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THE EQUATION $AX - XB = C$ WITHOUT A UNIQUE SOLUTION: THE AMBIGUITY WHICH BENEFITS APPLICATIONS

Abstract. This paper is a survey of the author's results regarding the equation $AX - XB = C$, in the case when it is without a unique solution. Sufficient conditions for the existence of infinitely many solutions are re-derived; methods for obtaining infinitely many solutions are revisited; some characterizations of the solution set are provided; the results are demonstrated on exact examples which require singularity of the initial equation.

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1. Introduction

Throughout the text, the notation used is a standard one; the letters V, W, V_1, V_2 and so on denote linear (vector) spaces over the field $\mathbb{C}$, unless stated differently. Letters A, B, C, X, Y, L, S and so on denote linear operators (linear transformations) defined in the afore-given vector spaces. The operator I stands for the identical operator on a fixed space. If an operator L is bounded (continuous), then it is understood that it is defined on the entire space; otherwise, if the operator L is unbounded, then we emphasize the set of vectors in which it exists (i.e. its domain), by denoting it $\mathcal{D}_L$. The operator L is densely defined in the normed space V_1 if $\overline{D_L} = V_1$, where the closure $\overline{\mathcal{D}_L}$ of $\mathcal{D}_L$ is understood in the topology induced by the given norm on V_1. The set of values which the operator L attains is called its range (or image), and is denoted as $\mathcal{R}(L)$. The set of all linear operators with their domains being subsets of V_1 and their ranges being subsets of V_2 is denoted as $L(V_1, V_2)$. The set of all bounded linear operators from V_1 to V_2 is denoted as $\mathcal{B}(V_1, V_2)$. Specially, if V_1 is a finite-dimensional vector space over $\mathbb{C}$ or $\mathbb{R}$, with $\dim V_1 = n$ and V_2 is a finite-dimensional vector space with $\dim V_2 = m$

over the same field, then $\mathcal{B}(V_1, V_2) = \mathbb{C}^{m\times n}$ or $\mathcal{B}(V_1, V_2) = \mathbb{R}^{m\times n}$, respectively. If $V_1 = V_2 = V$, we then write $L(V)$, $\mathcal{B}(V)$ and $\mathbb{C}^{n\times n}$, or respectively $\mathbb{R}^{n\times n}$.

Let V be a Banach space and let $L \in L(V)$ be a densely defined linear operator in V. The set of values $\lambda \in \mathbb{C}$ such that $(A - \lambda I)$ does not have a bounded inverse in $\mathcal{B}(V)$ defines the spectrum of L, denoted as $\sigma(L)$. The complement of $\sigma(L)$ in $\mathbb{C}$ defines the resolvent set for L, denoted as $\rho(L)$. The set of vectors $u \in V$ such that $Lu = 0$ defines the null-space for L, denoted as $\mathcal{N}(L)$. The value $\lambda \in \sigma(L)$, such that $\mathcal{N}(L - \lambda I)$ is nontrivial (i.e. there exists some $u \neq 0$ such that $Lu = \lambda u$) is an eigenvalue for L and the non-zero $u \in \mathcal{N}(L - \lambda I)$ is the corresponding eigenvector for L. The set of all eigenvalues for operator L define its points spectrum, denoted as $\sigma_p(L)$. The value $\lambda \in \sigma(L)$ is an approximate eigenvalue for L if there exists a normed sequence $(x_n)_{n\in\mathbb{N}} \subset D_L$ such that $\|x_n\| = 1$ for every n and

$$(L - \lambda I)x_n \to 0, \quad n \to +\infty.$$

In that sense, the set of all approximate eigenvalues for L defines its approximate point spectrum, denoted as $\sigma_{\text{app}}(L)$. The set of complex numbers λ such that $\mathcal{R}(L - \lambda I)$ is not dense in V denotes the defect spectrum for L,

$$\sigma_d(L) = \{\lambda \in \mathbb{C} : \overline{\mathcal{R}(L - \lambda I)} \neq V\} \subset \sigma(L).$$

The set of complex numbers λ such that $L - \lambda I$ is not "onto" in V represents the approximate defect spectrum for L,

$$\sigma_\delta(L) = \{\lambda \in \mathbb{C} : \mathcal{R}(L - \lambda I) \neq \mathcal{H}\} \subset \sigma(L).$$

Recall that, if the operator L is bounded, then $\sigma(L)$ is a compact, non-empty subset of $\mathbb{C}$ while $\rho(L)$ is an unbounded (non-empty) subset of $\mathbb{C}$. If the operator L is unbounded, then it can have an empty spectrum or an empty resolvent set. If L is a square matrix then $\sigma(L)$ consists of its eigenvalues. For given Hilbert spaces V and W, for arbitrary $L \in L(V, W)$, the unique closed (if such exists) $L^* \in L(W, V)$ which satisfies

$$\langle Lu, v\rangle = \langle u, L^*v\rangle,$$

for every $u \in \mathcal{D}_L$ and every $v \in \mathcal{D}_{L^*}$, denotes the Hilbert-conjugate (or adjoint) operator of the operator L.

1.1. On the equation $AX - XB = C$.

Let V_1 and V_2 be given Banach spaces. Equations of the form

$$AX - XB = C \tag{1.1}$$

are called Sylvester equations, where, in general, $A \in L(V_2)$, $B \in L(V_1)$ and $C \in L(V_1, V_2)$, are given linear operators. These expressions appear in many different branches of mathematics, physics and engineering, see [1, 2, 6, 8, 9, 20, 24, 25, 30, 33, 36, 38, 39, 41, 44, 45, 50–54, 57] and numerous references therein. Every individual application of the equation (1.1) requires unique mathematical, physical and technical assumptions for the spaces V_1 and V_2, and for the corresponding linear transformations A, B and C.

From mathematical perspective, it is significant to distinguish several important cases for these entities. Those are the case when V_1 and V_2 are finite-dimensional vector spaces (and the operators A, B and C are finite scalar matrices), the case when V_1 and V_2 are Banach or Hilbert spaces while A, B and C are appropriately provided bounded (continuous) linear operators, and finally, the case when V_1 and V_2 are Banach or Hilbert spaces while the operators A, B and C are unbounded.

The first result concerning solvability of such equations in matrices was established by J. J. Sylvester in 1884, hence the name of the equation.

Theorem 1.1. [61]*Let A, B and C be matrices of appropriate dimensions. The equation* (1.1) *has a unique solution X if and only if $\sigma(A) \cap \sigma(B) = \emptyset$.*

Proof. (see [6]). The square matrices $\mathbb{A}\colon X \mapsto AX$ and $\mathbb{B}\colon X \mapsto XB$ commute, therefore (see [30]) $\sigma(\mathbb{A} - \mathbb{B}) \subset \sigma(\mathbb{A}) - \sigma(\mathbb{B})$. Since $\sigma(A) = \sigma(\mathbb{A})$ and $\sigma(B) = \sigma(\mathbb{B})$, disjointness of spectra of A and B implies that $0 \notin \sigma(\mathbb{A} - \mathbb{B})$. Obviously, the Sylvester operator $S\colon X \mapsto AX - XB$ is the difference of operators $\mathbb{A}$ and $\mathbb{B}$, thus $0 \notin \sigma(S)$ and for every afore-given matrix C there exists a unique $X = S^{-1}(C)$ such that $S(X) = C$.

Conversely, assume that for every afore-given matrix C there exists a unique solution X to (1.1). If there exists a $\lambda \in \sigma(A) \cap \sigma(B)$ then $\bar{\lambda} \in \sigma(A^*)$, thus there exist (non-zero) eigenvectors u and v for B and A^*, respectively, which correspond to λ and $\bar{\lambda}$, respectively. Define $Cu := v$ and let X be a unique solution to the appropriate Sylvester equation. Then

$$\begin{aligned} 0 &= \lambda\langle Xu, v\rangle - \lambda\langle Xu, v\rangle = \langle Xu, \bar{\lambda}v\rangle - \lambda\langle Xu, v\rangle \\ &= \langle Xu, A^*v\rangle - \langle \lambda Xu, v\rangle = \langle AXu, v\rangle - \langle XBu, v\rangle \\ &= \langle (AX - XB)u, v\rangle = \langle Cu, v\rangle = \langle v, v\rangle = \|v\|^2 > 0, \end{aligned}$$

which is impossible. □

Its extension to bounded linear operators defined on Banach spaces was proved by Rosenblum in 1956. Notice that only one implication holds, rather than the equivalence.

Theorem 1.2. [52] *Let V_1 and V_2 be Banach spaces and let A, B and C be bounded linear operators defined on the appropriate spaces. The equation* (1.1) *has a unique solution if $\sigma(A) \cap \sigma(B) = \emptyset$.*

The proof of this statement provided in [6] relies on the lemma below.

Lemma 1.1. [6] *If $\mathbb{A}$ and $\mathbb{B}$ are commuting bounded linear operators on a Banach space V, then $\sigma(\mathbb{A} - \mathbb{B}) \subset \sigma(\mathbb{A}) - \sigma(\mathbb{B})$.*

Proof. Proof of Theorem 1.2 (see [6]). Observe the commuting operators $\mathbb{A}\colon X \mapsto AX$ and $\mathbb{B}\colon X \mapsto XB$, where $X \in \mathcal{B}(V_1, V_2)$ and $\mathbb{A}, \mathbb{B} \in \mathcal{B}(\mathcal{B}(V_1, V_2))$. Since A and B are bounded linear operators such that $\sigma(A) \cap \sigma(B) = \emptyset$ it follows that

$$0 \notin \sigma(\mathbb{A}) - \sigma(\mathbb{B}) \supset \sigma(\mathbb{A} - \mathbb{B}).$$

This proves that the Sylvester operator $S\colon X \mapsto AX - XB$, $S = \mathbb{A} - \mathbb{B}$ is invertible in $\mathcal{B}(\mathcal{B}(V_1, V_2))$. Subsequently, for every given $C \in \mathcal{B}(V_1, V_2)$ there exists a unique $X \in \mathcal{B}(V_1, V_2)$, given as $X = S^{-1}(C)$, such that $S(X) = C$. □

The above mentioned Lemma 1.1 is proved via Gelfand theory for commutative Banach algebras (see [6, 22], and [48]). For a unital complex semisimple commutative Banach Algebra $\mathcal{A}$, denote by $M_{\mathcal{A}}$ the maximal ideal space of $\mathcal{A}$, consisting of all complex homomorphisms $\varphi\colon \mathcal{A} \to \mathbb{C}$. Recall that $M_{\mathcal{A}}$ is a compact Hausdorff topological space (see [6, 14, 22, 48] and [56]). In that sense, for $a \in \mathcal{A}$, the Gelfand transform of a, denoted as $\widehat{a}$ is the continuous map $\widehat{a}\colon \varphi \mapsto \varphi(a)$ from $M_{\mathcal{A}}$ to $\mathbb{C}$ and the range of Gelfand transform at a is precisely the spectrum of a in $\mathcal{A}$ (see [22] or [48]), $\{\widehat{a}(\varphi)\colon \varphi \in M_{\mathcal{A}}\} = \sigma_{\mathcal{A}}(a)$. Thus, for $A \in \mathcal{A}^{n\times n}$, where $A = [a_{i,j}]_{n\times n}$, $a_{i,j} \in \mathcal{A}$, the Gelfand transform of A, denoted as $\widehat{A}$, is defined as $\widehat{A} := [\widehat{a}_{i,j}]_{n\times n}$.

Theorem 1.3. [56, Theorem 1.2.] *Let $\mathcal{A}$ be a commutative unital complex semisimple Banach algebra. Let $A \in \mathcal{A}^{n\times n}$ and $B \in \mathcal{A}^{m\times m}$ be such that*

$$\forall \phi \in M_{\mathcal{A}}, \quad \sigma(\widehat{A}) \cap \sigma(\widehat{B}) = \emptyset.$$

Then for every $C \in \mathcal{A}^{n\times m}$, there exists a unique $X \in \mathcal{A}^{n\times m}$ such that $AX - XB = C$.

However, unlike the matrix case, the converse statement does not hold for Banach algebras (or bounded linear operators on Banach spaces). It is rather trivial to provide a counterexample.

Example 1.1. [16, Example 1.1.] Let $V_1 = V_2$ be infinite dimensional Banach spaces and let $A = C = 0$. Assume that B is onto but is not injective. Then $\sigma(A) \cap \sigma(B) = \{0\}$, while the only solution to the equation $AX - XB = C \Leftrightarrow XB = 0$ is $X = 0$.

Lemma 1.2. [13, Lemma 1.2.2.] *Let $\mathcal{A}$ be a noncommutative unital Banach algebra that is infinite dimensional. Let a, b and $c \in \mathcal{A}$ such that $a = c = 0_{\mathcal{A}}$ and let b be a left zero divisor, which is not simultaneously a right zero divisor. Then $ax - xb = c$ has only one solution and that is $x = 0_{\mathcal{A}}$.*

Proof. Obviously $\sigma(a) = \{0\}$ while $0 \in \sigma(b)$, since b is a left zero divisor. Furthermore,

$$ax - xb = c \Leftrightarrow xb = 0_{\mathcal{A}} \Leftrightarrow x = 0_{\mathcal{A}}. \qquad \square$$

From the above discussion, the equation (1.1) is said to be *regular* whenever A, B and C are bounded linear operators on the corresponding Banach spaces and $\sigma(A) \cap \sigma(B) = \emptyset$. The equation is said to be *singular* if it is not regular, i.e. if $\sigma(A) \cap \sigma(B) \neq \emptyset$. Additionally, there are several results which give a unique bounded solution to (1.1), while the operators are unbounded, consult [41, 46] and [50]. These results have a huge impact on mathematical physics and quantum mechanics.

Definition 1.1. [21, Definition 1.1] Let V be a Banach space. The mapping $S\colon \mathbb{R}_0^+ \to \mathcal{B}(V)$ is a C_0- semigroup (of bounded linear operators) on V if

- $S(0) = I$;
- $S(t + s) = S(t)S(s), \quad t, s \geqslant 0$;
- For every $u \in V$ $\|S(t)u - u\| \to 0$ when $t \to 0 + 0$.

Definition 1.2. [21, Definition 1.2] Let $(S(t))_{t\geqslant 0}$ be a C_0-semigroup on the Banach space V. A linear operator L, $\mathcal{D}_L \subset V$ and $\mathcal{R}(L) \subset V$ is the infinitesimal generator for $(S(t))_{t\geqslant 0}$ if $Lu = \lim_{t\to 0+0} t^{-1}(S(t)u - u)$ for every $u \in \mathcal{D}_L$.

Every C_0-semigroup of operators has a unique infinitesimal generator. Necessary and sufficient conditions for the operator L to generate a C_0-semigroup of operators are provided by the famous Hille–Yosida theorem:

Theorem 1.4. [21, Theorem 1.10.] *Let L be a linear operator defined on a linear subspace D_L of the Banach space V, w a real number, and $M > 0$. Then L generates a strongly continuous semigroup $(S(t))_{t\geqslant 0}$ that satisfies $\|S(t)\| \leqslant M\,\mathrm{e}^{wt}$ if and only if*

(a) *L is closed and D_L is dense in V,*
(b) *every real $r > w$ belongs to the resolvent set of L and for such r and for all positive integers n, $\|(rI - L)^{-n}\| \leqslant \frac{M}{(r-w)^n}$.*

Definition 1.3. [21] For the semigroup $(S(t))_{t\geqslant 0}$ generated by an operator L, the value $w(L)$ represents the semigroups growth limit, and is provided as

$$w(L) = \inf\{\lambda \in \mathbb{R} : \exists M > 0 \quad \text{such that} \quad \|S(t)\| \leqslant M\,\mathrm{e}^{\lambda t}, \quad \forall t \geqslant 0\}.$$

If $w(L) < 0$, then the semigroup $(S(t))_{t\geqslant 0}$ is called uniformly exponentially stable.

We now proceed to study the equation in its unbounded form

$$AXu - XBu = Cu, \quad u \in \mathcal{D}_B \cap \mathcal{D}_C. \tag{1.2}$$

Below we recall some results which give a bounded and possibly unique solution to (1.2).

Theorem 1.5. [41]*Let A and $-B$ be generators of $\mathcal{C}_0-$semigroups $(T(t))$ and $(S(t))$, $t \geqslant 0$, on Banach spaces V_2 and V_1, respectively and let C be an operator from V_1 to V_2. Let*

$$Q(t) : \mathcal{D}_B \subset V_1 \to V_2 : \quad Q(t)(f) := T(t)CS(t)(f), \quad t \geqslant 0,$$

$$R(t) : \mathcal{D}_B \subset V_1 \to V_2 : \quad R(t)(f) := -\int_0^t Q(s)f ds, \quad t \geqslant 0.$$

Assume that:

(1) The weak topology closure of $\{Q(t)f\}_{t\geqslant 0}$ contains zero, for every $f \in \mathcal{D}_B$;
(2) $R(t)$ has a continuous extension to a bounded linear operator, for every $t \geqslant 0$ and the family $\{R(t)\}_{t\geqslant 0}$ is relatively compact with respect to the weak topology.

Then the equation (4.1) has a bounded solution. Contrary, if (1.2) has a bounded solution then $R(t)$ is bounded, for every $t \geqslant 0$. Furthermore, if for every bounded linear operator Y from V_1 to V_2 the operator $T(t)YS(t)$ converges towards zero when $t \to +\infty$ in the weak (resp. strong, uniform) operator topology, then the solution X to the equation (1.2) is unique and $R(t)$ converges to X in the weak (resp. strong, uniform) topology.

Theorem 1.6. [41] *Let $w(A) + w(-B) < 0$ and assume the family $(R(t))_{t\geqslant 0}$ from the Theorem* 1.5 *to be uniformly exponentially stable. Then the equation* (1.2) *has a unique bounded solution.*

Remark 1.1. The premise that $w(A) + w(-B) < 0$ actually contains the assumption $\sigma(A) \cap \sigma(B) = \emptyset$. Recall the Hille–Yosida theorem.

However, if $\sigma(A) \cap \sigma(B) \neq \emptyset$, or the operators A and $-B$ do not generate C_0-semigroups, then the previous theorems cannot be applied. This is the reason why we study the singular Sylvester equation (1.2) with unbounded A, B and C.

1.2. Motivation: solvability without uniqueness. Though the equation (1.1) has found vast applications in the previously mentioned papers, there are some drawbacks in assuming that the equation is regular.

When $C = 0$ the equation is said to be homogeneous. If a homogeneous equation is regular, then there is only one solution to $AX = XB$ and that is the trivial one $X = 0$. However, if we observe the case where $\sigma(A) \cap \sigma(B) \neq \emptyset$ (this also includes the case when $A = B$), then finding all nontrivial solutions (if they exist) to the homogeneous equation is equivalent to finding all X such that $AX = XB$, or specially, $AX = XA$. These are known as the commutator problems, which cannot be studied in terms of regular Sylvester equations. The expression $AX - XB$ is often called derivation of A and B.

Example 1.2. Let A and B be bounded but not compact linear operators on Banach spaces V_2 and V_1, respectively. Is there a noncompact operator X such that the derivation $AX - XB$ is a compact operator from V_1 to V_2? This problem appears in several research papers, for example see [38] and [39].

Denote by $C(V)$ the set of all compact operators over a Banach space V. Recall that $C(V)$ is a closed two-sided ideal in $\mathcal{B}(V)$ and $\mathcal{B}(V)/C(V)$ is a nontrivial quotient operator algebra, called the Calkin algebra. By observing $\mathcal{B}(V_1)/C(V_1)$ and $\mathcal{B}(V_2)/C(V_2)$, one transforms the initial problem into solving the homogeneous Sylvester equation $\widehat{A}\widehat{X} = \widehat{X}\widehat{B}$ for $\widehat{X} \in \mathcal{B}(V_1, V_2)/C(V_1, V_2)$ (recall that $\widehat{A} = A + C(V_2)$ and $\widehat{B} = B + C(V_1)$). If $\sigma(\widehat{A}) \cap \sigma(\widehat{B}) = \emptyset$, then the only solution is $\widehat{X} = 0_{\mathcal{B}(V_1,V_2)/C(V_1,V_2)}$, that is, the only solution is the entire class of compact operators from V_1 to V_2. Therefore, finding a noncompact X such that $AX - XB$ is compact requires the premise $\sigma(\widehat{A}) \cap \sigma(\widehat{B}) \neq \emptyset$. This problem was solved by the author in [14], and Section 3 below gives these results.

More generally, assume that $C \neq 0$ and $\sigma(A) \cap \sigma(B) \neq \emptyset$ then one gets the inhomogeneous singular equation (1.1). Notice that even in the simplest case, when $A = B$, the singular equation is not necessarily solvable.

Lemma 1.3. *Let $\mathcal{A}$ be a unital Banach algebra, with* 1 *as its unity. Then 1 is not a commutator in $\mathcal{A}$, meaning that, there are no a, $x \in \mathcal{A}$ such that $ax - xa = 1$.*

Proof. The proof can be found in numerous books on functional analysis and operator theory, to name a few, see [19, 22, 23, 27, 49, 55, 63, 64, 67]. Let a, $x \in \mathcal{A}$ and let $\sigma(a)$, $\sigma(x)$ denote the spectra of a and x, respectively, in $\mathcal{A}$. Then $\sigma(ax) \cup \{0\} =$

$\sigma(xa) \cup \{0\}$. On the other hand, if $ax = 1 + xa$, then $\sigma(ax) = \sigma(1 + xa) = 1 + \sigma(xa) = \{1 + \lambda : \lambda \in \sigma(xa)\}$. Consequently,

$$\sigma(xa) \cup \{0\} = \sigma(ax) \cup \{0\} = \{1 + \lambda : \lambda \in \sigma(xa)\} \cup \{0\}.$$

The above set equality is impossible for nonempty compact subsets of $\mathbb{C}$. □

However, if we change the nature of the spaces and the given linear operators, then the commutator equation $ax - xa = 1$ is indeed solvable:

Example 1.3. Let P be the one-dimensional position operator and let Q be the one-dimensional momentum operator, obtained by the virtue of the Fourier transform from P. These are essentially self-adjoint (unbounded) linear operators and $\sigma(\bar{P}) = \sigma(\bar{Q}) = \mathbb{R}$, see [63] and [65]. Additionally, they satisfy the fundamental equation of quantum mechanics $PQ - QP = \frac{h}{2\pi i} I$, where h is the Planck's constant ($h = 6.62607004 \cdot 10^{-34}\text{m}^2\text{kg/s}$). In terms of Sylvester equations, if $A = B = P$ then $X = Q$ is a solution to the singular equation $AX - XA = I$ and vice versa. The spectral overlap $\sigma(\bar{A}) \equiv \sigma(\bar{B}) = \mathbb{R}$ is obvious, so the above equation is indeed a singular Sylvester equation with unbounded A and B. This emphasizes the importance of closed operators and their advantage over the bounded linear operators (in fact, the entire quantum mechanics is built on this equation). This equation was studied in [65] and partially in [10]. In Section 4.3 we show the results from [10].

