Nebojša Č. Dinčić and Dragan S. Djordjević

SURVEY ON REVERSE ORDER LAWS FOR THE MOORE–PENROSE INVERSE OF HILBERT SPACE OPERATORS

Abstract. This survey article contains results from seven original research papers published by the authors in past ten years on the topic of the reverse order laws for the Moore–Penrose inverse of the Hilbert space operators. Some proofs are improved, and there are some new results in the seventh section.

Mathematics Subject Classification (2020): Primary: 47-02; Secondary: 15A09; 47A56; 47A60; 47B15.

Keywords: Moore–Penrose inverse, weighted Moore–Penrose inverse, reverse order law, ternary power, Borel functional calculus.

University of Niš, Faculty of Sciences and Mathematics, Serbia
`nebojsa.dincic@pmf.edu.rs & ndincic@hotmail.com`
`dragandjordjevic70@gmail.com & dragan@pmf.ni.ac.rs`

Contents

1. Introduction

The theory of generalized inverses is a part of functional analysis and linear algebra which developed during the XX century. Its very beginnings were about practical problems on integral operators (Fredholm, 1903.) and differential equations (Hilbert, 1904. on generalized Green's function), and later attention was turned to complex matrix case (Moore, 1920; Penrose, 1955.), Banach and Hilbert space operators, or even rings with involution, Banach- and $C^{\star}$-algebras. For more on the history of generalized inverses please see [3] and the references therein.

1.1. About the Moore–Penrose generalized inverse. Let X and Y be Banach spaces, and let $\mathcal{L}(X,Y)$ stand for the set of all bounded linear operators from X to Y; we abbreviate $\mathcal{L}(X,X)$ as $\mathcal{L}(X)$. For $A \in \mathcal{L}(X,Y)$ we use $\mathcal{R}(A)$ and $\mathcal{N}(A)$ to denote the range and the null-space of A.

An operator $B \in \mathcal{L}(Y,X)$ is an inner inverse of $A \in \mathcal{L}(X,Y)$, if $ABA = A$ holds. In this case A is inner invertible, or relatively regular. It is well-known that A is inner invertible if and only if $\mathcal{R}(A)$ is closed in Y. If there is some $C \in \mathcal{L}(Y,X)$,

$C \neq 0$, such that $CAC = C$, then C is an outer generalized inverse of A, and A is outer regular. An operator which is both inner and outer generalized inverse of A is a reflexive generalized inverse of A.

Let $\mathcal{H}$ and $\mathcal{K}$ denote arbitrary Hilbert spaces. By A^* we denote the Hilbert-adjoint operator of given $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. Recall that operator $A \in \mathcal{L}(\mathcal{H})$ is Hermitian (or selfadjoint) if $A = A^*$, and normal if $AA^* = A^*A$. Hermitian operator $A \in \mathcal{L}(\mathcal{H})$ is positive if $\langle Ax, x\rangle > 0$ for all $x \in \mathcal{H}\backslash\{0\}$.

The fact that a Hilbert space $\mathcal{H}$ can be decomposed as a direct sum of two closed subspaces M and N will be denoted by $\mathcal{H} = M \oplus N$; if this direct sum is orthogonal, we write $\mathcal{H} = M \oplus^{\perp} N$, and then M is the orthogonal complement of N (with respect to $\mathcal{H}$), $M = N^{\perp}$. By P_M we denote the orthogonal projection corresponding to closed subspace M.

The Moore–Penrose inverse can be defined in several equivalent ways (see e.g. [57, p. 321], or [3, p. 336]).

Definition 1.1 (Moore)**.** If $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is a closed range operator, then $A^\dagger \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ is the unique operator satisfying

$$(M1)\ AA^\dagger = P_{\mathcal{R}(A)},\ (M2)\ A^\dagger A = P_{\mathcal{R}(A^*)}.$$

Definition 1.2 (Penrose)**.** The Moore–Penrose inverse of given $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is the operator $A^\dagger \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ (unique when it exists) satisfying the following so-called Penrose equations

$$(1)\ AA^\dagger A = A, \quad (2)\ A^\dagger AA^\dagger = A^\dagger, \quad (3)\ (AA^\dagger)^* = AA^\dagger, \quad (4)\ (A^\dagger A)^* = A^\dagger A.$$

Definition 1.3 (Desoer-Whalen)**.** If $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is a closed range operator, then $A^\dagger \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ is the unique operator satisfying

$$(DW1)\ AA^\dagger x = x,\ x \in \mathcal{N}(A)^\perp,\ (DW2)\ A^\dagger y = 0,\ y \in \mathcal{R}(A)^\perp.$$

Since closed subspaces of a Hilbert space are always complemented, the Moore–Penrose inverse of A exists if and only if $\mathcal{R}(A)$ is closed. Note that for $\mathcal{R}(A)$ not closed, $A^\dagger$ is an unbounded, but closed linear operator; we shall not consider this situation here. For the more general case, when A is a closed densely defined Hilbert space operator, see e.g. [32]. When the closed-range operator A is invertible, then its Moore–Penrose inverse $A^\dagger$ coincides with its ordinary inverse A^{-1}. In [12] one can find what happens when the first two equations are slightly altered.

Note that if for given operator $A \in \mathcal{L}(X, Y)$ there is some $B \in \mathcal{L}(Y, X)$ such that B satisfies Penrose equations $\theta \subset \{1, 2, 3, 4\}$, then we say that B is a θ-inverse of A, and use the notation $B = A^\theta$ for an element and $A\{\theta\}$ for such set of all θ-inverse of A. For $\theta = \{1\}$ we have an inner inverse $A^{(1)}$ from the set $A\{1\}$, for $\theta = \{2\}$ we have an outer inverse $A^{(2)}$ from the set $A\{2\}$, while we use $A^{(1,2)} \in A\{1, 2\}$ for reflexive generalized inverse case. It is clear that $A^{(1,2,3,4)}$ coincides with $A^\dagger$.

The closed-range operator A is EP (or equal-projection, or range-Hermitian) if $AA^\dagger = A^\dagger A$, or, equivalently, $\mathcal{R}(A) = \mathcal{R}(A^*)$. If $M \in \mathcal{L}(\mathcal{K})$ and $N \in \mathcal{L}(\mathcal{H})$ are positive (and invertible) operators, and $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is a closed-range operator, there exists the unique operator $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ such that

$$(1)\ ABA = A,\ \ (2)\ BAB = B,\ \ (3M)\ (MAB)^* = MAB,\ \ (4N)\ (NBA)^* = NBA.$$

Such B is denoted by $A^{\dagger}_{M,N}$ and is known as the weighted Moore–Penrose inverse of A with respect to the weights M and N. There is one useful result connecting the ordinary and the weighted Moore–Penrose inverse [29, Theorem 5]

$$A^{\dagger}_{M,N} = N^{-1/2}(M^{1/2}AN^{-1/2})^{\dagger}M^{1/2}.$$

Properties of the Moore–Penrose inverse. Throughout the survey $\mathcal{H}, \mathcal{K}, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$ denote arbitrary Hilbert spaces. In the next proposition, a lot of well-known and important facts and properties concerning the Moore–Penrose inverse are collected, especially those we are using in the proofs.

Proposition 1.1. *Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ be closed-range operator and let $M \in \mathcal{L}(\mathcal{K})$ and $N \in \mathcal{L}(\mathcal{H})$ be positive definite invertible operators. Then*

(1) $(A^{\dagger})^{\dagger} = A$, $(A^*)^{\dagger} = (A^{\dagger})^*$, $(\lambda A)^{\dagger} = \lambda^{-1}A^{\dagger}$ *for* $\lambda \neq 0$*;*
(2) $A^* = A^{\dagger}AA^* = A^*AA^{\dagger}$, $A = AA^*(A^*)^{\dagger} = (A^*)^{\dagger}A^*A$*;*
(3) $A^{\dagger} = A^*(AA^*)^{\dagger} = (A^*A)^{\dagger}A^*$, $(AA^*)^{\dagger} = (A^*)^{\dagger}A^{\dagger}$, $(A^*A)^{\dagger} = A^{\dagger}(A^*)^{\dagger}$*;*
(4) $\mathcal{R}(A) = \mathcal{R}(AA^{\dagger}) = \mathcal{R}(AA^*)$*;*
(5) $\mathcal{R}(A^{\dagger}) = \mathcal{R}(A^*) = \mathcal{R}(A^{\dagger}A) = \mathcal{R}(A^*A)$*;*
(6) $\mathcal{R}(I - A^{\dagger}A) = \mathcal{N}(A^{\dagger}A) = \mathcal{N}(A) = \mathcal{R}(A^*)^{\perp}$*;*
(7) $\mathcal{R}(I - AA^{\dagger}) = \mathcal{N}(AA^{\dagger}) = \mathcal{N}(A^{\dagger}) = \mathcal{N}(A^*) = \mathcal{R}(A)^{\perp}$.

Hermitian operators have some additional properties.

Proposition 1.2. *Let $H \in \mathcal{L}(\mathcal{H})$ be a Hermitian operator. Then we have*

i) $H = H^2H^{\dagger} = H^{\dagger}H^2$*;*
ii) $H^{\dagger} = H(H^2)^{\dagger} = (H^2)^{\dagger}H$*;*
iii) $(H^n)^{\dagger} = (H^{\dagger})^n$*, for any* $n \in \mathbb{N}_0$*;*
iv) *if* $H^2 = H$ *then* $H^{\dagger} = H$.

The important role the Moore–Penrose inverse plays in solving linear equations can be described by the next proposition [22, Theorem 1.3.2.].

Proposition 1.3. *Let $A \in \mathcal{L}(X, Y)$ have a closed range and let $b \in Y$. Then $x_0 = A^{\dagger}b$ is the best approximate solution of the linear equation $Ax = b$. Moreover, if M is the set of all best approximate solutions of the equation $Ax = b$, then* $x_0 = \min\{||x|| : x \in M\}$.

More on the theory of the generalized inverses an interested reader can find, for example, in the following books: [3, 6, 7, 22, 25, 35, 57]. For an overview of the applications of the Moore–Penrose inverse in physics, the reader is referred to [2].

1.2. About the reverse order laws. Let a, b be invertible elements of a semigroup with a unit. The rule $(ab)^{-1} = b^{-1}a^{-1}$ is called the reverse order rule (or reverse order law, ROL for short) for the ordinary inverse. We will consider the rule $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ and closely related results for Hilbert space operators A, B. In general, even in the complex matrix case, $A^{\dagger}A \neq I$ or $BB^{\dagger} \neq I$, so it is important to find necessary and sufficient conditions for this ROL (often called "basic" ROL) to hold. Theoretically, $(AB)^{\dagger}$ can be written in one of the forms

- $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$,

- $(AB)^\dagger = B^\dagger A^\dagger + X$,
- $(AB)^\dagger = B^\dagger Y A^\dagger$,

for some operator expressions X and Y depending on A or B. For example, we may consider $X = B^\dagger[(I - BB^\dagger)(I - A^\dagger A)]^\dagger A^\dagger$ or $Y = (A^\dagger ABB^\dagger)^\dagger$.

