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ON GENERALIZED SPECTRA OF OPERATORS ON HILBERT C^*-MODULES

Abstract. We consider generalized spectra in C^*-algebras of operators on Hilbert C^*-modules and give a description of such spectra of shift operators, unitary, self-adjoint and normal operators on the standard Hilbert C^*-module over a unital C^*-algebra. Moreover, we proceed further by considering the generalized spectra induced by various subclasses of semi-C^*-Fredholm operators and study the relationship between these spectra.

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Contents

1. Generalized spectra of operators over C^*-algebras

Throughout this paper $\mathcal{A}$ always stands for a unital C^*-algebras and $H_{\mathcal{A}}$ denotes the standard module over $\mathcal{A}$. Moreover, we let $B^a(H_{\mathcal{A}})$ denote the set of all $\mathcal{A}$-linear, bounded, adjointable operators on $H_{\mathcal{A}}$. For $\alpha \in \mathcal{A}$ we may let αI be the operator on $H_{\mathcal{A}}$ given by $\alpha I(x_1, x_2, \dots) = (\alpha x_1, \alpha x_2, \dots)$. It is straightforward to check that αI is an $\mathcal{A}$-linear operator on $H_{\mathcal{A}}$. Moreover, αI is bounded and $\|\alpha I\| = \|\alpha\|$. Finally, αI is adjointable and its adjoint is given by $(\alpha I)^* = \alpha^* I$.

Our starting question is the following: If $\mathcal{A}$ is a C^*-algebra, then for $\alpha \in \mathcal{A}$ could we consider the generalized spectra in $\mathcal{A}$ of operators in $B^a(H_{\mathcal{A}})$ by setting for every $F \in B^a(H_{\mathcal{A}})$

$$\sigma^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \text{ is not invertible in } B^a(H_{\mathcal{A}})\}?$$

The main topic in this paper will be to obtain generalization of some results from spectral theory of operators on Hilbert spaces in the setting of generalized spectra in C^*-algebras of operators on Hilbert C^*-modules.

We introduce first the following notion:
$\sigma^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \text{ is not invertible in } B^a(H_{\mathcal{A}})\}$,
$\sigma_p^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid \ker(F - \alpha I) \neq \{0\}\}$,
$\sigma_{rl}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \text{ is bounded below, but not surjective on } H_{\mathcal{A}}\}$,
$\sigma_{cl}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid \operatorname{Im}(F - \alpha I) \text{ is } \underline{\text{not}} \text{ closed}\}$.

It is understood that $F \in B^a(H_{\mathcal{A}})$. Recall that not all closed submodules of $H_{\mathcal{A}}$ are orthogonally complementable in $H_{\mathcal{A}}$, which differs from the situation of Hilbert spaces. It may happen that $\overline{\operatorname{Im}(F - \alpha I)} \oplus \operatorname{Im}(F - \alpha I)^{\perp} \subsetneqq H_{\mathcal{A}}$. However, if $\operatorname{Im}(F - \alpha I)$ is closed, then $\operatorname{Im}(F^* - \alpha^* I)$ is closed and we also have

$$H_{\mathcal{A}} = \operatorname{Im}(F - \alpha I) \oplus \ker(F^* - \alpha^* I) = \ker(F - \alpha I) \oplus \operatorname{Im}(F^* - \alpha^* I)$$

whenever $F \in B^a(H_{\mathcal{A}})$, which follows from the proof of [10] [11, Theorem 2.3.3].

Therefore, it is more convinient in this setting to work with $\sigma_{rl}^{\mathcal{A}}(F)$ and $\sigma_{cl}^{\mathcal{A}}(F)$ for $F \in B^a(H_{\mathcal{A}})$ instead of the residual and the continuous spectrum.

Note that we obviously have

$$\sigma^{\mathcal{A}}(F) = \sigma_p^{\mathcal{A}}(F) \cup \sigma_{rl}^{\mathcal{A}}(F) \cup \sigma_{cl}^{\mathcal{A}}(F) \quad \text{and} \quad \sigma^{\mathcal{A}}(F^*) = (\sigma^{\mathcal{A}}(F))^*.$$

The challenges which arise are the following:

1) $\mathcal{A}$ may be non commutative;

2) If $\mathcal{A}$ is a non trivial C^*-algebra, then there exists certainly nonzero non-invertible elements by the Gelfand-Mazur Theorem [8, Chapter VII, Theorem 8.1]. Moreover, even if $\alpha \in \mathcal{A} \cap G(\mathcal{A})$, we do not have in general that $\|\alpha^{-1}\| = \frac{1}{\|\alpha\|}$. Therefore, $\sigma^{\mathcal{A}}(F)$ may be unbounded. (However, $\sigma^{\mathcal{A}}(F)$ is always closed in $\mathcal{A}$).

In Section 2 we give description of the generalized spectra of shift operators, unitary, self-adjoint and normal operators on $H_{\mathcal{A}}$ and investigate some further properties of these spectra. Most of the results in this section are generalizations of the results from [13, Chapter 4].

In Section 3 we consider generalized Fredholm spectra of operators on the standard Hilbert C^*-module. Fredholm theory on Hilbert C^*-modules as a generalization of Fredholm theory on Hilbert spaces was started by Mishchenko and Fomenko in [12]. They have introduced the notion of a Fredholm operator on the standard module and proved that some of the main results from the classical Fredholm theory hold when one considers this generalization, such as the Atkinson theorem, openness of the set of Fredholm operators etc. In [1–5] we went further in this direction. We defined semi-Fredholm and semi-Weyl operators on the standard module and proved generalized versions in this setting of several results from the classical semi-Fredholm theory on Hilbert spaces. Now, various subclasses of semi-Fredholm operators on $H_{\mathcal{A}}$ induce various corresponding generalized spectra in $\mathcal{A}$ of operators in $B^a(H_{\mathcal{A}})$. In Section 3 we investigate several properties of such spectra and the relationship between them, as a continuation of the research presented in [3] and [5]. Most of the results in this section are generalizations in this setting of the results from [15, Section 2.2 and Section 2.3].

This paper contains some of the unpublished results from the doctoral dissertation by the author, see [7].

2. Generalized spectra of shift operators, unitary, self-adjoint and normal operators

We start with the following proposition.

Proposition 2.1. *Let $\mathcal{A}$ be a unital C^*-algebra, $\{e_k\}_{k\in\mathbb{N}}$ denote the standard orthonormal basis of $H_{\mathcal{A}}$ and S be the operator defined by $Se_k = e_{k+1}, k \in \mathbb{N}$, that is S is a unilateral shift and $S^*e_{k+1} = e_k$ for all $k \in \mathbb{N}$. If $\mathcal{A} = L^\infty((0,1),\mu)$ where μ is the Lebesgue measure, or if $\mathcal{A} = C([0,1])$, then $\sigma^{\mathcal{A}}(S) = \{\alpha \in \mathcal{A} \mid \inf|\alpha| \leqslant 1\}$, where in the case when $\mathcal{A} = L^\infty((0,1),\mu)$, we set*

$$\inf|\alpha| = \inf\{C > 0 \mid \mu(|\alpha|^{-1}([0,C])) > 0\} = \sup\{K > 0 \mid |\alpha| > K \text{ a.e. on } [0,1]\}.$$

Moreover, $\sigma_p^{\mathcal{A}}(S) = \emptyset$ in both cases.

Proof. We have two cases.
Case 1: In this case we consider $\mathcal{A} = C([0,1])$. Let $\alpha \in \mathcal{A}$ and suppose that $\inf|\alpha| < 1$. Since $|\alpha|$ is continuous, we may find an open interval $(t_1, t_2) \subseteq (0,1)$ such that $|\alpha(t)| < 1 - \epsilon$ for all $t \in (t_1, t_2)$, where $0 < \epsilon < 1 - \inf|\alpha|$. We may find some $g \in \mathcal{A}$ such that $\operatorname{supp} g \subseteq (t_1, t_2)$ and $0 \leqslant g \leqslant 1$. Consider

$$x_\alpha = (g, \bar{\alpha}g, \bar{\alpha}^2 g, \dots).$$

Then, obviously, $x_\alpha \in H_{\mathcal{A}}$ and $\langle(\alpha I - S)e_k, x_\alpha\rangle = \bar{\alpha}^k g - \bar{\alpha}^k g = 0$. Hence $x_\alpha \in \operatorname{Im}(\alpha I - S)^\perp$ and $x_\alpha \neq 0$, which gives that $\alpha \in \sigma^{\mathcal{A}}(S)$. Therefore,

$$\{\alpha \in \mathcal{A} \mid \inf|\alpha| < 1\} \subseteq \sigma^{\mathcal{A}}(S).$$

Since $\sigma^{\mathcal{A}}(S)$ is closed in the norm topolgy in $\mathcal{A}$, it follows that

$$\{\alpha \in \mathcal{A} \mid \inf|\alpha| \leqslant 1\} \subseteq \sigma^{\mathcal{A}}(S).$$

On the other hand, if $\alpha \in \mathcal{A}$ and $\inf|\alpha| > 1$, then α is invertible and we have that $\sup|\alpha^{-1}| = \|\alpha^{-1}\| < 1$. It follows that $\|\alpha^{-1}S\| \leqslant \|\alpha^{-1}\|\|S\| < 1$, so $\alpha I - S = \alpha(I - \alpha^{-1}S)$ is invertible in $B^a(H_{\mathcal{A}})$. Next, suppose that $(\alpha I - S)(x) = 0$ for some $\alpha \in \mathcal{A}$ and $x \in H_{\mathcal{A}}$. This gives the following system of equations coordinatewise: $\alpha x_1 = 0, \alpha x_2 - x_1 = 0, \alpha x_3 - x_2 = 0, \dots$. Since $\alpha x_1 = 0$, we deduce that $x_{1|\operatorname{supp}\alpha} = 0$. However, since $\alpha x_2 - x_1 = 0$, it follows that $x_{1|(\operatorname{supp}\alpha)^c} =$ also. Hence $x_1 = 0$. However, then $\alpha x_2 = 0$ and $\alpha x_3 - x_2 = 0$. Using the same argument we obtain that $x_2 = 0$. Proceeding inductively, we obtain that $x_k = 0$ for all k, so $x = 0$. Since $\alpha \in \mathcal{A}$ was arbitrary chosen, we conclude that $\sigma_p^{\mathcal{A}}(S) = \emptyset$.
Case 2: In this case we consider $\mathcal{A} = L^\infty((0,1),\mu)$. Let $\alpha \in \mathcal{A}$ and assume that $\inf|\alpha| < 1$. This means that $\mu(|\alpha|^{-1}([0, 1-\epsilon])) > 0$, where $0 < \epsilon < 1 - \inf|\alpha|$. Set $M_\epsilon = |\alpha|^{-1}([0, 1-\epsilon])$, then $\chi_{M_\epsilon} \neq 0$. Letting χ_{M_ϵ} play the role of the function g in the previous proof, (which is possible since $x_\alpha = (\chi_{M_\epsilon}, \bar{\alpha}\chi_{M_\epsilon}, \bar{\alpha}^2\chi_{M_\epsilon}, \dots) \in H_{\mathcal{A}}$ because $|\alpha| \leqslant 1 - \epsilon$ on M_ϵ), we deduce by the same arguments that

$$\sigma^{\mathcal{A}}(S) = \{\alpha \in \mathcal{A} \mid \inf|\alpha| \leqslant 1\}.$$

Next, assume that $(\alpha I - S)(x) = 0$ for some $\alpha \in \mathcal{A}$ and $x \in H_{\mathcal{A}}$. As in the previous proof we get the system of equations $\alpha x_1 = 0$, $\alpha x_2 - x_1 = 0$, $\alpha x_3 - x_2 = 0, \dots$. The first equation gives that $x_1 = 0$ a.e. on $|\alpha|^{-1}(0,\infty)$, whereas the second equation gives $x_1 = 0$ a.e. on $\alpha^{-1}(\{0\})$. Hence $x_1 = 0$. Proceeding inductively as in the previous proof, we get $x = 0$, hence $\sigma_p^{\mathcal{A}}(S)$ is empty also in this case. □

Lemma 2.1. *Let $\mathcal{A} = B(H)$, $T \in B(H)$ and suppose that T is invertible. Then the equation $(T \cdot I - S)x = y$ has a solution in $H_{\mathcal{A}}$ for all $e_k, k \in \mathbb{N}$, if and only if the sequence $(T^{-1}, T^{-2}, \dots, T^{-k}, \dots)$ belongs to $H_{\mathcal{A}}$.*

Proof. For $k = 1$, if $(T \cdot I - S)x = e_1$, then we must have $TB_1 = I$, where $x = (B_1, B_2, \dots)$. Hence $B_1 = T^{-1}$. Next, $TB_2 - B_1 = 0$, so $TB_2 = B_1 = T^{-1}$ which gives $B_2 = T^{-2}$. Proceeding inductively, we obtain that $B_k = T^{-k}$ for all k. So the equation $(T \cdot I - S)x = e_1$ has a solution in $H_{\mathcal{A}}$ if and only if the sequence

$(T^{-1}, T^{-2}, \dots)$ belongs to $H_{\mathcal{A}}$. Now, if $(T^{-1}, T^{-2}, \dots) \in H_{\mathcal{A}}$, then the sequence $x^{(k)}$ in $H_{\mathcal{A}}$ given by

$$x_n^{(k)} = \begin{cases} 0 & \text{if } \ n \in \{1, \dots, k-1\} \\ T^{-(n-k+1)} & \text{for } \ n \in \{k, k+1, \dots\} \end{cases}$$

is the solution of the equation $(T \cdot I - S)x = e_k$ for each $k \in \mathbb{N}$. □

Set $\tilde{\sigma}_{cl}^{\mathcal{A}}(S) = \{\alpha \in \sigma_{cl}^{\mathcal{A}}(S) \mid \overline{\operatorname{Im}(\alpha I - S)} = H_{\mathcal{A}}\}$. We have the following corollary.

