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SURVEY ON STAR PARTIAL ORDER IN INDEFINITE INNER PRODUCT SPACES

Abstract. This is a short survey on the star partial order for matrices which is considered in the spaces with indefinite metric. The specific geometry of these spaces effects even on the definition of a star partial order, demanding the existence of the Moore–Penrose inverse of a certain matrix. A characterization of this partial order is also given. Some of the interesting properties are generalized and illustrated by the appropriate examples.

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1. Introduction

The theory of partial orders has a great progress in the last century. The first results were given in the context of semigroups, then on the set of complex matrices and rings. Most of the partial orderings are based on generalized inverses, although there are some based on annihilators. Today we can find plenty of results on this topic. One of the most important is the monograph [15] and references therein. Also, valuable results on partial orders can be found in [1–3, 6, 7, 10, 12, 14, 16, 17, 22–26]. On the other hand, the matrix partial orderings are not studied much in indefinite inner product spaces. In this survey we give results published in [27] as well as new ones. Here we generalize the notion of the star partial order to indefinite settings, give its characterization in terms of matrices and their Moore–Penrose inverses and show some properties.

This survey is organized as follows. After the Introduction, in Section 2 we present the space C^n equipped with an indefinite inner product, give some basic notions and results concerning those spaces and emphasize the difference with the spaces with standard inner product. In Section 3 we give the definition of the Moore–Penrose inverse for matrices in indefinite setting. This kind of inverse is closely related to the star partial order for matrices, so we examine its properties. In Section 4 we generalize the notion of the star relation for matrices to indefinite case. With an additional condition-existence of the Moore–Penrose inverse of a certain matrix, we get that the star relation is a partial order on C^n. In the Section 5 some of the properties of matrices under the star partial order are presented.

The properties that we generalize here to the indefinite inner product case are given in [1–3, 7, 11, 12]. For arbitrary matrices $A, B \in C^{m\times n}$ ($A, B \in C^{n\times n}$ when needed) these are:

(1) $B^\dagger = B^*$ and $A \leqslant^* B \implies A^\dagger = A^*$,
(2) $B = B^2$ and $A \leqslant^* B \implies A = A^2$,
(3) $B = BB^*$ and $A \leqslant^* B \implies A = AA^*$,
(4) $B^*B^\dagger = B^\dagger B^*$ and $A \leqslant^* B \implies A^*A^\dagger = A^\dagger A^*$,
(5) $A \leqslant^* B \implies B^*A \leqslant^* B^*B$ and $AB^* \leqslant^* BB^*$,
(6) $A \leqslant^* B \implies A^*A \leqslant^* B^*B$ and $AA^* \leqslant^* BB^*$,
(7) $A \leqslant^* B \implies A^\dagger A \leqslant^* B^\dagger B$ and $AA^\dagger \leqslant^* BB^\dagger$,
(8) $A \leqslant^* B \Longleftrightarrow A^\dagger \leqslant^* B^\dagger$,
(9) $A \leqslant^* B$ and $AB = BA \implies A^2 \leqslant^* B^2$.

2. Indefinite inner product spaces

In this section we give a brief review of basic facts concerning the indefinite inner product spaces. More precisely, we give the definition of the indefinite inner product on C^n which is induced by some Hermitian and invertible matrix H. For the difference to a standard inner product the nonnegativity is not assumed in general, which makes the geometry of those spaces specific. In this part we show how it reflects to the orthogonality of subspaces. Most of results in this part are taken from [9].

Definition 2.1. [9] A function $[.\,,.]\colon \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$ is called *an indefinite inner product in* $\mathbb{C}^n$ if the following axioms are satisfied:

(1) linearity in the first argument: $[\alpha x_1 + \beta x_2, y] = \alpha[x_1, y] + \beta[x_2, y]$ for all $x_1, x_2, y \in \mathbb{C}^n$ and all $\alpha, \beta \in \mathbb{C}$;
(2) antisymmerty: $[x, y] = \overline{[y, x]}$ for all $x, y \in \mathbb{C}^n$;
(3) nondegeneracy: if $[x, y] = 0$ for all $y \in \mathbb{C}^n$, then $x = 0$.

We will consider the space C^n equipped with an indefinite inner product. There is a bijection between the set of all indefinite inner products on C^n and the set of all Hermitian invertible $n \times n$ matrices. So we can consider the indefinite inner product induced by arbitrary Hermitian invertible matrix $H \in C^{n\times n}$ by

$$[x, y] = \langle Hx, y\rangle, \quad x, y \in C^n$$

where $\langle .\,, .\rangle$ denotes the standard Euclidean scalar product on C^n. Such a space is called an *indefinite inner product space* (IIPS). A matrix H is called *a weight.* Unlike the standard inner product, the indefinite one does not assume axiom of a positivity, so we can find some $x \in C^n$, $x \neq 0$ such that $[x, x] < 0$.

In indefinite inner product spaces the orthogonal companion of a subset S of C^n is a subspace $S^{[\perp]}$ in C^n defined by

$$S^{[\perp]} = \{x \in C^n \mid [x, y] = 0 \text{ for all } y \in S\}.$$

If the indefinite inner product is induced by a matrix H we usually say that $S^{[\perp]}$ is an H-orthogonal complement of S. Unlike in the Euclidean inner product space, in indefinite case we have one interesting fact that will be of a great importance in the sequel. Namely, the orthogonal companion of a subspace S in C^n is not a direct complement in C^n in general.

A subspace $S \subset C^n$ is nondegenerate if there is no vector $x \in S$, $x \neq 0$ which is H-orthogonal on all vectors from S. In other words, a subspace $S \in C^n$ is nondegenerate if and only if $S \cap S^{[\perp]} = 0$.

Theorem 2.1. [9, Theorem 2.2.2] *$S^{[\perp]}$ is a direct complement to S in C^n if and only if S is nondegenerate.*

By $M = S[\dot{\perp}]T$ we denote that a subspace M is a direct sum of mutually orthogonal subspaces S and T.

How the existence of the Moore–Penrose inverse of a matrix reflects on the orthogonality of appropriate subspaces of C^n is shown in [13] and [27] and will be explicitly stated in the next section.

