Martin Z. Ljubenović* and **Dragan S. Djordjević****

A SURVEY OF WEAK MAJORIZATION RELATIONS ON $\ell^1(I)^+$ AND THEIR LINEAR PRESERVERS

Abstract. A survey about the most important properties of extended three weak majorization relations $\prec_w$, $\prec_s$ and $\prec^{ws}$ determined by stochastic operators on the discrete Lebesgue space $\ell^1(I)$, is presented. In the second part, linear preservers of considered majorization relations, when I is an infinite set, are characterized and many examples with concrete matrix forms of linear preservers are given.

Mathematics Subject Classification (2020): Primary: NN-02; Secondary: 47B60; 15B51; 60E15; 15A86; 39B62; 47B38.

Keywords: weak majorization, submajorization, weak supermajorization, linear preserver, stochastic operators, permutation and partial permutation.

*University of Niš, Faculty of Mechanical Engineering, Serbia
`martinljubenovic@gmail.com`

**University of Niš, Faculty of Sciences and Mathematics, Serbia
`dragandjordjevic70@gmail.com & dragan@pmf.ni.ac.rs`

Contents

1. Introduction

Theory of majorization as a research tool for developing new mathematical inequalities plays an important role in various mathematical disciplines [1–3, 5, 16, 39, 40] as well as in the other branches of science like physics [33,41], quantum mechanics [18,38], statistics and economy. We recommend excellent monographs [17,19,34] which contain the most important results in the finite-dimensional majorization theory that was widely studied in past by many famous mathematicians.

There is a big step forward towards developing infinite dimensional extensions of various majorization relations [6, 8, 10, 21, 25, 28, 31, 37] and there exist sagnificant applications established in [4,13,14,22,23,32,40]. Some majorization generalizations are obtained using deep styding of stochastic operators [6, 12, 14, 15, 25, 28, 31] on sequence spaces [7,12] and on discrete Lebesgue spaces of functions [6,24]. Standard majorization relation is extended in [6] using doubly stochastic operators on discrete Lebesgue spaces $\ell^p(I)$, $p \in [1, \infty)$ while weak majorization relations was extended in [25, 28, 31] by several types of stochastic operators on $\ell^p(I)$.

There have been outstanding occupation in the linear preserver problems regard to majorization relations, matrices and operators. A couple of interesting articles which investigate linear preserver problems of majorization relations are [1, 6, 7, 9–11, 26, 27, 29–31].

This paper will survey some recent research published in papers [25–29, 31] of the following three weakened majorization relations on $\ell^1(I)^+$: weak majorization, submajorization and weak supermajorization. Precisely, in Section 2 we introduce different types of stochastic operators and we define mentioned relations on positive cone $\ell^1(I)^+$. These extensions are introduced by equivalents of majorization relations presented in Theorem 1.3 and Theorem 1.4 which holds only on $(\mathbb{R}^n)^+$. Hence, obtained extensions are in accordance with finite-dimensional case, that is, if we choose $I = \{1, 2, \dots, n\}$ than basic Definition 1.1 and Definition 1.2 coincide with Definition 2.2 of extended weak majorization relations. The aim of the Section 2 is to provide that these majorization relations may be considered as partial orders. We recall that in the finite-dimensional majorization theory these relations are pre-orders, that is, they are reflexive and transitive. Also, weakened anti-symmetricity holds in sense that if $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ are mutually majorized by one of considered majorization relations, than coordinates of the vector x are a permutation of coordinates of the vector y. Through the couple of lemmas and theorems (with complete proofs), it is provided that similar holds for three extended weak majorizations on $\ell^1(I)^+$, by Corollaries 2.2 and 2.6. The goal of Section 3 is to present linear preservers forms of weak majorization, submajorization and weak supermajorization on $\ell^1(I)^+$. All necessary results for this characterization of preservers are presented with appropriate proofs. The shape of linear preservers of all three considered weak majorization relations is collected in Corollary 3.1. At the end of the paper additional examples of linear preservers with matrix representations are given.

We emphasize that all definitions, all results and all examples presented in this survey paper are previously published in some mathematical high quality journal.

1.1. Majorization relations on $\mathbb{R}^n$ and stochastic matrices.

Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. We denote by $x^\downarrow = (x_1^\downarrow, x_2^\downarrow, \dots, x_n^\downarrow)$ the *decreasing rearrangement* of components of vector x that is, $x_1^\downarrow \geqslant x_2^\downarrow \geqslant \dots \geqslant x_n^\downarrow$. Similarly, $x^\uparrow = (x_1^\uparrow, x_2^\uparrow, \dots, x_n^\uparrow)$ is the *increasing rearrangement* of components of vector x, that is, $x_1^\uparrow \leqslant x_2^\uparrow \leqslant \dots \leqslant x_n^\uparrow$.

We present widely used definitions of (standard) majorization and weak majorization on $\mathbb{R}^n$.

Definition 1.1. [1, 17, 19, 34, 35] For two vectors $x, y \in \mathbb{R}^n$, the vector x is *weakly majorized* by the vector y if the following inequalities hold $\sum_{i=1}^k x_i^\downarrow \leqslant \sum_{i=1}^k y_i^\downarrow$, $k = 1, 2, \dots, n$.We denote it by $x \prec_w y$.

Moreover, if additionally holds $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$,than the vector x is *majorized* by the vector y (or vector y *majorizes* vector x) and we denote it by $x \prec y$.

For example, if for a positive vector $a = (a_1, a_2, \dots, a_n)$ the summation $\sum_{i=1}^n a_i = 1$ holds, than $(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}) \prec (a_1, a_2, \dots, a_n) \prec (1, 0, \dots, 0)$.

Also,

$$\Big(\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n}\Big) \prec \Big(\frac{1}{n-1},\frac{1}{n-1},\dots,\frac{1}{n-1},0\Big) \prec \Big(\frac{1}{2},\frac{1}{2},\dots,0\Big) \prec (1,0,\dots,0).$$

Definition 1.2. [1, 17, 19, 34] For two vectors $x, y \in \mathbb{R}^n$, the vector x is *weakly supermajorized* by the vector y and we denote it by $x \prec^{ws} y$, if the following inequalities hold $\sum_{i=1}^k x_i^{\uparrow} \geqslant \sum_{i=1}^k y_i^{\uparrow}$ $(k = 1, 2, \dots, n)$.

It is easy to see that $x_i^{\uparrow} = x_{n-i+1}^{\downarrow}$, $1 \leqslant i \leqslant n$, which implies

$$\sum_{i=1}^{k} x_i^{\uparrow} = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n-k} x_i^{\downarrow}.$$

Hence, we conclude that the (standard) majorization $x \prec y$ holds if and only if the weak supermajorization $x \prec^{ws} y$ and the additional statement $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$holds. Thus, the majorization relation $\prec$ implies the weak majorization $\prec_w$ by Definition 1.1 and imples weak supermajorization $\prec^{ws}$ by the above argument.

All three above introduced majorization relations are reflexive and transitive, that is, they are pre-orders. Strictly speaking, these relations neither symmetric nor anti-symmetric. However, $x \prec_w$ and $y \prec_w x$ implies $x = Py$ for some permutation matrix P. Precisely, vectors x and y have the same coordinates but at different places. If we identify all vectors which are permutations of each other, then we may consider all three relations "$\prec_s$", "$\prec^{ws}$" and "$\prec$" as partial orders. The aim of the first part in this survey paper is to present published results which provide that discussed extensions of majorization relations on $\ell^1(I)^+$ may be considered as partial orders in some sense (see Corollary 2.2 and Corollary 2.6).

A square $n \times n$ non-negative matrix $A = (a_{ij})$ is called doubly stochastic if all of its row sums and all of its column sums are equal 1. A square $n \times n$ matrix $D = (d_{ij})$ with non-negative entries is called doubly substochastic if there is a doubly stochastic matrix $A = (a_{ij})$ such that

$$d_{ij} \leqslant a_{ij}, \qquad 1 \leqslant i \leqslant n, \quad 1 \leqslant j \leqslant n. \tag{1.1}$$

As a direct consequence, it follows that all row sums and column sums of the doubly substochastic matrix D are less than or equal to 1. Von Neumann [36] provides the converse in the next theorem.

Theorem 1.1. [36], [34, Theorem I.2.C.1] *For every $n \times n$ non-negative matrix $D = (d_{ij})$ with*

$$\sum_{i=1}^{n} d_{ij} \leqslant 1, \quad \forall j \in \{1, 2, \dots, n\} \quad \textit{and} \quad \sum_{j=1}^{n} d_{ij} \leqslant 1, \quad \forall i \in \{1, 2, \dots, n\}, \tag{1.2}$$

there exists a doubly stochastic matrix $A = (a_{ij})$ such that $d_{ij} \leqslant a_{ij}$, for all $i, j \in \{1, 2, \dots, n\}$.

Thus, the statement (1.2) may be considered as an alternative definition of a doubly substochastic matrix. However, the last result of von Neumann cannot be generalized to infinite matrices considered as bounded linear operators on the discrete Lebesgue space $\ell^1(I)$, when I is an infinite set (good examples are shift operators, see (2.12)).

A $n \times n$ matrix $\tilde{D} = (\tilde{d}_{ij})$ with non-negative real entries is called doubly superstochastic, if there is a doubly stochastic matrix $D = (d_{ij})$ such that

$$\tilde{d}_{ij} \geqslant d_{ij} \tag{1.3}$$

for each i and j. This fact implies that each of its row sums and each of its column sums are grater than or equal to 1. We note that the converse is not true, that is there are matrices which satisfy the last rows and columns conditions but there is no doubly stochastic matrix such that (1.3) holds. For example, let

$$\begin{bmatrix} 0 & \frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \end{bmatrix}.$$

All of its row sums and all of its column sums are grater than or equal to 1 but there is no doubly stochastic matrix which satisfies (1.3).

Now, we present two the most used equivalents of the majorization relations $\prec$.

Theorem 1.2. [34, Teorema I.1.A.3] *For two vectors $x, y \in \mathbb{R}^n$, the next statements are equivalent*

i) $x \prec y$;
ii) $x = Ay$ *for a doubly stochastic matrix* A;
iii) $\sum_{i=1}^n \varphi(x_i) \leqslant \sum_{i=1}^n \varphi(y_i)$ *for every continuous convex function* $\varphi\colon \mathbb{R} \to \mathbb{R}$.

The next equivalents of weak majorization relations which are determined by stochastic matrices holds only for positive vectors x and y.

Theorem 1.3. [34, Teorema I.1.A.4] *For two positive vectors $x, y \in (\mathbb{R}^n)^+$, the next statements are equivalent:*

i) $x \prec_w y$;
ii) $x = Ay$ *for a doubly substochastic matrix* A.

Theorem 1.4. [34, Teorema I.1.A.5] *For two positive vectors $x, y \in (\mathbb{R}^n)^+$, the next statements are equivalent:*

i) $x \prec^{ws} y$;
ii) $x = Ay$ *for a doubly superstochastic matrix* A.

In this survey paper we will present already published results in [25–31], where above equivalents are used for generalization of three weak majorization relations on positive cone $\ell^1(I)^+$ and present the shape of their linear preservers.

1.2. Linear preservers of majorization relations on $\mathbb{R}^n$.

We give the definition of a linear preserver for an arbitrary relation ρ.

Definition 1.3. Let ρ be a relation defined on the linear vector space $\mathcal{V}$. A linear operator $A\colon \mathcal{V} \to \mathcal{V}$ is *a linear preserver* of ρ if $x\rho y$ implies $(Ax)\,\rho\,(Ay)$.

Now, we characterize bounded linear operators on the finite-dimensional space $\mathbb{R}^n$ which preserve some majorization relation. For instance, a linear preserver of the weak majorization $\prec_w$ may be defined as bounded linear operator $T\colon \mathbb{R}^n \to \mathbb{R}^n$ which satisfies that $x \prec_w y$ implies $(Ax) \prec_w (Ay)$. Similarly, we may introduce definitions of linear preservers for the other majorization relations. The next result is provided in [1] and may be find in monographs [17, 34].

Theorem 1.5. [1, Proposition 2.7] *A bounded linear operator $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear preserver of the majorization ($\prec$) if and only if one of the following statements hold for every $x \in \mathbb{R}^n$*

i) $T(x) = \operatorname{tr}(x)a$, *for some $a \in \mathbb{R}^n$;*
ii) $T(x) = \beta P(x) + \gamma \operatorname{tr}(x)e$, *for some $\beta, \gamma \in \mathbb{R}$ and for some permutation $P\colon \mathbb{R}^n \to \mathbb{R}^n$.*

A bounded linear operator $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is *positive* if the matrix of this operator contains all positive elements (greater than or equal to 0) with standard basis in $\mathbb{R}^n$.

Corollary 1.1. [20] *Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a positive bounded linear operator. The next statements are equivalent:*

i) *operator T preserve majorization ($\prec$);*
ii) *operator T preserve weak majorization ($\prec_w$);*
iii) *operator T preserve weak supermajorization ($\prec^{ws}$).*

Using above result, we present as corollary the forms of linear preservers of the weak majorization ($\prec_w$) and the weak supermajorization ($\prec^{ws}$).

Theorem 1.6. *A positive bounded linear operator $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear preserver of the weak majorization ($\prec_w$) or weak supermajorization ($\prec^{ws}$) if and only if one of the following statements hold for every $x \in \mathbb{R}^n$*

i) $T(x) = \operatorname{tr}(x)a$, *for some $a \in \mathbb{R}^n$;*
ii) $T(x) = \beta P(x) + \gamma \operatorname{tr}(x)e$, *for some $\beta, \gamma \in \mathbb{R}$ and for some permutation $P\colon \mathbb{R}^n \to \mathbb{R}^n$.*

Thus, when $n = 3$, linear preserver of considered three majorization relations may have one of the following matrix forms

$$\text{i)}\quad T = \begin{bmatrix} a_1 & a_1 & a_1 \\ a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 \end{bmatrix} \qquad \text{ii)}\quad T = \beta \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \gamma \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The aim of this survey paper is to present the shape of linear preservers of some weak majorization relations on the positive cone $\ell^1(I)^+$ of the discrete Lebesgue space $\ell^1(I)$.

2. Majorization relations determined by stochastic operators on $\ell^1(I)$

Discussed majorization relations and stochastic operators on $\ell^1(I)$ as well as all result given in this section are introduced and provided in papers [6, 25, 28, 31].

2.1. Discrete Lebesque space $\ell^1(I)$. We will consider real function $f\colon I \to \mathbb{R}$ defined on an arbitrary chosen non-empty set I. If the set I is finite, than the set of all such functions may be identified with real vectors from the finite-dimensional real vector space $\mathbb{R}^n$. Finite-dimensional majorization theory was deep studied in the past with lot of applications in science. If the set I contains all integers $I = \mathbb{N}$, (or more general when I is an arbitrary chosen a countable set) than the function $f\colon I \to \mathbb{R}$ is a real sequence. The set I may be uncountable, which is probably the most interesting case presented in this survey paper.

Let I be an arbitrary non-empty set. The function $f\colon I \to \mathbb{R}$ is summable if there exists a real number σ with the following property:
For given $\epsilon > 0$ we can find a finite set $J_0 \subseteq I$ such that $\left|\sigma - \sum_{j\in J} f(j)\right| \leqslant \epsilon$ whenever J is a finite set and $J_0 \subseteq J$. Then σ is called the sum of f and we denote it by $\sigma = \sum_{i\in I} f(i)$.

We denote by $\ell^1(I)$ the Banach space of all functions $f\colon I \to \mathbb{R}$ such that $\|f\|_1 := \sum_{i\in I} |f(i)| < \infty$. Each function $f \in \ell^1(I)$ has a representation

$$f = \sum_{i\in I} f(i) e_i.$$

Functions $e_i\colon I \to \mathbb{R}$ are defined by Kronecker delta, i.e., $e_i(j) = \delta_{ij}$, $i, j \in I$.

For each $g \in \ell^\infty(I)$, where $\ell^\infty(I)$ is the Banach space with supremum norm, the rule $f \to \langle f, g\rangle := \sum_{i\in I} f(i)g(i)$ defines a bounded linear functional on $\ell^1(I)$. This map $\langle\cdot,\cdot\rangle\colon \ell^1(I) \times \ell^\infty(I) \to \mathbb{R}$ is called the dual pairing.

The cone of positive functions is denoted by $\ell^1(I)^+ := \{f \in \ell^1(I) : f(i) \geqslant 0,\ \forall i \in I\}$.

2.2. Bounded linear operators on $\ell^1(I)$. Let $\mathbb{A} = \{a_{ij} : i, j \in \mathbb{N}\}$ be a family defined by

$$a_{ij} = \begin{cases} i^2, & i = j,\\ 0, & \text{otherwise.}\end{cases}$$

This family has a (infinite) matrix representation

$$\mathbb{A} = \begin{bmatrix} 1^2 & 0 & \dots & 0 & \dots\\ 0 & 2^2 & \dots & 0 & \dots\\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & n^2 & \dots\\ \vdots & \vdots & & \vdots & \ddots \end{bmatrix}.$$

If we use the standard definition for matrix operators

$$Af := \sum_{i\in\mathbb{N}} \Big(\sum_{j\in\mathbb{N}} a_{ij} f(j)\Big) e_i, \tag{2.1}$$

then we form the linear operator A with domain $\ell^1(\mathbb{N})$ for which $A(\ell^1(\mathbb{N})) \supsetneq \ell^1(\mathbb{N})$. Truly, if we choose $f \in \ell^1(\mathbb{N})$, to be $f(i) = \frac{1}{i^2}$, $i \in \mathbb{N}$ we obtain $Af = e = (1, 1, \dots, 1 \dots) \in \ell^\infty(\mathbb{N}) \setminus \ell^1(\mathbb{N})$.

Thus, the above example shows that there are infinite matrices which do not generate operators on from $\ell^1(I)$ to $\ell^1(I)$ by matrix definition (2.1). Because of this, we will identify all families which may be considered as bounded linear operators on $\ell^1(I)$, defined by (2.1). These families will be candidates for stochastic operators. For this purpose, we will going to present the next theorem and its proof provided in [28].

Theorem 2.1. [28, Theorem 3.1] *Let* $\mathbb{A} = \{a_{ij} : i, j \in I\}$ *be a family of real numbers, where* I *is an arbitrary non-empty set. This family defines a unique bounded linear operator* $A\colon \ell^1(I) \to \ell^1(I)$ *with*

$$\langle Ae_j, e_i\rangle := a_{ij}, \quad \forall i, j \in I \tag{2.2}$$

if

$$M := \sup_{j\in I} \sum_{i\in I} |a_{ij}| < \infty. \tag{2.3}$$

Conversely, each bounded linear operator $A\colon \ell^1(I) \to \ell^1(I)$ *satisfies* (2.3) *where* $a_{ij} := \langle Ae_j, e_i\rangle$. *Moreover,* $\|A\|_1 = M$.

Proof. Let $A\colon \ell^1(I) \to \ell^1(I)$ be a bounded linear operator and let $a_{ij} := \langle Ae_j, e_i\rangle$, $\forall i, j \in I$. Using linearity and continuity of A, we get $Af = \sum_{j\in I} f(j)Ae_j \in \ell^1(I)$, since f has representation $f = \sum_{j\in I} f(j)e_j$. We define linear functionals

$$A_i(f) := Af(i) = \langle Af, e_i\rangle = \sum_{j\in I} f(j)\langle Ae_j, e_i\rangle = \sum_{j\in I} f(j)a_{ij} < \infty, \ \ \forall f \in \ell^1(I),$$

for every $i \in I$. Since, $|A_i(f)| \leqslant \|A(f)\|_1 \leqslant \|A\|_1\|f\|_1$, we get $A_i \in (l^1(I))^*$, $\forall i \in I$. Moreover, $\sum_{i\in I} |A_i(f)| = \|Af\|_1 \leqslant \|A\|_1\|f\|_1$, $\forall f \in \ell^1(I)$. Choosing $f = e_j$, for any $j \in I$, we get

$$\sum_{i\in I} |a_{ij}| = \sum_{i\in I} |\langle Ae_j, e_i\rangle| = \sum_{i\in I} |A_i(e_j)| \leqslant \|A\|_1,$$

which implies (2.3) and $M \leqslant \|A\|_1$.

Let $\mathbb{A} = \{a_{ij} : i, j \in I\}$ be an arbitrary chosen family which satisfies (2.3). Now,

$$\sum_{j\in I}\sum_{i\in I} |a_{ij}f(j)| = \sum_{j\in I} |f(j)| \sum_{i\in I} |a_{ij}| \leqslant M\|f\|_1.$$

We may change the order of summation by the Fubini theorem

$$\sum_{i\in I} |\sum_{j\in I} a_{ij}f(j)| \leqslant \sum_{i\in I}\sum_{j\in I} |a_{ij}f(j)| = \sum_{j\in I}\sum_{i\in I} |a_{ij}f(j)| \leqslant M\|f\|_1, \tag{2.4}$$

$\forall f \in \ell^1(I)$. Hence, if we define an operator A on $\ell^1(I)$ by

$$Af := \sum_{i\in I} \Big(\sum_{j\in I} a_{ij}f(j)\Big)e_i, \tag{2.5}$$

then this operator is well-defined and $A(\ell^1(I)) \subseteq \ell^1(I)$, by (2.4). Linearity of operator A is clear. Also, by (2.4), we obtain

$$\|Af\|_1 = \sum_{i\in I} |\sum_{j\in I} a_{ij} f(j)| \leqslant M\|f\|_1$$

so A is a bounded operator and $\|A\|_1 \leqslant M$, so $\|A\|_1 = M$. Now, it is easy to verify that

$$\langle Ae_j, e_i\rangle = \sum_{r\in I}\left(\sum_{s\in I} a_{rs} e_j(s)\right)\langle e_r, e_i\rangle = \sum_{r\in I} a_{rj}\langle e_r, e_i\rangle = a_{ij}$$

for every $i \in I$, and for every $j \in I$.

If we suppose that there exist another one operator $A_1\colon \ell^1(I) \to \ell^1(I)$ such that (2.2) holds, then we have

$$\langle A_1 f, e_i\rangle = \sum_{j\in I} f(j)\langle A_1 e_j, e_i\rangle = \sum_{j\in I} f(j) a_{ij} = \sum_{j\in I} f(j)\langle Ae_j, e_i\rangle = \langle Af, e_i\rangle$$

thus, $A_1 = A$. □

Thus, under the condition (2.3) the family $\mathbb{A} = \{a_{ij} : i, j \in I\}$ may be considered as bounded linear operator $A\colon \ell^1(I) \to \ell^1(I)$ defined as matrix operator by (2.5), by above theorem. Since each bounded linear operator A on $\ell^1(I)$ forms a family $\mathbb{A} = \{a_{ij} : i, j \in I\}$, where $a_{ij} = \langle Ae_j, e_i\rangle$ for each $i, j \in I$ which satisfies (2.3), we may identify this family $\mathbb{A}$ and the appropriate operator A, so in the rest of the paper we will use the same letter A for both without confusion, in order to simplify notation. Moreover, always holds $a_{ij} = \langle Ae_j, e_i\rangle = Ae_j(i)$, $\forall i, j \in I$.

We note that if $I = \mathbb{N}$ than bounded linear operator $A\colon \ell^1(\mathbb{N}) \to \ell^1(\mathbb{N})$ has standard (infinite) matrix definition (2.1) and representation

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & \dots \\ a_{21} & a_{22} & \dots & a_{2n} & \dots \\ \vdots & \vdots & \ddots & \vdots & \\ a_{n1} & a_{n2} & \dots & a_{nn} & \dots \\ \vdots & \vdots & & \vdots & \ddots \end{bmatrix}.$$

where sequences $a_j := (a_{ij})_{i\in\mathbb{N}}$ are in $\ell^1(\mathbb{N})$ for all $j \in \mathbb{N}$ and $\sup_{j\in I} |a_j|_1 < \infty$.

2.3. Stochastic operators on $\ell^1(I)$.

Results presented in this subsection are published in papers [25, 28, 31].

Now, definitions of stochastic operators on $\ell^1(I)$ are presented below.

