)$ be linear bounded operator on Banach space X and $x, y \in X$. Consider the equation

$$Ax - \lambda x = y. \tag{13.1}$$

It holds

- $\lambda \notin \sigma(A)$ implies that equation (13.1) has a solution,
- $\lambda \notin \sigma_\epsilon(A)$ implies that equation (13.1) has a stable solution.

We generalize the pseudospectrum and the condition spectrum, and we formulate (p, q)-pseudospectrum and (p, q)-condition spectrum as follows

Definition 13.3. ((p, q)-pseudospectrum) Let $\epsilon > 0$. The $(p, q)-\epsilon$-pseudospectrum of an element $a \in \mathcal{A}$ is defined as

$$\Lambda_\epsilon(a) = \left\{ z \in \mathbb{C} \mid a - z \notin \mathcal{A}^{(2)}_{p,q} \text{ or } \|(a - z)^{(2)}_{p,q}\| \geqslant \epsilon \right\}.$$

Definition 13.4. ((p, q)-condition spectrum) Let $0 < \epsilon < 1$. The $(p, q) - \epsilon$-condition spectrum of an element $a \in \mathcal{A}$ is defined as

$$\sigma_{(p,q)-\epsilon}(a) = \left\{ z \in \mathbb{C} \mid a - z \notin \mathcal{A}^{(2)}_{p,q} \text{ or } \|(a - z)^{(2)}_{p,q}\| \cdot \|a - z\| \geqslant 1/\epsilon \right\}.$$

Notice that the uniqueness of $a^{(2)}_{p,q}$ allows us to consider the (p, q)-pseudospectrum and (p, q)-condition spectrum.

If $x = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_u \in \mathcal{A}$ relative to the idempotent $u \in \mathcal{A}$, then the norm of x can be define as

$$\|x\| = \max\{\|a\|, \|b\|\}.$$

Now, we state an auxiliary result.

Lemma 13.1. *Let $x = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_u \in \mathcal{A}$ relative to the idempotent $u \in \mathcal{A}$, $p_1, q_1 \in (u\mathcal{A}u)^\bullet$ and $p_2, q_2 \in ((1-u)\mathcal{A}(1-u))^\bullet$ and let $p = p_1+p_2 \in \mathcal{A}$ and $q = q_1+q_2 \in \mathcal{A}$. Then $x \in \mathcal{A}^{(2)}_{p,q}$ if and only if $a \in (u\mathcal{A}u)^{(2)}_{p_1,q_1}$ and $b \in ((1-u)\mathcal{A}(1-u))^{(2)}_{p_2,q_2}$. If $x \in \mathcal{A}^{(2)}_{p,q}$, then*

$$x^{(2)}_{p,q} = \begin{bmatrix} a^{(2)}_{p_1,q_1} & 0 \\ 0 & b^{(2)}_{p_2,q_2} \end{bmatrix}_u.$$

Proof. By Lemma 9.2 we obtain that p and q are idempotents. If $a \in (u\mathcal{A}u)^{(2)}_{p_1,q_1}$ and $b \in ((1-u)\mathcal{A}(1-u))^{(2)}_{p_2,q_2}$, by Theorem 9.4, we obtain $x \in \mathcal{A}^{(2)}_{p,q}$. If $x \in \mathcal{A}^{(2)}_{p,q}$, there exists the element $y = \begin{bmatrix} a_1 & c \\ d & b_1 \end{bmatrix}_u \in \mathcal{A}$ such that $y = x^{(2)}_{p,q}$. The equation $yxy = y$ is equivalent to equations

$$a_1aa_1 + cbd = a_1, \quad a_1ac + cbb_1 = c, \quad daa_1 + b_1bd = d, \quad dac + b_1bb_1 = b_1.$$

Also, $yx = p$ is equivalent to

$$a_1a = p_1, \quad cb = 0, \quad da = 0, \quad b_1b = p_2$$

and $1 - xy = q$ is equivalent to

$$u - aa_1 = q_1, \quad ac = 0, \quad bd = 0, \quad (1 - u) - bb_1 = q_2.$$

The equations $a_1ac + cbb_1 = c$, $cb = 0$ and $ac = 0$ imply $c = 0$. Analogously, $daa_1 + b_1bd = d$, $da = 0$ and $bd = 0$ imply $d = 0$. Now, we have the equations

$$a_1aa_1 = a_1, \quad a_1a = p_1, \quad u - aa_1 = q_1,$$
$$b_1bb_1 = b_1, \quad b_1b = p_2, \quad (1-u) - bb_1 = q_2$$

proving $a_1 = a^{(2)}_{p_1,q_1}$ and $b_1 = b^{(2)}_{p_2,q_2}$. Furthermore, if $x \in \mathcal{A}^{(2)}_{p,q}$, then

$$x^{(2)}_{p,q} = \begin{bmatrix} a^{(2)}_{p_1,q_1} & 0 \\ 0 & b^{(2)}_{p_2,q_2} \end{bmatrix}_u . \qquad \square$$

As a corollary, we have the following result for the invertibility of an element $x = \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]_u \in \mathcal{A}$ relative to the idempotent $u \in \mathcal{A}$.