The previous examples show that simply assuming $\sigma(A) \cap \sigma(B) \neq \emptyset$ does not guarantee neither solvability nor un-solvability of the initial equation (1.1). Thus more detailed analysis is required in this case. In addition to the already mentioned problems which favor the singular setting $\sigma(A) \cap \sigma(B) \neq \emptyset$, below we mention some applications which only require solvability of the Sylvester equation, discarding (non)uniqueness of the solution.

Example 1.4 (Roth's removal rule)**.** Consider the 2×2 bounded operator matrix $\left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right]$ defined on $V_2 \times V_1$. When is this matrix block-diagonal? The diagonalization problem is essential in applied operator theory and matrix analysis, as it drastically simplifies computational procedures, such as computation of the matrix (or operator) sign function, linear model reductions, invariant subspaces characterization etc. consult [3, 5, 6, 8, 20, 29, 30, 37, 41, 45, 46, 50, 51, 54, 58, 59, 64].

One way to block-diagonalize the operator matrix $\left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right]$ is to prove its similarity to $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$. Recall that for every $Y \in \mathcal{B}(V_1, V_2)$, the operator marix $\left[\begin{smallmatrix} I_2 & Y \\ 0 & I_1 \end{smallmatrix}\right]$ is invertible in $\mathcal{B}(V_2 \times V_1)$, with $\left[\begin{smallmatrix} I_2 & -Y \\ 0 & I_1 \end{smallmatrix}\right]$ being its inverse. Now if the Sylvester equation $AX - XB = -C$ is solvable for some (not necessarily unique) $X \in \mathcal{B}(V_1, V_2)$, then the below equality holds

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix},$$

that is, the initial operator matrix $\left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right]$ is similar to the block-diagonal operator matrix $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$. Simple application of mathematical induction generalizes this statement to n-dimensional upper triangular operator matrices, consult [6].

Example 1.5 (Fréchet derivatives)**.** For the given Banach spaces V_1 and V_2, let U be an open subspace of V_1 and let $x \in U$ such that the ball $U(x,t)$ and point x at radius $t > 0$ is contained in U, see [18]. A function $f\colon U \to V_2$ is said to be Fréchet differentiable at x if there exists a linear operator $L \in \mathcal{B}(V_1, V_2)$ defined at point $y \in V_1$ as

$$Df_x(y) = \lim_{t\to 0} \frac{f(x+ty) - f(x)}{t} = Ly.$$

Specially, let $V_1 = V_2 = V$ be a Banach space, A, $B \in \mathcal{B}(V)$ and let $f(A) = A^2$. Then the Fréchet derivative of f at point A is a linear operator in $\mathcal{B}(\mathcal{B}(V))$, defined at point B as the expression

$$Df_A(B) = \lim_{t\to 0} \frac{(A+tB)^2 - A^2}{t} = \lim_{t\to 0} \frac{A^2 + tAB + tBA + t^2B^2 - A^2}{t} = AB + BA.$$

For a given $C \in \mathcal{B}(V)$, observe the abstract linear ODE

$$Df_A(B) = C. \tag{1.3}$$

When is (1.3) *solvable? If there are infinitely many solutions, is there a way of extracting one particular solution?* This was answered in [14] and these results are shown in Section 3.

More generally, let $\mathcal{A}$ be a unital C^*-algebra and let $a \in \mathcal{A}$. Define $g(a)$ as $g(a) = a^*a$. Direct computation shows that g is Fréchet differentiable at a and at point $x \in \mathcal{A}$ it takes the value

$$\begin{aligned} Dg_a(x) &= \lim_{t\to 0} \frac{(a+tx)^*(a+tx) - a^*a}{t} \\ &= \lim_{t\to 0} \frac{a^*a + tx^*a + ta^*x + t^2x^*x - a^*a}{t} = x^*a + a^*x. \end{aligned}$$

Let $\mathcal{A}$ be a unital C^*-algebra, $a \in \mathcal{A}$ and let $g(a) = a^*a$. When does there exist an $x \in \mathcal{A}$ such that $Dg_a(x) = 1_{\mathcal{A}}$? This was answered in [12] and these results are demonstrated in Section 5.

For a moment, assume that the singular equation (1.1) is solvable. Since the Sylvester operator $S\colon X \mapsto AX - XB$ is linear in X, it is trivial to see that the set of all solutions to (1.1) can be characterized as

$$\{X:\ AX - XB = C\} = X_p + \{X_h:\ AX_h = X_hB\},$$

where X_p is one particular (fixed) solution to (1.1), while $\{X_h\}$ are all solutions to the homogeneous equation $AX_h = X_hB$. This will be heavily exploited in the further text.

2. The singular equation in matrices

In this section we consider the simplest singular case, and that is when V_1 and V_2 are finite-dimensional vector spaces over $\mathbb{C}$ while A, B and C are scalar matrices of appropriate dimensions. We revisit some results from [15], with a comment that some proofs are corrected and more precise.

At this point we assume that $\sigma(A) \cap \sigma(B) \neq \emptyset$. Denote by σ the spectral intersection of matrices A and B: $\{\lambda_1, \dots, \lambda_s\} =: \sigma = \sigma(A) \cap \sigma(B)$.

For more elegant notation, we introduce $E_B^k = \mathcal{N}(B - \lambda_k I)$ and $E_A^k = \mathcal{N}(A - \lambda_k I)$ whenever $\lambda_k \in \sigma$. Different eigenvalues generate linearly independent eigenvectors, so the spaces E_B^k form a direct sum. Put $E_B := \sum_{k=1}^{s} E_B^k$. It is a closed subspace of V_1 and there exists $E_B^\perp$ such that $V_1 = E_B \oplus E_B^\perp$. With respect to that decomposition, denote $B_E := BP_{E_B}$, $B_1 := BP_{E_B^\perp}$ and $C_1 := CP_{E_B^\perp}$. In that sense, the upper triangular splitting of the matrix B holds:

$$B = \begin{bmatrix} B_E & B_0 \\ 0 & B_{11} \end{bmatrix} : \begin{bmatrix} E_B \\ E_B^\perp \end{bmatrix} \to \begin{bmatrix} E_B \\ E_B^\perp \end{bmatrix}, \quad \text{where } B_1 = \begin{bmatrix} B_0 \\ B_{11} \end{bmatrix}. \tag{2.1}$$

Notice that E_B is a B-invariant subspace of V_1, that is, $B(E_B) = E_B$, and consequently $E_B \subset \mathcal{R}(B)$, while $E_B^\perp$ is B_{11}-invariant subspace of V_1. Additionally, B_{11} is a square matrix from $\mathcal{B}(E_B^\perp)$.

Lemma 2.1. *With respect to the previous notation, if*

$$B_0 \colon \mathcal{N}(B_{11} - \lambda I_{E_B^\perp}) \to \mathcal{R}(B_E - \lambda I_{E_B}), \quad \textit{for every} \quad \lambda \in \sigma(B_{11}), \tag{2.2}$$

then $\sigma(B_{11}) \subset \sigma(B)$.

Proof. Let $\lambda \in \sigma(B_{11})$ be arbitrary. Then for every $v \in \mathcal{N}(B_{11} - \lambda I_{E_B^\perp})$ there exists a vector $u \in E_B$ such that $(B_E - \lambda I_{E_B})(u) = B_0 v$, i.e. $B_E(-u) = -\lambda I_{E_B} u - B_0 v$. It is not difficult to see that

$$B \begin{bmatrix} -u \\ v \end{bmatrix} = \begin{bmatrix} B_E & B_0 \\ 0 & B_{11} \end{bmatrix} \begin{bmatrix} -u \\ v \end{bmatrix} = \begin{bmatrix} B_E(-u) + B_0 v \\ \lambda v \end{bmatrix} = \lambda \begin{bmatrix} -u \\ v \end{bmatrix},$$

so $\lambda \in \sigma(B)$, with $[-u \quad v]^T$ being the corresponding eigenvector for B. □

Theorem 2.1 (Existence of solutions)**.** *[15, Theorem 2.1.] With respect to the previous notation, let B be such that* (2.2) *holds. Additionally, if the the condition*

$$C \colon E_B^k \to \mathcal{R}(A - \lambda_k I) \tag{2.3}$$

holds for every $k = \overline{1, s}$ then there exist infinitely many solutions X to the matrix equation

$$AX - XB = C. \tag{2.4}$$

Proof. Recall notation from the previous paragraph. Respectively, the matrix B has the uppper triangular decomposition (2.1)

$$B = \begin{bmatrix} B_E & B_0 \\ 0 & B_{11} \end{bmatrix} : \begin{bmatrix} E_B \\ E_B^\perp \end{bmatrix} \to \begin{bmatrix} E_B \\ E_B^\perp \end{bmatrix}.$$

We first conduct analysis on E_B. For every $k \in \{1, \dots, s\}$ let $N_k \in \mathcal{B}(E_B^k, E_A^k)$ be an arbitrary linear mapping. For every $u \in E_B^k$, by the assumption (2.3), there exists a unique $d_u \in (E_A^k)^\perp$ such that $(A - \lambda_k I) d_u = Cu$. Define

$$X_{E(N_k)}^k \colon u \mapsto N_k u + d_u, \quad u \in E_B^k,$$

which is trivially a solution to $AY - YB_E = CP_{E_B}$ observed on E_B^k: for any $u_k \in E_B^k$ we have

$$AX^k_{E(N_k)}u_k - X^k_{E(N_k)}Bu_k = AX^k_{E(N_k)}u_k - \lambda_k X^k_{E(N_k)}u_k$$
$$= (A - \lambda_k I_{V_2})X^k_{E(N_k)}u_k = (A - \lambda_k I_{V_2})(N_k u_k + d_{u_k}) = Cu_k.$$

Adding them together gives $X_{E(N_1,\dots,N_s)} := \sum_{k=1}^{s} X^k_{E(N_k)}$, which is well defined on E_B as eigenvectors which correspond to different eigenvalues are linearly independent. Direct verification shows that $X_{E(N_1,\dots,N_s)}$ is a solution to $AY - YB_E = CP_{E_B}$ observed on E_B.

Now observe the complemented space $E_B^\perp$. By construction it follows that $\sigma = \sigma(B_E)$ and for every $\mu \in \sigma$ the eigenspace $\mathcal{N}(B_E - \mu I_{E_B})$ is just a formal projection of $\mathcal{N}(B - \mu I)$, i.e. $\mathcal{N}(B_E - \mu I_{E_B}) + 0_{E_B^\perp} = \mathcal{N}(B - \mu I)$. By the virtue of condition (2.2), Lemma 2.1 states that $\sigma(B_{11}) \subset \sigma(B)$, thus for every $\mu \in \sigma(B_{11})$ and for every corresponding eigenvector $u \in \mathcal{N}(B_{11} - \mu I_{E_B^\perp})$ there exists a vector $v \in E_B$ such that $(B_E - \mu I_{E_B})v = B_0 u$ and in that case $[-v \quad u]^T$ is an eigenvector for B which corresponds to μ. Assume that $\sigma(B_{11}) \cap \sigma(B_E) \neq \emptyset$ and denote by μ_0 their shared eigenvalue. As previously explained, for every eigenvector $u \in \mathcal{N}(B_{11} - \mu_0 I_{E_B^\perp})$ there exists a $v \in E_B$ such that $[-v \quad u]^T$ is an element in $\mathcal{N}(B - \mu_0 I)$. However, since $\mu_0 \in \sigma(B_E)$ it follows that $\mathcal{N}(B - \mu_0 I) = \mathcal{N}(B_E - \mu_0 I_{E_B}) + 0_{E_B^\perp}$, thus $u = 0_{E_B^\perp}$ which is impossible. Ergo, $\sigma(B_E) \cap \sigma(B_{11}) = \emptyset$ and finally $\sigma(B_{11}) \cap \sigma(A) = \emptyset$. Observe the reduced Sylvester equation on $E_B^\perp$

$$AXP_{E_B^\perp} - XB_1 = C_1 \Leftrightarrow AX_1 - X_1 B_{11} = C_1 + X_{E(N_1,\dots,N_s)}B_0, \tag{2.5}$$

where $B_{11} \in \mathcal{B}(E_B^\perp)$, $A \in \mathcal{B}(V_2)$, $C_1 + X_{E(N_1,\dots,N_s)}B_0 \in \mathcal{B}(E_B^\perp, V_2)$ are known matrices such that $\sigma(B_{11}) \cap \sigma(A) = \emptyset$, while $X_1 \in \mathcal{B}(E_B^\perp, V_2)$ is the sought solution. By the Sylvester theorem, there exists a unique $X_{1(N_1,\dots,N_s)} \in \mathcal{B}(E_B^\perp, V_2)$ such that (2.5) holds. Finally, it follows that

$$X = \begin{bmatrix} X_{E(N_1,\dots,N_s)} & X_{1(N_1,\dots,N_s)} \end{bmatrix} : \begin{bmatrix} E_B \\ E_B^\perp \end{bmatrix} \to V_2 \tag{2.6}$$

is an infinite family of solutions to the eq. (2.4). □

Remark 2.1. In the paper [15], Theorem 2.1. has a slightly different formulation, which is inaccurate: it assumes that $\sigma(B_{11}) \subset \sigma(B)$ by default, and does not have the additional assumption (2.2). Since in general this is not true, here we include the assumption (2.2). Additionally, proof of [15, Theorem 2.1.] contains the case when $\sigma(B_{11}) \cap \sigma(A) = \{0\}$, leading to more parametric solutions which require another solvability condition: $\mathcal{N}(C_1)^\perp = \mathcal{R}(B_{11})$. We now know that this case will never occur, thanks to Lemma 2.1, so this condition is obsolete.

Reading the proof of Theorem 2.1, we notice the following questions:

Question 1.: Is every solution to the equation (2.4) of the form (2.6)?

Question 2.: Under which conditions is the solution to (2.4) unique?

Both of these questions have affirmative answers, which is justified by the analysis of the following *eigen-problem associated with the given Sylvester equation*:

Assume that $\emptyset \neq \sigma = \sigma(A) \cap \sigma(B)$ *and let* $N_\lambda \in \mathcal{B}(E_B^\lambda, E_A^\lambda)$ *be arbitrary, for every* $\lambda \in \sigma$. *Define* $N_\sigma := \sum_{\lambda\in\sigma} N_\lambda$. *Find a solution* X *to the Sylvester equation such that the following eigen-problem is uniquely solved:*

$$\begin{cases} AX - XB = C \\ Xu_\lambda := P_{(E_A^\lambda)^\perp}(A-\lambda I)^{-1}Cu_\lambda + N_\lambda u_\lambda, \quad u_\lambda \in E_B^\lambda,\ \lambda \in \sigma. \end{cases} \tag{2.7}$$

Theorem 2.2 (Uniqueness of the solution to the eigen-problem)**.** [15, Theorem 2.2.] *With respect to the previous notation, assume that* (2.2) *holds.*

(1) If the condition (2.3) *holds for every shared eigenvalue* $\lambda \in \sigma$, *then the solution* X *depends only on the choice for matrix* N_σ, *that is, for fixed* N_σ, *there exists a unique solution* X *such that* (2.7) *holds.*

(2) Conversely, for every solution X *to* (2.4) *and for every shared eigenvalue* λ *for matrices* A *and* B, *there exists a unique quotient class*

$$(A-\lambda I)^{-1}C(\mathcal{N}(B-\lambda I)) \oplus \mathcal{N}(A-\lambda I)$$

such that X *is the unique solution to the quotient eigen-problem* (2.7).

Proof. Recall notation from proof of Theorem 2.1.

(1) The first statement of the theorem is proved directly. Namely, take $V_1 = E_B \oplus E_B^\perp$, $B = B_E \oplus B_1$, $V_2 = E_A \oplus E_A^\perp$, $A = A_E \oplus A_1$ like in Theorem 2.1. Then there exists $X = X_E \oplus X_1$, which is a solution to (2.4). By construction, since $\sigma(B_{11}) \cap \sigma(A) = \emptyset$, there exists a uniquely determined X_1 in $\mathcal{B}(E_B^\perp, V_2)$ while X_E^λ is uniquely determined in the class $\mathcal{B}(E_B/E_B^\lambda, V_2/E_A^\lambda)$ for every $\lambda \in \sigma$. Varying λ in σ completes the proof.

(2) Conversely, let X be a solution to the eq. (2.4). Let λ be one of the shared eigenvalues for A and B and fix u as a corresponding eigenvector for B. Then $XBu = \lambda Xu$. Hence $AXu - XBu = (A-\lambda I)Xu = Cu$. Split Xu into the orthogonal sum $Xu = v_1 + v_2$, where $v_1 \in \mathcal{N}(A-\lambda I)$ and $v_2 \in (\mathcal{N}(A-\lambda I))^\perp$. Then v_2 is the sought expression $P_{\mathcal{N}(A-\lambda I)^\perp}(A-\lambda I)^{-1}Cu$ and $Xu \in v_2 + \mathcal{N}(A-\lambda I)$. Condition (2.3) follows immediately. Repeating the same procedure for every shared eigenvalue for A and B completes the proof. ☐

Corollary 2.1 (Number of solutions)**.** [15, Corollary 2.1.] *Let* Σ *be the set of all* N_σ *introduced in the eigen-problem associated with given Sylvester equation* (2.7), *that is*

$$\Sigma = \left\{ N_\sigma : \quad N_\sigma = \sum_{\lambda\in\sigma} N_\lambda, \quad N_\lambda \in \mathcal{B}(E_B^\lambda, E_A^\lambda), \quad \lambda \in \sigma(A)\cap\sigma(B) = \sigma \right\}.$$

Let S *be the set of all solutions to* (2.4) *which satisfy condition* (2.2)–(2.3). *Then* $|\Sigma| = |S|$.

Proof. For fixed $N_\sigma \in \Sigma$ there exits a unique $X \in S$ such that (2.7) holds. Further, for arbitrary $X \in S$ and arbitrary $\lambda \in \sigma$ there exist quotient classes E_A^λ and E_B^λ such that (2.7) holds. Define $N_\lambda : E_B^\lambda \to E_A^\lambda$ to be bounded. Then $N_\sigma = \sum_{\lambda\in\sigma} N_\lambda$. It follows that $N_\sigma \in \Sigma$. There is a one-to-one surjective correspondence $S \leftrightarrow \Sigma$. ☐

Remark 2.2. Due to Corollary 2.1, for fixed $N_\sigma \in \Sigma$, the solution $X_{(N_\sigma)} \in S$ can be referred to as a *particular solution.*

Corollary 2.2 (Size of a particular solution)**.** [15, Corollary 2.2.] *With the assumptions and notation from Theorem* 2.1*, Theorem* 2.2 *and Corollary* 2.1*, the norm of* $X_{(N_\sigma)}$ *is given as*

$$\begin{aligned}\|X_{(N_\sigma)}\|^2 &= \|X_E\|^2 + \|X_1\|^2 \\ &\leqslant \|N_\sigma\|^2 + \sum_{k=1}^{s} \|P_{(E_A^k)^\perp}(A - \lambda_k I)^{-1} C P_{E_B^k}\|^2 + \|X_1\|^2,\end{aligned}$$

where equality holds if and only if the sum $\sum_{k=0}^{s} E_B^k$ *is orthogonal.*

Proof. Taking the same decomposition as in Theorem 2.1, let $X_{(N_\sigma)} = X_E + X_1$. Since X_E annihilates $E_B^\perp$ and X_1 annihilates E_B, it follows that

$$\|X_{(N_\sigma)}\|^2 = \|X_E + X_1\|^2 = \|X_E\|^2 + \|X_1\|^2.$$

By the same argument, taking

$$\|X_E\|^2 \leqslant \|N_\sigma\|^2 + \sum_{k=1}^{s} \|P_{(E_A^k)^\perp}(A - \lambda_k I)^{-1} C P_{E_B^k}\|^2,$$

where the equality holds if and only if the sum $\sum_{k=1}^{s} E_B^k$ is orthogonal. □

Recall that the equation (2.4) is said to be homogeneous when $C = 0$. This brings our attention to the set of all X, such that $AX = XB$. The following corollary speaks about cardinality of such a set.

Corollary 2.3. [16, Corollary 2.4.] *Let* $\lambda_1, \ldots, \lambda_s$ *be the* s *different common non-zero eigenvalues for square matrices* A *and* B*. For every* $k = \overline{1, s}$*, let* E_B^k *be the eigenspace for* B *which corresponds to* λ_k *and let* E_A^k *be the eigenspace for* A *which corresponds to* λ_k*. For every* $k = \overline{1, s}$*, put* $q_B^k := \dim E_B^k$ *and* $q_A^k := \dim E_A^k$*. There are at least* $\prod_{k=1}^{s} \left(q_A^k\right)^{q_B^k}$ *different non-zero solutions to the homogeneous equation* (2.4)*, acting from* $\sum_{k=1}^{s} E_B^k$ *to* $\sum_{k=1}^{s} E_A^k$*, which are non-zero on every eigenspace* E_B^k*,* $\quad k = \overline{1, s}$*.*

3. Generalization to bounded linear operators

In this section we extend Theorem 2.1 to the space of bounded linear operators over Banach spaces. We now return to the general case, where V_1 and V_2 are Banach spaces and A, B and C are accordingly provided bounded linear operators, such that $\sigma(A) \cap \sigma(B) \neq \emptyset$. Recall Lemma 1.2: if $\sigma(A) \cap \sigma(B) = \emptyset$ then the Sylvester operator $S\colon X \mapsto AX - XB$ is invertible in $\mathcal{B}(\mathcal{B}(V_1, V_2))$ thus for every $C \in \mathcal{B}(V_1, V_2)$ the solution X is uniquely determined as $X = S^{-1}(C) \in \mathcal{B}(V_1, V_2)$. However, our premise allows the possibility that $0 \in \sigma(S)$, and in that case the solution X is not uniquely determined (if it exists at all!).

Thus we try to go around this obstacle, by redirecting the problem into the Banach algebra $\mathcal{B}(\mathcal{B}(V_1, V_2))$. A bounded linear operator $J \in \mathcal{B}(V_1, V_2)$ is said to be a bounded embedding if it is linear, continuous and one-one. If a bounded embedding J has a closed range in V_2, then it has a bounded inverse which is defined as $J^{-1}\colon \mathcal{R}(J) \to V_1$, $J^{-1}(Ju) = u$, for every $u \in V_1$ (bounded inverse theorem, see [48] or [64]).