Definition 2.1. [6, 25, 28, 31] Let $A\colon \ell^1(I) \to \ell^1(I)$ be a bounded linear operator, where I is a non-empty set. The operator A is called

i) a *permutation*, if there exists a bijection $\theta\colon I \to I$ for which $Ae_j = e_{\theta(j)}$, for each $j \in I$; [6, Definition 2.1]
ii) *a partial permutation* for sets $I_1 \subset I$ and $I_2 \subset I$, if there exists a bijection $\theta\colon I_1 \to I_2$ for which $Ae_j = e_{\theta(j)}$ whenever $j \in I_1$, and $Ae_j = 0$, otherwise; [25, Definition 3.1]

iii) *positive* if $Ag \in \ell^1(I)^+$ holds, whenever $g \in \ell^1(I)^+$;

iv) *doubly stochastic*, if A is positive,

$$(\forall i \in I) \quad \sum_{j\in I}\langle Ae_j, e_i\rangle = 1 \quad \text{and} \quad (\forall j \in I) \quad \sum_{i\in I}\langle Ae_j, e_i\rangle = 1; \tag{2.6}$$

[6, Definition 2.1]

v) *doubly substochastic*, if A is positive,

$$(\forall i \in I) \quad \sum_{j\in I}\langle Ae_j, e_i\rangle \leqslant 1 \quad \text{and} \quad (\forall j \in I) \quad \sum_{i\in I}\langle Ae_j, e_i\rangle \leqslant 1; \tag{2.7}$$

[25, Definition 3.1]

vi) *increasable doubly substochastic*, if there is doubly stochastic operator $A_1 : \ell^1(I) \to \ell^1(I)$ such that

$$(\forall i \in I) \quad (\forall j \in I) \quad \langle Ae_j, e_i\rangle \leqslant \langle A_1e_j, e_i\rangle; \tag{2.8}$$

[31, Definition 3.1]

vii) *doubly superstochastic*, if there is doubly stochastic operator $A_1 : \ell^1(I) \to \ell^1(I)$ such that

$$(\forall i \in I) \quad (\forall j \in I) \quad \langle Ae_j, e_i\rangle \geqslant \langle A_1e_j, e_i\rangle. \tag{2.9}$$

[28, Definition 3.1]

The next very useful theorem is provided in [6] by Bahrami, Bayati and Manjegani.

Theorem 2.2. [6, Theorem 2.2.] *Let I and J be two arbitrary non-empty sets. There exists a positive bounded linear operator $D : \ell^1(J) \to \ell^1(I)$ which satisfies*

$$(\forall i \in I) \quad \sum_{j\in J}\langle Ae_j, e_i\rangle = 1 \quad \text{and} \quad (\forall j \in J) \quad \sum_{i\in I}\langle Ae_j, e_i\rangle = 1 \tag{2.10}$$

if and only if $\mathrm{card}(J) = \mathrm{card}(I)$*, where* $\mathrm{card}(I)$ *denotes the cardinal number of the set I.*

Remark 2.1. Using above theorem we can see that for a bounded linear operator $D : \ell^1(J) \to \ell^1(I)$ which satisfies (2.10), have to be $\mathrm{card}(J) = \mathrm{card}(I)$, so in this case we may identify I and J, and without losing of generality, we may consider operator D with domain $\ell^1(I)$ instead of $\ell^1(J)$. Because of this, we present the definition of a doubly stochastic operator by (2.6). We note that doubly substochastic operators in (2.7) may be defined from $\ell^1(J)$ to $\ell^1(I)$ such that $\mathrm{card}(I) \neq \mathrm{card}(J)$ since above theorem does not hold for them. For more details, see [25].

Remark 2.2. We note that, at first glance, the above definition of the doubly superstochastic operator on $\ell^1(I)$ is slightly different in comparison with original definition presented in [28]. Namely, in [28, Definition 3.1] we can see that for a family of positive real numbers $A = \{a_{ij} : i, j \in I\}$ we require condition (2.3) and additional condition $\sup_{i\in I}\sum_{j\in I}|a_{ij}| < \infty$. The last condition is surplus, by Theorem 2.1, because we consider only bounded linear operators on $\ell^1(I)$ instead of $\ell^p(I)$ for all $p \in [1, \infty]$ which is theme considered in [28].

Families which may be considered as bounded linear operators on $\ell^p(I)$ are studied in [28] and the definition of doubly superstochastic operators (i.e. families) in presented. In this case, the family A is called *doubly superstochastic*, if there is a doubly stochastic operator A_1 on $\ell^p(I)$ such that

$$a_{ij} \geqslant \langle A_1 e_j, e_i\rangle, \quad \forall i, j \in I. \tag{2.11}$$

Now, we can see that Definition 2.1, item vii) is in accordance with (2.11), by Theorem 2.1.

The set of all permutations, partial permutations, doubly stochastic, doubly substochastic, increasable doubly substochastic and doubly superstochastic operators on $\ell^1(I)$ are denoted by $P(\ell^1(I))$, $pP(\ell^1(I))$, $DS(\ell^1(I))$, $DSS(\ell^1(I))$, $iDSS(\ell^1(I))$ and $DSPS(\ell^1(I))$, respectively.

It is easy to see that by Definition 2.1 that

$$\begin{aligned} DS(\ell^1(I)) &\subsetneq iDSS(\ell^1(I)),\\ DS(\ell^1(I)) &\subsetneq DSPS(\ell^1(I)),\\ iDSS(\ell^1(I)) &\subseteq DSS(\ell^1(I)). \end{aligned}$$

When I is a finite set, the equality $iDSS(\ell^1(I)) = DSS(\ell^1(I))$ holds, by Theorem 1.1. When I is infinite, $iDSS(\ell^1(I)) \subsetneq DSS(\ell^1(I))$ holds. Left and right shift operators are good examples which provide that $iDSS(\ell^1(\mathbb{N}))$ is a proper subset of $DSS(\ell^1(\mathbb{N}))$. It is easy to see that the left and the right shift operators $L, R\colon \ell^1(\mathbb{N}) \to \ell^1(\mathbb{N})$ defined by

$$\langle Le_j, e_i\rangle := l_{ij} = \begin{cases} 1, & j-i=1,\\ 0, & \text{otherwise}\end{cases} \qquad \langle Re_j, e_i\rangle := l_{ij} = \begin{cases} 1, & i-j=1,\\ 0, & \text{otherwise}\end{cases}$$

with matrix forms

$$L = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots\\ 0 & 0 & 1 & 0 & \dots\\ 0 & 0 & 0 & 1 & \dots\\ 0 & 0 & 0 & 0 & \dots\\ 0 & 0 & 0 & 0 & \dots\\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad R = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots\\ 1 & 0 & 0 & 0 & \dots\\ 0 & 1 & 0 & 0 & \dots\\ 0 & 0 & 1 & 0 & \dots\\ 0 & 0 & 0 & 1 & \dots\\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{2.12}$$

are doubly substochastic by (2.7), but there is no doubly stochastic operator A such that

$$\langle Le_j, e_i\rangle \leqslant \langle Ae_j, e_i\rangle, \ \forall i, j \in \mathbb{N} \quad \text{or} \quad \langle Re_j, e_i\rangle \leqslant \langle Ae_j, e_i\rangle, \ \forall i, j \in \mathbb{N}.$$

We recall that the definition (2.12) of the matrix operators L and R is supported by Theorem 2.1. We conclude that shift operators L and R are doubly substochastic, but they are not doubly substochastic in the sense of definition (1.1) which is extended in (2.8) and called increasable doubly substochastic. We note that increasable doubly substochastic matrix $I \times I$ is introduced in [14, Definition 2.2].

Lemma 2.1. [31, Lemma 3.3] *For each increasable doubly substochastic operator* $D \in iDSS(\ell^1(I))$ *there are two operators* $D_1 \in DS(\ell^1(I))$ *and* $D_2 \in DSS(\ell^1(I))$ *such that* $D_1 = D + D_2$.

Proof. Follows directly by (2.8). □

The next lemma follows directly by Theorem 2.1. Constructive proofs of the next lemma statements are presented in papers [6, 25, 28, 31].

Lemma 2.2. *The norm of*

i) $A \in DSS(\ell^1(I))$ *is at most* 1; [25, Lemma 3.3]
ii) $A \in iDSS(\ell^1(I))$ *is at most* 1; [31]
iii) $A \in DS(\ell^1(I))$ *is at most* 1; [6, Lemma 2.5]
iv) $A \in DSPS(\ell^1(I))$ *is at least* 1. [28, Theorem 3.6]

The next result follows directly by Theorem 2.1. Please see [25, Theorem 3.2] for complete proof without using mentioned theorem.

Theorem 2.3. [25, Theorem 3.2] *Let* $\{a_{ij} : i, j \in I\}$ *be a family of non-negative real numbers, with*

$$\forall j \in I, \quad \sum_{i \in I} a_{ij} \leqslant 1, \qquad \forall i \in I, \quad \sum_{j \in I} a_{ij} \leqslant 1,$$

where I *is an arbitrary chosen non-empty sets. Then there is a unique doubly substochastic operator* $A \in DSS(\ell^1(I))$ *such that*

$$\forall i \in I, \quad \forall j \in I, \quad \langle Ae_j, e_i \rangle = a_{ij}.$$

2.4. Weak majorization relations on $\ell^1(I)^+$.

The next definitions of majorization relations are introduced in references [6, 25, 28, 31].

Definition 2.2. For two functions $f, g \in \ell^1(I)^+$,

i) the function f is *weakly majorized* by g, if there exists a doubly substochastic operator $D \in DSS(\ell^1(I))$ such that $f = Dg$, which is denoted by $f \prec_w g$; [25, Definition 4.1]
ii) the function f is *submajorized* by g, if there exists an increasable doubly substochastic operator $D \in iDSS(\ell^1(I))$ such that $f = Dg$, which is denoted by $f \prec_s g$; [31, Definition 3.6]
iii) the function f is *weakly supermajorized* by g, if there exists a doubly superstochastic operator $D \in DSPS(\ell^1(I))$ such that $f = Dg$, which is denoted by $f \prec^{ws} g$. [28, Definition 4.1]

For two functions $f, g \in \ell^1(I)$,

iv) the function f is *majorized* by g, if there exists a doubly substochastic operator $D \in DS(\ell^1(I))$ such that $f = Dg$, which is denoted by $f \prec g$. [6, Definition 3.1]

It is easy to check using Lemma 2.1 that for two fixed functions $f, g \in \ell^1(I)^+$ the relation $f \prec_s g$ holds if and only if there exists $h \in \ell^1(I)^+$ such that $f \leqslant h$ and $h \prec g$, where $\leqslant$ is the entrywise order ($f(i) \leqslant h(i)$, $\forall i \in I$).

Clearly, $f \prec g$ implies $f \prec^{ws} g$ for two functions $f, g \in \ell^1(I)^+$, because $DS(\ell^1(I)) \subset DSPS(\ell^1(I))$. Similarly, for two functions $f, g \in \ell^1(I)^+$, relation $f \prec$ implies $f \prec_s g$ which implies $f \prec_w g$, by $DS(\ell^1(I)) \subset iDSS(\ell^1(I)) \subset DSS(\ell^1(I))$. The opposite direction is not true in general which is provided in [31].

Example 2.1. [31] Let $g = (g_1, g_2, g_3, \dots, g_n, \dots) \in \ell^1(\mathbb{N})$, where $g_i > 0$, $\forall i \in \mathbb{N}$ and $g_i < g_j$, whenever $i > j$. Suppose that $f := Rg = (0, g_1, g_2, g_3, \dots, g_n, \dots)$, where R is the right shift operator with the matrix form (2.12). It is easy to see that $f \prec_w g$, because $R \in DSS(\ell^1(\mathbb{N}))$.

We will show that $f \not\prec_s g$. Suppose that $f = Dg$ holds for some $D\colon \ell^1(\mathbb{N}) \to \ell^1(\mathbb{N}) \in DSS(\ell^1(I))$. We have $0 = f_1 = \sum_{j=1}^{\infty} \langle De_j, e_1\rangle g_j$ so $\langle De_j, e_1\rangle = 0$ holds $\forall j \in \mathbb{N}$, since g is a decreasing sequence with non-zero elements. Further,

$$g_1 = f_2 = \sum_{j=1}^{\infty} \langle De_j, e_2\rangle g_j \leqslant \sum_{j=1}^{\infty} \langle De_j, e_2\rangle g_1 \leqslant g_1.$$

The above statement holds only when $\langle De_1, e_2\rangle = 1$ and $\langle De_j, e_2\rangle = 0$, whenever $j \neq 1$. Because $D \in DSS(\ell^1(\mathbb{N}))$ it follows that $\langle De_1, e_i\rangle = 0$, whenever $i \neq 2$. Hence,

$$g_2 = f_3 = \sum_{j=1}^{\infty} \langle De_j, e_3\rangle g_j = 0 + \sum_{j=2}^{\infty} \langle De_j, e_3\rangle g_j \leqslant \sum_{j=2}^{\infty} \langle De_j, e_3\rangle g_2 \leqslant g_2.$$

Using the similar procedure as above, we obtain $\langle De_2, e_3\rangle = 1$ and $\langle De_j, e_3\rangle = 0$, whenever $j \neq 1$. Therefore, it follows that $\langle De_2, e_i\rangle = 0$, $i \neq 3$, since $D \in DSS(\ell^1(\mathbb{N}))$. Continuing this process we get $D = R \notin iDSS(\ell^1(\mathbb{N}))$. Thus $f \not\prec_s g$.

The next result connects extended weak majorization and increasing convex function.

Lemma 2.3. [25, Lemma 4.2] *Let $b \in (0, \infty]$ and suppose $\varphi\colon [0, b) \to [0, \infty)$ is a continuous increasing convex function with $\varphi(0) = 0$. For two functions $f, g \in \ell^1(I)^+$, where $f(i), g(i) < b$, $\forall i \in I$, if $f \prec_w g$ then $\sum_{i\in I} \varphi(f(i)) \leqslant \sum_{i\in I} \varphi(g(i))$.*

Proof. Suppose $f \prec_w g$, where $f, g \in \ell^1(I)^+$, $f(i), g(i) < b$. Therefore, there is $D \in DSS(\ell^1(I))$ such that $f = Dg$. Since, g may be represented by $g = \sum_{j\in I} g(j)e_j$ then $Dg = \sum_{j\in I} g(j)De_j$ so $f(i) = \langle Dg, e_i\rangle = \sum_{j\in I} g(j)\langle De_j, e_i\rangle$.

We define $r_i(A)$ to be

$$r_i(A) := 1 - \sum_{j\in J} \langle Ae_j, e_i\rangle, \quad \forall i \in I.$$

Since φ is a continuous convex function, we obtain

$$\begin{aligned}\varphi(f(i)) &= \varphi\Bigg(\sum_{j\in I} g(j)\langle De_j, e_i\rangle + 0 \cdot r_i(D)\Bigg)\\ &\leqslant \sum_{j\in I} \varphi(g(j))\langle De_j, e_i\rangle + \varphi(0) r_i(D) = \sum_{j\in I} \varphi(g(j))\langle De_j, e_i\rangle\end{aligned}$$

by Jensen's inequality. Now, changing the order of summation we get

$$\sum_{i\in I}\varphi(f(i))\leqslant\sum_{i\in I}\sum_{j\in I}\varphi(g(j))\langle De_j,e_i\rangle=\sum_{j\in I}\sum_{i\in I}\varphi(g(j))\langle De_j,e_i\rangle\leqslant\sum_{j\in I}\varphi(g(j)).\quad\square$$

Corollary 2.1. [25, Corollary 4.3] *Let $f,g\in\ell^1(I)^+$ and suppose $f\prec_w g$ and $g\prec_w f$. Then, for every continuous increasing convex function $\varphi\colon[0,b)\to[0,\infty)$ with $\varphi(0)=0$ is*

$$\sum_{i\in I}\varphi(f(i))=\sum_{i\in I}\varphi(g(i)) \tag{2.13}$$

where $b\in(0,\infty]$ and $f(i),g(i)<b$, $\forall i\in I$.

The next example shows that (2.13) don't imply $f\prec g$ and $g\prec f$, $\forall f,g\in\ell^1(I)^+$, in general. Using the same particularly chosen functions $f,g\in\ell^1(\mathbb{N})^+$ we can see that weaker relations $f\prec_w g$ and $g\prec_w f$ holds.

Example 2.2. [25, Example 4.1] Let $r>1$. We will consider two functions $f,g\in\ell^1(\mathbb{N})^+$,

$$f(k)=\frac{1}{k^r}\quad\text{and}\quad g(k+1)=\frac{1}{k^r},\quad k\in\mathbb{N},\ g(1)=0.$$

As we know, functions $f=\sum_{k=1}^{\infty}f(k)e_k=\sum_{k=1}^{\infty}\frac{1}{k^r}e_k$ and $g=\sum_{k=1}^{\infty}\frac{1}{k^r}e_{k+1}$ form two hyperharmonic series which represents the norm of these functions. Let $\varphi\colon[0,b)\to[0,\infty)$ be a continuous increasing convex function with $\varphi(0)=0$, where $b\in(0,\infty]$ and $f(i),g(i)<b$, $\forall i\in I$. We get

$$\sum_{k=1}^{\infty}\varphi(f(k))=\sum_{k=1}^{\infty}\varphi(\frac{1}{k^r})=\sum_{k=1}^{\infty}\varphi(g(k)).$$

Moreover, if $\varphi(0)\neq 0$ than (2.13) holds, too. Truly,

$$\sum_{k=1}^{\infty}\varphi(f(k))=\sum_{k=1}^{\infty}\varphi(\frac{1}{k^r})=\sum_{k=1}^{\infty}\varphi(g(k))=\infty.$$

Suppose that $g=Df$, where $D\colon\ell^1(\mathbb{N})\to\ell^1(\mathbb{N})$ is an arbitrary chosen bounded linear operator. It follows that $g=Df=\sum_{k=1}^{\infty}f(k)De_k$ and

$$0=g(1)=\sum_{k=1}^{\infty}f(k)\langle De_k,e_1\rangle=\sum_{k=1}^{\infty}\frac{1}{k^r}\langle De_k,e_1\rangle$$

which implies that $\langle De_k,e_1\rangle=0$, $\forall k\in\mathbb{N}$. Thus, D is not doubly stochastic, hence $g\not\prec f$.

Further, a family $\{a_{ij}:i,j\in\mathbb{N}\}$ defined by

$$a_{ij}=\begin{cases}1, & i-j=1,\\ 0, & \textit{otherwise},\end{cases}$$

represents a bounded linear operator A which satisfies $g = Af$. Clearly, $A \in DSS(\ell^1(\mathbb{N}))$, so $g \prec_w f$. Similarly, the family $\{b_{ij} : i, j \in \mathbb{N}\}$,

$$b_{ij} = \begin{cases} 1, & j - i = 1, \\ 0, & \textit{otherwise}, \end{cases}$$

generates a operator $B \in DSS(\ell^1(\mathbb{N}))$. Thus $f = Bg$, that is, $f \prec_w g$. Clearly, above operators A and B are shift operators.

Let $f \in \ell^1(I)^+$ be arbitrary chosen. We define two disjoint sets I_f^0 and I_f^+ to be

$$I_f^0 := \{i \in I : f(i) = 0\} \quad \text{and} \quad I_f^+ := \{i \in I : f(i) > 0\} \tag{2.14}$$

which we will use in the next results. Clearly, the set I_f^+ is at most countable, by definition of the Banach space $\ell^1(I)$. The set I_f^0 may be finite, countable or uncountable depends on the cardinality of I and the definition of f. Obviously, $I = I_f^0 \cup I_f^+$.

We refer reader to [25] for more results between weak majorization $\prec_w$ and convex or concave functions.

The next result provides a weak anti-symmetricity for weak majorization $\prec_w$ on $\ell^1(I)^+$.

Theorem 2.4. [25, Theorem 4.2] *Let $f, g \in \ell^1(I)^+$. The next statements are equivalent:*

i) *$f \prec_w g$ and $g \prec_w f$;*
ii) *There exist $P \in pP(\ell^1(I))$ for sets I_f^+ and I_g^+ such that $g = Pf$.*

Proof. Let $f \in \ell^1(I)^+$. The set $\{f(i) : i \in I\}$ is bounded, so we may define inductively a family $\{I_f^n : n \in \mathbb{N}\}$ of the disjoint finite subsets of I_f^+ to be

$$\begin{aligned} I_f^1 &:= \left\{ i \in I_f^+ : f(i) = \max\{f(j) : j \in I_f^+\} \right\} \\ I_f^n &:= \left\{ i \in I_f^+ : f(i) = \max\left\{ f(j) : j \in I_f^+ \setminus \bigcup_{k=1}^{n-1} I_f^k \right\} \right\} \end{aligned} \tag{2.15}$$

whenever $n \geqslant 2$. Clearly, $I_f^+ = \bigcup_{k=1}^{\infty} I_f^k$. If $I_f^r \neq \emptyset$, for any $r \in \mathbb{N}$, then we set

$$f_r := f(j), \quad \text{where} \quad j \in I_f^r. \tag{2.16}$$

If $I_f^r = \emptyset$, then $I_f^k = \emptyset$, $\forall k \geqslant r$, so we set $f_k := 0$, $\forall k \geqslant r$. Also, if $s > r$ then $f_s \leqslant f_r$. The last inequality is strict if I_f^r is non-empty.

Let $f \prec_w g$ and $g \prec_w f$ for an arbitrary functions $f, g \in \ell^1(I)^+$. In the sequel, we will consider a continuous increasing convex function $\varphi_\lambda \colon [0, \infty) \to [0, \infty)$, $\lambda \geqslant 0$ defined by $\varphi_\lambda(t) := (t - \lambda) \cdot \chi_{[\lambda, \infty)}(t)$. Clearly, $\varphi_\lambda(t) = 0$, for all $t \leqslant \lambda$. In particular, $\varphi_\lambda(0) = 0$. Now, using Corollary 2.1 we have

$$\sum_{i \in I} \varphi_\lambda(f(i)) = \sum_{i \in I} \varphi_\lambda(g(i)), \quad \forall \lambda \geqslant 0. \tag{2.17}$$

In particular, for $\lambda = 0$ we have $\sum_{i \in I_f^+} f(i) = \sum_{i \in I} f(i) = \sum_{i \in I} g(i) = \sum_{i \in I_g^+} g(i)$ so the sets I_f^+ and I_g^+ are simultaneous empty or non-empty. if $I_f^+ = I_g^+ = \emptyset$

the work is done. Suppose that I_f^+, I_g^+ are two non-empty sets. Then, I_f^1, I_g^1 have to be non-empty, too. Suppose that $f_1 \neq q_1$, for instance, $f_1 > g_1$, and let $\lambda := \min\{f_1, g_1\} = g_1$. Hence, using the continuous convex function φ_λ, we conclude $\sum_{j\in I} \varphi_\lambda(g(j)) = 0 < f_1 - \lambda \leqslant \sum_{j\in I} \varphi_\lambda(f(j))$ since there is at least one $k \in I$ such that $f(k) = f_1 > \lambda$, and $\varphi_\lambda(f(k)) = f_1 - \lambda$. This is a contradiction with (2.17), so $f_1 = g_1$. Further, if we set $\mu := \max\{f_2, g_2\}$, using the convex function φ_μ and (2.17) we get

$$\operatorname{card}(I_f^1)(f_1 - \mu) = \sum_{j\in I} \varphi_\mu((f(j)) = \sum_{j\in I} \varphi_\mu((g(j)) = \operatorname{card}(I_g^1)(g_1 - \mu).$$

It is provided that $f_1 = g_1$ and $\operatorname{card}(I_f^1) = \operatorname{card}(I_g^1)$.

We will provide by the induction that

$$f_k = g_k \quad \text{and} \quad \operatorname{card}(I_f^k) = \operatorname{card}(I_g^k), \quad \forall k \in \mathbb{N}. \tag{2.18}$$

Assume that (2.18) holds for every $k = 1, \dots, n$. Suppose that $I_f^{n+1} = \emptyset$. If $I_g^{n+1} \neq \emptyset$ then

$$\sum_{i\in I} \varphi_0(f(i)) = \sum_{j=1}^{n} \operatorname{card}(I_f^j) f_j = \sum_{j=1}^{n} \operatorname{card}(I_g^j) g_j < \sum_{j=1}^{n+1} \operatorname{card}(I_g^j) g_j \leqslant \sum_{i\in I} \varphi_0(g(i)),$$

because $I_f^{n+1} = \emptyset$ implies $I_f^j = \emptyset$, for every $j \geqslant n+1$. This contradicts (2.17) for $\lambda = 0$, so $I_f^j = I_g^j = \emptyset$ and $f_j = g_j = 0$ for every $j \geqslant n+1$.