The reverse order law for the generalized inverse of products is an interesting class of fundamental problems in the theory of generalized inverses. Together with reviving the interests for generalized inverses during the 1950s, it started with considering the conditions when the ROL for a product of two singular complex matrices A and B

$$(AB)^\dagger = B^\dagger A^\dagger \tag{1.1}$$

holds. Greville [24] was the first who gave 1966. necessary and sufficient conditions for the reverse order law (1.1) to hold. He proved the equivalence

$$(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow A^\dagger ABB^* A^* = BB^* A^* \wedge BB^\dagger A^* AB = A^* AB,$$

which is further equivalent to

$$\mathcal{R}(A^* AB) \subset \mathcal{R}(B) \wedge \mathcal{R}(BB^* A^*) \subset \mathcal{R}(A^*).$$

We also should mention the result from 1963. by Arghiriade [1]

$$(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow A^* ABB^* \text{ is EP matrix.}$$

The reverse order law for triple matrix product, which has the form

$$(ABC)^\dagger = C^\dagger B^\dagger A^\dagger,$$

(A, B, C are matrices of compatible dimensions) was considered in 1986. by Hartwig [26], and by Tian [49] in 1992. Note that, even if P and Q are two invertible linear operators, in general $(PAQ)^\dagger \neq Q^{-1} A^\dagger P^{-1}$. By checking all four Penrose equations we see that

$$(PAQ)^\dagger = Q^{-1} A^\dagger P^{-1} \Leftrightarrow [AA^\dagger, P^* P] = 0 \wedge [A^\dagger A, QQ^*] = 0,$$

where brackets denote the commutator: $[S, T] = ST - TS$. In [50] Tian considered n matrix product case: $(A_1 A_2 \dots A_n)^\dagger = A_n^\dagger \dots A_2^\dagger A_1^\dagger$.

The reverse order law for weighted Moore–Penrose inverse of the matrix product of the form $(AB)^\dagger_{M,L} = B^\dagger_{N,L} A^\dagger_{M,N}$, was considered in 1998. by Sun and Wei [44], while the triple product was investigated by Wang [56].

One can consider, so-called, weaker ROLs, for example

$$(AB)^\dagger = B^\dagger (A^\dagger ABB^\dagger)^\dagger A^\dagger, \text{ or } (AB)^\dagger = B^\dagger A^\dagger - B^\dagger[(I - BB^\dagger)(I - A^\dagger A)]^\dagger A^\dagger.$$

Further, some ROLs are equivalent, although they look quite different

$$(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow (ABB^\dagger)^\dagger = BB^\dagger A^\dagger \wedge (A^\dagger AB)^\dagger = B^\dagger A^\dagger A;$$

$$(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow (AB)^\dagger = B^\dagger (A^\dagger ABB^\dagger)^\dagger A^\dagger \wedge (A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A.$$

Some ROLs are identities, for example,

$$(AB)^\dagger = (A^\dagger AB)^\dagger (ABB^\dagger)^\dagger,$$

$$(AB)^\dagger = (A^\dagger AB)^\dagger (AB(A^\dagger AB)^\dagger)^\dagger.$$

Let us explain how some mixed-type ROL can be constructed. We start from some identity, for example, $T = TT^\dagger T$. Now we have

$$AB = AA^\dagger ABB^\dagger B = A(A^\dagger ABB^\dagger)B.$$

If we formally apply triple ROL (of the form $(PQR)^\dagger = R^\dagger Q^\dagger P^\dagger$), we have

$$(AB)^\dagger = B^\dagger (A^\dagger ABB^\dagger)^\dagger A^\dagger. \tag{1.2}$$

Because of the pure formal application of the triple ROL, the last expression need not be an identity, so it is important to find the necessary and sufficient conditions under which it holds.

It appears that Galperin and Waksman [23] were the first who studied the ROL (1.2) in 1980, followed by Izumino [28] in 1982 for Hilbert space operators. The paper [15] deals with Hilbert space settings, while the paper [36] further generalizes those results to the rings with involution.

The reverse order law for other generalized inverses was also investigated, especially for Drazin inverse[1]. Again, it was Greville [24] who was the first to show that $(AB)^D = B^D A^D$, under the condition that A and B commute. The sufficient and necessary conditions for the Drazin inverse of the product od 2 and n matrices were given, subsequently, by H. Tian [45] in 1999, and Wang [56]. Djordjević [19] studied the reverse order law for the outer generalized inverse with prescribed range and null-space in the form $(AB)^{(2)}_{K,L} = B^{(2)}_{T,S} A^{(2)}_{M,N}$. The reverse order law for outer generalized inverse of the product of n matrices with prescribed range and null-space, as far as we know, is not investigated yet.

More general reverse order laws were considered, for so-called θ-inverses ($\theta \subset \{1, 2, 3, 4\}$); one should mention the results due to Wei [58, 59], De Pierro and Wei [11], Wei and Guo [60] and Werner [61, 62].

Along with the reverse order laws, the forward order law was also investigated in the form $(AB)^\dagger = A^\dagger B^\dagger$. The forward order law appears to be more unnatural, because it does not hold even for ordinary inverse, except in some special cases, cf. [8] and the references therein.

1.3. Proving methods. While proving in the complex matrix case highly depends on the various rank identities (for example, in the papers of Y. Tian) or singular-value decomposition (for example, in the papers of Baksalarys and Trenkler), in the Hilbert space settings those methods are inapplicable. Therefore, extending the results from complex matrix case to infinite dimensional Hilbert spaces settings is far from trivial (see, for example, Theorem 5.1).

Our method, based on the collection of lemmas, provides us with simpler expressions that can easily be dealt with, unlike the original expressions. The main idea is the matrix form of the operator according to appropriate orthogonal decompositions of the Hilbert spaces. The decompositions are chosen such that the operator

---

[1]The index of a matrix $A \in \mathbb{C}^{n\times n}$ is the smallest nonnegative integer such that $r(A^k) = r(A^{k+1})$. For every matrix $A \in \mathbb{C}^{n\times n}$ of index k there is a unique matrix A^D such that (1^k) $A^k A^D A = A^k$, (2) $A^D A A^D = A^D$, (5) $AA^D = A^D A$, and it is known as Drazin inverse. In the particular case $k = 1$ it reduces to the group inverse, usually denoted by $A^\#$.

matrix has as many zeros as possible, or has some other advantages, for example, the invertibility of some of its entries.

The next two lemmas are well-known, but we prove them on the spot.

Lemma 1.1. *Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. Then A has a closed range if and only if AA^* (resp. A^*A) has a closed range. In such case, $A^\dagger = A^*(AA^*)^\dagger = (A^*A)^\dagger A^*$.*

Proof. Recall well-known result $\mathcal{R}(A)$ is closed iff $\mathcal{R}(A^*)$ is closed. Note that $\mathcal{R}(AA^*) = A(\mathcal{R}(A^*)) \subseteq \mathcal{R}(A)$. Let $y \in \mathcal{R}(A)$, then there is some $x \in \mathcal{H}_1$ such that $y = Ax$. The space $\mathcal{H}_1$ can be decomposed in the orthogonal direct sum $\mathcal{H}_1 = \overline{\mathcal{R}(A^*)} \oplus \mathcal{N}(A)$, so for such x there are unique $x_1 \in \overline{\mathcal{R}(A^*)}$ and $x_2 \in \mathcal{N}(A)$ such that $x = x_1 + x_2$. Hence, we have

$$y = Ax = A(x_1 + x_2) = Ax_1 \in A(\mathcal{R}(A^*)) = \mathcal{R}(AA^*),$$

so $\mathcal{R}(A) \subseteq \mathcal{R}(AA^*)$. Therefore, $\mathcal{R}(A) = \mathcal{R}(AA^*)$, which means that $A^\dagger$ exists iff $(AA^*)^\dagger$ exists.

We have $A^\dagger = A^*(AA^*)^\dagger$ by the Proposition 1.1. □

Lemma 1.2. *Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ have a closed range. Then A has the matrix decomposition with respect to the orthogonal decompositions of spaces $\mathcal{H} = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$ and $\mathcal{K} = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where A_1 is invertible. Moreover,

$$A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

Proof. Suppose that operator A has the matrix decomposition with respect to the orthogonal decompositions of spaces $\mathcal{H} = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$ and $\mathcal{K} = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

which means

$$\begin{aligned} &A_1 \colon \mathcal{R}(A*) \to \mathcal{R}(A^*), && A_2 \colon \mathcal{N}(A) \to \mathcal{R}(A), \\ &A_3 \colon \mathcal{R}(A^*) \to \mathcal{N}(A^*), && A_4 \colon \mathcal{N}(A) \to \mathcal{N}(A^*). \end{aligned}$$

It must be $A_2 = 0$ and $A_4 = 0$ (because their domain is $\mathcal{N}(A)$), and $A_3 = 0$ because its range is $\mathcal{N}(A^*)$ and we know that $\mathcal{R}(A) \cap \mathcal{N}(A^*) = \{0\}$.

Since A_1 is injective (because $\mathcal{N}(A_1) = \mathcal{N}(A) \cap \mathcal{R}(A^*) = \{0\}$) and onto, we conclude that A_1 is invertible. Hence indeed

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where A_1 is invertible.

By checking all four Penrose equations, we see that $A^\dagger$ is really of the form as stated in the Lemma. □

Lemma 1.3. [21] *Let $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$ have a closed range. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be closed and mutually orthogonal subspaces of $\mathcal{H}$, such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be closed and mutually orthogonal subspaces of $\mathcal{K}$, such that $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$. Then the operator A has the following matrix representations with respect to the orthogonal sums of subspaces $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$, and $\mathcal{K} = \mathcal{R}(A) \oplus \mathcal{N}(A^*) = \mathcal{K}_1 \oplus \mathcal{K}_2$:*

(a)
$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1A_1^* + A_2A_2^*$ maps $\mathcal{R}(A)$ into itself and $D > 0$. Also,

$$A^\dagger = \begin{bmatrix} A_1^*D^{-1} & 0 \\ A_2^*D^{-1} & 0 \end{bmatrix}.$$

(b)
$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \end{bmatrix},$$

where $D = A_1^*A_1 + A_2^*A_2$ maps $\mathcal{R}(A^*)$ into itself and $D > 0$. Also,

$$A^\dagger = \begin{bmatrix} D^{-1}A_1^* & D^{-1}A_2^* \\ 0 & 0 \end{bmatrix}.$$

Here A_i denotes different operators in any of these two cases.