Corollary 2.1. *Let $\mathcal{A}$ be a commutative unital C^*-algebra. Then*

$$\sigma^{\mathcal{A}}(S) = (\mathcal{A} \setminus G(\mathcal{A})) \cup \{\alpha \in G(\mathcal{A}) | (\alpha^{-1}, \alpha^{-2}, \dots, \alpha^{-k}, \dots) \notin H_{\mathcal{A}}\} \cup \tilde{\sigma}_{cl}^{\mathcal{A}}(S).$$

Proof. Since $\mathcal{A}$ is commutative, then the set of right invertible elemnts coincides with $G(\mathcal{A})$. Hence we can apply the arguments from the proof of Lemma 2.1. □

Corollary 2.2. *Let $\mathcal{A}$ be a unital C^*-algebra. If $1_{\mathcal{A}}$ denotes the unit in $\mathcal{A}$, then $1_{\mathcal{A}} \in \sigma^{\mathcal{A}}(S)$.*

Proof. We obviously have that the sequence $(1_{\mathcal{A}}, 1_{\mathcal{A}}, 1_{\mathcal{A}}, \dots) = (1_{\mathcal{A}}^{-1}, 1_{\mathcal{A}}^{-2}, 1_{\mathcal{A}}^{-3}, \dots)$ is not an element of $H_{\mathcal{A}}$. Then we apply the arguments from the proof of Lemma 2.1. □

Example 2.1. We may consider a weighted shift S_w on $H_{\mathcal{A}}$ given by $S_w(x)_{j+1} = w_j x_j$, where $w = (w_1, w_2, \dots)$ is a bounded sequence in $\mathcal{A}$. In this case, if α has a common right annihilator as w_j for some $j \in \mathbb{N}$, then the sequence having this right annihilator in its j-th coordinate and 0 elsewhere belongs to the kernel of $\alpha I - S_w$. Hence $\alpha \in \sigma^{\mathcal{A}}(S_w)$ in this case.

Example 2.2. Let $\mathcal{A} = L^\infty((0,1), \mu)$. Set

$$\tilde{S}(f_1, f_2, \dots) = (f_1 \chi_{(0, \frac{1}{2})}, f_2 \chi_{(0, \frac{1}{2})} + f_1 \chi_{(\frac{1}{2}, 1)}, f_3 \chi_{(0, \frac{1}{2})} + f_2 \chi_{(\frac{1}{2}, 1)}, \dots).$$

Then $\tilde{S}$ has the matrix $\left[\begin{smallmatrix} 1 & 0 \\ 0 & S \end{smallmatrix}\right]$ with respect to the decomposition

$$(H_{\mathcal{A}} \cdot \chi_{(0, \frac{1}{2})}) \oplus (H_{\mathcal{A}} \cdot \chi_{(\frac{1}{2}, 1)}).$$

It follows that

$$\sigma^{\mathcal{A}}(\tilde{S}) = \{\alpha \in \mathcal{A} \mid \inf\{C > 0 \mid \mu(|\alpha|^{-1}([0, C]) \cap (\tfrac{1}{2}, 1))\} \leqslant 1\} \\ \cup \{\alpha \in \mathcal{A} \mid (\alpha - 1) \cdot \chi_{(0, \frac{1}{2})} \text{ is not invertible in } L^\infty((0, \tfrac{1}{2}), \mu)\}.$$

Proposition 2.2. *Let $\alpha \in \mathcal{A}$. We have*

1. *If $\alpha I - F$ is bounded below and $F \in B^a(H_{\mathcal{A}})$, then $\alpha \in \sigma_{rl}^{\mathcal{A}}(F)$ if and only if $\alpha^* \in \sigma_p^{\mathcal{A}}(F^*)$,*
2. *If $F, D \in B^a(H_{\mathcal{A}})$ and $D = U^* F U$ for some unitary operator U, then*

$$\sigma^{\mathcal{A}}(F) = \sigma^{\mathcal{A}}(D), \ \ \sigma_p^{\mathcal{A}}(F) = \sigma_p^{\mathcal{A}}(D), \ \ \sigma_{cl}^{\mathcal{A}}(F) = \sigma_{cl}^{\mathcal{A}}(D) \quad \text{and} \quad \sigma_{rl}^{\mathcal{A}}(F) = \sigma_{rl}^{\mathcal{A}}(D).$$

Proof. 1) Suppose first that $F - \alpha I$ is bounded below and $\alpha \in \sigma_{rl}^{\mathcal{A}}(F)$. Then it follows that $\operatorname{Im}(F - \alpha I)$ is closed. Hence, by [10] [11, Theorem 2.3.3] we have that $H_{\mathcal{A}} = \operatorname{Im}(F - \alpha I) \oplus \operatorname{Im}(F - \alpha I)^{\perp}$ which gives that $\operatorname{Im}(F - \alpha I)^{\perp} \neq \{0\}$ as $\operatorname{Im}(F - \alpha I) \neq H_{\mathcal{A}}$. Since $\operatorname{Im}(F - \alpha I)^{\perp} = \ker(F^* - \alpha^* I)$, it follows that $\alpha^* \in \sigma_p^{\mathcal{A}}(F^*)$. Conversely, suppose that $\alpha^* \in \sigma_p^{\mathcal{A}}(F^*)$ and that $F - \alpha I$ is bounded below. Then, again, $\operatorname{Im}(F - \alpha I)$ is closed and moreover, $\operatorname{Im}(F - \alpha I)^{\perp} = \ker(F^* - \alpha^* I) \neq \{0\}$. It follows that $\alpha \in \sigma_{rl}^{\mathcal{A}}(F)$. It is straightforward to prove the statement 2. □

Now we are going to describe the generalized spectrum of a unitary operator on $H_{\mathcal{A}}$.

Proposition 2.3. *Let $U \in B^a(H_{\mathcal{A}})$ be unitary. Then*

$$\sigma^{\mathcal{A}}(U) \subseteq \{\alpha \in \mathcal{A} \mid \|\alpha\| \geqslant 1\},$$
$$\sigma^{\mathcal{A}}(U) \cap G(\mathcal{A}) \subseteq \{\alpha \in G(\mathcal{A}) \mid \|\alpha^{-1}\|, \|\alpha\| \geqslant 1\}.$$

Proof. We have $\alpha I - U = ((\alpha I)U^* - I)U$ and $\|U^*\| = \|U\| = 1$. □

Consider again the orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$ for $H_{\mathcal{A}}$. We may enumerate this basis by indexes in $\mathbb{Z}$. Then we get orthonormal basis $\{e_j\}_{j \in \mathbb{Z}}$ for $H_{\mathcal{A}}$ and we can consider a bilateral shift operator V with respect to this basis, i.e. $Ve_k = e_{k+1}$ all $k \in \mathbb{Z}$, which gives $V^* e_k = e_{k-1}$ for all $k \in \mathbb{Z}$.

Proposition 2.4. *Let V be the bilateral shift operator on $H_{\mathcal{A}}$. Then the following holds:*

(1) *If $\mathcal{A} = C([0,1])$, then $\sigma^{\mathcal{A}}(V) = \{f \in \mathcal{A} \mid |f|([0,1]) \cap \{1\} \neq \emptyset\}$,*
(2) *If $\mathcal{A} = L^{\infty}((0,1), \mu)$, then*

$$\sigma^{\mathcal{A}}(V) = \{f \in \mathcal{A} \mid \mu(|f|^{-1}((1-\epsilon, 1+\epsilon))) > 0 \ \forall \epsilon > 0\}.$$

In both cases $\sigma_p^{\mathcal{A}}(V) = \emptyset$.

Proof. Case 1: In this case we consider $\mathcal{A} = C([0,1])$. Suppose that $\alpha \in \mathcal{A}$ and $|\alpha(\tilde{t})| = 1$ for some $\tilde{t} \in [0,1]$. Choose a function $y \in \mathcal{A}$ such that $y(\tilde{t}) = 1$. If $\alpha I - V$ is surjective, then there exists an $x \in H_{\mathcal{A}}$ such that $(\alpha I - V)x = e_1 \cdot y$. Now, $x(\tilde{t}) \in l_2$ since $x \in H_{\mathcal{A}}$. If we let $\tilde{V}$ denote the ordinary bilateral shift on l_2, we get that $\alpha(\tilde{t})x(\tilde{t}) - \tilde{V}(x(\tilde{t})) = (1, 0, 0, \dots)$, since $y(\tilde{t}) = 1$. However, this is not possible since $|\alpha(\tilde{t})| = 1$ (for more details, see [13, Chapter 4, Proposition 19]). We conclude that $\alpha I - V$ can not be surjective, so $\alpha \in \sigma^{\mathcal{A}}(V)$. On the other hand, if $\alpha \in \mathcal{A}$ and $|\alpha|([0,1]) \cap \{1\} = \emptyset$, then either $|\alpha(t)| \geqslant C > 1$ or $|\alpha(t)| \leqslant K < 1$ for all $t \in [0,1]$ and some constants C or K (here we use that $|\alpha|$ is continuous). If $|\alpha(t)| \geqslant C > 1$ for all $t \in [0,1]$, then α is invertible in $\mathcal{A}$ and $\|\alpha^{-1}\| \leqslant \frac{1}{C} < 1$. Since $\|V\| = 1$, it follows that $\alpha \notin \sigma^{\mathcal{A}}(V)$. If $|\alpha(t)| \leqslant K < 1$ for all $t \in [0,1]$, then $\|\alpha\| \leqslant K < 1$, so, by Proposition 2.3 it follows then that $\alpha \notin \sigma^{\mathcal{A}}(V)$. Hence

$$\sigma^{\mathcal{A}}(V) = \{\alpha \in \mathcal{A} \mid |\alpha|([0,1]) \cap \{1\} \neq \emptyset\}.$$

Next, if $(\alpha I - V)x = 0$ for some $x \in H_{\mathcal{A}}$, then we must have $\alpha(t)x(t) - \tilde{V}x(t) = 0$ for all $t \in [0,1]$. This means that $x(t) = 0$ for all $t \in [0,1]$ since $\sigma_p(\tilde{V}) = \emptyset$ by [13, Chapter 4, Proposition 19].