Now, assume that $I_f^{n+1}, I_g^{n+1} \neq \emptyset$ and suppose without loss of generality that $f_{n+1} > g_{n+1}$. Using the similar argument as above, if we set $\lambda := \min\{f_{n+1}, g_{n+1}\} = g_{n+1}$ we obtain

$$\sum_{j\in I} \varphi_\lambda(g(j)) = \sum_{j=1}^{n} \operatorname{card}(I_g^j) g_j < f_{n+1} - \lambda + \sum_{j=1}^{n} \operatorname{card}(I_f^j) f_j \leqslant \sum_{j\in I} \varphi_\lambda(f(j))$$

since there is at least one $k \in I$ such that $f(k) = f_{n+1}$. Again, this is a contradiction with (2.17), so $f_{n+1} = g_{n+1}$. On the other hand, if we set $\mu = \max\{f_{n+2}, g_{n+2}\}$ then

$$\sum_{j=1}^{n+1} \operatorname{card}(I_f^j)(f_j - \mu) = \sum_{j\in I} \varphi_\mu((f(j)) = \sum_{j\in I} \varphi_\mu((g(j)) = \sum_{j=1}^{n+1} \operatorname{card}(I_g^j)(g_j - \mu).$$

Now, have to be $\operatorname{card}(I_f^{n+1}) = \operatorname{card}(I_g^{n+1})$ by the induction hypothesis. Thus, (2.18) holds for every $n \in \mathbb{N}$.

Further, we define the bijections $\omega_k \colon I_f^k \to I_g^k$, $\forall k \in \mathbb{N}$ whenever $I_f^k \neq \emptyset$. Also, using the well-known facts

$$I_f^+ = \bigcup_{k=1}^{\infty} I_f^k, \quad \bigcap_{k=1}^{\infty} I_f^k = \emptyset \quad \text{and} \quad I_g^+ = \bigcup_{k=1}^{\infty} I_g^k, \quad \bigcap_{k=1}^{\infty} I_g^k = \emptyset$$

we define a bijection $\Omega \colon I_f^+ \to I_g^+$ such that $\Omega(i) := \omega_k(i) \in I_g^k$, when $i \in I_f^k$, for any $k \in \mathbb{N}$. It follows that $f(i) = g(\Omega(i))$, $\forall i \in I_f^+$. Finally, the operator

$P\colon \ell^1(I) \to \ell^1(I)$ defined by $Pe_j = e_{\Omega(j)}$, whenever $j \in I_f^+$, and $Pe_j = 0$ otherwise, represents a partial permutation for the sets I_f^+ and I_g^+.

Let $P \in pP(\ell^1(I))$ for sets I_f^+ and I_g^+ such that $g = Pf$, and let $\Omega\colon I_f^+ \to I_g^+$ is appropriate bijection. Obviously, $g \prec_w f$ because $P \in pP(\ell^1(I)) \subset DSS(\ell^1(I))$. We define $Q\colon \ell^1(I) \to \ell^1(I)$ to be $Qe_j = e_{\Omega^{-1}(j)}$, whenever $j \in I_g^+$ and $Qe_j = 0$, otherwise. Thus, $Q \in pP(\ell^1(I))$ and $Qg = f$ so $f \prec_w g$. □

In the finite dimensional case, for two arbitrary vectors $x, y \in \mathbb{R}^n$, $x \prec_w y$ and $y \prec_w x$ implies $y = Px$, for any $n \times n$ permutation matrix P. In the next example we choose two functions $f, g \in \ell^1(I)$ to show that $f \prec_w g$ and $g \prec_w f$ do not imply existence of permutation $P \in P(\ell^1(I))$, such that $g = Pf$.

Example 2.3. [25, Example 4.2] Suppose that $0 < r < 1$. We define two functions $f, g \in \ell^1(\mathbb{N})^+$ to be $f(k) = 1/r^k$, $g(1) = 0$, $g(k+1) = 1/r^k$, $k \in \mathbb{N}$. For the (shift) operator $A \in DSS(\ell^1(\mathbb{N}))$ given by the family $\{a_{ij} : i, j \in \mathbb{N}\}$,

$$a_{ij} = \begin{cases} 1, & i - j = 1, \\ 0, & \textit{otherwise}, \end{cases}$$

$g = Af$ holds, i.e., $g \prec_w f$. Similarly, the family $\{b_{ij} : i, j \in \mathbb{N}\}$,

$$b_{ij} = \begin{cases} 1, & j - i = 1, \\ 0, & \textit{otherwise}, \end{cases}$$

forms the (shift) operator $B \in DSS(\ell^1(\mathbb{N}))$ with $f = Bg$, thus $f \prec_w g$. On the other hand, $g(1) = 0 \neq f(i)$, $\forall i \in \mathbb{N}$. It follows that there is no permutation $P \in P(\ell^1(\mathbb{N}))$ such that $g = Pf$.

The next corollary ensures that weak majorization $\prec_w$ may be consider, in some sense, as a partial order on $\ell^1(I)^+$.

Corollary 2.2. [25, Corollary 4.4] *The majorization relation "$\prec_w$" in Definition 2.2 is reflexive and transitive relation i.e. "$\prec_w$" is a pre-order. In particular, if we identify all functions which are partial permutations of their positive elements, then we may consider "$\prec_w$" as a partial order.*

Proof. The relation "$\prec_w$" is reflexive, since the identity operator I is doubly substochastic, so for any $f \in \ell^1(I)^+$, we have $f = If$, that is, $f \prec_w f$.

Transitivity holds because the set $DSS(\ell^1(I))$ is closed under the composition, by [25, Theorem 3.1]. Precisely, if $f \prec_w g$ and $g \prec_w h$ then there are $A, B \in DSS(\ell^1(I))$ such that $f = Ag$ and $g = Bh$, so $f = ABh$. Since $DSS(\ell^1(I))$ is closed under the composition, hence $f \prec_w h$.

If we identify all function which are different up to the partial permutation of their positive elements, then it follows directly from Theorem 2.4 that "$\prec_w$" is antisymmetric. □

Now, we present results provided in [28] about close relationship between weak supermajorization on $\ell^1(I)^+$ and convex and concave functions.

Theorem 2.5. [28, Theorem 4.2] *Let* $b \in (0, \infty]$*, and suppose that* $\varphi\colon [0, b) \to \mathbb{R}$ *be a continuous increasing concave function. For two functions* $f, g \in \ell^1(I)^+$*, where* $f(i), g(i) < b$*,* $\forall i \in I$*, if* $f \prec^{ws} g$ *then* $\sum_{i\in I} \varphi(f(i)) \geqslant \sum_{i\in I} \varphi(g(i))$*.*

Proof. If we suppose that $f \prec^{ws} g$, than there exists a family $D \in DSPS(\ell^1(I))$ such that $f = Dg$. Since, $g \in \ell^1(I)^+$ has representation $g = \sum_{j\in I} g(j)e_j$, we get

$$f(i) = Dg(i) = \sum_{j\in I} g(j)\langle De_j, e_i\rangle, \quad \forall i \in I.$$

Suppose that φ is a continuous increasing concave function. Using Jensen's inequality, we obtain

$$\varphi(f(i)) = \varphi\bigg(\sum_{j\in I} g(j)\langle De_j, e_i\rangle\bigg) \geqslant \varphi\bigg(\sum_{j\in I} g(j)\langle \tilde{D}e_j, e_i\rangle\bigg) \geqslant \sum_{j\in I} \varphi(g(j))\langle \tilde{D}e_j, e_i\rangle,$$

where the operator $\tilde{D} \in DS(\ell^1(I))$ satisfies

$$(\forall i \in I) \quad (\forall j \in I) \qquad \langle De_j, e_i\rangle \geqslant \langle \tilde{D}e_j, e_i\rangle \tag{2.19}$$

by definition (2.9). Now, changing the order of summation we obtain

$$\sum_{i\in I} \varphi(f(i)) \geqslant \sum_{i\in I}\sum_{j\in I} \varphi(g(j))\langle \tilde{D}e_j, e_i\rangle = \sum_{j\in I} \varphi(g(j)) \sum_{i\in I}\langle \tilde{D}e_j, e_i\rangle = \sum_{j\in I} \varphi(g(j)).$$

We mention that if φ is not positive, because it is continuous increasing function we get that $\lim_{\to 0^+} \varphi(t) = a < 0$ so, $\sum_{i\in I} \varphi(f(i)) = \sum_{i\in I} \varphi(g(i)) = -\infty$, when $\operatorname{card}(I) \geqslant \aleph_0$. Similarly, when $\varphi(0) > 0$ it has to be $\sum_{j\in I} \varphi(g(j)) = \infty$, but also $\sum_{j\in I} \varphi(f(j)) = \infty$ by above consideration. □

Theorem 2.6. [28, Theorem 4.3] *Let* $b \in (0, \infty]$*. Suppose that* $\varphi\colon [0, b) \to \mathbb{R}$ *is a continuous decreasing convex function. For two functions* $f, g \in \ell^1(I)^+$*, where* $f(i), g(i) < b$*,* $\forall i \in I$*, if* $f \prec^{ws} g$ *then* $\sum_{i\in I} \varphi(f(i)) \leqslant \sum_{i\in I} \varphi(g(i))$*.*

Proof. Let $f \prec^{ws} g$. There is a family $D \in DSPS(\ell^1(I))$ such that $f = Dg$. If φ is a continuous decreasing convex function, then

$$\begin{aligned}\varphi(f(i)) = \varphi\bigg(\sum_{j\in I} g(j)\langle De_j, e_i\rangle\bigg) &\leqslant \varphi\bigg(\sum_{j\in I} g(j)\langle \tilde{D}e_j, e_i\rangle\bigg)\\ &\leqslant \sum_{j\in I} \varphi(g(j))\langle \tilde{D}e_j, e_i\rangle, \quad \forall i \in I,\end{aligned}$$

where $\tilde{D}$ satisfies (2.19). Hence,

$$\begin{aligned}\sum_{i\in I} \varphi(f(i)) &\leqslant \sum_{i\in I}\sum_{j\in I} \varphi(g(j))\langle \tilde{D}e_j, e_i\rangle\\ &= \sum_{j\in I} \varphi(g(j)) \sum_{i\in I}\langle \tilde{D}e_j, e_i\rangle = \sum_{j\in I} \varphi(g(j)).\end{aligned}$$

When $\mathrm{card}(I) \geqslant \aleph_0$, if φ is not negative, it is easy to see that

$$\sum_{i\in I} \varphi(f(i)) = \sum_{i\in I} \varphi(g(i)) = \infty$$

and if $\varphi(0) < 0$, then $\sum_{j\in I} \varphi(g(j)) = -\infty$ and $\sum_{j\in I} \varphi(f(j)) = -\infty$. □

Corollary 2.3. [28, Corollary 4.3] *Let $b \in (0, \infty]$, and let $f, g \in \ell^1(I)^+$ such that $f(i), g(i) < b$, $\forall i \in I$. If $f \prec^{ws} g$ and $g \prec^{ws} f$ then*

i) *for each continuous increasing concave function $\varphi\colon [0, b) \to \mathbb{R}$,*

$$\sum_{i\in I} \varphi(f(i)) = \sum_{i\in I} \varphi(g(i)).$$

ii) *for each continuous decreasing convex function $\varphi\colon [0, b) \to \mathbb{R}$,*

$$\sum_{i\in I} \varphi(f(i)) = \sum_{i\in I} \varphi(g(i)).$$

Corollary 2.4. [28, Corollary 4.4] *Let $f, g \in \ell^1(I)^+$. If $f \prec^{ws} g$ and $g \prec^{ws} f$ then $\sum_{i\in I} \varphi_\lambda(f(i)) = \sum_{i\in I} \varphi_\lambda(g(i))$, for every function $\varphi_\lambda\colon [0, \infty) \to [0, \infty)$ defined by*

$$\varphi_\lambda(t) := t \cdot \chi_{[0,\lambda)}(t) + \lambda \cdot \chi_{[\lambda,\infty)}(t), \tag{2.20}$$

where $\lambda \in [0, \infty)$.

The next example shows that the opposite direction of Corollary 2.3 does not hold. Also, we conclude that opposite directions of Theorems 2.5 and 2.6 does not valid, too.

Example 2.4. Let $r > 1$ and let $f, g \in \ell^1(\mathbb{N})^+$ be two functions defined by

$$f = \sum_{k=1}^{\infty} \frac{1}{k^r} e_{k+1} \quad \text{and} \quad g = \sum_{k=1}^{\infty} \frac{1}{k^r} e_k.$$

Obviously, $f(k + 1) = g(k)$, $k \in \mathbb{N}$ and $f(1) = 0$. Suppose that $\varphi\colon [0, b) \to \mathbb{R}$ be a an arbitrary chosen continuous increasing concave (decreasing convex) function such that $b \in (0, \infty]$ and $f(i), g(i) < b$, $\forall i \in I$. If $\varphi(0) = 0$ then

$$\sum_{k=1}^{\infty} \varphi(f(k)) = \sum_{k=1}^{\infty} \varphi\Big(\frac{1}{k^r}\Big) = \sum_{k=1}^{\infty} \varphi(g(k)).$$

Moreover, if $\lim_{x\to 0^+} \varphi(x) \neq 0$, than we obtain

$$\sum_{k=1}^{\infty} \varphi(f(k)) = \sum_{k=1}^{\infty} \varphi(g(k)) \qquad (= +\infty \vee -\infty).$$

Suppose that $D\colon \ell^1(\mathbb{N}) \to \ell^1(\mathbb{N})$ is a bounded linear operator such that $f = Dg$, that is, $f = Dg = \sum_{k=1}^{\infty} g(k) De_k$. Since

$$0 = f(1) = \sum_{k=1}^{\infty} g(k)\langle De_k, e_1\rangle = \sum_{k=1}^{\infty} \frac{1}{k^r} \langle De_k, e_1\rangle,$$

we get $\langle De_k, e_1\rangle = 0$, $\forall k \in \mathbb{N}$. We conclude $D \notin DSPS(\ell^1(I))$, thus $f \not\prec^{ws} g$.

In the next result we present that the weakened anti-symmetricity for weak supermajorization $\prec^{ws}$ on $\ell^1(I)^+$ holds.

Theorem 2.7. [28, Theorem 4.4] *For $f, g \in \ell^1(I)^+$, the following conditions are equivalent:*

i) *$f \prec^{ws} g$ and $g \prec^{ws} f$.*
ii) *There exists a permutation $P \in P(\ell^1(I))$ such that $f = Pg$.*

Proof. Let $f \in \ell^1(I)^+$ be arbitrary chosen. We define two disjoint sets I_f^0 and I_f^+ by (2.14). The set $\{f(i) : i \in I\}$ is bounded, so we may define inductively a family $\{I_f^n : n \in \mathbb{N}\}$ of the disjoint finite subsets of I_f^+ by (2.15). Clearly, $I_f^+ = \bigcup_{k=1}^{\infty} I_f^k$. We define the sequence $(f_r)_{r\in\mathbb{N}}$ in the same way as in Theorem 2.4, by (2.16). Without loss of generality, we suppose that $\mathrm{card}(I_f^+) = \aleph_0$ and $\mathrm{card}(I_g^+) = \aleph_0$, that is $I_f^r, I_g^r \neq \emptyset$, for all $r \in \mathbb{N}$.

In the sequel, we will consider a continuous increasing concave function $\varphi_\lambda \colon [0,\infty) \to [0,\infty)$, $\lambda \geqslant 0$ defined by (2.20). Fix $f, g \in \ell^1(I)^+$ such that $f \prec^{ws} g$ and $g \prec^{ws} f$. Therefore,

$$\sum_{i\in I} \varphi_\lambda(f(i)) = \sum_{i\in I} \varphi_\lambda(g(i)), \quad \forall \lambda > 0 \tag{2.21}$$

by Corollary 2.4. Without loss of generality, suppose that $f_1 \geqslant g_1$. Then

$$\sum_{i\in I_f^+} f(i) = \sum_{i\in I} \varphi_{f_1}(f(i)) = \sum_{i\in I} \varphi_{f_1}(g(i)) = \sum_{i\in I_g^+} g(i), \tag{2.22}$$

by (2.21), so the sets I_f^+ and I_g^+ are simultaneous empty or non-empty. Obviously, if $I_f^+ = \emptyset = I_g^+$ the proof is easy to check. Suppose that I_f^+, I_g^+ are non-empty sets. Now, there is $m \in \mathbb{N}$ such that $f_k \geqslant g_1$, $\forall k = 1, \ldots m$ and $f_{m+1} < g_1$. We obtain

$$\begin{aligned} \sum_{i\in I_g^+} g(i) &= \sum_{i\in I} \varphi_{g_1}(g(i)) = \sum_{i\in I} \varphi_{g_1}(f(i)) \\ &= \sum_{i\in I_f^+ \smallsetminus \bigcup_{k=1}^m I_f^k} f(i) + g_1 \sum_{k=1}^{m} \mathrm{card}(I_f^k). \end{aligned} \tag{2.23}$$

Combining (2.22) and (2.23) we obtain $f_1 = g_1$ and $m = 1$. Let $\lambda := \max\{f_2, g_2\}$. It follows that

$$\sum_{i\in I} \varphi_\lambda(f(i)) = \sum_{i\in I\smallsetminus I_f^1} f(i) + \mathrm{card}(I_f^1)\cdot\lambda, \tag{2.24}$$

$$\sum_{i\in I} \varphi_\lambda(g(i)) = \sum_{i\in I\smallsetminus I_g^1} g(i) + \mathrm{card}(I_g^1)\cdot\lambda\ . \tag{2.25}$$

Equalities (2.22), (2.24) and (2.25) imply

$$\sum_{i\in I\smallsetminus I_f^1} f(i) - \sum_{i\in I\smallsetminus I_g^1} g(i) = \lambda(\mathrm{card}(I_g^1) - \mathrm{card}(I_f^1)). \tag{2.26}$$

On the other hand, using (2.22) and (2.23) for $m = 1$, we get

$$\sum_{i\in I\smallsetminus I_f^1} f(i) - \sum_{i\in I\smallsetminus I_g^1} g(i) = g_1(\mathrm{card}(I_g^1) - \mathrm{card}(I_f^1)). \tag{2.27}$$

Since $\lambda < g_1$, we get that (2.26) and (2.27) imply $\mathrm{card}(I_g^1) = \mathrm{card}(I_f^1)$. Further, we will prove by induction that

$$f_k = g_k \quad \text{and} \quad \mathrm{card}(I_f^k) = \mathrm{card}(I_g^k), \qquad \forall k \in \mathbb{N}. \tag{2.28}$$

Suppose that (2.28) holds for every $k = 1, \dots, n$. Assume that $I_f^{n+1}, I_g^{n+1} \neq \emptyset$. Without loss of generality, suppose that $f_{n+1} \geqslant g_{n+1}$. It follows

$$\sum_{i\in I\smallsetminus \bigcup_{k=1}^n I_f^k} f(i) = \sum_{i\in I\smallsetminus \bigcup_{k=1}^n I_g^k} g(i) \tag{2.29}$$

by (2.21) and (2.28), choosing $\lambda = f_{n+1}$. Next, if we set $\lambda = g_{n+1}$ then

$$\sum_{i\in I} \varphi_\lambda(g(i)) = \sum_{i\in I\smallsetminus \bigcup_{k=1}^n I_g^k} g(i) + \lambda \sum_{k=1}^{n} \mathrm{card}(I_g^k) \tag{2.30}$$

$$\sum_{i\in I} \varphi_\lambda(f(i)) = \sum_{i\in I\smallsetminus \bigcup_{k=1}^{n+1} I_f^k} f(i) + \lambda\cdot \mathrm{card}(I_f^{n+1}) + \lambda \sum_{k=1}^{n} \mathrm{card}(I_f^k).$$

Using (2.21), (2.28), (2.29) and (2.30) we conclude that $f_{n+1} = g_{n+1}$ and

$$\sum_{i\in I\smallsetminus \bigcup_{k=1}^{n+1} I_f^k} f(i) - \sum_{i\in I\smallsetminus \bigcup_{k=1}^{n+1} I_g^k} g(i) = g_{n+1}(\mathrm{card}(I_g^{n+1}) - \mathrm{card}(I_f^{n+1})). \tag{2.31}$$

Let $\lambda := \max\{f_{n+2}, g_{n+2}\}$. Then

$$\begin{aligned} 0 &= \sum_{i\in I} \varphi_\lambda(f(i)) - \sum_{i\in I} \varphi_\lambda(g(i)) \\ &= \sum_{i\in I\smallsetminus \bigcup_{k=1}^{n+1} I_f^k} f(i) - \sum_{i\in I\smallsetminus \bigcup_{k=1}^{n+1} I_g^k} g(i) + \lambda(\mathrm{card}(I_f^{n+1}) - \mathrm{card}(I_g^{n+1})). \end{aligned} \tag{2.32}$$

Because $\lambda < g_{n+1}$ then $\mathrm{card}(I_f^{n+1}) = \mathrm{card}(I_g^{n+1})$, by (2.31) and (2.32). Thus, (2.28) holds.

Using above facts, there are bijections $\omega_k \colon I_f^k \to I_g^k$, $\forall k \in \mathbb{N}$ if $I_f^k \neq \emptyset$. We define a bijection $\Omega\colon I_f^+ \to I_g^+$ such that $\Omega(i) := \omega_k(i) \in I_g^k$,whenever $i \in I_f^k$, $k \in \mathbb{N}$. This map is well defined because

$$I_f^+ = \bigcup_{k=1}^{\infty} I_f^k, \quad \bigcap_{k=1}^{\infty} I_f^k = \emptyset \quad \text{and} \quad I_g^+ = \bigcup_{k=1}^{\infty} I_g^k, \quad \bigcap_{k=1}^{\infty} I_g^k = \emptyset.$$

We will provide that $\mathrm{card}(I_f^0) = \mathrm{card}(I_g^0)$. If I is uncountable, then I_f^0 and I_g^0 are uncountable, too. If I is finite, then there is a permutation P such that $f = Pg$, thus $\mathrm{card}(I_f^0) = \mathrm{card}(I_f^0) < \aleph_0$. Suppose that I is a countable set. If

$\operatorname{card}(I_f^0) = \operatorname{card}(I_g^0) = \aleph_0$, the work is done. Let $\operatorname{card}(I_f^0) < \aleph_0$. Without loss of generality suppose that $\operatorname{card}(I_f^0) \leqslant \operatorname{card}(I_g^0) \leqslant \aleph_0$. If $k \in I_g^0$ then

$$0 = g(k) = \sum_{j \in I} f(j)\langle D_1 e_j, e_k\rangle,$$

so $\langle D_1 e_j, e_k\rangle = 0$, $\forall j \in I_f^+$, whenever $k \in I_g^0$, where $D_1 \in DSPS(\ell^1(I))$ with $g = D_1 f$. Now, there is $\widetilde{D_1} \in DS(\ell^1(I))$ such that $\langle D_1 e_j, e_i\rangle \geqslant \langle \widetilde{D_1} e_j, e_i\rangle$, $\forall i, j \in I$. Now, we get $\langle \widetilde{D_1} e_j, e_k\rangle = 0$, $\forall j \in I_f^+$, whenever $k \in I_g^0$, so

$$\sum_{j \in I_f^0} \langle \widetilde{D_1} e_j, e_k\rangle = 1, \quad \forall k \in I_g^0.$$

Also,

$$\sum_{k \in I_g^0} \langle \widetilde{D_1} e_j, e_k\rangle \leqslant \sum_{k \in I} \langle \widetilde{D_1} e_j, e_k\rangle = 1, \quad \forall j \in I_f^0.$$

Hence,

$$\sum_{j \in I_f^0} \sum_{k \in I_g^0} \langle \widetilde{D_1} e_j, e_k\rangle \leqslant \operatorname{card}(I_f^0) \leqslant \operatorname{card}(I_g^0) = \sum_{k \in I_g^0} \sum_{j \in I_f^0} \langle \widetilde{D_1} e_j, e_k\rangle. \tag{2.33}$$

Since, $\operatorname{card}(I_f^0) < \aleph_0$, changing the order of summation in (2.33) we conclude that all equalities in (2.33) hold, so $\operatorname{card}(I_f^0) = \operatorname{card}(I_g^0)$. Now, using the above consideration there is a bijection $\Omega_0 \colon I_f^0 \to I_g^0$. Finally, we define a bijection $\Psi \colon I_f \to I_g$ such that

$$\Psi(i) := \begin{cases} \Omega(i), & i \in I_f^+ \\ \Omega_0(i), & i \in I_f^0. \end{cases}$$

Suppose that $P \in P(\ell^1(I))$, where $Pe_j = e_{\Psi(j)}$, for each $j \in I$. Fix $f = \sum_{j \in I} f(j) e_j \in \ell^1(I)$. We have $Pf = \sum_{j \in I} f(j) e_{\Psi(j)} = \sum_{k \in I} f(\Psi^{-1}(k)) e_k$, therefore

$$Pf(i) = \sum_{k \in I} f(\Psi^{-1}(k))\langle e_k, e_i\rangle = f(\Psi^{-1}(i)).$$

If $i \in I_g^+$, then there is $m \in I$ such that $i \in I_g^m$ and $\Psi^{-1}(i) = \omega_m^{-1}(i) \in I_f^m \subset I_f^+$. Therefore, $g(i) = f(\Psi^{-1}(i)) = Pf(i)$, $\forall i \in I_g^+$. Similarly, if $i \in I_g^0$, then $\Psi^{-1}(i) = \Omega_0^{-1}(i) \in I_f^0$. Thus, $Pf = g$.