Proof. a) Suppose that operator A has the matrix decomposition with respect to the orthogonal decompositions of spaces $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{K} = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}.$$

It must be $A_3 = 0$ and $A_4 = 0$, because their ranges are $\mathcal{N}(A^*)$ and we know that $\mathcal{R}(A) \cap \mathcal{N}(A^*) = \{0\}$; hence

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}.$$

By using Lemma 1.1 we have (we denote $D = A_1A_1^* + A_2A_2^*$)

$$A^\dagger = A^*(AA^*)^\dagger = \begin{bmatrix} A_1^* & 0 \\ A_2^* & 0 \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}^\dagger = \begin{bmatrix} A_1^*D^{-1} & A_2^*D^{-1} \\ 0 & 0 \end{bmatrix}.$$

It is clear that D maps $\mathcal{R}(A^*)$ into itself, and since

$$\langle Dx, x\rangle = \langle A_1A_1^*x, x\rangle + \langle A_2A_2^*x, x\rangle = \|A_1^*x\|^2 + \|A_2^*x\|^2,$$

for any $x \in \mathcal{H}\backslash\{0\}$, we have $D > 0$, as required. □

The following result is Proposition 2.1. from [28] (also can be found in [6, p. 127]), and it will be a useful tool for proving the existence of Moore–Penrose inverses of some terms.

Lemma 1.4. *Let $A \in \mathcal{L}(\mathcal{H}_2,\mathcal{H}_3)$ and $B \in \mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$ have closed ranges. Then AB has a closed range if and only if $A^\dagger ABB^\dagger$ has a closed range.*

Let us point out the difference between the following notations. If $A, B \in \mathcal{L}(X)$, then $[A, B] = AB - BA$ denotes the commutator of A and B. On the other hand, if $U \in \mathcal{L}(X, Z)$ and $V \in \mathcal{L}(Y, Z)$, then $[U \quad V] \colon \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right] \to Z$ denotes the matrix form of the corresponding operator.

Lemma 1.5. [21, Lemma 2.1] *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, let $C \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ have a closed range, and let $D \in \mathcal{L}(\mathcal{K})$ be Hermitian and invertible. Then $\mathcal{R}(DC) = \mathcal{R}(C)$ if and only if $[D, CC^\dagger] = 0$.*

Proof. $(\Rightarrow)$: We consider the orthogonal decompositions $\mathcal{H} = \mathcal{R}(C^*) \oplus \mathcal{N}(C)$ and $\mathcal{K} = \mathcal{R}(C) \oplus \mathcal{N}(C^*)$. Then the operators C and D have the corresponding matrix forms as follows

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{bmatrix},$$

where C_1 is invertible, and

$$D = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{bmatrix},$$

where $D_3 = D_2^*$. It follows that

$$DC = \begin{bmatrix} D_1C_1 & 0 \\ D_3C_1 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^* \end{bmatrix}.$$

Hence, $\mathcal{R}(DC) = \mathcal{R}(C)$ implies $D_3 = 0$ and $D_2 = 0$, so $D = \left[\begin{smallmatrix} D_1 & 0 \\ 0 & D_4 \end{smallmatrix}\right]$. Since D is Hermitian and invertible, we obtain that D_1 and D_4 are also Hermitian and invertible. Since $C^\dagger = \left[\begin{smallmatrix} C_1^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$, we obtain that $DCC^\dagger = CC^\dagger D$ holds.

$(\Leftarrow)$: If D is invertible and $DCC^\dagger = CC^\dagger D$, then

$$\mathcal{R}(DC) = \mathcal{R}(DCC^\dagger) = \mathcal{R}(CC^\dagger D) = \mathcal{R}(CC^\dagger) = \mathcal{R}(C). \qquad \square$$

We shall also use the following result, which is given in [10] for complex matrices case and here is extended to the bounded linear Hilbert space operators.

Lemma 1.6. *Let $\mathcal{H}_i$, $i = \overline{1, 4}$, be arbitrary Hilbert spaces, and let $C \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, $X \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$, and $B \in \mathcal{L}(\mathcal{H}_3, \mathcal{H}_4)$ be closed range operators such that BXC is closed range. Then*

$$C(BXC)^\dagger B = X^\dagger \tag{1.3}$$

if and only if

$$\mathcal{R}(B^*BX) = \mathcal{R}(X) \quad \textit{and} \quad \mathcal{N}(XCC^*) = \mathcal{N}(X). \tag{1.4}$$

Proof. We consider the following orthogonal decompositions

$$\mathcal{H}_1 = \mathcal{R}(C^*) \oplus \mathcal{N}(C), \qquad \mathcal{H}_2 = \mathcal{R}(X^*) \oplus \mathcal{N}(X),$$
$$\mathcal{H}_3 = \mathcal{R}(X) \oplus \mathcal{N}(X^*), \qquad \mathcal{H}_4 = \mathcal{R}(B) \oplus \mathcal{N}(B^*);$$

then the operators have the following matrix forms

$$C = \begin{bmatrix} C_1 & 0 \\ C_2 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix},$$

where X_1 is invertible, and $D = B_1B_1^* + B_2B_2^*$ and $E = C_1^*C_1 + C_2^*C_2$ are positive invertible operators.

Now, eq. 1.3 is equivalent to the following four equations

$$C_1(B_1X_1C_1)^\dagger B_1 = X_1^{-1}, \tag{1.5}$$

$$C_1(B_1X_1C_1)^\dagger B_2 = 0, \tag{1.6}$$

$$C_2(B_1X_1C_1)^\dagger B_1 = 0, \tag{1.7}$$

$$C_2(B_1X_1C_1)^\dagger B_2 = 0. \tag{1.8}$$

If we multiply (1.5) and (1.6) from the right by B_1^* and B_2^*, respectively, and add them, we have $C_1(B_1X_1C_1)^\dagger D = X_1^{-1}B_1^*$. If we multiply (1.7) and (1.8) from the right by B_1^* and B_2^*, respectively, and add them, we obtain

$$C_2(B_1X_1C_1)^\dagger = 0, \tag{1.9}$$

because D is invertible. If we multiply (1.6) and (1.8) from the left by C_1^* and C_2^* respectively, and add them, we obtain

$$(B_1X_1C_1)^\dagger B_2 = 0, \tag{1.10}$$

because E is invertible.

Recall well-known fact

$$PQ = 0 \Leftrightarrow \mathcal{R}(Q) \subset \mathcal{N}(P), \tag{1.11}$$

which we will use in the proof. If we apply this and Proposition 1.1, part (5), to (1.9) we have: $\mathcal{R}(C_1^*X_1^*B_1^*) = \mathcal{R}((B_1X_1C_1)^\dagger) \subset \mathcal{N}(C_2)$, which is equivalent to

$$C_2C_1^*X_1^*B_1^* = 0. \tag{1.12}$$

Also, if we apply (1.11) and Proposition 1.1, part (7), to (1.10) we have

$$\mathcal{R}(B_2) \subset \mathcal{N}((B_1X_1C_1)^\dagger) = \mathcal{N}(C_1^*X_1^*B_1^*),$$

which is equivalent to

$$C_1^*X_1^*B_1^*B_2 = 0. \tag{1.13}$$

From (1.12) and (1.13) it is rather easy to obtain (1.6)–(1.8), only property 3 from Proposition 1.1 is used

$$\begin{aligned}
C_1(B_1X_1C_1)^\dagger B_2 &= C_1(C_1^*X_1^*B_1^*B_1X_1C_1)^\dagger C_1^*X_1^*B_1^*B_2 = 0,\\
C_2(B_1X_1C_1)^\dagger B_1 &= C_2C_1^*X_1^*B_1^*(B_1X_1C_1C_1^*X_1^*B_1^*)^\dagger B_1 = 0,\\
C_2(B_1X_1C_1)^\dagger B_2 &= C_2C_1^*X_1^*B_1^*(B_1X_1C_1C_1^*X_1^*B_1^*)^\dagger B_2 = 0.
\end{aligned}$$

Therefore, (1.3) is equivalent to (1.5), (1.12) and (1.13).

Now we consider the condition (1.4). In what follows we use the following observation

$$T \text{ onto} \Rightarrow TT^* \text{ invertible and } T^\dagger \text{ is the right inverse of } T \text{ i.e. } TT^\dagger = I. \tag{1.14}$$

With our decompositions, the first expression in (1.4) is equivalent to

$$\mathcal{R}\left(\begin{bmatrix} B_1^*B_1X_1 & 0 \\ B_2^*B_1X_1 & 0 \end{bmatrix}\right) = \mathcal{R}\left(\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}\right).$$

Since

$$\begin{aligned}\mathcal{R}\left(\begin{bmatrix} B_1^*B_1X_1 & 0 \\ B_2^*B_1X_1 & 0\end{bmatrix}\right) &= (B_1^*B_1X_1)\mathcal{R}(X^*) \oplus (B_2^*B_1X_1)\mathcal{R}(X^*) \\ &= (B_1^*B_1)\mathcal{R}(X) \oplus (B_2^*B_1)\mathcal{R}(X) \\ &= \mathcal{R}(B_1^*B_1X_1) \oplus \mathcal{R}(B_2^*B_1X_1) \\ &= \mathcal{R}(X) = \mathcal{R}(X_1),\end{aligned}$$

it must be $\mathcal{R}(B_1^*B_1X_1) = \mathcal{R}(X_1)$ and $\mathcal{R}(B_2^*B_1X_1) = \{0\}$. The latter is equivalent to $B_2^*B_1 = 0$, while the former is because of (1.14) equivalent to $B_1^*B_1$ be onto, which is further equivalent to $(B_1^*B_1)^\dagger$ is right inverse of $B_1^*B_1$, i.e.

$$I_{\mathcal{R}(X)} = B_1^*B_1(B_1^*B_1)^\dagger = B_1^*(B_1^*)^\dagger = (B_1^\dagger B_1)^* = B_1^\dagger B_1.$$

In a similar way, second expression in (1.4) is equivalent to (we used well-known fact $\mathcal{R}(T^*)^\perp = \mathcal{N}(T)$)

$$\mathcal{R}\left(\begin{bmatrix} C_1C_1^*X_1^* & 0 \\ C_2C_1^*X_1^* & 0\end{bmatrix}\right) = \mathcal{R}\left(\begin{bmatrix} X_1^* & 0 \\ 0 & 0\end{bmatrix}\right).$$

Since

$$\begin{aligned}\mathcal{R}\left(\begin{bmatrix} C_1C_1^*X_1^* & 0 \\ C_2C_1^*X_1^* & 0\end{bmatrix}\right) &= (C_1C_1^*X_1^*)\mathcal{R}(X) \oplus (C_2C_1^*X_1^*)\mathcal{R}(X) \\ &= (C_1C_1^*)\mathcal{R}(X^*) \oplus (C_2C_1^*)\mathcal{R}(X^*) \\ &= \mathcal{R}(C_1C_1^*X_1^*) \oplus \mathcal{R}(C_2C_1^*X_1^*) \\ &= \mathcal{R}(X^*) = \mathcal{R}(X_1^*),\end{aligned}$$

it must be $\mathcal{R}(C_1C_1^*X_1^*) = \mathcal{R}(X_1^*)$ and $\mathcal{R}(C_2C_1^*X_1^*) = \{0\}$. The latter is equivalent to $C_2^*C_1 = 0$, while the former is because of (1.14) equivalent to $C_1C_1^*$ be onto, which is further equivalent to $(C_1C_1^*)^\dagger$ is right inverse of $C_1C_1^*$, i.e.

$$I_{\mathcal{R}(X^*)} = C_1C_1^*(C_1C_1^*)^\dagger = C_1C_1^\dagger.$$

Hence, (1.4) is equivalent to $B_1^\dagger B_1 = I_{\mathcal{R}(X)},\ B_2^*B_1 = 0,\ C_1C_1^\dagger = I_{\mathcal{R}(X^*)},\ C_2^*C_1 = 0$.