Lemma 13.2. *Let $x = \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]_u \in \mathcal{A}$ relative to the idempotent $u \in \mathcal{A}$. Then $x \in \mathcal{A}^{-1}$ if and only if $a \in (u\mathcal{A}u)^{-1}$ and $b \in ((1-u)\mathcal{A}(1-u))^{-1}$. If $x \in \mathcal{A}^{-1}$, then*

$$x^{-1} = \begin{bmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{bmatrix}_u .$$

Therefore, for the spectrum of an element $x = \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]_u \in \mathcal{A}$, the following holds

$$\sigma(x) = \sigma(a) \cup \sigma(b).$$

We investigate whether the similar property holds for the pseudospectrum and condition spectrum. We formulate the following results.

Theorem 13.1. *Let $x = \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]_u \in \mathcal{A}$ relative to the idempotent $u \in \mathcal{A}$, $\epsilon > 0$, $p_1, q_1 \in (u\mathcal{A}u)^\bullet$ and $p_2, q_2 \in ((1-u)\mathcal{A}(1-u))^\bullet$ and let $p = p_1 + p_2 \in \mathcal{A}$ and $q = q_1 + q_2 \in \mathcal{A}$. Then*

$$\Lambda_{(p,q)-\epsilon}(x) = \Lambda_{(p_1,q_1)-\epsilon}(a) \cup \Lambda_{(p_2,q_2)-\epsilon}(b).$$

Proof. Let $z \in \Lambda_{(p,q)-\epsilon}(x)$. Then $x - z \notin \mathcal{A}^{(2)}_{p,q}$ or $\|(x-z)^{(2)}_{p,q}\| \geqslant \epsilon$. If $x - z = \left[\begin{smallmatrix} a-zu & 0 \\ 0 & b-z(1-u) \end{smallmatrix}\right]_u \notin \mathcal{A}^{(2)}_{p,q}$, by Lemma 13.1, we obtain that $a - zu \notin (u\mathcal{A}u)^{(2)}_{p_1,q_1}$ or $b - z(1-u) \notin ((1-u)\mathcal{A}(1-u))^{(2)}_{p_2,q_2}$. It implies $z \in \Lambda_{(p_1,q_1)-\epsilon}(a)$ or $z \in \Lambda_{(p_2,q_2)-\epsilon}(b)$, so $z \in \Lambda_{(p_1,q_1)-\epsilon}(a) \cup \Lambda_{(p_2,q_2)-\epsilon}(b)$. If $x - z = \left[\begin{smallmatrix} a-zu & 0 \\ 0 & b-z(1-u) \end{smallmatrix}\right]_u \in \mathcal{A}^{(2)}_{p,q}$, we have

$$(x-z)^{(2)}_{p,q} = \begin{bmatrix} (a-zu)^{(2)}_{p_1,q_1} & 0 \\ 0 & (b-z(1-u))^{(2)}_{p_2,q_2} \end{bmatrix}_u \quad \text{and}$$
$$\|(x-z)^{(2)}_{p,q}\| = \max\{\|(a-zu)^{(2)}_{p_1,q_1}\|, \|(b-z(1-u))^{(2)}_{p_2,q_2}\|\} \geqslant \epsilon.$$

By Lemma 13.1, we conclude that

$$a - zu \in (u\mathcal{A}u)^{(2)}_{p_1,q_1} \text{ and } b - z(1-u) \in ((1-u)\mathcal{A}(1-u))^{(2)}_{p_2,q_2}.$$

The assumption $\max\{\|(a-zu)^{(2)}_{p_1,q_1}\|, \|(b-z(1-u))^{(2)}_{p_2,q_2}\|\} \geqslant \epsilon$ implies that either $\|(a-zu)^{(2)}_{p_1,q_1}\| \geqslant \epsilon$ or $\|(b-z(1-u))^{(2)}_{p_2,q_2}\| \geqslant \epsilon$ holds. It follows that $z \in \Lambda_{(p_1,q_1)-\epsilon}(a)$ or $z \in \Lambda_{(p_2,q_2)-\epsilon}(b)$, so $z \in \Lambda_{(p_1,q_1)-\epsilon}(a) \cup \Lambda_{(p_2,q_2)-\epsilon}(b)$. We have proved $\Lambda_{(p,q)-\epsilon}(x) \subset \Lambda_{(p_1,q_1)-\epsilon}(a) \cup \Lambda_{(p_2,q_2)-\epsilon}(b)$.