Since $Q = P^{-1} \in P(\ell^1(I)) \subset DSPS(\ell^1(I))$, we obtain $Pf = g$ and $Qg = f$, i.e. $g \prec^{ws} f$ and $f \prec^{ws} g$, thus the opposite direction is provided. □

If we put $g \prec f$ instead of $g \prec_w f$ we can see in the next theorem that the stronger condition *ii)* in comparison with Theorem 2.4 holds.

Theorem 2.8. [25, Theorem 4.3] *Let $f, g \in \ell^1(I)^+$. The next statements are equivalent:*

i) *$f \prec_w g$ and $g \prec f$;*
ii) *There exist $P \in P(\ell^1(I))$ such that $g = Pf$.*

Proof. Let $f \prec_w g$ and $g \prec f$ for some $f, g \in \ell^1(I)^+$. Theorem 2.4 holds for this assumptions, so we will use parts of its proof. There is $D \in DS(\ell^1(I))$ such that $g = Df$. Without loss of generality, assume $I_f^n \neq \emptyset \neq I_g^n$, $\forall n \in \mathbb{N}$. We prove by the induction that

$$\forall i \in I_g^n, \quad \sum_{j \in I_f^n} \langle De_j, e_i \rangle = 1, \quad \text{and} \quad \forall j \in I_f^n, \quad \sum_{i \in I_g^n} \langle De_j, e_i \rangle = 1. \tag{2.34}$$

Firstly, suppose that $n = 1$, and $k \in I_g^1$. Then

$$\begin{aligned} g_1 = g(k) = \sum_{j \in I} f(j) \langle De_j, e_k \rangle &= \sum_{j \in I_f^1} f_1 \langle De_j, e_k \rangle + \sum_{j \in I \setminus I_f^1} f(j) \langle De_j, e_k \rangle \\ &\leqslant \sum_{j \in I_f^1} f_1 \langle De_j, e_k \rangle + \sum_{j \in I \setminus I_f^1} f_1 \langle De_j, e_k \rangle = f_1. \end{aligned}$$

Since $f_1 = g_1$, we get $\sum_{j \in I \setminus I_f^1} \langle De_j, e_k \rangle = 0$. By (2.18) we have $\mathrm{card}(I_f^1) = \mathrm{card}(I_g^1)$. On the other hand,

$$\mathrm{card}(I_g^1) = \sum_{i \in I_g^1} \sum_{j \in I_f^1} \langle De_j, e_i \rangle = \sum_{j \in I_f^1} \sum_{i \in I_g^1} \langle De_j, e_i \rangle$$

therefore

$$\mathrm{card}(I_f^1) = \sum_{j \in I_f^1} \sum_{i \in I_g^1} \langle De_j, e_i \rangle.$$

Now, $\sum_{i \in I_g^1} \langle De_j, e_i \rangle = 1$, $\forall j \in I_f^1$, thus (2.34) holds for $n = 1$. Suppose that (2.34) holds for every $k = 1, \ldots, n$ and let $r \in I_g^{n+1}$. This means if $j \in I_f^k$, $1 \leqslant k \leqslant n$, then $\langle De_j, e_r \rangle = 0$ by the induction hypothesis. Now, we get

$$\begin{aligned} g_{n+1} = g(r) &= \sum_{j \in I} f(j) \langle De_j, e_r \rangle \\ &= \sum_{j \in I_f^{n+1}} f_{n+1} \langle De_j, e_r \rangle + \sum_{j \in I \setminus \bigcup_{m=1}^{n+1} I_f^m} f(j) \langle De_j, e_r \rangle \\ &\leqslant \sum_{j \in I_f^{n+1}} f_{n+1} \langle De_j, e_r \rangle + \sum_{j \in I \setminus \bigcup_{m=1}^{n+1} I_f^m} f_{n+1} \langle De_j, e_r \rangle = f_{n+1}. \end{aligned}$$

Since $f_{n+1} = g_{n+1}$, we get $\sum_{j \in I \setminus I_f^{n+1}} \langle De_j, e_r \rangle = 0$. Again, by (2.18) we know that $\mathrm{card}(I_f^{n+1}) = \mathrm{card}(I_g^{n+1})$. In the other side,

$$\mathrm{card}(I_g^{n+1}) = \sum_{i \in I_g^{n+1}} \sum_{j \in I_f^{n+1}} \langle De_j, e_i \rangle = \sum_{j \in I_f^{n+1}} \sum_{i \in I_g^{n+1}} \langle De_j, e_i \rangle$$

therefore $\mathrm{card}(I_f^{n+1}) = \sum_{j \in I_f^{n+1}} \sum_{i \in I_g^{n+1}} \langle De_j, e_i \rangle$. Now, it has to be

$$\sum_{i \in I_g^{n+1}} \langle De_j, e_i \rangle = 1, \quad \forall j \in I_f^{n+1}$$

thus (2.34) holds. However, (2.34) implies that

$$\sum_{j\in I_g^+}\langle De_j, e_i\rangle = 1, \quad \forall i\in I_f^+, \tag{2.35}$$

$$\sum_{i\in I_f^+}\langle De_j, e_i\rangle = 1, \quad \forall j\in I_g^+. \tag{2.36}$$

Thus, if $i_0\in I_f^0 = I\setminus I_f^+$ we have $\langle De_j, e_{i_0}\rangle = 0$, $\forall j\in I_g^+$ by (2.36). Similarly, if $j_0\in I_g^0 = I\setminus I_g^+$, we have $\langle De_{j_0}, e_i\rangle = 0$, $\forall i\in I_f^+$ by (2.35). Now, it is easy to conclude that new map $D_1\colon \ell^1(I_f^0)\to \ell^1(I_g^0)$ defined by $\langle D_1e_j, e_i\rangle := \langle De_j, e_i\rangle$ is doubly stochastic. Therefore, there is a bijection $\Omega_0\colon I_f^0\to I_g^0$ by [6, Theorem 2.2.]. Finally, we define a bijection $\Psi\colon I\to I$ to be

$$\Psi(k) := \begin{cases}\Omega(k), & k\in I_f^+\\ \Omega_0(k), & k\in I_f^0.\end{cases}$$

Suppose that $P\in P(\ell^1(I))$ correspond to the bijection Ψ. Now,

$$Pf = \sum_{k\in I} f(k)e_{\Psi(k)} = \sum_{j\in I} f(\Psi^{-1}(j))e_j.$$

If $j\in I_g^0$ then $\Psi^{-1}(j) = \Omega_0^{-1}(j)\in I_f^0$ so $f(\Psi^{-1}(j)) = 0 = g(j)$. Also, if $j\in I_g^+$ then there is $n\in\mathbb{N}$ such that $j\in I_g^n$ and $\Psi^{-1}(j) = \Omega^{-1}(j) = \omega_n^{-1}(j)\in I_f^n$, hence $f(\Psi^{-1}(j)) = f_n = g_n = g(j)$. Thus, $g = Pf$, because $g(i) = Pf(i)$, $\forall i\in I$.

If we assume that $P\in P(\ell^1(I))$ such that $g = Pf$ we get $f = P^{-1}g$, where $P^{-1}\in P(\ell^1(I))$. Because $P(\ell^1(I))\subset DS(\ell^1(I))\subset DSS(\ell^1(I))$, we get $f\prec_w g$ and $g\prec f$. □

We provide using the next result that the weakened anti-symmetricity for the relation $\prec_s$ holds.

Lemma 2.4. [31, Lemma 3.7] *If $f\prec_s g$ and $g\prec_w f$, then $f\prec g$.*

Proof. Let $\{I_f^n : n\in\mathbb{N}\}$ be a family of disjoint finite subsets of I_f^+ for arbitrary chosen $f\in\ell^1(I)^+$ defined by (2.15). We define the sequence $(f_r)_{r\in\mathbb{N}}$ the same as in Theorem 2.4, by (2.16).

Let $f\prec_s g$ and $g\prec_w f$. Since $f\prec_s g$ implies $f\prec_w g$, there exists $P\in pP(\ell^1(I))$ for sets I_f^+ and I_g^+ such that $f = Pg$, by Theorem 2.4. Because of this, it is easy to see that $f_i = g_i$ and $\mathrm{card}(I_f^i) = \mathrm{card}(I_g^i)$, $\forall i\in\mathbb{N}$.

Since $f\prec_s g$, there is $D\in iDSS(\ell^1(I))$ such that $f = Dg$ so using Lemma 2.1 there exist two operators $D_1\in DS(\ell^1(I))$ and $D_2\in DSS(\ell^1(I))$ such that $D_1 = D + D_2$. We will provide that $D_2 g = 0$.

Let $i\in I_f^1$. Then

$$f_1 = f(i) = \sum_{j\in I} g(j)De_j(i) = \sum_{j\in I_g^1} g_1 De_j(i) + \sum_{j\in I\setminus I_g^1} g(j)De_j(i)$$

$$\leqslant \sum_{j\in I_g^1} g_1 De_j(i) + \sum_{j\in I\smallsetminus I_g^1} g_1 De_j(i) \leqslant g_1 = f_1.$$

It follows that $\sum_{j\in I_g^1} De_j(i) = 1$ and $\sum_{j\in I\smallsetminus I_g^1} De_j(i) = 0$, by $D \in iDSS(\ell^1(I))$. Thus, for every $i \in I_f^1$ we obtain $D_1e_j(i) = De_j(i)$ and $D_2e_j(i) = 0, \quad \forall j \in I$.

Since,

$$\operatorname{card}(I_g^1) = \operatorname{card}(I_f^1) = \sum_{i\in I_f^1}\sum_{j\in I_g^1} De_j(i) = \sum_{j\in I_g^1}\sum_{i\in I_f^1} De_j(i),$$

we get $\sum_{i\in I_f^1} De_j(i) = 1$, $\forall j \in I_g^1$ and $\sum_{i\in I\smallsetminus I_f^1} De_j(i) = 0$, $\forall j \in I_g^1$. Thus, $\forall j \in I_g^1$ we have $D_1e_j(i) = De_j(i)$ and $D_2e_j(i) = 0$, $\forall i \in I$.

Let $k \in I_f^2$. Using above arguments we obtain

$$\begin{aligned}
f_2 = f(k) &= \sum_{j\in I} g(j)De_j(k) \\
&= \sum_{j\in I_g^2} g_2De_j(k) + \sum_{j\in I_g^1} g(j)De_j(k) + \sum_{j\in I\smallsetminus\{I_g^1\cup I_g^2\}} g(j)De_j(k) \\
&= \sum_{j\in I_g^2} g_2De_j(k) + 0 + \sum_{j\in I\smallsetminus\{I_g^1\cup I_g^2\}} g(j)De_j(k) \\
&\leqslant \sum_{j\in I_g^2} g_2De_j(k) + \sum_{j\in I\smallsetminus\{I_g^1\cup I_g^2\}} g_2De_j(k) \leqslant g_2 = f_2.
\end{aligned}$$

Now, we get $\sum_{j\in I_g^2} De_j(k) = 1$ and $\sum_{j\in I\smallsetminus I_g^2} De_j(k) = 0$, by $D \in iDSS(\ell^1(I))$. Thus, $D_1e_j(k) = De_j(k)$ and $D_2e_j(k) = 0$, $\forall j \in I$, for each $k \in I_f^2$.

Similarly as above, using $\operatorname{card}(I_g^2) = \operatorname{card}(I_f^2)$ and changing the order of summation, we obtain

$$\operatorname{card}(I_g^2) = \sum_{j\in I_g^2}\sum_{i\in I_f^2} De_j(i),$$

so we conclude

$$\sum_{i\in I_f^2} De_j(i) = 1, \quad \forall j \in I_g^2 \quad \text{and} \quad \sum_{i\in I\smallsetminus I_f^2} De_j(i) = 0, \quad \forall j \in I_g^2.$$

Thus $\forall j \in I_g^2$ we have $D_1e_j(i) = De_j(i)$ and $D_2e_j(i) = 0$, $\forall i \in I$.

Continuing this process, we get for arbitrary chosen $n \in \mathbb{N}$ that

$$\begin{aligned}
&D_2e_j(k) = 0, \quad \forall k \in I_f^n,\ \forall j \in I, \\
&D_2e_j(i) = 0, \quad \forall j \in I_g^n,\ \forall i \in I,
\end{aligned} \tag{2.37}$$

hold. Finally, using (2.37) we obtain

$$D_2g(i) = \sum_{j\in I} g(j)D_2e_j(i) = \sum_{j\in I_g^+} g(j)D_2e_j(i) = \sum_{n=1}^{\infty}\sum_{j\in I_g^n} g(j)D_2e_j(i) = 0,$$

for each $i \in I$. Now, $f = Dg = (D_1 - D_2)g = D_1g$, that is $f \prec g$. ☐

As direct consequence of above presented results we get the following corollary that provides a weak anti-symmetricity for several majorization relations on $\ell^1(I)^+$.

We note that equivalence of statements i) and vi) is provided in [6, Theorem 3.5] independently of presented results in this paper.

Corollary 2.5. *The next statements are equivalent:*

i) $f \prec g$ *and* $g \prec f$;
ii) $f \prec_s g$ *and* $g \prec_s f$;
iii) $f \prec_s g$ *and* $g \prec_w f$;
iv) $f \prec g$ *and* $g \prec_w f$;
v) $f \prec^{ws} g$ *and* $g \prec^{ws} f$;
vi) *There exists* $P \in P(\ell^1(I))$ *such that* $g = Pf$.

Proof. Since $g \prec f$ implies $g \prec_s f$ and $g \prec_s f$ implies $g \prec_w f$, we get implications i) $\Rightarrow$ ii) $\Rightarrow$ iii).

Let statement iii) holds. Using Lemma 2.4 we get that $f \prec g$, so the statement iv) holds. Statement iv) implies vi) by Theorem 2.8.

Suppose that statement vi) is valid. Since, $g = Pf$ for $P \in P(\ell^1(I)) \subset DS(\ell^1(I))$ it follows that $f = P^{-1}g$, where $P^{-1} \in P(\ell^1(I)) \subset DS(\ell^1(I))$. Thus, i) holds. Statement v) is equivalent with vi) by Theorem 2.7. □

The next example shows that condition $f \prec_s g$ in the above corollary, statement iii) can not be replaced by $f \prec_w g$.

Example 2.5. [31] Let $f \in \ell^1(\mathbb{N})$ defined by $f(i) = 1/i^2$, $i \in \mathbb{N}$ that is, $f = \left(1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16} \dots\right)$. Using right shift operator R, we get that

$$g := Rf = \left(0, 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16} \dots\right).$$

Similarly, $f = Lg$. It follows that f and g are mutually weakly majorized so they are partial permutations of each other by Theorem 2.4. However, $g(1) = 0 \notin f(I)$. Thus there is no permutation $P \in \ell^1(I)$ to be $f = Pg$.

The next corollary ensures that submajorization $\prec_s$ and weak supermajorization $\prec^{ws}$ may be consider, in some sense, as partial orders on $\ell^1(I)^+$.

Corollary 2.6. *The submajorization "$\prec_s$" and the weak supermajorization "$\prec^{ws}$" are reflexive and transitive relations i.e. they are a pre-orders. If we identify all functions which are permutations of each other, then we may consider these relations as partial orders.*

Proof. Reflexivity is straightforward. Transitivity follows because sets $iDSS(\ell^1(I))$ and $DSPS(\ell^1(I))$ are closed under the composition. These facts are provided in [28, Theorem 3.7] and [31, Lemma 3.4]. If we identify all functions which are permutations of each other, then relations $\prec_s$ and $\prec^{ws}$ are anti-symmetric, by Corollary 2.5. □

3. Linear preservers of weak majorization relations on $\ell^1(I)^+$, when I is an infinite set

Firstly, in accordance with Definition 1.3 we reformulate definitions of linear preservers of weak majorization relations on $\ell^1(I)^+$. These definitions are introduced in papers [27, 29, 31].

Definition 3.1. A bounded linear operator $T\colon \ell^1(I) \to \ell^1(I)$ is called linear preserver

i) of weak majorization on $\ell^1(I)^+$, if T preserves the weak majorization relation, that is, $Tf \prec_w Tg$, whenever $f \prec_w g$, where $f, g \in \ell^1(I)^+$; [27, Definition 3.1]
ii) of submajorization on $\ell^1(I)^+$, if T preserves the submajorization relation, that is, $Tf \prec_s Tg$, whenever $f \prec_s g$, where $f, g \in \ell^1(I)^+$; [31, Definition 4.1]
iii) the weak supermajorization on $\ell^1(I)^+$, if T preserves the weak supermajorization relation, that is, $Tf \prec^{ws} Tg$, whenever $f \prec^{ws} g$, where $f, g \in \ell^1(I)^+$. [29, Definition 3.1]

The set of all linear preservers of the weak majorization ($\prec_w$), the submajorization ($\prec_s$) and the weak supermajorization ($\prec^{ws}$) on $\ell^1(I)^+$ are denoted by $\mathcal{P}_w(\ell^1(I)^+)$, $\mathcal{P}_s(\ell^1(I)^+)$ and $\mathcal{P}^{ws}(\ell^1(I)^+)$, respectively.

The first result gives basic properties of linear preservers of weak supermajorization, provided in [27, 29, 31].

Lemma 3.1. [27, 29, 31] *Let $\lambda \in \mathbb{R}, \lambda \geqslant 0$. Then*

i) $\lambda K \in \mathcal{P}_w(\ell^1(I)^+)$, *for each* $K \in \mathcal{P}_w(\ell^1(I)^+)$;
ii) $\lambda K \in \mathcal{P}_s(\ell^1(I)^+)$, *for each* $K \in \mathcal{P}_s(\ell^1(I)^+)$;
iii) $\lambda K \in \mathcal{P}^{ws}(\ell^1(I)^+)$, *for each* $K \in \mathcal{P}^{ws}(\ell^1(I)^+)$;
iv) $K_1K_2 \in \mathcal{P}_w(\ell^1(I)^+)$, *for each* $K_1, K_2 \in \mathcal{P}_w(\ell^1(I)^+)$;
v) $K_1K_2 \in \mathcal{P}_s(\ell^1(I)^+)$, *for each* $K_1, K_2 \in \mathcal{P}_s(\ell^1(I)^+)$;
vi) $K_1K_2 \in \mathcal{P}^{ws}(\ell^1(I)^+)$, *for each* $K_1, K_2 \in \mathcal{P}^{ws}(\ell^1(I)^+)$;
vii) *If* $K \in \mathcal{P}_w(\ell^1(I)^+)$, *then* $Ke_j(i) \geqslant 0$, $\forall i, j \in I$;
viii) *If* $K \in \mathcal{P}_s(\ell^1(I)^+)$, *then* $Ke_j(i) \geqslant 0$, $\forall i, j \in I$;
ix) *If* $K \in \mathcal{P}^{ws}(\ell^1(I)^+)$, *then* $Ke_j(i) \geqslant 0$, $\forall i, j \in I$.

Proof. Statement iii): Let $K \in \mathcal{P}^{ws}(\ell^1(I)^+)$ and let $\lambda \in \mathbb{R}$, $\lambda \geqslant 0$. Then $f \prec^{ws} g$ implies $Kf \prec^{ws} Kg$, i.e., there is $D \in DSPS(\ell^1(I))$ such that $Kf = DKg$. Since $\lambda Kf = \lambda DKg = D(\lambda Kg)$, we get $(\lambda K)f \prec^{ws} (\lambda K)g$, so $\lambda K \in \mathcal{P}^{ws}(\ell^1(I)^+)$.

Statement vi): Let $f \prec^{ws} g$ for some $f, g \in \ell^1(I)^+$. Since $K_2 \in \mathcal{P}^{ws}(\ell^1(I)^+)$ we get $K_2 f \prec^{ws} K_2 g$. Because $K_1 \in \mathcal{P}^{ws}(\ell^1(I)^+)$ it follows that $K_1K_2f \prec^{ws} K_1K_2g$.

Statement ix): We suppose contrary that there is at least one pair $i_0, j_0 \in I$ such that for the preserver K we have $\langle Ke_{j_0}, e_{i_0}\rangle < 0$. Since $e_{j_0} \prec^{ws} e_k$ does not imply $Ke_{j_0} \prec^{ws} Ke_k$ because $Ke_{j_0} \notin \ell^1(I)^+$, we have that K is not a linear preserver which is impossible. Thus, iv) holds.

The other statements may be provided using similar methods. □

In the sequel, we will consider a map $P_\theta\colon \ell^1(I) \to \ell^1(I)$ defined by

$$P_\theta(f) := \sum_{k\in I} f(k)e_{\theta(k)}, \quad f \in \ell^1(I), \tag{3.1}$$

where $\theta\colon I \to I$ is a one-to-one function. Obviously, P_θ is a bounded linear operator on $\ell^1(I)$ with norm $\|P\| = 1$. Moreover, if θ is a surjective, then P is a permutation.

We will present a result that for a doubly (substochastic or increasable substochastic) superstochastic operator D on $\ell^1(I)$ and for a family of operators P_θ defined by (3.1), which are determined by one-to-one functions θ with mutually disjoint images, there is at least one doubly (substochastic or increasable substochastic) superstochastic operator S such that $P_\theta D = SP_\theta$, for every θ.