Case 2: Let now $\mathcal{A} = L^\infty((0,1),\mu)$ and $\alpha \in \mathcal{A}$ be such that $\mu(|\alpha|^{-1}((1-\epsilon,1+\epsilon))) > 0$ for all $\epsilon > 0$. If $(\alpha I - V)x = e_0$ for some $x \in H_{\mathcal{A}}$, then we must have $\alpha x_k - x_{k-1} = 0$ for all $k \neq 0$ and $\alpha x_0 - x_{-1} = 1_{\mathcal{A}}$. For small $\epsilon > 0$ set $M_\epsilon = |\alpha|^{-1}((1-\epsilon,1+\epsilon))$, $M_\epsilon^- = |\alpha|^{-1}((1-\epsilon,1))$ and $M_\epsilon^+ = |\alpha|^{-1}((1,1+\epsilon))$, so $M_\epsilon = M_\epsilon^- \cup M_\epsilon^+$ and $\chi_{M_\epsilon} \neq 0$. From the first equation above we get $x_k = \alpha^{-(k+1)}x_{-1}$ for all $k \leqslant -1$. Moreover, $x_k = \alpha^{-k}x_0$ for all $k \geqslant 0$ a.e. on any subset of $(0,1)$ on which $|\alpha|$ is bounded below, thus in particular on M_ϵ. Hence $x_k\chi_{M_\epsilon} = x_0\alpha^{-k}\chi_{M_\epsilon}$ for all $k \geqslant 0$ where for all k we let $\alpha^{-k}\chi_{M_\epsilon}$ denote the function given by

$$\alpha^{-k}\chi_{M_\epsilon}(t) = \begin{cases} \alpha^{-k}(t) & \text{for } t \in M_\epsilon, \\ 0 & \text{else.} \end{cases}$$

Since $x \in H_{\mathcal{A}}$, it follows that $x_k\chi_{M_\epsilon^+} = 0$ for all $k \leqslant -1$ and $x_k\chi_{M_\epsilon^-} = 0$ for all $k \geqslant 0$. Setting this into the second equation above, we get $\alpha x_0\chi_{M_\epsilon^+} - x_{-1}\chi_{M_\epsilon^-} = \chi_{M_\epsilon}$, which gives $x_0\chi_{M_\epsilon} = \alpha^{-1}\chi_{M_\epsilon^+}$ and $x_{-1}\chi_{M_\epsilon} = -\chi_{M_\epsilon^-}$. Hence $x_k\chi_{M_\epsilon} = \alpha^{-(k+1)}\chi_{M_\epsilon^+}$ for all $k \geqslant 0$ and $x_k\chi_{M_\epsilon} = -\alpha^{-(k+1)}\chi_{M_\epsilon^-}$ for all $k \leqslant -1$. This gives $|x_k| \geqslant (1+\epsilon)^{-(k+1)}\chi_{M_\epsilon^+}$ for all $k \geqslant 0$ and $|x_k| \geqslant (1-\epsilon)^{-(k+1)}\chi_{M_\epsilon^-}$ for all $k \leqslant -1$. Since this holds for all $\epsilon > 0$ and moreover, we have that either $\chi_{M_\epsilon^-}$ or $\chi_{M_\epsilon^+}$ is non-zero (because χ_{M_ϵ} is non-zero for all $\epsilon > 0$), we get that the infinite sum $\sum_{k\in\mathbb{Z}} x_k^* x_k$ diverge in $\mathcal{A}$, otherwise $\| \sum_{k\in\mathbb{Z}} x_k^* x_k \| \geqslant \min\big\{ \sum_{k=0}^{\infty} \frac{1}{(1+\epsilon)^{k+1}}, \sum_{k=0}^{\infty}(1-\epsilon)^k \big\}$ for all $\epsilon > 0$, a contradiction. Hence x can not be an element of $H_{\mathcal{A}}$. We conclude that $e_0 \notin \operatorname{Im}(\alpha I - V)$, so $\alpha \in \sigma^{\mathcal{A}}(V)$ in this case.

On the other hand, if $\mu(|\alpha|^{-1}((1-\epsilon,1+\epsilon))) = 0$ for $\alpha \in \mathcal{A}$ and some $\epsilon > 0$, then we have $(0,1) = N_\epsilon^- \cup N_\epsilon^+$, where

$$N_\epsilon^- = |\alpha|^{-1}((0,1-\epsilon)) \quad \text{and} \quad N_\epsilon^+ = |\alpha|^{-1}((1+\epsilon,+\infty)).$$

Since the decomposition $H_{\mathcal{A}} = H_{\mathcal{A}} \cdot \chi_{N_\epsilon^+} \oplus H_{\mathcal{A}} \cdot \chi_{N_\epsilon^-}$ clearly reduces the operator $\alpha I - V$ and the restrictions of $\alpha I - V$ on both these submodules are invertible, (as the restriction of V to both these submodules acts as a unitary operator on these submodules), it follows that $\alpha I - V$ is invertible, so $\alpha \notin \sigma^{\mathcal{A}}(V)$. □

Example 2.3. Let $\{\alpha_1, \alpha_2, \dots\}$ be a sequence in a unital C^*-algebra $\mathcal{A}$ such that each α_k is a unitary element of $\mathcal{A}$. Then the operator V defined by

$$V(x_1, x_2, \dots) = (\alpha_1 x_1, \alpha_2 x_2, \dots)$$

is a unitary operator on $H_{\mathcal{A}}$. If $\mathcal{A} = C([0,1])$ or if $\mathcal{A} = L^\infty((0,1),\mu)$ and J_1, J_2 are two closed subintervals of $(0,1)$ such that $J_1 \cap J_2 = \emptyset$, then we may easily find a function $\beta \in \mathcal{A}$ such that $\beta = \alpha_1$ on J_1 and $|\beta(t)| > 1$ for all $t \in J_2$. Hence $\|\beta\| > 1$, but we also have $\beta \in \sigma^{\mathcal{A}}(V)$ since $\ker(\beta I - V) \neq \{0\}$. Similarly, if $\mathcal{A} = B(H)$ where H is an infinite-dimensional Hilbert space, then we may easily find two closed suspaces H_1 and H_2 such that $H_1 \perp H_2$ and $T \in B(H)$ satisfying $T_{|_{H_1}} = \alpha_{1|_{H_1}}$ and $\|T_{|_{H_2}}\| > 1$. Hence, again $T \in \sigma^{\mathcal{A}}(V)$ and $\|T\| > 1$. So, if V is a unitary operator on $H_{\mathcal{A}}$, we do <u>not</u> have in general that

$$\sigma^{\mathcal{A}}(V) \subseteq \{\alpha \in \mathcal{A} \mid \|\alpha\| = 1\}.$$

Next we are going to describe and investigate some properties of generalized spectra of self-adjoint operators on $H_{\mathcal{A}}$.

Lemma 2.2. *Let $\mathcal{A}$ be a commutative C^*-algebra. If F is a self-adjoint operator on $H_{\mathcal{A}}$, then $\sigma_p^{\mathcal{A}}(F)$ is a self-adjoint subset of $\mathcal{A}$, that is $\alpha \in \sigma_p^{\mathcal{A}}(F)$ if and only if $\alpha^* \in \sigma_p^{\mathcal{A}}(F)$.*

Proof. Since $F - \alpha I$ and $F - \alpha^* I = F^* - \alpha^* I$ mutually commute because $\mathcal{A}$ is commutative, we can deduce that $\|(F-\alpha I)x\| = \|(F-\alpha^* I)x\|$ for all $x \in H_{\mathcal{A}}$. □

Example 2.4. Let $\mathcal{A} = C([0,1])$ or $\mathcal{A} = L^\infty((0,1),\mu)$. If G is the operator on $H_{\mathcal{A}}$ given by $G(f_1, f_2, \dots) = (g_1 f_1, g_2 f_2, \dots)$, where $\{g_1, g_2, \dots\}$ is a bounded sequence of real valued functions in $\mathcal{A}$, then G is a self-adjoint operator. Suppose that there are two mutually disjoint, closed subintervals J_1 and J_2 of $(0,1)$ such that $g_{1_{|J_1}} \neq 0$ and $g_{1_{|J_2}} = 0$. Set $\tilde{g} = i g_1$. Then, if we choose a function f in $\mathcal{A}$ such that supp $f \subseteq J_2$, we get that $(\tilde{g}I - G)(f, 0, 0, \dots) = 0$. However, $\tilde{g} \neq \overline{\tilde{g}}$, so we do <u>not</u> have that $\sigma_p^{\mathcal{A}}(G)$ is included in the set of self-adjoint elements of $\mathcal{A}$.

Example 2.5. Let $\mathcal{A} = B(H)$ where H is a separable infinite-dimensional Hilbert space and let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis for H. If P denotes the orthogonal projection onto Span$\{e_1\}$, then the operator $P \cdot I$ is a self-adjoint operator on $H_{\mathcal{A}}$. Now, if S is the unilateral shift operator on H with respect to the orthonormal basis $\{e_j\}$, then $S - P$ is injective whereas $S^* - P$ is <u>not</u> injective because $(S^* - P)(e_1 + e_2) = 0$. It follows that $(S-P)\cdot I$ is an injective operator on $H_{\mathcal{A}}$, whereas $(S^* - P)\cdot I = ((S-P)\cdot I)^*$ is <u>not</u> an injective operator on $H_{\mathcal{A}}$, since $(S^* - P) \cdot I(Q, 0, 0, 0, \dots) = 0$, where Q is the orthogonal projection onto Span$\{e_1 + e_2\}$. Hence, if $\mathcal{A} = B(H)$, we do not have in general that $\sigma_p^{\mathcal{A}}(F)$ is a self-adjoint subset of $\mathcal{A}$ when $F = F^*$. It follows that the assumption that $\mathcal{A}$ is commutative is indeed necessary in Lemma 2.2.

Lemma 2.3. *Let $\mathcal{A}$ be a commutative C^*-algebra. If F is a self-adjoint operator on $H_{\mathcal{A}}$ and $\alpha \in \mathcal{A} \setminus \sigma_p^{\mathcal{A}}(F)$, then* $\operatorname{Im}(F - \alpha I)^\perp = \{0\}$*. Hence, if $\alpha \in \mathcal{A}$ and $F - \alpha I$ is bounded below, then $\alpha \in \mathcal{A} \setminus \sigma^{\mathcal{A}}(F)$.*

Proof. Suppose that $\alpha \in \mathcal{A} \setminus \sigma_p^{\mathcal{A}}(F)$. If $y \in \operatorname{Im}(F - \alpha I)^\perp$, then $y \in \ker(F^* - \alpha^* I)$. By the proof of Lemma 2.2 we obtain that $(F - \alpha I)y = 0$. Since $\alpha \notin \sigma_p^{\mathcal{A}}(F)$ by the choice of α, we get that $y = 0$. Thus, $\operatorname{Im}(F - \alpha I)^\perp = \{0\}$, when $\alpha \in \mathcal{A} \setminus \sigma_p^{\mathcal{A}}(F)$. Suppose next that $\alpha \in \mathcal{A}$ is such that $F - \alpha I$ is bounded below. Then $\alpha \in \mathcal{A} \setminus \sigma_p^{\mathcal{A}}(F)$, so from the previous arguments we deduce that $\operatorname{Im}(F - \alpha I)^\perp = \{0\}$. Moreover, since $\operatorname{Im}(F - \alpha I)$ is then closed and $F - \alpha I \in B^a(H_{\mathcal{A}})$, from [10] [11, Theorem 2.3.3] it follows that $\operatorname{Im}(F - \alpha I)$ is orthogonally complementable in $H_{\mathcal{A}}$. However, since $\operatorname{Im}(F - \alpha I)^\perp = \{0\}$, we must have that $\operatorname{Im}(F - \alpha I) = H_{\mathcal{A}}$. Hence $F - \alpha I$ is invertible in $B^a(H_{\mathcal{A}})$, so α is in $\mathcal{A} \setminus \sigma^{\mathcal{A}}(F)$. □

Corollary 2.3. *Let $\mathcal{A}$ be a unital commutative C^*-algebra and F be a self-adjoint operator on $H_{\mathcal{A}}$. If $\alpha \in \mathcal{A}$ and $\alpha - \alpha^* \in G(\mathcal{A})$, then $F - \alpha I$ is invertible. In this case,*

$$\|(F - \alpha I)^{-1}\| \leqslant 2\|(\alpha - \alpha^*)^{-1}\|.$$

Proof. If $\alpha \in \mathcal{A}$, then, since $\mathcal{A}$ is commutative, we get

$$\langle x, Fx - \alpha Ix\rangle - \langle Fx - \alpha Ix, x\rangle = \alpha^*\langle x, x\rangle - \langle x, x\rangle\alpha = (\alpha^* - \alpha)\langle x, x\rangle.$$

From the triangle inequality and the Cauchy-Schwartz inequality for the inner product we obtain $\|(\alpha - \alpha^*)\langle x, x\rangle\| \leqslant 2\|x\|\|Fx - \alpha Ix\|$. Since $(\alpha - \alpha^*)$ is invertible by assumption, we get from this inequality

$$\begin{aligned}\|x\|^2 = \|\langle x, x\rangle\| &\leqslant \|(\alpha - \alpha^*)^{-1}\|\|(\alpha - \alpha^*)\langle x, x\rangle\| \\ &\leqslant 2 \cdot \|x\|\|(F - \alpha I)x\|\|(\alpha - \alpha^*)^{-1}\|,\end{aligned}$$

which gives

$$\frac{\|x\|}{2 \cdot \|(\alpha - \alpha^*)^{-1}\|} \leqslant \|(F - \alpha I)x\|$$

for all $x \in H_{\mathcal{A}}$. From Lemma 2.3 it follows that $F - \alpha I$ is invertible. □

Remark 2.1. Let $\mathcal{A} = C([0,1])$ or $\mathcal{A} = L^\infty((0,1),\mu)$. As we have seen in Example 2.4, the operator $\tilde{g}I - G$ is not invertible, whereas $\tilde{g} - \overline{\tilde{g}} = 2ig_1 \neq 0$. Therefore, it is not sufficient only to assume that $\alpha - \alpha^* \neq 0$, so the requirement that $a - a^*$ is invertible is indeed necessary in Corollary 2.3.