Now, let $z \in \Lambda_{(p_1,q_1)-\epsilon}(a) \cup \Lambda_{(p_2,q_2)-\epsilon}(b)$. It follows

$$a - zu \notin (u\mathcal{A}u)^{(2)}_{p_1,q_1} \text{ or } \|(a-zu)^{(2)}_{p_1,q_1}\| \geqslant \epsilon \quad \text{or}$$
$$b - z(1-u) \notin ((1-u)\mathcal{A}(1-u))^{(2)}_{p_2,q_2} \text{ or } \|(b-z(1-u))^{(2)}_{p_2,q_2}\| \geqslant \epsilon.$$

If either $a - zu \notin (u\mathcal{A}u)^{(2)}_{p_1,q_1}$ or $b - z(1-u) \notin ((1-u)\mathcal{A}(1-u))^{(2)}_{p_2,q_2}$, by Lemma 13.1, it follows $x - z \notin \mathcal{A}^{(2)}_{p,q}$. So, $z \in \Lambda_{(p,q)-\epsilon}(x)$. On the other hand, if

$$a - zu \in (u\mathcal{A}u)^{(2)}_{p_1,q_1} \text{ and } b - z(1-u) \in ((1-u)\mathcal{A}(1-u))^{(2)}_{p_2,q_2},$$

it holds either $\|(a-zu)^{(2)}_{p_1,q_1}\| \geqslant \epsilon$ or $\|(b-z(1-u))^{(2)}_{p_2,q_2}\| \geqslant \epsilon$. Therefore, $\|(x-z)^{(2)}_{p,q}\| = \max\{\|(a-zu)^{(2)}_{p_1,q_1}\|, \|(b-z(1-u))^{(2)}_{p_2,q_2}\|\} \geqslant \epsilon$. This proves that $z \in \Lambda_{(p,q)-\epsilon}(x)$. The inclusion $\Lambda_{(p_1,q_1)-\epsilon}(a) \cup \Lambda_{(p_2,q_2)-\epsilon}(b) \subset \Lambda_{(p,q)-\epsilon}(x)$ has been proved. □

Theorem 13.2. *Let $x = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_u \in \mathcal{A}$ relative to the idempotent $u \in \mathcal{A}$, $0 < \epsilon < 1$, $p_1, q_1 \in (u\mathcal{A}u)^\bullet$ and $p_2, q_2 \in ((1-u)\mathcal{A}(1-u))^\bullet$ and let $p = p_1 + p_2 \in \mathcal{A}$ and $q = q_1 + q_2 \in \mathcal{A}$. Then*

$$\sigma_{(p_1,q_1)-\epsilon}(a) \cup \sigma_{(p_2,q_2)-\epsilon}(b) \subset \sigma_{(p,q)-\epsilon}(x).$$

Proof. Let $z \in \sigma_{(p_1,q_1)-\epsilon}(a) \cup \sigma_{(p_2,q_2)-\epsilon}(b)$. These imply

$$a - zu \notin (u\mathcal{A}u)^{(2)}_{p_1,q_1} \text{ or } \|(a-zu)^{(2)}_{p_1,q_1}\| \cdot \|a - zu\| \geqslant 1/\epsilon$$

or

$$b - z(1-u) \notin ((1-u)\mathcal{A}(1-u))^{(2)}_{p_2,q_2} \text{ or } \|(b-z(1-u))^{(2)}_{p_2,q_2}\| \cdot \|b - z(1-u)\| \geqslant 1/\epsilon.$$

If either $a - zu \notin (u\mathcal{A}u)^{(2)}_{p_1,q_1}$ or $b - z(1-u) \notin ((1-u)\mathcal{A}(1-u))^{(2)}_{p_2,q_2}$, by Lemma 13.1, it follows $x - z \notin \mathcal{A}^{(2)}_{p,q}$. Then, we have $z \in \sigma_{(p,q)-\epsilon}(x)$. On the other hand, if

$$a - zu \in (u\mathcal{A}u)^{(2)}_{p_1,q_1} \text{ and } b - z(1-u) \in ((1-u)\mathcal{A}(1-u))^{(2)}_{p_2,q_2},$$

it holds either

$$\|(a-zu)^{(2)}_{p_1,q_1}\| \cdot \|a - zu\| \geqslant 1/\epsilon \text{ or } \|(b-z(1-u))^{(2)}_{p_2,q_2}\| \cdot \|b - z(1-u)\| \geqslant 1/\epsilon.$$

