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# REVIEW OF PARTIAL ORDERS IN RINGS DEFINED BY GENERALIZED INVERSES

*Abstract*. A matrix $A^-$ is a generalized inverse of matrix $A$ if $AA^-A = A$ holds. Let $G$ be a function which assigns to each matrix $A$ the specific subset of the set of all generalized inverses of $A$. Then we say that $A$ is below matrix $B$ if $AA^- = BA^-$ and $A^-A = A^-B$ for some $A^- \in G(A)$. The minus, star, sharp, core and dual core partial orders are some of the well known matrix partial orders defined by appropriate choices of function $G$.

This article reviews the recent known results concerning generalizations of matrix partial orders to the setting of arbitrary rings with or without involution. The article mainly consists of the published results of the authors.

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## 1. Introduction

The roots of the matrix partial orders come from the papers from the middle of the last century in which several partial orders were defined in the context of semigroups. In 1952 Wagner introduced the notion of inverse semigroup and natural partial ordering on it [80]:

$$a < b \iff a^{-1}a = a^{-1}b,$$

where $a^{-1}$ is unique inverse of $a$ in the sense that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. Clifford and Preston [14] and Lyapin [43] defined the partial ordering on the set of idempotents of an arbitrary semigroup:

$$e < f \iff e = ef = fe.$$

In 1977, Drazin defined the binary relation on semigroup $S$ with a proper involution [23]:

$$a < b \iff a^*a = a^*b \text{ and } aa^* = ba^*.$$

This relation is a partial order and it is known as the star partial order. If $a$ and $b$ are Moore–Penrose invertible elements then this relation coincides with the relation

$$a < b \iff aa^\dagger = ba^\dagger \text{ and } a^\dagger a = a^\dagger b.$$

See the next section for the definition of Moore-penrose inverse $a^\dagger$. In 1980, Hartwig [33] and Nambooripad [61] independently introduced the following relation on semigroup called minus partial order

$$a < b \iff aa^{(1)} = ba^{(1)} \text{ and } a^{(1)}a = a^{(1)}b,$$

for some $a^{(1)}$ such that $aa^{(1)}a = a$. This relation is a partial order relation on regular semigroup. Later on Mitsch [55] extended the definition of minus order to the arbitrary semigroups $S$:

$$a < b \iff xb = xa = a = ay = by \text{ for some } x, y \in S^1.$$

Mitsch's order is a partial order relation on arbitrary semigroup and coincides with minus partial order when $S$ is regular.

Thereafter, the other orderings, such as sharp, core and one sided orders, are introduced on the the set of complex matrices. Recently, many new matrix partial orders have been introduced: partial order based on spectrally orthogonal decomposition (see [31]), partial order based on core-nilpotent decompostions (see [81]), partial order based on DMP relation (see [18]), partial orders based on weighted generalized inverses (see [58]), partial order based on G-outer inverse (see [56,60]), partial orders in indefinite inner product spaces (see [75]).

Although the star and minus partial orders were originally defined on semigroups, the most of the theory deals with matrix partial orders. But recently, a lot of papers consider a number of various generalized inverses in rings-some inverses are extensions of known matrix generalized inverses, some of them are new. Consequently, the corresponding partial orders in rings was studied, see for example [9, 12, 42, 46, 47, 62, 76, 79, 82–84]. Moreover, some partial orders based on generalized inverses were generalized to more abstract contexts such as regular modules and Baer bimodules (see [8, 78]). Possible applications of matrix partial orders to control systems and autonomous systems are considered in [15, 36].

Our aim is to review the recent results concerning different types of partial orders in rings particularly in von Neumann regular rings. Using generalized inverses of elements, the definitions of the minus, star, sharp, core and dual core partial orders are extended from the set of complex matrices to an arbitrary ring (with or without an involution). It turns out that the most important properties of the matrix partial orders based on generalized inverses stay valid in the ring case.

We will not here study the partial orders for operators on Banach or Hilbert space. They have been studied in [1, 16, 17, 39, 57, 59, 66, 68, 74]. Also, we will not here discuss the lattice properties of partial orders and their preservers. Lattice properties of varius partial orders for Hilbert space operators and for elements in Rickart *-ring are studied by Djikić in papers [19, 19–21] and by Cirulis in [13]. Many authors studied the preservers of partial orders [10, 11, 26, 28, 30, 74].

This paper mainly consists of PhD dissertation [64] of the first author, but there are also results from the other authors. Of course, there are plenty of results that are not included in this survey.

## 2. Preliminaries

Recall that the binary relation $\leqslant$ on the set $A$ is a partial order relation if for all $a, b, c \in A$ it satisfies

- reflexivity: $a \leqslant a$;
- antisymmetry: if $a \leqslant b$ and $b \leqslant a$ then $a = b$;
- transitivity: if $a \leqslant b$ and $b \leqslant c$ then $a \leqslant c$.

The relation $\leqslant$ is pre-order if it is reflexive and transitive.

The underlying field is always the field of complex numbers. The set of all complex matrices of order $m \times n$ is denoted by $M_{m\times n}$. When $m = n$, the algebra of squared matrices is denoted by $M_n$. For $A \in M_{m\times n}$, we write $\operatorname{Im} A$, $\operatorname{Ker} A$ and $A^*$ for the column space, the null space and the conjugate transpose of $A$, respectively. We say that a matrix $A \in M_n$ has the index less or equal one, $\operatorname{ind} A \leqslant 1$, if $\operatorname{rank} A = \operatorname{rank} A^2$. The set of all complex matrices of order $n$ with $\operatorname{ind} A \leqslant 1$ is denoted by $I_{1,n}$:

$$I_{1,n} = \{A \in M_n : \operatorname{ind} A \leqslant 1\}.$$

The elementary fact is that any matrix can be considered as the linear operator (linear transformation) among two vector spaces of finite dimensions, and vice versa.

Recall that a matrix $A \in M_n$ is regular if and only if $\operatorname{rank} A = n$, in which case it has the unique inverse $A^{-1}$. Among other properties, the inverse matrix of $A$ satisfies the following equations:

$$\begin{array}{lll} (1)\ AXA = A, & (4)\ (XA)^* = XA, & (7)\ AX^2 = X, \\ (2)\ XAX = X, & (5)\ AX = XA, & (8)\ A^2X = A, \\ (3)\ (AX)^* = AX, & (6)\ XA^2 = A, & (9)\ X^2A = X. \end{array}$$

Suppose now that $A \in M_{m\times n}$ is an arbitrary matrix. By a generalized inverse of $A$ we shall mean a matrix $X$ which satisfies some of the equations above. (It is natural to require that at least one of the first two equations is satisfied.) Let $A\{i, j, \dots k\}$ denote the set of all matrices $X \in M_{n\times m}$ which satisfies the equations $(i)$, $(j)$, ..., $(k)$. The matrix $X \in A\{i, j, \dots, k\}$ is called the $\{i, j, \dots, k\}$-inverse of $A$ and $X$ is also denoted by $A^{(i,j,\dots,k)}$.

We use the same notations when $A$ is an element of a ring.

**Definition 2.1.** [6] Let $A \in M_{m\times n}$. A matrix $X \in M_{n\times m}$ is called

(1) an inner inverse (or $g$-inverse) of $A$ if $X \in A\{1\}$;
(2) an outer inverse of $A$ if $X \in A\{2\}$;
(3) a reflexive inverse of $A$ if $X \in A\{1, 2\}$;
(4) the Moore–Penrose inverse (or MP inverse for short) of A if $X \in A\{1, 2, 3, 4\}$, and in this case we use the mark $X = A^\dagger$;

If $n = m$ then a matrix $X$ is called

(5) the group inverse of $A$ if $X \in A\{1, 2, 5\}$, and in this case we use the mark $X = A^\#$.

It is known that every matrix has the unique Moore–Penrose inverse. The matrix $A \in M_n$ has the group inverse if and only if $\operatorname{ind} A \leqslant 1$ in which case the group inverse is unique. The characterizations, minimal, spectral and other properties of generalized inverses can be found in [6].

Recently, two new matrix generalized inverses are introduced. They are core inverse $A^{\textcircled{\#}}$ and dual core inverse $A_{\textcircled{\#}}$ of a square matrix $A$.

**Definition 2.2.** [5] A matrix $A^{\textcircled{\#}} \in M_n$ is the core inverse of $A \in M_n$ if it satisfies $AA^{\textcircled{\#}} = P_A$ and $\operatorname{Im} A^{\textcircled{\#}} \subseteq \operatorname{Im} A$, where $P_A$ is orthogonal projection on $\operatorname{Im} A$.

The core inverse exists if and only if $\operatorname{ind} A \leqslant 1$ in which case it is unique. Note that the core inverse of $A$ coincide with the $g$-inverse $A^-_{\rho^*\chi}$ of $A$ which is originally discussed by Rao and Mitra in [72].

The dual core inverse is defined analogously.

**Definition 2.3.** [5] A matrix $A_{\textcircled{\#}} \in M_n$ is the dual core inverse of $A \in M_n$ if it satisfies $A_{\textcircled{\#}}A = P_{A^*}$ and $\operatorname{Im} A^{\textcircled{\#}} \subseteq \operatorname{Im} A^*$.

The dual core inverse exists if and only if $\operatorname{ind} A \leqslant 1$ in which case it is unique. It is easily seen that $A^{\textcircled{\#}}$ and $A_{\textcircled{\#}}$ are reflexive inverses of $A$.

The concept of most of the matrix partial orders based on generalized inverses is the same.

**Definition 2.4.** Let $A$ and $B$ be two complex matrices of the same order $m \times n$. Then we define:

- the minus partial order [33]: $A <^- B$ if there exists an $A^{(1)} \in A\{1\}$ such that $AA^{(1)} = BA^{(1)}$ and $A^{(1)}A = A^{(1)}B$;
- the star partial order [23]: $A <^* B$ if $AA^\dagger = BA^\dagger$ and $A^\dagger A = A^\dagger B$.

If $m = n$ then we define:

- the sharp partial order [51]: $A <^\# B$ if $A^\#$ exists and $AA^\# = BA^\#$ and $A^\# A = A^\# B$;
- the core partial order [5]: $A <^{\textcircled{\#}} B$ if $A^{\textcircled{\#}}$ exists and $AA^{\textcircled{\#}} = BA^{\textcircled{\#}}$ and $A^{\textcircled{\#}}A = A^{\textcircled{\#}}B$;
- the dual core partial order [5]: $A <_{\textcircled{\#}} B$ if $A_{\textcircled{\#}}$ exists and $AA_{\textcircled{\#}} = BA_{\textcircled{\#}}$ and $A_{\textcircled{\#}}A = A_{\textcircled{\#}}B$.

For thorough treatment of matrix partial orders, we refer the reader to monograph [54], articles [2, 3, 5, 7, 25, 35, 45, 50, 77] and the references given there. It is interesting that there is a generalization of minus partial that does not include the notion of generalized inverse. Šemrl first defined in [74] the partial order relation on the algebra of all bounded operators on Hilbert space. It is defined even for operators that do not have generalized inverse and it coincides with minus partial order when the space is finite-dimensional. Using the same idea, the sharp partial order was defined for the operators on Banach space in [24]. Following Šerml approach, using the notion of annihilators instead of generalized inverses, the minus, star, sharp and core partial orders are generalized to Rickart rings and Rickart $*$-rings in [12, 22, 40, 46–48, 65].

From now on, $(R, +, \cdot)$ will be an arbitrary ring with identity 1. An element $a \in R$ is called von Neumann regular (regular for short) if there exists $x \in R$ such that $axa = a$. In that case $x$ is called an inner generalized inverse (g-inverse for short) of $a$. The reflexive and group inverse of $a$ are defined with the same equations as in the matrix case. In the case when $R$ is a ring with an involution $*$, the Moore–Penrose inverse is defined with the equations (1), (2), (3) and (4) as in the matrix case.

The mark $a\{1\}$ stands for the set of all generalized inverses of $a$ and $a\{1,2\}$ for the set of all reflexive generalized inverses of $a$. The set of all regular elements of $R$ will be denoted by $R^{(1)}$.

A ring $R$ is von Neumann regular (regular for short) if every element of $R$ is regular. For theory concerning regular rings we refer the reader to [29]. Note that a ring of all $n \times n$ complex matrices is regular.

An element $e \in R$ is idempotent (self-adjoint idempotent) if $e^2 = e$ ($e = e^* = e^2$). The set of all idempotents in $R$ is denoted by $E(R)$, while the set of all self-adjoint idempotents is denoted by $\tilde{E}(R)$. We say that idempotents $e, f \in R$ are (mutually) orthogonal if $ef = fe = 0$. The idempotents $e_1, e_2, \dots, e_n \in R$ are called orthogonal if they are orthogonal in pairs. If $e_1, e_2, \dots, e_n \in R$ are orthogonal idempotents such that

$$1 = e_1 + e_2 + \cdots + e_n \tag{2.1}$$

then the equality (2.1) is called a decomposition of the identity of the ring $R$. The decomposition of the identity (2.1) is orthogonal if $e_1, \dots, e_n$ are self-adjoint.

The following observations is crucial for the studying of all partial orders in rings and in the operator case too (see [71]). Let $1 = e_1 + \cdots + e_m$ and $1 = f_1 + \cdots + f_n$ be two decompositions of the identity of a ring $R$. For any $x \in R$ we have

$$x = 1 \cdot x \cdot 1 = \Big(\sum_{i=1}^{m} e_i\Big) x \Big(\sum_{j=1}^{n} f_j\Big) = \sum_{i=1}^{m}\sum_{j=1}^{n} e_i x f_j = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix}_{e \times f},$$

where $x_{ij} = e_i x f_j$. It is easy to see that this sum defines a decomposition of $R$ into a direct sum of abelian groups $e_i R f_j := \{e_i x f_j : x \in R\}$:

$$R = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} e_i R f_j. \tag{2.2}$$

If $y = [y_{ij}]_{e \times f}$ then $x + y$ can be interpreted as addition of two matrices over $R$. Let $1 = g_1 + \cdots + g_k$ be another decompositions of the identity of $R$ and let $z = [z_{jl}]_{f \times g}$, $z_{jl} = f_j z g_l$. As $f_i f_j = 0$ for $i \neq j$, one can verify that $xz$ can be interpreted as multiplication of two matrices over $R$. If in addition $R$ is a ring with an involution $*$ then we have

$$x^* = \begin{bmatrix} x_{11}^* & \cdots & x_{m1}^* \\ \vdots & \ddots & \vdots \\ x_{1n}^* & \cdots & x_{mn}^* \end{bmatrix}_{f^* \times e^*}, \tag{2.3}$$

where the above representation is with respect to decompositions of the identity $1 = f_1^* + \cdots + f_n^*$ and $1 = e_1^* + \cdots + e_m^*$.

When $m = n$ and $e_i = f_i$, $i = \overline{1,n}$, the decomposition (2.2) is known as the two-sided Peirce decomposition of the ring $R$, [37].

When we have two idempotents $e, f \in R$ then they induce two decompositions $1 = e + (1-e)$ and $1 = f + (1-f)$. We will then write $x \in R$ in the following way

$$x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}_{e \times f}.$$

## 3. Partial orders in rings defined by generalized inverses

In this section we will discuss the most important partial orders based on generalized inverses.

### 3.1. Minus partial order.

Let $R$ be a ring with identity 1.

**Definition 3.1.** [33] Let $a, b \in R$. Then $a$ is said to be below $b$ under the minus partial order, denoted by $a <^- b$, if $a \in R^{(1)}$ and $ax = bx$, $xa = xb$ for some $x \in a\{1\}$.

A closer insight into the essence of the minus partial order for matirces gives the following well-known result.

**Theorem 3.1.** [54] *Let $A, B \in M_{m\times n}$ be two nonzero matrices and let* $\operatorname{rank}(A) = a$ *and* $\operatorname{rank}(B) = b$. *The following conditions are equivalent:*

(i) $A <^- B$;

(ii) *There exist invertible matrices $R \in M_m$ and $S \in M_n$ such that*

$$A = R \begin{bmatrix} I_a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} S \quad \text{and} \quad B = R \begin{bmatrix} I_a & 0 & 0 \\ 0 & I_{b-a} & 0 \\ 0 & 0 & 0 \end{bmatrix} S, \tag{3.1}$$

*where $I_a$ and $I_{b-a}$ are identity matrices of order $a$ and $b-a$ respectively.*

(iii) $B\{1\} \subseteq A\{1\}$;

(iv) $\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(B-A)$;

(v) *There exist idempotent matrices $P \in M_m$ and $Q \in M_n$ such that*

$$A = PB = BQ.$$

*The relation $<^-$ is the partial order relation on the set $M_{m\times n}$.*

The matrix decompositions (3.1) are crucial in proving many properties of minus partial order.

The underlying relation of all partial orders based on generalized inverses is the space pre-order relation, denoted by $<^s$. For $A, B \in M_{m\times n}$ we define, see [53]:

$$A <^s B \ \text{ if } \ \operatorname{Im} A \subseteq \operatorname{Im} B \ \text{ and } \ \operatorname{Ker} B \subseteq \operatorname{Ker} A.$$

It is easy to show that for $A, B \in M_n$ the following holds:

$$\operatorname{Im} A \subseteq \operatorname{Im} B \ \Leftrightarrow \ AM_n \subseteq BM_n$$
$$\operatorname{Ker} B \subseteq \operatorname{Ker} A \ \Leftrightarrow \ M_nA \subseteq M_nB.$$

So, the equivalent algebraic version of the space pre-order relation is

**Definition 3.2.** If $a, b \in R$ then $a <^s b$ if $aR \subseteq bR$ and $Ra \subseteq Rb$.

It is clear that $a <^s b \Leftrightarrow a \in bR \cap Rb$. Also, it is easy to prove that $a <^- b$ implies $a <^s b$.

**Theorem 3.2.** *The relation $<^s$ is pre-order on $R$.*

*Proof.* The reflexivity is obvious. Let $a <^s b$ and $b <^s c$. Then $a = bx = yb$ and $b = cz = wc$ for some $x, y, z, w \in R$. We have $a = bx = czx \in cR$ and $a = yb = ywc \in Rc$, so $a <^s c$. Therefore, the relation $<^s$ is transitive, and thus it is pre-order. □

The next result was originally proved in the matrix case, [4].

**Theorem 3.3.** [63] *Let $a, b \in R$. Then the following conditions are equivalent:*

(i) $a <^s b$;
(ii) $a = bb^{(1)}a = ab^{(1)}b$, *for all* $b^{(1)} \in b\{1\}$;
(iii) $a = bb^{(1)}ab^{(1)}b$, *for all* $b^{(1)} \in b\{1\}$.

*Any of* (i)–(iii) *implies*

(iv) $ab^{(1)}a$ *is invariant under the choices of* $b^{(1)} \in b\{1\}$.

*If $R$ is a regular ring then all four conditions* (i)–(iv) *are equivalent.*

*Proof.* When $a = 0$ or $b = 0$, the proof is trivial. Suppose that $a \neq 0$ and $b \neq 0$.

(i) $\Longrightarrow$ (ii): Since $a <^s b$ we have $aR \subseteq bR$, so there is $x \in R$ such that $a = bx$. Hence $a = bb^{(1)}bx = bb^{(1)}a$ for all $b^{(1)} \in b\{1\}$. Similarly, $a = ab^{(1)}b$, $\forall b^{(1)} \in b\{1\}$.

(ii) $\Longrightarrow$ (iii) $\Rightarrow$ (i) is trivial.