($\Leftarrow$:) We show that $B_1^\dagger B_1 = I_{\mathcal{R}(X)},\ B_2^*B_1 = 0,\ C_1C_1^\dagger = I_{\mathcal{R}(X^*)},\ C_2^*C_1 = 0$ imply (1.5)–(1.8).

From $B_2^*B_1 = 0$ and $C_2^*C_1 = 0$ we immediately have (1.12) and (1.13), hence (1.6)–(1.8). Let us show that (1.5) holds as well

$$\begin{aligned}C_1(B_1X_1C_1)^\dagger B_1 &= X_1^{-1}X_1C_1(B_1X_1C_1)^\dagger B_1X_1X_1^{-1} \\ &= X_1^{-1}B_1^\dagger B_1X_1C_1(B_1X_1C_1)^\dagger B_1X_1C_1C_1^\dagger X_1^{-1} \\ &= X_1^{-1}B_1^\dagger B_1X_1C_1C_1^\dagger X_1^{-1} \\ &= X_1^{-1}X_1X_1^{-1} \\ &= X_1^{-1}.\end{aligned}$$

($\Rightarrow$:) From $C_1(B_1X_1C_1)^\dagger B_1 = X_1^{-1}$ we have

$$X_1^{-1} = C_1(B_1X_1C_1)^\dagger B_1 = C_1C_1^\dagger C_1(B_1X_1C_1)^\dagger B_1B_1^\dagger B_1 = C_1C_1^\dagger X_1^{-1}B_1^\dagger B_1,$$

from where we conclude that $C_1C_1^\dagger = I_{\mathcal{R}(X^*)}$ and $B_1^\dagger B_1 = I_{\mathcal{R}(X)}$.

Let us prove that (1.12) and (1.13) imply $C_2^*C_1 = 0$ and $B_2^*B_1 = 0$, respectively. We use already proven fact (1.5) implies $C_1C_1^\dagger = I_{\mathcal{R}(X^*)}$ and $B_1^\dagger B_1 = I_{\mathcal{R}(X)}$

$$B_1X_1C_1C_2^* = 0 \Rightarrow B_1^\dagger B_1X_1C_1C_2^* = 0 \Rightarrow X_1C_1C_2^* = 0 \Rightarrow C_1C_2^* = 0,$$

$$B_2^*B_1X_1C_1 = 0 \Rightarrow B_2^*B_1X_1C_1C_1^\dagger = 0 \Rightarrow B_2^*B_1X_1 = 0 \Rightarrow B_2^*B_1 = 0. \qquad \square$$

2. Reverse order law for the Moore–Penrose inverse

In this section we present some results concerning the reverse order law for the Moore–Penrose inverse. Direct motivation were some results for complex matrix case, published in 2004 by Tian [46]. They are generalized to the Hilbert space settings and published in 2010 in [21].

Theorem 2.1. *Let $A \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$ and $B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be such that A, B, AB have closed ranges. Then the following statements are equivalent:*

(a) $ABB^\dagger A^\dagger AB = AB$;
(b) $B^\dagger A^\dagger ABB^\dagger A^\dagger = B^\dagger A^\dagger$;
(c) $A^\dagger ABB^\dagger = BB^\dagger A^\dagger A$;
(d) $A^\dagger ABB^\dagger$ *is an idempotent;*
(e) $BB^\dagger A^\dagger A$ *is an idempotent;*
(f) $B^\dagger (A^\dagger ABB^\dagger)^\dagger A^\dagger = B^\dagger A^\dagger$;
(g) $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$.

Notice that $A^\dagger ABB^\dagger$ has a closed range, according to Lemma 1.4. Moreover, A^*ABB^* also has a closed range

$$\mathcal{R}(B^*A^*A) = B^*(\mathcal{R}(A^*A)) = B^*(\mathcal{R}(A^*)) = \mathcal{R}((AB)^*)$$

is closed, so

$$\mathcal{R}(A^*ABB^*) = A^*A(\mathcal{R}(BB^*)) = A^*A(\mathcal{R}(B)) = \mathcal{R}(A^*AB) = \mathcal{R}((B^*A^*A)^*)$$

is closed.

Proof. Using Lemma 1.2 we conclude that the operator B has the following matrix form

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where B_1 is invertible. Then

$$B^\dagger = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix}.$$

From Lemma 1.3 it follows that the operator A has the following matrix form

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}.$$

where $D = A_1A_1^* + A_2A_2^*$ is invertible and positive in $\mathcal{L}(\mathcal{R}(A))$. Then

$$A^\dagger = \begin{bmatrix} A_1^*D^{-1} & 0 \\ A_2^*D^{-1} & 0 \end{bmatrix}.$$

Notice the following

$$BB^\dagger = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$

$$AA^\dagger = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

and

$$A^\dagger A = \begin{bmatrix} A_1^* D^{-1} A_1 & A_1^* D^{-1} A_2 \\ A_2^* D^{-1} A_1 & A_2^* D^{-1} A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}.$$

We obtain

$$A^\dagger ABB^\dagger = \begin{bmatrix} A_1^* D^{-1} A_1 & 0 \\ A_2^* D^{-1} A_1 & 0 \end{bmatrix}, \qquad BB^\dagger A^\dagger A = \begin{bmatrix} A_1^* D^{-1} A_1 & A_1^* D^{-1} A_2 \\ 0 & 0 \end{bmatrix}.$$

Consider the following chain of equivalencies, which is related to the statement of (a)

$$ABB^\dagger A^\dagger AB = AB \Leftrightarrow \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1^* D^{-1} A_1 & A_1^* D^{-1} A_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 B_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} A_1 A_1^* D^{-1} A_1 B_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 B_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Leftrightarrow A_1 A_1^* D^{-1} A_1 = A_1. \tag{2.1}$$

Consequently, the statement (a) is equivalent to (2.1).

Notice that (2.1) is equivalent to

$$A_1^* D^{-1} A_1 A_1^* = A_1^*. \tag{2.2}$$

We consider also the statement (b)

$$B^\dagger A^\dagger ABB^\dagger A^\dagger = B^\dagger A^\dagger$$

$$\Leftrightarrow \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1^* D^{-1} A_1 & 0 \\ A_2^* D^{-1} A_1 & 0 \end{bmatrix} \begin{bmatrix} A_1^* D^{-1} & 0 \\ A_2^* D^{-1} & 0 \end{bmatrix} = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1^* D^{-1} & 0 \\ A_2^* D^{-1} & 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} B_1^{-1} A_1^* D^{-1} A_1 A_1^* D^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B_1^{-1} A_1^* D^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Leftrightarrow B_1^{-1} A_1^* D^{-1} A_1 A_1^* D^{-1} = B_1^{-1} A_1^* D^{-1} \Leftrightarrow \text{(2.2)}.$$

Thus, (a)$\Leftrightarrow$(2.1)$\Leftrightarrow$(2.2)$\Leftrightarrow$(b).

In the case of statement (c) we have

$$A^\dagger ABB^\dagger = BB^\dagger A^\dagger A \Leftrightarrow \begin{bmatrix} A_1^* D^{-1} A_1 & 0 \\ A_2^* D^{-1} A_1 & 0 \end{bmatrix} = \begin{bmatrix} A_1^* D^{-1} A_1 & A_1^* D^{-1} A_2 \\ 0 & 0 \end{bmatrix}$$

$$\Leftrightarrow A_1^* D^{-1} A_2 = 0 \Leftrightarrow A_2^* D^{-1} A_1 = 0. \tag{2.3}$$

Thus, if (c) holds, i.e. $A_2^*D^{-1}A_1 = 0$, then it is obvious that $A_2A_2^*D^{-1}A_1 = 0$, so (2.1) also holds because of

$$\begin{aligned}(A_1A_1^* + A_2A_2^*)D^{-1} = I_{\mathcal{R}(A)} &\Rightarrow A_1A_1^*D^{-1}A_1 + A_2A_2^*D^{-1}A_1 = A_1\\ &\Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1.\end{aligned}$$

On the other hand, suppose that (2.1) holds. Then $A_2A_2^*D^{-1}A_1 = 0$, and we have

$$A_2A_2^*D^{-1}A_1 = 0 \Rightarrow \mathcal{R}(D^{-1}A_1) \subset \mathcal{N}(A_2A_2^*) = \mathcal{N}(A_2^*) \Rightarrow A_2^*D^{-1}A_1 = 0,$$

so (2.3) is satisfied. Consequently, (c) also holds. We have just proved (c)⇔(2.3)⇔(2.1)⇔(a).

A straightforward computation shows that (d) is equivalent to

$$\begin{aligned}A_1^*D^{-1}A_1A_1^*D^{-1}A_1 &= A_1^*D^{-1}A_1\\ A_2^*D^{-1}A_1A_1^*D^{-1}A_1 &= A_2^*D^{-1}A_1\end{aligned} \tag{2.4}$$

If the statement (2.1) holds, then obviously (2.4) is satisfied. On the other hand, suppose that (2.4) holds. Then multiply the first equation of (2.4) by A_1 from the left side, and multiply the second equation of (2.4) by A_2 from the left side. The sum of these two new equations leads to the equation (2.1).

Notice that (e) is also equivalent to (2.4). Consequently, (d)⇔(2.4)⇔(2.2)⇔(e).