Theorem 3.1. [14, Theorem 3.1.] *Let V_1 and V_2 be Banach spaces and let $A \in \mathcal{B}(V_2)$, $B \in \mathcal{B}(V_1)$ and $C \in \mathcal{B}(V_1, V_2)$ be given bounded linear operators. Assume that there exists a bounded embedding $J \in \mathcal{B}(V_1, V_2)$ with a closed complemented range in V_2 and denote by Q the projector from V_2 onto $\mathcal{R}(J)$. Define operators $\widehat{C}$ and $\widehat{S}$ in $\mathcal{B}(\mathcal{B}(V_1, V_2))$ as $\widehat{C}(L) := CJ^{-1}QL$ and $\widehat{S}(L) := AL - LB$, for every $L \in \mathcal{B}(V_1, V_2)$. If the equation*

$$\widehat{S}\widehat{X} = \widehat{C} \tag{3.1}$$

is solvable in $\mathcal{B}(\mathcal{B}(V_1, V_2))$, then there exists a solution $X \in \mathcal{B}(V_1, V_2)$ to (1.1).

Proof. In addition to $\widehat{S}$ and $\widehat{C}$, define the following operators

$$\widehat{A} \in \mathcal{B}(\mathcal{B}(V_1, V_2)), \quad \widehat{A}(L) := AL, \qquad L \in \mathcal{B}(V_1, V_2),$$
$$\widehat{B} \in \mathcal{B}(\mathcal{B}(V_1, V_2)), \quad \widehat{B}(L) := LB, \qquad L \in \mathcal{B}(V_1, V_2).$$

We immediately have

$$\widehat{S}(L) = AL - LB = (\widehat{A} - \widehat{B})(L),$$
$$\widehat{A}\widehat{B}(L) = \widehat{A}(LB) = ALB = \widehat{B}(AL) = \widehat{B}\widehat{A}(L),$$

for every $L \in \mathcal{B}(V_1, V_2)$. Thus $\widehat{S} = \widehat{A} - \widehat{B}$, while $\widehat{A}$ and $\widehat{B}$ commute. Further, notice that $J^{-1}QJ = I_{V_1}$. If (3.1) is solved for some $\widehat{X} \in \mathcal{B}(\mathcal{B}(V_1, V_2))$ then $\widehat{S}\widehat{X}(L) = \widehat{C}(L)$ for all $L \in \mathcal{B}(V_1, V_2)$, thus $\widehat{S}\widehat{X}(J) = \widehat{C}(J) = C$ which implies $\widehat{A}(\widehat{X}(J)) - \widehat{B}(\widehat{X}(J)) = \widehat{C}(J)$. Consequently, (1.1) is solved by the operator $\widehat{X}(J)$. □

Corollary 3.1. [14, Corollary 3.1.] *Let $B \in \mathcal{B}(V_1)$, $A \in \mathcal{B}(V_2)$ and $C \in \mathcal{B}(V_1, V_2)$. If V_1 is a closed, complemented subspace of V_2 and Q is the corresponding projector from V_2 onto V_1, $\mathcal{R}(Q) = V_1$, then there exists a solution to* (1.1) *if eq.* (3.1) *is solvable, where $\widehat{C}(L) := CQL$ for every $L \in \mathcal{B}(V_1, V_2)$.*

Proof. Let $I \in \mathcal{B}(V_2)$. The mapping $I \restriction_{V_1}\colon V_1 \to V_2$ is the one-one bounded embedding of V_1 into V_2, with a closed and complemented range in V_2: $\mathcal{R}(I \restriction_{V_1}) = V_1 = Q(V_2)$. It has a bounded inverse $(I \restriction_{V_1})^{-1} : \mathcal{R}(Q) = V_1 \to V_1$, again defined as $(I \restriction_{V_1})^{-1} = I \restriction_{V_1}$. Thus instead of J from Theorem 3.1 we observe $I \restriction_{V_1}$, while the rest of the proof is the same. □

In order to solve (3.1) we turn to the generalized inverses (see [19]). Recall that, if $\widehat{S} = 0$, then the equation is solvable if and only if $\widehat{C} = 0$ and in that case, every $\widehat{X}$ is a solution to (3.1). However, recall that, if $\widehat{S} \neq 0$ then it is outer regular i.e. there exists an outer inverse $\widehat{S}^{(2)} \neq 0$ such that $\widehat{S}^{(2)}\widehat{S}\widehat{S}^{(2)} = \widehat{S}^{(2)}$. Precisely, the outer inverse $\widehat{S}^{(2)}$ can be obtained in the following way.

Since $\widehat{S} \neq 0$, it follows that there exists an $L \in \mathcal{B}(V_1, V_2)$ such that $\widehat{S}L = Z \in \mathcal{B}(V_1, V_2)$ and $Z \neq 0$. In that sense, observe the Kato decompositions of the space

$\mathcal{B}(V_1, V_2) = \text{span}\{L\} + M$ and $\mathcal{B}(V_1, V_2) = \text{span}\{Z\} + N$. The operator $\widehat{S}$ has the decomposition

$$\widehat{S} = \begin{bmatrix} \widehat{S}_{11} & \widehat{S}_{12} \\ 0 & \widehat{S}_{22} \end{bmatrix} : \begin{bmatrix} \text{span}\{L\} \\ M \end{bmatrix} \to \begin{bmatrix} \text{span}\{Z\} \\ N \end{bmatrix},$$

where $\widehat{S}_{11}$ is invertible. Respectively, the operator $\widehat{S}^{(2)}$ can be chosen to be

$$\widehat{S}^{(2)} = \begin{bmatrix} \widehat{S}_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \text{span}\{Z\} \\ N \end{bmatrix} \to \begin{bmatrix} \text{span}\{L\} \\ M \end{bmatrix}.$$

Now observe the equation (3.1). Since $\widehat{S} \neq 0$ there exists an $\widehat{S}^{(2)} \in \mathcal{B}(\mathcal{B}(V_1, V_2))$ such that:

$$\widehat{S}^{(2)} = \begin{bmatrix} \widehat{S}_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R \\ \mathcal{N}(\widehat{S}^{(2)}) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(\widehat{S}^{(2)}) \\ T \end{bmatrix},$$

where $\mathcal{B}(V_1, V_2) = R \oplus \mathcal{N}(\widehat{S}^{(2)})$ and simultaneously $\mathcal{B}(V_1, V_2) = \mathcal{R}(\widehat{S}^{(2)}) \oplus T$. Respectively, the operator $\widehat{S}$ has the representation

$$\widehat{S} = \begin{bmatrix} \widehat{S}_1 & 0 \\ 0 & N_{\widehat{S}^{(2)}} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(\widehat{S}^{(2)}) \\ T \end{bmatrix} \to \begin{bmatrix} R \\ \mathcal{N}(\widehat{S}^{(2)}) \end{bmatrix}.$$

It follows that $\widehat{S}\widehat{S}^{(2)} = P_R$.

Proposition 3.1. *Assume that $\widehat{S}$ is outer regular. If there exists an outer inverse for $\widehat{S}$, $\widehat{S}^{(2)}$, such that $\mathcal{R}(\widehat{C}) \subset R$, where $\mathcal{B}(V_1, V_2) = R \oplus \mathcal{N}(\widehat{S}^{(2)})$, then the equation* (3.1) *is solvable, and one of its solutions is $\widehat{X} = \widehat{S}^{(2)}\widehat{C}$.*

Normally, such equations are solved via inner inverses, or via Drazin inverse. However, these inverses require, among other things, closedness of $\mathcal{R}(\widehat{S})$ and complementedness of $\mathcal{N}(\widehat{S})$ and $\mathcal{R}(\widehat{S})$ (see [19]). Since this is effectively not easy to verify, we turn to the outer inverses, as any non-zero operator is outer-regular.

3.1. Connections to Fredholm theory. Below we generalize Theorem 3.1 to the case when V_1 cannot be embedded into V_2. Recall that a bounded linear operator $J \in \mathcal{B}(V_1, V_2)$ is said to be left upper-semi Fredholm if $\alpha(J) := \dim \mathcal{N}(J)$ is finite and $\mathcal{R}(J)$ is a closed, complemented subspace in V_2 (equivalently, J is left invertible in the Calkin algebra $\mathcal{B}(V_1)/C(V_1)$), see [68–70]. The set of all left upper-semi Fredholm operators from V_1 to V_2 is denoted as $\Phi^{\ell}(V_1, V_2)$. Notice that this is a direct generalization of bounded invertible embeddings from V_1 to V_2.

Let $B \in \mathcal{B}(V_1)$ be a bounded linear operator and let $\lambda \in \sigma(B)$ be its Riesz point. This means that V_1 can be decomposed into into a direct sum $V_1 = E_B(\lambda) \oplus F_B(\lambda)$, where $E_B(\lambda)$ is a closed, B-invariant subspace of V_1 and $B - \lambda I$ is invertible on $E_B(\lambda)$, while $F_B(\lambda)$ is the finite-dimensional B-invariant subspace of V_1, such that $B - \lambda I$ is nilpotent on $F_B(\lambda)$, see [68–70]. Fact that $F_B(\lambda)$ is a finite-dimensional B-invariant subspace of V_1 implies that $B_{F(\lambda)} := B \restriction_{F_B(\lambda)} : F_B(\lambda) \to F_B(\lambda)$ is a finite square matrix. Recall the core-nilpotent decomposition for the matrix $B_{F(\lambda)}$:

there exists an $p \in \mathbb{N}$ such that $F_B(\lambda) = \mathcal{R}((B_{F(\lambda)})^p) \oplus \mathcal{N}((B_{F(\lambda)})^p)$ and

$$B_{F(\lambda)} = \begin{bmatrix} B_{F(\lambda),1} & 0 \\ 0 & B_{F(\lambda),2} \end{bmatrix} : \begin{bmatrix} \mathcal{R}((B_{F(\lambda)})^p) \\ \mathcal{N}((B_{F(\lambda)})^p) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}((B_{F(\lambda)})^p) \\ \mathcal{N}((B_{F(\lambda)})^p) \end{bmatrix},$$

with $B_{F(\lambda),1}$ being invertible while $B_{F(\lambda),2}$ being nilpotent with index not greater than p.

Definition 3.1. [14, Definition 3.1.] Let $B \in \mathcal{B}(V_1)$ and let $\lambda \in \sigma(B)$ be a Riesz point of B and let $E_B(\lambda)$ and $F_B(\lambda)$ be the B-invariant subspaces of V_1 as described above. Let operator $L \in \mathcal{B}(V_1, V)$, for some Banach space V, be given such that $\alpha(L) = \dim \mathcal{N}(L) < +\infty$. Then operator L decomposes operator B at point λ in the Riesz sense if $B_N := B \restriction_{\mathcal{N}(L)}$ has the property that

$$F_B(\lambda) = \mathcal{R}^\infty(B_N) \oplus \mathcal{N}^\infty(B_N). \tag{3.2}$$

Example 3.1. [14, Example 3.1.] Let V_1 be an arbitrary Banach space. Let $B \in \mathcal{B}(V_1)$ be a given bounded linear operator and let $\lambda \in \sigma(B)$ be its Riesz point. Then $F_B(\lambda)$ is the finite-dimensional B-invariant subspace of V_1, which allows the decomposition described as above $F_B(\lambda) = \mathcal{R}((B_{F(\lambda)})^p) \oplus \mathcal{N}((B_{F(\lambda)})^p)$ for some $p \geqslant \operatorname{asc}(B_{F(\lambda)}) = \operatorname{dsc}(B_{F(\lambda)})$. Observe the natural quotient mapping (which is a bounded linear operator) $Q_F \colon V_1 \to V_1/F_B(\lambda)$. In that sense, denote by V the quotient space $V_1/F_B(\lambda)$. Then Q_F is a bounded linear operator from V_1 to V, with $\mathcal{N}(Q_F)$ being precisely the space $F_B(\lambda) = \mathcal{R}((B_{F(\lambda)})^p) \oplus \mathcal{N}((B_{F(\lambda)})^p)$. Further, $B_N := B \restriction_{F_B(\lambda)}$ provides a decomposition of $F_B(\lambda)$:

$$F_B(\lambda) = \mathcal{R}((B_{F(\lambda)})^p) \oplus \mathcal{N}((B_{F(\lambda)})^p) = \mathcal{R}^\infty(B_N) \oplus \mathcal{N}^\infty(B_N).$$

By Definition 3.1 we see that Q_F decomposes the operator B at point λ in the Riesz sense.

Example 3.2. Let L be any bounded linear operator such that $\mathcal{N}(L) = F_B(\lambda)$. Then B restricted to $\mathcal{N}(L)$ is precisely $B_{F(\lambda)}$ and the core-nilpotent decomposition applies. Thus, every such L decomposes operator B at point λ in the Reisz sense. Specially, any left upper-semi Fredholm operator $J \in \Phi^\ell(V_1, V_2)$, such that $\mathcal{N}(J) = F_B(\lambda)$ by default decomposes B at point λ in the Reisz sense.

Lemma 3.1. [14, Lemma 3.1.] *Let U be a finite-dimensional subspace of the Banach space V_2 and let $A \in \mathcal{B}(V_2)$. Then there exists either trivial $\{0_{V_2}\}$ or a finite-dimensional A-invariant subspace of V_2, denoted as F_A, such that*

$$F_A \subset U + \mathcal{R}(A \restriction_U).$$

Precisely, in [14] it is shown that the finite-dimensional A-invariant subspace F_A is $F_A = \mathcal{N}^\infty(A \restriction_U) \oplus \mathcal{R}^\infty(A \restriction_U)$.

Theorem 3.2. [14, Theorem 3.2.] *Let $B \in \mathcal{B}(V_1)$ such that $\lambda \in \sigma(B)$ is a Riesz point of B and let $F_B(\lambda)$ be the corresponding finite dimensional B-invariant subspace of V_1. Let F_A be the finite-dimensional subspace of V_2 defined as*

$$F_A = \mathcal{R}^\infty(A \restriction_{\mathcal{R}(C\restriction_{F_B(\lambda)})}) \oplus \mathcal{N}^\infty(A \restriction_{\mathcal{R}(C\restriction_{F_B(\lambda)})}). \tag{3.3}$$

Let $J \in \Phi^{\ell}(V_1, V_2)$ be such that it decomposes B at point λ in the Riesz sense. If $C(F_B(\lambda)) = F_A$ and the finite matrices $B\restriction_{F_B(\lambda)}$, $C\restriction_{F_B(\lambda)}$ and $A\restriction_{F_A}$ satisfy conditions (2.2)–(2.3), then there exist infinitely many solutions to (1.1) if and only if

$$AX_1 - X_1B_1 = C_1 \tag{3.4}$$

is solvable on $E_B(\lambda)$, where $V_1 = F_B(\lambda) \oplus E_B(\lambda)$ and $B_1 = B\restriction_{E_B(\lambda)}$, $C_1 = C\restriction_{E_B(\lambda)}$.

Proof. Let $J \in \Phi^{\ell}(V_1, V_2)$ be a left upper semi-Fredholm operator which decomposes B at point λ in the Riesz sense. Recall that $F_B(\lambda)$ allows the decomposition given by (3.2), that is, $F_B(\lambda) = \mathcal{N}^{\infty}(B\restriction_{\mathcal{N}(J)}) \oplus \mathcal{R}^{\infty}(B\restriction_{\mathcal{N}(J)})$. Define $B_{F(\lambda)} := B\restriction_{F_B(\lambda)}$ and $C_{F(\lambda)} := C\restriction_{F_B(\lambda)}$. Then $\mathcal{R}(C_{F(\lambda)})$ is a finite-dimensional subspace of V_2 and Lemma 3.1 implies that F_A (provided by (3.3)) is an $A-$invarinat finite-dimensional subsapce of V_2 as well. Consequently $A_{F_A} := A\restriction_{F_A}$ satisfies $A_{F_A} \colon F_A \to F_A$. Observe the finite-dimensional spaces $F_B(\lambda)$ and F_A, and the finite-dimensional matrices defined on them, that is,

$$B_{F(\lambda)} \in \mathcal{B}(F_B(\lambda)), \quad C_{F_B(\lambda)} \in \mathcal{B}(F_B(\lambda), F_A), \quad A_{F_A} \in \mathcal{B}(F_A).$$

If they satisfy conditions (2.2)–(2.3) then there exist infinitely many solutions $X_{F_B(\lambda)}$ to $A_{F_A}X_{F_B(\lambda)} - X_{F_B(\lambda)}B_{F_B(\lambda)} = C_{F_B(\lambda)}$.

To complete the proof, note that

$$V_1 = \mathcal{N}(J) \oplus V_{11} = \mathcal{N}(J) \oplus (F_B(\lambda) \cap V_{11}) \oplus E_B(\lambda) = F_B(\lambda) \oplus E_B(\lambda),$$

and each subspace is closed. Let $J_1 = J\restriction_{V_{11}}$ and $J_2 = J_1\restriction_{E_B(\lambda)}$. Since $\mathcal{R}(J)$ is closed and J_1 is injective, with $\mathcal{R}(J) = \mathcal{R}(J_1)$, it follows that

$$\mathcal{R}(J_1) = J_1(F_B(\lambda) \cap V_{11}) \oplus \mathcal{R}(J_2),$$

thus $\mathcal{R}(J_2)$ is closed as well and because J_2 is injective, J_2 has a bounded inverse from $\mathcal{R}(J_2)$ to $E_B(\lambda)$. By assumption, J is a left upper semi-Fredholm operator, so there exists a bounded projection Q_1 from V_2 onto $\mathcal{R}(J) = \mathcal{R}(J_1)$. However, since $\mathcal{R}(J_1\restriction_{F_B(\lambda)\cap V_{11}})$ is finite dimensional, it follows that there exists a bounded projection Q_2 from V_2 onto $\mathcal{R}(J_2)$, so J_2 is a bounded embedding of $E_B(\lambda)$ into V_2, with a closed range, which is complemented in V_2. Further, since $V_1 = F_B(\lambda) \oplus E_B(\lambda)$ and λ is a Riesz point for the operator B, it follows that $E_B(\lambda)$ is a closed, B-invariant subspace of V_1. Since J_2 is a bounded embedding from $E_B(\lambda)$ to V_2 with a closed and complemented range in V_2, if the equation (3.4) has a solution X_1 (sufficient conditions for its existence are obtained in Theorem 3.1), then $X := X_{F_B(\lambda)} \oplus X_1$ defined on $F_B(\lambda) \oplus E_B(\lambda) = V_1$ is a solution to (1.1). Conversely, assume (1.1) is solvable with a solution X defined on V_1, and let all decompositions provided in the statement of the theorem hold. Decompose $V_1 = F_B(\lambda) \oplus E_B(\lambda)$ which are B-invariant subspaces of V_1. Since the matrices $B_{F_B(\lambda)}$, A_{F_A} and $C_{F_B(\lambda)}$ satisfy the above-derived relations, it follows that $X\restriction_{F_B(\lambda)}$ is one of the solutions to the matrix equation $A_{F_A}X_{F_B(\lambda)} - X_{F_B(\lambda)}B_{F_B(\lambda)} = C_{F_B(\lambda)}$. On the other hand, the operator $X\restriction_{E_B(\lambda)}$ indeed solves the equation (3.4) and the proof is complete. □

Remark 3.1. If the space F_A from (3.3) is trivial, $F_A = \{0_{V_2}\}$, then we can still apply Theorem 3.2 to the homogeneous equation $AX = XB$. Namely, the condition $C(F_B(\lambda)) = F_A = \{0_{V_2}\}$ as well as conditions (2.3) are indeed satisfied when $C = 0$.

Recall that $\lambda \in \sigma(B)$ is a Riesz point for B if and only if $\lambda \in \sigma_p(B)$ and it has finite geometric multiplicity. If B does not have such an eigenvalue, then we can, under certain conditions, observe the set of its approximate eigenvalues in a similar way. The problem of transferring approximate eigenvalues into proper eigenvalues was firstly studied by Berberian in [4], which was applied to Fredholm theory by Buoni, Harte and Wickstead in [7] and [28]. In what follows, we briefly recap the construction from [7], in order to make it applicable to our problem, which is finding the Riesz points of given operators.

For $1 \leqslant i \leqslant 2$, let $\ell_\infty(V_i)$ we denote the Banach space of bounded sequences in V_i, equipped with the supremum norm. By $m(V_i)$ we denote the subspace of $\ell_\infty(V_i)$ which consists of those bounded sequences in V_i such that each subsequence has a convergent subsequence, or, equivalently (see [7, p. 310]) every element from the space $m(V_i)$ is totally bounded. Now introduce $\mathcal{P}(V_i) = \ell_\infty(V_i)/m(V_i)$, equipped with the supremum norm. This defines a Banach space, with a norm which is precisely the measure of non compactness of the given sequence $(x_n)_{n\in\mathbb{N}} \in \mathcal{P}(V_i)$: $\|(x_n)_{n\in\mathbb{N}}\| = q((x_n)_{n\in\mathbb{N}})$, where

$$q((x_n)_{n\in\mathbb{N}}) = \inf\{\delta \geqslant 0 : \ (x_n)_{n\in\mathbb{N}} \text{ has a finite } \delta\text{-net}\}.$$

We now proceed to observe bounded linear operators defined on $P(V_1)$. For any $(x_n)_{n\in\mathbb{N}} \in \ell_\infty(V_1)$ and every bounded operator $L \in \mathcal{B}(V_1, V_2)$, one defines an $L_\infty \in \mathcal{B}(\ell_\infty(V_1), \ell_\infty(V_2))$ as

$$L_\infty((x_n)_{n\in\mathbb{N}}) := (Lx_n)_{n\in\mathbb{N}} \in \ell_\infty(V_2).$$

If an operator sends every bounded sequence into a bounded sequence with a convergent subsequence, then that operator is said to be compact (this is one of the equivalents to the definition of a compact operator. In that sense, if an operator $L \in \mathcal{B}(V_1, V_2)$ is a bounded linear operator which is not a compact operator, then there exists a bounded but not totally bounded sequence $(x_n)_{n\in\mathbb{N}} \in \ell_\infty(V_1)$ such that $L_\infty((x_n)_{n\in\mathbb{N}})$ is a bounded but not a totally bounded sequence in $\ell_\infty(V_2)$. This shows that the set of bounded linear operators from $P(V_1)$ to $P(V_2)$ is induced by those bounded linear operators from V_1 to V_2 which are not compact. More precisely, if $L \in \mathcal{B}(V_1, V_2)$ is not a compact linear operator then there exists an $(x_n)_{n\in\mathbb{N}} \in P(V_1)$ which is not zero in the quotient space $P(V_1)$, such that $L_\infty((x_n)_{n\in\mathbb{N}}) \in P(V_2)$ is not zero in that quotient space. To make the notation more consistent, for a given $L \in \mathcal{B}(V_1, V_2)$ we denote by $P(L)$ the corresponding element from $\mathcal{B}(P(V_1), P(V_2))$, where

$$P(L)((x_n)_{n\in\mathbb{N}}) := (L_\infty((x_n)_{n\in\mathbb{N}}))/m(V_2),$$

for any $(x_n)_{n\in\mathbb{N}} \in P(V_1)$. Notice that upper semi-Fredholm operators play a crucial role here, because a bounded linear operator is upper semi-Fredholm if and only if it sends bounded but not totally bounded sequences into bounded but not totally bounded sequences [7, Theorem 2]. We state some fundamental results obtained in [7] and [28].