(iii) $\Longrightarrow$ (iv): Fix $h \in b\{1\}$. For every $b^{(1)} \in b\{1\}$ we have

$$ab^{(1)}a = (bhahb)b^{(1)}(bhahb) = bhahbhahb,$$

which does not depend on $b^{(1)}$.

Suppose now that $R$ is regular ring.

(iv) $\Longrightarrow$ (i): Fix $h \in b\{1\}$ and let $e_1 = bh$, $e_2 = 1 - bh$, $f_1 = hb$, $f_2 = 1 - hb$. Then $1 = e_1 + e_2$ and $1 = f_1 + f_2$ are two decompositions of the identity of the ring $R$. If $b^{(1)} \in b\{1\}$ then $f_1 b^{(1)} e_1 = hbb^{(1)}bh = hbh$. If $f_1 x e_1 = hbh$ then $bxb = bhbxbhb = b(f_1 x e_1)b = b(hbh)b = b$, so $x \in b\{1\}$. Therefore, $b^{(1)} \in b\{1\}$ if and only if

$$b^{(1)} = \begin{bmatrix} hbh & x_{12} \\ x_{21} & x_{22} \end{bmatrix}_{f \times e}, \tag{3.2}$$

where $x_{ij} \in f_i R e_j$ are arbitrary. Now,

$$\begin{aligned} ab^{(1)}a &= [af_1 \quad af_2]_{1 \times f} \begin{bmatrix} hbh & x_{12} \\ x_{21} & x_{22} \end{bmatrix}_{f \times e} \begin{bmatrix} e_1 a \\ e_2 a \end{bmatrix}_{e \times 1} \\ &= ahbha + ax_{21}a + ax_{12}a + ax_{22}a \end{aligned}$$

does not depend on $x_{21}, x_{12}, x_{22}$ and thus equal to $ahbha$. Setting $x_{12} = x_{22} = 0$ it follows that $ax_{21}a = 0$ for all $x_{21} \in f_2 R e_1$, that is $af_2 x e_1 a = 0$ for all $x \in R$. Multiplying this equation by $e_1$ from the left and by $f_2$ from the right we obtain $(e_1 a f_2) x (e_1 a f_2) = 0$. Since $R$ is regular, we can choose $x = (e_1 a f_2)^{(1)} \in (e_1 a f_2)\{1\}$.

Hence, $e_1af_2 = 0$. Similarly, $e_2af_1 = 0$ and $e_2af_2 = 0$ so we conclude that $a = e_1af_1 = bhahb$. This implies $a <^s b$. □

**Corollary 3.1.** [69] *Let $a, b \in R$. If $a <^s b$ and $b\{1\} \cap a\{1,2\} \neq \emptyset$ then $a = b$.*

*Proof.* Let $a <^s b$ and $x \in b\{1\} \cap a\{1,2\}$. By Theroem 3.3 we have

$$b = bxb = b(xax)b = (bxa)xb = axb = a.$$ □

From the following result we can see that the condition $a <^- b$ is symmetric in $a$ and $b - a$.

**Lemma 3.1.** [63] *Let $a, b \in R^{(1)}$. The following conditions are equivalent:*

(i) $a <^- b$;
(ii) $a = ab^{(1)}a = ab^{(1)}b = bb^{(1)}a$ *for all* $b^{(1)} \in b\{1\}$;
(iii) $b - a = (b - a)b^{(1)}(b - a) = (b - a)b^{(1)}b = bb^{(1)}(b - a)$ *for all* $b^{(1)} \in b\{1\}$;
(iv) $b - a <^- b$.

*Proof.* (i) $\Longrightarrow$ (ii): Let $a <^- b$ and $b^{(1)} \in b\{1\}$. The proof of $a = ab^{(1)}b = bb^{(1)}a$ follows from Theorem 3.3. As $a <^- b$, there exists $a^{(1)} \in a\{1\}$ such that $aa^{(1)} = ba^{(1)}$ and $a^{(1)}a = a^{(1)}b$. For any $b^{(1)} \in b\{1\}$ we have

$$ab^{(1)}a = aa^{(1)}ab^{(1)}aa^{(1)}a = aa^{(1)}bb^{(1)}ba^{(1)}a = aa^{(1)}ba^{(1)}a = aa^{(1)}aa^{(1)}a = a.$$

(ii) $\Longrightarrow$ (i): Let $x = b^{(1)}ab^{(1)}$. Then $x \in a\{1\}$ and

$$ax = ab^{(1)}ab^{(1)} = ab^{(1)} = bb^{(1)}ab^{(1)} = bx.$$

Likewise, $xa = xb$, and hence $a <^- b$.

(ii) $\Longleftrightarrow$ (iii) is a matter of direct computation.

(iii) $\Longleftrightarrow$ (iv) follows from the equivalence of (i) and (ii). □

**Theorem 3.4.** [55] *The relation $<^-$ is a partial order relation on $R^{(1)}$. If $R$ is a regular ring then $<^-$ is a partial order relation on $R$.*

*Proof.* Let $a, b, c \in R^{(1)}$. Since $a \in R^{(1)}$, there exists $a^- \in a\{1\} \neq \emptyset$. From $aa^- = aa^-$ and $a^-a = a^-a$, we have $a <^- a$, so $<^-$ is reflexive. Suppose that $a <^- b$ and $b <^- a$. From $a <^- b$ it follows that there is $a^- \in a\{1\}$ such that $aa^- = ba^-$ and $a^-a = a^-b$. From $b <^- a$, by Lemma 3.1, we have $ba^-b = b$. Therefore, $a = aa^-a = ba^-b = b$, so $<^-$ is antisymmetric. Finally suppose that $a <^- b$ and $b <^- c$. Hence, there are $a^- \in a\{1\}$ and $b^- \in b\{1\}$ such that $aa^- = ba^-$, $a^-a = a^-b$ and $bb^- = cb^-$, $b^-b = b^-c$. Let $g = b^-ab^-$. From Lemma 3.1, we obtain $a = bb^-a = ab^-b = ab^-a$, so

$$aga = a$$
$$ag = ab^-ab^- = ab^- = bb^-ab^- = cb^-ab^- = cg$$
$$ga = b^-ab^-a = b^-a = b^-ab^-b = b^-ab^-c = gc.$$

Therefore, $a <^- c$, so $<^-$ is transitive. It follows that $<^-$ is a partial order relation on $R^{(1)}$. If $R$ is regular then $R = R^{(1)}$, so $<^-$ is a partial order relation on regular ring $R$. □

In fact, $<^-$ is a partial order relation on every regular semigroup, see [55].

**Remark 3.1.** Let $p, q \in E(R)$ and suppose that $a = paq$ is invertible element. Then $ab = ba = 1$ for some $b \in R$. We have $p = pab = p(paq)b = paqb = ab = 1$ and similarly, $q = 1$. It follows that if $p \neq 1$ or $q \neq 1$ and $a = paq$ then $a$ is not invertible. But we can still consider the invertibility of $a$ as an element of $pRq$ in the following way.

**Definition 3.3** (From [64])**.** Let $p, q \in E(R)$. We say that an element $a \in R$ is $(p, q)$-invertible if $a \in pRq$ and there exists $a' \in qRp$ such that $aa' = p$ and $a'a = q$. In this case we say that $a'$ is the $(p, q)$-inverse of $a$.

**Remark 3.2.** The following properties are easy to show. For $a \in R$ there exist $p, q \in E(R)$ such that $a$ is $(p, q)$-invertible if and only if $a$ is $g$-invertible. Moreover, $(p, q)$-inverse is unique when it exists. An element $a'$ is $(p, q)$-inverse of $a$ if and only if $a$ is $(q, p)$-inverse of $a'$. Furthermore, if $a$ is $(p, q)$-invertible and $b \in pR$ then the equation $ax = b$ has the unique solution in the set $qR$, namely $x = a'b$. Similarly, if $b \in Rq$ then the equation $xa = b$ has the unique solution in the set $Rp$, $x = ba'$.

The idea of the following theorem and its proof is based on algebraic technique and it differs from all techniques which are used in studying the matrix partial orders based on $g$-inverses.

**Theorem 3.5.** [63] *Let $a, b \in R^{(1)}$. Then the following conditions are equivalent:*

(i) $a <^- b$;

(ii) *There exist decompositions of the identity of the ring $R$*

$$1 = e_1 + e_2 + e_3, \quad 1 = f_1 + f_2 + f_3 \tag{3.3}$$

*with respect to which $a$ and $b$ have the following matrix forms:*

$$a = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e \times f}, \quad b = \begin{bmatrix} a & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e \times f}, \tag{3.4}$$

*where $a \in eRf$ is $(e, f)$-invertible and $b - a \in e_2Rf_2$ is $(e_2, f_2)$-invertible.*

*Proof.* The cases $a = 0$ or $b = 0$ are trivial.

(i) $\Longrightarrow$ (ii): Fix $h \in b\{1\}$ and set

$$\begin{aligned} e_1 &= ah, \quad e_2 = (b-a)h, \quad e_3 = 1 - bh, \\ f_1 &= ha, \quad f_2 = h(b-a), \quad f_3 = 1 - hb. \end{aligned} \tag{3.5}$$

Since $a <^- b$, by Lemma 3.1, we have $a = aha = ahb = bha$. Hence,

$$\begin{aligned} (ah)(ah) &= ah, \quad (ah)(bh) = ah, \quad (bh)(ah) = ah, \quad (bh)(bh) = bh, \\ (ha)(ha) &= ha, \quad (ha)(hb) = ha, \quad (hb)(ha) = ha, \quad (hb)(hb) = hb. \end{aligned}$$

It follows that $1 = e_1 + e_2 + e_3$ and $1 = f_1 + f_2 + f_3$ are two decompositions of the identity of the ring $R$.

From $e_1af_1 = ahaha = a$ and $e_2(b-a)f_2 = (b-a)h(b-a)h(b-a) = b - a$ we conclude that $a$ and $b$ have the matrix forms given by (3.4).

It is easy to show that $x_a = f_1he_1 = hah$ is $(e_1, f_1)$-inverse of $a$ and that $x_{b-a} = f_2he_2 = h(b-a)h$ is $(e_2, f_2)$-inverse of $b-a$.

(ii) $\Longrightarrow$ (i): Fix $a^{(1)} \in a\{1\}$ and set

$$x = f_1a^{(1)}e_1 = \begin{bmatrix} f_1a^{(1)}e_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{f\times e}.$$

It is easily seen that $x \in a\{1\}$, $ax = bx$ and $xa = xb$. □

When it is the case as in Theorem 3.5, we say that the decompositions (3.3), where idempotents are defined by (3.5), are *standard decompositions*. Note that the matrix representations (3.4) are analogous with matrix representations given in (3.1). As we will see later, the idea of decomposition of the identity of the ring induced by the condition $a <^- b$ is universal and it can be applied to all partial orders based on generalized inverses.

**Theorem 3.6.** [63] *Let $a, b \in R^{(1)}$ such that $a \neq 0$, $b \neq 0$ and $a <^- b$. With respect to the standard decompositions* (3.3) *defined by* (3.5)*, $a\{1\}$ is given by $a^{(1)} = [x_{ij}]_{f\times e}$, where $x_{11} = x_a$ is $(e_1, f_1)$-inverse of $a$ and $x_{ij} \in f_iRe_j$, $(i, j) \neq (1, 1)$ are arbitrary.*

*The set $b\{1\}$ is given by*

$$b^{(1)} = \begin{bmatrix} x_a & 0 & x_{13} \\ 0 & x_{b-a} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}_{f\times e}, \tag{3.6}$$

*where $x_{b-a}$ is $(e_2, f_2)$-inverse of $b-a$ and $x_{ij} \in f_iRe_j$ are arbitrary.*

*Proof.* It is easy to check that element from (3.6) belongs to $b\{1\}$. Let $b^{(1)} = [f_ib^{(1)}e_j]_{f\times e} \in b\{1\}$ be artitrary. From

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times f} = b = bb^{(1)}b = \begin{bmatrix} ab^{(1)}a & ab^{(1)}(b-a) & 0 \\ (b-a)b^{(1)}a & (b-a)b^{(1)}(b-a) & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times f}$$

it follows that $ab^{(1)}a = a$, $ab^{(1)}(b-a) = 0$, $(b-a)b^{(1)}a = 0$ and $(b-a)b^{(1)}(b-a) = b-a$. Multiplying $ab^{(1)}a = a$ by $x_a$ from the both sides yields $f_1b^{(1)}e_1 = x_a$. Multiplying $ab^{(1)}(b-a) = 0$ by $x_a$ from the left and by $x_{b-a}$ from the right, we get $f_1b^{(1)}e_2 = 0$. Similarly, we can obtain $f_2b^{(1)}e_1 = 0$ and $f_2b^{(1)}e_2 = x_{b-a}$. Therefore, the set $b\{1\}$ is given by (3.6). The characterization of $a\{1\}$ can be proved analogously. □

The following theorem gives the characterization of minus partial order by the inclusion of the appropriate subsets of $g$-inverses. This type of result is typical for many partial order relations based on generalized inverses.

**Theorem 3.7.** [9, 63] *Let $R$ be a regular ring and $a, b \in R$. Then the following conditions are equivalent:*

(i) $a <^- b$; (ii) $b\{1\} \subseteq a\{1\}$; (iii) $b\{1, 2\} \subseteq a\{1\}$.

*The implications* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *stay valid even for an arbitrary ring and* $a, b \in R^{(1)}$.

*Proof.* (i) $\Longrightarrow$ (ii): It follows from Lemma 3.1 (i) $\Rightarrow$ (ii).

(ii) $\Longrightarrow$ (iii) is trivial.

(iii) $\Longrightarrow$ (i): Let us follow the notation of Theorem 3.3. Let us show that $b^{(1,2)} \in b\{1,2\}$ if and only if

$$b^{(1,2)} = \begin{bmatrix} hbh & x_{12} \\ x_{21} & x_{21}bx_{12} \end{bmatrix}_{f\times e}, \tag{3.7}$$

where $x_{12} \in f_1Re_2$ and $x_{21} \in f_2Re_1$ are arbitrary. In the proof of Theorem 3.3 we showed that $b^{(1)} \in b\{1\}$ if and only if $b^{(1)}$ is of the form (3.2). Hence $b^{(1,2)} \in b\{1,2\}$ has the same form. We have $x_{22} = x_{21}bx_{12}$, because

$$\begin{aligned}\begin{bmatrix} hbh & x_{12} \\ x_{21} & x_{22} \end{bmatrix}_{f\times e} &= b^{(1,2)} = b^{(1,2)}bb^{(1,2)} = \begin{bmatrix} hbh & x_{12} \\ x_{21} & x_{22} \end{bmatrix}_{f\times e} \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}_{e\times f} b^{(1,2)} \\ &= \begin{bmatrix} f_1 & 0 \\ x_{21}b & 0 \end{bmatrix}_{f\times f} \begin{bmatrix} hbh & x_{12} \\ x_{21} & x_{22} \end{bmatrix}_{f\times e} = \begin{bmatrix} hbh & x_{12} \\ x_{21} & x_{21}bx_{12} \end{bmatrix}_{f\times e}.\end{aligned}$$

Conversely, if $b^{(1,2)}$ is of the form (3.7) then it is easy to show that $b^{(1,2)} \in b\{1,2\}$. From this we have

$$a = ab^{(1,2)}a = ahbha + ax_{21}a + ax_{12}a + ax_{21}bx_{12}a,$$

for all $x_{12} \in f_1Re_2$ and $x_{21} \in f_2Re_1$. As in the proof of Theorem 3.3 we can prove that

$$e_1af_2 = e_2af_1 = 0 \text{ and } a = ahbha = af_1he_1a. \tag{3.8}$$

By (3.8) we obtain

$$a = (e_1+e_2)(af_1he_1a)(f_1+f_2) = e_1(af_1he_1a)f_1 = e_1af_1 = e_1a = af_1.$$

Let $x = f_1hahe_1$. Then

$$\begin{aligned} x &= he_1af_1h = he_1ah = haf_1h = hah, \\ he_1 &= f_1h = hbh \in b\{1,2\} \subseteq a\{1\}, \end{aligned}$$

so

$$\begin{aligned} axa &= a(f_1hahe_1)a = (af_1ha)he_1a = ahe_1a = a \\ ax &= ahe_1ah = ah = e_1ah = bhah = bx, \\ xa &= haf_1ha = ha = haf_1 = hahb = xb. \end{aligned}$$

By definition, $a <^- b$ holds. □

We survey some other known characterization of minus partial order.

**Theorem 3.8.** [9, 38] and [63] *Let* $a, b \in R^{(1)}$. *The following conditions are equivalent:*

(i) $a <^- b$.

(ii) *There exist* $p, q \in E(R)$ *such that*

$$b = \begin{bmatrix} a & 0 \\ 0 & b-a \end{bmatrix}_{p \times q}. \tag{3.9}$$

(iii) *There exist* $p, q \in E(R)$ *such that* $a = pb = bq$.
(iv) $bR = aR \oplus (b-a)R$.
(v) $Rb = Ra \oplus R(b-a)$.
(vi) $aR \cap (b-a)R = \{0\} = Ra \cap R(b-a)$.

*Proof.* (i) $\Rightarrow$ (ii): Let $h \in b^{(1)}$. Setting $p = e_1 = bh$ and $q = f_1 = hb$, the representation (3.9) follows from Theorem 3.5.

(ii) $\Rightarrow$ (iii): From (3.9) we have $a = pbq$ and $b - a = (1-p)b(1-q)$. From the first equation we get $a = pa = aq$. Multiplying the second equation by $p$ from the left, we obtain $p(b-a) = 0$, so $pb = pa = a$. Similarly, we obtain $a = bq$.

(iii) $\Rightarrow$ (i): From $a = pb = bq$, we have $a = pa = aq$ Let $g = qa^{(1)}p$, where $a^{(1)} \in a\{1\}$ is arbitrary. It is easy to show that $g \in a\{1\}$, $ag = bg$ and $ga = gb$, so $a <^- b$.

(ii) $\Rightarrow$ (iv): Since $b = a + (b-a)$, we have $bR \subseteq aR + (b-a)R$. From (3.9) we have $a = pbq$ and $b - a = (1-p)b(1-q)$, so $a = pa = aq$, $(1-p)a = a(1-q) = 0$, $(1-p)(b-a) = (b-a)(1-q) = b-a$ and $p(b-a) = (b-a)q = 0$. Now, for arbitrary $x, y \in R$, we obtain $bqx = (a+(b-a))qx = ax$ and $b(1-q)y = (a+(b-a))(1-q)y = (b-a)y$. Therefore, $aR + (b-a)R \subseteq bR$. If $ax = (b-a)y$ for some $x, y \in R$ then $pax = p(b-a)y$, so $ax = 0$. It follows that $bR = aR \oplus (b-a)R$.

(ii) $\Rightarrow$ (v), (vi): The proof is similar to the proof of (ii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (iii): From $bR = aR \oplus (b-a)R$, we have $a = bx$ for some $x \in R$. Let $h \in b\{1\}$. Then $bha = bhbx = bx = a$ and

$$(b-a)ha = bha - aha = a - aha \in aR \cap (b-a)R = \{0\},$$

so $aha = a$. Also

$$\begin{aligned} ah(b-a) &= (b-(b-a))h(b-a) = bh(b-a) - (b-a)h(b-a) \\ &= bhb - bha - (b-a)h(b-a) = b - a - (b-a)h(b-a) \\ &= (b-a)(1 - h(b-a)) \in aR \cap (b-a)R = \{0\}, \end{aligned}$$

so $ahb = aha = a$. Let $p = ah$ and $q = ha$. Then $p, q \in E(R)$ and $a = pb = bq$.