In order to establish (f), we proceed as follows. Let $Q = A^\dagger ABB^\dagger$. From Lemma 1.4 we know that Q has a closed range. We use the formula $Q^\dagger = Q^*(QQ^*)^\dagger = (Q^*Q)^\dagger Q^*$. Hence,

$$\begin{aligned}(A^\dagger ABB^\dagger)^\dagger &= (BB^\dagger A^\dagger AA^\dagger ABB^\dagger)^\dagger BB^\dagger A^\dagger A = (BB^\dagger A^\dagger ABB^\dagger)^\dagger BB^\dagger A^\dagger A\\ &= \begin{bmatrix} A_1^*D^{-1}A_1 & 0\\ 0 & 0\end{bmatrix}^\dagger \begin{bmatrix} A_1^*D^{-1}A_1 & A_1^*D^{-1}A_2\\ 0 & 0\end{bmatrix}\\ &= \begin{bmatrix} (A_1^*D^{-1}A_1)^\dagger & 0\\ 0 & 0\end{bmatrix} \begin{bmatrix} A_1^*D^{-1}A_1 & A_1^*D^{-1}A_2\\ 0 & 0\end{bmatrix}\\ &= \begin{bmatrix} (A_1^*D^{-1}A_1)^\dagger A_1^*D^{-1}A_1 & (A_1^*D^{-1}A_1)^\dagger A_1^*D^{-1}A_2\\ 0 & 0\end{bmatrix}.\end{aligned}$$

We get

$$B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger - B^\dagger A^\dagger = 0 \Leftrightarrow \begin{bmatrix} B_1^{-1}(A_1^*D^{-1}A_1)^\dagger A_1^*D^{-1} - B_1^{-1}A_1^*D^{-1} & 0\\ 0 & 0\end{bmatrix} = 0$$

$$\Leftrightarrow (A_1^*D^{-1}A_1)^\dagger A_1^* = A_1^*. \tag{2.5}$$

We need to prove (2.1)⇔(2.5). Let $P = A_1^*D^{-1}A_1$. Obviously, $P^* = P$.

(2.1)⇒(2.5): It is clear that $P^2 = A_1^*D^{-1}A_1A_1^*D^{-1}A_1 = A_1^*D^{-1}A_1 = P$, so P is an orthogonal projector, hence $P^\dagger = P$. Therefore

$$(A_1^*D^{-1}A_1)^\dagger A_1^* = A_1^*D^{-1}A_1A_1^* = A_1^*.$$

(2.5)⇒(2.1): In this case we have

$$\begin{aligned} A_1A_1^*D^{-1}A_1 &= A_1(A_1^*D^{-1}A_1)^\dagger A_1^*D^{-1}((A_1^*D^{-1}A_1)^\dagger A_1^*)^* \\ &= A_1(A_1^*D^{-1}A_1)^\dagger A_1^*D^{-1}A_1(A_1^*D^{-1}A_1)^\dagger \\ &= A_1(A_1^*D^{-1}A_1)^\dagger = A_1. \end{aligned}$$

We have just proved (f)⇔(2.1)⇔(a).

To prove (g)⇔(f), we use the fact that is already proved for (f), i.e. for $(A^\dagger ABB^\dagger)^\dagger$. Thus, we have

$$(A^\dagger ABB^\dagger)^\dagger - BB^\dagger A^\dagger A = 0 \Leftrightarrow \begin{cases} (A_1^*D^{-1}A_1)^\dagger A_1^*D^{-1}A_1 = A_1^*D^{-1}A_1, \\ (A_1^*D^{-1}A_1)^\dagger A_1^*D^{-1}A_2 = A_1^*D^{-1}A_2. \end{cases}$$

It is easy to conclude that (g)⇔(f). □

Now we prove the following result.

Theorem 2.2. *Let $A \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$ and $B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be such that A, B, AB have closed ranges. Then the following statements hold:*

(a) $AB(AB)^\dagger = ABB^\dagger A^\dagger \Leftrightarrow A^*AB = BB^\dagger A^*AB \Leftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,3\}$;

(b) $(AB)^\dagger AB = B^\dagger A^\dagger AB \Leftrightarrow ABB^* = ABB^*A^\dagger A \Leftrightarrow \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,4\}$;

(c) *The following statements are equivalent*
  (1) $(AB)^\dagger = B^\dagger A^\dagger$;
  (2) $AB(AB)^\dagger = ABB^\dagger A^\dagger$ *and* $(AB)^\dagger AB = B^\dagger A^\dagger AB$;
  (3) $A^*AB = BB^\dagger A^*AB$ *and* $ABB^* = ABB^*A^\dagger A$;
  (4) $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ *and* $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$.

Proof. The operators A and B have the same matrix representations as in the previous theorem. The following products will be useful

$$AB = \begin{bmatrix} A_1B_1 & 0 \\ 0 & 0 \end{bmatrix}, (AB)^\dagger = \begin{bmatrix} (A_1B_1)^\dagger & 0 \\ 0 & 0 \end{bmatrix}, B^\dagger A^\dagger = \begin{bmatrix} B_1^{-1}A_1^*D^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

We find the equivalent expressions for our statements in terms of A_1, A_2 and B_1.

(a)
1. $AB(AB)^\dagger = ABB^\dagger A^\dagger \Leftrightarrow A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1}$. Here $A_1B_1(A_1B_1)^\dagger$ is Hermitian, so $[A_1A_1^*, D^{-1}] = 0$.
2. $A^*AB = BB^\dagger A^*AB \Leftrightarrow A_2^*A_1 = 0$.
3. Notice that $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ if and only if $BB^\dagger A^*AB = A^*AB$, so $2 \Leftrightarrow 3$.
4. If we check the Penrose equations, we see: $B^\dagger A^\dagger \in (AB)\{1,2,3\} \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[A_1A_1^*, D^{-1}] = 0$.

Now, we prove the following: $1 \Leftrightarrow 2$, $4 \Rightarrow 2$ and $1 \Rightarrow 4$.

We prove $1 \Leftrightarrow 2$. Notice that

$$A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1} \Leftrightarrow (A_1B_1)^\dagger = (A_1B_1)^\dagger A_1A_1^*D^{-1}.$$

Now, there is a chain of equivalences

$$\begin{aligned}&(A_1B_1)^\dagger = (A_1B_1)^\dagger A_1A_1^* D^{-1}\\ &\quad\Leftrightarrow (A_1B_1)^\dagger(A_1A_1^* + A_2A_2^*) = (A_1B_1)^\dagger A_1A_1^*\\ &\quad\Leftrightarrow (A_1B_1)^\dagger A_2A_2^* = 0 \Leftrightarrow \mathcal{R}(A_2A_2^*) \subset \mathcal{N}((A_1B_1)^\dagger)\\ &\quad\Leftrightarrow \mathcal{R}(A_2) \subset \mathcal{N}((A_1B_1)^*) \Leftrightarrow B_1^*A_1^*A_2 = 0 \Leftrightarrow A_1^*A_2 = 0,\end{aligned}$$

Therefore, we have just proved that $1 \Leftrightarrow 2$.

Let us prove $1 \Rightarrow 4$. If we multiply $A_1B_1(A_1B_1)^\dagger = A_1A_1^*D^{-1}$ by A_1B_1 from the right hand side, we get $A_1A_1^*D^{-1}A_1 = A_1$. Thus, 4 holds.

Finally, we prove $4 \Rightarrow 2$. If $A_1A_1^*D^{-1}A_1 = A_1$ and $[A_1A_1^*, D^{-1}] = 0$, then $A_1A_1^*A_1 = DA_1 = A_1A_1^*A_1 + A_2A_2^*A_1$, implying that $A_2A_2^*A_1 = 0$. Hence, $\mathcal{R}(A_1) \subset \mathcal{N}(A_2A_2^*) = \mathcal{N}(A_2^*)$, so $A_2^*A_1 = 0$. Thus, 2 holds.

Notice that the equivalence $3 \Leftrightarrow 4$ is proved in [22], also.

(b)
1. $(AB)^\dagger AB = B^\dagger A^\dagger AB \Leftrightarrow (A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$. Moreover, $(A_1B_1)^\dagger A_1B_1$ is Hermitian, so $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$.
2. $ABB^* = ABB^*A^\dagger A \Leftrightarrow A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1B_1B_1^*$ and $A_1B_1B_1^*A_1^*D^{-1}A_2 = 0$.
3. Notice that $\mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*)$ if and only if $A^\dagger ABB^*A^* = BB^*A^*$, which is equivalent to $ABB^*A^\dagger A = ABB^*$. Hence, $2\Leftrightarrow 3$.
4. From the Penrose equations we see that: $B^\dagger A^\dagger \in (AB)\{1,2,4\} \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$.

We prove $1 \Rightarrow 4 \Rightarrow 2 \Rightarrow 1$.

Suppose that 1 holds. If we multiply $(A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$ by A_1B_1 from the left hand side, we obtain $A_1 = A_1A_1^*D^{-1}A_1$. Furthermore, it is clear that $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$ holds. Therefore, $1 \Rightarrow 4$.

Let 4 hold. Obviously, $A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1A_1^*D^{-1}A_1B_1B_1^* = A_1B_1B_1^*$. Thus, the first equality of 2 holds. The second equality of 2 also holds, since $A_1^*D^{-1}A_2 = 0 \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$, which is shown in the proof of the Theorem 2.1. Here we use again $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$. Consequently, $4 \Rightarrow 2$.

In order to prove that $2 \Rightarrow 1$, we multiply $A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1B_1B_1^*$ by $(A_1B_1)^\dagger$ from the left side. It follows that $B_1^*A_1^*D^{-1}A_1 = (A_1B_1)^\dagger A_1B_1B_1^*$, so $(A_1B_1)^\dagger A_1B_1 = B_1^*A_1^*D^{-1}A_1(B_1^*)^{-1}$ which is equivalent to $(A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$. Hence, $2 \Rightarrow 1$.

Notice that $3 \Leftrightarrow 4$ is also proved in [22].

Finally, part (c) follows from parts (a) and (b). □

We also prove the following result.

Theorem 2.3. *Let $A \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$ and $B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be such that A, B, AB have closed ranges. Then we have:*

(a) $AB(AB)^\dagger A = ABB^\dagger \Leftrightarrow A^*ABB^\dagger = BB^\dagger A^*A \Leftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,3\}$;

(b) $B(AB)^\dagger AB = A^\dagger AB \Leftrightarrow A^\dagger ABB^* = BB^*A^\dagger A \Leftrightarrow \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1,2,4\}$;

(c) *The following three statements are equivalent*
  (1) $(AB)^\dagger = B^\dagger A^\dagger$;
  (2) $AB(AB)^\dagger A = ABB^\dagger$ *and* $B(AB)^\dagger AB = A^\dagger AB$;
  (3) $A^*ABB^\dagger = BB^\dagger A^*A$ *and* $A^\dagger ABB^* = BB^*A^\dagger A$.

Proof. The operators A and B have the same matrix representations as in the previous theorem. First, we find equivalent expressions, in the terms of A_1, A_2 and B_1, for our assumptions.