Theorem 3.3. [7, Theorem 2] *If $T\colon V_1 \to V_2$ is a bounded linear operator between Banach spaces V_1 and V_2, then the following are equivalent:*

(a) *$P(T)\colon \mathcal{P}(V_1) \to \mathcal{P}(V_2)$ is one-one;*
(b) *$T\colon V_1 \to V_2$ is upper semi-Fredholm;*
(c) *$P(T)\colon \mathcal{P}(V_1) \to \mathcal{P}(V_2)$ is bounded below.*

In analogy to $Lx = 0 \Rightarrow x = 0$ whenever L is injective, the implication

$$TU \text{ is compact} \Rightarrow U \text{ is compact}$$

defines T as an *essentially one-one* operator. In analogy to the reverse order law in dual spaces, the implication

$$UT \text{ is compact} \Rightarrow U \text{ is compact}$$

defines T as an *essentially dense* operator.

Theorem 3.4. [7, Theorem 4] *Let T be a bounded operator between two Banach spaces. Then the following implications hold:*

(a) *T is left upper semi-Fredholm$\Rightarrow$ T is upper semi-Fredholm $\Rightarrow$ T is essentially one-one;*
(b) *T is right lower semi-Fredholm $\Rightarrow$ T is lower semi-Fredholm $\Rightarrow$ T is essentially dense.*

Applying [7, Theorem 2] stated above, we see that

$$\sigma_{\text{app}}(L) = \sigma_p(P(L)) = \sigma_{\text{app}}(P(L)).$$

We proceed to generalize the statement from Theorem 3.2.

Theorem 3.5. [14, Theorem 3.5.] *Define $\mathcal{P}(V_1)$, $\mathcal{P}(V_2)$, $P(B)$, $P(C)$ and $P(A)$ as described above.*

(a) *Assume there exists an $J \in \Phi^{\ell}(V_1, V_2)$. Define the operators $\widehat{P(S)}$ and $\widehat{P(C)}$ in $\mathcal{B}(\mathcal{B}(\mathcal{P}(V_1), \mathcal{P}(V_2)))$ as*

$$\begin{aligned}\widehat{P(S)}(P(L)) &:= P(A)P(L) - P(L)P(B),\\ \widehat{P(C)}(P(L)) &:= P(C)P(J)^{-1}P_{\mathcal{R}(P(J))}P(L),\end{aligned}$$

for every $P(L) \in \mathcal{B}(\mathcal{P}(V_1), \mathcal{P}(V_2))$. If the equation $\widehat{P(S)}\widehat{P(X)} = \widehat{P(C)}$ is solvable in $\mathcal{B}(\mathcal{B}(\mathcal{P}(V_1), \mathcal{P}(V_2)))$, then exists a solution to the quotient equation

$$P(A)P(X) - P(X)P(B) = P(C). \tag{3.5}$$

(b) *Let $\lambda \in \sigma_{\text{app}}(B)$ such that it is a Riesz point for $P(B)$ and denote by $F_{P(B)}(\lambda)$ the corresponding finite-dimensional $P(B)$-invariant subspace of $\mathcal{P}(V_1)$. Let $F_{P(A)}$ be defined as in* (3.3) *with respect to $P(C)(F_{P(B)}(\lambda))$. Assume there exists an upper semi-Fredholm operator $P(\varphi) \in \Phi^{\ell}(\mathcal{P}(V_1), \mathcal{P}(V_2))$ which decomposes $P(B)$ at point λ in the Riesz sense. If $P(C)(F_{P(B)}(\lambda)) = F_{P(A)}$ and the finite matrices $P(B)\restriction_{F_{P(B)}(\lambda)}$, $P(C)\restriction_{F_{P(B)}(\lambda)}$ and $P(A)\restriction_{F_{P(A)}}$ satisfy conditions* (2.2)–(2.3)*, then* *there exist infinitely many solutions to* (3.5) *iff* $P(A)P(X_1)-P(X_1)P(B_1) = P(C_1)$ *is solvable for* $P(X_1)$ *on* $\mathcal{P}(V_{12})$*, where* $\mathcal{P}(V_1) = F_{P(B)}(\lambda) \oplus E_{P(B)}(\lambda)$*,* $P(B_1) = P(B) \restriction_{E_{P(B)}(\lambda)}$ *and* $P(C_1) = P(C) \restriction_{E_{P(B)}(\lambda)}$.

Proof. (a) From the discussion above, operator J defines an injective $P(J)$, with closed and complemented range in $\mathcal{P}(V_2)$, so Theorem 3.1 applies to (3.5).

(b) Similarly, all the conditions of Theorem 3.2 hold, so (3.5) has infinitely many solutions. □

Corollary 3.2. [14, Corollary 3.2.] *Let* $\lambda \in \sigma_{\text{app}}(B)$ *such that* λ *is a Riesz point of* $P(B)$ *and assume that* $P(\varphi) \in \Phi_+(\mathcal{P}(V_1), \mathcal{P}(V_2))$ *is an upper semi-Fredholm operator which decomposes* $P(B)$ *at point* λ *in the Riesz sense. Then* $\varphi \notin \Phi_+(V_1, V_2)$.

Proof. Assume that $P(\varphi)$ is an upper semi-Fredholm operator, which decomposes $P(B)$ at point λ in the Riesz sense. If $\varphi \in \Phi_+(V_1, V_2)$, then (by [7]) $P(\varphi)$ is one-one, that is, $\mathcal{N}(P(\varphi)) = \{0\}$. But by assumption, $P(\varphi)$ decomposes $P(B)$ at point λ in the Riesz sense, so the finite-dimensional part (as in (3.2)) is equal to zero:

$$F_{P(B)}(\lambda) = \{0\}.$$

Then $P(B) - \lambda$ is invertible in $\mathcal{P}(V_1)$, which contradicts the fact that $\lambda \in \sigma_p(P(B))$. □

3.2. Some applications to compact operators. Below we illustrate how our results answer some questions regarding compact derivations and compact extensions. We emphasize that these results rely heavily on the fact that the appropriate Sylvester equation is singular (recall Section 1).

Corollary 3.3. [14, Corollary 3.3.] *Let* A *and* B *be bounded linear operators on Banach spaces* V_2 *and* V_1*, respectively, such that* B *has a complemented null-space. In addition, assume that if* $\mathcal{N}(B)$ *intersects with* $\mathcal{R}(B)$*, then that intersection is closed and complemented in* $\mathcal{N}(B)$*,*

$$V_1 = V_{12} \oplus \mathcal{N}(B), \quad \mathcal{N}(B) = \mathcal{N}_1(B) \oplus (\mathcal{N}(B) \cap \mathcal{R}(B)).$$

If there exists a $J \in \mathcal{B}(\mathcal{N}_1(B), \mathcal{N}(A)) \smallsetminus \mathcal{C}(\mathcal{N}_1(B), \mathcal{N}(A))$[1] *then there exists an* $X \in \mathcal{B}(V_1, V_2) \smallsetminus \mathcal{C}(V_1, V_2)$ *such that* $AX - XB \in \mathcal{C}(V_1, V_2)$.

Proof. Let

$$V_1 = V_{12} \oplus \mathcal{N}(B) = V_{12} \oplus (\mathcal{N}(B) \cap \mathcal{R}(B)) \oplus N_1(B).$$

Notice that $B\colon V_1 \to \mathcal{R}(B) \subset V_{12} \oplus (\mathcal{N}(B) \cap \mathcal{R}(B))$. Choose arbitrary $K \in \mathcal{C}(V_{12} \oplus (\mathcal{N}(B) \cap \mathcal{R}(B)), V_2)$. Then

$$X := \begin{bmatrix} K & 0 \\ 0 & J \end{bmatrix} : \begin{bmatrix} V_{12} \oplus (\mathcal{N}(B) \cap \mathcal{R}(B)) \\ \mathcal{N}_1(B) \end{bmatrix} \to \begin{bmatrix} K(V_{12} \oplus (\mathcal{N}(B) \cap \mathcal{R}(B))) \\ \mathcal{N}(A) \end{bmatrix}$$

trivially satisfies

$$AX - XB = AKP_{V_{12} \oplus (\mathcal{N}(B) \cap \mathcal{R}(B))} + AJP_{\mathcal{N}_1(B)} - KB \in \mathcal{C}(V_1, V_2)$$

and $X \notin \mathcal{C}(V_1, V_2)$. □

---

[1] It suffices to choose J to be a (lower or upper) semi-Fredholm operator.

When are compact operators L and T equivalent after extension?

Two operators T and L defined on two different Banach spaces V_1 and V_2 respectively, are said to be *equivalent after extension*, if they can both be extended to $V_1 \oplus V_2$, $\tilde{T} := T + I_{V_2}$, $\tilde{L} = I_{V_1} + L$, and in addition satisfy $\tilde{T} = U\tilde{L}V$, for some bounded and invertible linear operators U and V on $V_1 \oplus V_2$ (see [31,32] and [64]). Specially, if $V_1 = \{0\}$ or $V_2 = \{0\}$, then T and L, which are equivalent after extension, are said to be equivalent after one-sided extension. Note that if T and L are compact operators which are equivalent after extension, then $\tilde{T}$ and $\tilde{L}$ are Fredholm operators. Thus it suffices to find an invertible U such that $\tilde{L}U = U\tilde{T}$. This is now solved by Theorem 3.1, Theorem 3.2, Theorem 3.5, Corollary 3.1 or Corollary 3.3. A necessary condition is in that case (see below) $\mathcal{I}_L = \mathcal{I}_T$.

Definition 3.2. [31, Definition 2.1.] Let $T \in \mathcal{B}(V_1, V_2)$ be a Banach space operator. For any Banach spaces Z_1 and Z_2, we define

$$\mathcal{I}_T(Z_1, Z_2) := \bigcup_{n\in\mathbb{N}} \left\{ \sum_{j=1}^{n} R_j T R_j' : \ R_j \in \mathcal{B}(Y, Z_2),\ R_j' \in \mathcal{B}(Z_1, X) \right\}.$$

Denote by $\mathcal{I}_T$ the (proper) class $\bigcup_{Z_1,Z_2} \mathcal{I}_T(Z_1, Z_2)$, and refer to $\mathcal{I}_T$ as the operator ideal generated by T.

Theorem 3.6. [31, Theorem 2.5.] *Let $T \in \mathcal{B}(V_1)$ and $L \in \mathcal{B}(V_2)$ be non-zero compact Banach space operators. If T and L are equivalent after extension, then $\mathcal{I}_T = \mathcal{I}_L$.*

3.3. Estimating the solution set. The previous results guarantee solvability of (1.1), but characterizing the entire solution set is a rather difficult task. In what follows we revisit some results obtained in [11], which speak of algebraic and topological properties of this set. Recall that the solution set can be described as $X_p + \{X_h\}$, where X_p is one particular solution and $\{X_h\}$ is the solution set to the homogeneous Sylvester equation

$$AX - XB = 0. \tag{3.6}$$

Since the previous results offer the existence conditions for at least one particular solution X_p, we proceed to characterize the solution set for the equation (3.6).

In what follows, we fix the Banach spaces V_1 and V_2 and the bounded linear operators $A \in \mathcal{B}(V_2)$, $B \in \mathcal{B}(V_1)$ and $Z \in \mathcal{B}(V_1, V_2)$ such that $Z \neq 0$. Define $n-$th power of AZB in $\mathcal{B}(V_1, V_2)$ by

$$(AZB)^n := A^n Z B^n, \quad n \in \mathbb{N}_0.$$

Put

$$\mathcal{A}_{AZB} := \overline{\{p(AZB) : p \in P[\mathbb{C}]\}}. \tag{3.7}$$

It can be shown that any complex function f holomorphic in the Cauchy domain which contains the compact set $\sigma(A) \cup \sigma(B)$ can be obtained as the uniform limit of polynomials p_n, thus

$$f(AZB) = f(A)Zf(B) = \lim_n p_n(A)Zp_n(B) = \lim_n p_n(AZB).$$

Multiplication in $\mathcal{A}_{AZB}$ is defined as multiplication of functions,

$$f(AZB) \cdot g(AZB) = (f \cdot g)(AZB).$$

Theorem 3.7. [11, Theorem 2.1.] *Assume $\|A\|$ and $\|B\|$ to be smaller than one. Let n, $m \in \mathbb{N}_0$ such that $0 \leqslant n < m$ and let $\mathcal{A}_{AXB}$ be provided as in* (3.7). *Then*

(1) *The ordered triple $(\mathcal{A}_{AZB}, \|\cdot\|, +)$ is a separable Banach subspace of $\mathcal{B}(V_1, V_2)$. The ordered triple $(\mathcal{A}_{AZB}, +, \cdot)$ is a commutative algebra with the unity Z. The ordered quadruple $(\mathcal{A}_{AZB}, \|\cdot\|, +, \cdot)$ is not necessarily a normed algebra, i.e. the sub-multiplicativity $\|ab\| \leqslant \|a\|\|b\|$ does not necessarily hold.*

(2) *The inequality $\|(AZB)^m\| \leqslant \|(AZB)^n\|$ holds, where the equality is obtained iff $(AZB)^k = 0$, for some $k \in \{0, \ldots, n\}$.*

(3) *The series*

$$\sum_{j=0}^{+\infty} (AZB)^{m \cdot j} \tag{3.8}$$

converges in $\mathcal{A}_{AZB}$. The operator $Z - (AZB)^m$ is invertible in $\mathcal{A}_{AZB}$ and its inverse is given as (3.8).

Recall the commutative multiplication operators $\mathbb{A}$ and $\mathbb{B}$ defined as $\mathbb{A}(Z) := AZ$ and $\mathbb{B}(Z) = ZB$. The following lemma obviously holds:

Lemma 3.2. [11, Lemma 2.2.] *With respect to the previous notation, the algebra $\mathcal{A}_{AZB}$ is isometrically isomorphic to $\overline{\{p(\mathbb{A} \circ \mathbb{B})(Z),\ p \in P[\mathbb{C}]\}}$.*

For a given $L \in \mathcal{B}(V)$, the set $[L]$ represents the set of all operators from $\mathcal{B}(V)$ which commute with L. Consequently, $L^n \in [L]$, for every $n \in \mathbb{N}_0$. Define

$$[\mathcal{A}_{AZB}] := [A] \cdot \mathcal{A}_{AZB} \cdot [B] = \{CDE : C \in [A], D \in \mathcal{A}_{AZB}, E \in [B]\},$$
$$\mathcal{B}_{AZB} := \mathcal{B}(V_2) \cdot \mathcal{A}_{AZB} \cdot \mathcal{B}(V_1) = \{FGH : F \in \mathcal{B}(V_2), G \in \mathcal{A}_{AZB}, H \in \mathcal{B}(V_1)\}.$$

Now one takes natural extension of the multiplication from $\mathcal{A}_{AZB}$ to $[\mathcal{A}_{AZB}]$ and $\mathcal{B}_{AZB}$. More precisely, let $C_1D_1E_1$, $C_2D_2E_2 \in [\mathcal{A}_{AZB}]$ and let $F_1G_1H_1$, $F_2G_2H_2 \in \mathcal{B}_{AZB}$. Then

$$(C_1D_1E_1) \cdot (C_2D_2E_2) := (C_1 \cdot C_2) \cdot (D_1 \cdot D_2) \cdot (E_1 \cdot E_2),$$
$$(F_1G_1H_1) \cdot (F_2G_2H_2) := (F_1 \cdot F_2) \cdot (G_1 \cdot G_2) \cdot (H_1 \cdot H_2).$$

The previous construction allows us to show how large the solution set for (3.6) really is.

Theorem 3.8. [11, Theorem 3.4] *Let A, X and B be provided such that $AX = XB$. Then for every $Y \in \mathcal{A}_{AXB}$ it follows that $AY = YB$. In other words, every element from $\mathcal{A}_{AXB}$ is a solution to the homogeneous Sylvester equation* (3.6).

Proof. First observe the basis of $\mathcal{A}_{AXB}$: X, AXB, A^2XB^2,.... Given the way A, B and X are provided, it follows that

$$A(A^nXB^n) = A(A^{n-1}XB^{n+1}) = (A^nXB^n)B, \quad n \in \mathbb{N},$$

so $(AXB)^n$ is a solution to (3.6), for every $n \in \mathbb{N}$. Further, every finite linear combination of the basis elements is a solution to (3.6). This proves that $p_n(AXB)$ is a solution to the homogeneous Sylvester equation, for every $p_n \in P[\mathbb{C}]$. One should note that, in the bounded-operator case, the set of solutions to the equation $AX - XB = 0$ is closed. This is directly verifiable. Nevertheless, let f be a holomorphic function on some Cauchy domain Ω, $\sigma(A)$, $\sigma(B) \subset \Omega$, given as the limit of some complex polynomials $f(z) = \lim_{n\to+\infty} p_n(z)$, $z \in \Omega$. Then Lemma 3.2 applies, giving $f(AXB) = \lim_{n\to+\infty} p_n(AXB)$ and

$$\begin{aligned} Af(AXB) &= A(\lim_{n\to\infty} p_n(AXB)) \\ &= \lim_{n\to\infty} Ap_n(AXB) = \lim_{n\to\infty} (p_n(AXB)B) = f(AXB)B, \end{aligned}$$

so $\mathcal{A}_{AXB}$ is contained in the set of solutions to the homogeneous Sylvester equation (3.6). □

Corollary 3.4. [11, Corollary 3.3.] *Let A, B and X be provided such that $AX = XB$, and let $\mathbb{A}$ and $\mathbb{B}$ be provided as in Lemma 3.2. Then $\mathcal{A}_{AXB}$ is isomorphic to*

$$\overline{\{p(\mathbb{A}^2)(X) : p \in P[\mathbb{C}]\}} \tag{3.9}$$

and to

$$\overline{\{p(\mathbb{B}^2)(X) : p \in P[\mathbb{C}]\}}. \tag{3.10}$$

In order for $AX = XB$ to be solvable for a nonzero X, it is required for (3.9) *and* (3.10) *to be isomorphic to each other.*

If $\{X_i\}_{i\in I}$ is a family of different solutions to the inhomogeneous eq. (1.1), how do the operator algebras $\mathcal{A}_{AX_iB}$ behave? We recall the following result from [33]:

Lemma 3.3. [33, Lemma 2.1.] *Assume X is a solution to* (1.1). *Then for any $k \geqslant 1$*

$$A^kX - XB^k = \sum_{i=0}^{k-1} A^{k-1-i}CB^i. \tag{3.11}$$

This lemma shows us that any solution to (1.1) satisfies the corollary below.

Corollary 3.5. [11, Corollary 3.2.] *Let A, B, C and X be provided such that* (1.1) *holds. Then $C \in [\mathcal{A}_{AXB}]$ and for every $k \in \mathbb{N}_0$,*

$$A^kX - XB^k \in [\mathcal{A}_{ACB}].$$

Proof. The first claim follows directly $C = AX - XB \in [\mathcal{A}_{AXB}]$. When $k = 0$ then $X - X = 0 \in [\mathcal{A}_{ACB}]$. When $k \geqslant 1$, then (3.11) applies, and

$$A^kX - XB^k = A^{k-1}C + \cdots + CB^{k-1}.$$

Since $A^\ell \in [A]$ and $B^s \in [B]$, for every s, $\ell \in \{0, \dots, k-1\}$, it follows that every addend on the right-hand-side is in $[\mathcal{A}_{ACB}]$, and so is $A^kX - XB^k$. □

4. Closed operators

In this section we study the initial equation (1.1) under the premise that V_1 and V_2 are Banach spaces, while A and B are closed densely defined linear operators and where C is an arbitrary densely defined linear operator from V_1 with values in V_2. We now proceed to study the equation in its unbounded form (1.2). For easier reference, we are writing it here once again:

$$AXu - XBu = Cu, \qquad u \in \mathcal{D}_B \cap \mathcal{D}_C. \tag{4.1}$$

At this point, we drop the assumption that A and $-B$ generate C_0-semigroups on V_1 and V_2, respectively. Instead, we assume that A and B are arbitrary closed operators, which have nonempty point spectra and we assume that

$$\sigma(A) \cap \sigma(B) = \sigma_p(A) \cap \sigma_p(B).$$

We introduce the notion of weak solutions.

Definition 4.1. [16, Definition 2.1.] Linear operator X is a weak solution to the equation (4.1) if

(1) $\mathcal{D}_C \cap \mathcal{D}_B \neq \emptyset$
(2) $\mathcal{D}_X \subset \mathcal{D}_B \cap \mathcal{D}_C$, $\mathcal{R}(X) \subset \mathcal{D}_A$ and $\mathcal{D}_X$ is B-invariant subspace of V_1.
(3) for every $u \in \mathcal{D}_X$ $AX(u) - XB(u) = C(u)$.

The weak solution to the equation $AX = XB$ is defined analogously. We start our analysis with the homogeneous equation and then transform the inhomogeneous (4.1) into a homogeneous one.

4.1. The homogeneous equation.

Let V be an arbitrary vector space over the field F and let I be an arbitrary index set. Recall that a set of different vectors $\{a_i\}_{i\in I}$ from V is said to be Hamel or algebraic basis for V if every vector $a \in V$ can be represented as a unique finite linear combination of vectors from $\{a_i\}_{i\in I}$:

$$(\forall a \in V)\ (\exists! n \in \mathbb{N})\ (\exists! a_1, \dots, a_n \in \{a_i\}_{i\in I})\ (\exists! \alpha_1, \dots, \alpha_n \in F)\ a = \sum_{k=1}^{n} \alpha_k a_k.$$

It is known that every vector space has a Hamel basis; this is a direct corollary from the axiom of choice. Uniqueness of the representation of every vector from V with regards to its Hamel basis $\{a_i\}_{i\in I}$ yields that $\{a_i\}_{i\in I}$ are linearly independent vectors. Additionally, all Hamel bases of the same vector space have the same cardinality. Recall that, even if S is an infinite set of vectors, $\mathrm{Lin}(S)$ or $\mathrm{span}(S)$ stands a finite linear span of vectors from S.