Many classes of matrices that appear in a definite case have their analogue ones in indefinite inner product spaces. In this survey we deal with the adjoint matrix, or more precisely, H-adjoint matrix of an arbitrary matrix $A \in C^{n\times n}$ and denote it by $A^{[*]}$. For every matrix $A \in C^{n\times n}$ there is the unique matrix $A^{[*]}$ satisfying

$$[A^{[*]}x, y] = [x, Ay], \text{ for all } x, y \in C^n.$$

It is obvious that $A^{[*]} = H^{-1}A^*H$.

We can deal with a more general case. A matrix $A \in C^{m\times n}$ can be considered as a linear transformation from C^n to C^m. Let $N \in C^{n\times n}$ and $M \in C^{m\times m}$ be Hermitian invertible matrices that induce indefinite inner products on C^n and C^m, respectively. The adjoint of a matrix $A \in C^{m\times n}$ is a matrix $A^{[*]} \in C^{n\times m}$ defined by $A^{[*]} = N^{-1}A^*M$.

For adjoint matrices in indefinite inner product spaces we are giving the next familiar result.

Theorem 2.2. [13, Proposition 1] *Let $A, B \in C^{m\times n}$ and $C \in C^{n\times p}$. Then:*

(i) $(A^{[*]})^{[*]} = A$,
(ii) $(AB)^{[*]} = B^{[*]}A^{[*]}$,
(iii) $(A + C)^{[*]} = A^{[*]} + C^{[*]}$,
(iv) *If C^n is an indefinite inner product space with a weight $H \in C^{n\times n}$, then $H^{[*]} = H$.*

Another interesting result that is a consequence of a geometry of an indefinite inner product spaces is the next one. For an arbitrary matrix $A \in C^{m\times n}$ $R(A^{[*]}A) = R(A^{[*]})$ and $R(AA^{[*]}) = R(A)$ does not hold in general. Also, the similar statement for the ranks does not hold either.

We are familiar with the implication that is widely used in many proofs of the results in the theory of general inverses of matrices. If $A^*A = 0$, where A^* is a conjugate-transpose matrix of a matrix $A \in C^{m\times n}$, then $A = 0$ [18]. It does not hold in indefinite case.

The next example illustrates last two statements.

Example 2.1. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then $A^{[*]}A = 0$ but $A \neq 0$.

A great and overall survey of the theory of linear algebra with indefinite inner product can be found in "*Indefinite Linear Algebra and Applications*" by Gohberg, Lancaster and Rodman. More results of indefinite inner product spaces can be found in [5, 8, 9, 19–21, 27].

3. Moore–Penrose inverse in indefinite inner product spaces

The star partial order is closely related to the Moore–Penrose inverses. So in this section we are giving a notion and basic properties of this inverse in indefinite inner product spaces. We emphasize its differences to the one in Euclidean case, and analyze how it reflects on the partial order. The most important one is the existence of the Moore–Penrose inverse for matrices. As in Euclidean spaces it always exists, it is not a case in indefinite settings and in most of the following results the existence has to be assumed. A wide theory of general inverses of matrices in Euclidean space, including the Moore–Penrose inverse are presented in [4].

The definition and basic properties for the Moore–Penrose inverse in nondegenerate indefinite inner product spaces are given by K. Kamaraj and K. C. Siavakumar in [13]. It was shown that in nondegenerate indefinite inner product spaces the Moore–Penrose inverse for a matrix $A \in C^{m\times n}$ is the unique matrix $A^{[\dagger]} \in C^{n\times m}$ that satisfies following equations:

$$AA^{[\dagger]}A = A, \quad A^{[\dagger]}AA^{[\dagger]} = A^{[\dagger]}, \quad AA^{[\dagger]} = (AA^{[\dagger]})^{[*]} \quad \text{and} \quad A^{[\dagger]}A = (A^{[\dagger]}A)^{[*]}.$$

Necessary and sufficient conditions for the existence of the Moore–Penrose inverse are given in the next Theorem.

Theorem 3.1. [13, Theorem 1] *Let $A \in C^{m\times n}$. Then $A^{[\dagger]}$ exists if and only if*

$$r(A) = r(A^{[*]}A) = r(AA^{[*]}).$$

If $A^{[\dagger]}$ exists, then it is unique.

Some of the basic properties of the Moore–Penrose inverses for matrices are quite similar to those in Euclidean space and they are listed in the next theorem.

Theorem 3.2. *Let $\lambda \in C$ and $A \in C^{m\times n}$. If $A^{[\dagger]}$ exists, then the following properties hold:*

(i) $(A^{[\dagger]})^{[\dagger]} = A$;

(ii) $(A^{[\dagger]})^{[*]} = (A^{[*]})^{[\dagger]}$;

(iii) $(\lambda A)^{[\dagger]} = \lambda^{[\dagger]}A^{[\dagger]}$, *where* $\lambda^{[\dagger]} = \begin{cases} \lambda^{-1}, & \lambda \neq 0, \\ 0, & \lambda = 0 \end{cases}$;

(iv) $A^{[*]} = A^{[*]}AA^{[\dagger]}$ *and* $A^{[*]} = A^{[\dagger]}AA^{[*]}$;

(v) $(A^{[*]}A)^{[\dagger]} = A^{[\dagger]}(A^{[\dagger]})^{[*]}$ *and* $(AA^{[*]})^{[\dagger]} = (A^{[*]})^{[\dagger]}A^{[\dagger]}$;

(vi) $A^{[\dagger]} = (A^{[*]}A)^{[\dagger]}A^{[*]} = A^{[*]}(AA^{[*]})^{[\dagger]}$.

In [9] the authors showed the connection between the images and kernels of matrices and their adjoint matrices in indefinite settings. Here we give the Theorem and the proof for the sake of completeness.

Theorem 3.3. [9, Theorem 4.1.1] *Let $A \in C^{m\times n}$. Then*

$$R(A^{[*]}) = (N(A))^{[\perp]} \text{ and } N(A^{[*]}) = (R(A))^{[\perp]}.$$

Proof. Let $x \in R(A^{[*]})$ so that $x = A^{[*]}y$ for some $y \in C^n$. Then for every $z \in N(A)$: $[x, z] = [A^{[*]}y, z] = [y, Az] = 0$, and it follows that

$$R(A^{[*]}) \subseteq (N(A))^{[\perp]}. \tag{3.1}$$

Now,

$$\dim(R(A^{[*]})) = \dim(R(A^*)) = n - \dim(N(A)) = \dim(N(A))^{[\perp]},$$

so that equality must obtain in (3.1).