Without loss of generality, assume that $\|(a-zu)^{(2)}_{p_1,q_1}\| \cdot \|a - zu\| \geqslant 1/\epsilon$ holds. Therefore,

$$\begin{aligned} &\|(x-z)^{(2)}_{p,q}\|\|x-z\| \\ &= \max\{\|(a-zu)^{(2)}_{p_1,q_1}\|, \|(b-z(1-u))^{(2)}_{p_2,q_2}\|\} \cdot \max\{\|a-zu\|, \|b-z(1-u)\|\} \\ &\geqslant \|(a-zu)^{(2)}_{p_1,q_1}\| \cdot \|a-zu\| \geqslant 1/\epsilon. \end{aligned}$$

This proves that $z \in \sigma_{(p,q)-\epsilon}(x)$. □

The next example shows that the converse inclusion is not true in the previous theorem.

Example 13.1. Let $0 < \epsilon < 1$, $z \in \mathbb{C}$ and $u \in \mathcal{A}^\bullet$ such that $\|u\| < \frac{1}{\sqrt{\epsilon}}$ and $\|1-u\| < \frac{1}{\sqrt{\epsilon}}$. Let $x = \left[\begin{smallmatrix} (\epsilon^2+z)u & 0 \\ 0 & (\epsilon+z)(1-u) \end{smallmatrix}\right]_u \in \mathcal{A}$ relative to the idempotent $u \in \mathcal{A}$. Then

$$z \in \sigma_{(1,0)-\epsilon}(x), \text{ but } z \notin (\sigma_{(u,0)-\epsilon}((\epsilon^2+z)u) \cup \sigma_{(1-u,0)-\epsilon}((\epsilon+z)(1-u))).$$

Proof. For idempotents $u \in \mathcal{A}$ and $1-u \in \mathcal{A}$, we have $\|u\| \geqslant 1$ and $\|1-u\| \geqslant 1$. There exists the inverse

$$(x-z)^{(2)}_{1,0} = \begin{bmatrix} \frac{1}{\epsilon^2}u & 0 \\ 0 & \frac{1}{\epsilon}(1-u) \end{bmatrix}_u$$

as well as inverses

$$((\epsilon^2+z)u - zu)^{(2)}_{u,0} = (\epsilon^2 u)^{(2)}_{u,0} = \frac{1}{\epsilon^2}u$$

and

$$((\epsilon+z)(1-u) - z(1-u))^{(2)}_{1-u,0} = (\epsilon(1-u))^{(2)}_{1-u,0} = \frac{1}{\epsilon}(1-u).$$

Now, we have

$$\begin{aligned}\|(x-z)^{(2)}_{1,0}\|\|x-z\| &= \max\{\|u/\epsilon^2\|, \|((1-u)/\epsilon\|\} \cdot \max\{\|\epsilon^2 u\|, \|\epsilon(1-u)\|\} \\ &= \|u/\epsilon^2\| \cdot \|\epsilon(1-u)\| \geqslant |1/\epsilon^2| \cdot |\epsilon| \geqslant 1/\epsilon,\end{aligned}$$

but also

$$\|(\epsilon^2 u)^{(2)}_{u,0}\| \cdot \|\epsilon^2 u\| = \|u/epsilon^2\| \cdot \|\epsilon^2 u\| = \|u\|^2 < 1/\epsilon$$

and

$$\|(\epsilon(1-u))^{(2)}_{1-u,0}\| \cdot \|\epsilon(1-u)\| = \|(1-u)/\epsilon\| \cdot \|\epsilon(1-u)\| = \|1-u\|^2 < 1/\epsilon.$$

Therefore,

$$z \in \sigma_{(1,0)-\epsilon}(x), \text{ but } z \notin (\sigma_{(u,0)-\epsilon}((\epsilon^2+z)u) \cup \sigma_{(1-u,0)-\epsilon}((\epsilon+z)(1-u))). \quad \square$$

If $x \in \mathcal{A}$ is invertible, $p = 1$ and $q = 0$, then $x^{-1} = x^{(2)}_{p,q}$.

As corollaries of Theorems 13.1 and 13.2, we formulate the following results for the pseudospectrum and the condition spectrum.

Theorem 13.3. *Let* $x = \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]_u \in \mathcal{A}$ *relative to the idempotent* $u \in \mathcal{A}$ *and* $\epsilon > 0$. *Then* $\Lambda_\epsilon(x) = \Lambda_\epsilon(a) \cup \Lambda_\epsilon(b)$.

Theorem 13.4. *Let* $x = \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]_u \in \mathcal{A}$ *relative to the idempotent* $u \in \mathcal{A}$ *and* $0 < \epsilon < 1$. *Then* $\sigma_\epsilon(a) \cup \sigma_\epsilon(b) \subset \sigma_\epsilon(x)$.

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