At this point we assume V_1 and V_2 to be linear (vector) spaces and $A \in L(V_2)$, $B \in L(V_1)$ to be both one-to-one (injective). We will return to the case of closed operators in Banach spaces later. We also assume that there exists $W < \mathcal{D}_B < V_1$ which is a B-invariant subspace of V_1. Let $\mathcal{U} = \{u_i\}_{i\in I}$ be an algebraic basis of W. Further, since $\{u_i\}_{i\in I}$ is an algebraic (Hamel) basis for W, it follows that $\{B(u_i)\}_{i\in I}$ is an algebraic (Hamel) basis for $B(W)$. The operator B is injective, so $\mathrm{card}(\{u_i\}_{i\in I}) = \mathrm{card}(\{B(u_i)\}_{i\in I})$. Therefore, there exists a linear bijection

$T_W : \{B(u_i)\}_{i\in I} \to \{u_i\}_{i\in I}$, such that for each $i \in I$ there exists a unique $j \in I$ so that $T_W B(u_j) = u_i$. For every $u \in \mathcal{U}$, we define the class of u as

$$[u] = \{(T_W B)^n(u) : n \in \mathbb{Z}\}.$$

It is not difficult to see that $\{[u_i] : i \in I\}$ forms a partition of $\mathcal{U}$. In that sense, let $[\mathcal{U}] := \{[u_i] : i \in I\}$. Conversely, for every $[u] \in [\mathcal{U}]$, fix one $u_0 \in [u]$ which is the generator for its entire equivalence class:

$$[u_0] = \{(T_W B)^n(u_0) : n \in \mathbb{Z}\} = [u].$$

Define $\cdot_B : [u] \times [u] \to [u]$:

$$(\forall n, m \in \mathbb{Z})\ (T_W B)^n(u_0) \cdot_B (T_W B)^m(u_0) := (T_W B)^{n+m}(u_0).$$

The following result is straightforward.

Lemma 4.1. [16, Lemma 2.1.] *Let $u \in \mathcal{U}$.*

1) *If $[u]$ has a finite number of different elements, say k of them, then $([u], \cdot_B)$ is isomorphic to $(\mathbb{Z}_k, +_k)$;*
2) *If $[u]$ has infinitely many different elements, then $([u], \cdot_B)$ is isomorphic to $(\mathbb{Z}, +)$.*

Let $Z < \mathcal{D}_A < V_2$ be an A-invariant subspace of V_2 and let $\mathcal{V} = \{v_j\}_{j\in J}$ be an algebraic basis for Z. Let $S_Z \in L(A(Z), Z)$ be a bijective linear operator, such that $S_Z(\mathcal{V}) \subset (\mathcal{V})$. For every $v \in \mathcal{V}$, define $[v]$ using $S_Z A$, in the analogous way we defined $[u]$, using $T_W B$, when $u \in \mathcal{U}$. On every $[v]$ define $\cdot_A$ using $S_Z A$ in the analogous way we defined $\cdot_B$ using $T_W B$ on every $[u]$.

Corollary 4.1. [16, Corollary 2.1.] *For every $v \in \mathcal{V}$, $([v], \cdot_A)$ is isomorphic to exactly one of the elements in $\{(\mathbb{Z}, +), (\mathbb{Z}_k, +_k)\}$.*

Remark 4.1. The aforementioned isomorphisms between elements of $\{([u], \cdot_B), ([v], \cdot_A)\}$ and elements of $\{(\mathbb{Z}, +), (\mathbb{Z}_k, +_k)\}$ will be denoted as "$\cong$".

Theorem 4.1 (The shifted injective homogeneous equation)**.** [16, Theorem 2.1.] *Let V_1 and V_2 be vector spaces and let $B \in L(\mathcal{D}_B, V_1)$, $A \in L(\mathcal{D}_A, V_2)$ be one-to-one linear operators, where $\mathcal{D}_B \subset V_1$ and $\mathcal{D}_A \subset V_2$, and let $W \subset \mathcal{D}_B$ be a B-invariant subspace of V_1 and let $Z \subset \mathcal{D}_A$ be an $A-$ invariant subspace of V_2. Let T_W and S_Z be provided as in previous discussion. Then there exists a linear operator $X \in L(W, Z)$ which is a weak solution to the equation*

$$XT_W B = S_Z AX, \tag{4.2}$$

defined on W.

Proof. Let $\mathcal{U}$ and $\mathcal{V}$ be the algebraic bases for W and Z, respectively, on which T_W and S_Z are respectively defined. We define X in the following manner. For a fixed $[u] \in [\mathcal{U}]$, if there exists an $v \in \mathcal{V}$ such that $([u], \cdot_B) \cong ([v], \cdot_A)$, then X maps one fixed generator $u_0 \in [u]$ into the generator v_0 for $[v]$, and consecutively $X : (T_W B)^m u_0 \mapsto (S_Z A)^m v_0$. If no such $([v], \cdot_A)$ exists then $X([u]) := 0_{V_2}$. Either way, it is directly verifiable that $S_Z A(X(u')) = X(T_W B(u'))$ holds for every $u' \in [u]$.

Now assume the previous procedure was done for every $[u] \in [\mathcal{U}]$. Then for arbitrary $u \in W$ decompose the vector u into a finite linear combination of elements from $\mathcal{U}$

$$u = \sum_{k=1}^{n} \alpha_k u_k, \quad u_k \in \mathcal{U},\ \alpha_k \in \mathbb{C},\ k = \overline{1,n},\ n \in \mathbb{N}.$$

Then obviously $[X(u) := \sum_{k=1}^{n} \alpha_k X(u_k)$. It is clear that X is well defined linear operator from W to Z and is a solution to (4.2). By construction, we have that $\mathcal{D}_X = W \subset \mathcal{D}(T_W B)$, $\mathcal{R}(X) = Z \subset \mathcal{D}(S_Z A)$ and W is $T_W B$-invariant, so X is indeed a weak solution. □

Remark 4.2. Note that the solution X is not uniquely determined.

We generalize the previous statement to Banach spaces and closed operators A and B.

Theorem 4.2 (The homogeneous equation)**.** [16, Theorem 2.3.] *Let V_1 and V_2 be given Banach spaces, $B \in L(V_1)$ and $A \in L(V_2)$ closed operators, such that $\mathcal{N}(B)$ and $\mathcal{N}(A)$ are complemented in V_1 and V_2, respectively. If $(\sigma_p(B) \cap \sigma_p(A)) \setminus \{0\} \neq \emptyset$ then the homogeneous equation*

$$AX - XB = 0 \tag{4.3}$$

has a non-trivial weak solution.

Proof. The space V_1 can be split into a direct sum $V_1 = \mathcal{N}(B) \oplus V_1'$. Respectively, for every $u \in \mathcal{D}_B$ there exists a unique $u_1 \in \mathcal{N}(B)$ and a unique $u_2 \in V_1' \cap \mathcal{D}_B$ such that $u = u_1 + u_2$. Denote $V_1(B) := V_1' \cap \mathcal{D}_B$ and define $B_1 \colon V_1(B) \to V_1$ as: $B_1(u_2) := B(u)$. This way, B_1 is one-to-one, so $0 \notin \sigma_p(B_1)$. Note that $\sigma_p(B) \setminus \{0\} \equiv \sigma_p(B_1)$.

Assume the same thing is done with A and the Banach space V_2: $V_2 = \mathcal{N}(A) \oplus V_2'$, put $V_2(A) := \mathcal{D}_A \cap V_2'$ and $A_1 : V_2(A) \to V_2$ defined as $A_1(v_2) := A(v)$, whenever $v \in \mathcal{D}_A$ and $v = v_1 + v_2$, $v_1 \in \mathcal{N}(A)$, and $v_2 \in V_2(A)$. Now A_1 is one-to-one and $0 \notin \sigma_p(A_1)$. Also note that $\sigma_p(A) \setminus \{0\} = \sigma_p(A_1)$. Now the condition of the theorem yields that $\sigma_p(A_1) \cap \sigma_p(B_1) \neq \emptyset$. Denote that spectral intersection by $\{\lambda_i\}_{i \in I}$, for some index set I, where $\lambda_i = \lambda_j \Rightarrow i = j$. Let $u_i \in \mathcal{D}(B_1)$ and $v_i \in \mathcal{D}(A_1)$ such that $B_1 u_i = \lambda_i u_i$ and $A_1 v_i = \lambda_i v_i$, whenever $i \in I$. It follows that $\{u_i\}_{i \in I}$ and $\{v_i\}_{i \in I}$ are families of linearly independent vectors. Now put $\mathcal{U} := \{u_i\}_{i \in I}$ and $\mathcal{V} = \{v_i\}_{i \in I}$. Trivially, $W := \mathrm{Lin}(\mathcal{U})$ is a B_1-invariant subspace of V_1 and $Z := \mathrm{Lin}(\mathcal{V})$ is an A_1- invariant subspace of V_2.

For each $i \in I$ define bounded linear operators on $\mathrm{Lin}(u_i)$ and $\mathrm{Lin}(v_i)$ respectively as $T_i(u) := \lambda_i^{-1} u$ and $S_i(v) := \lambda_i^{-1} v$. Finally put $T_W(u_i) := T_i(u_i)$ and $S_Z(v_i) := S_i(v_i)$.

Since $\mathrm{Lin}(u_i) \cap \mathrm{Lin}(u_j) = \{0\}$ whenever $i \neq j$, it follows that T_W is a correctly defined operator on $\oplus_{i \in I} \mathrm{Lin}(u_i)$ (which is an eigenspace for B_1 and therefore for B). Analogously, S_Z is correctly defined operator on $\oplus_{i \in I} \mathrm{Lin}(v_i)$. Now all conditions of Theorem 4.1 are satisfied, so there exists a linear operator X_1 from W to Z such that $X_1 T_W B_1 = S_Z A_1 X_1$ holds.

Further, we see that $(\overline{[u_i]}, \cdot_{B_1}) \cong (\overline{[v_i]}, \cdot_{A_1}) \cong (\mathbb{Z}_1, +_1)$ (singletons) for every $i \in I$. For $u \in W$, such that $u = \sum_{k=0}^{n} \alpha_k u_k$ where $u_k \in \{u_i\}_{i\in I}$, we have:

$$\begin{aligned} S_Z A_1 X_1(u) &= S_Z A_1 X_1\left(\sum_{k=1}^{n} \alpha_k u_k\right) = \sum_{k=1}^{n} \alpha_k S_Z A_1 X_1(u_k) \\ &= \sum_{k=1}^{n} \alpha_k S_Z A_1(v_k) = \sum_{k=1}^{n} \alpha_k S_Z(\lambda_k v_k) = \sum_{k=1}^{n} \alpha_k v_k = \sum_{k=1}^{n} \alpha_k X_1(u_k) \\ &= \sum_{k=1}^{n} \alpha_k X_1 T_W(\lambda_k u_k) = \sum_{k=1}^{n} \alpha_k X_1 T_W B_1(u_k) \\ &= X_1 T_W B_1\left(\sum_{k=1}^{n} \alpha_k u_k\right) = X_1 T_W B_1(u). \end{aligned}$$

Since S_Z and T_W are injective and act in the same way on the corresponding spaces, it directly follows that $A_1 X_1 u = X_1 B_1 u$. Therefore, $X_1 \in L(W, Z)$ where $\mathcal{D}_X = W < \mathcal{D}_B$ and $\mathcal{R}(X) = Z < \mathcal{R}(A)$, so X is a weak solution to the equation $A_1 X_1 = X_1 B_1$.

Let $N \in L(\mathcal{N}(B), \mathcal{N}(A))$ be arbitrary. Put

$$X = \begin{bmatrix} N & 0 \\ 0 & X_1 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(B) \\ W \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(A) \\ Z \end{bmatrix}.$$

It follows that X is a weak solution to (4.3). □

Remark 4.3. When constructing injective operators A_1 and B_1 one encounters problem of losing the information about the null-spaces of A and B. However, this property is not as restrictive as it may seem at the first sight. In particular, suppose that $\{0\} = \sigma_p(A) \cap \sigma_p(B)$. Then $\mathcal{N}(B)$ and $\mathcal{N}(A)$ are the corresponding eigenspaces of B and A, respectively, which correspond to the shared eigenvalue $\lambda = 0$. But then arbitrary operator $N \in \mathcal{L}(\mathcal{N}(B), \mathcal{N}(A))$ (provided in the proof of Theorem 4.2) is the desired map that maps 0-eigenspace of B into the corresponding 0-eigenspace of A. In other words, one could simply put $X := \left[\begin{smallmatrix} N & 0 \\ 0 & 0 \end{smallmatrix}\right]$. However, this case is somewhat irrelevant because both XB and AX vanish on $\mathcal{N}(B)$. Nevertheless, Theorem 4.2 holds even if $\sigma_p(A) \cap \sigma_p(B) = \{0\}$.

Recall that every closed subspace M of a given Hilbert space $\mathcal{H}$ has a topological complement N. Furthermore, N can be provided such that M and N form an orthogonal sum, i.e. $\mathcal{H} = M \oplus^{\perp} N$.

Corollary 4.2. [16, Corollary 2.2.] *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, $A \in L(\mathcal{H}_2)$ and $B \in L(\mathcal{H}_1)$ closed operators. If $(\sigma_p(A) \cap \sigma_p(B)) \neq \emptyset$ then the homogeneous equation* (4.3) *has a non-trivial weak solution.*

The previous theorem gives weak solutions to the equation (1.1), which are defined on finite linear combinations of the corresponding eigenvectors. In what follows we extend this result to a summable family of eigenvectors.

Recall that, when solving PDEs and ODEs, one usually uses the Fourier separation method and then solves the corresponding Sturm–Liuville eigenfunction problem. As a result, the solution to the given PDE or ODE is represented as a superposition of countably-many summable eigenfunctions.

Definition 4.2. Let V be a Banach space over the field F. A Schauder basis is an ordered sequence $\{b_n\}_{n\in\mathbb{N}}$ of elements from V such that for every element $v \in V$ there exists a unique sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of scalars in F such that $v = \sum_{n\in\mathbb{N}} \alpha_n b_n$, where the convergence is understood in the norm topology

$$\lim_{n\to+\infty} \|v - \sum_{k=1}^{n} \alpha_n b_n\| = 0.$$

From the uniqueness of the representation of v in $\{b_n\}_{n\in\mathbb{N}}$ it follows that $\{b_n\}_{n\in\mathbb{N}}$ is a family of linearly independent vectors. There is no exact criterion which yields when does a given Banach space have a Schauder basis. However, the necessary condition is obtained in the following two well-known theorems.

Theorem 4.3. *Let V be a Banach space. Then its Hamel (algebraic) basis is either finite or has the cardinality of at least c (continuum).*

Theorem 4.4. *Let V be a Banach space and suppose it has a Schauder basis. Then V must be separable.*

Contrary, if the provided Banach space V is separable, it does not imply that it has a Schauder basis. Counterexample was provided by P. Enflo [35] in 1973. It is well-known fact that ℓ_∞ space does not have a Schauder basis.

We again start with arbitrary linear spaces V_1 and V_2 and one-to-one operators $B \in L(\mathcal{D}_B, V_1)$, $\mathcal{D}_B \subset V_1$ and $A \in L(\mathcal{D}_A, V_2)$, $\mathcal{D}_A \subset V_2$ defined on them. Suppose there exists a $W < \mathcal{D}_B$ which is a B-invariant subspace of V_1, such that it allows a Schauder basis $\mathcal{W} = \{w_n : n \in \mathbb{N}\}$. It is not difficult to see that there exists a bijective operator $T \in L(B(W), W)$ such that $T(w_n) \in \mathcal{W}$, for every $n \in \mathbb{N}$, because B is assumed to be one-to-one. Now for every $w \in \mathcal{W}$ define

$$[w] = \{(TB)^n(w) : n \in \mathbb{Z}\}$$

and define binary operation $\cdot_B$ on $[w]$ as

$$(\forall n, m \in \mathbb{Z})(TB)^n(w) \cdot_B (TB)^m(w) := (TB)^{n+m}(w).$$

Lemma 4.1 yields that $([w], \cdot_B)$ is isomorphic to exactly one element from the set $\{(\mathbb{Z}, +), (\mathbb{Z}_k, +_k), k \in \mathbb{N}\}$.

Analogously, assume there exists a $Z < \mathcal{D}_A$ which is an A-invariant subspace of V_2, which allows a Schauder basis $\mathcal{Z} = \{z_n : n \in \mathbb{N}\}$ and define the bijective operator $S \in L(A(Z), Z)$ such that for every $n \in \mathbb{N}$ it follows that $S(z_n) \in \mathcal{Z}$. For every $z \in \mathcal{Z}$, define $[z] = \{(ZA)^n(z) : n \in \mathbb{Z}\}$ and define $\cdot_A$ on every class $[z]$ as

$$(\forall n, m \in \mathbb{Z})(SA)^n(w) \cdot_A (SA)^m(w) := (SA)^{n+m}(w).$$

Lemma 4.1 yields that $([z], \cdot_A)$ is isomorphic to exactly one element from the set $\{(\mathbb{Z}, +), (\mathbb{Z}_k, +_k), k \in \mathbb{N}\}$.

The following corollaries are immediate consequences of Theorem 4.1 and Theorem 4.2, respectively.

Corollary 4.3 (The shifted injective homogeneous equation)**.** [16, Corollary 2.5.] *With regards to the previous notation, there exists $X \in L(W_0, Z)$ which is a weak solution to the equation $XTB = SAX$, defined on*

$$W_0 := \left\{ \sum_{n\in\mathbb{N}} \alpha_n w_n : w_n \in \mathcal{W}, \alpha_n \in \mathbb{C}, n \in \mathbb{N} \text{ and } \sum_{n\in\mathbb{N}} \alpha_n X(w_n) \text{ converges in } Z \right\}.$$

Corollary 4.4 (The homogeneous equation)**.** [16, Corollary 2.6.] *Let V_1 and V_2 be given Banach spaces, $B \in L(V_1)$ and $A \in L(V_2)$ closed operators, such that $\mathcal{N}(B)$ and $\mathcal{N}(A)$ are complemented in V_1, V_2, respectively, i.e. $V_1 = \mathcal{N}(B) \oplus V_1'$ and $V_2 = \mathcal{N}(A) \oplus V_2'$. If $\sigma_p(B) \cap \sigma_p(A) \neq \emptyset$ and the corresponding eigenvectors form Schauder bases for some $S_1 < \mathcal{D}_B \cap V_1'$ and $S_2 < \mathcal{D}_A \cap V_2'$, respectively, then the homogeneous equation $AX - XB = 0$ has a non-trivial weak solution, defined on some subset of S_1.*

4.2. The inhomogeneous equation.

In this section we provide weak solutions to the inhomogeneous Sylvester equation (1.1), i.e. the case where $C \neq 0$. The lemma below illustrates how to reduce the inhomogeneous equation to the homogeneous one.

Lemma 4.2. [16, Lemma 2.2.] *Let V_1 and V_2 be Banach spaces, $B, \Psi_1 \in L(V_1)$, $A, \Psi_2 \in L(V_2)$ closed operators and $C \in L(V_1, V_2)$, such that for every $u \in \mathcal{D}(\Psi_1) \cap \mathcal{R}(\Psi_1) \cap \mathcal{D}_C$ we have $C(u) \in \mathcal{D}(\Psi_2)$ and*

$$\Psi_2 C(u) - C\Psi_1(u) = C(u). \tag{4.4}$$

Suppose $\mathcal{D}(\Psi_2) \cap \mathcal{D}_A \neq \emptyset$ and $\mathcal{D}(\Psi_1) \cap \mathcal{D}_B \neq \emptyset$. Finally, we require that $\mathcal{N}(A - \Psi_2)$ and $\mathcal{N}(B - \Psi_1)$ have topological complements and

$$(\sigma_p(A - \Psi_2) \cap \sigma_p(B - \Psi_1)) \setminus \{0\} \neq \emptyset. \tag{4.5}$$

Then for every $Y \in L(\mathcal{D}(Y), \mathcal{R}(Y))$, $\mathcal{D}(Y) = \mathcal{D}(\Psi_1) \cap \mathcal{R}(\Psi_1) \cap \mathcal{D}_C$, $\mathcal{R}(Y) \subset \mathcal{D}(\Psi_2) \cap \mathcal{D}_A$, which is a weak solution to

$$\Psi_2 Y - Y\Psi_1 = 0, \tag{4.6}$$

the operator $X := Y + C$ is a weak solution to the inhomogeneous Sylvester equation (1.1) *iff it is a weak solution to the homogeneous equation*

$$(A - \Psi_2)X - X(B - \Psi_1) = 0. \tag{4.7}$$

Proof. Assume there exists Y such that the equation (4.6) is satisfied. Put $X := Y + C$. By applying Theorem 4.2, we see that (4.5) yields that there exists a non-trivial weak solution X to the equation (4.7). Finally, we verify that

$$\begin{aligned} &(A - \Psi_2)X - X(B - \Psi_1) = 0 \\ \Leftrightarrow\ &AX - XB = \Psi_2 X - X\Psi_1 \\ \Leftrightarrow\ &AX - XB = \Psi_2 Y + \Psi_2 C - Y\Psi_1 - C\Psi_1 = C. \end{aligned}$$ □

Remark 4.4. Such Ψ_1 and Ψ_2 always exist, e.g. $\Psi_2 = (\alpha + 1)I$, $\Psi_1 = \alpha I$ for any $\alpha \in \mathbb{C}$.

Let V_1 and V_2 be Banach spaces, $B \in L(V_1)$, $A \in L(V_2)$ closed operators such that $\mathcal{N}(B)$ and $\mathcal{N}(A)$ are complemented in V_1 and V_2, respectively (the complements are denoted respectively by V_1' and V_2'). The projector from V_2 to V_2' will be denoted as $P_{V_2'}$. Let $C \in L(V_1, V_2)$ be such that $\mathcal{D}_C \cap \mathcal{D}_B \neq \emptyset$ and $C(\mathcal{D}_C \cap \mathcal{D}_B) \subset \mathcal{R}(A)$. We assume that $(\sigma_p(B) \cap \sigma_p(A)) \smallsetminus \{0\} \neq \emptyset$ and label such intersection as $\sigma \equiv (\sigma_p(B) \cap \sigma_p(A)) \smallsetminus \{0\}$.