The proof of the second relation is similar. □

In [13] it is shown that for a matrix $A \in C^{m\times n}$ subspaces $R(A)$ and $N(A^{[*]})$ are not complementary in general, although they are orthogonal. These subspaces are complementary in C^m under the assumption of the existence of the Moore–Penrose inverse of a matrix A.

Theorem 3.4. [13, Theorem 8] *Let $A \in C^{m\times n}$ such that $A^{[\dagger]}$ exists. Then $R(A)$ and $N(A^{[*]})$ are orthogonal complementary subspaces of C^m.*

Proof. Let $x \in R(A)$ and $y \in N(A^{[*]})$. Then for some z, $[x, y] = [Az, y] = [z, A^{[*]}y] = 0$. Thus, $R(A)$ and $N(A^{[*]})$ are mutually orthogonal. Let $x \in R(A) \cap N(A^{[*]})$, then for some $y \in C^n$,

$$x = Ay = AA^{[\dagger]}Ay = AA^{[\dagger]}x = (A^{[\dagger]})^{[*]}A^{[*]}x = 0.$$

Thus, $R(A) \cap N(A^{[*]}) = \{0\}$. Since $rank(A) = rank(A^{[*]})$, the dimensions of $N(A^{[*]})$ and $N(A)$ are equal. Then by the rank nullity dimension theorem, $R(A) \oplus N(A^{[*]}) = C^m$. This completes the proof. □

Example 3.1. Taking the matrices A and H given in Example 2.1. it easily follows that $R(A) = N(A^{[*]})$, i.e. $R(A)$ and $N(A^{[*]})$ are not complementary in C^m.

As a generalization of this results we showed that the existence of the Moore–Penrose inverse of an arbitrary matrix $A \in C^{m\times n}$ is the necessary and sufficient condition for $R(A^{[*]})$ and $N(A)$ (i.e. $R(A)$ and $N(A^{[*]})$) being direct complements in appropriate spaces. This result is published in [27].

Theorem 3.5. *Let $A \in C^{m\times n}$. Then $R(A)$ and $N(A^{[*]})$ are orthogonal complementary subspaces of C^m and $R(A^{[*]})$ and $N(A)$ are orthogonal complementary subspaces of C^n if and only if $A^{[\dagger]}$ exists.*

Proof. $(:\Leftarrow)$ The first part is a consequence of the Theorem 3.4.

It is easy to check that $R(A^{[*]})$ and $N(A)$ are orthogonal subspaces in C^n. We prove that they are complementary. Let $x \in N(A) \cap R(A^{[*]})$. Then $Ax = 0$ and $x = A^{[*]}y$ for some vector $y \in C^n$.

As $A^{[*]} = A^{[*]}AA^{[\dagger]}$ it follows that

$$\begin{aligned} x &= A^{[*]}y = A^{[*]}AA^{[\dagger]}y = A^{[*]}(A^{[\dagger]})^{[*]}A^{[*]}y \\ &= A^{[*]}(A^{[\dagger]})^{[*]}x = (A^{[\dagger]}A)^{[*]}x = A^{[\dagger]}Ax = 0. \end{aligned}$$

(:⇒) Let $A \in C^{m\times n}$ be a matrix such that $R(A)$ and $N(A^{[*]})$ are orthogonal complementary subspaces of C^m and $R(A^{[*]})$ and $N(A)$ are orthogonal complementary subspaces of C^n.

Let $y \in R(A)$, then $y = Ax$ for some $x \in C^n$. Under the assumption $C^n = N(A)[\dot{\perp}]R(A^{[*]})$ it follows that $x = p + q$ for some $p \in N(A)$ and $q \in R(A^{[*]})$.

Thus we have $y = Ax = A(p+q) = Aq = AA^{[*]}t$, for some vector $t \in C^n$. It follows that $R(A) \subseteq R(AA^{[*]})$. The opposite always holds true so we have $R(A) = R(AA^{[*]})$ and so $r(A) = r(AA^{[*]})$.

In the same manner it can be achieved the second part, i.e. $R(A^{[*]}) = R(A^{[*]}A)$ and so $r(A^{[*]}) = r(A^{[*]}A)$.

According to Theorem 2.1, the Moore–Penrose inverse for a matrix $A \in C^{m\times n}$ exists. This completes the proof. □

It turns out that the existence of the Moore–Penrose inverse is a sufficient condition for the implication that we mentioned at the beginning $A^{[*]}A = 0 \implies A = 0$, i.e. we give the next lemma.

Lemma 3.1. [27] *Let $A \in C^{m\times n}$. If $A^{[\dagger]}$ exists then $A^{[*]}A = 0$ implies $A = 0$.*

Proof. Let $A \in C^{m\times n}$ such that $A^{[\dagger]}$ exists. Assume that $A^{[*]}A = 0$. A multiplication by $A^{[\dagger]}$ from the right hand side gives $A^{[*]} = A^{[*]}AA^{[\dagger]} = 0$, which is equivalent to $A = 0$. □

4. Definition and characterization of the star partial order in indefinite inner product spaces

The star relation ($\leqslant *$) on semigroup S with involution is introduced by Drazin in 1978. in [7] as: for $a, b \in S$

$$a \leqslant^* b \iff a^*a = a^*b \quad \text{and} \quad aa^* = ba^*.$$

If the involution is proper on S then this relation is a partial order and it is called *the star partial order*. Remind that the involution is *proper* on S if for all $a, b \in S$

$$a^*a = a^*b = b^*a = b^*b \Longrightarrow a = b.$$

Applied on the algebra M_n the star partial ordering for matrices is defined in a similar way: Let $A, B \in C^{n\times n}$ be arbitrary matrices. Then

$$A \leqslant^* B \iff A^*A = A^*B \quad \text{and} \quad AA^* = BA^*.$$

In the same paper Drazin showed that

$$A \leqslant^* B \iff A^\dagger A = A^\dagger B \quad \text{and} \quad AA^\dagger = BA^\dagger,$$

which is often taken for a definition of the star matrix partial order. Similarly, we also have:

$$A \leqslant^* B \iff A^\dagger A = B^\dagger A \quad \text{and} \quad AA^\dagger = AB^\dagger.$$

These characterizations are equivalent to

$$A \leqslant^* B \iff AA^\dagger B = A = BA^\dagger A \iff B^\dagger AA^\dagger = A^\dagger = A^\dagger AB^\dagger,$$

which is given by Hartwig in [10].