(v) $\Rightarrow$ (iii): The proof is similar to the proof of (iv) $\Rightarrow$ (iii).

(vi) $\Rightarrow$ (iii): Let $h \in b\{1\}$, $p = ah$ and $q = ha$. From

$$\begin{aligned} a(1-hb) &= (b-(b-a))(1-hb) = b - bhb - (b-a)(1-hb) \\ &= (b-a)(hb-1) \in aR \cap (b-a)R = \{0\}, \end{aligned}$$

we obtain $a = ahb = pb$. Similarly, we can prove that $a = bq$. □

We can characterize the all elements $x$ which appears in the Definition 3.1. Let

$$a\{1\}_b = \{x \in a\{1\} : ax = bx, xa = xb\} \text{ and}$$
$$a\{1,2\}_b = \{x \in a\{1,2\} : ax = bx, xa = xb\}.$$

The following theorem gives the explicit characterizations of these sets. The same characterizations in the complex matrix case were proved in [52] and [53].

**Theorem 3.9.** [63] *Let* $a, b \in R^{(1)}$ *and let* $a <^- b$. *Then*

(i) $a\{1\}_b = \{b^{(1)} - b^{(1)}(b-a)b^{(1)} : b^{(1)} \in b\{1\}\}$;

(ii) $a\{1,2\}_b = \{b^{(1)}ab^{(1)} : b^{(1)} \in b\{1\}\} = \{b^{(1,2)}ab^{(1,2)} : b^{(1,2)} \in b\{1,2\}\}$.

*Proof.* (i): Let $S$ be the set on the right-hand side of (i). Since $a <^- b$, we have that $a$, $b$ and $b^{(1)}$ have the representations given by (3.4) and (3.6). Therefore $x \in S$ if and only if

$$x = b^{(1)} - b^{(1)}(b-a)b^{(1)} = \begin{bmatrix} x_a & 0 & x_{13} \\ 0 & 0 & 0 \\ x_{31} & 0 & z \end{bmatrix}_{f\times e}, \tag{3.10}$$

for some elements $x_{13} \in f_1Re_3$, $x_{31} \in f_3Re_1$ and $z \in f_3Re_3$ ($z = x_{33} - x_{32}(b-a)x_{23}$). A trivial verification shows that $axa = a$, $ax = bx$ and $xa = xb$, i.e. $x \in a\{1\}_b$.

Assume now that $x \in a\{1\}_b$. Then $x \in a\{1\}$ and hence $x = [x_{ij}]_{f\times e}$, where $x_{11} = x_a$. From $ax = bx$ and $xa = xb$ we obtain $x_{12} = x_{21} = x_{22} = x_{23} = x_{32} = 0$. It is easy to check that $x = b^{(1)} - b^{(1)}(b-a)b^{(1)} \in S$ where

$$b^{(1)} = \begin{bmatrix} x_a & 0 & x_{13} \\ 0 & x_{b-a} & 0 \\ x_{31} & 0 & x_{33} \end{bmatrix}_{f\times e}.$$

(ii): The proof of (ii) is similar. □

The following theorem is the generalization of Theorem 3.5.6. in [54] where $a$, $b$ are complex matrices.

**Theorem 3.10.** [63] *Let* $a, b \in R^{(1)}$ *and* $a <^- b$.

(i) *For every* $a^{(1)} \in a\{1\}_b$ *there exists* $b^{(1)} \in b\{1\}$ *such that*

$$b^{(1)}a = a^{(1)}a, \quad ab^{(1)} = aa^{(1)}; \tag{3.11}$$

(ii) *For every* $b^{(1)} \in b\{1\}$ *there exists* $a^{(1)} \in a\{1\}_b$ *such that* (3.11) *holds.*

*Proof.* (i): From the proof of Theorem 3.9 we know that $a^{(1)} \in a\{1\}_b$ has matrix representation given in (3.10). Then (3.11) holds for

$$b^{(1)} = \begin{bmatrix} x_a & 0 & x_{13} \\ 0 & x_{b-a} & x'_{23} \\ x_{31} & x'_{32} & x'_{33} \end{bmatrix}_{f\times e},$$

where $x'_{23} \in f_2Re_3$, $x'_{32} \in f_3Re_2$, $x'_{33} \in f_3Re_3$ are arbitrary.

(ii): Any $b^{(1)} \in b\{1\}$ is of the form (3.6). The element

$$a^{(1)} = \begin{bmatrix} x_a & 0 & x_{13} \\ 0 & 0 & 0 \\ x_{31} & 0 & x'_{33} \end{bmatrix}_{f\times e}$$

where $x'_{33} \in f_3Re_3$ is arbitrary, has desired properties. □

As we proved in Theorem 3.8, the condition $a <^- b$ is equivalent to $a = pb = bq$ for some $p, q \in E(R)$. The following theorem gives an explicit formula for the idempotents $p$ and $q$. For the complex matrix case see [54, Theorems 3.5.13–3.5.18].

**Theorem 3.11.** [63] *Let $a, b \in R^{(1)}$ such that $a <^- b$. Then the class of all idempotents $p \in E(R)$ such that $a = pb$ is given by*

$$p = \begin{bmatrix} e_1 & 0 & x_{13}(e_3 - p_{33}) \\ 0 & 0 & x_{23}p_{33} \\ 0 & 0 & p_{33} \end{bmatrix}_{e\times e}, \tag{3.12}$$

*where $p_{33} \in e_3Re_3$ is an arbitrary idempotent and $x_{13} \in e_1Re_3$, $x_{23} \in e_2Re_3$ are arbitrary elements. The class of all idempotents $p$ such that $a = pb$ and $pR = aR$ is given by $\{aa^{(1)} : a^{(1)} \in a\{1\}_b\}$. Every idempotent $p$ which satisfies $a = pb$ can be written as $p = p_1 + p_2$, where $p_1 \in E(R)$ with $a = p_1b$, $p_1R = aR$, and $p_2 \in E(R)$ with $p_1p_2 = p_2p_1 = p_2a = p_2b = 0$.*

*The class of all idempotents $q \in R$ such that $a = bq$ is given by*

$$q = \begin{bmatrix} f_1 & 0 & 0 \\ 0 & 0 & 0 \\ (f_3 - q_{33})x_{31} & q_{33}x_{32} & q_{33} \end{bmatrix}_{f\times f},$$

*where $q_{33} \in f_3Rf_3$ is an arbitrary idempotent and $x_{31} \in f_3Rf_1$, $x_{32} \in f_3Rf_2$ are arbitrary elements. The class of all idempotents $q$ such that $a = bq$ and $Rq = Ra$ is given by $\{a^{(1)}a : a^{(1)} \in a\{1\}_b\}$. Every idempotent $q$ which satisfies $a = bq$ can be written as $q = q_1 + q_2$, where $q_1 \in E(R)$ with $a = bq_1$, $Rq_1 = Ra$, and $q_2 \in E(R)$ with $q_1q_2 = q_2q_1 = aq_2 = bq_2 = 0$.*

*Proof.* If $p$ is of the form (3.12) then $p$ is an idempotent and $a = pb$. Let $p$ be an idempotent such that $a = pb$. Suppose that $p = [p_{ij}]_{e\times e}$, $i, j = \overline{1,3}$, with respect to standard decomposition $1 = e_1 + e_2 + e_3$. From $a = pb$, using (3.4), we obtain $p_{11}a = a$, and $p_{12}(b - a) = p_{21}a = p_{22}(b - a) = p_{31}a = p_{32}(b - a) = 0$. Now, it is easy to see that $p_{11} = e_1$ and $p_{12} = p_{21} = p_{22} = p_{31} = p_{32} = 0$. Condition $p = p^2$ implies $p_{23} = p_{23}p_{33}$, $p_{13} = p_{13} + p_{13}p_{33}$ and $p_{33} = p_{33}^2$. Hence $p_{13}p_{33} = 0$, so $p_{13} = p_{13}(e_3 - p_{33})$. Thus, $p$ has the form (3.12).

Since $a = e_1a$ and $e_1 = ax_a$, we have $aR = e_1R$. By (3.12) we see that $p$ is idempotent such that $a = pb$ and $pR = aR = e_1R$ if and only if $p_{33} = 0$ if and only if

$$p = \begin{bmatrix} e_1 & 0 & x_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e},$$

where $x_{13} \in e_1Re_3$ is arbitrary. From (3.10) we see that $aa^{(1)}$ where $a^{(1)} \in a\{1\}_b$ has the above form.

According to the above, we can take

$$p_1 = \begin{bmatrix} e_1 & 0 & x_{13}(e_3 - p_{33}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}, \quad p_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x_{23}p_{33} \\ 0 & 0 & p_{33} \end{bmatrix}_{e\times e}.$$

The results concerning idempotent $q$ can be proved similarly. □

**Remark 3.3.** Let $M_{m\times n}(R)$ be the set of all $m\times n$ matrices whit elements from the ring $R$. Due to von Neumann, if $R$ is regular ring then the ring $M_{n\times n}(R)$ is also regular. Moreover, every matrix $A\in M_{m\times n}(R)$ is $g$-invertible when $R$ is regular. It is not difficult to verify that, under appropriate context, all previous results from the present subsection are valid in the case when $a,b\in M_{m\times n}(R)$.

**3.2. Core and dual core inverse.** The core and dual core partial orders are defined by core and dual core inverses Definition 2.2 and Definition 2.3. Thus, we have to extend the notions of core and dual core inverse to the ring case and we will give the most important properties of these inverses. But we will omit the proofs because the main focus of this article is on partial orders.

For a matrix $A$, let us denote by $A^{(1,2)}_{T,S}$ the $\{1,2\}$ inverse of $A$ with $\operatorname{Im}A=T$ and $\operatorname{Ker}A=S$. This inverse is unique when it exists [6]. It is known that [6]

$$A^\dagger=A^{(1,2)}_{\operatorname{Im}A^*,\operatorname{Ker}A^*}\quad\text{and}\quad A^\#=A^{(1,2)}_{\operatorname{Im}A,\operatorname{Ker}A}.$$

Since

$$A^{\textcircled{\#}}=A^\#AA^\dagger\quad\text{and}\quad A_{\textcircled{\#}}=A^\dagger AA^\#\quad(\text{see [5]}),\tag{3.13}$$

it is easy to see that $A^{\textcircled{\#}}=A^{(1,2)}_{\operatorname{Im}A,\operatorname{Ker}A^*}$ and $A_{\textcircled{\#}}=A^{(1,2)}_{\operatorname{Im}A^*,\operatorname{Ker}A}$.

The algebraic characterizations of core and dual core inverse is given in the next lemma.

**Lemma 3.2.** [67] *A matrix $X\in M_n$ is the core inverse of $A\in M_n$ if and only if $AXA=A$, $XM_n=AM_n$ and $M_nX=M_nA^*$. A matrix $Y\in M_n$ is the dual core inverse of $A\in M_n$ if and only if $AYA=A$, $YM_n=A^*M_n$ and $M_nY=M_nA$.*

*Proof.* Suppose that $X$ is the core inverse of $A$. Since $\operatorname{Im}X\subseteq\operatorname{Im}A$, it is clear that $XM_n\subseteq AM_n$. By (3.13), we see that $AXA=A$ and $XA=A^\#A$, so $A=XA^2$, hence $AM_n\subseteq XM_n$. Also, $A^*=A^*(AX)^*=A^*AX$ so $M_nA^*\subseteq M_nX$. Finally, $X=A^\#AA^\dagger=A^\#(A^\dagger)^*A^*$ implies $M_nX\subseteq M_nA^*$. Conversely, suppose that $A=AXA$, $XM_n=AM_n$ and $M_nX=M_nA^*$. It follows that $\operatorname{Im}X\subseteq\operatorname{Im}A$ and there exist $V\in M_n$ such that $X=VA^*$. It is now clear that $(AX)^2=AX$, and $X=VA^*=VA^*X^*A^*=XX^*A^*$. Therefore $AX=AX(AX)^*$ which is self-adjoint, so $AX=P_A$. The proof of the result about the dual core inverse is similar. □

Based on above lemma, the notions of core and dual core inverse can be generalized from $M_n$ to the ring case.

**Definition 3.4.**[67] Let $R$ be a ring with involution and $a\in R$. An element $a^{\textcircled{\#}}\in R$ satisfying

$$aa^{\textcircled{\#}}a=a,\quad a^{\textcircled{\#}}R=aR\quad\text{and}\quad Ra^{\textcircled{\#}}=Ra^*$$

is called the core inverse of $a$. An element $a_{\textcircled{\#}}\in R$ satisfying

$$aa_{\textcircled{\#}}a=a,\quad a_{\textcircled{\#}}R=a^*R\quad\text{and}\quad Ra_{\textcircled{\#}}=Ra$$

is called the dual core inverse of $a$.

Let $a \in R$. It is easy to see that $x = a^\dagger$ if and only if $axa = a$, $xR = a^*R$ and $Rx = Ra^*$. Also, $x = a^\#$ if and only if $axa = a$, $xR = aR$ and $Rx = Ra$. The similarity with definition 3.4 is obvious.

The following theorems gives crucial characterizations of core and dual core inverses.

**Theorem 3.12.** [67] *Let $a \in R$. The following assertions are equivalent:*

(i) *$a$ is core invertible.*

(ii) *There exist $x \in R$ such that*

$$(1)\ axa = a \quad (2)\ xax = x \quad (3)\ (ax)^* = ax \quad (6)\ xa^2 = a \quad (7)\ ax^2 = x.$$

(iii) [73] *There exist $x \in R$ such that*

$$(3)\ (ax)^* = ax \quad (6)\ xa^2 = a \quad (7)\ ax^2 = x.$$

(iv) *There exist $p \in \tilde{E}(R)$ and $q \in E(R)$ such that $pR = aR$, $qR = aR$ and $Rq = Ra$.*

*If the previous assertions are valid then $x = a^{\textcircled{\#}}$, $a^{\textcircled{\#}}$ is unique and idempotents $p$ and $q$ are unique. Moreover, $qa^{(1)}p$ is invariant under the choice of $a^{(1)} \in a\{1\}$ and*

$$a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{p\times q}, \quad a^{\textcircled{\#}} = \begin{bmatrix} qa^{(1)}p & 0 \\ 0 & 0 \end{bmatrix}_{q\times p}.$$

**Theorem 3.13.** [67] *Let $a \in R$. The following assertions are equivalent:*

(i) *$a$ is dual core invertible.*

(ii) *There exists $x \in R$ such that*

$$(1)\ axa = a \quad (2)\ xax = x \quad (4)\ (xa)^* = xa \quad (8)\ a^2x = a \quad (9)\ x^2a = x.$$

(iii) (See [73].) *There exist $x \in R$ such that*

$$(4)\ (xa)^* = xa \quad (8)\ a^2x = a \quad (9)\ x^2a = x.$$

(iv) *There exist $r \in \tilde{E}(R)$ and $q \in E(R)$ such that $Rr = Ra$, $qR = aR$ and $Rq = Ra$.*

*If the previous assertions are valid then $x = a_{\textcircled{\#}}$, $a_{\textcircled{\#}}$ is unique and idempotents $r$ and $q$ are unique. Moreover, $ra^{(1)}q$ is invariant under the choice of $a^{(1)} \in a\{1\}$ and*

$$a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{q\times r}, \quad a_{\textcircled{\#}} = \begin{bmatrix} ra^{(1)}q & 0 \\ 0 & 0 \end{bmatrix}_{r\times q}.$$

Similar to Theorems 3.12 and 3.13 there are analogous results for MP and group inverses, see [67, Theorems 2.11 and 2.12].

It should be noted that $g$-invertibility of $a$ implies MP invertibility of $a$ in the case when $R$ is $C^*$-algebra (see [32]) or Rickart $*$-ring. Also,

$$R^\# \cap R^\dagger = R^{\textcircled{\#}} \cap R_{\textcircled{\#}} \quad \text{and} \quad R^{\textcircled{\#}} \cup R_{\textcircled{\#}} \subseteq R^\#, \tag{3.14}$$

where $R^\#$, $R^\dagger$, $R^{\textcircled{\#}}$, $R_{\textcircled{\#}}$ stands for the set of all group, MP, core, dual core invertible elements, respectively.

The idempotents $p$, $q$ and $r$ appearing in Theorems 3.12 and 3.13 satisfy the following equations (see [67]):

$$\begin{aligned} q &= aa^{\#} = a^{\#}a = a^{\textcircled{\#}}a = aa_{\textcircled{\#}} \\ p &= aa^{\dagger} = aa^{\textcircled{\#}} \\ r &= a^{\dagger}a = a_{\textcircled{\#}}a. \end{aligned} \tag{3.15}$$

Also, $pq = q$, $qp = p$, $rq = r$, $qr = q$. Moreover,

$$q^*p = (pq)^* = q^*, \quad pq^* = (qp)^* = p, \quad q^*r = (rq)^* = r, \quad rq^* = (qr)^* = q^*.$$

We can summarize the all matrix forms of $a$ and its $g$-inverses:

$$\begin{aligned} a &= \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{q\times q} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{q\times r} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{p\times r} \\ a^{\#} &= \begin{bmatrix} a^{\#} & 0 \\ 0 & 0 \end{bmatrix}_{q\times q} = \begin{bmatrix} a^{\#} & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} = \begin{bmatrix} a^{\#} & 0 \\ 0 & 0 \end{bmatrix}_{q\times r} = \begin{bmatrix} a^{\#} & 0 \\ 0 & 0 \end{bmatrix}_{p\times r} \\ a^{\dagger} &= \begin{bmatrix} a^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}_{r\times p} = \begin{bmatrix} a^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}_{q^*\times p} = \begin{bmatrix} a^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}_{r\times q^*} = \begin{bmatrix} a^{\dagger} & 0 \\ 0 & 0 \end{bmatrix}_{q^*\times q^*} \\ a^{\textcircled{\#}} &= \begin{bmatrix} a^{\textcircled{\#}} & 0 \\ 0 & 0 \end{bmatrix}_{q\times p} = \begin{bmatrix} a^{\textcircled{\#}} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} a^{\textcircled{\#}} & 0 \\ 0 & 0 \end{bmatrix}_{q\times q^*} = \begin{bmatrix} a^{\textcircled{\#}} & 0 \\ 0 & 0 \end{bmatrix}_{p\times q^*} \\ a_{\textcircled{\#}} &= \begin{bmatrix} a_{\textcircled{\#}} & 0 \\ 0 & 0 \end{bmatrix}_{r\times q} = \begin{bmatrix} a_{\textcircled{\#}} & 0 \\ 0 & 0 \end{bmatrix}_{q^*\times q} = \begin{bmatrix} a_{\textcircled{\#}} & 0 \\ 0 & 0 \end{bmatrix}_{r\times r} = \begin{bmatrix} a_{\textcircled{\#}} & 0 \\ 0 & 0 \end{bmatrix}_{q^*\times r}. \end{aligned}$$

Also, the following hold:

(1) If $a \in R^{\#}$ then $a^{\#}$ is $(q,q)$-inverse of $a$.
(2) If $a \in R^{\dagger}$ then $a^{\dagger}$ is $(p,r)$-inverse of $a$.
(3) If $a \in R^{\textcircled{\#}}$ then $a^{\textcircled{\#}}$ is $(p,q)$-inverse of $a$.
(4) If $a \in R_{\textcircled{\#}}$ then $a_{\textcircled{\#}}$ is $(q,r)$-inverse of $a$.