Theorem 3.1. [26, 29, 31] *Suppose that*

$$\Theta := \{\theta_j\colon I \xrightarrow{1-1} I \mid j \in I_0,\ \theta_j(I) \cap \theta_j(I) = \emptyset,\ i \neq j\} \tag{3.2}$$

is a family of one-to-one maps on I with mutually disjoint images, where I_0 is at most a countable set. If

i) *$D \in DSS(\ell^1(I))$, then there is at least one $S \in DSS(\ell^1(I))$ such that $P_\theta D = SP_\theta$, $\forall\theta \in \Theta$;* [26, Theorem 3.2]
ii) *$D \in iDSS(\ell^1(I))$, then there is at least one $S \in iDSS(\ell^1(I))$ such that $P_\theta D = SP_\theta$, $\forall\theta \in \Theta$;* [31, Theorem 4.2]
iii) *$D \in DSPS(\ell^1(I))$, then there is at least one $S \in DSPS(\ell^1(I))$ such that $P_\theta D = SP_\theta$, $\forall\theta \in \Theta$.* [29, Theorem 3.3]

Proof. Proof of i): We define a family $\mathcal{S} = \{s_{ij} \mid i, j \in I\}$ to be

$$s_{ij} := \begin{cases} \langle De_{\theta^{-1}(j)}, e_{\theta^{-1}(i)}\rangle, & i, j \in \theta(I),\ \theta \in \Theta, \\ 0, & i \in \theta(I), j \notin \theta(I),\ \theta \in \Theta, \\ a, & i \notin \cup_{\theta\in\Theta}\theta(I),\ j = i, \\ 0, & i \notin \cup_{\theta\in\Theta}\theta(I),\ j \neq i, \end{cases}$$

where $0 \leqslant a \leqslant 1$ is arbitrary chosen. We obtain that $\sum_{j\in I} s_{ij} \leqslant 1$, $\forall i \in I$. Using this definition, the above family $\mathcal{S}$ may be represented in the similar way,

$$s_{ij} := \begin{cases} \langle De_{\theta^{-1}(j)}, e_{\theta^{-1}(i)}\rangle, & i, j \in \theta(I),\ \theta \in \Theta, \\ 0, & j \in \theta(I), i \notin \theta(I),\ \theta \in \Theta, \\ a, & j \notin \cup_{\theta\in\Theta}\theta(I),\ j = i, \\ 0, & j \notin \cup_{\theta\in\Theta}\theta(I),\ j \neq i. \end{cases}$$

It is easy to see that $\sum_{i\in I} s_{ij} \leqslant 1$, $\forall j \in I$. The family $\mathcal{S}$ satisfies conditions of Theorem 2.3. It follows that there is $S \in DSS(\ell^1(I))$ such that $[\langle Se_j, e_i\rangle = s_{ij}$, $\forall i, j \in I$. We will show that $P_\theta D = SP_\theta$, $\forall\theta \in \Theta$. Fix $\theta \in \Theta$. Since $D(e_k) = \sum_{i\in I}\langle De_k, e_i\rangle e_i$, we have

$$P_\theta D(e_k) = \sum_{i\in I}\langle De_k, e_i\rangle P_\theta(e_i) = \sum_{i\in I}\langle De_k, e_i\rangle e_{\theta(i)}. \tag{3.3}$$

On the other hand,

$$SP_\theta(e_k) = S(e_{\theta(k)}) = \sum_{r\in\theta(I)} s_{r,\theta(k)} e_r + \sum_{r\notin\theta(I)} s_{r,\theta(k)} e_r.$$

By the definition of S, we have that $s_{r,\theta(k)} = 0$, whenever $r \notin \theta(I)$, so we get

$$SP_\theta(e_k) = \sum_{r\in\theta(I)} s_{r,\theta(k)} e_r = \sum_{r\in\theta(I)} \langle De_k, e_{\theta^{-1}(r)}\rangle e_r = \sum_{i\in I}\langle De_k, e_i\rangle e_{\theta(i)}. \tag{3.4}$$

Combining (3.3) and (3.4), we conclude $SP_\theta(e_k) = P_\theta D(e_k)$, $\forall k \in I$. Choose arbitrary $f = \sum_{k\in I} f(k)e_k \in \ell^1(I)$. Now,

$$\begin{aligned} SP_\theta(f) &= SP_\theta\Big(\sum_{k\in I} f(k)e_k\Big) = \Big(\sum_{k\in I} f(k) SP_\theta(e_k)\Big) \\ &= \Big(\sum_{k\in I} f(k) P_\theta D(e_k)\Big) = P_\theta D(f). \end{aligned}$$

Proof of ii): Let $D \in iDSS(\ell^1(I))$. There are two operators $D_1 \in DS(\ell^1(I))$ and $D_2 \in DSS(\ell^1(I))$ such that

$$D_1 = D + D_2 \tag{3.5}$$

by Lemma 2.1. Now, using [6, Lemma 4.2] there exists operator $S_1 \in DS(\ell^1(I))$ such that $P_\theta D_1 = S_1 P_\theta$, $\forall \theta \in \Theta$. Actually, we can see in the proof of above mentioned theorem that operator S_1 is defined by

$$\langle S_1 e_j, e_i\rangle = \begin{cases} \langle D_1 e_{\theta^{-1}(j)}, e_{\theta^{-1}(i)}\rangle, & i, j \in \theta(I) \text{ for some } \theta \in \Theta, \\ 1, & i, j \notin \cup_{\theta\in\Theta}\theta(I) \text{ and } j = i, \\ 0, & \text{otherwise.} \end{cases}$$

Using statement $i)$ there is an operator $S_2 \in DSS(\ell^1(I))$ such that $P_\theta D_2 = S_2 P_\theta$, $\forall \theta \in \Theta$, defined by

$$\langle S_2 e_j, e_i\rangle = \begin{cases} \langle D_2 e_{\theta^{-1}(j)}, e_{\theta^{-1}(i)}\rangle, & i, j \in \theta(I), \text{ for some } \theta \in \Theta, \\ a, & i, j \notin \cup_{\theta\in\Theta}\theta(I), \text{ and } j = i, \\ 0, & \text{otherwise,} \end{cases}$$

where $0 \leqslant a \leqslant 1$. Clearly, operator S_2 is not uniquely determined. Let $S := S_1 - S_2$. The operator S has the representation

$$\langle Se_j, e_i\rangle = \begin{cases} \langle D_1 e_{\theta^{-1}(j)}, e_{\theta^{-1}(i)}\rangle - \langle D_2 e_{\theta^{-1}(j)}, e_{\theta^{-1}(i)}\rangle, & i, j \in \theta(I) \text{ for some } \theta \in \Theta, \\ 1 - a, & i, j \notin \cup_{\theta\in\Theta}\theta(I) \text{ and } j = i, \\ 0, & \text{otherwise.} \end{cases} \tag{3.6}$$

Obviously, $S \in iDSS(\ell^1(I))$ by the above representation (3.6) and the decomposition (3.5). Now,

$$P_\theta D = P_\theta(D_1 - D_2) = P_\theta D_1 - P_\theta D_2 = S_1 P_\theta - S_2 P_\theta = SP_\theta.$$

Proof of iii): Let $S = \{s_{ij} \mid i,j \in I\}$ to be a family defined by

$$s_{ij} := \begin{cases} \langle De_{\theta^{-1}(j)}, e_{\theta^{-1}(i)}\rangle, & i,j \in \theta(I), \text{ for some } \theta \in \Theta, \\ b, & i,j \notin \bigcup_{\theta\in\Theta}\theta(I), \ j = i, \\ 0, & \text{otherwise,} \end{cases} \tag{3.7}$$

where $b \geqslant 1$ is arbitrary chosen.

The family S satisfies the condition (2.3). Truly, if $j \in \theta(I)$ is arbitrary chosen, for some $\theta \in \Theta$, then using definition (3.7) we get

$$\sum_{i\in I}|s_{ij}| = \sum_{i\in\theta(I)}|s_{ij}| + \sum_{i\in I\setminus\theta(I)}|s_{ij}| = \sum_{i\in\theta(I)}\langle De_{\theta^{-1}(j)}, e_{\theta^{-1}(i)}\rangle \leqslant \sup_{j\in I}\sum_{i\in I}|d_{ij}| < \infty,$$

because operator D satisfies conditions (2.3). If $j \in I \setminus \bigcup_{\theta\in\Theta}\theta(I)$ then $\sum_{i\in I}|s_{ij}| = b \geqslant 1$, thus it follows that $\sup_{j\in I}\sum_{i\in I}|s_{ij}| < \infty$. We conclude that the family S may be considered as bounded linear operator on $\ell^1(I)$ defined by (2.5), by Theorem 2.1. Using (3.7) we obtain that $\sum_{j\in I} s_{ij} \geqslant 1$, $\forall i \in I$ and $\sum_{i\in I} s_{ij} \geqslant 1$, $\forall j \in I$.

We have to show that there is a doubly stochastic operator $\tilde{S} \in DS(\ell^1(I))$ such that $s_{ij} \geqslant \langle \tilde{S}e_j, e_i\rangle$, $\forall i,j \in I$, hence it implies that S is a doubly superstochastic. Because $D \in DSPS(\ell^1(I))$, we get there is a operator $\tilde{D} \in DS(\ell^1(I))$ with $\langle De_j, e_i\rangle \geqslant \langle \tilde{D}e_j, e_i\rangle$, $\forall i,j \in I$.

We define a family $\tilde{D} = \{\tilde{d}_{ij} \mid i,j \in I\}$ to be

$$\tilde{s}_{ij} := \begin{cases} \langle \tilde{D}e_{\theta^{-1}(j)}, e_{\theta^{-1}(i)}\rangle, & i,j \in \theta(I), \text{ for some } \theta \in \Theta, \\ 1, & i,j \notin \bigcup_{\theta\in\Theta}\theta(I), \ j = i, \\ 0, & \text{otherwise,} \end{cases} \tag{3.8}$$

We get that the family $\tilde{S}$ defines a doubly stochastic operator on $\ell^1(I)$ defined by (2.5), because it is easy to see that $(\forall i \in I)\ \sum_{j\in I}\tilde{s}_{ij} = 1$ and $(\forall j \in I)\ \sum_{i\in I}\tilde{s}_{ij} = 1$. Next, if $i,j \in \theta(I)$ for some $\theta \in \Theta$, then by above definitions (3.7) and (3.8) we have

$$s_{ij} = \langle De_{\theta^{-1}(j)}, e_{\theta^{-1}(i)}\rangle \geqslant \langle \tilde{D}e_{\theta^{-1}(j)}, e_{\theta^{-1}(i)}\rangle = \tilde{s}_{ij}, \quad \text{and} \quad s_{ii} = b \geqslant 1 = \tilde{s}_{ii},$$

when $i \notin \bigcup_{\theta\in\Theta}\theta(I)$. Hence $S \in DSPS(\ell^1(I))$.

We will show that $P_\theta D = DP_\theta$, $\forall\theta \in \Theta$. Choose an arbitrary function $\theta \in \Theta$. We get

$$DP_\theta(e_j) = D(e_{\theta(j)}) = \sum_{l\in\theta(I)} d_{l,\theta(j)}e_l + \sum_{l\notin\theta(I)} d_{l,\theta(j)}e_l.$$

Using the definition of the operator D, we have that $d_{l,\theta(j)} = 0$, when $l \notin \theta(I)$, so we conclude that

$$DP_\theta(e_j) = \sum_{l\in\theta(I)} d_{l,\theta(j)}e_l = \sum_{l\in\theta(I)}\langle De_j, e_{\theta^{-1}(l)}\rangle e_l = \sum_{i\in I} De_j(i)e_{\theta(i)}. \tag{3.9}$$

Also, using $D(e_j) = \sum_{i\in I} De_j(i)e_i$, we obtain

$$P_\theta D(e_j) = \sum_{i\in I} De_j(i)P_\theta(e_i) = \sum_{i\in I} De_j(i)e_{\theta(i)}. \tag{3.10}$$

Combining (3.9) and (3.10), we get $DP_\theta(e_j) = P_\theta D(e_j)$, $\forall j \in I$. Then

$$DP_\theta(f) = DP_\theta\left(\sum_{j\in I} f(j)e_j\right) = \left(\sum_{j\in I} f(j)DP_\theta(e_j)\right)$$
$$= \left(\sum_{j\in I} f(j)P_\theta D(e_j)\right) = P_\theta D(f),$$

for arbitrary choosen function $f = \sum_{j\in I} f(j)e_j \in \ell^1(I)$. □

The next result is provided in [26] and [29] which gives one possible form of considered linear preservers.

Theorem 3.2. [26, Theorem 3.3], [29, Theorem 3.4] *Let I_0 be at most countable subset of an infinite set I. Suppose that Θ is a family of one-to-one maps on I with disjoint images, defined by* (3.2). *If $\lambda \in \ell^1(I_0)^+$, then*

$$T := \sum_{j\in I_0} \lambda_j P_{\theta_j} \in \mathcal{P}^{ws}(\ell^1(I)^+) \quad \cap \mathcal{P}_w(\ell^1(I)^+). \tag{3.11}$$

Proof. Let $T = \sum_{j\in I_0} \lambda_j P_{\theta_j}$ for some $f \in \ell^1(I)$. We claim that T is bounded. Since the family Θ is contains only functions with disjoint images, we get

$$\|Tf\| = \sum_{j\in I_0}\sum_{i\in I}(\lambda_j f(i)) = \|f\| \sum_{j\in I_0} \lambda_j = \|\lambda\|\|f\|.$$

We conclude that T is a bounded linear operator on $\ell^1(I)$ with norm $\|T\| = \|\lambda\|$.

Let $f \prec_w g$, that is, $f = Dg$ for some $D \in DSS(\ell^1(I))$. Now, there is a $S \in DSS(\ell^1(I))$ such that $P_\theta D = SP_\theta$, $\forall\theta \in \Theta$, by Theorem 3.1. We get

$$\begin{aligned} Tf &= \sum_{k\in I_0} \lambda_k P_{\theta_k}(f) = \sum_{k\in I_0} \lambda_k P_{\theta_k}(Dg) \\ &= \sum_{k\in I_0} \lambda_k SP_{\theta_k}(g) = S\left(\sum_{k\in I_0} \lambda_k P_{\theta_k}(g)\right) = S(Tg). \end{aligned} \tag{3.12}$$

It follows that $Tf \prec_w Tg$, that is $T \in \mathcal{P}_w(\ell^1(I))^+$.

Suppose that $f \prec^{ws} g$. It follows that $f = Dg$ for some $D \in DSPS(\ell^1(I))$. There is an operator $S \in DSPS(\ell^1(I))$ such that $P_\theta D = SP_\theta$, for each $\theta \in \Theta$, by Theorem 3.1. Now, using linearity and continuity of operator D, we obtain using (3.12) that $Tf \prec^{ws} Tg$, so $T \in \mathcal{P}^{ws}(\ell^1(I)^+$. □

Previously, we showed that operators defined by (3.11) are linear preservers of weak majorization and of weak supermajorization on $\ell^1(I)^+$, by Theorem 3.2. The set of all operators (3.11) we will denote by $\mathcal{A}(\ell^1(I)^+)$. We can find linear preservers of these two relation which do not have form (3.11).

Example 3.1. [27, Example 3.1], [29, Example 3.6]Suppose that $h \in \ell^1(I)^+$ and let

$$T_h(f) := h\sum_{i\in I} f(i), \quad \forall f \in \ell^1(I). \tag{3.13}$$

Obviously, T_h is a bounded linear operator with norm $\|T_h\| = \|h\|$. Since $T_h(e_j) = h$, $\forall j \in I$, we say that columns of T_h are mutually equal.

Suppose that $f \prec_w g$ and $f_1 \prec^{ws} g_1$, where $f, f_1, g, g_1 \in \ell^1(I)^+$. It follows that $f = Dg$ and $f_1 = D_1 g_1$, where $D \in DSS(\ell^1(I))$ and $D_1 \in DSPS(\ell^1(I))$. It is easy to see that $\|f\| = \sum_{i\in I} f(i) \leqslant \sum_{i\in I} g(i) = \|g\|$. Truly, changing the order of summation, we obtain

$$\|f\| = \sum_{i\in I} f(i) = \sum_{i\in I}\sum_{j\in I} g(j)\langle De_j, e_i\rangle = \sum_{j\in I} g(j) \sum_{i\in I} \langle De_j, e_i\rangle \leqslant \|g\|.$$

Now, $T_h(f) = h\|f\| \leqslant h\|g\| = T_h(g)$. Let $m = \frac{\|f\|}{\|g\|}$. Then $R := m\mathcal{I} \in DSS(\ell^1(I))$ (where $\mathcal{I}$ stands for the identity operator) satisfies $T_h(f) = RT_h(g)$, thus $T \in \mathcal{P}_w(\ell^1(I)^+)$.

Similarly as above, using the definition of operator D_1, we have that there is an operator $\widetilde{D_1} \in DS(\ell^1(I))$ such that $D_1 e_j(i) \geqslant \widetilde{D_1} e_j(i)$, so changing the order of summation we get

$$\begin{aligned} \|f\| &= \sum_{i\in I} f(i) = \sum_{i\in I}\sum_{j\in I} g(j) D_1 e_j(i) \\ &\geqslant \sum_{i\in I}\sum_{j\in I} g(j)\widetilde{D_1} e_j(i) = \sum_{j\in I} g(j) \sum_{i\in I} \widetilde{D_1} e_j(i) = \|g\|. \end{aligned} \tag{3.14}$$

Thus, $T_h(f) = h\|f\| \geqslant h\|g\| = T_h(g)$. Let $\alpha = \frac{\|f\|}{\|g\|} \geqslant 1$. Let $Q := \alpha\mathcal{I} \in DSPS(\ell^1(I))$. Since

$$QT_h(g) = \|g\|Qh = \alpha\|g\|h = \|f\|h = T_h(f),$$

we get $T_h(f) \prec^{ws} T_h(g)$, that is, $T_h \in \mathcal{P}^{ws}(\ell^1(I)^+)$.

Similarly we may provide that $T \in \mathcal{P}_s(\ell^1(I)^+)$.

We denote by $\mathcal{B}(\ell^1(I)^+)$ the set of all bounded linear operators defined by (3.13). Clearly, $\mathcal{B}(\ell^1(I)^+)$ is a convex cone. The last example shows that operator T_h defined by (3.13) preserves relations $\prec_w$, $\prec_s$ and $\prec^{ws}$, that is $\mathcal{B}(\ell^1(I)^+)$ is a subset of any of sets $\mathcal{P}_w(\ell^1(I)^+)$, $\mathcal{P}_s(\ell^1(I)^+)$ and $\mathcal{P}^{ws}(\ell^1(I)^+)$ and it has different structure than preservers from class $\mathcal{A}(\ell^1(I)^+)$. Also, it is easy to see that the set $\mathcal{A}(\ell^1(I)^+) \cap \mathcal{B}(\ell^1(I)^+)$ contains only null operator.

The next example shows that the sum of two operators from two different classes $\mathcal{A}(\ell^1(I)^+)$ and $\mathcal{B}(\ell^1(I)^+)$ is not necessarily in $\mathcal{P}_w(\ell^1(I)^+) \cup \mathcal{P}_s(\ell^1(I)^+) \cup \mathcal{P}^{ws}(\ell^1(I)^+)$.

Example 3.2. [27, Example 3.2], [29, Example 3.7] Suppose that operator T_{e_r} is defined by (3.13), where $h = e_r$. Clearly, $e_r \prec_w e_k$, $e_r \prec_s e_k$ and $e_r \prec^{ws} e_k$, for all $k, r \in I$. We note that identity operator $\mathcal{I}$ is in $\mathcal{A}(\ell^1(I)^+)$, obviously. Also, $T_{e_r}(e_k) = e_r$, $\forall k \in I$. Now, $4(T_{e_r} + \mathcal{I})(e_r) = 2e_r$. On the other hand, $(T_{e_r} + \mathcal{I})(e_k) = e_r + e_k$. However, $2e_r \not\prec_w e_r + e_k$ and $2e_r \not\prec_s e_r + e_k$, when $k \neq r$. Truly, it has to be

$$\langle 2e_r, e_r\rangle = 2 = \langle De_r, e_r\rangle + \langle De_k, e_r\rangle,$$

thus, $D \notin DSS(\ell^1(I))$, that is $T_{e_r} + \mathcal{I} \notin \mathcal{P}_w(\ell^1(I)^+) \cup \mathcal{P}_s(\ell^1(I)^+)$.

We claim that $2e_r \not\prec^{ws} e_r + e_k$. Suppose contrary that there is an operator $D \in DSPS(\ell^1(I))$ such that $D(e_r+e_k) = 2e_r$. It follows that $De_r(r)+De_k(r) = 2$. Since $2e_r(t) = 0$, for each $t \neq r$, we get

$$De_r(t) = De_k(t) = 0, \quad \text{for every} \quad t \in I \setminus \{r\}. \tag{3.15}$$

It has to be

$$De_r(r) = De_k(r) = 1. \tag{3.16}$$

However, using (3.15) and (3.16) we conclude that there is no $\tilde{D} \in DS(\ell^1(I))$ such that $De_j(i) \geqslant \tilde{D}e_j(i)$, $\forall i, j \in I$, so we have a contradiction, thus $D \notin DSPS(\ell^1(I))$, so $T_{e_r} + \mathcal{I} \notin \mathcal{P}^{ws}(\ell^1(I)^+)$.

The next result gives sufficient conditions that the sum of two arbitrary chosen bounded linear operators form two disjoint classes $\mathcal{A}(\ell^1(I)^+)$ and $\mathcal{B}(\ell^1(I)^+)$ is a linear preserver of weak majorization and weak supermajorization on $\ell^1(I)^+$.

Theorem 3.3. [27, Theorem 3.2], [29, Theorem 3.8] *Let I be an infinite set. If $A \in \mathcal{A}(\ell^1(I)^+)$ and $B \in \mathcal{B}(\ell^1(I)^+)$ are chosen to be*

$$Af(i_2) = Bf(i_1) = 0, \quad \forall i_1 \in I_1, \quad \forall i_2 \in I_2, \quad \forall f \in \ell^1(I)^+, \tag{3.17}$$

where $I_1, I_2 \subset I$, $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = I$, then $A+B \in \mathcal{P}_w(\ell^1(I)^+) \cap \mathcal{P}^{ws}(\ell^1(I)^+)$.

Proof. Choose an arbitrary operators $A \in \mathcal{A}(\ell^1(I)^+)$ and $B \in \mathcal{B}(\ell^1(I)^+)$, such that (3.17) holds. It follows that the operator A has the form $A = \sum_{j \in I_0} \lambda_j P_{\theta_j}$, where family Θ is presented in (3.2). Without lose of generality, we suppose that $\lambda_j > 0$, $\forall j \in I_0$. Now,

$$\begin{aligned} Af(i_2) &= \sum_{i \in I} f(i) Ae_i(i_2) = \sum_{i \in I} f(i) \sum_{j \in I_0} \lambda_j P_{\theta_j} e_i(i_2) \\ &= \sum_{i \in I} f(i) \sum_{j \in I_0} \lambda_j e_{\theta_j(i)}(i_2), \quad \forall f \in \ell^1(I). \end{aligned}$$

Using the theorem assumption $Af(i_2) = 0$ we get that

$$e_{\theta_j(i)}(i_2) = \langle e_{\theta_j(i)}, e_{i_2} \rangle = 0, \quad \forall i \in I, \quad \forall j \in I_0, \quad \forall i_2 \in I_2,$$

so $\left(\bigcup_{j \in I_0} \theta_j(I) \right) \cap I_2 = \emptyset$.

Suppose that $f \prec_w g$, for some $f, g \in \ell^1(I)$, that is, $f = Dg$, where $D \in DSS(\ell^1(I))$. Now, there is a $\tilde{D} \in DSS(\ell^1(I))$ such that $P_{\theta_k} D = \tilde{D} P_{\theta_k}$, $\forall k \in I_0$, by Theorem 3.1. It is easy to conclude that $AD = \tilde{D}A$.

We can find in the proof of the Theorem 3.1 the definition of this operator $\tilde{D}$ in the following form

$$\langle \tilde{D}e_j, e_i \rangle = \begin{cases} \langle De_{\theta^{-1}(j)}, e_{\theta^{-1}(i)} \rangle, & i, j \in \theta(I),\ \theta \in \Theta, \\ 0, & i \in \theta(I), j \notin \theta(I),\ \theta \in \Theta, \\ a, & i \notin \cup_{\theta \in \Theta} \theta(I),\ j = i, \\ 0, & i \notin \cup_{\theta \in \Theta} \theta(I),\ j \neq i, \end{cases}$$

and

$$\langle \tilde{D}e_j, e_i\rangle = \begin{cases} \langle De_{\theta^{-1}(j)}, e_{\theta^{-1}(i)}\rangle, & i,j \in \theta(I),\ \theta\in\Theta, \\ 0, & j\in\theta(I), i\notin\theta(I),\ \theta\in\Theta, \\ a, & j\notin \cup_{\theta\in\Theta}\theta(I),\ j=i, \\ 0, & j\notin \cup_{\theta\in\Theta}\theta(I),\ j\neq i. \end{cases}$$

Obviously, operator $\tilde{D}$ is not unique. Also, as we know, $f = Dg$ implies

$$\|f\| = \sum_{i\in I} f(i) \leqslant \sum_{i\in I} g(i) = \|g\|,$$

therefore, choosing $a := \frac{\|f\|}{\|g\|} \leqslant 1$ we get

$$BDg = Bf = h\sum_{i\in I} f(i) = ah\sum_{i\in I} g(i) = aBg,$$

for some $h \in \ell^1(I)^+$. Also, using the above definition of the operator $\tilde{D}$, we get that $\tilde{D}e_j = ae_j$, $\forall j\in I_2$ which implies $\tilde{D}Bl = aBl$, for every $l\in\ell^1(I)^+$. Therefore,

$$\begin{aligned} \tilde{D}(A+B)g &= \tilde{D}Ag + \tilde{D}Bg = \tilde{D}Ag + aBg = ADg + BDg \\ &= (A+B)Dg = (A+B)f. \end{aligned}$$

Thus, $(A+B)f \prec_w (A+B)g$, so $T := A+B \in \mathcal{P}_w(\ell^1(I)^+)$.