Example 2.6. Let $\mathcal{A} = M_2(\mathbb{C})$ and $T_1, T_2 \in \mathcal{A}$ be given by $T_1 = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, $T_2 = \left[\begin{smallmatrix} 0 & i \\ i & i \end{smallmatrix}\right]$. Then T_1 is self-adjoint and $T_2 - T_2^* = 2i\left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, so $T_2 - T_2^*$ is invertible. Now, $T_1 - T_2 = \left[\begin{smallmatrix} 2 & 1-i \\ 1-i & -i \end{smallmatrix}\right]$, so $\det(T_1 - T_2) = 0$, which gives that $T_1 - T_2$ is not invertible. Hence the operator $F := T_1 \cdot I$ is a self-adjoint operator on $H_{\mathcal{A}}$, but $F - T_2 \cdot I = (T_1 - T_2) \cdot I$ is <u>not</u> invertible. This shows that the assumption that $\mathcal{A}$ is commutative in Corollary 2.3 is indeed necessary.

For a self-adjoint operator F on $H_{\mathcal{A}}$, set

$$M(F) = \sup\{\|\langle Fx, x\rangle\| \mid \|x\| = 1\} \text{ and } m(F) = \inf\{\|\langle Fx, x\rangle\| \mid \|x\| = 1\}.$$

We have the following corollary.

Corollary 2.4. *If $\mathcal{A} = C([0,1])$ and F is a self-adjoint operator on $H_{\mathcal{A}}$, then*

$$\sigma^{\mathcal{A}}(F) \subseteq \{f \in \mathcal{A} \mid |f|([0,1]) \cap [m, M] \neq \emptyset\}.$$

If $\mathcal{A} = L^\infty((0,1),\mu)$ and F is a self-adjoint operator on $H_{\mathcal{A}}$, then

$$\sigma^{\mathcal{A}}(F) \subseteq \{f \in \mathcal{A} \mid \mu(|f|^{-1}([m - \epsilon, M + \epsilon])) > 0 \textit{ for all } \epsilon > 0\}.$$

Proof. Let $\mathcal{A} = L^\infty((0,1),\mu)$, F be a self-adjoint operator on $H_{\mathcal{A}}$ and $\alpha \in \mathcal{A}$ be such that there exists an $\epsilon = \epsilon(\alpha)$ with the property that $\mu(|\alpha|^{-1}([m - \epsilon, M + \epsilon])) = 0$. Then $(0,1) = M_1 \cup M_2$, where M_1 and M_2 are Lebesgue measurable, mutually disjoint subsets of $(0,1)$ satisfying $|\alpha|\chi_{M_1} \geqslant (M + \epsilon)\chi_{M_1}$ and $|\alpha|\chi_{M_2} \leqslant (m - \epsilon)\chi_{M_2}$ a.e. Hence, for all $x \in H_{\mathcal{A}}$, we have

$$\langle (F - \alpha I)x, x\rangle = \langle (F - \alpha I)x, x\rangle \cdot \chi_{M_1} + \langle (F - \alpha I)x, x\rangle \cdot \chi_{M_2}.$$

Now, we have

$$\begin{aligned}\|\langle (F - \alpha I)x, x\rangle\| &\geqslant \|\langle (F - \alpha I)x, x\rangle\chi_{M_1}\| \\ &\geqslant \|\bar{\alpha}\langle x, x\rangle\chi_{M_1}\| - \|\langle Fx, x\rangle\chi_{M_1}\|\end{aligned}$$

$$
\begin{aligned}
&= \|\bar{\alpha}\chi_{M_1}\langle x,x\rangle\chi_{M_1}\| - \|\chi_{M_1}\langle Fx,x\rangle\chi_{M_1}\| \\
&= \|\bar{\alpha}\chi_{M_1}\langle x,x\rangle\chi_{M_1}\| - \|\langle Fx\cdot\chi_1, x\cdot\chi_{M_1}\rangle\| \\
&\geqslant (M+\epsilon)\|\langle x,x\rangle\chi_{M_1}\| - \|\langle F(x\cdot\chi_{M_1}), x\cdot\chi_{M_1}\rangle\| \\
&\geqslant (M+\epsilon)\|\langle x,x\rangle\chi_{M_1}\| - M\|\langle x\cdot\chi_{M_1}, x\cdot\chi_{M_1}\rangle\| \\
&= (M+\epsilon)\|\langle x,x\rangle\chi_{M_1}\| - M\|\chi_{M_1}\langle x,x\rangle\chi_{M_1}\| = \epsilon\|\langle x,x\rangle\chi_{M_1}\|
\end{aligned}
$$

(where we have used that

$$
\|\langle Fy,y\rangle\| = \|y\|^2\Big\|\Big\langle F\Big(\frac{y}{\|y\|}\Big), \frac{y}{\|y\|}\Big\rangle\Big\| \leqslant \|\langle y,y\rangle\|M).
$$

Similarly we obtain

$$
\begin{aligned}
\|\langle (F-\alpha I)x,x\rangle\| &\geqslant \|\langle (F-\alpha I)x,x\rangle\chi_{M_2}\| \geqslant \|\langle Fx,x\rangle\chi_{M_2}\| - \|\bar{\alpha}\langle x,x\rangle\chi_{M_2}\| \\
&= \|\langle F(x\cdot\chi_{M_2}), x\cdot\chi_{M_2}\rangle\| - \|\bar{\alpha}\langle x,x\rangle\chi_{M_2}\| \\
&\geqslant m\|\langle x\cdot\chi_{M_2}, x\cdot\chi_{M_2}\rangle\| - (m-\epsilon)\|\langle x,x\rangle\chi_{M_2}\| = \epsilon\|\langle x,x\rangle\chi_{M_2}\|.
\end{aligned}
$$

Hence $\|\langle (F-\alpha I)x,x\rangle\| \geqslant \epsilon\max\{\|\langle x,x\rangle\chi_{M_2}\|, \|\langle x,x\rangle\chi_{M_1}\|\} = \epsilon\|\langle x,x\rangle\|$. Thus, $\|(F-\alpha I)x\|\|x\| \geqslant \|\langle (F-\alpha I)x,x\rangle\| \geqslant \epsilon\|x\|^2$ for all $x \in H_{\mathcal{A}}$. It follows that $F-\alpha I$ is bounded below, hence, from Lemma 2.3 we deduce that $F-\alpha I$ is invertible in $B^a(H_{\mathcal{A}})$. The proof in the case when $\mathcal{A} = C([0,1])$ is similar, but more simple, because if $\alpha \in \mathcal{A}$ and $|\alpha|([0,1]) \cap [m,M] = \emptyset$, then by the continuity of $|\alpha|$ we must either have that $|\alpha| < m$ or $|\alpha| > M$ that on the whole interval $[0,1]$. Moreover, there exists then an $\epsilon > 0$ such that $|\alpha| \leqslant m-\epsilon$ or $|\alpha| \geqslant M+\epsilon$ on the whole $[0,1]$. Hence we may proceed in the same way as in the above proof. ☐

Finally, we are going to study the properties of generalized spectra of normal operators on $H_{\mathcal{A}}$.

Lemma 2.4. *Let $\mathcal{A}$ be a commutative unital C^*-algebra and F be a normal operator on $H_{\mathcal{A}}$, that is $FF^* = F^*F$. If $\alpha_1, \alpha_2 \in \sigma_p^{\mathcal{A}}(F)$ and $\alpha_1 - \alpha_2$ is not a zero divisor in $\mathcal{A}$, then* $\ker(F-\alpha_1 I) \perp \ker(F-\alpha_2 I)$.

Proof. Since F commutes with F^* and $\mathcal{A}$ is a commutative unital C^*-algebra, then $F-\alpha_2 I$ and $F^*-\alpha_2^* I$ mutually commute. Hence $\ker(F-\alpha_2 I) = \ker(F^*-\alpha_2^* I)$. For $x_1 \in \ker(F-\alpha_1 I)$ and $x_2 \in \ker(F-\alpha_2 I) = \ker(F^*-\alpha_2^* I)$, we get

$$
\langle x_2,x_1\rangle(\alpha_1-\alpha_2) = \langle x_2,x_1\rangle\alpha_1 - \alpha_2\langle x_2,x_1\rangle = \langle x_2,Fx_1\rangle - \langle F^*x_2,x_1\rangle = 0
$$

(where we have used that $\mathcal{A}$ is commutative, so $\langle x_2,x_1\rangle\ \alpha_2 = \alpha_2\langle x_2,x_1\rangle$). Since $(\alpha_1-\alpha_2)$ is not a zero divisor by assumption, it follows that $\langle x_2,x_1\rangle = 0$. ☐

Example 2.7. Let $\mathcal{A} = C([0,1])$ or $\mathcal{A} = L^\infty((0,1),\mu)$ and consider the self-adjoint operator G from Example 2.4. For any function f in $\mathcal{A}$ with the support contained in J_2, we have $(f,0,0,\dots) \in \ker G \cap \ker(\tilde{g}I-G)$. However, $\tilde{g} = ig_1 \neq 0$ and $f \neq 0$, but $\tilde{g}$ is <u>not</u> invertible in $\mathcal{A}$, so it is not sufficient only to assume that $\alpha_1-\alpha_2 \neq 0$ and the assumption that $\alpha_1-\alpha_2$ is not a zero divisor in $\mathcal{A}$ is indeed necessary.

Example 2.8. Let $\mathcal{A} = B(H)$ and $T \in \mathcal{A}$ be a normal and invertible operator. If H_1 and H_2 are two closed subspaces of H such that $H = H_1 \tilde{\oplus} H_2$ and $H_1 \neq H_2^{\perp}$ (that is H_1 and H_2 are not mutually orthogonal), then $T\sqcap$ and $T(1-\sqcap)$ are elements of $\sigma_p^{\mathcal{A}}(T \cdot I)$, where $\sqcap$ stands for the skew projection onto H_1 along H_2. Moreover, the operator $T \cdot I$ is normal operator on $H_{\mathcal{A}}$ and $T\sqcap - T(1-\sqcap)$ is invertible in $\mathcal{A}$ because $T\sqcap - T(1-\sqcap)$ has the matrix $\left[\begin{smallmatrix} T & 0 \\ 0 & -T \end{smallmatrix}\right]$ with respect to the decomposition $H = H_1 \tilde{\oplus} H_2 \to T(H_1) \tilde{\oplus} T(H_2) = H$. However, if P_1 and P_2 denote the orthogonal projections onto H_1 and H_2, respectively, then, for all j,

$$e_j \cdot P_1 \in \ker(T\sqcap \cdot I - T \cdot I) \quad \text{and} \quad e_j \cdot P_2 \in \ker(T(1-\sqcap) \cdot I - T \cdot I),$$

since $\sqcap P_1 = P_1$ and $(1-\sqcap)P_2 = P_2$. Moreover, $P_1 P_2 \neq 0$. So the assumption that $\mathcal{A}$ is commutative is indeed necessary in Lemma 2.4.