**3.3. Star, sharp, core and dual core partial order.** We have already gave the definitions of star, sharp, core and dual core partial orders of matrices in the preliminary section, see Definition 2.4. Although the conditions $A <^{\#} B$, $A <^{\textcircled{\#}} B$ and $A <_{\textcircled{\#}} B$ does not require that $\operatorname{ind} B \leqslant 1$, we always suppose this because this is crucial for developing some properties of these relations such as transitivity.

To have a better insight into the partial orders based on generalized inverses, we will give the most important characterizations of these relations in the complex matrix case. They show us their substantial meanings. The representations (3.16), (3.17) and (3.19) are especially important.

Let $\operatorname{rank}(A) = a$ and $\operatorname{rank}(B) = b$.

**Theorem 3.14.** [54] *For $A, B \in M_{m\times n}$ the following conditions are equivalent:*

(i) $A <^* B$;
(ii) $AA^* = BA^*$ *and* $A^*A = A^*B$;

(iii) *There exist unitary matrices* $U \in M_m$ *and* $V \in M_n$, *and positive definite diagonal matrices* $D_a \in M_a$ *and* $D_{b-a} \in M_{b-a}$ *such that*

$$A = U \begin{bmatrix} D_a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^* \text{ and } B = U \begin{bmatrix} D_a & 0 & 0 \\ 0 & D_{b-a} & 0 \\ 0 & 0 & 0 \end{bmatrix} V^*; \tag{3.16}$$

(iv) $B\{1,3,4\} \subseteq A\{1,3,4\}$;

(v) *There exist self-adjoint idempotents* $P \in M_m$ *and* $Q \in M_n$ *such that*

$$A = PB = BQ.$$

*The relation* $<^*$ *is the partial order relation on* $M_{m\times n}$.

**Theorem 3.15.** [54] *For* $A, B \in I_{1,n}$ *the following conditions are equivalent:*

(i) $A <^{\#} B$;

(ii) $A^2 = AB = BA$;

(iii) *There exists invertible matrix* $R \in M_n$ *such that*

$$A = R \begin{bmatrix} D_a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} R^{-1} \text{ and } B = R \begin{bmatrix} D_a & 0 & 0 \\ 0 & D_{b-a} & 0 \\ 0 & 0 & 0 \end{bmatrix} R^{-1}, \tag{3.17}$$

*where* $D_a \in M_a$ *and* $D_{b-a} \in M_{b-a}$ *are invertible matrices.*

(iv) $B\{1,5\} \subseteq A\{1,5\}$;

(v) *There exists idempotent* $P \in M_n$ *such that* $A = PB = BP$.

*The relation* $<^{\#}$ *is a partial order relation on* $I_{1,n}$.

**Theorem 3.16.** [5] and [34]

*Let* $A \in M_n$ *be a matrix of rank* $r$. *Then* $A$ *can be written in the form*

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*, \tag{3.18}$$

*where* $U \in M_n$ *is unitary,* $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_r)$ *is diagonal where* $\sigma_1 \geqslant \dots, \geqslant \sigma_r > 0$ *are all singular values of* $A$ *with their multiplicity, and matrices* $K \in M_r$ *and* $L \in M_{r\times n-r}$ *satisfy* $KK^* + LL^* = I_r$.

*The matrix* $A$ *has core inverse if and only if* $\operatorname{ind} A \leqslant 1$. *In that case* $K$ *is invertible and*

$$A^{\#\!\!\!\bigcirc} = U \begin{bmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

*Let* $A, B \in I_{1,n}$ *and let the decomposition of* $A$ *be as in* (3.18). *Then* $A <^{\#\!\!\!\bigcirc} B$ *if and only if*

$$B = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & Z \end{bmatrix} U^*, \tag{3.19}$$

*where* $\Sigma K$ *is invertible and* $Z \in M_{n-r}$ *satisfies* $\operatorname{ind} Z \leqslant 1$.

*The relation* $<^{\#\!\!\!\bigcirc}$ *is a partial order relation on* $I_{1,n}$.

Of course, the analogous result hold for dual core partial order. Additional results for the core partial order on matrices can be found in [27].

Let now $R$ be a ring with involution. The four relations are defined in the same way as in the matrix case.

**Definition 3.5** (From [70])**.** Let $a, b \in R$.
If $a, b \in R^\dagger$ then $a <^* b$ if $aa^\dagger = ba^\dagger$ and $a^\dagger a = a^\dagger b$.
If $a, b \in R^\#$ then $a <^\# b$ if $aa^\# = ba^\#$ and $a^\# a = a^\# b$.
If $a, b \in R^{\textcircled{\#}}$ then $a <^{\textcircled{\#}} b$ if $aa^{\textcircled{\#}} = ba^{\textcircled{\#}}$ and $a^{\textcircled{\#}} a = a^{\textcircled{\#}} b$.
If $a, b \in R_{\textcircled{\#}}$ then $a <_{\textcircled{\#}} b$ if $aa_{\textcircled{\#}} = ba_{\textcircled{\#}}$ and $a_{\textcircled{\#}} a = a_{\textcircled{\#}} b$.

We will give the basic characterizations.

**Theorem 3.17.** [23] *Let $a, b \in R^\dagger$. The following conditions are equivalent:*

(i) $a <^* b$; (iii) $aa^\dagger = ab^\dagger$ *and* $a^\dagger a = b^\dagger a$; (v) $a^\dagger <^* b^\dagger$.
(ii) $aa^* = ba^*$ *and* $a^* a = a^* b$; (iv) $a^* <^* b^*$;

*Proof.* (i) $\Leftrightarrow$ (ii): By definition, $a <^* b$ if and only if $aa^\dagger = ba^\dagger$ and $a^\dagger a = a^\dagger b$. Let us prove that $aa^\dagger = ba^\dagger$ is equivalent to $aa^* = ba^*$. If $aa^\dagger = ba^\dagger$ then

$$ba^* = b(aa^\dagger a)^* = b(a^\dagger a)^* a^* = ba^\dagger aa^* = aa^\dagger aa^* = aa^*.$$

If $aa^* = ba^*$ then

$$ba^\dagger = ba^\dagger aa^\dagger = b(a^\dagger a)^* a^\dagger = ba^*(a^\dagger)^* a^\dagger = aa^*(a^\dagger)^* a^\dagger = a(a^\dagger a)^* a^\dagger = aa^\dagger aa^\dagger = aa^\dagger.$$

Similarly, $a^\dagger a = a^\dagger b$ is equivalent to $a^* a = a^* b$.

(i) $\Leftrightarrow$ (iii): Suppose that $a <^* b$ that is $aa^\dagger = ba^\dagger$ and $a^\dagger a = a^\dagger b$. Then

$$\begin{aligned} ab^\dagger &= aa^\dagger ab^\dagger = aa^\dagger bb^\dagger = (aa^\dagger)^*(bb^\dagger)^* = (bb^\dagger aa^\dagger)^* = (bb^\dagger ba^\dagger)^* = (ba^\dagger)^* \\ &= (aa^\dagger)^* = aa^\dagger. \end{aligned}$$

Similarly, $b^\dagger a = a^\dagger a$. If (iii) holds then

$$ba^\dagger = ba^\dagger aa^\dagger = bb^\dagger aa^\dagger = (aa^\dagger bb^\dagger)^* = (ab^\dagger bb^\dagger)^* = (ab^\dagger)^* = (aa^\dagger)^* = aa^\dagger$$

and similarly $a^\dagger b = a^\dagger a$.

(iv) $\Leftrightarrow$ (i): Recall that $(a^\dagger)^* = (a^*)^\dagger$. Note that $a^* <^* b^*$ if and only if $a^*(a^*)^\dagger = b^*(a^*)^\dagger$ and $(a^*)^\dagger a^* = (a^*)^\dagger b^*$ if and only if $a^\dagger a = a^\dagger b$ and $aa^\dagger = ba^\dagger$ if and only if $a <^* b$.

(v) $\Leftrightarrow$ (iii): Recall that $(a^\dagger)^\dagger = a$. We have $a^\dagger <^* b^\dagger$ if and only if $a^\dagger(a^\dagger)^\dagger = b^\dagger(a^\dagger)^\dagger$ and $(a^\dagger)^\dagger a^\dagger = (a^\dagger)^\dagger b^\dagger$ if and only if $a^\dagger a = b^\dagger a$ and $aa^\dagger = ab^\dagger$. □

**Theorem 3.18.** [70]. See also [54] *Let $a, b \in R^\#$. The following conditions are equivalent:*

(i) $a <^\# b$; (ii) $a^2 = ab = ba$; (iii) $aa^\# = ab^\# = b^\# a$; (iv) $a^\# <^\# b^\#$.

*Moreover,*

$$\begin{aligned} &a^\# a = a^\# b \Leftrightarrow a^2 = ab \Leftrightarrow aa^\# = ab^\# \quad \textit{and} \\ &aa^\# = ba^\# \Leftrightarrow a^2 = ba \Leftrightarrow a^\# a = b^\# a. \end{aligned} \tag{3.20}$$

*Proof.* (i) $\Leftrightarrow$ (ii): If $a <^{\#} b$ then $aa^{\#} = ba^{\#}$ and $a^{\#}a = a^{\#}b$. We have

$$ba = baa^{\#}a = ba^{\#}a^2 = aa^{\#}a^2 = a^2.$$

Similarly $a^{\#}a = a^{\#}b$ implies $a^2 = ab$. Conversely, if $a^2 = ba$ then

$$ba^{\#} = baa^{\#}a^{\#} = a^2a^{\#}a^{\#} = aa^{\#}.$$

Similarly, $a^2 = ab$ implies $a^{\#}a = a^{\#}b$, so $a <^{\#} b$.

(ii) $\Rightarrow$ (iii): Since $a = a^2a^{\#} = a^3(a^{\#})^2$ and $a^3 = a^2a = baa = b^2a$ we have

$$b^{\#}a = b^{\#}a^3(a^{\#})^2 = b^{\#}b^2a(a^{\#})^2 = ba(a^{\#})^2 = a^2(a^{\#})^2 = a^{\#}a.$$

Similarly, $ab^{\#} = aa^{\#}$.

(iii) $\Rightarrow$ (ii): Since $a = a^{\#}a^2 = b^{\#}a^2 = (b^{\#})^2a^3$ we have

$$ba = b(b^{\#})^2a^3 = b^{\#}a^3 = a^2.$$

Similarly, $ab = a^2$.

(iii) $\Leftrightarrow$ (iv) follows by definition of the sharp order and the elementary property of group inverse $(a^{\#})^{\#} = a$.

The equivalences (3.20) follow from the previous part of the proof. □

**Lemma 3.3.** [70] and [23] *Let $a, b \in R^{\textcircled{\#}} \cap R^{\dagger}$. Then*

(i) $a^{\textcircled{\#}}a = a^{\textcircled{\#}}b \;\Leftrightarrow\; a^*a = a^*b \;\Leftrightarrow\; a^{\dagger}a = a^{\dagger}b$;

(ii) $aa^{\textcircled{\#}} = ba^{\textcircled{\#}} \;\Leftrightarrow\; a^2 = ba \;\Leftrightarrow\; aa^{\#} = ba^{\#}$;

*Proof.* (i): Suppose that $a^{\textcircled{\#}}a = a^{\textcircled{\#}}b$. Using the equations that characterize core inverse (Theorem 3.12), we obtain

$$a^*b = (aa^{\textcircled{\#}}a)^*b = a^*(aa^{\textcircled{\#}})^*b = a^*aa^{\textcircled{\#}}b = a^*aa^{\textcircled{\#}}a = a^*a.$$

If $a^*a = a^*b$ then

$$a^{\textcircled{\#}}b = a^{\textcircled{\#}}(aa^{\textcircled{\#}})^*b = a^{\textcircled{\#}}(a^{\textcircled{\#}})^*a^*b = a^{\textcircled{\#}}(a^{\textcircled{\#}})^*a^*a = a^{\textcircled{\#}}a.$$

The equivalence of $a^*a = a^*b$ and $a^{\dagger}a = a^{\dagger}b$ is proved in Theorem 3.17.

(ii): Recall that $a(a^{\textcircled{\#}})^2 = a^{\textcircled{\#}}$ and $a^{\textcircled{\#}}a^2 = a$. If $aa^{\textcircled{\#}} = ba^{\textcircled{\#}}$ then

$$ba = ba^{\textcircled{\#}}a^2 = aa^{\textcircled{\#}}aa = a^2.$$

Conversely, suppose that $a^2 = ba$. Then

$$ba^{\textcircled{\#}} = ba(a^{\textcircled{\#}})^2 = a^2(a^{\textcircled{\#}})^2 = aa^{\textcircled{\#}}.$$

The equivalence of $a^2 = ba$ and $aa^{\#} = ba^{\#}$ is proved in previous theorem. □

**Theorem 3.19.** [70] *Let $a, b \in R^{\textcircled{\#}} \cap R^{\dagger}$. The following statements are equivalent:*

(i) $a <^{\textcircled{\#}} b$;

(ii) $a^{\dagger}a = a^{\dagger}b$ *and* $aa^{\#} = ba^{\#}$;

(iii) $a^*a = a^*b$ *and* $a^2 = ba$.

*Any of* (i)–(iii) *implies*

(iv) $aa^{\textcircled{\#}} = ab^{\textcircled{\#}}$ *and* $a^{\textcircled{\#}}a = b^{\textcircled{\#}}a$.

*Proof.* The equivalence of (i), (ii) i (iii) is a direct consequence of Lemma 3.3. Let us prove that (i) implies (iv). Suppose that $a <^{\textcircled{\#}} b$, that is $aa^{\textcircled{\#}} = ba^{\textcircled{\#}}$ and $a^{\textcircled{\#}}a = a^{\textcircled{\#}}b$. We have

$$\begin{aligned} ab^{\textcircled{\#}} &= aa^{\textcircled{\#}}ab^{\textcircled{\#}} = aa^{\textcircled{\#}}bb^{\textcircled{\#}} = (aa^{\textcircled{\#}})^*bb^{\textcircled{\#}} = (ba^{\textcircled{\#}})^*(bb^{\textcircled{\#}})^* \\ &= (bb^{\textcircled{\#}}ba^{\textcircled{\#}})^* = (ba^{\textcircled{\#}})^* = (aa^{\textcircled{\#}})^* = aa^{\textcircled{\#}}, \\ b^{\textcircled{\#}}a &= b^{\textcircled{\#}}aa^{\textcircled{\#}}a = b^{\textcircled{\#}}ba^{\textcircled{\#}}a = b^{\textcircled{\#}}ba(a^{\textcircled{\#}})^2a \quad (\text{follows from } a^{\textcircled{\#}} = a(a^{\textcircled{\#}})^2) \\ &= b^{\textcircled{\#}}bb(a^{\textcircled{\#}})^2a = b(a^{\textcircled{\#}})^2a \quad (\text{follows from } b^{\textcircled{\#}}b^2 = b) \\ &= a(a^{\textcircled{\#}})^2a = a^{\textcircled{\#}}a \quad (\text{follows from } a^{\textcircled{\#}} = a(a^{\textcircled{\#}})^2). \end{aligned}$$ □

In the same manner we can prove the following theorem.

**Theorem 3.20.** *(From [70].) Let $a, b \in R_{\textcircled{\#}} \cap R^\dagger$. The following statements are equivalent:*

(i) $a <_{\textcircled{\#}} b$;

(ii) $aa^\dagger = ba^\dagger$ *and* $a^\# a = a^\# b$;

(iii) $aa^* = ba^*$ *and* $a^2 = ab$.

*Any of* (i)–(iii) *implies*

(iv) $a_{\textcircled{\#}}a = b_{\textcircled{\#}}a$ *and* $aa_{\textcircled{\#}} = ab_{\textcircled{\#}}$.

The crucial techniques for developing the properties of star, sharp, core and dual core partial orders involve the idempotents induced by the condition $a < b$, where $<$ stands for one of these four relations.

**Theorem 3.21.** [70]

(i) *If $a, b \in R^\dagger$ then $a <^* b$ if and only if there exist self-adjoint idempotents $p, r \in \tilde{E}(R)$ such that $a = pb = br$.*

(ii) *If $a, b \in R^\#$ then $a <^\# b$ if and only if there exists idempotent $q \in E(R)$ such that $a = qb = bq$.*

(iii) *If $a, b \in R^{\textcircled{\#}}$ then $a <^{\textcircled{\#}} b$ if and only if there exists self-adjoint idempotent $p \in \tilde{E}(R)$ and idempotent $q \in E(R)$ such that $qa = a$ and $a = pb = bq$.*

(iv) *If $a, b \in R_{\textcircled{\#}}$ then $a <_{\textcircled{\#}} b$ if and only if there exists self-adjoint idempotent $r \in \tilde{E}(R)$ and idempotent $q \in E(R)$ such that $aq = a$ and $a = br = qb$.*

*Proof.* To avoid repetition, we will prove only the statement (iv). Let $a, b \in R_{\textcircled{\#}}$ and $a <_{\textcircled{\#}} b$. Set $r = a_{\textcircled{\#}}a$ and $q = aa_{\textcircled{\#}}$. Then $r = r^2 = r^*$, $q = q^2$ and $aq = a^2a_{\textcircled{\#}} = a$. Also

$$br = ba_{\textcircled{\#}}a = aa_{\textcircled{\#}}a = a \quad \text{and} \quad qb = aa_{\textcircled{\#}}b = aa^{\textcircled{\#}}a = a.$$

Conversely, suppose there are self-adjoint idempotent $r$ and idempotent $q$ such that $aq = a$ and $a = br = qb$. We obtain $ar = brr = br = a$ so

$$ba^* = b(ar)^* = bra^* = aa^*, \quad ab = (aq)b = aa = a^2.$$

By Theorem 3.20 we conclude that $a <^{\textcircled{\#}} b$. □

It should be noted that in the previous theorem we can make that idempotents with the same mark are the same. Actually, if $a, b \in R^{\dagger} \cap R^{\#}$ then we can choose idempotents $p$, $q$ and $r$ such that they are the same as those given by equations (3.15).

We have already emphasized the importance of the representations (3.16), (3.17) and (3.19). The following four theorems give the generalization of these representations. We will give only the proof for the core partial order because the proofs for $<^*$, $<^{\#}$ and $<_{\textcircled{\#}}$ can be derived similarly.

**Theorem 3.22.** [70] *Let $a, b \in R^{\dagger}$. If $a <^* b$ then there exist orthogonal decompositions of the identity of the ring $R$*

$$1 = e_1 + e_2 + e_3 \quad \textit{and} \quad 1 = g_1 + g_2 + g_3$$

*with respect to which $a$ and $b$ have the following matrix representations*

$$a = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e \times g}, \quad b = \begin{bmatrix} a & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e \times g},$$

*where $a \in e_1 R g_1$ is $(e_1, g_1)$-invertible and $b - a \in e_2 R g_2$ is $(e_2, g_2)$-invertible.*

*Conversely, if*

$$b = \begin{bmatrix} a & 0 \\ 0 & b-a \end{bmatrix}_{e \times g},$$

*for some idempotents $e, g \in E(R)$ such that $a^{\dagger} e = a^{\dagger}$ and $g a^{\dagger} = a^{\dagger}$ then $a <^* b$.*

**Theorem 3.23.** [70] *Let $a, b \in R^{\#}$. If $a <^{\#} b$ then there exists decomposition of the identity of the ring $R$*

$$1 = f_1 + f_2 + f_3$$

*with respect to which $a$ and $b$ have the following matrix representations*

$$a = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{f \times f}, \quad b = \begin{bmatrix} a & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & 0 \end{bmatrix}_{f \times f},$$

*where $a \in f_1 R f_1$ is $(f_1, f_1)$-invertible and $b - a \in f_2 R f_2$ is $(f_2, f_2)$-invertible.*

*Conversely, if*

$$b = \begin{bmatrix} a & 0 \\ 0 & b-a \end{bmatrix}_{f \times f},$$

*for some idempotent $f \in E(R)$ such that $a^{\#} f = f a^{\#} = a^{\#}$ (equivalently $af = fa = a$) then $a <^{\#} b$.*

The advantage of representations (3.21) given in the following Theorem 3.24 compared to representations (3.19) lies in the fact that they have more zeros and all nonzero entries are invertible in the sense of Definition 3.3.