First, we give the definition of the star relation and then of the partial order in indefinite inner product spaces.

Definition 4.1. The star relation ($\leqslant^{[*]}$) for matrices in indefinite inner product spaces is defined as follows: Let $A, B \in \mathbb{C}^{m\times n}$. Then $A \leqslant^{[*]} B$ if $AA^{[*]} = BA^{[*]}$ and $A^{[*]}A = A^{[*]}B$.

This relation on indefinite inner product settings is not a partial ordering. Moreover, it is not even a preorder in that case because it is neither antisymmetric nor a transitive one. The next two examples show that.

Example 4.1. Let $H = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ be a matrix that induces the indefinite inner product on C^n, $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and $B = 2A$. Then $AA^{[*]} = BA^{[*]}$ and $A^{[*]}A = A^{[*]}B$, i.e. $A \leqslant^{[*]} B$.

Also, $BB^{[*]} = AB^{[*]}$ and $B^{[*]}B = B^{[*]}A$, i.e. $B \leqslant^{[*]} A$.

But it does not follow that $A = B$.

Example 4.2. Let

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix}.$$

It is easy to show that $A \leqslant^{[*]} B$ and $B \leqslant^{[*]} C$. Also, $AA^{[*]} = CA^{[*]}$, but $A^{[*]}A \neq A^{[*]}C$. Thus, it is not true that $A \leqslant^{[*]} C$. So the transitivity does not hold.

Thus, in Definition 4.1 we will demand the existence of the Moore–Penrose inverse of a matrix A. It turns out that the star relation in that case defines a partial order for matrices in indefinite inner product spaces. More precisely, we have the next definition.

Definition 4.2. Let $A, B \in \mathbb{C}^{m\times n}$. Then $A \leqslant^{[*]} B$ if

(1) $A^{[\dagger]}$ exists,
(2) $AA^{[*]} = BA^{[*]}$ and $A^{[*]}A = A^{[*]}B$.

Before we show that the star relation defined in Definition 4.2 is a partial order, we give some results that will be useful and can be found in [27].

Theorem 4.1. *Let $A, B \in \mathbb{C}^{m\times n}$. Then $A \leqslant^{[*]} B$ if and only if $A^{[\dagger]}$ exists and $AA^{[\dagger]} = BA^{[\dagger]}$ and $A^{[\dagger]}A = A^{[\dagger]}B$.*

Proof. Let $A \leqslant^{[*]} B$. By Definition 4.2 the Moore–Penrose inverse of a matrix A exists. Also $A^{[*]}A = A^{[*]}B$ and $AA^{[*]} = BA^{[*]}$ hold. From the existence of $A^{[\dagger]}$ and the property $A^{[*]}AA^{[\dagger]} = A^{[*]}$ the first equality becomes

$$A^{[*]}AA^{[\dagger]}A = A^{[*]}AA^{[\dagger]}B, \quad \text{i.e.,} \quad A^{[*]}A(A^{[\dagger]}A - A^{[\dagger]}B) = 0,$$

which is satisfied if and only if

$$R(A^{[\dagger]}A - A^{[\dagger]}B) \subseteq N(A^{[*]}A) = N(A) = R(A^{[*]})^{[\perp]}.$$

Thus, $y \in R(A^{[\dagger]}A - A^{[\dagger]}B) \implies y \in R(A^{[*]})^{[\perp]}$. So, there is some $x \in C^n$ such that $y = (A^{[\dagger]}A - A^{[\dagger]}B)x$. Now, for every $z \in C^n$ we have

$$\begin{aligned} 0 = [y, A^{[*]}z] &= [(A^{[\dagger]}A - A^{[\dagger]}B)x, A^{[*]}z] \\ &= [(AA^{[\dagger]}A - AA^{[\dagger]}B)x, z] = [(A - AA^{[\dagger]}B)x, z] \end{aligned}$$

and so $A - AA^{[\dagger]}B = 0$. Thus, $A = AA^{[\dagger]}B$. By multiplying the last equality by $A^{[\dagger]}$ from the left side we obtain $A^{[\dagger]}A = A^{[\dagger]}B$.

The other equality can be proved in the similar way. Let $AA^{[*]} = BA^{[*]}$. Then

$$AA^{[\dagger]}AA^{[*]} = BA^{[\dagger]}AA^{[*]}, \quad \text{i.e.,} \quad AA^{[*]}AA^{[\dagger]} = AA^{[*]}(A^{[\dagger]})^{[*]}B^{[*]},$$

and so $AA^{[*]}(AA^{[\dagger]} - (A^{[\dagger]})^{[*]}B^{[*]}) = 0$. Now, we have

$$R(AA^{[\dagger]} - (A^{[\dagger]})^{[*]}B^{[*]}) \subseteq N(AA^{[*]}) = N^{[*]} = R(A)^{[\perp]}.$$

Now, $y \in R(AA^{[\dagger]} - (A^{[\dagger]})^{[*]}B^{[*]}) \implies y \in R(A)^{[\perp]}$. Thus, for $x \in C^n$ such that $y = (AA^{[\dagger]} - (A^{[\dagger]})^{[*]}B^{[*]})x$ and an arbitrary $z \in C^n$ we have:

$$\begin{aligned} 0 = [y, Az] &= [(AA^{[\dagger]} - (A^{[\dagger]})^{[*]}B^{[*]})x, Az] \\ &= [(A^{[*]} - (A^{[\dagger]}A)^{[*]}B^{[*]})x, z] = [(A^{[*]} - A^{[\dagger]}AB^{[*]})x, z]. \end{aligned}$$

This implies $A^{[*]} - A^{[\dagger]}AB^{[*]} = 0$, i.e., $A^{[*]} = A^{[\dagger]}AB^{[*]}$. Taking adjoint of both sides gives $A = BA^{[\dagger]}A$, and so $AA^{[\dagger]} = BA^{[\dagger]}$. □

Now, we could give a characterization similar to the appropriate one in Euclidean case:

$$A \leqslant^{[*]} B \iff AA^{[\dagger]}B = A = BA^{[\dagger]}A.$$

The similar result $AA^{[\dagger]} = AB^{[\dagger]}$ and $A^{[\dagger]}A = AB^{[\dagger]}$ still cannot be achieved without assuming the existence of $B^{[\dagger]}$.

The next example illustrates this and can be found in [27].