Lemma 2.5. *Let $\mathcal{A}$ be a commutative C^*-algebra and F be a normal operator on $H_{\mathcal{A}}$. Then $\sigma_{rl}^{\mathcal{A}}(F) = \emptyset$, hence $\sigma^{\mathcal{A}}(F) = \sigma_p^{\mathcal{A}}(F) \cup \sigma_{cl}^{\mathcal{A}}(F)$.*

Proof. Suppose that $\alpha \in \sigma_{rl}^{\mathcal{A}}(F)$. Then $F - \alpha I$ is bounded below. Again, since $F - \alpha I$ and $F^* - \alpha^* I$ mutually commute, we get that $\ker(F - \alpha I) = \ker(F^* - \alpha^* I) = \{0\}$. Next, since $\operatorname{Im}(F - \alpha I)$ is closed, by [10] [11, Theorem 2.3.3] we have that

$$H_{\mathcal{A}} = \ker(F^* - \alpha^* I) \oplus \operatorname{Im}(F - \alpha I) = \operatorname{Im}(F - \alpha I).$$

So $F - \alpha I$ is surjective, thus invertible, which gives that $\sigma_{rl}^{\mathcal{A}}(F) = \emptyset$. □

Example 2.9. Let $\mathcal{A} = B(H)$ and S, P be as in Example 2.5. Then $P \cdot I$ is a normal operator on $H_{\mathcal{A}}$ being self-adjoint and $(S - P) \cdot I$ is bounded below on $H_{\mathcal{A}}$. Indeed, we have that $\|(S - P)x\| \geqslant \|x\|$ for all $x \in H$, hence $m(S - P) \geqslant 1$. Therefore, since

$$T^*(S - P)^*(S - P)T \geqslant (m(S - P))^2 T^* T$$

for all $T \in B(H)$, it is not hard to see that $(S - P) \cdot I$ is bounded below on $H_{\mathcal{A}}$. However, $\operatorname{Im}((S - P) \cdot I)^{\perp} = \ker((S^* - P) \cdot I)$ and $\ker((S^* - P) \cdot I) \neq \{0\}$ as we have seen in Example 2.5. Hence $P \cdot I$ is a normal operator on $H_{\mathcal{A}}$ and $S \in \sigma_{rl}^{\mathcal{A}}(P \cdot I)$, which shows that the assumption that $\mathcal{A}$ is commutative is indeed necessary in Lemma 2.5. Moreover, this also shows that the assumption that $\mathcal{A}$ is commutative is indeed necessary in Lemma 2.3 as well, because $S \in \mathcal{A} \setminus \sigma_p^{\mathcal{A}}(P \cdot I)$, however, $\operatorname{Im}((S - P) \cdot I)^{\perp} \neq \{0\}$.

The next lemma is a generalization of [8, Chapter XI, Proposition 1.1]. For $F \in B^a(H_{\mathcal{A}})$, set

$$\sigma_a^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \ \text{ is not bounded below}\},$$
$$\sigma_l^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \ \text{ is not left invertible in } \ B^a(H_{\mathcal{A}})\},$$
$$\sigma_r^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \ \text{ is not right invertible in } \ B^a(H_{\mathcal{A}})\}.$$

Lemma 2.6. *Let $F \in B^a(H_{\mathcal{A}})$. Then the following statements are equivalent.*

a) $\alpha \in \mathcal{A} \setminus \sigma_a^{\mathcal{A}}(F)$.
b) $\alpha \in \mathcal{A} \setminus \sigma_l^{\mathcal{A}}(F)$.
c) $\alpha^* \in \mathcal{A} \setminus \sigma_r^{\mathcal{A}}(F^*)$.

d) $\mathrm{Im}(\alpha^* I - F^*) = H_{\mathcal{A}}$.

Proof. This proof is similar to the proof of [8, Chapter XI, Proposition 1.1]. Indeed, if $F - \alpha I$ is bounded below, then $\mathrm{Im}(F - \alpha I)$ is orthogonally complementable in $H_{\mathcal{A}}$ by [10] [11, Theorem 2.3.3]. The operator $F - \alpha I$ is invertible viewed as an operator from $H_{\mathcal{A}}$ onto $\mathrm{Im}(F - \alpha I)$. This follows by the Banach open mapping theorem. Hence $(F - \alpha I)^{-1} \in B^a(\mathrm{Im}(F - \alpha I), H_{\mathcal{A}})$. Let P denote the orthogonal projection onto $\mathrm{Im}(F-\alpha I)$, then $(F-\alpha I)^{-1}P$ is a left inverse of $F-\alpha I$ in $B^a(H_{\mathcal{A}})$. Next, $F - \alpha I$ has left inverse if and only if $F^* - \alpha^* I$ has right inverse in $B^a(H_{\mathcal{A}})$, so $(b) \Rightarrow (c)$. Part $(c) \Rightarrow (d)$ is obvious. Finally, if $\mathrm{Im}(\alpha^* I - F^*) = H_{\mathcal{A}}$, then $\ker(F-\alpha I) = \mathrm{Im}(F^*-\alpha^* I)^{\perp} = \{0\}$. Moreover, from the proof of [10] [11, Theorem 2.3.3] we have that $\mathrm{Im}(F - \alpha I)$ is closed since $\mathrm{Im}(F^* - \alpha^* I)$ is closed. Therefore, $F - \alpha I$ is bounded below. □

The next two propositions can be proved in exactly the same way as for operators on Hilbert spaces, see [13, Chapter 4, Proposition 20] and [13, Chapter 4, Proposition 21].

Proposition 2.5. *For $F \in B^a(H_{\mathcal{A}})$, we have that $\sigma_a^{\mathcal{A}}(F)$ is a closed subset of $\mathcal{A}$ in the norm topology and $\sigma^{\mathcal{A}}(F) = \sigma_a^{\mathcal{A}}(F) \cup \sigma_{rl}^{\mathcal{A}}(F)$.*

Proof. The statement follows since $M^a(H_{\mathcal{A}})$ is open in $B^a(H_{\mathcal{A}})$ in the norm topology. Next, if $F - \alpha_0 I$ is bounded below, it is easy to see that either $\alpha_0 \in \sigma_{rl}^{\mathcal{A}}(F)$ or $F - \alpha_0 I$ is invertible. □

Proposition 2.6. *Let $\mathcal{A}$ be a commutative C^*-algebra. If $F \in B^a(H_{\mathcal{A}})$, then $\partial\sigma^{\mathcal{A}}(F) \subseteq \sigma_a^{\mathcal{A}}(F)$. Moreover, if M is a closed submodule of $H_{\mathcal{A}}$ invariant with respect to F and $F_0 = F_{|_M}$, then we have $\partial\sigma^{\mathcal{A}}(F_0) \subseteq \sigma_a^{\mathcal{A}}(F)$ and $\sigma^{\mathcal{A}}(F_0) \cap \rho^{\mathcal{A}}(F) = \sigma_{rl}^{\mathcal{A}}(F_0)$, where $\rho^{\mathcal{A}}(F) = \mathcal{A} \setminus \sigma^{\mathcal{A}}(F)$.*

Proof. Let $\alpha_0 \in \partial\sigma^{\mathcal{A}}(F)$. Then there exists a sequence $\{\alpha_n\} \subseteq \mathcal{A} \setminus \sigma^{\mathcal{A}}(F)$ such that $\alpha_n \to \alpha_0$ in $\mathcal{A}$, hence $F - \alpha_n I \longrightarrow F - \alpha_0 I$ in the norm. From a well known result for operators on Banach spaces stated in [13, Chapter 4, Proposition 12], there exists a subsequence α_{n_k} such that $\|(F-\alpha_{n_k} I)^{-1}\| \longrightarrow \infty$ as $k \longrightarrow \infty$ since $F-\alpha_0 I$ is not invertible. Hence, there exists a sequence of unit vectors $\{x_k\} \subseteq H_{\mathcal{A}}$ such that $\|(F - \alpha_{n_k} I)^{-1} x_k\| \longrightarrow \infty$ as $k \longmapsto \infty$. For each k, set $y_k = (F - \alpha_{n_k} I)^{-1} x_k$ and $v_k = \frac{y_k}{\|y_k\|}$. Then we have that

$$\|(F - \alpha_0 I)v_k\| \leqslant \|(\alpha_0 - \alpha_{n_k})I v_k\| + \|(F - \alpha_{nk} I)v_k\| \leqslant \|\alpha_0 - \alpha_{n_k}\| + \frac{1}{\|y_k\|},$$

which gives that $\|(F - \alpha_0 I)v_k\| \longrightarrow 0$, so $\alpha_0 \in \sigma_a^{\mathcal{A}}(F)$. This shows the first statement in the proposition. However, then we have that

$$\partial\sigma^{\mathcal{A}}(F_0) \subseteq \sigma_a^{\mathcal{A}}(F_0) \subseteq \sigma_a^{\mathcal{A}}(F) \subseteq \sigma^{\mathcal{A}}(F).$$ □

Example 2.10. We may also consider the operators on $H_{\mathcal{A}}$ defined by

$$W(e_k) = e_{2k} \quad \text{and} \quad W'(e_k) = e_{2k-1} \quad \text{for all} \quad k \in \mathbb{N}.$$

Also for these operators we have $\sigma^{\mathcal{A}}(W) = \sigma^{\mathcal{A}}(W') = \{\alpha \in \mathcal{A} \mid \inf|\alpha| \leqslant 1\}$ in the case when $\mathcal{A} = C([0,1])$ or when $\mathcal{A} = L^\infty((0,1),\mu)$. Suppose now that $\mathcal{A} = L^\infty((0,1),\mu)$ and consider the operator F on $H_{\mathcal{A}}$ given by

$$F(f_1, f_2, f_3, \dots) = (\chi_{(0,\frac{1}{2})}f_1, \chi_{(\frac{1}{2},1)}f_1, \chi_{(0,\frac{1}{2})}f_2, \chi_{(\frac{1}{2},1)}f_2, \dots).$$

It follows that F has the matrix $\left[\begin{smallmatrix} W' & 0 \\ 0 & W \end{smallmatrix}\right]$ with respect to the decomposition

$$H_{\mathcal{A}} = (H_{\mathcal{A}} \cdot \chi_{(0,\frac{1}{2})}) \oplus (H_{\mathcal{A}} \cdot \chi_{(\frac{1}{2},1)}) \xrightarrow{F} (H_{\mathcal{A}} \cdot \chi_{(0,\frac{1}{2})}) \oplus (H_{\mathcal{A}} \cdot \chi_{(\frac{1}{2},1)}) = H_{\mathcal{A}}.$$

Therefore, $\sigma^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid \inf|\alpha| \leqslant 1\}$. Next we have that

$$\sigma_p^{\mathcal{A}}(W) = \emptyset, \sigma_p^{\mathcal{A}}(W') = \{\alpha \in \mathcal{A} \mid \alpha = 1 \text{ on some closed subinterval } J \subseteq [0,1]\}$$

in the case when $\mathcal{A} = C([0,1])$ and

$$\sigma_p^{\mathcal{A}}(W') = \{\alpha \in \mathcal{A} \mid \mu(\{t \in (0,1) \mid \alpha(t) = 1\}) > 0\}$$

in the case when $\mathcal{A} = L^\infty((0,1),\mu)$. Hence, we get that

$$\sigma_p^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid \mu(\{t \in (0, \frac{1}{2}) \mid \alpha(t) = 1\}) > 0\}.$$

Consider next the operators

$$Z(e_j) = \begin{cases} e_k & \text{when } j = 2k \\ 0 & \text{else} \end{cases}, \quad k \in \mathbb{N}; \quad Z'(e_j) = \begin{cases} e_k & \text{when } j = 2k-1 \\ 0 & \text{else} \end{cases}, \quad k \in \mathbb{N}$$

Then $\sigma^{\mathcal{A}}(Z) = \sigma^{\mathcal{A}}(Z') = \{\alpha \in \mathcal{A} \mid \inf|\alpha| \leqslant 1\}$. This follows since $Z = W^*$ and $Z' = W'^*$. Moreover, we have

$$\sigma_p^{\mathcal{A}}(Z) = \{\alpha \in \mathcal{A} \mid \inf|\alpha| < 1\}$$

both in the case when $\mathcal{A} = C([0,1])$ and when $\mathcal{A} = L^\infty((0,1),\mu)$. In the case when $\mathcal{A} = L^\infty((0,1),\mu)$ we have that

$$\sigma_p^{\mathcal{A}}(Z') = \{\alpha \in \mathcal{A} \mid \inf|\alpha| < 1 \text{ or } \mu(\{t \in (0,1) \mid \alpha(t) = 1\}) > 0\}$$

and in the case when $\mathcal{A} = C([0,1])$, we have that

$$\sigma_p^{\mathcal{A}}(Z') = \{\alpha \in \mathcal{A} \mid \inf|\alpha| < 1 \text{ or } \alpha = 1 \text{ on some closed subinterval } J \subseteq [0,1]\}.$$

Let the operator D on $H_{\mathcal{A}}$ be given by

$$D(g_1, g_2, g_3, \dots) = (g_1\chi_{(0,\frac{1}{2})} + g_2\chi_{(\frac{1}{2},1)}, g_3\chi_{(0,\frac{1}{2})} + g_4\chi_{(\frac{1}{2},1)}, \dots)$$

when $\mathcal{A} = L^\infty((0,1),\mu)$. Then $D = F^*$ and D has the matrix $\left[\begin{smallmatrix} Z' & 0 \\ 0 & Z \end{smallmatrix}\right]$ with respect to the decomposition $H_{\mathcal{A}} \cdot \chi_{(0,\frac{1}{2})} \oplus H_{\mathcal{A}} \cdot \chi_{(\frac{1}{2},1)}$. It follows that

$$\sigma^{\mathcal{A}}(D) = \{\alpha \in \mathcal{A} \mid \inf|\alpha| \leqslant 1\},$$
$$\sigma_p^{\mathcal{A}}(D) = \big\{\alpha \in \mathcal{A} \mid \inf|\alpha| < 1 \text{ or } \mu\big(\big\{t \in \big(0, \tfrac{1}{2}\big) \mid \alpha(t) = 1\big\}\big) > 0\big\}.$$

3. Generalized Fredholm spectra of operators over C^*-algebras

We recall first the following definitions.