**Theorem 3.24.** [70] *Let $a, b \in R^{\textcircled{\#}}$. If $a <^{\textcircled{\#}} b$ then there exists orthogonal decomposition of the identity of the ring $R$*

$$1 = e_1 + e_2 + e_3$$

*and there exists decomposition of the identity of the ring $R$*

$$1 = f_1 + f_2 + f_3$$

*with respect to which $a$ and $b$ have the following matrix representations*

$$a = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times f}, \quad b = \begin{bmatrix} a & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times f}, \tag{3.21}$$

*where $a \in e_1Rf_1$ is $(e_1, f_1)$-invertible and $b - a \in e_2Rf_2$ is $(e_2, f_2)$-invertible.*

*Conversely, if*

$$b = \begin{bmatrix} a & 0 \\ 0 & b-a \end{bmatrix}_{e\times f}, \tag{3.22}$$

*for some idempotents $e, f \in E(R)$ such that $a^{\textcircled{\#}}e = a^{\textcircled{\#}}$ and $fa^{\textcircled{\#}} = a^{\textcircled{\#}}$ ($fa^{\textcircled{\#}} = a^{\textcircled{\#}}$ $\Leftrightarrow$ $fa = a$) then $a <^{\textcircled{\#}} b$.*

*Proof.* The idea of the proof is the same as in Theorem 3.5 where we proved the similar result for the minus partial order. Suppose that $a <^{\textcircled{\#}} b$. Then $aa^{\textcircled{\#}} = ba^{\textcircled{\#}}$ and $a^{\textcircled{\#}}a = a^{\textcircled{\#}}b$. Let

$$e_1 = ab^{\textcircled{\#}}, \quad e_2 = (b-a)b^{\textcircled{\#}}, \quad e_3 = 1 - bb^{\textcircled{\#}}$$
$$f_1 = b^{\textcircled{\#}}a, \quad f_2 = b^{\textcircled{\#}}(b-a), \quad f_3 = 1 - b^{\textcircled{\#}}b.$$

From Theorem 3.19 we have $e_1 = aa^{\textcircled{\#}}$ and $f_1 = a^{\textcircled{\#}}a$. Since $a <^{\textcircled{\#}} b$ we know that $a <^{-} b$, so by Lemma 3.1, we have

$$a = ab^{\textcircled{\#}}b = bb^{\textcircled{\#}}a = ab^{\textcircled{\#}}a. \tag{3.23}$$

Therefore

$$ab^{\textcircled{\#}}ab^{\textcircled{\#}} = ab^{\textcircled{\#}}, \quad bb^{\textcircled{\#}}bb^{\textcircled{\#}} = bb^{\textcircled{\#}}, \quad ab^{\textcircled{\#}}bb^{\textcircled{\#}} = ab^{\textcircled{\#}}, \quad bb^{\textcircled{\#}}ab^{\textcircled{\#}} = ab^{\textcircled{\#}},$$
$$(ab^{\textcircled{\#}})^* = (aa^{\textcircled{\#}})^* = aa^{\textcircled{\#}} = ab^{\textcircled{\#}}, \quad (bb^{\textcircled{\#}})^* = bb^{\textcircled{\#}}.$$

It follows $e_ie_j = 0$, for $i \neq j$ and $e_i^2 = e_i = e_i^*$, so $1 = e_1 + e_2 + e_3$ is an orthogonal decomposition of the identity of $R$. In the same manner we can prove that $1 = f_1 + f_2 + f_3$ is a decempisition of the identity of $R$. Because of the uniqueness of matrix representation of an element, to show the matrix forms (3.21), it is necessary and sufficient to show that $e_1af_1 = a$, $e_1bf_1 = a$ and $e_2bf_2 = b - a$. Using (3.23), we get

$$\begin{aligned} e_1bf_1 &= ab^{\textcircled{\#}}bb^{\textcircled{\#}}a = ab^{\textcircled{\#}}a = a, \\ e_2bf_2 &= (b-a)b^{\textcircled{\#}}bb^{\textcircled{\#}}(b-a) = (b-a)b^{\textcircled{\#}}(b-a) \\ &= bb^{\textcircled{\#}}b - bb^{\textcircled{\#}}a - ab^{\textcircled{\#}}b + ab^{\textcircled{\#}}a = b - a - a + a = b - a. \end{aligned}$$

Note that from $e_1bf_1 = a$ follows $e_1af_1 = a$. Let us prove that $a \in e_1Rf_1$ is $(e_1, f_1)$-invertible and that $b - a \in e_2Rf_2$ is $(e_2, f_2)$-invertible. From $e_1 = aa^{\textcircled{\#}}$ and $f_1 = a^{\textcircled{\#}}a$, it is easy to see that $a^{\textcircled{\#}} \in f_1Re_1$ is $(e_1, f_1)$-inverse of $a$. Since

$$b^{\textcircled{\#}}(b-a)b^{\textcircled{\#}} = b^{\textcircled{\#}} - b^{\textcircled{\#}}ab^{\textcircled{\#}} = b^{\textcircled{\#}} - a^{\textcircled{\#}}aa^{\textcircled{\#}} = b^{\textcircled{\#}} - a^{\textcircled{\#}},$$

and $e_2 = (b-a)b^{\textcircled{\#}}$, $f_2 = b^{\textcircled{\#}}(b-a)$ it follows that $b^{\textcircled{\#}} - a^{\textcircled{\#}} = f_2b^{\textcircled{\#}} = b^{\textcircled{\#}}e_2$, that is

$$b^{\textcircled{\#}} - a^{\textcircled{\#}} = f_2(b^{\textcircled{\#}} - a^{\textcircled{\#}})e_2 \in f_2Re_2.$$

It is easy to show that $(b-a)(b^{\textcircled{\#}} - a^{\textcircled{\#}}) = e_2 - ba^{\textcircled{\#}} + aa^{\textcircled{\#}} = e_2$ and similarly $(b^{\textcircled{\#}} - a^{\textcircled{\#}})(b-a) = f_2$. Therefore $b^{\textcircled{\#}} - a^{\textcircled{\#}}$ is $(e_2, f_2)$-inverse of $b - a$.

Conversely, suppose that $b$ have the form (3.22) with $a^{\textcircled{\#}}e = a^{\textcircled{\#}}$ and $fa^{\textcircled{\#}} = a^{\textcircled{\#}}$. Then $ebf = a$ and $(1-e)b(1-f) = b-a$. Multiplying the last equation by $a^{\textcircled{\#}}e = a^{\textcircled{\#}}$ from the left, we obtain $0 = a^{\textcircled{\#}}(b-a)$, that is $a^{\textcircled{\#}}a = a^{\textcircled{\#}}b$. Similarly, multiplying $(1-e)b(1-f) = b-a$ by $fa^{\textcircled{\#}} = a^{\textcircled{\#}}$ from the right, we obtain $0 = (b-a)a^{\textcircled{\#}}$, that is $aa^{\textcircled{\#}} = ba^{\textcircled{\#}}$, so $a <^{\textcircled{\#}} b$. The equivalence of $fa^{\textcircled{\#}} = a^{\textcircled{\#}}$ and $fa = a$ follows from equations $a^{\textcircled{\#}}a^2 = a$ and $a(a^{\textcircled{\#}})^2 = a^{\textcircled{\#}}$ from Theorem 3.12. □

**Theorem 3.25.** [70] *Let $a, b \in R_{\textcircled{\#}}$. If $a <_{\textcircled{\#}} b$ then there exists orthogonal decomposition of the identity of the ring $R$*

$$1 = g_1 + g_2 + g_3$$

*and there exists decomposition of the identity of the ring $R$*

$$1 = f_1 + f_2 + f_3$$

*with respect to which $a$ and $b$ have the following matrix representations*

$$a = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{f \times g}, \quad b = \begin{bmatrix} a & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & 0 \end{bmatrix}_{f \times g},$$

*where $a \in f_1Rg_1$ is $(f_1, g_1)$-invertible and $b - a \in f_2Rg_2$ is $(f_2, g_2)$-invertible.*

*Conversely, if*

$$b = \begin{bmatrix} a & 0 \\ 0 & b-a \end{bmatrix}_{f \times g},$$

*for some idempotents $f, g \in E(R)$ such that $a_{\textcircled{\#}}f = a_{\textcircled{\#}}$ ($a_{\textcircled{\#}}f = a_{\textcircled{\#}} \Leftrightarrow af = a$) and $ga_{\textcircled{\#}} = a_{\textcircled{\#}}$ then $a <_{\textcircled{\#}} b$.*

Note that if $a, b \in R^{\dagger} \cap R^{\#}$ then idempotents $e_i$ which appear in Theorem 3.22 and idempotents $e_i$ which appear in Theorem 3.24 are the same; the idempotents $f_i$ which appear in Theorems 3.23, 3.24 and 3.25 are the same; idempotents $g_i$ which appear in Theorems 3.22 and 3.25 are the same.

Furthermore, under the notation of Theorem 3.24, from its proof, we have that $a <^{\#\!\!\!\bigcirc} b$ implies

$$b^{\#\!\!\!\bigcirc} = \begin{bmatrix} a^{\#\!\!\!\bigcirc} & 0 & 0 \\ 0 & b^{\#\!\!\!\bigcirc} - a^{\#\!\!\!\bigcirc} & 0 \\ 0 & 0 & 0 \end{bmatrix}_{f \times e}. \tag{3.24}$$

Also (see [70]),

$$a <^{*} b \Leftrightarrow b - a <^{*} b, \quad a <^{\#} b \Leftrightarrow b - a <^{\#} b, \quad a <^{-} b \Leftrightarrow b - a <^{-} b.$$

But $a <^{\#\!\!\!\bigcirc} b$ does not imply $b - a <^{\#\!\!\!\bigcirc} b$ even in complex matrix case, see [5].

## 3.4. The characterization of partial orders by set inclusion.

As in the Theorems 3.14 (iv) and 3.15 (iv), we can characterize the condition $a < b$, where $<$ belongs to $\in \{<^{*}, <^{\#}, <^{\#\!\!\!\bigcirc}, <_{\#\!\!\!\bigcirc}\}$ by the inclusion of the appropriate subsets of generalized inverses.

**Lemma 3.4.** [70]

(i) *Let* $b \in R^{\dagger}$. *Then*

$$b\{1,3\} = \left\{ \begin{bmatrix} b^{\dagger} & 0 \\ x_3 & x_4 \end{bmatrix}_{r \times p} : x_3 \in (1-r)Rp,\ x_4 \in (1-r)R(1-p) \right\},$$

$$b\{1,4\} = \left\{ \begin{bmatrix} b^{\dagger} & x_2 \\ 0 & x_4 \end{bmatrix}_{r \times p} : x_2 \in rR(1-p),\ x_4 \in (1-r)R(1-p) \right\},$$

$$b\{1,3,4\} = \left\{ \begin{bmatrix} b^{\dagger} & 0 \\ 0 & x_4 \end{bmatrix}_{r \times p} : x_4 \in (1-r)R(1-p) \right\},$$

*where* $r = b^{\dagger}b$ *and* $p = bb^{\dagger}$.

(ii) *Let* $b \in R^{\#}$. *Then*

$$b\{1,6\} = b\{6\} = \left\{ \begin{bmatrix} b^{\#} & x_2 \\ 0 & x_4 \end{bmatrix}_{q \times q} : x_2 \in qR(1-q),\ x_4 \in (1-q)R(1-q) \right\},$$

$$b\{1,8\} = b\{8\} = \left\{ \begin{bmatrix} b^{\#} & 0 \\ x_3 & x_4 \end{bmatrix}_{q \times q} : x_3 \in (1-q)Rq,\ x_4 \in (1-q)R(1-q) \right\},$$

$$b\{1,5\} = b\{1,6,8\} = \left\{ \begin{bmatrix} b^{\#} & 0 \\ 0 & x_4 \end{bmatrix}_{q \times q} : x_4 \in (1-q)R(1-q) \right\},$$

*where* $q = bb^{\#} = b^{\#}b$.

(iii) *Let* $b \in R^{\#\!\!\!\bigcirc}$. *Then*

$$b\{1,3\} = \left\{ \begin{bmatrix} b^{\#\!\!\!\bigcirc} & 0 \\ x_3 & x_4 \end{bmatrix}_{q \times p} : x_3 \in (1-q)Rp,\ x_4 \in (1-q)R(1-p) \right\},$$

$$b\{1,6\} = b\{6\} = \left\{ \begin{bmatrix} b^{\#\!\!\!\bigcirc} & x_2 \\ 0 & x_4 \end{bmatrix}_{q \times p} : x_2 \in qR(1-p),\ x_4 \in (1-q)R(1-p) \right\},$$

$$b\{1,3,6\} = b\{3,6\} = \left\{ \begin{bmatrix} b^{\#\!\!\!\!\bigcirc} & 0 \\ 0 & x_4 \end{bmatrix}_{q\times p} : x_4 \in (1-q)R(1-p) \right\},$$

*where* $q = b^{\#\!\!\!\!\bigcirc} b = b^{\#} b$ *and* $p = bb^{\#\!\!\!\!\bigcirc}$. *Moreover, if* $b \in R^{\dagger}$ *then* $p = bb^{\dagger}$.

(iv) *Let* $b \in R_{\#\!\!\!\!\bigcirc}$. *Then*

$$b\{1,4\} = \left\{ \begin{bmatrix} b_{\#\!\!\!\!\bigcirc} & x_2 \\ 0 & x_4 \end{bmatrix}_{r\times q} : x_2 \in rR(1-q),\, x_4 \in (1-r)R(1-q) \right\},$$

$$b\{1,8\} = b\{8\} = \left\{ \begin{bmatrix} b_{\#\!\!\!\!\bigcirc} & 0 \\ x_3 & x_4 \end{bmatrix}_{r\times q} : x_3 \in (1-r)Rq,\, x_4 \in (1-r)R(1-q) \right\},$$

$$b\{1,4,8\} = b\{4,8\} = \left\{ \begin{bmatrix} b_{\#\!\!\!\!\bigcirc} & 0 \\ 0 & x_4 \end{bmatrix}_{r\times q} : x_4 \in (1-r)R(1-q) \right\},$$

*where* $r = b_{\#\!\!\!\!\bigcirc} b$ *and* $q = bb_{\#\!\!\!\!\bigcirc} = bb^{\#}$. *Moreover, if* $b \in R^{\dagger}$ *then* $r = b^{\dagger} b$.

*Proof.* (iii): First, if $b \in R^{\#\!\!\!\!\bigcirc}$ then, by (3.14), $b \in R^{\#}$. Let $q = b^{\#\!\!\!\!\bigcirc} b$ and $p = bb^{\#\!\!\!\!\bigcirc}$. Then $q = b^{\#\!\!\!\!\bigcirc} b^2 b^{\#} = bb^{\#} = b^{\#} b$. Note that if $xb^2 = b$ then

$$bxb = bxb^2 b^{\#} = bbb^{\#} = b.$$

Therefore, $b\{1,6\} = b\{6\}$ and $b\{1,3,6\} = b\{3,6\}$. Since $p = bb^{\#\!\!\!\!\bigcirc}$ and $q = b^{\#\!\!\!\!\bigcirc} b = bb^{\#} = b^{\#} b$ we have $pbq = b$ and $qbq = b$, so

$$b = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}_{q\times q}.$$

Let us prove the first equality. Suppose that $x \in b\{1,3\}$ and $x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}_{q\times p}$. From $bxb = b$ it follows

$$\begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} = b = bxb = \begin{bmatrix} bx_1 b & 0 \\ 0 & 0 \end{bmatrix}_{p\times q}.$$

Hence $x_1 = qx_1 p = b^{\#\!\!\!\!\bigcirc} bx_1 bb^{\#\!\!\!\!\bigcirc} = b^{\#\!\!\!\!\bigcirc} bb^{\#\!\!\!\!\bigcirc} = b^{\#\!\!\!\!\bigcirc}$. Since $x \in b\{1,3\}$ we have $(bx)^* = bx$. In view of (2.3),

$$\begin{bmatrix} bb^{\#\!\!\!\!\bigcirc} & bx_2 \\ 0 & 0 \end{bmatrix}_{p\times p} = bx = (bx)^* = \begin{bmatrix} (bb^{\#\!\!\!\!\bigcirc})^* & 0 \\ (bx_2)^* & 0 \end{bmatrix}_{p\times p}.$$

It follows that $bx_2 = 0$. Multiplying this equation by $b^{\#\!\!\!\!\bigcirc}$ from the left side we get $0 = b^{\#\!\!\!\!\bigcirc} bx_2 = qx_2 = x_2$, since $x_2 \in qR(1-p)$. It follows that $x$ belongs to the set on left side of the first equality. Conversely, if $x = \left[\begin{smallmatrix} b^{\#\!\!\!\!\bigcirc} & 0 \\ x_3 & x_4 \end{smallmatrix}\right]_{q\times p}$ then $bxb = b$ and $(bx)^* = bx$. Thus, we show the first equality in (iii).

Suppose now that $x \in b\{6\}$ and $x = \left[\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right]_{q\times p}$. We have

$$\begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}_{q\times q} = b = xb^2 = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}_{q\times p} \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}_{q\times q} = \begin{bmatrix} x_1 b^2 & 0 \\ x_3 b^2 & 0 \end{bmatrix}_{q\times q}.$$

Hence $x_1 b^2 = b$ and $x_3 b^2 = 0$. Multiplying the first equation by $b^{\#}$ from the right we get $x_1 b = q$. Multiplying the last equality by $b^{\textcircled{\#}}$ from the right we obtain $x_1 p = qb^{\textcircled{\#}}$. Since $x_1 \in qRp$ and $qb^{\textcircled{\#}} = b^{\textcircled{\#}} bb^{\textcircled{\#}} = b^{\textcircled{\#}}$, we have $x_1 = b^{\textcircled{\#}}$. Similarly, from $x_3 b^2 = 0$ we derive $x_3 = 0$. Conversely, if $x = \begin{bmatrix} b^{\textcircled{\#}} & x_2 \\ 0 & x_4 \end{bmatrix}_{q\times p}$ then it is easy to show that $xb^2 = b$. We have proved the second equality in (iii).

Note that $b\{1,3,6\} = b\{1,3\} \cap b\{6\}$. Now, the proof of the third equation follows from the first two equations.

(i), (ii), (iv): These equalities can be proved in a similar manner. We will only prove that if $b \in R^{\#}$ then $b\{1,5\} = b\{1,6,8\}$. If $bxb = b$ and $bx = xb$ then $xb^2 = bxb = b$ and $b^2 x = bxb = b$. If $b = bxb = xb^2 = b^2 x$ then $bx = b^{\#} b^2 x = b^{\#} b$ and $xb = xb^2 b^{\#} = bb^{\#}$, so $bx = xb$. □

**Theorem 3.26.** [70]

(i) *Let $a, b \in R^{\dagger}$. The following conditions are equivalent:*
  (a) $a <^{*} b$,
  (b) $b\{1,3\} \subseteq a\{1,3\}$ *and* $b\{1,4\} \subseteq a\{1,4\}$,
  (c) $b\{1,3,4\} \subseteq a\{1,3,4\}$.