(a)
1. $AB(AB)^\dagger A = ABB^\dagger \Leftrightarrow A_1B_1(A_1B_1)^\dagger A_1 = A_1 \wedge A_1B_1(A_1B_1)^\dagger A_2 = 0$. The first equality on the right side of the equivalence always holds, so: $AB(AB)^\dagger A = ABB^\dagger \Leftrightarrow A_1B_1(A_1B_1)^\dagger A_2 = 0$.
2. $A^*ABB^\dagger = BB^\dagger A^*A \Leftrightarrow A_1^*A_2 = 0$.
3. $\mathcal{R}(A^*AB) \subset \mathcal{R}(B) \Leftrightarrow BB^\dagger A^*AB = A^*AB \Leftrightarrow A_2^*A_1 = 0$ (see the proof of Theorem 2.2, the part (a) 2 and 3).
4. $B^\dagger A^\dagger \in (AB)\{1,2,3\} \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[A_1A_1^*, D^{-1}] = 0$ (see Theorem 2.2 (a) 4.).

To prove that $1 \Leftrightarrow 2$, we see that

$$\begin{aligned} & A_1B_1(A_1B_1)^\dagger A_2 = 0 \\ & \Leftrightarrow \mathcal{R}(A_2) \subset \mathcal{N}((A_1B_1)(A_1B_1)^\dagger) = \mathcal{N}((A_1B_1)^\dagger) \\ & \qquad\qquad = \mathcal{N}((A_1B_1)^*) = \mathcal{N}(B_1^*A_1^*) = \mathcal{N}(A_1^*) \\ & \Leftrightarrow A_1^*A_2 = 0. \end{aligned}$$

Now, we prove that $2 \Leftrightarrow 4$. If $[A_1A_1^*, D^{-1}] = 0$, then $A_1A_1^*D^{-1}A_1 = A_1 \Leftrightarrow A_1A_1^*A_1 = DA_1 \Leftrightarrow A_2A_2^*A_1 = 0 \Leftrightarrow A_1^*A_2A_2^* = 0 \Leftrightarrow \mathcal{R}(A_2A_2^*) \subset \mathcal{N}(A_1^*) \Leftrightarrow \mathcal{R}(A_2) \subset \mathcal{N}(A_1^*) \Leftrightarrow A_1^*A_2 = 0$. On the other hand, if $A_1^*A_2 = 0$, then $A_1A_1^*D = A_1A_1^*A_1A_1^*$ is Hermitian, so $A_1A_1^*$ commutes with D. This implies $[A_1A_1^*, D^{-1}] = 0$ and $A_1A_1^*D^{-1}A_1 = A_1$.

From Theorem 2.2 we know that $3 \Leftrightarrow 4$.

(b)
1. $B(AB)^\dagger AB = A^\dagger AB \Leftrightarrow B_1(A_1B_1)^\dagger A_1 = A_1^*D^{-1}A_1 \wedge A_2^*D^{-1}A_1 = 0$.
2. $A^\dagger ABB^* = BB^*A^\dagger A \Leftrightarrow [B_1B_1^*, A_1^*D^{-1}A_1] = 0$ and $A_1^*D^{-1}A_2 = 0$.
3. $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow A_1B_1B_1^*A_1^*D^{-1}A_2 = 0 \wedge A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1B_1B_1^*$ (Theorem 2.2 (b), parts 2 and 3).
4. $B^\dagger A^\dagger \in (AB)\{1,2,4\} \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$ (Theorem 2.2 (b) part 4).

$1 \Rightarrow 4$: We multiply the expression $B_1(A_1B_1)^\dagger A_1 = A_1^*D^{-1}A_1$ by A_1 from the left side, and by B_1 from the right side, and thus obtain $A_1A_1^*D^{-1}A_1 = A_1$. Also, we obtain that $(A_1B_1)^\dagger A_1B_1 = B_1^{-1}A_1^*D^{-1}A_1B_1$ is Hermitian, so $A_1^*D^{-1}A_1B_1B_1^*$ is Hermitian, hence $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$.

$4 \Rightarrow 1$: If 4 holds, then it is easy to see that $B_1^{-1}A_1^*D^{-1}A_1B_1(A_1B_1)^\dagger$ is the Moore–Penrose inverse of A_1B_1 (check the Penrose equations). This implies $B_1(A_1B_1)^\dagger A_1 = A_1^*D^{-1}A_1$. Now, we obtain that $A_1 = A_1A_1^*D^{-1}A_1$. From $(A_1A_1^* + A_2A_2^*)D^{-1}A_1 = A_1$ it follows that $A_2A_2^*D^{-1}A_1 = 0$, so $\mathcal{R}(D^{-1}A_1) \subset \mathcal{N}(A_2A_2^*) = \mathcal{N}(A_2^*)$, and $A_2^*D^{-1}A_1 = 0$.

$2 \Rightarrow 3$: If 2 holds, then $A_1B_1B_1^*A_1^*D^{-1}A_2 = 0$ is trivially satisfied. Moreover, $A_1B_1B_1^*A_1^*D^{-1}A_1 = A_1B_1B_1^*$ is equivalent to $A_1A_1^*D^{-1}A_1 = A_1$, which follows from $A_1^*D^{-1}A_2 = 0$.

$3 \Rightarrow 2$: From the proof of Theorem 2.2, part (b) 4, it follows that $[B_1B_1^*, A_1^*D^{-1}A_1] = 0$. Now, as usual, we get that $A_2A_2^*D^{-1}A_1 = 0$, so $A_1^*D^{-1}A_2 = 0$.

$2 \Leftrightarrow 4$: Obvious.

Part (c) follows from parts (a) and (b). □

Remark that some of the results from Theorem 2.2 and 2.3 can be found in [20]. We also prove the following result.

Theorem 2.4. *Let $A \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$ and $B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be such that A, B, AB have closed ranges. The following statements hold.*

(a) $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger \Leftrightarrow B^\dagger(ABB^\dagger)^\dagger = B^\dagger A^\dagger \Leftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$.

(b) $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A \Leftrightarrow (A^\dagger AB)^\dagger A^\dagger = B^\dagger A^\dagger \Leftrightarrow \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$.

(c) *The following three statements are equivalent*
   (1) $(AB)^\dagger = B^\dagger A^\dagger$;
   (2) $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger$ *and* $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A$;
   (3) $B^\dagger(ABB^\dagger)^\dagger = B^\dagger A^\dagger$ *and* $(A^\dagger AB)^\dagger A^\dagger = B^\dagger A^\dagger$.

Notice that $ABB^\dagger$ and $A^\dagger AB$ have closed ranges. This is explained in further proof.

Proof. The operators A and B have the same matrix representations as in the previous theorem.

(a) Notice that $\mathcal{R}(ABB^\dagger) = \mathcal{R}(AB)$ is closed, so there exists $(ABB^\dagger)^\dagger$.
   1. $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger \Leftrightarrow A_1^\dagger = A_1^*D^{-1}$ (the existence of $A_1^\dagger$ follows from the assumptions).
   2. $B^\dagger(ABB^\dagger)^\dagger = B^\dagger A^\dagger \Leftrightarrow A_1^\dagger = A_1^*D^{-1}$, so $1 \Leftrightarrow 2$.
   3. $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \Leftrightarrow A_1A_1^*D^{-1}A_1 = A_1$ and $[A_1A_1^*, D^{-1}] = 0$ (see Theorem 2.2, (a) parts 3 and 4).

   $1 \Rightarrow 3$: If $A_1^\dagger = A_1^*D^{-1}$, then $A_1^\dagger D = A_1^*$ and $A_1A_1^\dagger = A_1A_1^*D^{-1}$ is Hermitian, so $[A_1A_1^*, D^{-1}] = 0$. Moreover, $A_1A_1^\dagger A_2A_2^* = 0$. We conclude $\mathcal{R}(A_2A_2^*) \subset \mathcal{N}(A_1A_1^\dagger) = \mathcal{N}(A_1^*)$, so $A_1^*A_2A_2^* = 0$ and $A_2^*A_1 = 0$. Now, $(A_1A_1^* + A_2A_2^*)A_1 = A_1A_1^*A_1$, so $A_1 = D^{-1}A_1A_1^*A_1 = A_1A_1^*D^{-1}A_1$.

   $3 \Rightarrow 1$: If 3 holds, then it is easy to see that $A_1^*D^{-1}$ is the Moore–Penrose inverse of A_1 (check the Penrose equations).

(b) We see that $\mathcal{R}((A^\dagger AB)^*) = \mathcal{R}(B^*A^\dagger A) = \mathcal{R}(B^*A^*) = \mathcal{R}((AB)^*)$ is closed, so $(A^\dagger AB)^\dagger$ exists. Notice that

$$B^\dagger A^\dagger A = \begin{bmatrix} B_1^{-1}A_1^*D^{-1}A_1 & B_1^{-1}A_1^*D^{-1}A_2 \\ 0 & 0 \end{bmatrix}$$

and $A^\dagger AB = \left[\begin{smallmatrix} A_1^* D^{-1} A_1 B_1 & 0 \\ A_2^* D^{-1} A_1 B_1 & 0 \end{smallmatrix}\right]$. Using the formula $T^\dagger = (T^*T)^\dagger T^*$, we obtain that $(A^\dagger AB)^\dagger$ is

$$\begin{bmatrix} (B_1^* A_1^* D^{-1} A_1 B_1)^\dagger B_1^* A_1^* D^{-1} A_1 & (B_1^* A_1^* D^{-1} A_1 B_1)^\dagger B_1^* A_1^* D^{-1} A_2 \\ 0 & 0 \end{bmatrix}.$$

1. $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A \Leftrightarrow (B_1^* A_1^* D^{-1} A_1 B_1)^\dagger B_1^* A_1^* D^{-1} A_1 = B_1^{-1} A_1^* D^{-1} A_1$ and $(B_1^* A_1^* D^{-1} A_1 B_1)^\dagger B_1^* A_1^* D^{-1} A_2 = B_1^{-1} A_1^* D^{-1} A_2$.
2. $(A^\dagger AB)^\dagger A^\dagger = B^\dagger A^\dagger \Leftrightarrow B_1 (B_1^* A_1^* D^{-1} A_1 B_1)^\dagger B_1^* A_1^* = A_1^*$.
3. $\mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*) \Leftrightarrow A_1 A_1^* D^{-1} A_1 = A_1 \wedge [B_1 B_1^*, A_1^* D^{-1} A_1] = 0$.

$1 \Rightarrow 2$: We multiply the first equality of 1 by A_1^* from the right side, and we multiply the second equality of 1 by A_2^* from the right side. By summing the obtained equalities we obtain 2.

$2 \Rightarrow 1$: This is obvious.