Suppose that $f \prec^{ws} g$, for some $f, g\in\ell^1(I)^+$. It follows that $f = Qg$, where $Q\in DSPS(\ell^1(I))$. There is at least one operator $D\in DSPS(\ell^1(I))$ such that $P_{\theta_j}Q = DP_{\theta_j}$, $\forall j\in I_0$ by Theorem 3.1, and $AQ = DA$, by (3.12) . Actually, the operator D is not unique, which is obvious by its definition:

$$d_{ij} := \begin{cases} \langle Qe_{\theta^{-1}(j)}, e_{\theta^{-1}(i)}\rangle, & i,j\in\theta(I),\ \text{for some } \theta\in\Theta, \\ b, & i,j\notin \bigcup_{\theta\in\Theta}\theta(I),\ j=i, \\ 0, & \text{otherwise}, \end{cases}$$

(see proof of Theorem 3.1). Using (3.14) and choosing $b := \frac{\|f\|}{\|g\|} \geqslant 1$, we get

$$BQg = Bf = h\sum_{j\in I} f(j) = bh\sum_{j\in I} g(j) = bBg,$$

for some $h\in\ell^1(I)^+$. Also $De_j = be_j$, $\forall j\in I_2$, by definition of D, so using (3.17) we obtain $DB\tilde{f} = \sum_{j\in I_2} B\tilde{f}(j)De_j = bB\tilde{f}$ for every $\tilde{f}\in\ell^1(I)^+$. Now, we conclude

$$\begin{aligned} D(A+B)g &= DAg + DBg = DAg + bBg = AQg + BQg \\ &= (A+B)Qg = (A+B)f. \end{aligned}$$

It follows that $(A+B)f \prec^{ws} (A+B)g$, thus $A+B\in\mathcal{P}^{ws}(\ell^1(I)^+)$. □

In the rest of the paper, our aim is to prove the opposite direction of the last theorem, that is, to prove that every linear preserver of weak majorization and weak supermajorization on $\ell^1(I)^+$ may be represent by the sum of two operators from classes $A\in\mathcal{A}(\ell^1(I)^+)$ and $B\in\mathcal{B}(\ell^1(I)^+)$, which satisfies conditions (3.17). For this purpose, we need the following results.

Lemma 3.2. [27, Lemma 3.1] *Let $u = \{u_j\} \in \mathbb{R}^n$ and let $\{u_{ij} \mid i \in I_0, j = 1, \ldots, n\}$ be a family of real numbers, where I_0 is at most a countable index set. If*

$$\sum_{j=1}^{n} \alpha_j u_j \in \left\{ \sum_{j=1}^{n} \alpha_j u_{ij} \mid i \in I_0 \right\}, \tag{3.18}$$

for all $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_j > 0$ for each $j = 1, \ldots, n$, then there exists $k \in I_0$ such that $u_j = u_{kj}$, for each $j = 1, \ldots, n$.

Proof. Let $\in (\mathbb{R}^{++})^n := \{(\alpha_1, \alpha_2, \ldots, \alpha_n) \mid \alpha_j > 0, \forall j = 1, \ldots, n\}$, and denote $u_i := (u_{i1}, u_{i2}, \ldots, u_{in})$, for all $i \in I_0$. Using the standard inner product $(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, we may reformulate (3.18) in the following way:

$$(\alpha, u) \in \{(\alpha, u_i) \mid i \in I_0\}, \quad \forall \alpha \in (\mathbb{R}^{++})^n.$$

For each $\alpha \in (\mathbb{R}^{++})^n$, there is a $k \in I_0$ such that $\alpha \in (u - u_k)^\perp$, where $v^\perp$ stands for the set of all orthogonal vectors of the vector v, i.e., $v^\perp := \{w \in \mathbb{R}^n \mid (v, w) = 0\}$. Now, we obtain $(\mathbb{R}^{++})^n \subseteq \bigcup_{i \in I_0} (u - u_i)^\perp$.

Assume contrary $u \neq u_i$, for each $i \in I_0$. It follows that $(u - u_i)^\perp$ is a proper subspace of the Banach space $\mathbb{R}^n$, for every $i \in I_0$. We recall that all these proper subspaces are nowhere dense sets in $\mathbb{R}^n$.

Suppose that $x = (x_1, x_2, \ldots, x_n) \in (\mathbb{R}^{++})^n$, and we define $r > 0$ to be $r := \min\{x_1, x_2, \ldots, x_n\}$. We get that the ball $K := B(x, \frac{r}{2}) \subset \mathbb{R}^n$ is also contained in $(\mathbb{R}^{++})^n \subseteq \bigcup_{i \in I_0} (u - u_i)^\perp$. This is a contradiction with respect to the Baire category theorem, because at the most a countable union of nowhere dense sets in the Banach space $\mathbb{R}^n$ cannot contain a ball. □

The next result stands that for every "row" of linear preserver of weak majorization on $\ell^1(I)^+$ and for finite number of elements of this "row", where at least one is non-zero, there is another one "row" such that this elements appear on same places, but permuted.

Lemma 3.3. [27, Lemma 3.2] *Let $T \colon \ell^1(I) \to \ell^1(I) \in \mathcal{P}_w(\ell^1(I)^+)$ and define $J := \{j_1, j_2, \ldots, j_n\}$, where $j_i \in I$, $i = 1, \ldots, n$ are arbitrary chosen. Suppose that $\delta \colon J \to J$ is a bijection. For every $r \in I$ for which there is at least one $j_0 \in J$ such that $\langle Te_{j_0}, e_r \rangle > 0$, there exists $s \in I$ such that*

$$\langle Te_j, e_r \rangle = \langle Te_{\delta(j)}, e_s \rangle, \quad \forall j \in J. \tag{3.19}$$

Proof. Choose $\alpha = (\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_n}) \in (R^{++})^n$. Since,

$$\sum_{j \in J} \alpha_j e_j \prec_w \sum_{j \in J} \alpha_j e_{\delta(j)} \quad \text{and} \quad \sum_{j \in J} \alpha_j e_{\delta(j)} \prec_w \sum_{j \in J} \alpha_j e_j$$

we obtain

$$f \prec_w g \quad \text{and} \quad g \prec_w f,$$

where

$$f := \sum_{j \in J} \alpha_j T e_j \quad \text{and} \quad g := \sum_{j \in J} \alpha_j T e_{\delta(j)}.$$

Obviously, $r \in I_f^+$ and $f(r) = \sum_{j\in J} \alpha_j \langle Te_j, e_r\rangle > 0$, by the lemma assumption. Now, there is $P \in pP(\ell^1(I))$ for positive elements of f and g with indexes in I_f^+ and I_g^+, respectively, by Theorem 2.4. More precisely,

$$\sum_{j\in J} \alpha_j \langle Te_j, e_r\rangle \in \Big\{ \sum_{j\in J} \alpha_j \langle Te_{\delta(j)}, e_i\rangle \mid i \in I_g^+ \Big\}.$$

Since $\mathrm{card}(I_g^+) \leqslant \aleph_0$, using Lemma 3.2 we obtain that there is a $s \in I_g^+$ such that (3.19) holds. □

Similarly as above we may provide the next result related to preservers of $\prec_s$ and $\prec^{ws}$.

Lemma 3.4. [29, Lemma 3.9], [31, Lemma 4.7] *Suppose that J is a finite subset of I and let $\delta\colon J \to J$ be a bijection. If*

$$T \in \mathcal{P}^s(\ell^1(I)^+) \cup \mathcal{P}^{ws}(\ell^1(I)^+) \tag{3.20}$$

then for every $u \in I$ there exists $v \in I$ such that

$$Te_m(u) = Te_{\delta(m)}(v), \quad \forall m \in J. \tag{3.21}$$

Proof. Let $\mathrm{card}(J) = n \in \mathbb{N}$.

Clearly, functions $\sum_{m\in J} a_m e_m$ and $\sum_{m\in J} a_m e_{\delta(m)}$ are mutually majorized by two majorization relations $\prec_s$ and $\prec^{ws}$, so using the lemma assumption (3.20) we get

$$\sum_{m\in J} a_m Te_m \prec_s \sum_{m\in J} a_m Te_{\delta(m)} \quad \text{and} \quad \sum_{m\in J} a_m Te_{\delta(m)} \prec_s \sum_{m\in J} a_m Te_m$$

or

$$\sum_{m\in J} a_m Te_m \prec^{ws} \sum_{m\in J} a_m Te_{\delta(m)} \quad \text{and} \quad \sum_{m\in J} a_m Te_{\delta(m)} \prec^{ws} \sum_{m\in J} a_m Te_m$$

for every $a = (a_{m_1}, a_{m_2}, \dots, a_{m_n}) \in (\mathbb{R}^{++})^n$. Now, using Corollary 2.5, functions $h := \sum_{m\in J} a_m Te_m$ and $h_\delta := \sum_{m\in J} a_m Te_{\delta(m)}$ are different up to the permutation, i.e.,

$$\sum_{m\in J} a_m Te_m(u) \in \Big\{ \sum_{m\in J} a_m Te_{\delta(m)}(k) \mid k \in I_0 \Big\},$$

where $I_0 := I_{h_\delta}^+ \cup \{r\}$ with $r \in I_{h_\delta}^0$. Since $I_{h_\delta}^+$ is a countable set, using Lemma 3.2 we get that there is a $v \in I$ such that (3.21) holds. □

The next lemma claims that if one "row" of a preserver contains more than one non-zero elements, then these elements are the same.

Lemma 3.5. [27, Lemma 3.3], [29, Lemma 3.10], [31, Lemma 4.8] *Let*

$$T \in \mathcal{P}_w(\ell^1(I)^+) \cup \mathcal{P}_s(\ell^1(I)^+) \cup \mathcal{P}^{ws}(\ell^1(I)^+)$$

where I is an infinite set. If there are two distinct $n, m \in I$ such that $\langle Te_n, e_r\rangle > 0$ and $\langle Te_m, e_r\rangle > 0$ for some $r \in I$, then $\langle Te_n, e_r\rangle = \langle Te_m, e_r\rangle$.

Proof. Suppose that there exist $m, n, r \in I$ such that $\langle Te_m, e_r\rangle > 0$, $\langle Te_n, e_r\rangle > 0$ and

$$\langle Te_m, e_r\rangle \neq \langle Te_n, e_r\rangle \tag{3.22}$$

Let $a_1, a_2 > 0$ and let $l \in I \setminus \{m, n\}$ be arbitrary chosen. Clearly, functions $a_1e_m + a_2e_n$ and $a_1e_m + a_2e_l$ are mutually majorized by all three majorization relations $\prec_w$, $\prec_s$ and $\prec^{ws}$.

Because $T \in \mathcal{P}_w(\ell^1(I)^+) \cup \mathcal{P}_s(\ell^1(I)^+) \cup \mathcal{P}^{ws}(\ell^1(I)^+)$ we obtain

$$a_1Te_m + a_2Te_n \prec_w a_1Te_m + a_2Te_l \quad \text{and} \quad a_1Te_m + a_2Te_l \prec_w Ta_1e_m + a_2Te_n,$$

or

$$a_1Te_m + a_2Te_n \prec_s a_1Te_m + a_2Te_l \quad \text{and} \quad a_1Te_m + a_2Te_l \prec_s Ta_1e_m + a_2Te_n$$

or

$$a_1Te_m + a_2Te_n \prec_{ws} a_1Te_m + a_2Te_l \quad \text{and} \quad a_1Te_m + a_2Te_l \prec_{ws} Ta_1e_m + a_2Te_n.$$

It follows that

$$a_1\langle Te_m, e_r\rangle + a_2\langle Te_n, e_r\rangle \in \{a_1\langle Te_m, e_j\rangle + a_2\langle Te_l, e_j\rangle \mid j \in I\}.$$

by Theorem 2.4 and Corollary 2.5.

Since $a_1Te_m + a_2Te_l \in \ell^1(I)^+$, the above set is at most countable, so using Lemma 3.2 for $n = 2$ we get that for l there is $k \in I$ such that

$$\langle Te_m, e_k\rangle = \langle Te_m, e_r\rangle \quad \text{and} \quad \langle Te_l, e_k\rangle = \langle Te_n, e_r\rangle. \tag{3.23}$$

On the other hand, it is clear that for fixed $c \in \mathbb{R}$, $c \neq 0$ holds

$$\operatorname{card}\{i \in I \mid \langle Te_m, e_i\rangle = \mathrm{c}\} < \aleph_0. \tag{3.24}$$

Since I is an infinite set and $l \in I \setminus \{m, n\}$ is arbitrary chosen, using (3.23) and (3.24) there is $s \in I$ and there is a sequence $(t_i)_{i\in\mathbb{N}}$ of distinct elements $t_i \in I$ such that

$$\langle Te_m, e_s\rangle = \langle Te_m, e_r\rangle \quad \text{and} \quad \langle Te_{t_i}, e_s\rangle = \langle Te_n, e_r\rangle, \quad \forall i \in \mathbb{N}.$$

Let $\{\Phi_j\}_{j\in\mathbb{N}}$ be a family where $\Phi_j := \{t_1, t_2, \dots t_j\}$ for every $j \in \mathbb{N}$. We define bijections $\gamma_j : \{t_1, t_2, \dots t_j\} \cup \{m\} \to \{t_1, t_2, \dots t_j\} \cup \{m\}$ in the following way:

$$\gamma_j(x) := \begin{cases} t_j, & x = m, \\ m, & x = t_j \\ x, & x \in \{t_1, t_2, \dots t_{j-1}\}. \end{cases}$$

For each $j \in \mathbb{N}$ there exists $r_j \in I$ such that

$$\langle Te_m, e_{r_j}\rangle = \langle Te_{\gamma_j(t_j)}, e_{r_j}\rangle = \langle Te_{t_j}, e_s\rangle = \langle Te_n, e_r\rangle, \tag{3.25}$$

$$\langle Te_{t_j}, e_{r_j}\rangle = \langle Te_{\gamma_j(m)}, e_{r_j}\rangle = \langle Te_m, e_s\rangle = \langle Te_m, e_r\rangle, \tag{3.26}$$

by Lemma 3.4. Also, for each $x \in \{t_1, t_2, \dots t_{j-1}\}$ we get

$$\langle Te_x, e_{r_j}\rangle = \langle Te_{\gamma_j(x)}, e_{r_j}\rangle = \langle Te_x, e_s\rangle = \langle Te_n, e_r\rangle, \tag{3.27}$$

again by Lemma 3.4.

If we provide that the set $\{r_j \mid j \in \mathbb{N}\}$ is countable, we obtain using (3.25) that $\langle Te_m, e_{r_j}\rangle = \langle Te_n, e_r\rangle$, for all $j \in \mathbb{N}$ which is a contradiction with (3.24).

Let $k_1 < k_2$ for some integers k_1, k_2 and suppose that $r_{k_1} = r_{k_2}$. Since bijections γ_{k_1} and γ_{k_2} are different, using (3.26) for γ_{k_1} we obtain $\langle Te_{t_{k_1}}, e_{r_{k_1}}\rangle = \langle Te_m, e_r\rangle$. Because $k_1 < k_2$ we get using (3.27) for γ_{k_2} that

$$\langle Te_{t_{k_1}}, e_{r_{k_1}}\rangle = \langle Te_{t_{k_1}}, e_{r_{k_2}}\rangle = \langle Te_n, e_r\rangle.$$

Combine above two facts, we get $\langle Te_m, e_r\rangle = \langle Te_{t_{k_1}}, e_{r_{k_1}}\rangle = \langle Te_n, e_r\rangle$ which is a contradiction with the assumption at the beginning of the proof (3.22). Thus, $r_{k_1} \neq r_{k_2}$ for all $k_1, k_2 \in \mathbb{N}$, so the set $\{r_j \; j \in I\}$ is a countable. □

3.1. Linear preservers of weak majorization and weak supermajorization on $\ell^1(I)^+$, when I is an infinite set. The goal of this part is to prove Theorem 3.6 which gives the proper form of two considered linear preservers on $\ell^1(I)^+$. In order to do this, we need the next results provided in [27, 29].

The next lemma claims that if one "row" of a preserver of considered relations contains more than one non-zero elements, then all elements in this "row" are the same.

Lemma 3.6. [27, Lemma 3.4], [29, Lemma 3.11] *Let*

$$T \in \mathcal{P}_w(\ell^1(I)^+) \cup \mathcal{P}^{ws}(\ell^1(I)^+),$$

where I is an infinite set. If there are two distinct $k, l \in I$ such that $Te_k(i) > 0$ and $Te_l(i) > 0$, for some $i \in I$, then the set $\{Te_j(i) \mid j \in I\}$ is a singleton.

Proof. Firstly, it easy to see that if $Te_j(i) > 0$, for some $j \in I$, then $Te_j(i) = Te_k(i) = Te_l(i)$, by Lemma 3.5.

Suppose contrary that there is a $m \in I$ such that $Te_m(i) = 0$. Let

$$M := \{j \in I \mid Te_k(j) = Te_l(j) = \tilde{k},\}$$

where $\tilde{k} := Te_k(i) = Te_l(i)$. Obviously, the set M is a finite nonempty set. We chose an arbitrary $n \in I \setminus \{k, l, m\}$, and define a bijection

$$\delta_n(x) := \begin{cases} k, & x = k, \\ l, & x = l, \\ n, & x = m, \\ m, & x = n. \end{cases}$$

Now, using Lemma 3.4 there is $i_n \in I$ such that

$$\begin{aligned} Te_n(i_n) &= Te_{\delta_n(m)}(i_n) = Te_m(i) = 0, \\ Te_k(i_n) &= Te_{\delta_n(k)}(i_n) = Te_k(i) = \tilde{k}, \\ Te_l(i_n) &= Te_{\delta_n(l)}(i_n) = Te_l(i) = \tilde{k}. \end{aligned}$$

Now, it is easy to conclude that $i_n \in M \subset I$. Because $\operatorname{card}(M) < \aleph_0$, we get that there exists $s \in M$ and for this s there is a sequence of distinct elements $(m_j)_{j\in\mathbb{N}}$ such that $Te_{m_j}(s) = 0$, $\forall j \in \mathbb{N}$. Similarly as in the proof of the last result, we define bijections $\phi_j\colon \Phi_j \cup \{k,l\} \to \Phi_j \cup \{k,l\}$, correspond to sets $\Phi_j := \{m_1, m_2, \dots m_j\}$, $j \in \mathbb{N}$, defined by

$$\phi_j(x) := \begin{cases} k, & x = k, \\ l, & x = m_j \\ m_j & x = l \\ x, & x \in \Phi_j \setminus \{m_j\}. \end{cases}$$

Again using Lemma 3.4 for each $j \in I$, we can find $s_j \in I$ such that

$$\begin{aligned} Te_k(s_j) = Te_{\phi_j(k)}(s_j) = Te_k(s) = \tilde{k}, \\ Te_{m_j}(s_j) = Te_{\phi_j(l)}(s_j) = Te_l(s) = \tilde{k}, \\ Te_l(s_j) = Te_{\phi_j(m_j)}(s_j) = Te_{m_j}(s) = 0, \end{aligned} \tag{3.28}$$

$$Te_x(s_j) = Te_{\phi_j(x)}(s_j) = Te_x(s) = 0, \quad \forall x \in \Phi_j \setminus \{m_j\}. \tag{3.29}$$

If we suppose that there exist integers $a < b$ such that $s_a = s_b$, then using bijection ϕ_a we get $Te_{m_a}(s_a) = \tilde{k}$ by (3.28), and using bijection ϕ_b we obtain $Te_{m_a}(s_a) = Te_{m_a}(s_b) = 0$,by (3.29), which is contradiction with $\tilde{k} > 0$. Thus, $s_a \neq s_b$, whenever $a \neq b$. We get that the set $\{s_j \in I : j \in I\} \subset \{i \in I : Te_k(i) = \tilde{k}\}$ is infinite, which is impossible by $Te_k \in \ell^1(I)^+$. □

Theorem 3.4. [27, Theorem 3.1], [29, Theorem 3.12] *Let $A\colon \ell^1(I) \to \ell^1(I)$ be a bounded linear operator, where I is an infinite set. Then, $A \in \mathcal{A}(\ell^1(I)^+)$ if and only if functions Ae_j and Ae_k are mutually majorized by $\prec_w$ or by $\prec^{ws}$, and for each $i \in I$ there is at most one $j \in I$ such that $Ae_j(i) > 0$.*

Proof. Suppose that operator $A \in \mathcal{A}(\ell^1(I)^+)$ is defined by $A := \sum_{j\in I_0} \lambda_j P_{\theta_j}$ where the set Θ is defined in (3.2) and $\theta_j \in \Theta$, for all $j \in I_0$. Because $A \in \mathcal{P}^{ws}(\ell^1(I)^+) \cap \mathcal{P}_w(\ell^1(I)^+)$ by Theorem 3.2, we get that for arbitrary $k, j \in I$

$$e_j \prec^w e_k, \quad e_k \prec^w e_j, \quad e_j \prec^{ws} e_k \quad \text{and} \quad e_k \prec^{ws} e_j$$

implies

$$Ae_j \prec^w Ae_k, \quad Ae_k \prec^w Ae_j, \quad Ae_j \prec^{ws} Ae_k \quad \text{and} \quad Ae_k \prec^{ws} Ae_j.$$

Using the fact that the family Θ defined by (3.2) contains maps with disjoint images, if $s \notin \bigcup_{j\in I_0}(\theta_j(I))$, then we get $P_{\theta_j} e_l(s) = 0$, $\forall j \in I_0$, thus $Ae_l(s) = 0$, for every $l \in I$. If $s \in \bigcup_{j\in I_0}(\theta_j(I))$, then there is exactly one ordered pair (j_s, r_s), where $j_s \in I_0$ and $r_s \in I$, such that $\theta_{j_s}(r_s) = s$, and $\theta_{j_0}(r_0) \neq s$ for each pair (j_0, r_0) with $(j_0, r_0) \neq (j_s, r_s)$. Therefore,

$$Ae_r(s) = \sum_{j\in I_0} \lambda_j P_{\theta_j} e_r(s) = \sum_{j\in I_0} \lambda_j e_{\theta_j(r)}(s) = \lambda_{j_s} e_{\theta_{j_s}(r)}(s) = 0,$$

when $r \neq r_s$. Thus, each "row" contains at most one non-zero element, so the second part holds.

Suppose that $A\colon \ell^1(I) \to \ell^1(I)$ is a bounded linear operator. If $A := 0$ then $A \in \mathcal{A}(\ell^1(I)^+)$, obviously. Let $A \neq 0$. It follows that there is $k, l \in I$ such that $Ae_k(l) > 0$ which implies that $Ae_j \neq 0$ using the theorem assumptions that "columns" of the operator A is mutually weakly supermajorized and submajorized so they are different up to the permutation by Theorem 2.5. More precisely, there exist permutations $P_j \in P(\ell^1(I))$ corresponding to bijections $\omega_j \colon I \to I$ such that $P_j A e_k = A e_j$, for every $j \in I$.

We define a family Θ of maps θ_j, $j \in I_0$ defined by $\theta_j(i) = \omega_i(j)$, $\forall i \in I$ that is, $\Theta := \{\theta_j \colon I \to I \mid j \in I_0\}$, where $I_0 := I^+_{Ae_k}$, is at most a countable set. Clearly, θ_i are one-to-one maps. To show that maps θ_i have mutually disjoint images $\theta_i(I)$, assume that for some $a, b \in I_0$, $a \neq b$ there exist $j_a, j_b \in I$ such that

$$i_0 := \theta_a(j_a) = \theta_b(j_b), \quad \text{so} \quad \omega_{j_a}(a) = \omega_{j_b}(b).$$

Since $a, b \in I^+_{Ae_k}$, it is $Ae_k(a) > 0$ and $Ae_k(b) > 0$. Also,

$$Ae_{j_a}(i_0) = \langle Ae_{j_a}, e_{\omega_{j_a}(a)} \rangle = \langle Ae_{j_a}, P_{j_a} e_a \rangle = \langle P^{-1}_{j_a} Ae_{j_a}, e_a \rangle = Ae_k(a) > 0.$$

Similarly, $Ae_{j_b}(i_0) = Ae_k(b) > 0$, which is a contradiction with theorem's assumptions.