Definition 3.1. [9] [11, Definition 2.7.1] Let M be an abelian monoid. Consider the Cartesian product $M \times M$ and its quotient monoid with respect to the equivalence relation

$$(m,n) \sim (m',n') \Leftrightarrow \exists p,q : (m,n)+(p,p) = (m',n')+(q,q).$$

This quotient monoid is a group, which is denoted by $S(M)$ and is called the symmetrization of M. Consider now the additive category $\mathcal{P}(\mathcal{A})$ of projective modules over a unital C^*-algebra $\mathcal{A}$ and denoted by $[\mathcal{M}]$ the isomorphism class of an object $\mathcal{M}$ from $\mathcal{P}(\mathcal{A})$. The set $\phi(\mathcal{P}(\mathcal{A}))$ of these classes has the structure of an Abelian monoid with respect to the operation $[\mathcal{M}]+[\mathcal{N}] = [\mathcal{M} \oplus \mathcal{N}]$. In this case the group $S(\phi(\mathcal{P}(\mathcal{A})))$ is denoted by $K(\mathcal{A})$ or $K_0(\mathcal{A})$ and is called the K-group of $\mathcal{A}$ or the Grothendieck group of the category $\mathcal{P}(\mathcal{A})$.

As regards the K-group $K_0(\mathcal{A})$, it is worth mentioning that it is not true in general that $[M] = [N]$ implies that $M \cong N$ for two finitely generated Hilbert modules M,N over $\mathcal{A}$. If $K_0(\mathcal{A})$ satisfies the property that $[N] = [M]$ implies that $N \cong M$ for any two finitely generated, Hilbert modules M,N over $\mathcal{A}$, then $K_0(\mathcal{A})$ is said to satisfy "the cancellation property", see [14, Section 6.2].

Definition 3.2. [12], [11, Definition 2.7.4] A (bounded $\mathcal{A}$-linear) operator $F\colon H_{\mathcal{A}} \to H_{\mathcal{A}}$ is called (adjointable) $\mathcal{A}$-Fredholm if

1) it is adjointable;
2) there exists a decomposition of the domain, $H_{\mathcal{A}} = \mathcal{M}_1 \tilde{\oplus} \mathcal{N}_1$, and the range $H_{\mathcal{A}} = \mathcal{M}_2 \tilde{\oplus} \mathcal{N}_2$ (where $\mathcal{M}_1, \mathcal{M}_2, \mathcal{N}_1, \mathcal{N}_2$ are closed $\mathcal{A}$-modules and $\mathcal{N}_1, \mathcal{N}_2$ have a finite number of generators), such that F has the matrix form $F = \left[\begin{smallmatrix} F_1 & 0 \\ 0 & F_2 \end{smallmatrix}\right]$ with respect to these decompositions and $F_1 = F_{|\mathcal{M}_1} : \mathcal{M}_1 \to \mathcal{M}_2$ is an isomorphism.

Definition 3.3. [12], [11, Definition 2.7.8] Let the conditions of Definition 3.2 hold. We define the index of F by $\operatorname{index} F = [\mathcal{N}_1] - [\mathcal{N}_2] \in K_0(\mathcal{A})$.

Next we recall the definition of semi-$\mathcal{A}$-Fredholm and semi-$\mathcal{A}$-Weyl operators on $H_{\mathcal{A}}$.

Definition 3.4. [1, Definition 2.1] Let $F \in B^a(H_{\mathcal{A}})$. We say that F is an upper semi-$\mathcal{A}$-Fredholm operator if there exists a decomposition

$$H_{\mathcal{A}} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_{\mathcal{A}}$$

with respect to which F has the matrix $\left[\begin{smallmatrix} F_1 & 0 \\ 0 & F_4 \end{smallmatrix}\right]$, where F_1 is an isomorphism, M_1, M_2, N_1, N_2 are closed submodules of $H_{\mathcal{A}}$ and N_1 is finitely generated. Similarly, we say that F is a lower semi-$\mathcal{A}$-Fredholm operator if all the above conditions hold except that in this case we assume that N_2 (and not N_1) is finitely generated.

Set

$$\mathcal{M}\Phi_+(H_{\mathcal{A}}) = \{F \in B^a(H_{\mathcal{A}}) \mid F \text{ is upper semi-}\mathcal{A}\text{-Fredholm}\},$$

$$\mathcal{M}\Phi_-(H_\mathcal{A}) = \{F \in B^a(H_\mathcal{A}) \mid F \text{ is lower semi-}\mathcal{A}\text{-Fredholm}\},$$
$$\mathcal{M}\Phi(H_\mathcal{A}) = \{F \in B^a(H_\mathcal{A}) \mid F \text{ is } \mathcal{A}\text{-Fredholm operator on } H_\mathcal{A}\}.$$

Next we set $\mathcal{M}\Phi_\pm(H_\mathcal{A}) = \mathcal{M}\Phi_+(H_\mathcal{A}) \cup \mathcal{M}\Phi_-(H_\mathcal{A})$. Notice that if M, N are two arbitrary Hilbert modules C^*-modules, the definition above could be generalized to the classes $\mathcal{M}\Phi_+(M,N)$ and $\mathcal{M}\Phi_-(M,N)$.

Definition 3.5. [1, Definition 5.1] Let $F \in \mathcal{M}\Phi(H_\mathcal{A})$.

We say that $F \in \tilde{\mathcal{M}}\Phi_+^-(H_\mathcal{A})$ if there exists a decomposition

$$H_\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_\mathcal{A}$$

with respect to which F has the matrix $\left[\begin{smallmatrix} F_1 & 0 \\ 0 & F_4 \end{smallmatrix}\right]$, where F_1 is an isomorphism, N_1, N_2 are closed, finitely generated and $N_1 \preceq N_2$, that is N_1 is isomorphic to a closed submodule of N_2. We define similarly the class $\tilde{\mathcal{M}}\Phi_-^+(H_\mathcal{A})$, the only difference in this case is that $N_2 \preceq N_1$. Then we set

$$\mathcal{M}\Phi_+^-(H_\mathcal{A}) = (\tilde{\mathcal{M}}\Phi_+^-(H_\mathcal{A})) \cup (\mathcal{M}\Phi_+(H_\mathcal{A}) \setminus \mathcal{M}\Phi(H_\mathcal{A})),$$
$$\mathcal{M}\Phi_-^+(H_\mathcal{A}) = (\tilde{\mathcal{M}}\Phi_-^+(H_\mathcal{A})) \cup (\mathcal{M}\Phi_-(H_\mathcal{A}) \setminus \mathcal{M}\Phi(H_\mathcal{A})).$$

Further, we define $\mathcal{M}\Phi_0(H_\mathcal{A})$ to be the set of all $F \in \mathcal{M}\Phi(H_\mathcal{A})$ for which there exists an $\mathcal{M}\Phi$-decomposition $H_\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_\mathcal{A}$, where $N_1 \cong N_2$.

Definition 3.6. [1, Definition 5.6] Let $F \in \mathcal{M}\Phi_+(H_\mathcal{A})$.

We say that $F \in \mathcal{M}{\Phi_+^-}'(H_\mathcal{A})$ if there exists a decomposition

$$H_\mathcal{A} = M_1 \tilde{\oplus} N_1 \xrightarrow{F} M_2 \tilde{\oplus} N_2 = H_\mathcal{A}$$

with respect to which $F = \left[\begin{smallmatrix} F_1 & 0 \\ 0 & F_4 \end{smallmatrix}\right]$, where F_1 is an isomorphism, N_1 is closed, finitely generated and $N_1 \preceq N_2$. Similarly, we define the class $\mathcal{M}{\Phi_-^+}'(H_\mathcal{A})$, only in this case $F \in \mathcal{M}\Phi_-(H_\mathcal{A})$, N_2 is finitely generated and $N_2 \preceq N_1$.

Such operators will be called semi-$\mathcal{A}$-Weyl operators throughout the paper.

Then we introduce the following definition.

Definition 3.7. We set $ms_\Phi(F) = \inf\{\|\alpha\| \mid \alpha \in \mathcal{A}, F - \alpha I \notin \mathcal{M}\Phi(H_\mathcal{A})\}$,

$$ms(F) = \inf\{\|\alpha\| \mid \alpha \in \mathcal{A}, F - \alpha I \notin \mathcal{M}\Phi_\pm(H_\mathcal{A})\},$$
$$ms_+(F) = \inf\{\|\alpha\| \mid \alpha \in \mathcal{A}, F - \alpha I \notin \mathcal{M}\Phi_+(H_\mathcal{A})\},$$
$$ms_-(F) = \inf\{\|\alpha\| \mid \alpha \in \mathcal{A}, F - \alpha I \notin \mathcal{M}\Phi_-(H_\mathcal{A})\}.$$

It follows that

$$ms_\Phi(F) = \max\{\epsilon \geqslant 0 \mid \|\alpha\| < \epsilon \Rightarrow F - \alpha I \in \mathcal{M}\Phi(H_\mathcal{A})\},$$
$$ms_+(F) = \max\{\epsilon \geqslant 0 \mid \|\alpha\| < \epsilon \Rightarrow F - \alpha I \in \mathcal{M}\Phi_+(H_\mathcal{A})\},$$
$$ms_-(F) = \max\{\epsilon \geqslant 0 \mid \|\alpha\| < \epsilon \Rightarrow F - \alpha I \in \mathcal{M}\Phi_-(H_\mathcal{A})\},$$
$$ms(F) = \max\{\epsilon \geqslant 0 \mid \|\alpha\| < \epsilon \Rightarrow F - \alpha I \in \mathcal{M}\Phi_\pm(H_\mathcal{A})\}.$$

From [11, Lemma 2.7.10] and [1, Theorem 4.1] it follows that

$$ms_\Phi(F) > 0 \Leftrightarrow F \in \mathcal{M}\Phi(H_\mathcal{A}), \quad ms_+(F) > 0 \Leftrightarrow F \in \mathcal{M}\Phi_+(H_\mathcal{A}),$$

$$ms_-(F) > 0 \Leftrightarrow F \in \mathcal{M}\Phi_-(H_\mathcal{A}), \quad ms(F) > 0 \Leftrightarrow F \in \mathcal{M}\Phi_\pm(H_\mathcal{A}).$$

From [1, Corollary 2.11] it follows that

$$ms_+(F) = ms_-(F^*), \quad ms_\Phi(F) = ms_\Phi(F^*), \quad ms(F) = ms(F^*).$$

We have the following lemma.