Example 4.3. Let matrix

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

induce the indefinite inner product in C^n,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

It is easy to verify that $A^{[\dagger]}$ exists as $r(A) = r(AA^{[*]}) = r(A^{[*]}A)$. Also, $AA^{[*]} = BA^{[*]}$ and $A^{[*]}A = A^{[*]}B$, so $A \leqslant^{[*]} B$, but $B^{[\dagger]}$ does not exist.

Theorem 4.2. *Let $A, B \in \mathbb{C}^{m\times n}$ such that $B^{[\dagger]}$ exists. Then*

$$A \leqslant^{[*]} B \iff A^{[\dagger]}A = B^{[\dagger]}A \quad \textit{and} \quad AA^{[\dagger]} = AB^{[\dagger]}.$$

Proof. From $A \leqslant^{[*]} B$ it follows that $A^{[\dagger]}$ exists, too. Replacing $B^{[*]} = B^{[\dagger]}BB^{[*]}$ in $A^{[*]}A = B^{[*]}A$, we get

$$A^{[*]}A = B^{[\dagger]}BB^{[*]}A \tag{4.1}$$

As $B^{[*]}A = A^{[*]}A$ and $BA^{[*]} = AA^{[*]}$, (4.1) becomes

$$A^{[*]}A = B^{[\dagger]}AA^{[*]}A. \tag{4.2}$$

Multiplication of (4.2) by $A^{[\dagger]}(A^{[\dagger]})^{[*]}$ on the right side gives

$$A^{[*]}(A^{[\dagger]})^{[*]} = B^{[\dagger]}AA^{[*]}(A^{[\dagger]})^{[*]},$$

which is equivalent to

$$(A^{[\dagger]}A)^{[*]} = B^{[\dagger]}A(A^{[\dagger]}A)^{[*]}.$$

Now it follows that $A^{[\dagger]}A = B^{[\dagger]}AA^{[\dagger]}A$, or, equivalently, $A^{[\dagger]}A = B^{[\dagger]}A$.

The second part can be achieved in similar way, so $AA^{[\dagger]} = AB^{[\dagger]}$. □

Finally, we can give the next characterization for matrices $A, B \in \mathbb{C}^{m\times n}$ when $B^{[\dagger]}$ exists.

$$A \leqslant^{[*]} B \Longleftrightarrow A^{[\dagger]} = B^{[\dagger]}AA^{[\dagger]} = A^{[\dagger]}AB^{[\dagger]}. \tag{4.3}$$

Theorem 4.3. *The star relation defines the partial order in indefinite inner product spaces.*

Proof. The reflexivity condition for an arbitrary matrix $A \in \mathbb{C}^{m\times n}$ is obviously satisfied.

Now, assume that $A \leqslant^{[*]} B$ and $B \leqslant^{[*]} A$, for $A, B \in \mathbb{C}^{m\times n}$. From Definition 4.1 it follows that $A^{[\dagger]}$ and $B^{[\dagger]}$ exist and from Theorem 4.1 we have

$$A = BA^{[\dagger]}A \quad \text{and} \quad A = AA^{[\dagger]}B; \tag{4.4}$$

$$B = BA^{[\dagger]}B \quad \text{and} \quad B = BB^{[\dagger]}A. \tag{4.5}$$

Also, from $BA^{[\dagger]} = AA^{[\dagger]}$ it follows (by multiplying with $BB^{[\dagger]}$ on the left side) that $BA^{[\dagger]} = BB^{[\dagger]}AA^{[\dagger]}$. Thus, using (4.3), (4.4) and (4.5), we achieve

$$A = BA^{[\dagger]}A = BB^{[\dagger]}AA^{[\dagger]}A = BB^{[\dagger]}A = B.$$

So, the antisymmetry holds.

Let us prove the transitivity. Assume that matrices $A, B, C \in \mathbb{C}^{m\times n}$ are in the star relation $A \leqslant^{[*]} B$ and $B \leqslant^{[*]} C$. Then $A^{[\dagger]}$ exists and the next equalities hold:

$$A^{[*]}A = A^{[*]}B \quad \text{and} \quad AA^{[*]} = BA^{[*]};$$

$$B^{[*]}B = B^{[*]}C \quad \text{and} \quad BB^{[*]} = CB^{[*]}.$$

Then

$$\begin{aligned} A^{[*]}C &= A^{[\dagger]}AA^{[*]}C = A^{[\dagger]}AB^{[*]}C = A^{[\dagger]}AB^{[*]}B \\ &= A^{[\dagger]}AA^{[*]}B = A^{[\dagger]}AA^{[*]}A = A^{[*]}A. \end{aligned}$$

Similarly, we have

$$CA^{[*]} = CA^{[*]}AA^{[\dagger]} = CB^{[*]}AA^{[\dagger]} = BB^{[*]}AA^{[\dagger]}$$
$$= BA^{[*]}AA^{[\dagger]} = AA^{[*]}AA^{[\dagger]} = AA^{[*]}.$$

Thus, the relation ($\leqslant^{[*]}$) defines the partial order, which we call the *star partial order* in indefinite inner product spaces. □

5. Properties of the star partial order in indefinite inner product spaces

In this section we give some interesting properties of matrices under the star partial order in indefinite case. In [3, 7, 11, 12] the authors collected results with inheriting properties under this partial order. They showed that if $A \leqslant^* B$ for some matrices $A, B \in C^{m\times n}$ and matrix B has a certain property (partial isometry, idempotency, orthogonal projectors) then the matrix A has that property, too. We generalize these and other properties to indefinite inner product spaces.

Theorem 5.1. *Let $A, B \in \mathbb{C}^{m\times n}$. The next implications hold:*
(1) *If $B^{[\dagger]}$ exists and $B^{[\dagger]} = B^{[*]}$ and $A \leqslant^{[*]} B$ then $A^{[\dagger]} = A^{[*]}$,*
(2) *$B = B^2$ and $A \leqslant^{[*]} B \implies A = A^2$,*
(3) *$B = BB^{[*]}$ and $A \leqslant^{[*]} B \implies A = AA^{[*]}$.*

Proof. (1) Let a matrix $B \in C^{m\times n}$ be a partial isometry in indefinite inner product spaces such that its Moore–Penrose inverse exists and let a matrix $A \in C^{m\times n}$ be such that $A \leqslant^{[*]} B$. A multiplication of $B^{[\dagger]} = B^{[*]}$ by A on the left side gives

$$AB^{[\dagger]} = AB^{[*]}. \tag{5.1}$$

From Theorem 4.2 and from Definition 4.1 $A^{[\dagger]}$ exists and the equality (5.1) becomes $AA^{[\dagger]} = AA^{[*]}$, which, after the multiplication by $A^{\dagger]}$ on the left side, becomes $A^{[\dagger]} = A^{[*]}$.