$2 \Rightarrow 3$: If we multiply $B_1 (B_1^* A_1^* D^{-1} A_1 B_1)^\dagger B_1^* A_1^* = A_1^*$ from the left by $B_1^* A_1^* D^{-1} A_1$, and by $D^{-1} A_1 B_1$ from right side, we get $A_1^* D^{-1} A_1 = A_1^* D^{-1} A_1 A_1^* D^{-1} A_1$. Now, $A_1^* D^{-1} A_1$ is the orthogonal projection onto a subspace of $\mathcal{R}(A_1^*)$, so it follows that $A_1 A_1^* D^{-1} A_1 = A_1$.

Since $(B_1^* A_1^* D^{-1} A_1 B_1)^\dagger B_1^* A_1^* D^{-1} A_1 B_1 = B_1^{-1} A_1^* D^{-1} A_1 B_1$ is Hermitian, we obtain $[B_1 B_1^*, A_1^* D^{-1} A_1] = 0$.

$3 \Rightarrow 2$: Using the formula $T^\dagger = (T^*T)^\dagger T^*$, we have

$$(B_1^* A_1^* D^{-1} A_1 B_1)^\dagger B_1^* A_1^* D^{-1/2} = (D^{-1/2} A_1 B_1)^\dagger,$$

which means that

$$B_1 (B_1^* A_1^* D^{-1/2} D^{-1/2} A_1 B_1)^\dagger B_1^* A_1^* = B_1 (D^{-1/2} A_1 B_1)^\dagger D^{1/2}.$$

We wish to show that 3 implies $B_1 (D^{-1/2} A_1 B_1)^\dagger D^{1/2} = A_1^*$. This means that we will show $(D^{-1/2} A_1 B_1)^\dagger = B_1^{-1} A_1^* D^{-1/2}$, by proving that the last expression satisfies all four Penrose equations provided that the conditions from 3 are valid. Hence,

$$\begin{aligned} D^{-1/2} A_1 B_1 B_1^{-1} A_1^* D^{-1/2} D^{-1/2} A_1 B_1 &= D^{-1/2} A_1 A_1^* D^{-1} A_1 B_1 \\ &= D^{-1/2} A_1 B_1, \end{aligned}$$

$$\begin{aligned} B_1^{-1} A_1^* D^{-1/2} D^{-1/2} A_1 B_1 B_1^{-1} A_1^* D^{-1/2} &= B_1^{-1} A_1^* D^{-1} A_1 A_1^* D^{-1/2} \\ &= B_1^{-1} A_1^* D^{-1/2}, \end{aligned}$$

$$D^{-1/2} A_1 B_1 B_1^{-1} A_1^* D^{-1/2} = D^{-1/2} A_1 A_1^* D^{-1/2} \text{ is Hermitian},$$

$$B_1^{-1} A_1^* D^{-1/2} D^{-1/2} A_1 B_1 = B_1^{-1} A_1^* D^{-1} A_1 B_1 \text{ is Hermitian,}$$
$$\text{since } [B_1 B_1^*, A_1^* D^{-1} A_1] = 0.$$

(c) Follows from (a) and (b). □

Theorem 2.5. *Let $A \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$ and $B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be such that A, B, AB have closed ranges. Then we have*

(a) $B^\dagger = (AB)^\dagger A \Leftrightarrow \mathcal{R}(B) = \mathcal{R}(A^* AB)$.

(b) $A^\dagger = B(AB)^\dagger \Leftrightarrow \mathcal{R}(A^*) = \mathcal{R}(BB^*A^*)$.

Proof. The conclusions follow from Lemma 1.6 if we take a) $X := B,\ B := A,\ C := I$; b) $X := A,\ C := B,\ B := I$. Remark that in [21] the proof was different. □

Finally, we prove the following results.

Theorem 2.6. *Let $A \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$ and $B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be such that A, B, AB have closed ranges. Then we have*
(a) $(AB)^\dagger = (A^\dagger AB)^\dagger A^\dagger \Leftrightarrow \mathcal{R}(AA^*AB) = \mathcal{R}(AB)$;
(b) $(AB)^\dagger = B^\dagger(ABB^\dagger)^\dagger \Leftrightarrow \mathcal{R}(B^*B(AB)^*) = \mathcal{R}((AB)^*)$.

Remark that $A^\dagger AB$ and $ABB^\dagger$ have closed ranges.

Proof. (a) Notice that

$$\mathcal{R}((A^\dagger AB)^*) = \mathcal{R}(B^*A^\dagger A) = B^*\mathcal{R}(A^\dagger A) = B^*\mathcal{R}(A^*) = \mathcal{R}((AB)^*)$$

is closed, so $\mathcal{R}(A^\dagger AB)$ is closed. First, let we see how our conditions look like in the terms of their components.

1. Let us denote $T = A^\dagger AB$. We find $T^\dagger$ as follows

$$\begin{aligned} T^\dagger &= (T^*T)^\dagger T^* \\ &= \begin{bmatrix} (B_1^*A_1^*D^{-1}A_1B_1)^\dagger B_1^*A_1^*D^{-1}A_1 & (B_1^*A_1^*D^{-1}A_1B_1)^\dagger B_1^*A_1^*D^{-1}A_2 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Now, it is easy to see that $(AB)^\dagger = (A^\dagger AB)^\dagger A^\dagger$ is equivalent with

$$(A_1B_1)^\dagger = (B_1^*A_1^*D^{-1}A_1B_1)^\dagger B_1^*A_1^*D^{-1} = (D^{-1/2}A_1B_1)^\dagger D^{-1/2}.$$

2. It is obvious that $AA^*AB = \begin{bmatrix} DA_1B_1 & 0 \\ 0 & 0 \end{bmatrix}$, so 2 holds if and only if $\mathcal{R}(DA_1B_1) = \mathcal{R}(A_1B_1)$.

$1 \Rightarrow 2$: The third Penrose equation for $(A_1B_1)^\dagger = (D^{-1/2}A_1B_1)^\dagger D^{-1/2}$ implies that $A_1B_1(D^{-1/2}A_1B_1)^\dagger D^{-1/2}$ is Hermitian, so we have the following equivalences

$$\begin{aligned} &A_1B_1(D^{-1/2}A_1B_1)^\dagger D^{-1/2} \text{ is Hermitian} \\ &\Leftrightarrow D^{-1/2}A_1B_1(D^{-1/2}A_1B_1)^\dagger D^{-1} \text{ is Hermitian} \\ &\Leftrightarrow [D, D^{-1/2}A_1B_1(D^{-1/2}A_1B_1)^\dagger] = 0 \\ &\Leftrightarrow D^{1/2}A_1B_1(D^{-1/2}A_1B_1)^\dagger = D^{-1/2}A_1B_1(D^{-1/2}A_1B_1)^\dagger D \\ &\Leftrightarrow DA_1B_1(D^{-1/2}A_1B_1)^\dagger = A_1B_1(D^{-1/2}A_1B_1)^\dagger D. \end{aligned}$$

Now,

$$\mathcal{R}(DA_1B_1) = \mathcal{R}(DA_1B_1(A_1B_1)^\dagger) = \mathcal{R}(A_1B_1(A_1B_1)^\dagger D) = \mathcal{R}(A_1B_1).$$

$2 \Rightarrow 1$: If $\mathcal{R}(DA_1B_1) = \mathcal{R}(A_1B_1)$, then we apply Lemma 1.5 to obtain $[D, A_1B_1(A_1B_1)^\dagger] = 0$. Now, from the previous implication, it follows that $A_1B_1(D^{-1/2}A_1B_1)^\dagger D^{-1/2}$ is Hermitian. Notice that the operator $(D^{-1/2}A_1B_1)^\dagger D^{-1/2}A_1B_1$ is an orthogonal projection onto

$$\mathcal{R}((A_1B_1)^*D^{-1/2}) \subset \mathcal{R}((A_1B_1)^*),$$

so $A_1B_1(D^{-1/2}A_1B_1)^\dagger D^{-1/2}A_1B_1 = A_1B_1$. Finally, it is not difficult to verify that $(A_1B_1)^\dagger = (D^{-1/2}A_1B_1)^\dagger D^{-1/2}$ holds.

(b) According to (a), we have the following equivalences

$$(AB)^\dagger = (A^\dagger AB)^\dagger A^\dagger \Leftrightarrow \mathcal{R}(AA^*AB) = \mathcal{R}(AB)$$
$$(B^*A^*)^\dagger = (A^*)^\dagger(B^*A^\dagger A)^\dagger \Leftrightarrow \mathcal{R}(AA^*AB) = \mathcal{R}(A)$$
$$(\text{now take } A' = B^* \text{ and } B' = A^*)$$
$$(A'B')^\dagger = B'^\dagger(ABB'^\dagger)^\dagger \Leftrightarrow \mathcal{R}(BB'^*B'^*A'^*) = \mathcal{R}(B'^*A'^*).$$ □

We remark that those results are further generalized to the C^*-star algebras settings e.g. in the papers [37] and [63].

3. Basic reverse order law and its equivalencies

Recall that basic ROL for the Moore–Penrose inverse of closed-range operators $A \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$, $B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is the expression $(AB)^\dagger = B^\dagger A^\dagger$.

As mentioned in the Introduction, we have the following classical results

$$\begin{aligned}(AB)^\dagger = B^\dagger A^\dagger &\Leftrightarrow A^\dagger ABB^*A^* = BB^*A^* \wedge BB^\dagger A^*AB = A^*AB \quad [24]\\ &\Leftrightarrow \mathcal{R}(A^*AB) \subset \mathcal{R}(B) \wedge \mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*)\\ &\Leftrightarrow A^*ABB^* \text{ is Hermitian range matrix} \quad [1]\end{aligned}$$

We should mention the paper of Tian [48] from 2007, which is generalized and extended for the Hilbert space operator case in our paper [16] from 2012. Those results are presented in this section.

3.0.1. Auxiliary results. It is well-known that Hermitian operators and the Moore–Penrose inverse agree very well, which can be exploited for obtaining particularly useful forms for some operator expressions.

Lemma 3.1. *Let H be a Hermitian bounded linear operator with a closed range. Then $(\forall n \in \mathbb{N})$ $(H^n)^\dagger = (H^\dagger)^n$.*

Proof. For $n = 1$ we actually have a well-known identity for Moore–Penrose inverse. For other values of n, it is easy to check all four Penrose equations, using the fact that $H = H^\dagger H^2 = H^2 H^\dagger$, which follows from Proposition 1.1 for any Hermitian operator H. □

Remark 3.1. The Hermitian operator $H \in \mathcal{L}(\mathcal{H})$ is closed-range if and only if $0 \notin acc(\sigma(H))$, i.e. 0 is not accumulation point of spectrum $\sigma(H)$ of operator H. According to the spectral mapping theorem, if Hermitian operator H is closed-range, then H^n is also closed-range for arbitrary positive integer n. This remark justifies the existence of the Moore–Penrose inverse through this section.