Lemma 3.1. *Let $F \in B^a(H_\mathcal{A})$. If $ms_+(F) > 0$ and $ms_-(F) > 0$, then $ms_+(F) = ms_-(F)$.*

Proof. Since $ms_+(F)$ and $ms_-(F)$ are strictly positive by assumption, then, by [1, Corollary 2.4], $F \in \mathcal{M}\Phi_+(H_\mathcal{A}) \cap \mathcal{M}\Phi_-(H_\mathcal{A}) = \mathcal{M}\Phi(H_\mathcal{A})$. If $ms_+(F) > ms_-(F)$, then, obviously, there exists an $\alpha \in \mathcal{A}$ such that $\|\alpha\| \in (ms_-(F), ms_+(F))$, and $(F - \alpha I) \in \mathcal{M}\Phi_+(H_\mathcal{A}) \setminus \mathcal{M}\Phi_-(H_\mathcal{A})$. However, if we consider the map $f\colon [0,1] \to B^a(H_\mathcal{A})$ given by $f(t) = F - t\alpha I$, then f is continuous. Since $\|\alpha\| < ms_+(F)$, it follows that $f([0,1]) \subseteq \mathcal{M}\Phi_+(H_\mathcal{A}) \subseteq \mathcal{M}\Phi_\pm(H_\mathcal{A})$. By [1, Corollary 4.3] we deduce that $f(1) \in \mathcal{M}\Phi(H_\mathcal{A})$ since $f(0) \in \mathcal{M}\Phi(H_\mathcal{A})$. However, we have that $f(1) = F - \alpha I \notin \mathcal{M}\Phi_-(H_\mathcal{A})$. Since $\mathcal{M}\Phi(H_\mathcal{A}) \subseteq \mathcal{M}\Phi_-(H_\mathcal{A})$, we get a contradiction. Thus, $ms_+(F) = ms_-(F)$ in this case. Similarly, if $ms_-(F) \geqslant ms_+(F)$, we can show that actually $ms_-(F) = ms_+(F)$. □

Lemma 3.2. *Let $F \in B^a(H_\mathcal{A})$. Then*

1) $ms_\Phi(F) = \min\{ms_+(F), ms_-(F)\}$,
2) $ms(F) = \max\{ms_+(F), ms_-(F)\}$.

Proof. First we prove 1). If $0 = \min\{ms_+(F), ms_-(F)\}$, then either $ms_+(F) = 0$ or $ms_-(F) = 0$. Suppose that $ms_+(F) = 0$. Then, by the above arguments, since $\mathcal{M}\Phi_+(H_\mathcal{A})$ is open, we must have that $F \notin \mathcal{M}\Phi_+(H_\mathcal{A})$. Hence $F \notin \mathcal{M}\Phi(H_\mathcal{A})$, so $ms_\Phi(F) = 0$. Similarly, if $ms_-(F) = 0$, it follows that $ms_\Phi(F) = 0$, since $\mathcal{M}\Phi_-(H_\mathcal{A})$ is open and $\mathcal{M}\Phi(H_\mathcal{A}) \subseteq \mathcal{M}\Phi_-(H_\mathcal{A})$. Suppose now that

$$0 < \min\{ms_+(F), ms_-(F)\} = ms_+(F).$$

By Lemma 3.1 we have $ms_+(F) = ms_-(F)$. Applying [1, Corollary 2.4] we easily deduce that $ms_\Phi(F) = ms_+(F) = ms_-(F)$.

Next we prove 2). If $\max\{ms_+(F), ms_-(F)\} = 0$, then $F \notin \mathcal{M}\Phi_\pm(H_\mathcal{A})$, hence $ms(F) = 0$, as in the proof of [15, (2.3.8.2.)]. Suppose that $0 < \max\{ms_+(F), ms_-(F)\} = ms_+(F)$. Obviously, we have that $ms(F) \geqslant ms_+(F)$. If $ms(F) > ms_+(F)$, then for any $r \in (ms_+(F), ms(F))$, the set

$$C_r := \{F - \alpha I \mid \alpha \in \mathcal{A} \mid \|\alpha\| \leqslant r\}$$

would intersect both $\mathcal{M}\Phi_+(H_\mathcal{A})$ and $\mathcal{M}\Phi_-(H_\mathcal{A}) \setminus \mathcal{M}\Phi_+(H_\mathcal{A})$, which are both open by [1, Theorem 4.1] and [6, Remark 3.3.4]. Hence the sets $\mathcal{M}\Phi_+(H_\mathcal{A}) \cap C_r$ and $(\mathcal{M}\Phi_-(H_\mathcal{A}) \setminus \mathcal{M}\Phi_+(H_\mathcal{A})) \cap C_r$ would form a separation of C_r, since $C_r \subseteq \mathcal{M}\Phi_\pm(H_\mathcal{A})$. Indeed, since $r > \max\{ms_+(F), ms_-(F)\}$, we can not have that $C_r \subseteq \mathcal{M}\Phi_+(H_\mathcal{A})$ or $C_r \subseteq \mathcal{M}\Phi_-(H_\mathcal{A})$. On the other hand, since $r < ms(F)$, we must have that $C_r \subseteq \mathcal{M}\Phi_\pm(H_\mathcal{A})$. Therefore, it follows that $C_r \cap \mathcal{M}\Phi_+(H_\mathcal{A}) \neq \emptyset$ and $C_r \cap (\mathcal{M}\Phi_-(H_\mathcal{A}) \setminus \mathcal{M}\Phi_+(H_\mathcal{A})) \neq \emptyset$. This is a contradiction since C_r is connected.

Hence we must have $ms(F) = ms_+(F)$. The case when $\max\{ms_+(F), ms_-(F)\} = ms_-(F)$ can be treated analogously. □

Definition 3.8. Let $F \in B^a(H_{\mathcal{A}})$. We set

$$\begin{aligned}\sigma_{ew}^{\mathcal{A}}(F) &= \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\Phi_0(H_{\mathcal{A}})\},\\ \sigma_{euf}^{\mathcal{A}}(F) &= \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\Phi_+(H_{\mathcal{A}})\},\\ \sigma_{elf}^{\mathcal{A}}(F) &= \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\Phi_-(H_{\mathcal{A}})\},\\ \sigma_{ek}^{\mathcal{A}}(F) &= \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\Phi_\pm(H_{\mathcal{A}})\},\\ \sigma_{ef}^{\mathcal{A}}(F) &= \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\Phi(H_{\mathcal{A}})\}.\end{aligned}$$

Lemma 3.3. *Let $F \in B^a(H_{\mathcal{A}})$ and suppose that $K_0(\mathcal{A})$ satisfies the cancellation property. Then $\sigma^{\mathcal{A}}(F) = \sigma_{ew}^{\mathcal{A}}(F) \cup \sigma_p^{\mathcal{A}}(F) \cup \sigma_{cl}^{\mathcal{A}}(F)$.*

Proof. It suffices to show " $\subseteq$ ". Suppose that $\alpha \in \sigma^{\mathcal{A}}(F) \setminus (\sigma_{cl}^{\mathcal{A}}(F) \cup \sigma_{ew}^{\mathcal{A}}(F))$. Then $\operatorname{Im}(F - \alpha I)$ is closed and $(F - \alpha I) \in \mathcal{M}\Phi_0(H_{\mathcal{A}})$. By Theorem [11, Theorem 2.3.3] the operator $F - \alpha I$ has the matrix $\left[\begin{smallmatrix} (F-\alpha I)_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ with respect to the decomposition

$$H_{\mathcal{A}} = \ker(F - \alpha I)^\perp \tilde{\oplus} \ker(F - \alpha I) \xrightarrow{F-\alpha I} \operatorname{Im}(F - \alpha I) \tilde{\oplus} \operatorname{Im}(F - \alpha I)^\perp = H_{\mathcal{A}},$$

where $(F - \alpha I)_1$ is an isomorphism by the Banach open mapping theorem. Since we have $(F - \alpha I) \in \mathcal{M}\Phi_0(H_{\mathcal{A}})$, then it holds that

$$0 = \operatorname{index}(F - \alpha I) = [\ker(F - \alpha I)] - [\operatorname{Im}(F - \alpha I)^\perp],$$

so $[\ker(F-\alpha I)] = [\operatorname{Im}(F-\alpha I)^\perp]$. If $[\ker(F-\alpha I)] = 0$, then $\ker(F-\alpha I) = \{0\}$, since $K_0(\mathcal{A})$ satisfies the cancellation property by assumption. By the same reason we would have $\operatorname{Im}(F-\alpha I)^\perp = \{0\}$, so $F-\alpha I$ is then invertible, which is a contradiction, since $\alpha \in \sigma^{\mathcal{A}}(F)$. Thus, we must have $\ker(F - \alpha I) \neq \{0\}$, so $\alpha \in \sigma_p^{\mathcal{A}}(F)$. □

Example 3.1. Let $\mathcal{A} = B(H)$, where H is an infinite-dimensional, separable Hilbert space. If H_1 is any infinite-dimensional subspace of H, then there exists an isometric isomorphism U of H onto H_1. Set $\tilde{U}$ to be the operator on $\mathcal{A}$ given by $\tilde{U}(F) = JUF$ for all $F \in \mathcal{A}$ where J is the inclusion of H_1 into H. Then $\tilde{U} \in B^a(\mathcal{A})$ and moreover, $\tilde{U}$ is an isometry. Put T to be the operator with the matrix $\left[\begin{smallmatrix} 1 & 0 \\ 0 & \tilde{U} \end{smallmatrix}\right]$ with respect to the decomposition

$$H_{\mathcal{A}} = L_1^\perp \oplus L_1 \xrightarrow{T} L_1^\perp \oplus L_1 = H_{\mathcal{A}}.$$

Then $T \in B^a(H_{\mathcal{A}})$ and T is bounded below. Moreover,

$$\operatorname{Im} T^\perp = \operatorname{Span}_{\mathcal{A}}\{(P, 0, 0, 0, \dots)\},$$

where P is the orthogonal projection of H onto $H_1^\perp$. Obviously, $T \in \mathcal{M}\Phi_0(H_{\mathcal{A}})$ and moreover, T is bounded below, but T is not surjective, thus not invertible. Hence

$$0 \in (\sigma_{rl}^{\mathcal{A}}(T) \setminus \sigma_{ew}^{\mathcal{A}}(T)) \subseteq (\sigma^{\mathcal{A}}(T) \setminus (\sigma_{ew}^{\mathcal{A}}(T) \cup \sigma_p^{\mathcal{A}}(T) \cup \sigma_{cl}^{\mathcal{A}}(T))).$$

This shows that the assumption that $K_0(\mathcal{A})$ satisfies the cancellation property is indeed necessary in Lemma 3.3.

For $F \in B^a(H_{\mathcal{A}})$ we set

$$\mathcal{M}\Phi_+(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \in \mathcal{M}\Phi_+(H_{\mathcal{A}})\},$$
$$\mathcal{M}\Phi_-(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \in \mathcal{M}\Phi_-(H_{\mathcal{A}})\},$$
$$\mathcal{M}\Phi(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \in \mathcal{M}\Phi(H_{\mathcal{A}})\},$$
$$\mathcal{M}\Phi_\pm(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \in \mathcal{M}\Phi_\pm(H_{\mathcal{A}})\},$$
$$\mathcal{M}\Phi_0(F) = \{\alpha \in \mathcal{A} \mid F - \alpha I \in \mathcal{M}\Phi_0(H_{\mathcal{A}})\}.$$

The next two results are generalizations of [8, Chapter XI, Proposition 4.9].