(ii) *Let $a, b \in R^{\#}$. The following conditions are equivalent:*
  (a) $a <^{\#} b$,
  (b) $b\{6\} \subseteq a\{6\}$ *and* $b\{8\} \subseteq a\{8\}$,
  (c) $b\{1,5\} \subseteq a\{1,5\}$.

(iii) *Let $a, b \in R^{\textcircled{\#}}$. The following conditions are equivalent:*
  (a) $a <^{\textcircled{\#}} b$,
  (b) $b\{1,3\} \subseteq a\{1,3\}$ *and* $b\{6\} \subseteq a\{6\}$, (*See* [49] *for the complex matrix case.*)
  (c) $b\{3,6\} \subseteq a\{3,6\}$.

(iv) *Let $a, b \in R_{\textcircled{\#}}$. The following conditions are equivalent:*
  (a) $a <_{\textcircled{\#}} b$,
  (b) $b\{1,4\} \subseteq a\{1,4\}$ *and* $b\{8\} \subseteq a\{8\}$,
  (c) $b\{4,8\} \subseteq a\{4,8\}$.

*Proof.* (iii): (a) ⇒ (b) Suppose that $a, b \in R^{\textcircled{\#}}$ and $a <^{\textcircled{\#}} b$. From Theorem 3.24 we have

$$a = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times f}, \quad b = \begin{bmatrix} a & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times f},$$

where $e_1 = ab^{\textcircled{\#}} = aa^{\textcircled{\#}}$, $f_1 = b^{\textcircled{\#}} a = a^{\textcircled{\#}} a$. According to Lemma 3.4 (iii), it follows that $x \in a\{1,3\}$ if and only if

$$x = \begin{bmatrix} a^{\textcircled{\#}} & 0 & 0 \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}_{f\times e},$$

where $x_{ij}$ are arbitrary elements in appropriate subsets of $R$. Note that $e_1 + e_2 = bb^{\textcircled{\#}}$ and $f_1 + f_2 = b^{\textcircled{\#}} b$. By Lemma 3.4 (iii) and representation for $b^{\textcircled{\#}}$ given in

(3.24), it follows that $x \in b\{1,3\}$ if and only if

$$x = \begin{bmatrix} a^{\#} & 0 & 0 \\ 0 & b^{\#} - a^{\#} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix}_{f \times e}.$$

We conclude that $b\{1,3\} \subseteq a\{1,3\}$. Likewise, we obtain that $b\{6\} \subseteq a\{6\}$.

(b) $\Rightarrow$ (c) is trivial since $b\{3,6\} = b\{1,3,6\}$.

(c) $\Rightarrow$ (a) Suppose that $b\{3,6\} \subseteq a\{3,6\}$. Let $a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_{p \times q}$, where $p = bb^{\#} = p^*$ and $q = b^{\#}b$. From

$$b^{\#} = \begin{bmatrix} b^{\#} & 0 \\ 0 & 0 \end{bmatrix}_{q \times p} \in b\{3,6\} \subseteq a\{3,6\} = a\{1,3,6\}$$

we conclude $ab^{\#}a = a$, $ab^{\#} = (ab^{\#})^*$, $b^{\#}a^2 = a$. The condition $ab^{\#} = (ab^{\#})^*$ became in the matrix form

$$\begin{bmatrix} a_1 b^{\#} & 0 \\ a_3 b^{\#} & 0 \end{bmatrix}_{p \times p} = \begin{bmatrix} (a_1 b^{\#})^* & (a_3 b^{\#})^* \\ 0 & 0 \end{bmatrix}_{p \times p}.$$

Thus $a_1 b^{\#} = (a_1 b^{\#})^* = (b^{\#})^* a_1^*$ and $a_3 b^{\#} = 0$. When we multiply the latter equation by $b$ from the right we obtain $a_3 q = 0$, so $a_3 = 0$ since $a_3 \in (1-p)Rq$. Multiplying the equation $a_1 b^{\#} = (b^{\#})^* a_1^*$ by $b^*$ from the left and by $b$ from the right side we get $b^* a_1 q = b^*(b^{\#})^* a_1^* b$. Thus

$$b^* a_1 = (a_1 b^{\#} b)^* b = (a_1 q)^* b = a_1^* b. \tag{3.25}$$

Note that $qp = b^{\#}bbb^{\#} = (b^{\#}b^2)b^{\#} = bb^{\#} = p$ so $(1-q)(1-p) = 1-q$, ie. $1-q \in (1-q)R(1-p)$. Let $x = \begin{bmatrix} b^{\#} & 0 \\ 0 & 1-q \end{bmatrix}_{q \times p}$. By Lemma 3.4, $x \in b\{3,6\}$, so $x \in a\{3,6\}$. As $a_2(1-q) = a_2$ and $a_4(1-q) = a_4$, we get $ax = \begin{bmatrix} a_1 b^{\#} & a_2 \\ 0 & a_4 \end{bmatrix}_{p \times p}$. Therefore, the condition $ax = (ax)^*$ gives $a_2 = 0$. Next, $ab^{\#}a = a$ yields

$$\begin{bmatrix} a_1 b^{\#} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p \times q} = \begin{bmatrix} a_1 & 0 \\ 0 & a_4 \end{bmatrix}_{p \times q}.$$

Thus $a_4 = 0$ and $a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}_{p \times q} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{p \times q}$, thus $a = paq$. Therefore, the condition (3.25) reduces to

$$b^* a = a^* b. \tag{3.26}$$

Multiplying the equality $b^{\#}a^2 = a$ by $b$ from the left we obtain $pa^2 = ba$, i.e.

$$a^2 = ba. \tag{3.27}$$

Now, by Theorem 3.19, to prove $a <^{\textcircled{\#}} b$ it is sufficient to show that $a^*b = a^*a$. We have

$$\begin{aligned} a^*b &= (aa^{\textcircled{\#}}a)^*b = (aa(a^{\textcircled{\#}})^2a)^*(b^*)^* \quad (\text{by } a^{\textcircled{\#}} = a(a^{\textcircled{\#}})^2) \\ &= (b^*aa(a^{\textcircled{\#}})^2a)^* = (a^*ba(a^{\textcircled{\#}})^2a)^* \quad (\text{by } (3.26)) \\ &= (a^*a^2(a^{\textcircled{\#}})^2a)^* \quad (\text{by } (3.27)) \\ &= (a^*aa^{\textcircled{\#}}a)^* = (a^*a)^* = a^*a. \end{aligned}$$

(i), (ii), (iv): The equivalences in (i), (ii) and (iv) can be proved similarly. □

**Theorem 3.27.** *The relations $<^*$, $<^\#$, $<^{\textcircled{\#}}$ and $<_{\textcircled{\#}}$ are partial order relations on the sets $R^\dagger$, $R^\#$, $R^{\textcircled{\#}}$ and $R_{\textcircled{\#}}$, respectively.*

*Proof.* Reflexivity and transitivity follow by Theorem 3.26. Since $a < b$ implies $a <^- b$, where $< \in \{<^*, <^\#, <^{\textcircled{\#}}, <_{\textcircled{\#}}\}$ and $a <^- b$ is antisymmetric relation, it follows that $<$ is antisymmetric. □

### 3.5. When does minus partial order imply other partial orders?

If $< \in \{<^*, <^\#, <^{\textcircled{\#}}, <_{\textcircled{\#}}\}$ then it is clear that $a < b$ implies $a <^- b$. We will now see under what conditions the converse is true.

**Lemma 3.5.**[70] *Let $a, b \in R^\dagger$ and $a <^- b$. The following conditions are equivalent:*

(i) $a^*a = a^*b$

(ii) $a^*b$ *is self-adjoint*;

(iii) $a^\dagger a = a^\dagger b$;

(iv) $a^\dagger b$ *is self-adjoint*;

(v) $ab^\dagger$ *is self-adjoint.*

*Proof.* Since $a <^- b$, by Lemma 3.1 it follows that $a = ab^{(1)}a = bb^{(1)}a = ab^{(1)}b$ for any $b^{(1)} \in b\{1\}$.

(i) ⇔ (ii): It is clear that (i) implies (ii). Conversely, suppose that $a^*b$ is self-adjoint. Since $a <^- b$ there exists $a^{(1)} \in a\{1\}$ such that $aa^{(1)} = ba^{(1)}$. Therefore,

$$a^*a = a^*aa^{(1)}a = a^*ba^{(1)}a = (a^*b)^*a^{(1)}a = b^*aa^{(1)}a = b^*a,$$

so, taking adjoint, we obtain $a^*a = a^*b$.

(i) ⇔ (iii) follows from the proof of Theorem 3.17.

(iii) ⇔ (iv): It is clear that (iii) implies (iv). Conversely, suppose that $a^\dagger b = (a^\dagger b)^*$. Let $b^{(1)} \in b\{1\}$. We have,

$$\begin{aligned} a^\dagger a &= a^\dagger bb^{(1)}a = (a^\dagger b)^*b^{(1)}a = (a^\dagger aa^\dagger b)^*b^{(1)}a \\ &= (a^\dagger b)^*a^\dagger ab^{(1)}a = (a^\dagger b)^*a^\dagger a = (a^\dagger aa^\dagger b)^* = (a^\dagger b)^*, \end{aligned}$$

so $a^\dagger a = a^\dagger b$.

(i) ⇔ (v): If $a^*a = a^*b$ then

$$\begin{aligned} ab^\dagger &= aa^\dagger ab^\dagger = (aa^\dagger)^*ab^\dagger = (a^\dagger)^*a^*ab^\dagger = (a^\dagger)^*a^*bb^\dagger \\ &= (aa^\dagger)^*(bb^\dagger)^* = (bb^\dagger aa^\dagger)^* = (aa^\dagger)^* = aa^\dagger, \end{aligned}$$

so $ab^\dagger$ is self-adjoint. If $ab^\dagger$ is self-adjoint then

$$a^*b = (ab^\dagger a)^* b = a^*(ab^\dagger)^* b = a^* ab^\dagger b = a^* a. \qquad \square$$

Similarly, we have dual result.

**Lemma 3.6.** *Let $a, b \in R^\dagger$ and $a <^- b$. The following statements are equivalent:*

(i) $aa^* = ba^*$ (iii) $aa^\dagger = ba^\dagger$; (v) $b^\dagger a$ *is self-adjoint.*

(ii) $ba^*$ *is self-adjoint*; (iv) $ba^\dagger$ *is self-adjoint*;

**Lemma 3.7.** [70] *Let $a, b \in R^\#$. The following conditions are equivalent:*
(i) *$ab = ba$;*
(ii) *$a^2b = aba$ and $ba^2 = aba$.*

*Proof.* It is clear that (i) implies (ii). Suppose that $a^2b = aba$ and $ba^2 = aba$. If $q = aa^\#$ then $a = qaq$, so $a = \left[\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right]_{q\times q}$. Suppose that $b = \left[\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right] q \times q$. From $a^2b = aba$ it follows that $a^2b_1 = ab_1a$ and $a^2b_2 = 0$. Multiplying $a^2b_1 = ab_1a$ by $a^\#$ from the left we obtain $ab_1 = a^\# ab_1 a = qb_1a = b_1a$. Multiplying $a^2b_2 = 0$ by $(a^\#)^2$ from the left we obtain that $b_2 = 0$. Similarly, $ba^2 = aba$ implies $b_3 = 0$. Now, it is easy to see that $ab = ba$. $\square$

**Lemma 3.8.** [70] *Let $a, b \in R^\#$ and $a <^- b$. The following conditions are equivalent:*

(i) $ba = a^2$; (iii) $b(a^\#)^2 = a^\# ba^\#$; (v) $b^\#(a^\#)^2 = a^\# b^\# a^\#$.

(ii) $ba^2 = aba$; (iv) $b^\# a^2 = ab^\# a$;

*Proof.* Since $a <^- b$ there exists $a^{(1)} \in a\{1\}$ such that $a^{(1)}a = a^{(1)}a$ and $aa^{(1)} = ba^{(1)}$. In (3.2) in the proof of Theorem 3.3 we proved that $a^{(1)} \in a\{1\}$ if and only if

$$a^{(1)} = \begin{bmatrix} a^\# & x_2 \\ x_3 & x_4 \end{bmatrix}_{q\times q},$$

where $q = aa^\#$. Of course,

$$a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{q\times q} \quad \text{and} \quad a^\# = \begin{bmatrix} a^\# & 0 \\ 0 & 0 \end{bmatrix}_{q\times q}.$$

Since $a <^- b$ implies $b\{1\} \subseteq a\{1\}$, we have

$$ab^\# a = a. \tag{3.28}$$

(i) $\Leftrightarrow$ (ii): If $ba = a^2$ then $ba^2 = a^3 = aba$. Conversely, suppose that $ba^2 = aba$ and let $b = \left[\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right]_{q\times q}$. The condition $ba^2 = aba$ gives $b_1a^2 = ab_1a$ and $b_3a^2 = 0$. Multiplying the latter equation by $(a^\#)^2$ from the right we obtain $0 = b_3a^2(a^\#)^2 = b_3aa^\# = b_3q = b_3$. Also,

$$\begin{bmatrix} q & 0 \\ x_3a & 0 \end{bmatrix}_{q\times q} = a^{(1)}a = a^{(1)}b = \begin{bmatrix} a^\# b_1 & a^\# b_2 + x_2b_4 \\ x_3b_1 & x_3b_2 + x_4b_4 \end{bmatrix}_{q\times q},$$

so $q = a^\# b_1$. Multiplying this equation by $a$ from the left side we get $a = aa^\# b_1 = qb_1 = b_1$. Now it is easy to show that $ba = a^2$.

(i) $\Leftrightarrow$ (iii): If $a^2 = ba$ then $b(a^\#)^2 = ba(a^\#)^3 = a^2(a^\#)^3 = a^\#$ and $a^\# ba^\# = a^\# ba(a^\#)^2 = a^\# a^2(a^\#)^2 = a^\#$, so $b(a^\#)^2 = a^\# ba^\#$. Conversely, suppose that $b(a^\#)^2 = a^\# ba^\#$ and let $b = \left[\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right]_{q\times q}$. The condition, $b(a^\#)^2 = a^\# ba^\#$ reduces to $b_1(a^\#)^2 = a^\# b_1 a^\#$ and $b_3(a^\#)^2 = 0$. We have $b_3 = b_3 q = b_3 a^\# a = b_3(a^\#)^2 a^2 = 0$. Now the proof proceeds along the same lines as the proof of (ii) $\Rightarrow$ (i).

(i) $\Leftrightarrow$ (iv): If $a^2 = ba$ then

$$b^\# a^2 = b^\# a^2 aa^\# = b^\# baaa^\# = b^\# bbaa^\# = baa^\# = a^2 a^\# = a.$$

By (3.28), $ab^\# a = a$. Conversely, suppose that $b^\# a^2 = ab^\# a = a$. We deduce

$$ba = bb^\# a^2 = bb^\# aa^{(1)} aa = bb^\# ba^{(1)} aa = ba^{(1)} aa = aa^{(1)} aa = a^2.$$

(iv) $\Leftrightarrow$ (v): Since $a <^- b$ we have

$$a^\# b^\# a^\# = (a^\#)^2 ab^\# a(a^\#)^2 = (a^\#)^2 a(a^\#)^2 = (a^\#)^3. \tag{3.29}$$

Suppose that $b^\# a^2 = ab^\# a = a$. We get

$$b^\#(a^\#)^2 = b^\# a^2(a^\#)^4 = a(a^\#)^4 = (a^\#)^3,$$

so $a^\# b^\# a^\# = b^\#(a^\#)^2$. Conversely, suppose that $b^\#(a^\#)^2 = a^\# b^\# a^\#$. By (3.29), we have $b^\#(a^\#)^2 = (a^\#)^3$. Multiplying this equation by $a^4$ from the right we obtain $b^\# a^2 = a$. □

In the same manner we can prove the following lemma.

**Lemma 3.9.** [70] *Let $a, b \in R^\#$ and $a <^- b$. The following conditions are equivalent:*

(i) $ab = a^2$; (iii) $(a^\#)^2 b = a^\# ba^\#$; (v) $(a^\#)^2 b^\# = a^\# b^\# a^\#$.

(ii) $a^2 b = aba$; (iv) $a^2 b^\# = ab^\# a$;

**Theorem 3.28** (From [70]. See [51] and Theorem 5.2.10 in [54] for the complex matrix case)**.** *Let $a, b \in R^\dagger$ and $a <^- b$. The following conditions are equivalent:*

(i) $a <^* b$; (iv) $ab^\dagger$ *and* $b^\dagger a$ *are self-adjoint*;

(ii) $a^* b$ *and* $ba^*$ *are self-adjoint*; (v) $ba^\dagger b = a$;

(iii) $a^\dagger b$ *and* $ba^\dagger$ *are self-adjoint*; (vi) $b^\dagger ab^\dagger = a^\dagger$.

*Proof.* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) follows by Lemma 3.5, Lemma 3.6 and Theorem 3.17.

(i) $\Leftrightarrow$ (v): If $a <^* b$ then, by definition, $ba^\dagger b = aa^\dagger a = a$. Conversely, suppose that $a <^- b$ and $ba^\dagger b = a$. There exists $a^{(1)} \in a\{1\}$ such that $aa^{(1)} = ba^{(1)}$ and $a^{(1)} a = a^{(1)} b$. Post-multiplying $a = ba^\dagger b$ by $a^{(1)} aa^\dagger$ we get

$$aa^\dagger = ba^\dagger ba^{(1)} aa^\dagger = ba^\dagger aa^{(1)} aa^\dagger = ba^\dagger.$$

Similarly, we conclude that $a^\dagger a = a^\dagger b$.

(i) $\Leftrightarrow$ (vi): If $a <^* b$ then Theorem 3.17 gives $b^\dagger ab^\dagger = a^\dagger aa^\dagger = a^\dagger$.

Conversely, suppose that $a^\dagger = b^\dagger ab^\dagger$. Since $a <^- b$ we have $ab^\dagger a = a$. Post-multiplying $a^\dagger = b^\dagger ab^\dagger$ by $a$ we get $a^\dagger a = b^\dagger ab^\dagger a = b^\dagger a$. Similarly, pre-multiplying the same equation by $a$, we get $aa^\dagger = ab^\dagger ab^\dagger = ab^\dagger$. By Theorem 3.17, it follows that $a <^* b$. □

**Theorem 3.29** (From [70]. See [51] and Theorem 4.2.12 in [54] for the complex matrix case)**.** *Let $a, b \in R^\#$ and $a <^- b$. The following conditions are equivalent:*

(i) $a <^\# b$; (iii) $a^\# b = ba^\#$; (v) $a^\# b^\# = b^\# a^\#$; (vii) $ba^\# b = a$.
(ii) $ab = ba$; (iv) $ab^\# = b^\# a$; (vi) $b^\# ab^\# = a^\#$;

*Proof.* The equivalences of (i)–(v) follow by Lemma 3.7, Lemma 3.8, Lemma 3.9 and Theorem 3.18.