Theorem 4.5 (The inhomogeneous equation)**.** [16, Theorem 2.4.] *With regards to the previous notation, if σ contains two disjoint families of different non-zero elements $\{\mu_j\}_{j\in J} \cup \{\lambda_i\}_{i\in I} \subset \sigma$, where $\{\mu_j\}_{j\in J}$ and $\{\lambda_i\}_{i\in I}$ have following properties:*

(1) For every $j \in J$ let $u_j' \in \mathcal{D}_B \cap \mathcal{D}_C \cap V_1'$ such that $Bu_j' = \mu_j u_j'$ and $C(u_j') = 0$.

(2) For every $i \in I$, let $u_i \in \mathcal{D}_B \cap \mathcal{D}_C \cap V_1'$ such that $Bu_i = \lambda_i u_i$ and $C(u_i) \neq 0$, $C(u_i) \in \mathcal{R}(A - \lambda_i I)$ and $C(u_i)$ is linearly independent with the vectors from $\{(A - \lambda_k I)^{-1} P_{V_2'} C(u_k)\}_{k\in I}$. We also require that $\{C(u_i)\}_{i\in I}$ are linearly independent different vectors.

Then there exists a weak solution to the inhomogeneous equation (1.1)*, defined on*

$$(\mathcal{N}(B) \cap \mathcal{D}_C) \oplus (\mathrm{Lin}(\{u_j'\}_{j\in J})) \oplus (\mathrm{Lin}(\{u_i\}_{i\in I})).$$

Remark 4.5. Notice that $\mathrm{Lin}(\{u_j'\}_{j\in J}) \cap \mathrm{Lin}(\{u_i\}_{i\in I}) = \{0\}$, where u_j' and u_i are eigenvectors for B which correspond to different eigenvalues μ_j and λ_i of B. Therefore, the direct sum $\mathrm{Lin}(\{u_j'\}_{j\in J}) \oplus \mathrm{Lin}(\{u_i\}_{i\in I})$ exists. We now proceed to prove the stated theorem.

Proof. Since B and A are closed operators, the corresponding null spaces are closed subspaces in V_1, V_2, respectively. The subspaces $\mathcal{N}(B)$ and $\mathcal{N}(A)$ have topological complements, so V_1 and V_2 can be split into direct sums. Let $V_1 = \mathcal{N}(B) \oplus V_1'$ and $V_2 = \mathcal{N}(A) \oplus V_2'$ as stated in the theorem. Put $V_1(B) := V_1' \cap \mathcal{D}_B$ and $V_2(A) := V_2' \cap \mathcal{D}_A$. Define one-to-one operators $B_1 \in L(V_1(B), V_1)$ and $A_1 \in L(V_2(A), V_2)$ like in the proof of Theorem 4.2. We now have $\sigma_p(A_1) = \sigma_p(A) \smallsetminus \{0\}$ and $\sigma_p(B_1) = \sigma_p(B) \smallsetminus \{0\}$.

Let $u \in \mathcal{N}(B) \cap \mathcal{D}_C$. Since $C(u) \in \mathcal{R}(A) = \mathcal{R}(A_1)$ there exists a unique $v \in V_2(A)$ such that $C(u) = A_1 v = Av$. Put $N(u) := v$. It follows that

$$AN(u) - NB(u) = AN(u) = A(v) = C(u), \tag{4.8}$$

for every $u \in \mathcal{N}(B) \cap \mathcal{D}_C$.

Now observe V_1', V_2' and B_1 and A_1. We define closed one-to-one operators $\Psi_1^{(0)} \in L(\mathrm{Lin}(\{u_j'\}_{j\in J}), V_1)$ and $\Psi_2^{(0)} \in L(\mathrm{Lin}(\{v_j'\}_{j\in J}), V_2)$ such that $\Psi_1^{(0)} u_j' := \frac{\mu_j}{2} u_j'$, $\Psi_2^{(0)} v_j' := \frac{\mu_j}{2} v_j'$, for every $j \in J$. Now $\{\frac{\mu_j}{2}\}_{j\in J} \subset \sigma_p(\Psi_1^{(0)}) \cap \sigma_p(\Psi_2^{(0)}) \neq \emptyset$, and $\mu_i = \mu_j \Rightarrow i = j$. Since $\mathcal{N}(\Psi_1^{(0)}) = 0_{V_1}$ and $\mathcal{N}(\Psi_2^{(0)}) = 0_{V_2}$, then $\mathcal{N}(\Psi_1^{(0)})$ and $\mathcal{N}(\Psi_2^{(0)})$ have topological complements in V_1' and V_2', respectively. Now Theorem 4.2 implies that there exists a non-trivial weak solution

$$Y^{(0)} \in L(\mathrm{Lin}(\{u_j'\}_{j\in J}), \mathrm{Lin}(\{v_j'\}_{j\in J})),$$

such that

$$\Psi_2^{(0)} Y^{(0)} - Y^{(0)} \Psi_1^{(0)} = 0 \tag{4.9}$$

holds. Further, for every $j \in J$ we have $Y^{(0)}(u'_j) = v'_j$ (see proof of Theorem 4.2). Note that

$$0 \notin \{\mu_j/2\}_{j\in J} \subset \sigma_p(B_1 - \Psi_1^{(0)}) \cap \sigma(A_1 - \Psi_2^{(0)}) \neq \emptyset,$$

and $\{u'_j\}_{j\in J}$ and $\{v'_j\}_{j\in J}$ are the corresponding eigenvectors, respectively. Due to the assumption 1. of the theorem, $C(u'_j) = 0$, so $(Y^{(0)} + C)(u'_j) = v'_j$, for every $j \in J$. Since $B_1 - \Psi_1^{(0)}$ is one-to-one on $\mathrm{Lin}(\{u'_j\}_{j\in J})$ and $A_1 - \Psi_2^{(0)}$ is one-to-one on $\mathrm{Lin}(\{v'_j\}_{j\in J})$, we can apply Theorem 4.2 and conclude that $Y^{(0)} + C$ is a weak solution to the injective equation $(A_1 - \Psi_2^{(0)})X - X(B_1 - \Psi_1^{(0)}) = 0$, defined on $\mathrm{Lin}(\{u'_j\}_{j\in J})$. But then for every $u' \in \mathrm{Lin}(\{u'_j\}_{j\in J})$,

$$\begin{aligned}
0 &= (A_1 - \Psi_2^{(0)})(Y^{(0)} + C)(u') - (Y^{(0)} + C)(B_1 - \Psi_1^{(0)})(u') \\
&= A_1(Y^{(0)} + C)(u') - \Psi_2^{(0)}Y^{(0)}(u') - \Psi_2^{(0)}C(u') \\
&\quad - (Y^{(0)} + C)B_1(u') + Y^{(0)}\Psi_1^{(0)}(u') + C\Psi_1^{(0)}(u') \\
&= A_1(Y^{(0)} + C)(u') - (Y^{(0)} + C)B_1(u') \\
&\quad - (\Psi_2^{(0)}Y^{(0)} - Y^{(0)}\Psi_1^{(0)})(u') - (\Psi_2^{(0)}C - C\Psi_1^{(0)})(u') \\
&= A_1(Y^{(0)} + C)(u') - (Y^{(0)} + C)B_1(u') - C(u'),
\end{aligned}$$

where we used (4.9) and $\Psi_2^{(0)}C(u') - C\Psi_1^{(0)}(u') = 0 = C(u')$, $u' \in \mathrm{Lin}(\{u'_j\}_{j\in J})$. Put $X^{(0)} := C + Y^{(0)}$.

Condition 2. of the Theorem yields the following. For every $i \in I$, define

$$\Psi_1(u_i) := \tfrac{1}{2}B_1(u_i).$$

Then $\sigma_p(\Psi_1) \supset \{\frac{\lambda_i}{2}\}_{i\in I}$ and $\{u_i\}_{i\in I}$ are the corresponding eigenvectors. Also note that $\{\lambda_i/2\}_{i\in I} \subset \sigma_p(B_1 - \Psi_1)$ and u_i are the corresponding eigenvectors. Now define

$$\Psi_2(C(u_i)) := (1 + \lambda_i/2)C(u_i).$$

Further, since $C(u_i) \in \mathcal{R}(A - \lambda_i I)$, there exists a unique $v_i \in V'_2 \cap \mathcal{D}_A$ such that $v_i = (A_1 - \lambda_i I)^{-1}(C(u_i))$, that is, $(A_1 - \lambda_i I)v_i = C(u_i)$. Since $\{C(u_i)\}_{i\in I}$ are linearly independent vectors, it follows that $\{v_i\}_{i\in I}$ are linearly independent vectors. Define

$$\Psi_2(v_i) := \frac{\lambda_i}{2}v_i + C(u_i).$$

Since $\{C(u_i)\}_{i\in I}$ are linearly independent vectors with regards to $\{v_i\}_{i\in I}$, we conclude that Ψ_2 is well defined on $\mathrm{Lin}(\{C(u_i)\}_{i\in I}) \oplus \mathrm{Lin}(\{v_i\}_{i\in I})$. Now

$$\Psi_2(v_i - C(u_i)) = \frac{\lambda_i}{2}(v_i - C(u_i)).$$

In other words, $\{\frac{\lambda_i}{2}\}_{i\in I} \subset \sigma_p(\Psi_2)$ and $v_i - C(u_i)$ are the corresponding eigenvectors. Also

$$(A_1 - \Psi_2)v_i = A_1(v_i) - \frac{\lambda_i}{2}v_i - C(u_i) = A_1(v_i) - \frac{\lambda_i}{2}v_i - (A_1 - \lambda_i I)v_i = \frac{\lambda_i}{2}v_i,$$

so $\{\lambda_i/2\}_{i\in I} \subset \sigma_p(A_1 - \Psi_2)$ and v_i are the corresponding eigenvectors. Now

$$\left\{\frac{\lambda_i}{2}\right\}_{i\in I} \subset \sigma_p(A_1 - \Psi_2) \cap \sigma_p(B_1 - \Psi_1).$$

Since $\mathcal{N}(A_1 - \Psi_2) = 0_{V_2}$ and $\mathcal{N}(B_1 - \Psi_1) = 0_{V_1}$, it follows that $\mathcal{N}(A_1 - \Psi_2)$ and $\mathcal{N}(B_1 - \Psi_1)$ have topological complements in V_2' and V_1', respectively, so (applying Theorem 4.2) there exists an $X^{(1)}$, which is a weak solution to the equation (4.7), and it is defined as $X^{(1)}(u_i) := v_i$, (see proof of Theorem 4.2). Put

$$Y(u_i) := X^{(1)}(u_i) - C(u_i) = v_i - C(u_i).$$

We verify that (4.6) holds:

$$\begin{aligned}\Psi_2 Y(u_i) - Y\Psi_1(u_i) &= \Psi_2(v_i - C(u_i)) - Y\Big(\frac{\lambda_i}{2}u_i\Big)\\ &= \frac{\lambda_i}{2}v_i + C(u_i) - \Psi_2(C(u_i)) - \frac{\lambda_i}{2}u_i + \frac{\lambda_i}{2}C(u_i)\\ &= \Big(1 + \frac{\lambda_i}{2}\Big)C(u_i) - \Psi_2(C(u_i)) = 0.\end{aligned}$$

Finally, we verify that (4.4) holds:

$$\Psi_2 C(u_i) - C\Psi_1(u_i) - C(u_i) = \Big(1 + \frac{\lambda_i}{2}\Big)C(u_i) - \frac{\lambda_i}{2}C(u_i) - C(u_i) = 0. \tag{4.10}$$

Put $X = N \oplus X^{(0)} \oplus X^{(1)}$. Combining observations from (4.8) to (4.10), we see that X is a weak solution to (1.1), defined on

$$(\mathcal{N}(B) \cap \mathcal{D}_C) \oplus (\mathrm{Lin}(\{u_j'\}_{j\in J})) \oplus (\mathrm{Lin}(\{u_i\}_{i\in I})). \qquad \square$$

Remark 4.6. Once again, if $\sigma_p(A) \cap \sigma_p(B) = \{0\}$ then the solution is provided by the operator N from the equation (4.8).

Corollary 4.5 (The inhomogeneous equation)**.** [16, Corollary 2.7.] *Let V_1 and V_2 be Banach spaces, $B \in L(V_1)$, $A \in L(V_2)$ closed operators such that $\mathcal{N}(B)$ and $\mathcal{N}(A)$ are complemented in V_1, V_2, respectively. In that sense, put $V_1 = \mathcal{N}(B) \oplus V_1'$ and $V_2 = \mathcal{N}(A) \oplus V_2'$. Let $C \in L(V_1, V_2)$ such that $\mathcal{D}_C \cap \mathcal{D}_B \neq \{0\}$ and $C(\mathcal{D}_C \cap \mathcal{D}_B) \subset \mathcal{R}(A)$. If*

$$\{\mu_j\}_{j\in\mathbb{N}} \cup \{\lambda_i\}_{i\in\mathbb{N}} \subset (\sigma_p(B) \cap \sigma_p(A)) \setminus \{0\}$$

where $\{\mu_j\}_{j\in\mathbb{N}}$ and $\{\lambda_i\}_{i\in\mathbb{N}}$ are disjoint families of different elements with following properties:

(1) *For every $j \in \mathbb{N}$ let $u_j' \in \mathcal{D}_B \cap \mathcal{D}_C \cap V_1'$ such that $Bu_j' = \mu_j u_j'$ and $C(u_j') = 0$. Assume $\{u_j'\}_{j\in\mathbb{N}}$ to form a Schauder basis for some $S_J < \mathcal{D}_B \cap \mathcal{D}_C \cap V_1'$.*

(2) *For every $i \in \mathbb{N}$ let $u_i \in \mathcal{D}_B \cap \mathcal{D}_C \cap V_1'$ such that $Bu_i = \lambda_i u_i$ and $\{u_i\}_{i\in\mathbb{N}}$ forms a Schauder basis for some $S_I < \mathcal{D}_B \cap \mathcal{D}_C \cap V_1'$. Assume that $\{C(u_i)\}_{i\in\mathbb{N}}$ are linearly independent different non-zero vectors, which form a Schauder basis for some $S_C < \mathcal{R}(A) \cap \mathcal{R}(A - \lambda_i I)$, and vectors $\{P_{V_2'}(A - \lambda_i I)^{-1}C(u_i)\}_{i\in\mathbb{N}}$ to form a Schauder basis for some $S_V < \mathcal{D}_A \cap V_2'$, such that $S_C \cap S_V = \{0\}$.*

(3) *We require $S_J \cap S_I = \{0\}$.*

Then there exists a weak solution to the non-homogeneous equation (1.1)*, defined on* $(\mathcal{N}(B) \cap \mathcal{D}_C) \oplus (S_J) \oplus (S_I)$.

4.3. A special case of self-adjoint operators on Hilbert spaces. As shown by the previous results from this section, when we observe the unbounded singular Sylvester equation (4.1) we only obtain weak solutions, which exist on the corresponding eigenspaces of the shared eigenvalues. These results are "good enough" if we are interested in some nice enough differential operators, like the Sturm–Liouville operators, where we are only interested in behavior of operators in the observed eigenfunctions.

In this section we restrict to the case where V_1 and V_2 are separable Hilbert spaces and A and B are self-adjoint operators. By doing so, we extend the weak solutions obtained in the previous section to the largest amenable domains. We briefly recap some results about self-adjoint operators which are relevant for the paper, see [22, 42, 48] or [63]. To start, recall that if L is a self-adjoint operator, then $\sigma(L) = \sigma_{\mathrm{app}}(L) = \sigma_c(L) \cup \sigma_p(L)$, where $\sigma_c(L) = \sigma_{\mathrm{app}}(L) \setminus \sigma_p(L)$.

Theorem 4.6 (Spectral mapping theorem for self-adjoint operators)**.** *For a self-adjoint operator L, densely defined on a separable Hilbert space V, there exists a unique decomposition of identity, $(E_\lambda : \ \lambda \in \mathbb{R})$, consisting of orthogonal projectiors E_λ, such that*

(1) *The representation $L = \int_{-\infty}^{+\infty} \lambda \, \mathrm{d}\, E_\lambda$ holds, where $\mathcal{D}_L$ consists of those $x \in V$ such that the integral $\int_{-\infty}^{+\infty} \lambda^2 \, \mathrm{d}\, |E_\lambda x|^2$ converges.*
(2) *The function $\lambda \mapsto E_\lambda$ is strongly continuous from above. Furthermore, points of discontinuity of the function are precisely the eigenvalues for the operator L. In that case, if λ_0 is an eigenvalue of L, then $E_{\lambda_0} - E_{\lambda_0 - 0}$ is the orthogonal projector from V onto the eigenspace W_{λ_0} of L, which corresponds to λ_0.*
(3) *The operator L commutes with every E_λ. Furthermore, an operator S commutes with L if and only if it commutes with every projector E_λ.*

Separability of the space V, as well as density of the domain $\mathcal{D}_L$ play essential roles in the proof: important consequences follow immediately, which are applied in this paper as well.

Proposition 4.1. *With respect to the previous Theorem, the space V allows an orthogonal decomposition $V = \bigoplus_n V_n$, where V_n is an L-invariant subspace of V, such that $L_n := L(\mathcal{D}_L \cap V_n)$ is a bounded linear self-adjoint operator on V_n with $\mathcal{D}_{L_n} = \mathcal{D}_L \cap V_n$. In that case, $L = \bigoplus_n L_n$.*

Proposition 4.2. *Let V be a separable Hilbert space and let $V = \bigoplus_n V_n$ be an orthogonal sum of mutually orthogonal closed spaces V_n. If $(L_n)_n$ is a sequence of self-adjoint bounded linear operators, $L_n \in \mathcal{B}(V_n)$, then there exists a unique self-adjoint operator L densely defined in V, such that every V_n is L-invariant, and that L restricted to V_n coincides with L_n. The domain $\mathcal{D}_L$ consists of those vectors $x \in V$ such that the series $\sum_{n=1}^{+\infty} \|L_n x_n\|^2$ converges, where $x_n = P_{V_n} x$. If* $\sup\{\|L_n\| : n \in \mathbb{N}\}$ *is finite, then L is a bounded operator.*

Once again, we denote $\emptyset \neq \sigma(A) \cap \sigma(B) =: \sigma$. Throughout this section, for simpler notation, we assume that $\mathcal{D}_B \subset \mathcal{D}_C$.

We start with the simplest case, and that is when the spectral intersection occurs at point spectra, that is $\sigma = \sigma_p(A) \cap \sigma_p(B)$, The results obtained in this case are remarkably similar to those obtained in Theorem 2.1. Recall notation from Section 2: let $E_B^\lambda := \mathcal{N}(B-\lambda I)$ and $E_A^\lambda := \mathcal{N}(A-\lambda I)$ whenever $\lambda \in \sigma$. Different eigenvalues generate mutually orthogonal eigenvectors, so the spaces E_B^λ form an orthogonal sum. Put $E_B := \overline{\sum_\lambda E_B^\lambda}$. It is a closed subspace of V_1 and there exists $E_B^\perp$ such that $V_1 = E_B \oplus E_B^\perp$. Take $B = B_E \oplus B_1$ with respect to that decomposition and denote $C_0 = CP_{E_B^\perp}$.

Theorem 4.7 (The point spectrum case)**.** [10, Theorem 2.1.] *For given separable Hilbert spaces V_1 and V_2, let A and B be densely defined self-adjoint operators on V_2 and V_1 respectively, such that $\sigma(A) \cap \sigma(B) = \sigma_p(A) \cap \sigma_p(B) = \sigma$. Further, let $C \in L(V_1, V_2)$ be an arbitrary densely defined linear operator, such that $\mathcal{D}_B \subset \mathcal{D}_C$.*

(1) *If the condition*

$$C(\mathcal{N}(B-\lambda I)) \subset \mathcal{R}(A-\lambda I), \tag{4.11}$$

holds for every $\lambda \in \sigma$, then there exist infinitely many solutions X_E to the equation (4.1)*, defined on D_E*

$$\left\{u \in \mathcal{N}(B-\lambda I) : \ \lambda \in \sigma, \sum_{\lambda \in \sigma} P_{\mathcal{N}(A-\lambda I)^\perp} (A-\lambda I)^{-1} Cu \ \textit{converges} \right\}.$$

(2) *In addition, B_1 is a densely defined closed self-adjoint operator as well, $B_1 \colon \mathcal{D}_{B_1} \to E_B^\perp$. Assume that $\mathcal{D}_{B_1} \subset \mathcal{D}_{C_0}$, and that the following implication holds*

$$\begin{gathered} 0 \in \sigma(A) \cap \sigma(B_1) \Rightarrow 0 \in \sigma_p(A) \cap \sigma_p(B_1), \\ C(\mathcal{N}(B_1)) \subset \mathcal{R}(A). \end{gathered} \tag{4.12}$$

Then there exist infinitely many solutions X_1 to the eq. (4.1)*, defined on* (*with respect to inclusion*) *the largest subspace $\mathcal{D}_{X_1} \subset E_B^\perp$.*

(3) *The solutions $X := X_E + X_1$, obtained in parts* 1 *and* 2 *of this theorem, defined on their largest domains* (*with respect to inclusion*) $\mathcal{D}_E + \mathcal{D}_{X_1}$*, are unique in the quotient class of operators from*

$$L(V_1/(E_B + \mathcal{N}(B_1)), V_2/(E_A + \mathcal{N}(A))), \tag{4.13}$$

defined on the same domain.

Remark 4.7. This theorem is proved in a very similar manner in which Theorem 2.1 was proved. Furthermore, since B is self-adjoint, it follows that both E_B and $E_B^\perp \cap \mathcal{D}_B$ are B-invariant subspaces of V_1, thus the functional calculus for B_E and B_1 applies. In addition, Statement 3. and expression (4.13) naturally generalize the characterization of matrix solutions obtained in the eigen-problem (2.7).

Corollary 4.6 (Number of solutions)**.** [10, Corollary 2.1.] *With respect to the previous notation, let all assumptions from Theorem* 4.7 *hold. Denote by* Σ *and* Ω *the sets of linear operators such that*

$$\Sigma = \big\{N_\sigma :\ N_\sigma = \textstyle\bigoplus_{\lambda\in\sigma} N_\lambda,\ N_\lambda \in L(E_B^\lambda, E_A^\lambda),\ \lambda\in\sigma\big\},$$
$$\Omega = \{N_0 \in L(\mathcal{N}(B_1), \mathcal{N}(A))\}.$$

Let S *be the set of all solutions to* (4.1)*, which are defined on the largest domains possible. Then* $|\Omega|\cdot|\Sigma| = |S|$.