We will show that the operator A has the form (3.11). If we define $\lambda_i := Ae_k(i)$, $\forall i \in I_0$, and fixing $g = \sum_{j \in I} g(j) e_j \in \ell^1(I)^+$, then we obtain

$$\begin{aligned} Ag &= \sum_{j \in I} g(j) Ae_j = \sum_{i \in I} g(j) P_j Ae_k \\ &= \sum_{j \in I} g(j) \Big(\sum_{i \in I_0} Ae_k(i) P_j e_i \Big) \\ &= \sum_{j \in I} g(j) \sum_{i \in I_0} \lambda_i e_{\omega_j(i)} = \sum_{j \in I} g(j) \sum_{i \in I_0} \lambda_i e_{\theta_i(j)}. \end{aligned} \tag{3.30}$$

Further,

$$P_{\theta_i}(g) = \sum_{j \in I} g(j) P_{\theta_i} e_j = \sum_{j \in I} g(j) e_{\theta_i(j)}, \quad \text{by (3.1)} \tag{3.31}$$

There is a finite set $J_0 \subset I_0$ such that for each finite set $\widetilde{I_0} \supset J_0$, we have that $\sum_{j \in I_0 \smallsetminus \widetilde{I_0}} \lambda_j \leqslant \epsilon$, where $\epsilon > 0$ is arbitrarily chosen, so combining (3.30) and (3.31), we conclude

$$\begin{aligned} \Big\| Ag - \sum_{i \in \widetilde{I_0}} \lambda_i P_{\theta_i}(g) \Big\| &= \Big\| \sum_{j \in I} g(j) \sum_{i \in I_0 \smallsetminus \widetilde{I_0}} \lambda_i e_{\theta_i(j)} \Big\| = \sum_{j \in I} \sum_{i \in I_0 \smallsetminus \widetilde{I_0}} g(j) \lambda_i \\ &= \|g\| \sum_{i \in I_0 \smallsetminus \widetilde{I_0}} \lambda_i \leqslant \epsilon \|g\|, \end{aligned}$$

thus, $A = \sum_{i \in I_0} \lambda_i P_{\theta_i}$. □

The set $\mathcal{A}(\ell^1(I)^+)$ is neither a vector space nor a convex cone, which is verified by the following example. Precisely, for $F_1, F_2 \in \mathcal{A}(\ell^1(I)^+)$ it does not necessarily follows that $F = F_1 + F_2 \in \mathcal{A}(\ell^1(I)^+)$. Consequently the same holds for $\mathcal{P}_w(\ell^1(I)^+)$ and $\mathcal{P}^{ws}(\ell^1(I)^+)$).

Example 3.3. [27, Example 3.3], [29, Example 3.13] Let $\varphi_1, \varphi_k \colon \mathbb{N} \to \mathbb{N}$ are defined by $\varphi_1(j) = j$ and $\varphi_k(j) = kj$, $\forall j \in \mathbb{N}$, for some $k \in \mathbb{N} \setminus \{1\}$, and let $F(f) := P_{\varphi_1}(f) + P_{\varphi_k}(f)$. Now, $P_{\varphi_1}(e_1) = e_{\varphi_1(1)} = e_1$ and $P_{\varphi_k}(e_1) = e_{\varphi_k(1)} = e_k$, so $\langle F(e_1), e_k\rangle = \langle e_1 + e_k, e_k\rangle = 1$. Also, $\langle F(e_k), e_k\rangle = \langle e_k + e_{k^2}, e_k\rangle = 1$. On the other hand, $P_{\varphi_1}(e_2) = e_{\varphi_1(2)} = e_2$ and $P_{\varphi_k}(e_2) = e_{\varphi_k(2)} = e_{2k}$, so

$$\langle F(e_2), e_k\rangle = \langle e_2 + e_{2k}, e_k\rangle = 0.$$

Since $\langle F(e_k), e_k\rangle = \langle K(e_1), e_k\rangle = 1 > 0$ and $\langle F(e_2), e_k\rangle = 0$, it follows that $F \notin \mathcal{A}(\ell^1(I)^+)$, by Theorem 3.4.

The next result shows that an arbitrary chosen linear preserver of weak majorization or weak supermajorization on $\ell^1(I)^+$ may be decomposed as sum of two unique operators defined by (3.11) and (3.13).

Theorem 3.5. [27, Theorem 3.2], [29, Theorem 3.14] *Let I be an infinite set. If $T \in \mathcal{P}_w(\ell^1(I)^+) \cup \mathcal{P}^{ws}(\ell^1(I)^+)$ then there are unique operators $A \in \mathcal{A}(\ell^1(I)^+)$ and $B \in \mathcal{B}(\ell^1(I)^+)$ such that $T = A + B$. Moreover, these operators A and B satisfy condition* (3.17).

Proof. Let $T \in \mathcal{P}_w(\ell^1(I)^+) \cup \mathcal{P}^{ws}(\ell^1(I)^+)$. We define two sets $I_1, I_2 \subset I$ such that I_1 contains each $i \in I$ such that $\langle Te_j, e_i\rangle > 0$ for at most one $j \in I$ and $I_2 := I \setminus I_1$. In the other words, using Lemma 3.6 we get that the set I_2 contains all $k \in I$ for which $Te_j(k) = c > 0$, $\forall j \in I$.

Now, we define operators $A, B \colon \ell^1(I) \to \ell^1(I)$ by

$$Af(i) := \begin{cases} Tf(i) & i \in I_1, \\ 0, & i \in I_2. \end{cases} \quad \text{and} \quad Bf(i) := \begin{cases} Tf(i) & i \in I_2, \\ 0, & i \in I_1. \end{cases}$$

The operators A and B are bounded linear operators and $A + B = T$, obviously.

Next, we will show that $A \in \mathcal{A}_{pr}^{ws}(\ell^1(I)^+)$ and $B \in \mathcal{B}_{pr}^{ws}(\ell^1(I)^+)$.

Suppose that $I_2 = \emptyset$. It follows that there is no $i \in I$ such that $Te_{j_1}(i) > 0$ and $Te_{j_2}(i) > 0$. Also, for a preserver always holds that columns Ae_j and Ae_k are mutually majorized by appropriate majorization relation ($\prec_w$ or $\prec^{ws}$), so we obtain using Theorem 3.4 that $A \in \mathcal{A}(\ell^1(I)^+)$. Obviously, $B = 0 \in \mathcal{B}(\ell^1(I)^+)$.

Let $I_1, I_2 \neq \emptyset$. Next, we define

$$I_{Te_m}^1 := \{j \in I_{Te_m}^+ : Te_m(j) = \max\{Te_m(r) : r \in I_{Te_m}^+\}\},$$

$$I_{Te_m}^k := \left\{ j \in I_{Te_m}^+ : Te_m(j) = \max\left\{ Te_m(r) : r \in I_{Te_m}^+ \setminus \bigcup_{i=1}^{k-1} I_{Te_m}^i \right\}\right\}$$

when $k \geqslant 2$.

Case $T \in \mathcal{P}_w(\ell^1(I)^+)$. Fix $r, s \in I$. Since $T \in \mathcal{P}_w(\ell^1(I)^+)$, we obtain $Te_r \prec_w Te_s$ and $Te_s \prec_w Te_r$, that is, there is a partial permutation $P \in pP(\ell^1(I))$ corresponding to a bijection $\theta \colon I_{Te_r}^+ \to I_{Te_s}^+$ with $\langle Te_r, e_i\rangle = \langle Te_s, e_{\theta(i)}\rangle$, $\forall i \in I_{Te_r}^+$, by Theorem 2.4. Precisely, there are bijections $\omega_k \colon I_{Te_r}^k \to I_{Te_s}^k$, $\forall k \in \mathbb{N}$ if $I_f^k \neq \emptyset$, which determine the bijection $\theta \colon I_{Te_r}^+ \to I_{Te_s}^+$ by $\theta(i) := \omega_k(i) \in I_{Te_s}^k$, when $i \in I_{Te_r}^k$, for any $k \in \mathbb{N}$. Clearly, $\operatorname{card}(I_{Te_r}^k) = \operatorname{card}(I_{Te_s}^k)$, $\forall k \in \mathbb{N}$.

Since $\langle Te_r, e_i\rangle = \langle Te_s, e_i\rangle$, $\forall i \in I_2$, we get $\text{card}(I^k_{Te_r} \setminus I_2) = \text{card}(I^k_{Te_s} \setminus I_2)$, $\forall k \in \mathbb{N}$, so we may define bijections

$$\widetilde{\omega}_k \colon I^k_{Te_r} \setminus I_2 \to I^k_{Te_s} \setminus I_2, \quad \forall k \in \mathbb{N},$$

whenever $I^k_{Te_r} \neq \emptyset$. Now, there exist a bijection $\tilde{\theta}\colon I^+_{Te_r} \setminus I_2 \to I^+_{Te_s} \setminus I_2$ defined by $\tilde{\theta}(i) := \tilde{\omega}_k(i)$, when $i \in I^k_{Te_r} \setminus I_2$, for any $k \in \mathbb{N}$. It follows that there is a partial permutation $\tilde{P} \in pP(\ell^1(I))$ corresponding to bijection $\tilde{\theta}$ with

$$\langle Ae_r, e_i\rangle = \langle Te_r, e_i\rangle = \langle Te_s, e_{\tilde{\theta}(i)}\rangle = \langle Ae_s, e_{\tilde{\theta}(i)}\rangle, \quad \forall i \in I^+_{Te_r} \setminus I_2.$$

Thus, $Ae_r \prec_w Ae_s$ and $Ae_s \prec_w Ae_r$, so $A \in \mathcal{A}(\ell^1(I)^+)$ by Theorem 3.4.

Case $T \in \mathcal{P}^{ws}(\ell^1(I)^+)$. Fix $m, n \in I$. Hence $Te_m \prec^{ws} Te_n$ and $Te_n \prec^{ws} Te_m$, so functions Te_n and Te_m are permutation of each other, that is, there is a permutation $P \in P(\ell^1(I))$ corresponding to a bijection $\omega\colon I \to I$ with $\langle Te_m, e_i\rangle = \langle Te_n, e_{\theta(i)}\rangle$, $\forall i \in I$, by Theorem 2.7. Since

$$\text{card}(I^k_{Te_m}) = \text{card}(I^k_{Te_n}) \quad \text{and} \quad \text{card}(I^0_{Te_m}) = \text{card}(I^0_{Te_n}), \qquad \forall k \in \mathbb{N},$$

it is easy to conclude that bijection ω is determined by bijections

$$\omega_0 \colon I^0_{Te_m} \to I^0_{Te_n}, \quad \text{if} \quad I^0_{Te_m} \neq \emptyset$$
$$\omega_k \colon I^k_{Te_m} \to I^k_{Te_n}, \quad \text{if} \quad I^k_{Te_m} \neq \emptyset, \qquad k \in \mathbb{N},$$

in the following way:

$$\omega(i) := \begin{cases} \omega_k(i), & i \in I^k_{Te_m}, \\ \omega_0(i), & i \in I^0_{Te_m}. \end{cases}$$

Since for each $i \in I_2$, $Te_m(i) = Te_n(i)$, we obtain

$$\text{card}(I^k_{Te_m} \setminus I_2) = \text{card}(I^k_{Te_n} \setminus I_2), \quad \forall k \in \mathbb{N},$$

so we may define bijections

$$\tilde{\omega}_k \colon I^k_{Te_m} \setminus I_2 \to I^k_{Te_n} \setminus I_2, \quad \text{if} \quad I^k_{Te_m} \setminus I_2 \neq \emptyset, \qquad k \in \mathbb{N}.$$

by $\tilde{\omega}_k(i) = \omega_k(i)$, $\forall i \in I^k_{Te_m} \setminus I_2$. We form a bijection $\tilde{\omega}\colon I \to I$ defined by

$$\tilde{\omega}(i) := \begin{cases} \tilde{\omega}_k(i), & i \in I^k_{Te_m} \setminus I_2, \\ \omega_0(i), & i \in I^0_{Te_m}, \\ i, & i \in I_2. \end{cases}$$

It follows that the permutation $\tilde{P} \in P(\ell^1(I))$, which correspond to the bijection $\tilde{\omega}$ defined by $Pe_i = e_{\tilde{\omega}(i)}$, $\forall i \in I$ satisfies $PAe_m = Ae_n$. It follows that $Ae_m \prec^{ws} Ae_n$ and $Ae_n \prec^{ws} Ae_m$, $\forall m, n \in I$, so $A \in \mathcal{A}(\ell^1(I)^+)$ by Theorem 3.4.

The rest is valid for both cases:

To show that $B \in \mathcal{B}(\ell^1(I)^+)$, firstly we get

$$\begin{aligned} Be_m &= \sum_{i\in I} Be_m(i)e_i = \sum_{i\in I_1} Be_m(i)e_i + \sum_{i\in I_2} Be_m(i)e_i \\ &= \sum_{i\in I_2} Be_n(i)e_i = \sum_{i\in I} Be_n(i)e_i = Be_n, \end{aligned} \tag{3.32}$$

for fixed $m, n \in I$. Using (3.32) and defining $h := Be_r$ for some $r \in I$, we obtain

$$Bf = B\bigg(\sum_{j\in I} f(j)e_j\bigg) = h\bigg(\sum_{j\in I} f(j)\bigg), \tag{3.33}$$

thus $B \in \mathcal{B}(\ell^1(I)^+)$.

If $I_1 = \emptyset$, then using statements (3.32) and (3.33), when $I_2 = I$, we obtain that $B = T \in \mathcal{B}(\ell^1(I)^+)$ and $A = 0 \in \mathcal{A}(\ell^1(I)^+)$.

Assume that there is another one pair A_1, B_1 such that $T = A_1 + B_1$, where $A_1 \in \mathcal{A}(\ell^1(I)^+)$ and $B_1 \in \mathcal{B}(\ell^1(I)^+)$. We get $[A - A_1 = B_1 - B$. We know that $Be_m = Be_n$ and $B_1e_m = B_1e_n$, $\forall m, n \in I$. On the other hand, using Theorem 3.4, since for each $i \in I$, there is at most one $s \in I$ such that $Ae_s(i) > 0$, there is at least one $j_s \in I$ such that $Ae_{j_s}(i) = A_1e_{j_s}(i) = 0$. Using above arguments, we get

$$0 = (A - A_1)e_{j_s}(i) = (B_1 - B)e_{j_s}(i) = (B_1 - B)e_j(i), \quad \forall j \in I,$$

so we get $B = B_1$ and $A = A_1$. □

Now, it is clear why the sum of two operators T_{e_r} and I from Example 3.2 is not a linear preserver of both $\prec_w$ and $\prec^{ws}$ by above theorem.

All results provided above are collected below. The next theorem characterizes linear preservers of two majorization relations, weak majorization $\prec_w$ and weak supermajorization $\prec^{ws}$ on $\ell^1(I)^+$.

Theorem 3.6. [27, Theorem 3.3], [29, Theorem 3.15] *Let $T\colon \ell^1(I) \to \ell^1(I)$ be a bounded linear operator, where I is an infinite set. The following statements are equivalent:*

i) $T \in \mathcal{P}_w(\ell^1(I)^+)$;

ii) $T \in \mathcal{P}^{ws}(\ell^1(I)^+)$;

iii) *There are operators $A \in \mathcal{A}^{ws}_{pr}(\ell^1(I)^+)$ and $B \in \mathcal{B}^{ws}_{pr}(\ell^1(I)^+)$ and disjoint sets $I_1, I_2 \subset I$ with $I_1 \cup I_2 = I$ such that $T = A + B$ where A, B are chosen to be*

$$Af(i_2) = Bf(i_1) = 0, \quad \forall i_1 \in I_1, \quad \forall i_2 \in I_2, \quad \forall f \in \ell^1(I)^+;$$

iv) *There is at most countable set $I_0 \subset I$ and there is a family*

$$\Theta := \{\theta_j\colon I \xrightarrow{1-1} I \mid j \in I_0,\ \theta_i(I) \cap \theta_j(I) = \emptyset,\ i \neq j\}$$

of one-to-one maps, $\theta_j \in \Theta$, $\forall j \in I_0$, and $(\lambda_j)_{j\in I_0} \in \ell^1(I_0)^+$ such that

$$T = \sum_{j\in I_0} \lambda_j P_{\theta_j} + B_h,$$

where $B_h(f) := h\sum_{i\in I} f(i)$, for $h \in \ell^1(I)^+$ with $h(i) = 0$, $\forall i \in \bigcup_{j\in I_0} \theta_j(I)$;

v) *functions Ae_j and Ae_k are mutually majorized by $\prec_w$ or by $\prec^{ws}$, and for each $i \in I$, either there exists exactly one $j \in I$ with $Te_j(i) > 0$ or the set $\{Te_j(i) \mid j \in I\}$ is a singleton.*

Proof. We provide i) → v) → iii) and ii)→ v)→ iii), by Lemma 3.6 and Theorem 3.5. Also, Statement iii) implies ii) and i) by Theorem 3.3. Statements iv) and v) are equivalent by Theorem 3.4. □

3.2. Linear preservers of submajorization on $\ell^1(I)^+$, when I is an infinite set.

In the sequel, we will find the proper form of linear preservers of submajorization on $\ell^1(I)^+$, when I is an infinite set. Actually, we will present two set relations $\mathcal{P}_w(\ell^1(I)^+) \subset \mathcal{P}_s(\ell^1(I)^+)$ and $\mathcal{P}_s(\ell^1(I)^+) \subset \mathcal{P}_w(\ell^1(I)^+)$ provided in [31].

Theorem 3.7. [31, Theorem 4.6] *Let I be an infinite set. Then, $\mathcal{P}_w(\ell^1(I)^+) \subset \mathcal{P}_s(\ell^1(I)^+)$ holds.*

Proof. Let $T \in \mathcal{P}_w(\ell^1(I)^+)$. Using Theorem 3.6 statement iii), we get that there is a decomposition of the operator T as $T = A + B$, where $A \in \mathcal{A}(\ell^1(I)^+)$ and $B \in \mathcal{B}(\ell^1(I)^+)$, and where sets $I_1, I_2 \subset I$ are disjoint with $I_1 \cup I_2 = I$ and operators A, B are chosen to be

$$\langle Af, e_{i_2}\rangle = \langle Bf, e_{i_1}\rangle = 0, \quad \forall i_1 \in I_1, \quad \forall i_2 \in I_2, \quad \forall f \in \ell^1(I)^+. \tag{3.34}$$

Using Theorem 3.6 statement iv) we have $A = \sum_{k\in I_0} \lambda_k P_{\theta_k}$ where I_0 is at most a countable subset of I, $(\lambda_k)_{k\in I_0} \in \ell^1(I_0)^+$ and for every $k \in I_0$ we have

$$\theta_k \in \Theta = \{\theta_k \colon I \xrightarrow{1-1} I \mid k \in I_0,\ \theta_i(I) \cap \theta_j(I) = \emptyset,\ i \neq j\}.$$

If there is $i \in I_0$ such that $\lambda_i = 0$ then we will consider the set $I_0 \setminus \{i\}$ instead of I_0. Hence, we may assume that $\lambda_j > 0$, for every $j \in I_0$.

Let $f \prec_s g$ for fixed $f, g \in \ell^1(I)^+$. It follows that there exists $D \in iDSS(\ell^1(I))$ such that $f = Dg$. Using Theorem 3.1 we obtain that there is $S \in iDSS(\ell^1(I))$ such that $P_{\theta_k} D = S P_{\theta_k}$, $\forall k \in I_0$ holds. Clearly, the operator S is not unique and it is defined by (3.6), where $0 \leqslant a \leqslant 1$. Similarly as in Theorem 3.2, statement (3.12) we obtain that A preserve submajorization relation:

$$Af = ADg = \sum_{k\in I_0} \lambda_k P_{\theta_k}(Dg) = \sum_{k\in I_0} \lambda_k S P_{\theta_k}(g) = S(Ag). \tag{3.35}$$

Next, changing the order of summation we obtain

$$\begin{aligned} \|f\| &= \sum_{i\in I} |f(i)| = \sum_{i\in I} f(i) = \sum_{i\in I}\sum_{j\in I} \langle De_j, e_i\rangle g(j) \\ &= \sum_{j\in I}\sum_{i\in I} \langle De_j, e_i\rangle g(j) = \sum_{j\in I} g(j) \sum_{i\in I} \langle De_j, e_i\rangle \leqslant \|g\|. \end{aligned}$$

Thus, inequality $\|f\| \leqslant \|g\|$ holds. If we set $a := 1 - \frac{\|f\|}{\|g\|}$, using the above argument we get that $0 \leqslant a \leqslant 1$. Now, using (3.13) we get

$$BDg = Bf = h \sum_{i\in I} f(i) = h\|f\| = h(1-a)\|g\| = (1-a)Bg, \tag{3.36}$$

where $h := Be_j$, for some $j \in I$.

We will show that

$$SB = (1-a)B. \tag{3.37}$$

Firstly, we will show that $Se_k = (1-a)e_k$, for every $k \in I_2$. Fix $k \in I_2$. We have that $\langle Af, e_k\rangle = 0$, by (3.34). Since,

$$\begin{aligned}\langle Af, e_k\rangle &= \sum_{j\in I} f(j)\langle Ae_j, e_k\rangle = \sum_{j\in I} f(j)\sum_{i\in I_0}\lambda_i\langle P_{\theta_i}e_j, e_k\rangle\\ &= \sum_{j\in I} f(j)\sum_{i\in I_0}\lambda_i\langle e_{\theta_i(j)}, e_k\rangle.\end{aligned}$$

It follows that $k \notin \cup_{i\in I_0}\theta_i(I)$, because f is arbitrary fixed and λ is positive. Now, using the definition (3.6) of the operator S we get $Se_k = (1-a)e_k$. Now, using (3.34) we obtain

$$\begin{aligned}SBu = \Big(\sum_{j\in I} u_j\Big)Sh &= \Big(\sum_{j\in I} u_j\Big)\sum_{k\in I_2} h(k)Se_k\\ &= (1-a)\Big(\sum_{j\in I} u_j\Big)\sum_{k\in I_2} h(k)e_k = (1-a)Bu,\end{aligned}$$

for every $u \in \ell^1(I)^+$, so (3.37) is provided. Finally, using (3.35), (3.36) and (3.37) we obtain

$$Tf = (A+B)Dg = ADg + BDg = SAg + (1-a)Bg = SAg + SBg = STg.$$

Since, $S \in iDSS(\ell^1(I))$, we get $Tf \prec_s Tg$, that is $T \in \mathcal{P}_s(\ell^1(I)^+)$. □

Theorem 3.8. [31, Theorem 4.9] *Let I be an infinite set. Then, $\mathcal{P}_s(\ell^1(I)^+) \subset \mathcal{P}_w(\ell^1(I)^+)$ holds.*

Proof. Assume that $T \in \mathcal{P}_s(\ell^1(I)^+)$. Since $e_i \prec_s e_j$ and $e_j \prec_s e_i$, we get $Te_i \prec_s Te_j$ and $Te_j \prec_s Te_i$. Since, relation $\prec_s$ implies $\prec_w$, we have $Te_i \prec_w Te_j$ and $Te_j \prec_w Te_i$, so the first part of statement $v)$ in Theorem 3.6 is satisfied.

Suppose that there exist $m, n, r \in I$ such that $\langle Te_m, e_r\rangle > 0$ and $\langle Te_n, e_r\rangle > 0$. Using Lemma 3.5 we get that $\langle Te_m, e_r\rangle = \langle Te_n, e_r\rangle$. Precisely, all non-zero elements in one "row" have to be mutually equal.