(i) ⇒ (vi): Suppose that $a <^\# b$ i.e. $a^2 = ab = ba$. Hence $a^3 = aa^2 = a(ab) = a^2 b = abb = ab^2$. Similarly, $a^3 = b^2 a$. It follows that

$$b^\# ab^\# = b^\# a^3 (a^\#)^5 a^3 b^\# = b^\# b^2 a (a^\#)^5 ab^2 b^\# = ba(a^\#)^5 ab = a^2 (a^\#)^5 a^2 = a^\#.$$

(vi) ⇒ (iv): Suppose that $a^\# = b^\# ab^\#$. Recall that $ab^\# a = a$ so $a^\# a = b^\# ab^\# a = b^\# a$ and $aa^\# = ab^\# ab^\# = ab^\#$. Thus, $ab^\# = b^\# a$.

(i) ⇒ (vii): Suppose that $a^2 = ab = ba$. Then

$$ba^\# b = ba(a^\#)^3 ab = a^2 (a^\#)^3 a^2 = a.$$

(vii) ⇒ (i): Suppose that $ba^\# b = a$. Since $a <^- b$ there exists $a^{(1)} \in a\{1\}$ such that $aa^{(1)} = ba^{(1)}$ and $a^{(1)} a = a^{(1)} b$. We have

$$aa^\# = aa^{(1)} aa^\# = ba^\# ba^{(1)} aa^\# = ba^\# aa^{(1)} aa^\# = ba^\# aa^\# = ba^\#$$
$$a^\# a = a^\# aa^{(1)} a = a^\# aa^{(1)} ba^\# b = a^\# aa^{(1)} aa^\# b = a^\# aa^\# b = a^\# b,$$

so $a <^\# b$. □

As a consequence of Lemma 3.5, Lemma 3.8 and Theorem 3.19, we obtain the next result.

**Theorem 3.30.** [70] *Let $a, b \in R^{\textcircled{\#}}$ and $a <^- b$. The following conditions are equivalent:*

(i) $a <^{\textcircled{\#}} b$.
(ii) *One of the conditions* (i), (ii) *from Lemma* 3.5 *holds and one of the conditions* (i)–(v) *from Lemma* 3.8 *holds.*

*If in addition $a, b \in R^\dagger$ then the following conditions are equivalent:*

(i) $a <^{\textcircled{\#}} b$.
(ii) *One of the conditions* (iii)–(v) *from Lemma* 3.5 *holds and one of the conditions* (i)–(v) *from Lemma* 3.8 *holds.*

**Theorem 3.31.** [44] and [70] *Let $a, b \in R^{\textcircled{\#}}$ and $a <^- b$. The following conditions are equivalent:*

(i) $a <^{\textcircled{\#}} b$; (ii) $ba^{\textcircled{\#}} b = a$; (iii) $b^{\textcircled{\#}} ab^{\textcircled{\#}} = a^{\textcircled{\#}}$.

*Proof.* Since $a <^{-} b$, there exists $a^{(1)} \in a\{1\}$ such that $aa^{(1)} = ba^{(1)}$ and $a^{(1)}a = a^{(1)}b$. Also, from $a <^{-} b$, by Lemma 3.1, it follows that $ab^{\textcircled{\#}}b = bb^{\textcircled{\#}}a = ab^{\textcircled{\#}}a = a$.

(i) $\Leftrightarrow$ (ii): If $a <^{\textcircled{\#}} b$ then, by definition, $ba^{\textcircled{\#}}b = aa^{\textcircled{\#}}a = a$. Conversely, suppose that $a = ba^{\textcircled{\#}}b$. Multiplying this equality by $a^{\textcircled{\#}}aa^{(1)}$ from the left we get

$$a^{\textcircled{\#}}aa^{(1)}a = a^{\textcircled{\#}}aa^{(1)}ba^{\textcircled{\#}}b = a^{\textcircled{\#}}aa^{(1)}aa^{\textcircled{\#}}b = a^{\textcircled{\#}}aa^{\textcircled{\#}}b = a^{\textcircled{\#}}b.$$

Thus $a^{\textcircled{\#}}a = a^{\textcircled{\#}}b$. Similarly, multiplying $a = ba^{\textcircled{\#}}b$ by $a^{(1)}aa^{\textcircled{\#}}$ from the right we obtain $aa^{\textcircled{\#}} = ba^{\textcircled{\#}}$. Thus, $a <^{\textcircled{\#}} b$.

(i) $\Leftrightarrow$ (iii): If $a <^{\textcircled{\#}} b$ then, by Theorem 3.19, $aa^{\textcircled{\#}} = ab^{\textcircled{\#}}$ and $a^{\textcircled{\#}}a = b^{\textcircled{\#}}a$. Thus, $b^{\textcircled{\#}}ab^{\textcircled{\#}} = a^{\textcircled{\#}}aa^{\textcircled{\#}} = a^{\textcircled{\#}}$. Conversely, suppose that $a^{\textcircled{\#}} = b^{\textcircled{\#}}ab^{\textcircled{\#}}$. Let $p = ab^{\textcircled{\#}}$ and $q = b^{\textcircled{\#}}a$. We have $a = pb = bq$ and

$$a^{\textcircled{\#}}a = b^{\textcircled{\#}}ab^{\textcircled{\#}}a = b^{\textcircled{\#}}a \quad \text{and} \quad aa^{\textcircled{\#}} = ab^{\textcircled{\#}}ab^{\textcircled{\#}} = ab^{\textcircled{\#}}.$$

It follows that $p \in \tilde{E}(R)$ and $q \in E(R)$ such that $qa = a^{\textcircled{\#}}a^2 = a$. By Theorem 3.21, $a <^{\textcircled{\#}} b$. □

Of course, there are analogous results for dual core inverse.

**Theorem 3.32.** [70] *Let $a, b \in R_{\textcircled{\#}}$ and $a <^{-} b$. The following conditions are equivalent:*

(i) $a <_{\textcircled{\#}} b$.
(ii) *One of the conditions* (i), (ii) *from Lemma* 3.6 *holds and one of the conditions* (i)–(v) *from Lemma* 3.9 *holds.*

*If in addition $a, b \in R^{\dagger}$ then the following conditions are equivalent:*

(i) $a <_{\textcircled{\#}} b$.
(ii) *One of the conditions* (iii)–(v) *from Lemma* 3.6 *holds and one of the conditions* (i)–(v) *from Lemma* 3.9 *holds.*

**Theorem 3.33.** [70] *Let $a, b \in R^{\textcircled{\#}}$ and $a <^{-} b$. The following conditions are equivalent:*

$$\text{(i)}\ \ a <_{\textcircled{\#}} b; \quad \text{(ii)}\ \ ba_{\textcircled{\#}}b = a; \quad \text{(iii)}\ \ b_{\textcircled{\#}}ab_{\textcircled{\#}} = a_{\textcircled{\#}}.$$

## 4. Unified theory

The unification of matrix partial orders based on generalized inverses were done by Mitra in [53] and [54]. The aim of this subsection is to extend Mitra's approach to the ring case. The definitions and notations used in [54] can also be used in the ring case, so we will follow them.

**Definition 4.1.** [54] Let $\mathcal{P}(R)$ denote the class of all subsets of $R$. A g-map is a map $\mathcal{G}\colon R \to \mathcal{P}(R)$ such that for each $a \in R$, $\mathcal{G}(a) \subseteq a\{1\}$. The set

$$\Omega_{\mathcal{G}} = \{a \in R : \mathcal{G}(a) \neq \emptyset\}$$

is called the support of the g-map $\mathcal{G}$. Also, $\mathcal{G}_r(a) = \mathcal{G}(a) \cap a\{1, 2\}$.

**Definition 4.2.** [54] Let $\mathcal{G}\colon R \to \mathcal{P}(R)$ be a g-map. For $a, b \in R$, we say

$$a <^{\mathcal{G}} b \quad \text{if} \quad a \in \Omega_{\mathcal{G}},\ ga = gb \text{ and } ag = bg \quad \text{for some} \quad g \in \mathcal{G}(a).$$

The relation $<^{\mathcal{G}}$ is called $\mathcal{G}$-based order relation.

An element $a \in R$ is $\mathcal{G}$-maximal if for any $b \in R$, $a = b$ whenever $a <^{\mathcal{G}} b$.

**Definition 4.3.** [54] Let $\mathcal{G}\colon R \to \mathcal{P}(R)$ be a g-map and $a, b \in R$. The class

$$\tilde{\mathcal{G}}(a) = \{g : ga = a^{(1)}a,\ ag = aa^{(1)} \text{ for some } a^{(1)} \in \mathcal{G}(a)\}$$

is called the completion of $\mathcal{G}(a)$. Let $\mathcal{G}(a, b) = \{hah : h \in \mathcal{G}(b)\}$. A pair $(a, b)$ is said to satisfy the (T)-condition if $\mathcal{G}(a, b) \subseteq \mathcal{G}(a)$. The set $\mathcal{G}(a)$ is said to be semi-complete if $\mathcal{G}(a, a) \subseteq \mathcal{G}(a)$. If for each $a \in R$, $\mathcal{G}(a)$ is semi-complete, we say the g-map $\mathcal{G}$ is semi-complete.

We see that $\tilde{\mathcal{G}}(a) \subseteq a\{1\}$ and $\mathcal{G}(a) \subseteq \tilde{\mathcal{G}}(a)$ for each $a \in R$. Also, $\mathcal{G}(a)$ is semi-complete if and only if $(a, a)$ satisfy the (T)-condition.

It is clear that if $\mathcal{G}(a) = a\{1\}$ then $<^{\mathcal{G}}$ coincides with $<^-$; if $\mathcal{G}(a) = \{a^\dagger\}$ then $<^{\mathcal{G}}$ coincides with $<^*$, and so on. Also, it is easy to see that each of the g-maps $\mathcal{G}(a) = a\{1\}$, $\mathcal{G}(a) = \{a^\dagger\}$, $\mathcal{G}(a) = \{a^\#\}$, $\mathcal{G}(a) = \{a^{\textcircled{\#}}\}$ and $\mathcal{G}(a) = \{a_{\textcircled{\#}}\}$ is semi-complete.

It is clear that $\mathcal{G}$ relation is reflexive on its support. Also, $a <^{\mathcal{G}} b$ implies $a <^- b$ so $<^{\mathcal{G}}$ is antisymmetric.

In what follows, we need the following definitions. First, we say that idempotents $e, f \in E(R)$ are equivalent, written $e \sim f$, if there exist elements $x \in eRf$ and $y \in fRe$ such that $xy = e$ and $yx = f$. It is easy to see that $e \sim f$ if and only if there exist $x, y \in R$ such that $xy = e$ and $yx = f$. Also, $\sim$ is an equivalence relation on $E(R)$. Note that $e \sim 0$ implies $e = 0$.

**Definition 4.4.** [69] A ring $R$ is FD (finite dimensional) ring if for arbitrary $e, f \in E(R)$ the following holds:

$$e \sim f \quad \Longrightarrow \quad 1 - e \sim 1 - f.$$

This definition is directly inspired by the fact that an arbitrary vector space $V$ is finite dimensional if and only if for every idempotent linear transformations $P, Q\colon V \to V$ the following holds $P \sim Q \Rightarrow I - P \sim I - Q$. See [69] for the proof.

Note that the notion of FD ring is closely related with the notion of Dedekind finite ring. Recall that $R$ is Dedekind finite ring if for every idempotent $e \in E(R)$ the following holds $e \sim 1 \Rightarrow e = 1$, see [29, 41]. It is easy to see that every FD ring is Dedekind finite ring.

In the next theorem we will see that the decompositions similar to that in Theorems 3.5, 3.22, 3.23, 3.24 and 3.25 is also valid in the unified theory.

**Theorem 4.1.** [69] *Let $R$ be a ring and $\mathcal{G}\colon R \to \mathcal{P}(R)$ be a g-map. Let $a, b \in \Omega_{\mathcal{G}}$. Then the following holds:*

(i) *Suppose that $a <^{\mathcal{G}} b$. Fix $h \in \mathcal{G}(b)$ and set*

$$e_1 = ah, \quad e_2 = (b - a)h, \quad e_3 = 1 - bh$$

$$f_1 = ha, \quad f_2 = h(b-a), \quad f_3 = 1 - hb.$$

*Then* $1 = e_1 + e_2 + e_3$ *and* $1 = f_1 + f_2 + f_3$ *are two decompositions of the identity of the ring* $R$ *with respect to which* $a$ *and* $b$ *have the following matrix forms:*

$$a = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times f}, \quad b = \begin{bmatrix} a & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times f}, \tag{4.1}$$

*where* $a \in e_1Rf_1$ *is* $(e_1, f_1)$*-invertible and* $b - a \in e_2Rf_2$ *is* $(e_2, f_2)$*-invertible.*

(ii) *Conversely, if*

$$b = \begin{bmatrix} x_1 & x_2 \\ x_3 & b-a \end{bmatrix}_{e\times f}, \tag{4.2}$$

*where* $e, f \in E(R)$ *are idempotents such that* $ge = g$ *and* $fg = g$ *for some* $g \in \mathcal{G}(a)$ *then* $a <^{\mathcal{G}} b$.

*Proof.* The proof of (i) can be done in the same way as the proof of Theorem 3.5. To prove (ii) let $b$ have representation (4.2) where $e, f \in E(R)$ and $ge = g$ and $fg = g$ for some $g \in \mathcal{G}(a)$. From (4.2), it follows $(1-e)b(1-f) = b - a$. Multiplying this equation by $ge = g$ from the left, we obtain $ga = gb$. Similarly, multiplying it by $fg = g$ from the right, we obtain $ag = bg$. Thus, $a <^{\mathcal{G}} b$. □

In the next theorem we give the characterization of all elements $b$ such that $a <^{\mathcal{G}} b$, where $\mathcal{G}$ is arbitrary semi-complete g-map.

**Theorem 4.2.** [69] *Let* $\mathcal{G}\colon R \to \mathcal{P}(R)$ *be a semi-complete g-map and* $a \in \Omega_{\mathcal{G}}$. *Then*

$$\{b \in R : a <^{\mathcal{G}} b\} = \{a + (1-ag)d(1-ga) : g \in \mathcal{G}(a),\ d \in R\}. \tag{4.3}$$

*Proof.* Suppose that $a <^{\mathcal{G}} b$. Then $ag = bg$ and $ga = gb$ for some $g \in \mathcal{G}(a)$. Therefore, $(b-a)g = 0$ and $g(b-a) = 0$. Let $S$ denote the set on the right hand side of (4.3). It is easy to check that

$$b = a + b - a = a + (1-ag)(b-a)(1-ga),$$

so $b \in S$. Suppose now that $b \in S$, i.e. $b = a + (1-ag)d(1-ga)$ for some $g \in \mathcal{G}(a)$ and $d \in R$. Let $g' = gag$. Since $\mathcal{G}$ is semi-complete, we have $g' \in \mathcal{G}(a)$. As $aga = a$, it is easily seen that $bg' = ag'$ and $g'b = g'a$. Thus $a <^{\mathcal{G}} b$. □

**Corollary 4.1.** *Let* $\mathcal{G}\colon R \to \mathcal{P}(R)$ *be a semi-complete g-map and* $a \in \Omega_{\mathcal{G}}$, $b \in R$. *Then* $a <^{\mathcal{G}} b$ *if and only if there exists* $g \in \mathcal{G}(a)$ *such that*

$$b = \begin{bmatrix} a & 0 \\ 0 & v \end{bmatrix}_{p\times q},$$

*where* $p = ag$, $q = ga$ *and* $v \in (1-p)R(1-q)$ *is arbitrary.*

If $R$ is FD ring then the only maximal elements are invertible elements.

**Theorem 4.3.** [69] *Let* $R$ *be an FD ring and let* $\mathcal{G}\colon R \to \mathcal{P}(R)$ *be a semi-complete g-map. The element* $a \in \Omega_{\mathcal{G}}$ *is* $\mathcal{G}$*-maximal under the relation* $<^{\mathcal{G}}$ *if and only if* $a$ *is invertible.*

*Proof.* Suppose that $a \in \Omega_G$ is not invertible. Let us prove that $a$ is not $\mathcal{G}$-maximal. Let $a^{(1)} \in \mathcal{G}(a)$ and $g = a^{(1)}aa^{(1)}$. Then $gag = g$ and $g \in \mathcal{G}(a)$, since $G$ is semi-complete. Set $e = ag$ and $f = ga$. We have $e \sim f$. As $R$ is FD ring, we have $1 - e \sim 1 - f$. It is easy to prove that if $e = 1$ or $f = 1$ then $e = f = 1$, and hence $a$ is invertible. Therefore, $e \neq 1$ and $f \neq 1$. There exist $x \in (1 - e)R(1 - f)$ and $y \in (1 - f)R(1 - e)$ such that $xy = 1 - e$ and $yx = 1 - f$. Let $b = a + x$. Since $1 - e \neq 0$, we have $x \neq 0$ so $b \neq a$. Note that

$$(1 - f)g = (1 - ga)g = 0 \quad \text{and} \quad g(1 - e) = g(1 - ag) = 0.$$

Therefore, $xg = 0$ and $gx = 0$. It follows that $bg = ag$ and $gb = ga$ so $a <^{\mathcal{G}} b$, which means that $a$ is not $\mathcal{G}$-maximal.

On the other hand, suppose that $a \in \Omega_{\mathcal{G}}$ is invertible and suppose that $a <^{\mathcal{G}} b$ for some $b \in R$. Since $a$ is invertible, the only $g$-inverse of $a$ is $a^{-1}$, so $\mathcal{G}(a) = \{a^{-1}\}$. Now, $a <^{\mathcal{G}} b$ implies $aa^{-1} = ba^{-1}$. Thus, $a = b$. We prove that $a$ is maximal. □

We will now give the a necessary and sufficient conditions for a $\mathcal{G}$-based relation to be a partial order on $R$. It turns out that the sufficient conditions are easier to establish.

**Theorem 4.4.** [54] and [69] *Let $R$ be a ring and let $\mathcal{G}\colon R \to \mathcal{P}(R)$ be a g-map. Given any $a, b \in R$ suppose that $a <^{\mathcal{G}} b$ and $b$ is not maximal imply the pair $(a, b)$ satisfies the $(T)$-condition. Then the binary relation $<^{\mathcal{G}}$ is a partial order on $\Omega_G$.*

*Proof.* We have already noted that $<^{\mathcal{G}}$ is always reflexive and antisymmetric on $\Omega_G$. Suppose that $a <^{\mathcal{G}} b$ and $b <^{\mathcal{G}} c$. If $b$ is maximal element then $b = c$ so $a <^{\mathcal{G}} c$. So, suppose that $b$ is not maximal. Since $b <^{\mathcal{G}} c$, there is $h \in \mathcal{G}(b)$ such that $bh = ch$ and $hb = hc$. By the assumption of the theorem, as $a <^{\mathcal{G}} b$, the pair $(a, b)$ satisfies (T)-condition. Thus $g := hah \in \mathcal{G}(a)$. Since $a <^{\mathcal{G}} b$ imply $a <^{-} b$, we obtain $a = bha = ahb = aha$. It follows $ag = ahah = ah = bhah = chah = cg$. Similarly $ga = gc$, thus $a <^{\mathcal{G}} c$ and $<^{\mathcal{G}}$ is transitive. □

**Theorem 4.5.**[69] *Let $R$ be a ring and let $\mathcal{G}\colon R \to \mathcal{P}(R)$ be a semi-complete g-map. For $a, b \in R$, the following conditions are equivalent:*

(i) *If $a <^{\mathcal{G}} b$ and $b$ is not maximal then the pair $(a, b)$ satisfies the $(T)$-condition.*
(ii) *If $a <^{\mathcal{G}} b$ and $b$ is not maximal then $\tilde{\mathcal{G}}(b) \subseteq \tilde{\mathcal{G}}(a)$.*
(iii) *If $a <^{\mathcal{G}} b$ and $b$ is not maximal then $\mathcal{G}(b) \subseteq \tilde{\mathcal{G}}(a)$.*

*If one of the conditions* (i)–(iii) *is satisfied for each $a, b \in R$ then the relation $<^{\mathcal{G}}$ is a partial order on $\Omega_G$.*

*Proof.* In view of Theorem 4.4 it is sufficient to show the equivalence of (i), (ii) and (iii). If $\mathcal{G}(b) = \emptyset$ then all the conditions are satisfied. Suppose that $\mathcal{G}(b) \neq \emptyset$.