Proof. Proof follows directly from Theorem 4.7, because choices for solutions depend solely on N_λ and N_0, whenever $\lambda\in\sigma$. □

Remark 4.8. Due to Corollary 4.6, the solution $X_{(N_\sigma+N_0)}\in S$, with $N_\sigma\in\Sigma$ and $N_0\in\Omega$, can be referred to as a *particular solution.* Similarly to the matrix case, these particular solutions are highly unstable to small perturbations, because they depend on the choice of the corresponding eigenvectors.

We now investigate the general case, where the spectral intersection occurs in the approximate point spectra of A and B. Let $L\in\{A,B\}$, and assume that $\lambda\in\sigma_{\text{app}}(L)$, that is, there exists a sequence $(x_n)\subset\mathcal{D}_L$ such that $\|x_n\| = 1$ while $\|(L-\lambda I)x_n\|\to 0$ as $n\to\infty$. The main idea is to construct a set which resembles an approximate eigenspace with respect to λ, in order to apply the same method from the previous case.

To start, assume that L is a bounded normal operator on a Hilbert space V. Then for fixed μ and $\lambda\in\sigma_{\text{app}}(L)$, there exist two normed sequences (x_n) and (y_n), such that $\|(L-\lambda I)x_n\|$ and $\|(L-\mu I)y_n\|$ simultaneously tend to zero as n approaches infinity. Then for every n:

$$|(\mu-\lambda)\langle x_n, y_n\rangle| = |\langle\lambda x_n - Lx_n, y_n\rangle + \langle x_n, L^*y_n - \bar{\mu}y_n\rangle| \leqslant \|\lambda x_n - Lx_n\| + \|Ly_n - \nu y_n\|,$$

which tends to zero as $n\to+\infty$. This implies that approximate eigenvectors corresponding to different approximate eigenvalues tend to behave in an orthogonal manner, similarly to the exact eigenvectors corresponding to the actual different eigenvalues. This motivates the characterization of the approximate point spectrum of all bounded linear operators $L\in\mathcal{B}(V)$, which goes as the following (see [4]). Denote by $\ell_\infty(V)$ the space of all bounded sequences with values in V, equipped with the sup $-$norm. The set of all sequences which converge to zero is denoted by $c_0(V)$. It follows that c_0 is, with respect to the relative topology inherited from $\ell_\infty(V)$, a proper closed subspace, and defines a quotient space $\ell_\infty(V)/c_0(V)$ in a natural way. What is left is to enclose this space, in a manner that $\overline{\ell_\infty(V)/c_0(V)}$ forms a complete inner product space, with inner product defined via the generalized limits (called Banach limits) in $\ell_\infty(V)$ (see [4] for a more detailed construction). For a sequence $(x_n)_n\in\ell_\infty(V)$, a bounded linear operator $L\in\mathcal{B}(V)$ defines a bounded linear map on $\ell_\infty(V)$ as $L'((x_n)_n) := (Lx_n)_n\in\ell_\infty(V)$. Furthermore, it follows that $L'(x_n)\in c_0(V)$, whenever $(x_n)\in c_0(V)$. Hence, $L_0'\colon \ell_\infty(V)/c_0(V)\to\ell_\infty(V)/c_0(V)$ defines a bounded linear operator, such that $L_0'((x)_n/c_0(V)) := (L'(x_n))/c_0(V)$, for every $(x_n) \in \ell_\infty(V)$. This implies that $\|L\| = \|L_0'\|$, and that L_0' extends continuously to the entire space $\overline{\ell_\infty(V)/c_0(V)}$, and that extension is denoted again by L_0'.

Theorem 4.8. [4, Theorem 1] *For every $L \in \mathcal{B}(V)$, $\sigma_{\text{app}}(L) = \sigma_{\text{app}}(L_0') = \sigma_p(L_0')$.*

Combining the previous discussion with the spectral mapping theorem for self-adjoint operators (Theorem 4.6), we modify Theorem 4.8 and apply it to our own problem.

Lemma 4.3. [10, Lemma 2.1.] *Let V be a Hilbert space and let L be a densely defined bounded self-adjoint operator on V. Then there exists L_0' defined in the previous manner. For that L_0' we have $\sigma_{\text{app}}(L) = \sigma_p(L_0') = \sigma_{\text{app}}(L_0')$.*

Theorem 4.9 (The general case)**.** [10, Theorem 2.3] *Let $A \in L(V_1)$ and $B \in L(V_2)$ be closed densely defined self-adjoint operators on separable Hilbert spaces V_1 and V_2, with spectral resolutions of identities*

$$(4.14)\quad B = \int_{-\infty}^{+\infty} \mu \, \mathrm{d}\, F_\mu, \quad V_1 = \oplus_n V_{1n},\ B_n \colon V_{1n} \to V_{1n} \text{ is a bounded operator},$$

$$(4.15)\quad A = \int_{-\infty}^{+\infty} \lambda \, \mathrm{d}\, E_\lambda, \quad V_2 = \oplus_n V_{2n},\ A_n \colon V_{2n} \to V_{2n} \text{ is a bounded operator.}$$

Assume that $\sigma_{\text{app}}(B) \cap \sigma_{\text{app}}(A) =: \sigma \neq \emptyset$ and let $C \in L(V_1, V_2)$ be arbitrary densely defined linear operator, such that $\mathcal{D}_B \subset \mathcal{D}_C$. For every n, let operators $(B_n)_0'$ and $(A_n)_0'$ be defined as in the previous paragraph and let $(C_n)_0'$ be defined accordingly. If operators $(A_n)_0'$, $(B_n)_0'$ and $(C_n)_0'$ satisfy conditions (4.11)–(4.12) *from Theorem 4.7, then there exist infinitely many solutions to $A_0'X_0' - X_0'B_0' = C_0'$, defined on the largest subsets of $\overline{\ell_\infty(V_1)/c_0(V_1)}$ possible.*

Proof. The first step is to apply spectral decomposition as in (4.14) and (4.15). Now if $\sigma = \sigma_p(A) \cap \sigma_p(B)$, then Theorem 4.7 applies. Otherwise, apply Lemma 4.3 to each B_n and A_n, respectively. Then the problem is transferred to the first case, that is, the spectral intersection occurs in the point spectra. If the conditions (4.11)–(4.12) are satisfied, then Theorem 4.7 applies and the proof is complete. ☐

5. Lyapunov operator equations

Let $V_1 = V_2 = \mathcal{H}$ be a separable Hilbert space. For a (not necessarily bounded) self-adjoint operator A, when $B = -A$ and $C = I$ we get the symmetric Lyapunov operator equation

$$(5.1)\qquad AX + XA = I,$$

which is called symmetric because we are interested in finding symmetric solutions X. This is a special case of the proper Lyapunov operator equation

$$(5.2)\qquad A^*X + X^*A = I,$$

where A is an arbitrary closed densely defined linear operator on $\mathcal{H}$.

Similarly to the Sylvester equations, Lyapunov operator equation (5.1) is regular if there exists a unique bounded, stable, symmetric solution X, and contrary, the equation is singular if it is not regular. Obviously, if the self-adjoint operator A generates a C_0-semigroup of operators which is unifromily exponentially stable (equivalently, if the spectrum of A is negative) then Theorem 1.6 applies and there exists a unique bounded X which is a solution to (5.1). However, taking the Hilbert-conjugate of the equation, we get that $X^*A + AX^* = I$ holds, thus $X = X^*$. This logic also applies to a more general Lyapunov operator equation

$$AX + XA^* = \pm I, \tag{5.3}$$

where A is an arbitrary closed operator. In this case, if the spectrum of A is contained in the open left complex half-plane then $\mathrm{Re}(\sigma(A))$ is bounded from the right by the imaginary axis. Consequently, A generates a uniformly exponentially stable C_0-semigroup, and Theorem 1.6 states that there exists a unique solution to (5.3). Taking the Hilbert-conjugate of the equation once again gives $X = X^*$.

The equation (5.2) cannot explicitly be solved via Theorem 1.6, but the following decomposition applies (see [12]). If A is a closed operator then $A = U|A|$ is the polar decomposition for A, where $|A| = \sqrt{A^*A}$ and U is a partial isometry. In that sense, the eq. (5.2) becomes

$$|A|U^*X + X^*U|A| = I. \tag{5.4}$$

If the equation $|A|Y + Y|A| = I$ is regular, then there exists a unique solution $Y = \frac{1}{2}|A|^{-1}$ and consequently X from (5.4) is precisely of the form $X = UY$. Notice that the eq. $|A|Y + Y|A| = I$ is regular if and only if $\mathcal{N}(|A|) = \{0\}$. Conveniently, the polar decomposition $A = U|A|$ uniquely determines U if we require $\mathcal{N}(U) = \mathcal{N}(|A|)$. That being said, since both Y and U are uniquely determined when $|A|Y+Y|A| = I$ is a regular equation, then so is X from (5.2) when that is a regular equation.

The previous analysis shows that when the equations (5.1) and (5.3) are regular their solutions are symmetric, and when the equation (5.2) is regular, then its solution is $A-$symmetric (meaning that $X = UY$, where Y a symmetric operator and U is the partial isometry obtained from the polar decomposition of A). In fact, this property is crucial for papers which concern regular Lyapunov operator and matrix equations (see [6, 17, 60, 66]) and consequently, for the papers which concern Cauchy problems whose solutions are stable in the Lyapunov sense (see [6,26,34,40,41,43,50,66]). This is exactly what motivates us to investigate behavior of singular equations (5.1) and (5.2).

We start with obtaining a symmetric solution to (5.1) and then we work our way up to find $A-$symmetric solutions to the equation (5.2). We also prove that some of these results cannot be weakened. Notice that we can find infinitely many solutions to (5.1) via Theorem 4.7, but this does not guarantee that a single one of them will be symmetric. In order to find a symmetric solution to (5.1), we impose a natural assumption for the self-adjoint operator A, which is by far less restrictive than requiring it to be the infinitesimal generator of a C_0-semigroup:

Assumption 1. $(\mathcal{A}_1(L))$: Let L be a (not necessarily bounded) self-adjoint operator on the separable Hilbert space $\mathcal{H}$. The set $\Delta_L := \mathcal{D}_{L^2} \cap \mathcal{R}(L)$ is assumed to be dense in $\mathcal{H}$, where $\mathcal{R}(L)$ is the image of L and $\mathcal{D}_{L^2} = \{u \in \mathcal{D}_L : \ Lu \in \mathcal{D}_L\}$.

Theorem 5.1. [12, Theorem 3.1.] *Let A be a self-adjoint operator such that $\mathcal{A}_1(A)$ holds. Then there exists a symmetric $X\colon \Delta_A \to \mathcal{H}$ which is a solution to the Lyapunov operator equation* (5.1) *on Δ_A.*

Proof. Let $(P_\lambda)_{\lambda\in\mathbb{R}}$ be the spectral resolution for the self-adjoint operator A. Since A and A^2 commute on $\mathcal{D}_{A^3}$, it follows that A^2 commutes with every P_λ on $\mathcal{D}_{A^3}$ as well. Now let Y be any finite real linear combination of such orthogonal projectors P_λ (or any real summable family $Y = \sum_\lambda \alpha_\lambda P_\lambda$ of the projections P_λ, where $(\alpha_\lambda)_\lambda \in \ell^1(\mathbb{R})$). It follows that Y commutes with A on $\mathcal{D}_A$. In addition, Y is a symmetric operator on $\mathcal{D}_A$. Denote by Z the operator $Z := AY + YA - I$, defined on $\mathcal{D}_A$. It follows that $Z^* = AY^* + Y^*A - I \supset AY + YA - I = Z$, thus Z is a symmetric operator on $\mathcal{D}_A$. By construction, Z commutes with A on $\mathcal{D}_{A^2}$.

Case 1. Assume that zero is a regular point for the operator A. Then every projector P_λ (from the spectral resolution of A) commutes with A^{-1} and consequently, so does the operator Y (restricted to $\mathcal{R}(A)\cap\mathcal{D}_A$) and finally so does the operator Z (on $\mathcal{D}_{A^2}\cap\mathcal{R}(A)$). Let $X := Y - \frac{1}{2}A^{-1}Z$ be defined on $\mathcal{D}_X = \Delta_A$. Then X is symmetric on its domain and for every $u \in \Delta_A$ we have

$$(AX + XA)u = \big(AY - \tfrac{1}{2}Z + YA - \tfrac{1}{2}Z\big)u = (I + Z - Z)u = Iu.$$

Notice that X is in fact $X = Y - \frac{1}{2}A^{-1}Z = Y - \frac{1}{2}Y - \frac{1}{2}Y + \frac{1}{2}A^{-1} = \frac{1}{2}A^{-1}$. So in this case, X is additionally bounded and self-adjoint on $\mathcal{H}$.

Case 2. Now assume that zero is an eigenvalue for the operator A. Then $\mathcal{N}(A)$ is a nontrivial closed proper subspace of $\mathcal{H}$ and there exists an orthogonal projection Q_0 which maps $\mathcal{H}$ onto $\mathcal{N}(A)$ parallel with $\mathcal{N}(A)^\perp$. It is noteworthy to see that $Q_0 = P_\lambda - P_{\lambda-0}$ and that we can choose the operator Y to commute with Q_0. Respectively, $\mathcal{D}_A = Q_0\mathcal{D}_A \oplus (I - Q_0)\mathcal{D}_A$ and $A = 0 \oplus A_1$, where $0 = AQ_0$ and $A_1 = A(I - Q_0)$. In that sense, there exists an operator $A_1^{-1}\colon \mathcal{R}(A) \to (I - Q_0)\mathcal{D}_A$, defined as

$$(\forall u \in \mathcal{D}_A)\ \ (u = u_1 \oplus u_2,\ u_1 \in \mathcal{N}(A),\ u_2 \in \mathcal{D}_A \cap \mathcal{N}(A)^\perp)$$
$$A_1^{-1}(Au) = A_1^{-1}Au_2 = u_2.$$

This concludes that $A_1^{-1}A$ is the orthogonal projection $I - Q_0$ observed on $\mathcal{D}_A$. Analogously, for any $u \in \mathcal{R}(A)$ there exists a $v \in \mathcal{D}_A$ such that $Av = u$ and $v = v_1 \oplus v_2$, where $v_1 \in \mathcal{N}(A)$ and $v_2 \in \mathcal{D}_A \cap \mathcal{N}(A)^\perp$. Then

$$AA_1^{-1}u = AA_1^{-1}Av = Av_2 = Av = u,$$

thus proving that $AA_1^{-1} = I_{\mathcal{R}(A)}$. Decompose $\mathcal{R}(A)\cap\mathcal{D}_{A^2}$ into an orthogonal sum

$$\begin{aligned}\mathcal{R}(A)\cap\mathcal{D}_{A^2} &= Q_0(\mathcal{R}(A)\cap\mathcal{D}_{A^2}) \oplus (I - Q_0)(\mathcal{R}(A)\cap\mathcal{D}_{A^2})\\ &= (\mathcal{N}(A)\cap\mathcal{R}(A)) \oplus (\mathcal{N}(A)^\perp\cap\mathcal{R}(A)\cap\mathcal{D}_{A^2}).\end{aligned}$$

Restricted to $\mathcal{N}(A)^\perp\cap\mathcal{R}(A)\cap\mathcal{D}_{A^2}$, the mapping AA_1^{-1} is the same as $I_{\mathcal{R}(A)}$, which is on that particular subspace the same as $I - Q_0$. Thus for every $u \in \mathcal{N}(A)^\perp\cap\mathcal{R}(A)\cap\mathcal{D}_{A^2}$ we have

$$0 = (2Y(I - Q_0) - 2(I - Q_0)Y)u = (2YAA_1^{-1} - 2A_1^{-1}AY)u$$

$$
\begin{aligned}
&= (AYA_1^{-1} + YAA_1^{-1} - A_1^{-1} - A_1^{-1}AY - A_1^{-1}AY + A_1^{-1})u \\
&= ((AY + YA - I)A_1^{-1} - A_1^{-1}(AY + YA - I))u \\
&= (ZA_1^{-1} - A_1^{-1}Z)u.
\end{aligned}
$$

Similarly as before, take $X_1 := Y - \frac{1}{2}A_1^{-1}Z$, with its domain $D_{X_1} = \mathcal{N}(A)^\perp \cap \mathcal{R}(A) \cap \mathcal{D}_{A^2}$. Then for every $x \in \mathcal{D}_{X_1}$ we have

$$
(AX_1 + X_1A)u = \Big(AY - \frac{1}{2}AA_1^{-1}Z + YA - \frac{1}{2}A_1^{-1}ZA\Big)u = (I + Z - Z)u = Iu.
$$

To complete the proof, we conduct a similar methodology on $\mathcal{N}(A) \cap \mathcal{R}(A)$. Let $u \in \mathcal{N}(A) \cap \mathcal{R}(A)$. Then there exists a $v \in \mathcal{D}_A$ such that $u = Av$ and $v = v_1 \oplus v_2$, where $v_1 \in \mathcal{N}(A)$ and $v_2 \in \mathcal{N}(A)^\perp$. Then

$$
(AX + XA)u = u \Leftrightarrow AXAv = Av \Leftrightarrow AXAv_2 = Av_2.
$$

Thus we take $X_0 := A_1^{-1}$, with its domain $\mathcal{D}_{X_0} = \mathcal{R}(A) \cap \mathcal{N}(A)$. Then $X := X_1 \oplus X_0$ is an operator defined on

$$
\mathcal{D}_X = (\mathcal{N}(A)^\perp \cap \mathcal{R}(A) \cap \mathcal{D}_{A^2}) \oplus (\mathcal{R}(A) \cap \mathcal{N}(A)) = \Delta_A.
$$

To verify that X is indeed symmetric on its domain, take arbitrary u and $v \in \Delta_A$. Then $u = u_1 \oplus u_2$ and $v = v_1 \oplus v_2$, where u_1, $v_1 \in \mathcal{N}(A)^\perp \cap \mathcal{R}(A) \cap \mathcal{D}_{A^2}$ while u_2, $v_2 \in \mathcal{N}(A) \cap \mathcal{R}(A)$. Then

$$
\begin{aligned}
\langle Xu, v\rangle &= \langle X_1u_1, (v_1 + v_2)\rangle + \langle X_0u_2, (v_1 + v_2)\rangle \\
&= \Big\langle \Big(Y - \frac{1}{2}A_1^{-1}Z\Big)u_1, (v_1 + v_2)\Big\rangle + \langle A_1^{-1}u_2, (v_1 + v_2)\rangle \\
&= \langle Yu_1, v_1\rangle + \langle Yu_1, v_2\rangle - \frac{1}{2}\langle (A_1^{-1}Z)u_1, v_1\rangle - \frac{1}{2}\langle (A_1^{-1}Z)u_1, v_2\rangle \\
&\quad + \langle A_1^{-1}u_2, v_1\rangle + \langle A_1^{-1}u_2, v_2\rangle.
\end{aligned}
$$

Because Y is a symmetric operator, invariant under the decomposition $\mathcal{H} = \mathcal{N}(A) \oplus \mathcal{N}(A)^\perp$ (because it commutes with Q_0), it follows that

$$
\langle Yu_1, v_1\rangle + \langle Yu_1, v_2\rangle = \langle u_1, Yv_1\rangle.
$$

Further, since A_1^{-1} and Z commute on $\mathcal{N}(A)^\perp \cap \mathcal{R}(A) \cap \mathcal{D}_{A^2}$ and Z and A_1^{-1} are symmetric on their domains, we have

$$
\langle (A_1^{-1}Z)u_1, v_1\rangle = \langle (ZA_1^{-1})u_1, v_1\rangle = \langle (A_1^{-1})u_1, Zv_1\rangle = \langle u_1, (A_1^{-1}Z)v_1\rangle.
$$

Now recall that $\mathcal{R}(A_1^{-1}) \subset \mathcal{N}(A)^\perp$, thus $\langle (A_1^{-1}Z)u_1, v_2\rangle = 0 = \langle A_1^{-1}u_2, v_2\rangle$. Since $u_2 \in \mathcal{N}(A) \cap \mathcal{R}(A)$, there exists an $w \in \mathcal{D}_A$ such that $Aw = Aw_1 = u_2$, where $w = w_1 \oplus w_2$ and $w_1 \in \mathcal{D}_A \cap \mathcal{N}(A)^\perp$, $w_2 \in \mathcal{N}(A)$. Then $A_1^{-1}u_2 = w_1$. In addition, $v_1 \in \mathcal{N}(A)^\perp \cap \mathcal{D}_{A^2} \cap \mathcal{R}(A)$, thus there exists a $z \in \mathcal{D}_A$ such that $Az = Az_1 = v_1$, where $z = z_1 \oplus z_2$, $z_1 \in \mathcal{D}_A \cap \mathcal{N}(A)^\perp$ while $z_2 \in \mathcal{N}(A)$. Then

$$
\begin{aligned}
\langle A_1^{-1}u_2, v_1\rangle &= \langle w_1, v_1\rangle = \langle w_1, Az\rangle = \langle Aw_1, z\rangle \\
&= \langle u_2, z\rangle = \langle u_2, z_2\rangle = \langle AA_1^{-1}u_2, z_2\rangle = \langle w_1, Az_2\rangle = 0.
\end{aligned}
$$

This proves that

$$\langle Xu, v\rangle = \big\langle u_1, \big(Y - \tfrac{1}{2}A_1^{-1}Z\big)v_1\big\rangle = \langle u_1, X_1 v_1\rangle.$$

In the same manner, we have

$$\langle u, Xv\rangle = \langle X_1 u_1, v_1\rangle = \langle u_1, X_1 v_1\rangle = \langle Xu, v\rangle,$$

because X_1 is symmetric by construction on its domain.

Case 3. If zero is in the continuous part of the spectrum of A then A is not a regular operator, but it is injective from $\mathcal{D}_A$ onto $\mathcal{R}(A)$. In that sense, for every $z \in \mathcal{R}(A)$ there exists a unique $w \in \mathcal{D}_A$ such that $Aw = z$. Notice that the inverse map $z \mapsto w$ is not necessarily bounded, because A is not bounded from below. Thus we define X pointwise on elements of Δ_A: $X\colon z \mapsto \frac{1}{2}w$, where $z \in \Delta_A$ and $w \in \mathcal{D}_A$ is unique such that $Aw = z$. It follows that X is well defined on Δ_A, and though it is not bounded it is symmetric because A is self-adjoint. □

The assumption $\mathcal{A}_1$ is clearly sufficient. Below we show that it is also necessary when some natural conditions are imposed.