Proposition 3.1. *If $F \in B^a(H_{\mathcal{A}})$, then the components of $\mathcal{A} \setminus (\sigma^{\mathcal{A}}_{euf}(F) \cap \sigma^{\mathcal{A}}_{elf}(F))$ are either completely contained in one of the sets*

$$\mathcal{M}\Phi_+(F) \setminus \mathcal{M}\Phi(F), \quad \mathcal{M}\Phi_-(F) \setminus \mathcal{M}\Phi(F)$$

or they are completely contained in $\mathcal{M}\Phi(F)$ and in this case $\operatorname{index}(F - \alpha I)$ *is constant on them.*

Proof. Let C be a component of $\mathcal{A} \setminus (\sigma^{\mathcal{A}}_{euf}(F) \cap \sigma^{\mathcal{A}}_{elf}(F))$. Then either $C \cap \mathcal{M}\Phi_+(F) \neq \emptyset$ or $C \cap \mathcal{M}\Phi_-(F) \neq \emptyset$. Hence we must have that either $C \subseteq \mathcal{M}\Phi_-(F)$ or $C \subseteq \mathcal{M}\Phi_+(F)$ because otherwise the sets

$$C \cap \mathcal{M}\Phi_-(F) \quad \text{and} \quad C \cap (\mathcal{M}\Phi_+(F) \setminus \mathcal{M}\Phi_-(F))$$

would form a separation of C, which is a contradiction. Indeed, it follows straightforward from [1, Theorem 4.1] and [6, Remark 3.3.4] that the sets $\mathcal{M}\Phi_-(F)$ and $\mathcal{M}\Phi_+(F) \setminus \mathcal{M}\Phi_-(F)$ are open in the norm topology of $\mathcal{A}$. Assume that $C \subseteq \mathcal{M}\Phi_+(F)$. If $C \cap \mathcal{M}\Phi(F) \neq \emptyset$, then $C \subseteq \mathcal{M}\Phi(F)$ because otherwise the sets $\mathcal{M}\Phi(F)$ and $\mathcal{M}\Phi_+(F) \setminus \mathcal{M}\Phi(F)$ would form a separation of C, since it follows straightforward from [11, Lemma 2.7.10] and [1, Theorem 4.1] that $\mathcal{M}\Phi(F)$ and $\mathcal{M}\Phi_+(F) \setminus \mathcal{M}\Phi(F)$ are open. So, either $C \subseteq \mathcal{M}\Phi_+(F) \setminus \mathcal{M}\Phi(F)$ or $C \subseteq \mathcal{M}\Phi(F)$. Now, if $C \subseteq \mathcal{M}\Phi(F)$, then $\operatorname{index}(F - \alpha I)$ must be constant on C, since index is locally constant by [11, Lemma 2.7.10].

The case when $C \subseteq \mathcal{M}\Phi_-(F)$ can be treated similarly. □

Lemma 3.4. *Let $F \in B^a(H_{\mathcal{A}})$. If $\alpha \in \partial\sigma^{\mathcal{A}}(F) \setminus (\sigma^{\mathcal{A}}_{euf}(F) \cap \sigma^{\mathcal{A}}_{elf}(F))$, then $\alpha \in \mathcal{M}\Phi_0(F)$.*

Proof. Let $\alpha \in \partial\sigma^{\mathcal{A}}(F) \setminus (\sigma^{\mathcal{A}}_{euf}(F) \cap \sigma^{\mathcal{A}}_{elf}(F))$. Then $\alpha \in \mathcal{M}\Phi_\pm(F)$. Since $\alpha \in \partial\sigma^{\mathcal{A}}(F)$, each open neighbourhood of α in $\mathcal{A}$ intersects $\mathcal{M}\Phi_0(F)$ non-empty. Since $\mathcal{M}\Phi_+(F) \setminus \mathcal{M}\Phi(F)$ and $\mathcal{M}\Phi_-(F) \setminus \mathcal{M}\Phi(F)$ are open, it follows that α must be an element of $\mathcal{M}\Phi(F)$. Now, since $\alpha \in \partial\sigma^{\mathcal{A}}(F)$ and $\mathcal{M}\Phi(F) \setminus \mathcal{M}\Phi_0(F)$ is open (this follows from [6, Lemma 3.4.16], we must have that $\alpha \in \mathcal{M}\Phi_0(F)$. □

Now we consider the following spectra for $F \in B^a(H_{\mathcal{A}})$:

$$\sigma^{\mathcal{A}}_{e\tilde{a}}(F) = \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \tilde{\mathcal{M}}\Phi^-_+(H_{\mathcal{A}})\},$$
$$\sigma^{\mathcal{A}}_{ea}(F) = \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\Phi^-_+(H_{\mathcal{A}})\},$$
$$\sigma^{\mathcal{A}}_{e\tilde{b}}(F) = \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\tilde{\Phi}^+_-(H_{\mathcal{A}})\},$$

$$\sigma_{eb}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}\Phi_{-}^{+}(H_{\mathcal{A}})\},$$
$$\sigma_{ea'}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}{\Phi_{+}^{-}}'(H_{\mathcal{A}})\},$$
$$\sigma_{eb'}^{\mathcal{A}}(F) = \{\alpha \in \mathcal{A} \mid (F - \alpha I) \notin \mathcal{M}{\Phi_{-}^{+}}'(H_{\mathcal{A}})\}.$$

By [1, Remark 5.8] we have that

$$\mathcal{M}{\Phi_{+}^{-}}'(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi_{+}^{-}(H_{\mathcal{A}}) \quad \text{and} \quad \mathcal{M}{\Phi_{-}^{+}}'(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi_{-}^{+}(H_{\mathcal{A}}).$$

Hence, we get $\sigma_{ea}^{\mathcal{A}}(F) \subseteq \sigma_{ea'}^{\mathcal{A}}(F) \subseteq \sigma_{e\tilde{a}}^{\mathcal{A}}(F)$ and $\sigma_{eb}^{\mathcal{A}}(F) \subseteq \sigma_{eb'}^{\mathcal{A}}(F) \subseteq \sigma_{e\tilde{b}}^{\mathcal{A}}(F)$. We present the following proposition.

Proposition 3.2. *Let $F \in B^a(H_{\mathcal{A}})$. Then*

$$\partial\sigma_{e\tilde{a}}^{\mathcal{A}}(F) \subseteq \partial\sigma_{ea'}^{\mathcal{A}}(F) \subseteq \partial\sigma_{ea}^{\mathcal{A}}(F),$$
$$\partial\sigma_{e\tilde{b}}^{\mathcal{A}}(F) \subseteq \partial\sigma_{eb'}^{\mathcal{A}}(F) \subseteq \partial\sigma_{eb}^{\mathcal{A}}(F).$$

Proof. It suffices to show that

$$\partial\sigma_{e\tilde{a}}^{\mathcal{A}}(F) \subseteq \sigma_{ea'}^{\mathcal{A}}(F), \quad \partial\sigma_{ea'}^{\mathcal{A}}(F) \subseteq \sigma_{ea}^{\mathcal{A}}(F),$$
$$\partial\sigma_{e\tilde{b}}^{\mathcal{A}}(F) \subseteq \sigma_{eb'}^{\mathcal{A}}(F), \quad \partial\sigma_{eb'}^{\mathcal{A}}(F) \subseteq \sigma_{eb}^{\mathcal{A}}(F).$$

Suppose that $\alpha \in \partial\sigma_{e\tilde{a}}^{\mathcal{A}}(F) \setminus \sigma_{ea'}^{\mathcal{A}}(F)$. Then

$$\begin{aligned} F - \alpha I \in \mathcal{M}{\Phi_{+}^{-}}'(H_{\mathcal{A}}) \setminus \tilde{\mathcal{M}}\Phi_{+}^{-}(H_{\mathcal{A}}) &= \mathcal{M}{\Phi_{+}^{-}}'(H_{\mathcal{A}}) \setminus (\mathcal{M}{\Phi_{+}^{-}}'(H_{\mathcal{A}}) \cap \mathcal{M}\Phi(H_{\mathcal{A}})) \\ &= \mathcal{M}{\Phi_{+}^{-}}'(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}}) \\ &= \mathcal{M}{\Phi_{+}^{-}}'(H_{\mathcal{A}}) \cap (\mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}\Phi(H_{\mathcal{A}})), \end{aligned}$$

where in the first equality we apply [1, Proposition 5.7] and in the last equality we apply the fact that $\mathcal{M}{\Phi_{+}^{-}}'(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi_{+}(H_{\mathcal{A}})$ by definition. Now, by [1, Theorem 4.1] and [1, Lemma 5.9], we obtain that $\mathcal{M}{\Phi_{+}^{-}}'(H_{\mathcal{A}}) \setminus \tilde{\mathcal{M}}\Phi_{+}^{-}(H_{\mathcal{A}})$ is open in the norm topology. As $F - \alpha I$ is in $\mathcal{M}{\Phi_{+}^{-}}'(H_{\mathcal{A}}) \setminus \tilde{\mathcal{M}}\Phi_{+}^{-}(H_{\mathcal{A}})$, it follows that $\alpha \notin \partial\sigma_{e\tilde{a}}^{\mathcal{A}}(F)$, which is a contradiction. Thus we must have that $\partial\sigma_{e\tilde{a}}^{\mathcal{A}}(F) \subseteq \sigma_{ea'}^{\mathcal{A}}(F)$. Next suppose that $\alpha \in \partial\sigma_{ea'}^{\mathcal{A}}(F) \setminus \sigma_{ea}^{\mathcal{A}}(F)$. Since $\mathcal{M}{\Phi_{+}^{-}}'(H_{\mathcal{A}})$ is open by [1, Lemma 5.9], we must have that $\sigma_{ea'}^{\mathcal{A}}(F)$ is closed, hence $F - \alpha I \in \mathcal{M}\Phi_{+}^{-}(H_{\mathcal{A}}) \setminus \mathcal{M}{\Phi_{+}^{-}}'(H_{\mathcal{A}})$. Now, as $\mathcal{M}{\Phi_{+}^{-}}'(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi_{+}^{-}(H_{\mathcal{A}}) \subseteq \mathcal{M}\Phi_{+}(H_{\mathcal{A}})$, we get that

$$\mathcal{M}\Phi_{+}^{-}(H_{\mathcal{A}}) \setminus \mathcal{M}{\Phi_{+}^{-}}'(H_{\mathcal{A}}) = \mathcal{M}\Phi_{+}^{-}(H_{\mathcal{A}}) \cap (\mathcal{M}\Phi_{+}(H_{\mathcal{A}}) \setminus \mathcal{M}{\Phi_{+}^{-}}'(H_{\mathcal{A}})),$$

so by [6, Corollary 3.4.10] and [6, Lemma 3.4.16] we deduce that the set difference $\mathcal{M}\Phi_{+}^{-}(H_{\mathcal{A}}) \setminus \mathcal{M}{\Phi_{+}^{-}}'(H_{\mathcal{A}})$ is open in the norm topology. It follows that $\alpha \notin \partial\sigma_{ea'}^{\mathcal{A}}(F)$, which is a contradiction. We conclude then that $\partial\sigma_{ea'}^{\mathcal{A}}(F) \subseteq \sigma_{ea}^{\mathcal{A}}(F)$.

Similarly we can prove that $\partial\sigma_{e\tilde{b}}^{\mathcal{A}}(F) \subseteq \sigma_{eb'}^{\mathcal{A}}(F)$ and $\partial\sigma_{eb'}^{\mathcal{A}}(F) \subseteq \sigma_{eb}^{\mathcal{A}}(F)$. □

Corollary 3.1. *The sets $\mathcal{M}{\Phi_{+}^{-}}'(H_{\mathcal{A}}) \setminus \tilde{\mathcal{M}}\Phi_{+}^{-}(H_{\mathcal{A}})$, $\mathcal{M}\Phi_{+}^{-}(H_{\mathcal{A}}) \setminus \mathcal{M}{\Phi_{+}^{-}}'(H_{\mathcal{A}})$, $\mathcal{M}{\Phi_{-}^{+}}'(H_{\mathcal{A}}) \setminus \tilde{\mathcal{M}}{\Phi_{-}^{+}}'(H_{\mathcal{A}})$ and $\mathcal{M}\Phi_{-}^{+}(H_{\mathcal{A}}) \setminus \mathcal{M}{\Phi_{-}^{+}}'(H_{\mathcal{A}})$ are open.*

Example 3.2. Consider the Hilbert space $L^2((0,1),\mu)$. For every $f \in C([0,1])$ or $f \in L^\infty((0,1),\mu)$ we consider the multiplication operator M_f on $L^2((0,1),\mu)$, i.e. $M_f(g) = gf$ for all $g \in L^2((0,1),\mu)$. Then M_f is well defined, bounded linear operator on $L^2((0,1),\mu)$, $\|M_f\| \leqslant \|f\|_\infty$, and $M_f^* = M_{\bar{f}}$. If $F \in B(L^2(0,1),\mu)$, then the operators $F - M_f$, when f runs through $C([0,1])$ or $L^\infty((0,1),\mu)$, give rise to another kind of generalized spectra of F in $C([0,1])$ or in $L^\infty((0,1),\mu)$, respectively. Many of the results presented in this chapter have their natural analogue in this setting here. However, we should notice that, since $L^2((0,1),\mu)$ is an ordinary Hilbert space, we consider now generalized spectra in $C([0,1])$ or in $L^\infty((0,1),\mu)$ induced by the corresponding subclasses of the classical semi-Fredholm operators on $L^2((0,1),\mu)$.

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