(i) ⇒ (ii): Suppose that (i) is satisfied and let $a <^{\mathcal{G}} b$ and $b$ is not maximal. Let $h$ be an arbitrary element of the set $\tilde{\mathcal{G}}(b)$. This means that there exists $b^{(1)} \in \mathcal{G}(b)$ such that $hb = b^{(1)}b$ and $bh = bb^{(1)}$. Let $g = b^{(1)}ab^{(1)}$. Since $(a, b)$ satisfies the (T)-condition, we have $g \in \mathcal{G}(a)$. Since $a <^{\mathcal{G}} b$ implies $a <^{-} b$, we have $a = aua = aub = bua$ for each $u \in b\{1\}$. Therefore, $a = bha = ahb = ab^{(1)}a$. It follows that

$$ha = h(bha) = b^{(1)}bha = b^{(1)}a = b^{(1)}(ab^{(1)}a) = ga$$
$$ah = (ahb)h = ahbb^{(1)} = ab^{(1)} = (ab^{(1)}a)b^{(1)} = ag.$$

Therefore, $h \in \tilde{\mathcal{G}}(a)$, so $\tilde{\mathcal{G}}(b) \subseteq \tilde{\mathcal{G}}(a)$.

(ii) ⇒ (iii) is clear since $\mathcal{G}(b) \subseteq \tilde{\mathcal{G}}(b)$, for each $b \in R$.

(iii) ⇒ (i): Suppose that (iii) is satisfied and let $a <^{\mathcal{G}} b$ where $b$ is not maximal. For any $h \in \mathcal{G}(b) \subseteq \tilde{\mathcal{G}}(a)$ there exists $g \in \mathcal{G}(a)$ such that $ag = ah$ and $ga = ha$. Since $\mathcal{G}$ is semi-complete, we obtain $hah = gag \in \mathcal{G}(a)$, so the pair $(a, b)$ satisfies the (T)-condition. □

**Remark 4.1.** Note that if $b$ is maximal then all three conditions from Theorem 4.5 are satisfied.

**Corollary 4.2.** [69] *Let $R$ be a ring and let $\mathcal{G}\colon R \to \mathcal{P}(R)$ be a g-map such that $\mathcal{G}(x) = \{g_x\}$ where $g_x$ is a certain reflexive inverse of $x \in R$. For $a, b \in R$ the following conditions are equivalent:*

(i) *If $a <^{\mathcal{G}} b$ then $g_b a g_b = g_a$.*
(ii) *If $a <^{\mathcal{G}} b$ then $\tilde{\mathcal{G}}(b) \subseteq \tilde{\mathcal{G}}(a)$.*
(iii) *If $a <^{\mathcal{G}} b$ then $ag_a = ag_b$ and $g_a a = g_b a$.*

*If one of the conditions* (i)–(iii) *is satisfied for each $a, b \in R$ then the relation $<^{\mathcal{G}}$ is a partial order on $\Omega_{\mathcal{G}}$.*

*Proof.* Since for any $x \in \Omega_{\mathcal{G}}$, $g_x$ is a reflexive generalized inverse of $x$, it follows that $\mathcal{G}$ is semi-complete. Now, the proof follows by Theorem 4.5. □

Using Corollary 4.2 it is easy to show that the relations $<^{-}$, $<^{*}$, $<^{\#}$, $<^{\textcircled{\#}}$ and $<_{\textcircled{\#}}$ are partial orders.

In the next theorem we will see that under certain assumptions, the sufficient conditions given in Theorem 4.4 and Theorem 4.5, are also necessary for a $\mathcal{G}$-based relation to be a partial order on $R$.

**Theorem 4.6.** *Let $R$ be an FD ring and let $\mathcal{G}\colon R \to \mathcal{P}(R)$ be a semi-complete g-map. Then the following statements are equivalent:*

(i) *The relation $<^{\mathcal{G}}$ is a partial order on $\Omega_G$.*
(ii) *Let $a, b \in R$. If $a <^{\mathcal{G}} b$ then the pair $(a, b)$ satisfies the $(T)$-condition.*
(iii) *Let $a, b \in R$. If $a <^{\mathcal{G}} b$ then $\tilde{\mathcal{G}}(b) \subseteq \tilde{\mathcal{G}}(a)$.*
(iv) *Let $a, b \in R$. If $a <^{\mathcal{G}} b$ then $\mathcal{G}(b) \subseteq \tilde{\mathcal{G}}(a)$.*

*Proof.* (i) ⇒ (ii): Let $a, b \in R$ with $a <^{\mathcal{G}} b$. The case $\mathcal{G}(b) = \emptyset$ is trivial, so let $\mathcal{G}(b) \neq \emptyset$. We have to show that $b^{(1)}ab^{(1)} \in \mathcal{G}(a)$ for any $b^{(1)} \in \mathcal{G}(b)$. Let $h = b^{(1)}bb^{(1)}$. The map $\mathcal{G}$ is semi-complete, so $h \in \mathcal{G}(b) \subseteq b\{1\}$. Also, $hbh = h$ and

$$hah = b^{(1)}(bb^{(1)}a)b^{(1)}bb^{(1)} = b^{(1)}ab^{(1)}bb^{(1)} = b^{(1)}(ab^{(1)}b)b^{(1)} = b^{(1)}ab^{(1)}. \tag{4.4}$$

Let $e_i$ and $f_i$, $i = 1, 2, 3$ be as in Theorem 4.1. Then $a$ and $b$ have representations (4.1). Note that $bh \sim hb$. Since $R$ is FD ring, we have $e_3 = 1 - bh \sim 1 - hb = f_3$.

Therefore $e_3 = xy$ and $f_3 = yx$ for some $x \in e_3Rf_3$ and $y \in f_3Re_3$. Note that $hah = f_1h = he_1$ so $hah \in f_1Re_1$. Similarly $h(b-a)h \in f_2Re_2$. It follows that

$$h = hbh = hah + h(b-a)h = \begin{bmatrix} hah & 0 & 0 \\ 0 & h(b-a)h & 0 \\ 0 & 0 & 0 \end{bmatrix}_{f\times e}. \tag{4.5}$$

Let $c = b + x$ and $z = h + y$. We have

$$c = \begin{bmatrix} a & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & x \end{bmatrix}_{e\times f} \quad \text{and} \quad z = \begin{bmatrix} hah & 0 & 0 \\ 0 & h(b-a)h & 0 \\ 0 & 0 & y \end{bmatrix}_{f\times e}. \tag{4.6}$$

As we know $<^{\mathcal{G}}$ implies $<^{-}$, so $bha = ahb = aha = a$. Thus, we obtain

$$cz = \begin{bmatrix} ahah & 0 & 0 \\ 0 & (b-a)h(b-a)h & 0 \\ 0 & 0 & xy \end{bmatrix}_{e\times e} = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix}_{e\times e} = 1.$$

Similarly $zc = 1$, so $c$ is invertible and $z = c^{-1}$. Let us prove that $b <^{\mathcal{G}} c$. From (4.1), (4.5) and (4.6) we obtain

$$bh = ch = \begin{bmatrix} ahah & 0 & 0 \\ 0 & (b-a)h(b-a)h & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}.$$

In the same manner $hb = hc = f_1 + f_2$. Thus $b <^{\mathcal{G}} c$, by definition. Since $<^{\mathcal{G}}$ is transitive, we have $a <^{\mathcal{G}} c$. Therefore $ag = cg$ and $ga = gc$ for some $g \in \mathcal{G}(a)$. Thus, $a = aga = cgc$. It follows

$$\begin{aligned} g &= c^{-1}ac^{-1} = zaz \\ &= \begin{bmatrix} hah & 0 & 0 \\ 0 & h(b-a)h & 0 \\ 0 & 0 & y \end{bmatrix}_{f\times e} \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times f} \begin{bmatrix} hah & 0 & 0 \\ 0 & h(b-a)h & 0 \\ 0 & 0 & y \end{bmatrix}_{f\times e} \\ &= \begin{bmatrix} hahahah & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{f\times e} = hah. \end{aligned}$$

From (4.4) we obtain $b^{(1)}ab^{(1)} = g \in \mathcal{G}(a)$.

The proof of (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) follows by Theorem 4.5. □

**Corollary 4.3.** [69] *Let $R$ be an FD ring and let $\mathcal{G}\colon R \to \mathcal{P}(R)$ be a g-map such that $\mathcal{G}(x) = \{g_x\}$ where $g_x$ is a certain reflexive generalized inverse of $x \in R$. Then the following conditions are equivalent:*

(i) *The order relation $<^{\mathcal{G}}$ is a partial order on $\Omega_{\mathcal{G}}$.*
(ii) *Let $a, b \in R$. If $a <^{\mathcal{G}} b$ then $g_bag_b = g_a$.*
(iii) *Let $a, b \in R$. If $a <^{\mathcal{G}} b$ then $ag_a = ag_b$ and $g_aa = g_ba$.*
(iv) *Let $a, b \in R$. If $a <^{\mathcal{G}} b$ then $\tilde{\mathcal{G}}(b) \subseteq \tilde{\mathcal{G}}(a)$.*

*Proof.* Since for any $x \in \Omega_{\mathcal{G}}$, $g_x$ is a reflexive generalized inverse of $x$, it follows that $\mathcal{G}$ is semi-complete. Now, the proof follows by Theorem 4.6. $\square$

We have already seen in Theorem 4.6 that under certain conditions $a <^{\mathcal{G}} b$ implies $\mathcal{G}(b) \subseteq \tilde{\mathcal{G}}(a)$. It is natural to ask whether the converse is true.

**Theorem 4.7.**[69] *Let $R$ be a ring and let $\mathcal{G}\colon R \to \mathcal{P}(R)$ be a semi-complete g-map. For $a, b \in \Omega_{\mathcal{G}}$, if $\mathcal{G}(b) \subseteq \tilde{\mathcal{G}}(a)$ and $a <^s b$ then $a <^{\mathcal{G}} b$.*

*Proof.* Suppose that $\mathcal{G}(b) \subseteq \tilde{\mathcal{G}}(a)$ and $a <^s b$ and fix $h \in \mathcal{G}(b)$. Then $h \in \tilde{\mathcal{G}}(a)$ so there exists an $g \in \mathcal{G}(a)$ such that $ha = ga$ and $ah = ag$. As $a <^s b$, we have $a = xb = by$ for some $x, y \in R$. We conclude that $bha = bhby = by = a$. Similarly, $a = ahb$. Since $\mathcal{G}$ is semi-complete, we have $f := gag \in \mathcal{G}(a)$. It is easy to see that $faf = f$ and $fa = gaga = ga = ha$. It follows that $af = bhaf = bfaf = bf$. Similarly, $af = ah$ and $fa = fb$. By definition, $a <^{\mathcal{G}} b$. $\square$

We proved in Theorem 4.6 that under certain conditions $a <^{\mathcal{G}} b$ implies that $(a, b)$ satisfies (T)-condition i.e. for each $h \in G(b)$ there exists an $g \in \mathcal{G}(a)$ such that $hah = g$. Also, $a <^{\mathcal{G}} b$ implies $a <^- b$. We will see that the converse is true.

**Theorem 4.8.** [69] *Let $a, b \in R$ and let $\mathcal{G}\colon R \to \mathcal{P}(R)$ be a g-map. Suppose that $a <^- b$ and suppose that there exist $h \in \mathcal{G}(b)$ and $g \in \mathcal{G}_r(a)$ such that $hah <^s g$. Then $a <^{\mathcal{G}} b$.*

*Proof.* Suppose that $a <^- b$ and $hah <^s g$ for some $h \in \mathcal{G}(b)$ and $g \in \mathcal{G}_r(a)$. Thus, $hah = xg = gy$ for some $x, y \in R$. As $a <^- b$ we have $a = aha = ahb = bha$. We obtain

$$\begin{aligned} g &= gag = g(aha)g = gahahag = ga(hah)ag \\ &= ga(xg)ag = gax(gag) = gaxg = ga(xg) \\ &= ga(gy) = (gag)y = gy, \end{aligned}$$

so $hah = g$. Now

$$\begin{aligned} gb &= hahb = h(ahb) = ha = haha = (hah)a = ga \\ bg &= bhah = (bha)h = ah = ahah = a(hah) = ag. \end{aligned}$$

It follows that $a <^{\mathcal{G}} b$. $\square$

**Theorem 4.9.** [69] *Let $a, b \in R$ and let $\mathcal{G}\colon R \to \mathcal{P}(R)$ be a g-map. Then the following hold:*

(i) *If $\mathcal{G}$ is semi-complete and $a <^{\mathcal{G}} b$ then $a <^- b$ and $a = bgb$ for some $g \in \mathcal{G}_r(a)$.*
(ii) *If $b \in \Omega_{\mathcal{G}}$, $a <^- b$ and if there exists $g \in \mathcal{G}_r(a)$ such that $bgb <^s a$ then $a <^{\mathcal{G}} b$.*

*Proof.* (i): Suppose that $\mathcal{G}$ is semi-complete and $a <^{\mathcal{G}} b$. Then $a <^- b$ and there exists $a^{(1)} \in \mathcal{G}(a)$ such that $aa^{(1)} = ba^{(1)}$ and $a^{(1)}a = a^{(1)}b$. Set $g = a^{(1)}aa^{(1)}$. Since $\mathcal{G}$ is semi-complete, we obtain $g \in \mathcal{G}(a)$. It is easy to see that $bgb = a$.

(ii): Suppose that $a <^- b$ and suppose that there exists $g \in \mathcal{G}_r(a)$ such that $bgb <^s a$. Therefore, there exist $x, y \in R$ such that $bgb = xa = ay$. Fix $h \in \mathcal{G}(b)$.

From $a <^- b$, we obtain $a = aha = bha = ahb$. We have

$$\begin{aligned} a &= aga = (ahb)g(bha) = ah(bgb)ha = ah(xa)ha \\ &= ahx(aha) = ahxa = ah(xa) = ah(ay) = (aha)y = ay, \end{aligned}$$

so $bgb = a$. It follows that

$$\begin{aligned} bg &= bgag = bg(bha)g = (bgb)hag = ahag = ag \\ gb &= gagb = g(ahb)gb = gah(bgb) = gaha = ga, \end{aligned}$$

so $a <^{\mathcal{G}} b$. □

As we noted before, $A < B$ implies $B - A < B$, where $< \in \{<^-, <^\dagger, <^\#\}$. But, $B - A \not< B$ when $< \in \{<^{\textcircled{\#}}, <_{\textcircled{\#}}\}$.

**Theorem 4.10.** [53,69] *Let $R$ be an FD ring and let $\mathcal{G}\colon R \to \mathcal{P}(R)$ be a g-map such that $\mathcal{G}(x) = \{g_x\}$ where $g_x$ is a certain reflexive generalized inverse of $x$. Suppose that $<^{\mathcal{G}}$ is a partial order on $\Omega_{\mathcal{G}}$ and suppose that $a <^{\mathcal{G}} b$. Then $b - a <^{\mathcal{G}} b$ if and only if $g_{b-a} = g_b - g_a$.*

*Proof.* Suppose that $<^{\mathcal{G}}$ is a partial order on $\Omega_{\mathcal{G}}$. Suppose that $a <^{\mathcal{G}} b$ and $b - a <^{\mathcal{G}} b$. From Corollary 4.3 it follows that $g_b a g_b = g_a$ and $g_b(b-a)g_b = g_{b-a}$. Thus, $g_{b-a} = g_b b g_b - g_b a g_b = g_b - g_a$. On the other hand suppose that $a <^{\mathcal{G}} b$ and $g_{b-a} = g_b - g_a$. From Corollary 4.3, it follows that $ag_a = ag_b$ and $g_a a = g_b a$. Thus, we obtain

$$(b-a)g_{b-a} = (b-a)(g_b - g_a) = b(g_b - g_a) - ag_b + ag_a = bg_{b-a}.$$

In the same manner we obtain $g_{b-a}(b-a) = g_{b-a}b$ so $b - a <^{\mathcal{G}} b$. □

By Corollary 4.3, it follows that $a < b$ implies $\tilde{\mathcal{G}}(b) \subseteq \tilde{\mathcal{G}}(a)$, where $< \in \{<^-, <^\#, <^\dagger, <^{\textcircled{\#}}, <_{\textcircled{\#}}\}$ and $\mathcal{G}$ is appropriate $g$-map. We conclude this section by an explicit characterization of the set $\tilde{\mathcal{G}}(a)$.

**Theorem 4.11.** [53,69] *Let $\mathcal{G}\colon R \to \mathcal{P}(R)$ be a g-map such that $\mathcal{G}(x) = \{g_x\}$ where $g_x$ is a certain reflexive g-inverse of $x$. For $a \in \Omega_{\mathcal{G}}$ we have*

$$\tilde{\mathcal{G}}(a) = \left\{ \begin{bmatrix} g_a & 0 \\ 0 & g_4 \end{bmatrix}_{q\times p} : g_4 \in (1-q)R(1-p) \right\},$$

*where $p = ag_a$ and $q = g_a a$.*

*Proof.* Suppose that $g$ belongs to the set of the right hand side. Since $a = paq$ and $g_a = qg_a p$, we obtain

$$a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{p\times q} \quad \text{and} \quad g_a = \begin{bmatrix} g_a & 0 \\ 0 & 0 \end{bmatrix}_{q\times p}.$$

Now it is easy to see that $ga = g_a a$ and $ag = ag_a$, i.e. $g \in \tilde{\mathcal{G}}(a)$. Conversely, suppose that $g \in \tilde{\mathcal{G}}(a)$ and let $g = \left[\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix}\right]_{q\times p}$. From $ag = ag_a$ it follows that

$$\begin{bmatrix} ag_1 & ag_2 \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} ag_a & 0 \\ 0 & 0 \end{bmatrix}_{p\times p}.$$

Therefore, $ag_1 = ag_a$ and $ag_2 = 0$. Multiplying these equations by $g_a$ from the left side we obtain $qg_1 = qg_a$ and $qg_2 = 0$, respectively. Since $g_a \in qRp$, $g_1 \in qRp$ and $g_2 \in qR(1-p)$ we conclude that $g_1 = g_a$ and $g_2 = 0$. Similarly, from $ga = g_a a$ we obtain $g_3 = 0$. It follows that $g$ has desired form. □

Using Lemma 3.4, one can check the following characterizations of $\tilde{\mathcal{G}}(a)$ for various g-maps (see [53, 69]).

If $\mathcal{G}(a) = \{a^{\#}\}$ then $\tilde{\mathcal{G}}(a) = a\{1,5\}$.

If $\mathcal{G}(a) = \{a^{\dagger}\}$ then $\tilde{\mathcal{G}}(a) = a\{1,3,4\}$.

If $\mathcal{G}(a) = \{a^{\textcircled{\#}}\}$ then $\tilde{\mathcal{G}}(a) = a\{3,6\}$.

If $\mathcal{G}(a) = \{a_{\textcircled{\#}}\}$ then $\tilde{\mathcal{G}}(a) = a\{4,8\}$.

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