(2) Let $A \leqslant^{[*]} B$. Thus $A^{[\dagger]}$ exists. Assume that $B = B^2$, i.e. that a matrix B is idempotent. If we multiply the last equality by $A^{[\dagger]}$ from both sides, we get

$$A^{[\dagger]}BA^{[\dagger]} = A^{[\dagger]}BBA^{[\dagger]}. \tag{5.2}$$

By Theorem 4.1 and the definition of star partial order, (5.2) becomes $A^{[\dagger]} = A^{[\dagger]}AAA^{[\dagger]}$. After the multiplication by A both left and right, we get the desired property, i.e., $A = A^2$.

(3) Let $A \leqslant^{[*]} B$ and $B = BB^{[*]}$. We multiply the last equality by $A^{[*]}$ from the left side and apply $A^{[*]}A = A^{[*]}B$ and $AA^{[*]} = BA^{[*]}$ and get

$$A^{[*]}A = A^{[*]}AA^{[*]}. \tag{5.3}$$

Now, the multiplication of (5.3) by $(A^{[*]})^{[\dagger]}$ on the left and [13, Proposition 3] gives $A = AA^{[*]}$. □

Theorem 5.2. [27] *Let $A, B \in \mathbb{C}^{m\times n}$ such that $B^{[\dagger]}$ exist. The next implication holds:*

$$(B^{[*]}B^{[\dagger]} = B^{[\dagger]}B^{[*]} \quad \textit{and} \quad A \leqslant^{[*]} B) \implies A^{[*]}A^{[\dagger]} = A^{[\dagger]}A^{[*]}.$$

Proof. The existence of $A^{[\dagger]}$ follows from $A \leqslant^{[*]} B$. Let us multiply $B^{[*]}B^{[\dagger]} = B^{[\dagger]}B^{[*]}$ by A on both sides. Then we get $AB^{[*]}B^{[\dagger]}A = AB^{[\dagger]}B^{[*]}A$. From Definition 4.2 and Theorem 4.1 it follows that $AA^{[*]}A^{[\dagger]}A = AA^{[\dagger]}A^{[*]}A$. Multiplication of the last equality by $A^{[\dagger]}$ on both sides gives us the desired property

$$A^{[*]}A^{[\dagger]} = A^{[\dagger]}A^{[*]}.$$

□

Theorem 5.3. *Let $A, B \in \mathbb{C}^{m\times n}$. The next implication holds:*

$$A \leqslant^{[*]} B \implies (B^{[*]}A \leqslant^{[*]} B^{[*]}B \quad \text{and} \quad AB^{[*]} \leqslant^{[*]} BB^{[*]}).$$

Proof. For proving the first part $A \leqslant^{[*]} B \implies B^{[*]}A \leqslant^{[*]} B^{[*]}B$ we need to show the existence of the Moore–Penrose inverse of $B^{[*]}A$ and that the next two equalities:

$$B^{[*]}AA^{[*]}B = B^{[*]}BA^{[*]}B \quad \text{and} \quad A^{[*]}BB^{[*]}A = A^{[*]}BB^{[*]}B$$

hold. Let us prove the existence of $(B^{[*]}A)^{[\dagger]}$. According to the Theorem 3.1, this Moore–Penrose inverse exists if and only if $r(B^{[*]}A) = r(B^{[*]}AA^{[*]}B) = r(A^{[*]}BB^{[*]}A)$.

We have

$$\begin{aligned} r(B^{[*]}A) &= r(B^{[*]}AA^{[*]}(A^{[\dagger]})^{[*]}) \leqslant r(B^{[*]}AA^{[*]}) \\ &= r(B^{[*]}AA^{[*]}AA^{[\dagger]}) = r(B^{[*]}AA^{[*]}BA^{[\dagger]}) \\ &\leqslant r(B^{[*]}AA^{[*]}B) \leqslant r(B^{[*]}A). \end{aligned} \tag{5.4}$$

Thus, we proved that $r(B^{[*]}AA^{[*]}B) = r(B^{[*]}A)$.

From Definition 4.2 it follows that $B^{[*]}AA^{[*]}B = A^{[*]}BB^{[*]}A$, and, together with (5.4) it proves the existence of $(B^{[*]}A)^{[\dagger]}$.

Similarly, the existence of the Moore–Penrose inverse of $AB^{[*]}$ can be shown. As $A \leqslant^{[*]} B$, we have $A^{[*]}A = A^{[*]}B$. If we multiply it by $B^{[*]}B$ on the left side, we obtain $B^{[*]}BA^{[*]}A = B^{[*]}BA^{[*]}B$, which is equivalent to

$$B^{[*]}AA^{[*]}A = B^{[*]}BA^{[*]}A \quad \text{and} \quad B^{[*]}AA^{[*]}B = B^{[*]}BA^{[*]}B,$$

which we wanted to show.

Also, $A^{[*]}A = A^{[*]}B$. After the multiplication with $B^{[*]}B$ on the right side we get $A^{[*]}AB^{[*]}B = A^{[*]}BB^{[*]}B$, which is equivalent to $A^{[*]}BB^{[*]}A = A^{[*]}BB^{[*]}B$.

The second statement can be proved similarly. □

For the next result we need a theorem, given in [13] which claims that the existence of $A^{[\dagger]}$ implies the existence of $(AA^{[*]})^{[\dagger]}$ and $(A^{[*]}A)^{[\dagger]}$.