Theorem 5.2. [12, Theorem 3.2.] *Let A be a self-adjoint operator and let X be a densely defined symmetric operator, such that $\mathcal{D} := \mathcal{D}_A = \mathcal{D}_X$, $\mathcal{R}(A) \subset \mathcal{D}$ and $\mathcal{R}(X) \subset \mathcal{D}$. If X is a solution to* (5.1) *on $\mathcal{D}$ then $\mathcal{R}(A)$ is dense in $\mathcal{H}$.*

Proof. Assume that (5.1) is solvable for a symmetric X and that $\overline{\mathcal{R}(A)} \neq \mathcal{H}$. Then there exists a non-trivial W such that $\mathcal{H} = \overline{\mathcal{R}(A)} \oplus W$ and $W \perp \overline{\mathcal{R}(A)}$. If there exists a sequence $(w_n)_{n\in\mathbb{N}}$ in $W \cap \mathcal{D}$ such that $\|w_n\| = 1$ and $Aw_n \to 0$ when $n \to +\infty$, then

$$1 = \langle w_n, w_n\rangle = \underbrace{\langle AXw_n, w_n\rangle}_{0} + \langle XAw_n, w_n\rangle = \langle Aw_n, Xw_n\rangle = \langle w_n, AXw_n\rangle = 0$$

which is impossible. Ergo, A is bounded from below on $W \cap \mathcal{D}$, injective and $\mathcal{R}(A\restriction_{W\cap\mathcal{D}})$ is closed in $\mathcal{R}(A)$. In that case, for any $w \in W \cap \mathcal{D}$ we have $w = YAw$, where Y is the inverse for $A\restriction_{W\cap\mathcal{D}}$, $Y\colon \mathcal{R}(A\restriction_{W\cap\mathcal{D}}) \to W \cap \mathcal{D}$. Let u be arbitrary from $\mathcal{R}(A\restriction_{W\cap\mathcal{D}})$. Then $\|u\|^2 = \langle AYu, u\rangle = \langle Yu, Au\rangle = 0$, which gives $u = 0$. Consequently, $A\restriction_{W\cap\mathcal{D}} = 0\restriction_{W\cap\mathcal{D}}$, which is impossible by regularity of A on W. □

The following corollary immediately follows.

Corollary 5.1. [12, Corollary 3.1.] *Let A be a bounded self-adjoint operator on $\mathcal{H}$. The following statements hold:*

(a) *If $\mathcal{R}(A)$ is dense in $\mathcal{H}$ then there exists a symmetric solution X to* (5.1) *defined on $\mathcal{R}(A)$.*
(b) *There exists a bounded self-adjoint solution X to* (5.1) *if and only if A is invertible on $\mathcal{H}$. In that case, the solution X is also invertible on $\mathcal{H}$.*

Without loss of generality, if A is a bounded self-adjoint operator, then the bounded self-adjoint solution X to (5.1) can be chosen to be $X = \frac{1}{2}A^{-1}$. For any unital C^*−subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ which contains A, it follows that $\frac{1}{2}A^{-1} \in \mathcal{A}$, thus Gelfand-Naimark theorem proves the following statement.

Theorem 5.3. [12, Theorem 3.3.] *Let $\mathcal{A}$ be a unital C^*-algebra and let $a \in \mathcal{A}$ be a self-adjoint element. There exists a self-adjoint $x \in \mathcal{A}$ such that $ax + xa = 1_{\mathcal{A}}$ if and only if a is invertible and in that case x is invertible as well.*

We now return to the general equation (5.2). Recall the polar decomposition for A by which we obtained the equation (5.4). If A is a closed densely-defined operator in $\mathcal{H}$ then $|A| := \sqrt{A^*A}$ is a self-adjoint operator which is defined on the same domain $\mathcal{D}_{|A|} = \mathcal{D}_A$. In that sense, there exists a partial isometry U such that $A = U|A|$ and this representation is called the polar decomposition of A. If we choose $\mathcal{N}(|A|) = \mathcal{N}(U)$, then the partial isometry U is uniquely determined via $|A|$. The following statements immediately follow.

Theorem 5.4. [12, Theorem 3.4.] *Let $A\colon \mathcal{D}_A \to \mathcal{H}$ be a closed densely-defined linear operator on $\mathcal{H}$ and $A = U|A|$ be its polar decomposition. The following statements hold:*

(a) *If $|A|$ satisfies $\mathcal{A}_1(|A|)$ then there exists a solution X to* (5.2) *defined on $\Delta_{|A|}$, such that the decomposition $X = US$ holds, where S is a symmetric operator on $\Delta_{|A|}$ which solves $|A|S + S|A| = I$.*

(b) *Conversely, assume that there exists an X defined on $\mathcal{D}_A$ which solves the equation* (5.2) *and in addition allows the decomposition $X = US$, where S is a symmetric operator on $\mathcal{D}_A$, which solves $|A|S + S|A| = I$, $\mathcal{R}(S) \subset \mathcal{D}_A$ and $\mathcal{R}(A) \subset \mathcal{D}_A$. Then $\mathcal{R}(|A|)$ is dense in $\mathcal{H}$.*

Corollary 5.2. [12, Corollary 3.2] *Let A be a bounded linear operator on $\mathcal{H}$ with its polar decomposition $A = U|A|$. The following statements hold:*

(a) *If $\mathcal{R}(|A|)$ is dense in $\mathcal{H}$ there exists a solution X to* (5.2) *defined on $\mathcal{R}(|A|)$ which allows the decomposition $X = US$ where S is a symmetric operator on $\mathcal{R}(|A|)$ which solves $|A|S + S|A| = I$.*

(b) *There exists a bounded solution X to* (5.2)*, which allows the decomposition $X = US$ where S is a self-adjoint operator which solves $|A|S + S|A| = I$ if and only if A is invertible. In that case, the operator S is also invertible.*

If a is an invertible element in a unital C^*-algebra $\mathcal{A}$ then $|a| := \sqrt{a^*a}$ is also an invertible element in $\mathcal{A}$. In that case, $a = a|a|^{-1}|a|$ and $a|a|^{-1}$ is a unitary element in $\mathcal{A}$. Thus $a = u|a|$, where $u = a|a|^{-1}$ is unitary, is the polar decomposition of a in $\mathcal{A}$, consult [62].

Theorem 5.5. [12, Theorem 3.5.] *Let $a \in \mathcal{A}$ and $|a| = \sqrt{a^*a}$. The following statements are equivalent*

(a) *a is invertible in $\mathcal{A}$.*

(b) *There exists a self-adjoint solution $s \in \mathcal{A}$ to $|a|s + s|a| = 1$.*

(c) *There exists a unitary u such that $a = u|a|$ and us solves the equation*

$$a^*x + x^*a = 1_{\mathcal{A}}. \tag{5.5}$$

If any of these statements hold, the solution $x = us$ to (5.5) *is invertible in $\mathcal{A}$.*

5.1. Abstract Cauchy problems. Below we show how Lyapunov operator equations help study stability of solutions to Abstract Cauchy problems. We start with the simplest one

$$u'(t) = Au(t), \quad u(0) = u_0 \in \mathcal{D}_A \tag{5.6}$$

where $u \in C^1([0,+\infty),\mathcal{H})$, $u\colon t \mapsto u(t) \in \mathcal{D}_A$ for every $t \geqslant 0$, and A is a closed operator densely defined on the separable Hilbert space $\mathcal{H}$. Recall that any ACP can be solved in the classical or in the mild sense, see [21]. A function u is a classical solution to (5.6) if $u(0) = u_0$ and if it identically solves $u'(t) = Au(t)$, $t \geqslant 0$. A function u is said to be a mild solution to (5.6) if for every bounded linear functional $\psi \in \mathcal{H}'$ the equality $\psi(u(0) - u_0) = 0$ holds and $\psi(u'(t) - Au(t)) = 0$ when $t \geqslant 0$. Clearly every classical solution is also a mild solution, while the converse does not hold in general (see [21] or [47]). Applying the Riesz representation lemma (consult [21, 48] or [63]), it follows that u is a mild solution to (5.6) if for every $w \in \mathcal{H}$ one has $\langle u_0 - u(0), w\rangle = 0$ and $\langle u'(t) - Au(t), w\rangle = 0$ when $t \geqslant 0$.

A basic result in dynamical systems states (see [21, 48] and [63]) that for any initial $u_0 \in \mathcal{D}_A$, the ACP (5.6) has a unique classical solution given as $u(t) = S_t u_0$, which continuously depends on u_0, if and only if the operator A is the infinitesimal generator of a C_0-semigroup $(S_t)_{t\geqslant 0}$ on $\mathcal{H}$. This remarkable result tells plenty information about the operator A itself, about the C_0-semigroup $(S_t)_{t\geqslant 0}$ and finally, about the solution $u(t)$ to (5.6). In particular, the solution $u(t) = S_t u_0$ satisfies the upper bound estimate

$$\|u(t)\| \leqslant \|S_t\| \cdot \|u_0\| \leqslant M\, \mathrm{e}^{wt}\, \|u_0\|,$$

where $M \geqslant 1$ and w is the semigroup growth limit. Consequently, (see [26, 40, 43, 66]) if $\sigma(A)$ is contained strictly in the left complex half-plane, then all solutions to ACP (5.6) are stable in the Lyapunov sense, meaning that $\|u(t)\| \to 0$ when $t \to +\infty$, for any given $u_0 \in \mathcal{D}_A$ (notice that in this case the operator equation (5.1), (5.2) and (5.3) are all regular). Conversely, if there exists a $\lambda \in \sigma(A)$ such that $\operatorname{Re}\lambda > 0$, then all solutions to the linear ACP (5.6) are unstable in the Lyapunov sense, that is $\|u(t)\| \to +\infty$ when $t \to +\infty$ for any initial condition $u_0 \in \mathcal{D}_A$. For further results regarding C_0-semigroups and their applications to differential and abstract differential equations, consult [6, 21, 26, 34, 40, 43] and [66]. However, not all ACPs can be solved in this manner: there are abstract Cauchy problems which do not possess the classical infinitesimal generator for the corresponding C_0-semigroup of operators. An example can be found in [47], where the author has shown that some functional stochastic differential equations are indeed solvable, but the corresponding C_0-semigroup of expectations does not possess the classical infinitesimal generator.

At this point we show, by assuming that A is a self-adjoint operator densely defined on a separable Hilbert space $\mathcal{H}$, that the asymptotic behavior on the unit sphere of the expression (called the quadratic form)

$$\Big\langle A\frac{u(t)}{\|u(t)\|}, \frac{u(t)}{\|u(t)\|}\Big\rangle \quad \text{when} \quad t \to +\infty$$

gives an asymptotic upper bound for the unstable solution $u(t)$ to ACP (5.6). By doing so we omit the classical C_0-semigroup approach and the requirement that $\sigma(A)$ is bounded from the right. Consequently, the operator A need not be the infinitesimal generator in the classical sense. Theorem 5.6 below is an extension of a result obtained by Willimas in [66], where he had studied asymptotic behavior of stable solutions to the ACP (5.6) via regular operator equation (5.1).

The symbol $\lesssim$ stands for the asymptotic upper bound of given functions, i. e. for complex-valued functions f and g defined on $[0,+\infty)$, where $g(t) \neq 0$ for every large enough t, by $f(t) \lesssim g(t)$ we assume $\limsup_{t\to+\infty} |f(t)/g(t)| \leqslant 1$.

Theorem 5.6. [12, Theorem 4.2.] *Let A be a self-adjoint operator such that $\mathcal{A}_1(A)$ holds. Let $u\colon [0,+\infty) \to \mathcal{H}$ be a continuously-differentiable function with values in Δ_A which is a solution to* (5.6) *in the mild or classical sense. If there exists a non-decreasing real-valued function $h\colon [0,+\infty) \to \mathbb{R}$ such that*

$$|\langle Au(t), u(t)\rangle| \lesssim h(t) \cdot \|u(t)\|^2, \quad t \to +\infty$$

then

$$\|u(t)\| \lesssim \mathrm{e}^{t(h(t)+1/2)}. \tag{5.7}$$

Proof. By Theorem 5.1, there exists an X defined on Δ_A which is a symmetric linear operator and is a solution to Lyapunov operator equation (5.1). Define $f\colon [0,+\infty) \to \mathbb{R} + i\mathbb{R}^+$ as

$$f(t) := \langle (iI + X)u(t), u(t)\rangle = i\|u(t)\|^2 + \langle Xu(t), u(t)\rangle.$$

It follows that $|f(t)| \geqslant |\operatorname{Im}(f)(t)| = \|u(t)\|^2$. Differentiating f via t gives

$$\begin{aligned} f'(t) &= \langle (iI + X)u'(t), u(t)\rangle + \langle (iI + X)u(t), u'(t)\rangle \\ &= \langle (iI + X)Au(t), u(t)\rangle + \langle (iI + X)u(t), Au(t)\rangle \\ &= 2i\langle Au(t), u(t)\rangle + \langle (AX + XA)u(t), u(t)\rangle \\ &= 2i\langle Au(t), u(t)\rangle + \|u(t)\|^2. \end{aligned}$$

Let ln be the complex logarithm with a branch cut at $(-\infty, 0]$. Then $(\ln(f(t)))' = f'(t)/f(t)$, and

$$\begin{aligned} |(\ln(f(t)))'| &= \frac{|f'(t)|}{|f(t)|} \leqslant \|u(t)\|^{-2}\sqrt{4\langle Au(t), u(t)\rangle^2 + \|u(t)\|^4} \\ &= \sqrt{\frac{4\langle Au(t), u(t)\rangle^2}{\|u(t)\|^4} + 1} \leqslant 2\|u(t)\|^{-2}|\langle Au(t), u(t)\rangle| + 1 \\ &\lesssim 2h(t) + 1 \quad t \to +\infty. \end{aligned} \tag{5.8}$$

On the other hand we have

$$|\ln(f(t)) - \ln(f(0))| = \left|\int_0^t (\ln(f(s)))'\,\mathrm{d}\,s\right| \leqslant \int_0^t |(\ln(f(s)))'|\,\mathrm{d}\,s,$$

consequently,

$$|\ln|f(t)|| \leqslant |\ln f(t) - \ln f(0)| \leqslant \int_0^t |(\ln(f(s)))'|\,\mathrm{d}\,s \leqslant t \cdot \sup_{0\leqslant s\leqslant t} |(\ln(f(s)))'|$$

for every $t \geqslant 0$, so for sufficiently large t we obtain

$$\frac{|\ln|f(t)||}{t} \lesssim 2h(t) + 1. \tag{5.9}$$

Combined with the previous assessments (5.8)–(5.9), we get

$$\|u(t)\|^2 \leqslant |f(t)| \lesssim \mathrm{e}^{t(2h(t)+1)}, \quad t \to +\infty.$$

or equivalently $\|u(t)\| \lesssim \mathrm{e}^{t(h(t)+1/2)}$. □

Remark 5.1. Since the the function f is defined via scalar product, it really does not matter if u is a solution to (5.6) in the classical or mild sense.

Example 5.1. [12, Example 4.1.] Let $\mathcal{H} \cong L^2(\mathbb{R})$ and let $(X(t))_{t\geqslant 0}$ be a real-valued random process with normal distribution:

$$X(t) : \mathcal{N}(t, 1), \quad \rho_t(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-(x-t)^2/2}, \qquad x \in \mathbb{R}.$$

Define $(u(t))(x) := \mathrm{e}^{t^2/2} \rho_t(x)$, $x \in \mathbb{R}$, $t \geqslant 0$. In that case $\|u(t)\|_1 = \mathrm{e}^{t^2/2}$ and $\|u(t)\|_2 = \frac{1}{\sqrt{2\sqrt{\pi}}} \mathrm{e}^{t^2/2}$, thus $u(t) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ for every $t \geqslant 0$. Naturally define $u\colon [0, +\infty) \to L^2(\mathbb{R})$ as $u\colon t \mapsto u(t)$. We compute that

$$(u'(t))(x) = \partial_t\Big(\frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2+xt}\Big) = xu(t)(x) = (Au(t))(x),$$

therefore A is the sought self-adjoint operator, defined via $Af(x) \mapsto xf(x)$, $x \in \mathbb{R}$, $f \in L^2(\mathbb{R})$, with $\mathcal{D}_A = \{f \in L^2(\mathbb{R})\colon xf(x) \in L^2(\mathbb{R})\}$. In addition we have

$$\langle Au(t), u(t)\rangle_{L^2(\mathbb{R})} = \mathrm{e}^{t^2} \frac{1}{2\pi} \int_{-\infty}^{+\infty} x\, \mathrm{e}^{-(x-t)^2} \,\mathrm{d}\, x = t\frac{\mathrm{e}^{t^2}}{2\sqrt{\pi}} = t\|u(t)\|_2^2$$

so $h(t) = t$. Theorem 5.6 gives the estimate (5.7): $\|u(t)\|_2 \lesssim \mathrm{e}^{t(t+1/2)}$, which can be considered fairly precise asymptotically speaking.

Corollary 5.3. [12, Corollary 4.1.] *Let $(X(t))_{t\geqslant 0}$ be a random process over $\mathbb{R}^n$ ($n \in \mathbb{N}$) such that each random variable $X(t)$ has its probability density ρ_t. For a measurable real-valued function $\varphi \in L^1(\mathbb{R}^n)$, denote by M_φ the corresponding self-adjoint multiplication operator in $L^2(\mathbb{R}^n)$, defined on*

$$\mathcal{D}_{M_\varphi} = \{f \in L^2(\mathbb{R}^n)\colon \varphi f \in L^2(\mathbb{R}^n)\}.$$

Let $u\colon [0, +\infty) \to L^2(\mathbb{R}^n)$ be a continuously-differentiable mapping with the set of values

$$\mathcal{R}(u) = \{u(t)\colon t \in [0, +\infty)\} \subset \mathcal{D}_{M_{\varphi^2}} \cap \mathcal{R}(M_\varphi),$$

such that $\rho_t = \|u(t)\|_2^{-2} u(t)\overline{u(t)}$. If there exists a non-decreasing real-valued function h defined on $[0, +\infty)$, such that $|E(\varphi(X(t)))| \lesssim h(t)$, $t \to +\infty$ then the solution to ACP

$$(u'(t))(x) = \varphi(x) \cdot (u(t))(x), \quad x \in \mathbb{R}^n,$$
$$(u(0))(x) = u_0(x)$$

satisfies $\|u(t)\|_2 \lesssim \mathrm{e}^{t(h(t)+1/2)}$, for any initial data u_0 in $\mathcal{D}_{M_\varphi^2} \cap \mathcal{R}(M_\varphi)$.

An important generalization of ACP (5.6) is the inhomogeneous ACP of the form

$$u'(t) = Au(t) + v(t), \\ u(0) = u_0 \in \mathcal{D}_A \tag{5.10}$$

where $v, u \in C^1([0, +\infty), \mathcal{H})$, $u\colon t \mapsto u(t) \in \mathcal{D}_A$ and $v\colon t \mapsto v(t) \in \mathcal{D}_A$ for every $t \geqslant 0$, and A is a closed operator densely defined on the separable Hilbert space $\mathcal{H}$. Classical and mild solutions to (5.10) are defined as they were for ACP (5.6).

One way of homogenizing the (5.10) was done in [41], under the premise that A was the infinitesimal generator of a C_0-semigroup. In that paper, Roth's removal rule was applied and the ACP (5.10) was homogenized.

Under similar assumptions as before, we are going to derive an asymptotic upper growth bound for the solution to (5.10) without the C_0-semigroup theory. Again, we assume that A is a self-adjoint operator and the solution $u(t)$ to (5.10) is unstable in the Lyapunov sense. Consequently, it is natural to assume that

$$\|v(t)\| = o(\|u(t)\|), \quad t \to +\infty.$$

Recall that if zero is a regular point for A, then there exists a bounded inverse A^{-1}. In that case, notice that the symmetric solution X to (5.1) can be chosen to be precisely $\frac{1}{2}A^{-1}$, which is bounded and is defined on $\mathcal{R}(A)$. This motivates the following assumtion

Assumption 2. ($\mathcal{A}_2$): If X is a symmetric solution to (5.1), we assume that $v(t)$ from (5.10) is in $\mathcal{D}_X$, $\|v(t)\| = o(\|u(t)\|)$ and that $\|Xv(t)\| = o(\|u(t)\|)$ when $t \to +\infty$.

Theorem 5.7. [12, Theorem 4.3.] *Let A be a self-adjoint linear operator such that $\mathcal{A}_1(A)$ holds. Let $u, v\colon [0, +\infty) \to \mathcal{H}$ be continuously-differentiable functions with values in Δ_A, such that u is a solution to* (5.10) *(in the weak or classical sense) and that $\mathcal{A}_2$ holds. If there exists a non-decreasing real-valued function $h\colon [0, +\infty) \to \mathbb{R}$ such that $|\langle Au(t), u(t)\rangle| \lesssim h(t) \cdot \|u(t)\|^2$, $t \to +\infty$, then $\|u(t)\| \lesssim \mathrm{e}^{t(h(t)+1/2)}$.*

Proof. The proof is pretty much identical to the proof of Theorem 5.6. There exists an X defined on Δ_A, which is a symmetric linear operator and is a solution to Lyapunov operator equation (5.1). Again, define $f\colon [0, +\infty) \to \mathbb{R} + i\mathbb{R}^+$ as

$$f(t) := \langle (iI + X)u(t), u(t)\rangle = i\|u(t)\|^2 + \langle Xu(t), u(t)\rangle.$$

It follows that

$$|f(t)| \geqslant |\operatorname{Im}(f)(t)| = \|u(t)\|^2.$$

Differentiating f via t gives

$$\begin{aligned} f'(t) &= \langle (iI + X)u'(t), u(t)\rangle + \langle (iI + X)u(t), u'(t)\rangle \\ &= \langle (iI + X)(Au(t) + v(t)), u(t)\rangle + \langle (iI + X)u(t), Au(t) + v(t)\rangle \\ &= 2i\langle Au(t), u(t)\rangle + \|u(t)\|^2 + 2i \operatorname{Re}\langle v(t), u(t)\rangle + 2 \operatorname{Re}\langle u(t), Xv(t)\rangle. \end{aligned}$$

Let ln be the complex logarithm with a branch cut at $(-\infty, 0]$. Then $(\ln(f(t)))' = f'(t)/f(t)$ and

$$|(\ln(f(t)))'| = \frac{|f'(t)|}{|f(t)|} \leqslant \|u(t)\|^{-2}(\|u(t)\|^2 + |2\operatorname{Re}\langle u(t), Xv(t)\rangle|)$$
$$+ \|u(t)\|^{-2}(|2\operatorname{Re}\langle u(t), v(t)\rangle| + |2\langle Au(t), u(t)\rangle|) \lesssim 1 + 2h(t) + o(1), \ t \to +\infty.$$

The rest of the proof is the same. □

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