We claim that all elements in the "row" indexed by r are the same, that is, there is no zero element. Suppose contrary that there exists $l \in I$ such that $\langle Te_l, e_r\rangle = 0$. Fix $k \in I \setminus \{m, n, l\}$. We define a bijection

$$\theta_k(x) := \begin{cases} m, & x = m,\\ n, & x = n,\\ k, & x = l,\\ l, & x = k.\end{cases}$$

Now, applying Lemma 3.4 on bijection θ_k we get that there exists $i_k \in I$ such that

$$\langle Te_m, e_{i_k}\rangle = \langle Te_m, e_r\rangle, \quad \langle Te_n, e_{i_k}\rangle = \langle Te_n, e_r\rangle \quad \text{and} \quad \langle Te_k, e_{i_k}\rangle = \langle Te_l, e_r\rangle = 0.$$

It follows that

$$\operatorname{card}\{i \in I \mid \langle Te_m, e_i\rangle = \langle Te_n, e_i\rangle = \langle Te_m, e_r\rangle\} < \aleph_0. \tag{3.38}$$

by $Te_m, Te_n \in \ell^1(I)^+$. It is easy to see that i_k is contained in the above set. Now, since k is arbitrary chosen from infinite set $I \setminus \{m,n,l\}$, there exists at least one $s \in I$ (s is contained in the set considered in (3.38)) and there is a sequence $(t_i)_{i\in\mathbb{N}}$ of distinct elements $t_i \in I$ such that

$$\langle Te_m, e_s\rangle = \langle Te_m, e_r\rangle = \langle Te_n, e_s\rangle = \langle Te_n, e_r\rangle \quad \text{and} \quad \langle Te_{t_i}, e_s\rangle = 0, \qquad \forall i \in \mathbb{N}.$$

Now, for each $j \in \mathbb{N}$ we define bijections

$$\gamma_j \colon \{t_1, t_2, \dots t_j\} \cup \{m,n\} \to \{t_1, t_2, \dots t_j\} \cup \{m,n\},$$

by

$$\gamma_j(x) := \begin{cases} m, & x = m, \\ t_j, & x = n, \\ n, & x = t_j, \\ x, & x \in \{t_1, t_2, \dots t_{j-1}\}. \end{cases}$$

Similarly as in (3.25), (3.26) and (3.27), for each $j \in I$ and for the appropriate bijection γ_j, there exists $r_j \in I$ such that

$$\langle Te_m, e_{r_j}\rangle = \langle Te_m, e_s\rangle = \langle Te_m, e_r\rangle, \tag{3.39}$$

$$\langle Te_{t_j}, e_{r_j}\rangle = \langle Te_{\gamma_j(n)}, e_{r_j}\rangle = \langle Te_n, e_s\rangle = \langle Te_n, e_r\rangle, \tag{3.40}$$
$$\langle Te_n, e_{r_j}\rangle = \langle Te_{\gamma_j(t_j)}, e_{r_j}\rangle = \langle Te_{t_j}, e_s\rangle = 0,$$

again by Lemma 3.4. Also, for every $x \in \{t_1, t_2, \dots t_{j-1}\}$ we have

$$\langle Te_x, e_{r_j}\rangle = \langle Te_{\gamma_j(x)}, e_{r_j}\rangle = \langle Te_x, e_s\rangle = 0. \tag{3.41}$$

by Lemma 3.4.

Suppose that $k_1 < k_2$. We will show that $r_{k_1} \neq r_{k_2}$. Using (3.41) for $\gamma_{k_2}(x)$ we get that $\langle Te_{t_{k_1}}, e_{r_{k_2}}\rangle = 0$. However, using (3.40) for bijection $\gamma_{k_1}(x)$ we obtain $\langle Te_{t_{k_1}}, e_{r_{k_1}}\rangle = \langle Te_n, e_r\rangle > 0$, so we get that $\langle Te_{k_1}, e_{r_{k_1}}\rangle \neq \langle Te_{k_1}, e_{r_{k_2}}\rangle$, therefore $r_{k_1} \neq r_{k_2}$.

We get that sequence $\{r_j\}_{j\in\mathbb{N}}$ contains mutually district elements. Using (3.39) it follows that

$$\|Te_m\| = \sum_{i\in I}\langle Te_m, e_i\rangle \geqslant \sum_{j\in I}\langle Te_m, e_{r_j}\rangle = \sum_{i=1}^{\infty}\langle Te_m, e_r\rangle = +\infty,$$

which is impossible. Thus, there is no $l \in I$ such that $\langle Te_l, e_r\rangle = 0$, so the set $\{\langle Te_j, e_i\rangle \mid j \in I\}$ is a singleton. Now, we get that T satisfies statement v) of Theorem 3.6, that is $T \in \mathcal{P}_w(\ell^1(I)^+)$. □

Corollary 3.1. [27, Theorem 3.3], [29, Theorem 3.15], [31, Corollary 4.10] *Let $T\colon \ell^1(I) \to \ell^1(I)$ be a bounded linear operator, where I is an infinite set. The following statements are equivalent:*

i) *$T \in \mathcal{P}_w(\ell^1(I)^+)$;*
ii) *$T \in \mathcal{P}_s(\ell^1(I)^+)$;*
iii) *$T \in \mathcal{P}^{ws}(\ell^1(I)^+)$;*

iv) *There are operators $A \in \mathcal{A}_{pr}^{ws}(\ell^1(I)^+)$ and $B \in \mathcal{B}_{pr}^{ws}(\ell^1(I)^+)$ and disjoint sets $I_1, I_2 \subset I$ with $I_1 \cup I_2 = I$ such that $T = A + B$ where A, B are chosen to be*

$$Af(i_2) = Bf(i_1) = 0, \quad \forall i_1 \in I_1, \quad \forall i_2 \in I_2, \quad \forall f \in \ell^1(I)^+;$$

v) *There is at most countable set $I_0 \subset I$ and there is a family*

$$\Theta := \{\theta_j \colon I \xrightarrow{1-1} I \mid j \in I_0,\ \theta_i(I) \cap \theta_j(I) = \emptyset,\ i \neq j\}$$

of one-to-one maps, $\theta_j \in \Theta$, $\forall j \in I_0$, and $(\lambda_j)_{j\in I_0} \in \ell^1(I_0)^+$ such that

$$T = \sum_{j\in I_0} \lambda_j P_{\theta_j} + B_h,$$

where $B_h(f) := h \sum_{i\in I} f(i)$, for $h \in \ell^1(I)^+$ with $h(i) = 0$, $\forall i \in \cup_{j\in I_0}\theta_j(I)$;

vi) *functions Ae_j and Ae_k are mutually majorized by $\prec_w$ or by $\prec_s$ or by $\prec^{ws}$, and for each $i \in I$, either there exists exactly one $j \in I$ with $Te_j(i) > 0$ or the set $\{Te_j(i) \mid j \in I\}$ is a singleton.*

Proof. Statements i) and ii) are equivalent by Theorem 3.7 and Theorem 3.8. The rest follows by Theorem 3.6. □

3.3. More examples of linear preservers. The next example is provided in [29, Example 3.5].

Example 3.4. [29, Example 3.5] Fix $1 < m \in \mathbb{N}$ and let $\Theta := \{\theta_1, \theta_2, \dots, \theta_m\}$ be a family of one-to-one maps $\theta_n \colon \mathbb{N} \to \mathbb{N}$, defined by

$$\theta_n(k) = m^k + n, \quad \forall k \in \mathbb{N}, \qquad n = 1, 2, \dots, m.$$

Let $T := \sum_{n=1}^m \frac{1}{n^2} P_{\theta_n}$. We may represent T in the following way:

$$Tf = \Big[\underbrace{0,\dots,0}_{m\text{-times}}, \underbrace{f(1), \frac{f(1)}{2^2}\dots, \frac{f(1)}{m^2}}_{\text{m-times}}, \underbrace{0,\dots,0}_{\substack{m^2-2m\\ \text{times}}}, \underbrace{f(2), \frac{f(2)}{2^2}\dots, \frac{f(2)}{m^2}}_{\text{m-times}}, \underbrace{0,\dots,0}_{\substack{m^3-m^2-m\\ \text{times}}}, \underbrace{f(3), \frac{f(3)}{2^2}\dots, \frac{f(3)}{m^2}}_{\text{m-times}}, \dots\Big]^T$$

by the definition of the family Θ. The operator T is bounded since for every $f \in \ell^1(I)$ we get

$$\|Tf\| = \sum_{n=1}^m \sum_{k=1}^\infty \Big|\frac{1}{n^2} f(k)\Big| \leqslant \sum_{n=1}^m \frac{1}{n^2}\|f\| \leqslant \|f\| \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}\|f\|.$$

Since $(\lambda_j)_{j\in\mathbb{N}} = \left(\frac{1}{n^2}\right)_{j\in\mathbb{N}} \in \ell^1(\mathbb{N})^+$ and Θ has form (3.2), we conclude $T \in \mathcal{A}(\ell^1(\mathbb{N}))$, that is, $T \in \mathcal{P}_w(\ell^1(\mathbb{N})^+) \cup \mathcal{P}_s(\ell^1(\mathbb{N})^+) \cup \mathcal{P}^{ws}(\ell^1(\mathbb{N})^+)$ by Corollary 3.1.

In the above example, operator T contains a finite sum. In the next two examples, which are presented in [31], we present constructions with infinite sums of linear preservers from class $\mathcal{A}(\ell^1(I)^+)$.

Example 3.5. [31] We define maps $\theta_i \colon \mathbb{N} \to \mathbb{N}$ by

$$\theta_i(j) = i + 1 + \sum_{k=0}^{i+j-2} k, \quad \forall i, j \in \mathbb{N}. \tag{3.42}$$

Suppose that there exist $i_1, i_2, j_1, j_2 \in \mathbb{N}$ such that $\theta_{i_1}(j_1) = \theta_{i_2}(j_2)$.It follows that

$$i_1 + \sum_{k=0}^{i_1+j_1-2} k = i_2 + \sum_{k=0}^{i_2+j_2-2} k. \tag{3.43}$$

If $i_1 + j_1 < i_2 + j_2$ then, by (3.43), we get

$$i_1 - i_2 = \sum_{k=i_1+j_1-1}^{i_2+j_2-2} k \geqslant i_1 + j_1 - 1.$$

Because $j_1, i_2 \in \mathbb{N}$, we obtain using above arguments that $j_1 + i_2 \leqslant 1$ holds, which is impossible. Similarly, if $i_1 + j_1 > i_2 + j_2$ then we get $j_2 + i_1 \leqslant 1$ which is impossible. It follows that $i_1 + j_1 = i_2 + j_2$. Therefore, (3.43) gives $i_1 = i_2$ and $j_1 = j_2$. Thus, $\theta_{i_1}(\mathbb{N}) \cap \theta_{i_2}(\mathbb{N}) = \emptyset$, for all $i_1, i_2 \in \mathbb{N}$ with $i_1 \neq i_2$. Also, using definition (3.42) it is easy to see that maps θ_i are one-to-one for each $i \in \mathbb{N}$. Hence,

$$\Theta := \{\theta_k : \mathbb{N} \xrightarrow{1-1} \mathbb{N} \mid k \in I_0,\ \theta_i(\mathbb{N}) \cap \theta_j(\mathbb{N}) = \emptyset,\ i \neq j\}. \tag{3.44}$$

Let A be an operator defined by

$$A = \sum_{i=1}^{\infty} \lambda_i P_{\theta_i}, \tag{3.45}$$

where $\lambda = (\lambda_i)_{i\in\mathbb{N}} \in \ell^1(\mathbb{N})^+$ is an arbitrary fixed function. The operator A may be represented by an infinite matrix in the following way

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ \lambda_1 & 0 & 0 & 0 & 0 & \dots \\ 0 & \lambda_1 & 0 & 0 & 0 & \dots \\ \lambda_2 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 & 0 & 0 & 0 & \dots \\ \lambda_3 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \lambda_1 & 0 & \dots \\ 0 & 0 & \lambda_2 & 0 & 0 & \dots \\ 0 & \lambda_3 & 0 & 0 & 0 & \dots \\ \lambda_4 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \lambda_1 & \dots \\ 0 & 0 & 0 & \lambda_2 & 0 & \dots \\ 0 & 0 & \lambda_3 & 0 & 0 & \dots \\ 0 & \lambda_4 & 0 & 0 & 0 & \dots \\ \lambda_5 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Using (2.5) we obtain

$$Af := [0, \lambda_1 f_1, \lambda_1 f_2, \lambda_2 f_1, \lambda_1 f_3, \lambda_2 f_2, \lambda_3 f_1, \lambda_1 f_4, \lambda_2 f_3, \lambda_3 f_2, \lambda_4 f_1, \lambda_1 f_5, \lambda_2 f_4, \dots]^T$$

for each $f \in \ell^1(\mathbb{N})$. The operator A is a bounded linear operator on $\ell^1(\mathbb{N})$ by Theorem 2.1. We get that operator A satisfies the statement $v)$ of Corollary 3.1. Also, using the matrix representation of A we may conclude that it satisfies statement $vi)$ of Corollary 3.1. Therefore, the operator A preserves all three majorization relation $\prec_w$, $\prec_s$ and $\prec^{ws}$ on $\ell^1(\mathbb{N})^+$. Let

$$B(f) := h \sum_{i \in \mathbb{N}} f(i), \quad \forall f \in \ell^1(\mathbb{N}) \tag{3.46}$$

where $h \in \ell^1(\mathbb{N})^+$ defined by

$$h(j) := \begin{cases} a \geqslant 0, & j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Now, the operator $T := A + B$ may be represented by an infinite matrix

$$T = \begin{bmatrix} a & a & a & a & a & \dots \\ \lambda_1 & 0 & 0 & 0 & 0 & \dots \\ 0 & \lambda_1 & 0 & 0 & 0 & \dots \\ \lambda_2 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 & 0 & 0 & 0 & \dots \\ \lambda_3 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \lambda_1 & 0 & \dots \\ 0 & 0 & \lambda_2 & 0 & 0 & \dots \\ 0 & \lambda_3 & 0 & 0 & 0 & \dots \\ \lambda_4 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \lambda_1 & \dots \\ 0 & 0 & 0 & \lambda_2 & 0 & \dots \\ 0 & 0 & \lambda_3 & 0 & 0 & \dots \\ 0 & \lambda_4 & 0 & 0 & 0 & \dots \\ \lambda_5 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$Tf := \Big[a \sum_{i=1}^{\infty} f(i), \lambda_1 f_1, \lambda_1 f_2, \lambda_2 f_1, \lambda_1 f_3, \lambda_2 f_2, \lambda_3 f_1, \lambda_1 f_4, \lambda_2 f_3, \lambda_3 f_2, \\ \lambda_4 f_1, \lambda_1 f_5, \lambda_2 f_4, \dots \Big]^T$$

for each $f \in \ell^1(\mathbb{N})$. The operator T is a bounded linear operator on $\ell^1(\mathbb{N})$ by Theorem 2.1 because it satisfies (2.3). Now, T preserves all three majorization relation $\prec_w$, $\prec_s$ and $\prec^{ws}$ on $\ell^1(\mathbb{N})^+$, by Corollary 3.1.

We presented above linear preservers which have only one "row" with mutually equal non-zero elements. In the next example we give preservers where sets I_1 and I_2 in Corollary 3.1 are both countable, that is, where there are countable "rows" which are a singleton.

Example 3.6. [31] Let $\theta_i\colon \mathbb{N} \to \mathbb{N}$ be maps defined by

$$\theta_i(j) = i - 1 + \sum_{k=1}^{i+j-1} k, \quad \forall j \in \mathbb{N}$$

for any $i \in \mathbb{N}$. Similarly as in the previous example, we may provide that the family (3.44) contains one-to-one maps $\theta_i(j)$ with mutually disjoint images.

Fix a non-zero sequence $\mu = (\mu_i)_{i\in\mathbb{N}} \in \ell^1(\mathbb{N})^+$ and suppose that $h \in \ell^1(\mathbb{N})^+$ is defined by

$$h(j) := \begin{cases} \mu_i, & j = \sum_{k=2}^{i+1} k, \\ 0, & \text{otherwise.} \end{cases}$$

We define the operator $T := A_1 + B_1$, where operators A_1 and B_1 are determined as in (3.45) and (3.46), respectively.

Suppose contrary that there exists $r \in \cup_{i\in\mathbb{N}}\theta_i(\mathbb{N})$ such that $\langle h, e_r\rangle > 0$. It follows that $r = \theta_i(j)$ and $r = \sum_{k=2}^{n+1} k$ for some $i, j, n \in \mathbb{N}$. Now,

$$\sum_{k=2}^{n+1} k = r = i - 1 + \sum_{k=1}^{i+j-1} k$$

so $\sum_{k=1}^{n+1} k - \sum_{k=1}^{i+j-1} k = i \geqslant 1$.

Hence, $n + 1 > i + j - 1$, and so $i = \sum_{k=1}^{n+1} k - \sum_{k=1}^{i+j-1} k \geqslant i + j > i$ which is a contradiction. Thus, $T \in \mathcal{P}_s(\ell^1(I)^+)$ by statement $v)$ of Corollary 3.1, so the operator T may be represented by an infinite matrix in the following way

$$T = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & \dots \\ \mu_1 & \mu_1 & \mu_1 & \mu_1 & \mu_1 & \dots \\ 0 & \lambda_1 & 0 & 0 & 0 & \dots \\ \lambda_2 & 0 & 0 & 0 & 0 & \dots \\ \mu_2 & \mu_2 & \mu_2 & \mu_2 & \mu_2 & \dots \\ 0 & 0 & \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 & 0 & 0 & 0 & \dots \\ \lambda_3 & 0 & 0 & 0 & 0 & \dots \\ \mu_3 & \mu_3 & \mu_3 & \mu_3 & \mu_3 & \dots \\ 0 & 0 & 0 & \lambda_1 & 0 & \dots \\ 0 & 0 & \lambda_2 & 0 & 0 & \dots \\ 0 & \lambda_3 & 0 & 0 & 0 & \dots \\ \lambda_4 & 0 & 0 & 0 & 0 & \dots \\ \mu_4 & \mu_4 & \mu_4 & \mu_4 & \mu_4 & \dots \\ 0 & 0 & 0 & 0 & \lambda_1 & \dots \\ 0 & 0 & 0 & \lambda_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Acknowledgement. The research is financially supported by the Ministry of Education, Science and Technological Development, Republic of Serbia, Grant No. 451-03-68/2022-14/200109 and Grant No. 451-03-68/2022-14/200124.

References

[1] T. Ando, *Majorization, doubly stochastic matrices, and comparison of eigenvalues*, Linear Algebra Appl. **118** (1989), 163–248.

[2] J. Antezana, P. Massey, M. Ruiz, D. Stojanoff, *The Schur–Horn theorem for operators and frames with prescribed norms and frame operator*, Illinois J. Math. **51**(2) (2007), 537–560.

[3] J. Antezana, P. Massey, D. Stojanoff, *Jensen's inequality for spectral order and submajorization*, J. Math. Anal. Appl. **331**(1) (2007), 297–307.

[4] M. Argerami, *Majorisation and the Carpenter's theorem*, Integral Equations Oper. Theory **82**(1) (2016), 33–49.

[5] W. Arveson, R. V. Kadison, *Diagonals of self-adjoint operators*, in: D. Han, P. E. T. Jorgensen, D. R. Larson, eds., *Operator Theory, Operator Algebras, and Applications*, Contemp. Math. **414** (2006), 247–263.

[6] F. Bahrami, A. Bayati, S. M. Manjegani, *Linear preservers of majorization on $\ell^p(I)$*, Linear Algebra Appl. **436** (2012), 3177–3195.

[7] F. Bahrami, A. Bayati, S. M. Manjegani, *Majorization on ℓ^∞ and on its closed linear subspace c, and their linear preservers*, Linear Algebra Appl. **437** (2012), 2340–2358.

[8] A. Bayati, N. Eftekhari, *Convex majorization on discrete ℓ^p spaces*, Linear Algebra Appl. **474** (2015), 124–140.

[9] A. Bayati, S. M. Manjegani, *Some properties of operators preserving convex majorization on discrete ℓ^p spaces*, Linear Algebra Appl. **484** (2015), 130–140.

[10] N. Eftekhari, A. B. Eshkaftaki, *Isotonic linear operators on the space of all convergent real sequences*, Linear Algebra Appl. **506** (2016), 535–550.

[11] A. B. Eshkaftaki, M. H. Berenjegani, F. Bahrami, *DSS-weak majorization and its linear preservers on ℓ^p spaces*, Linear Multilinear Algebra **66**(10) (2018), 2076–2088.

[12] A. B. Eshkaftaki, *Doubly (sub)stochastic operators on ℓ^p spaces*, J. Math. Anal. Appl. **498**(1) (2021), 124923.

[13] A. B. Eshkaftaki, *Row-summable matrices with application to generalization of Schröder's and Abel's functional equations*, Aequat. Math. **96** (2022), 525–533.

[14] A. B. Eshkaftaki, *Increasable doubly substochastic matrices with application to infinite linear equations*, Linear Multilinear Algebra (2021), DOI: 10.1080/03081087.2021.1939253.

[15] A. B. Eshkaftaki, *Minimal completion of $I \times I$ doubly substochastic matrices*, Linear Multilinear Algebra **70**(11) (2022), 2065–2077.

[16] F. Hiai, *Majorization and stochastic maps in von Neumann algebras*, J. Math. Anal. Appl. **127**(1) (1987), 18–48.

[17] R, Bhatia, *Matrix Analysis*, Springer, 1997.

[18] G. Gour, D. Jennings, F. Buscemi, R. Duan, I. Marvian, *Quantum majorization and a complete set of entropic conditions for quantum thermodynamics*, Nature Communications **9** (2018), 5352.

[19] G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, second ed., Cambridge University Press, London and New York, 1952.

[20] A. M. Hasani, M. A. Vali, *Linear maps which preserve or strongly preserve weak majorization*, J. Inequal. Appl. **2007** (2008), 082910.

[21] V. Kaftal, G. Weiss, *An infinite dimensional Schur-Horn Theorem and majorization theory*, J. Funct. Anal. **259** (2010), 3115–3162.

[22] M. Kennedy, P. Skoufranis, *The Schur-Horn problem for normal operators*, Proc. London Math. Soc. **111**(2) (2015), 354–380.

[23] Y. Li, P. Busch, *Von Neumann entropy and majorization*, J. Math. Anal. Appl. **408**(1) (2013), 384–393.

[24] M. Z. Ljubenović, *Majorization and doubly stochastic operators*, Filomat **29**(9) (2015), 2087–2095.

[25] M. Z. Ljubenović, *Weak majorization and doubly substochastic operators on $\ell^p(I)$*, Linear Algebra Appl. **486** (2015), 295–316.

[26] M. Z. Ljubenović, D. S. Djordjević, *Linear preservers of weak majorization on $\ell^p(I)^+$, when $p \in (1, \infty)$*, Linear Algebra Appl. **497** (2016), 181–198.
[27] M. Z. Ljubenović, D. S. Djordjević, *Linear preservers of weak majorization on $\ell^1(I)^+$, when I is an infinite set*, Linear Algebra Appl. **517** (2017), 177–198.
[28] M. Z. Ljubenović, D. S. Djordjević, *Weak supermajorization and families as doubly superstochastic operators on $\ell^p(I)$*, Linear Algebra Appl. **532** (2017), 312–346.
[29] M. Z. Ljubenović, D. S. Djordjević, *Bounded linear operators that preserve the weak supermajorization relation on $\ell^1(I)^+$, when I is an infinite set*, Electron. J. Linear Algebra **37** (2018), 407–427.
[30] M. Z. Ljubenović, D. S. Rakić, D. S. Djordjević, *Linear preservers of DSS-weak majorization on discrete Lebesgue space $\ell^1(I)$, when I is an infinite set*, Linear Multilinear Algebra **69**(14) (2021), 2657–2673.
[31] M. Z. Ljubenović, D. S. Rakić, *Submajorization on $\ell^p(I)^+$ determined by increasable doubly substochastic operators and its linear preservers*, Banach J. Math. Anal. **15** (2021), 60.
[32] J. Loreaux, G. Weiss, *Majorization and a Schur-Horn theorem for positive compact operators, the nonzero kernel case*, J. Funct. Anal. **268**(3) (2015), 703–731.
[33] S. M. Manjegani, S. Moein, *Quasi doubly stochastic operator on ℓ^1 and Nielsen's theorem*, J. Math. Phys. **60** (2019), 103508.
[34] A. W. Marshall, I. Olkin, B. C. Arnold, *Inequalities: Theory of majorization and its applications*, second ed., Springer, 2011.
[35] L.Mirsky, *On a convex set of matrices*, Arch. Math. **10** (1959), 88–92.
[36] J. von Neumann, *A certain zero-sums two-person game equivalent to the optimal assignment problem*, Contributions to the Theory of Games **2** (1953), 5–12.
[37] A. Neumann, *An infinite-dimensional generalization of the Schur–Horn convexity theorem*, J. Funct. Anal. **161**(2) (1999), 418–451.
[38] M. A. Nielsen, *An Introduction of Majorization and its Applications to Quantum Mechanics*, Lecture Notes, Department of Physics, University of Queensland, Australia, 2002.
[39] C. P. Niculescu, I. Roventa, *An approach of majorization in spaces with a curved geometry*, J. Math. Anal. Appl. **270**(4) (2014), 1319–1360.
[40] R. Pereira, S. Plosker, *Extending a characterisation of majorization to infinite dimensions*, Linear Algebra Appl. **468** (2015), 80–86.
[41] J. M. Renes, *Relative submajorization and its use in quantum resource theories*, J. Math. Phys. **57** (2016), 122202.