Theorem 5.4. [13, Theorem 7] *Let $A \in \mathbb{C}^{m\times n}$. If $A^{[\dagger]}$ exists then both $(AA^{[*]})^{[\dagger]}$ and $(A^{[*]}A)^{[\dagger]}$ exist. In that case*

$$(AA^{[*]})^{[\dagger]} = (A^{[*]})^{[\dagger]}A^{[\dagger]} \quad \text{and} \quad (A^{[*]}A)^{[\dagger]} = A^{[\dagger]}(A^{[*]})^{[\dagger]}.$$

Theorem 5.5. *Let $A, B \in \mathbb{C}^{m\times n}$. The next implication holds:*

$$A \leqslant^{[*]} B \implies A^{[*]}A \leqslant^{[*]} B^{[*]}B \quad \text{and} \quad AA^{[*]} \leqslant^{[*]} BB^{[*]}.$$

Proof. Let $A \leqslant^{[*]} B$. It follows that $A^{[\dagger]}$ exists. Now, accordingly to Theorem 5.4 $(AA^{[*]})^{[\dagger]}$ and $(A^{[*]}A)^{[\dagger]}$ exist. We also have $A^{[*]}A = A^{[*]}B$ and $AA^{[*]} = BA^{[*]}$, as well as $A^{[*]}A = B^{[*]}A$ and $AA^{[*]} = AB^{[*]}$. Now,

$$B^{[*]}BA^{[*]}A = B^{[*]}AA^{[*]}A = A^{[*]}AA^{[*]}A.$$

Also, $A^{[*]}AB^{[*]}B = A^{[*]}AA^{[*]}B = A^{[*]}AA^{[*]}A$. The last two equalities are equivalent to $A^{[*]}A \leqslant^{[*]} B^{[*]}B$.

The proof for $AA^{[*]} \leqslant^{[*]} BB^{[*]}$ is analogue. □

In the previous theorem an implication cannot be replaced by an equivalence, i.e. $A \leqslant^{[*]} B$ implies $A^{[*]}A \leqslant^{[*]} B^{[*]}B$ and $AA^{[*]} \leqslant^{[*]} BB^{[*]}$, but the opposite does not hold. It is illustrated by the next example.

Example 5.1. [27, Example 3.1] Let $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $AA^{[*]} = A^{[*]}A = 0$, and so $A^{[*]}A \leqslant^{[*]} B^{[*]}B$ and $AA^{[*]} \leqslant^{[*]} BB^{[*]}$. But $A^{[*]}B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \neq A^{[*]}A$. Thus, it is not true that $A \leqslant^{[*]} B$.

Theorem 5.6. *Let $A, B \in \mathbb{C}^{m\times n}$ such that $B^{[\dagger]}$ exists. The next implication holds:*

$$A \leqslant^{[*]} B \implies (A^{[\dagger]}A \leqslant^{[*]} B^{[\dagger]}B \quad \textit{and} \quad AA^{[\dagger]} \leqslant^{[*]} BB^{[\dagger]}).$$

Proof. From $A \leqslant^{[*]} B$ it follows that $A^{[\dagger]}$ exists and

$$A^{[\dagger]}A = A^{[\dagger]}B \quad \text{and} \quad AA^{[\dagger]} = BA^{[\dagger]}. \tag{5.5}$$

It is obvious that the Moore–Penrose inverse of $A^{[\dagger]}A$ and $AA^{[\dagger]}$ exist. From (5.5) and (4.3) we have:

$$(A^{[\dagger]}A)^{[*]}A^{[\dagger]}A = (A^{[\dagger]}A)^{[*]}A^{[\dagger]}B = (A^{[\dagger]}A)^{[*]}A^{[\dagger]}AB^{[\dagger]}B = A^{[\dagger]}AB^{[\dagger]}B,$$
$$A^{[\dagger]}A(A^{[\dagger]}A)^{[*]} = B^{[\dagger]}AA^{[\dagger]}A(A^{[\dagger]}A)^{[*]} = B^{[\dagger]}AA^{[\dagger]}A = B^{[\dagger]}BA^{[\dagger]}A.$$

Now, from the last two lines $A^{[\dagger]}A \leqslant^{[*]} B^{[\dagger]}B$ follows.

The rest of the proof is analogue. □

The condition $A \leqslant^{[*]} B$ from Theorem 5.6 can be relaxed by $A^{[*]}A \leqslant^{[*]} B^{[*]}B$, although the assumption of the existence of $A^{[\dagger]}$ and $B^{[\dagger]}$ can not be omitted.

Drazin showed that in Euclidean spaces $A^{\dagger}A \leqslant^{*} B^{\dagger}B$ follows from $A^{*}A \leqslant^{*} B^{*}B$. In indefinite inner product spaces it is not the case even if we assume the existence of $(A^{[*]}A)^{[\dagger]}$ and $(B^{[*]}B)^{[\dagger]}$. Actually, the existence of $(A^{[*]}A)^{[\dagger]}$ does not imply the existence of $A^{[\dagger]}$. Example 5.1 proves it, too.

Theorem 5.7. *Let $A, B \in \mathbb{C}^{m\times n}$ and $A^{[\dagger]}$ and $B^{[\dagger]}$ exist. Then*

$$A^{[*]}A \leqslant^{[*]} B^{[*]}B \implies A^{[\dagger]}A \leqslant^{[*]} B^{[\dagger]}B.$$

Proof. Let $A^{[\dagger]}$ and $B^{[\dagger]}$ exist and $A^{[*]}A \leqslant^{[*]} B^{[*]}B$. According to Theorem 5.4, as $A^{[\dagger]}$ and $B^{[\dagger]}$ exist, $(AA^{[*]})^{[\dagger]}$ and $(A^{[*]}A)^{[\dagger]}$ exist too, and in that case, $(AA^{[*]})^{[\dagger]} = (A^{[*]})^{[\dagger]}A^{[\dagger]}$ and $(A^{[*]}A)^{[\dagger]} = A^{[\dagger]}(A^{[*]})^{[\dagger]}$.

Also, by [13, Corollary 2], $A^{[\dagger]} = A^{[*]}(AA^{[*]})^{[\dagger]} = (A^{[*]}A)^{[\dagger]}A^{[*]}$.

From $A^{[*]}A \leqslant^{[*]} B^{[*]}B$ by Theorem 5.6 it follows that

$$(A^{[*]}A)^{[\dagger]}A^{[*]}A \leqslant^{[*]} (B^{[*]}B)^{[\dagger]}B^{[*]}B,$$

which is equivalent to $A^{[\dagger]}A \leqslant^{[*]} B^{[\dagger]}B$. □

An analogue result also holds:

Theorem 5.8. *Let $A, B \in \mathbb{C}^{m\times n}$ and $A^{[\dagger]}$ and $B^{[\dagger]}$ exist. The next implication holds:*

$$AA^{[*]} \leqslant^{[*]} BB^{[*]} \implies AA^{[\dagger]} \leqslant^{[*]} BB^{[\dagger]}.$$

The next example illustrates the fact that the assumption of the existence of $A^{[\dagger]}$ and $B^{[\dagger]}$ can not be replaced by the existence of the $(A^{[*]}A)^{[\dagger]}$ and $(B^{[*]}B)^{[\dagger]}$.

Example 5.2. Let

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

It is easy to verify that $A^{[*]}A \leqslant^{[*]} B^{[*]}B$. Also, $A^{[\dagger]}$, $(A^{[*]}A)^{[\dagger]}$ and $(B^{[*]}B)^{[\dagger]}$ exist. But $B^{[\dagger]}$ do not exist, thus the implication does not hold.

The Moore–Penrose inverse for matrices is isotonic in indefinite inner product spaces, so we give the following theorem.

Theorem 5.9. *Let $A, B \in \mathbb{C}^{m\times n}$ such that $B^{[\dagger]}$ exists. Then*

$$A \leqslant^{[*]} B \iff A^{[\dagger]} \leqslant^{[*]} B^{[\dagger]}.$$

Proof. Let $A^{[\dagger]}$ and $B^{[\dagger]}$ exist and $A \leqslant^{[*]} B$. By Theorem 4.2, we have:

$$A \leqslant^{[*]} B \iff (AA^{[\dagger]} = AB^{[\dagger]} \quad \text{and} \quad A^{[\dagger]}A = B^{[\dagger]}A),$$

which is equivalent to

$$((A^{[\dagger]})^{[\dagger]}A^{[\dagger]} = (A^{[\dagger]})^{[\dagger]}B^{[\dagger]} \quad \text{and} \quad A^{[\dagger]}(A^{[\dagger]})^{[\dagger]} = B^{[\dagger]}(A^{[\dagger]})^{[\dagger]}). \tag{5.6}$$

Now, (5.6) is equivalent to $A^{[\dagger]} \leqslant^{[*]} B^{[\dagger]}$. □

Theorem 5.10. *Let $A, B \in \mathbb{C}^{n\times n}$. Then*

$$A \leqslant^{[*]} B \quad \text{and} \quad AB = BA \implies A^2 \leqslant^{[*]} B^2.$$

Proof. Let $A \leqslant^{[*]} B$ and $AB = BA$. Then

$$\begin{aligned} (A^2)^{[*]}B^2 &= A^{[*]}A^{[*]}BB = A^{[*]}A^{[*]}AB \\ &= A^{[*]}A^{[*]}BA = A^{[*]}A^{[*]}AA = (A^2)^{[*]}A^2, \\ B^2(A^2)^{[*]} &= BBA^{[*]}A^{[*]} = BAA^{[*]}A^{[*]} \\ &= ABA^{[*]}A^{[*]} = AAA^{[*]}A^{[*]} = A^2(A^2)^{[*]}. \end{aligned}$$

Thus, $A^2 \leqslant^{[*]} B^2$. □

Corollary 5.1. *Let $A, B \in \mathbb{C}^{n\times n}$ such that $A^{[\dagger]}, B^{[\dagger]}$ and $(B-A)^{[\dagger]}$ exist. Then the following statements are equivalent:*

(i) $A \leqslant^{[*]} B$; (iii) $(B-A)^{[\dagger]} \leqslant^{[*]} B^{[\dagger]}$;

(ii) $(B-A) \leqslant^{[*]} B$; (iv) $B^{[\dagger]} - A^{[\dagger]} \leqslant^{[*]} B^{[\dagger]}$;

Proof. Let $A^{[\dagger]}, B^{[\dagger]}$ and $(A-B)^{[\dagger]}$ exist.

(i) $\Rightarrow$ (ii) Assume that $A \leqslant^{[*]} B$ and so $A^{[*]}A = A^{[*]}B = B^{[*]}A$. Now we have:

$$\begin{aligned}(B-A)^{[*]}(B-A) &= (B^{[*]}-A^{[*]})(B-A) = B^{[*]}B - B^{[*]}A - A^{[*]}B + A^{[*]}A \\ &= B^{[*]}B - A^{[*]}B = (B-A)^{[*]}B.\end{aligned}$$

Similarly, from $AA^{[*]} = BA^{[*]} = AB^{[*]}$, it follows:

$$\begin{aligned}(B-A)(B-A)^{[*]} &= (B-A)(B^{[*]}-A^{[*]}) = BB^{[*]} - BA^{[*]} - AB^{[*]} + AA^{[*]} \\ &= BB^{[*]} - BA^{[*]} = B(B-A)^{[*]}.\end{aligned}$$

Thus, we have that $(B-A) \leqslant^{[*]} B$.

(ii) $\Rightarrow$ (i) Let $(B-A) \leqslant^{[*]} B$, i.e. $(B-A)^{[*]}(B-A) = (B-A)^{[*]}B$ and $(B-A)(B-A)^{[*]} = B(B-A)^{[*]}$. We get that

$$\begin{aligned}B^{[*]}B - B^{[*]}A - A^{[*]}B + A^{[*]}A &= B^{[*]}B - A^{[*]}B, \\ BB^{[*]} - BA^{[*]} - AB^{[*]} + AA^{[*]} &= BB^{[*]} - BA^{[*]},\end{aligned}$$

and therefore $A^{[*]}A = A^{[*]}B$ and $AA^{[*]} = BA^{[*]}$, i.e. $A \leqslant^{[*]} B$.

(ii) $\Leftrightarrow$ (iii) Follows directly from Theorem 5.9.

(i) $\Rightarrow$ (iv) Since $A \leqslant^{[*]} B$, by Theorem 5.9 it follows that $A^{[\dagger]} \leqslant^{[*]} B^{[\dagger]}$. Now, we can prove that $B^{[\dagger]} - A^{[\dagger]} \leqslant^{[*]} B^{[\dagger]}$ in a similar way as (i) $\Rightarrow$ (ii).

(iv) $\Rightarrow$ (i) Let $B^{[\dagger]} - A^{[\dagger]} \leqslant^{[*]} B^{[\dagger]}$. Similarly as in (ii) $\Rightarrow$ (i) it can be shown that $A^{[\dagger]} \leqslant^{[*]} B^{[\dagger]}$, which is, by Theorem 5.9 equivalent to $A \leqslant^{[*]} B$. □

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