Remark 3.2. According to the Lemma 3.1, if an operator T has the form $T = \left[\begin{smallmatrix}\star & \star\\ 0 & 0\end{smallmatrix}\right]$, where "$\star$" denotes arbitrary component, then

$$((T^*T)^\dagger)^n = (T^\dagger(T^\dagger)^*)^n = T^\dagger((T^\dagger)^*T^\dagger)^{n-1}(T^*)^\dagger = T^\dagger((TT^*)^\dagger)^{n-1}(T^\dagger)^*,$$

where TT^* has the following form ("inv." means some invertible operator)

$$TT^* = \begin{bmatrix} inv. & 0 \\ 0 & 0 \end{bmatrix},$$

which provides us with simplified computations.

Similarly, if an operator S has the form: $S = \left[\begin{smallmatrix} \star & 0 \\ \star & 0 \end{smallmatrix}\right]$, then

$$((SS^*)^\dagger)^n = ((S^\dagger)^* S^\dagger)^n = (S^\dagger)^*(S^\dagger(S^\dagger)^*)^{n-1}S^\dagger = (S^*)^\dagger((S^*S)^\dagger)^{n-1}S^\dagger,$$

where S^*S has the simple form $S^*S = \left[\begin{smallmatrix} \text{inv.} & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Those facts will be often used in the proof of our main result.

Proposition 3.1. *For closed-range operator $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and any $m \in \mathbb{N}$ we have*

(a) $((AA^*)^\dagger)^m (AA^*)^m = ((A^*)^\dagger A^\dagger)^m (AA^*)^m = AA^\dagger$;
(b) $(AA^*)^m ((AA^*)^\dagger)^m = (AA^*)^m ((A^*)^\dagger A^\dagger)^m = AA^\dagger$;
(c) $((A^*A)^\dagger)^m (A^*A)^m = (A^\dagger (A^*)^\dagger)^m (A^*A)^m = A^\dagger A$;
(d) $(A^*A)^m ((A^*A)^\dagger)^m = (A^*A)^m (A^\dagger (A^*)^\dagger)^m = A^\dagger A$.

Proof. Let us prove statement a). The case $m = 1$ is, by Proposition 1.1, true. For $m \geqslant 2$ we have

$$\begin{aligned} ((AA^*)^\dagger)^m (AA^*)^m &= ((AA^*)^\dagger)^{m-1} (AA^*)^\dagger AA^* (AA^*)^{m-1} \\ &= ((AA^*)^\dagger)^{m-1} (A^*)^\dagger A^* (AA^*)^{m-1} \\ &= ((AA^*)^\dagger)^{m-1} (AA^*)^{m-1} = \dots \\ &= (AA^*)^\dagger AA^* = (A^*)^\dagger A^* = (AA^\dagger)^* = AA^\dagger. \end{aligned}$$

In a completely analogous way, other three statements can be proved. □

3.0.2. Main result. We present 14 equivalent conditions to the basic ROL.

Theorem 3.1. *Let $A \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$ and $B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be bounded linear operators, such that A, B and AB have closed ranges. The following statements are equivalent*

(a) $(AB)^\dagger = B^\dagger A^\dagger$;
(b) $(AB)^\dagger = B^\dagger A^\dagger ABB^\dagger A^\dagger$;
(c) $((A^\dagger)^* B)^\dagger = B^\dagger A^*$;
(d) $(A(B^\dagger)^*)^\dagger = B^* A^\dagger$;
(e) $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger$ *and* $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A$;
(f) $(AB)^\dagger = B^\dagger (A^\dagger ABB^\dagger)^\dagger A^\dagger$ *and* $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$;
(g) $(AB)^\dagger = (A^\dagger AB)^\dagger A^\dagger$ *and* $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A$;
(h) $(AB)^\dagger = B^\dagger (ABB^\dagger)^\dagger$ *and* $(AAB^\dagger)^\dagger = BB^\dagger A^\dagger$;
(i) $(AB)^\dagger = (A^*AB)^\dagger A^*$ *and* $(A^*AB)^\dagger = B^\dagger (A^*A)^\dagger$;
(j) $(AB)^\dagger = B^*(ABB^*)^\dagger$ *and* $(ABB^*)^\dagger = (BB^*)^\dagger A^\dagger$;
(k) $(AB)^\dagger = B^*(A^*ABB^*)^\dagger A^*$ *and* $(A^*ABB^*)^\dagger = (BB^*)^\dagger (A^*A)^\dagger$;
(l) $(AB)^\dagger = B^*B(AA^*ABB^*B)^\dagger AA^*$ *and* $(AA^*ABB^*B)^\dagger = (BB^*B)^\dagger (AA^*A)^\dagger$;
(m) $(AB)^\dagger = B^*BB^*((A^*A)^2(BB^*)^2)^\dagger A^*AA^*$ *and* $((A^*A)^2(BB^*)^2)^\dagger = ((BB^*)^2)^\dagger ((A^*A)^2)^\dagger$;
(n) $\{B^{(1,3)}A^{(1,3)}\} \subseteq \{(AB)^{(1,3)}\}$ *and* $\{B^{(1,4)}A^{(1,4)}\} \subseteq \{(AB)^{(1,4)}\}$.

Proof. Let us say something about the existence of the Moore–Penrose inverse of various terms appearing in the formulas above. The existence of $(A^\dagger ABB^\dagger)^\dagger$ follows immediately from Lemma 1.4. It is easy to see the existence of $((A^\dagger)^*B)^\dagger$ and $(A(B^\dagger)^*)^\dagger$. We have

$$\mathcal{R}(B^*A^*A) = B^*(\mathcal{R}(A^*A)) = B^*(\mathcal{R}(A^*)) = \mathcal{R}((AB)^*)$$

is closed, which implies the existence of $(A^*AB)^\dagger$, $(A^\dagger AB)^\dagger$ and also of $(A^*ABB^*)^\dagger$, because of

$$\mathcal{R}(A^*ABB^*) = A^*A(\mathcal{R}(BB^*)) = A^*A(\mathcal{R}(B)) = \mathcal{R}(A^*AB) = \mathcal{R}((B^*A^*A)^*).$$

In a completely analogous way, one can prove the existence of expressions $(ABB^\dagger)^\dagger$ and $(ABB^*)^\dagger$.

Note that if A and B are closed-range operators, their product AB need not to be; more on this issue can be found on [4–6, 28]. In this paper we will not further investigate this problem, so in our results throughout this paper we always consider the case when AB is a closed-range operator.

First, we enlist some parts of the proof regardless of the decomposition we will use later.

(a)⇔(n): This is already proven in [22, Corollary 6.2.4].

(a)⇔(e): Already proven in [21, Theorem 2.4.c)].

(f)–(m)⇒(a): Those implications are proven on the same way: the second part of the statement is replaced onto the first one, and common identities (see Proposition 1.1 and Lemma 3.1) are applied if necessary. As a result, we yield statement (a). For illustration, we will present two specific cases:

(j)⇒(a): $(AB)^\dagger = B^*(ABB^*)^\dagger = B^*(BB^*)^\dagger A^\dagger = B^\dagger A^\dagger$.

(m)⇒(a): Here we will use Lemma 3.1 for $n = 2$.

$$\begin{aligned}(AB)^\dagger &= B^*BB^*((A^*A)^2(BB^*)^2)^\dagger = B^*BB^*((BB^*)^2)^\dagger((A^*A)^2)^\dagger A^*AA^* \\ &= B^*BB^*(BB^*)^\dagger(BB^*)^\dagger(A^*A)^\dagger(A^*A)^\dagger A^*AA^* \\ &= B^*BB^\dagger(BB^*)^\dagger(A^*A)^\dagger A^\dagger AA^* = B^*(BB^*)^\dagger(A^*A)^\dagger A^* = B^\dagger A^\dagger.\end{aligned}$$

Let us continue the proof.

(a)⇒(b): $B^\dagger A^\dagger = (AB)^\dagger = (AB)^\dagger AB(AB)^\dagger = B^\dagger A^\dagger ABB^\dagger A^\dagger$.

(a)⇒(g): $(AB)^\dagger = B^\dagger A^\dagger = B^\dagger A^\dagger AA^\dagger = (A^\dagger AB)^\dagger A^\dagger$, according to the already proven statement (e).

(a)⇒(h): $(AB)^\dagger = B^\dagger A^\dagger = B^\dagger BB^\dagger A^\dagger = B^\dagger(A^\dagger BB^\dagger)^\dagger$, according to the already proven statement (e).

For the rest of the proof, we will use the following operator decompositions.

Using Lemma 1.2, we conclude that the operator B has the following matrix form

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where B_1 is invertible. Then

$$B^\dagger = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix},$$

From Lemma 1.3 also follows that the operator A has the following matrix form

$$A=\begin{bmatrix}A_1 & A_2\\ 0 & 0\end{bmatrix}:\begin{bmatrix}\mathcal{R}(B)\\ \mathcal{N}(B^*)\end{bmatrix}\to\begin{bmatrix}\mathcal{R}(A)\\ \mathcal{N}(A^*)\end{bmatrix},$$

where $D=A_1A_1^*+A_2A_2^*$ is invertible and positive in $\mathcal{L}(\mathcal{R}(A))$. Then

$$A^\dagger=\begin{bmatrix}A_1^*D^{-1} & 0\\ A_2^*D^{-1} & 0\end{bmatrix}:\begin{bmatrix}\mathcal{R}(A)\\ \mathcal{N}(A^*)\end{bmatrix}\to\begin{bmatrix}\mathcal{R}(B)\\ \mathcal{N}(B^*)\end{bmatrix}.$$

(a)⇔(c)⇔(d): Easy computations shows that statements (a), (c) and (d) are equivalent to: $(A_1B_1)^\dagger=B_1^{-1}A_1^*D^{-1}$, $(D^{-1}A_1B_1)^\dagger=B_1^{-1}A_1^*$ and $(A_1(B_1^*)^{-1})^\dagger=B_1^*A_1^*D^{-1}$, respectively. Each of them is further equivalent to the following

$$A_1A_1^*D^{-1}A_1=A_1,\quad [A_1A_1^*,D^{-1}]=0,\quad [B_1B_1^*,A_1^*D^{-1}A_1]=0.$$

Proving the statements: (a)⇒(f) and (a)⇒(i)–(j) are very similar, so we will show it only on the case (a)⇒(i). Using the decomposition described above, it is easy to conclude that (i) becomes

$$(A_1B_1)^\dagger=(D^{1/2}A_1B_1)^\dagger D^{1/2},$$
$$(D^{1/2}A_1B_1)^\dagger D^{-1/2}A_i=B_1^{-1}A_1^*D^{-2}A_i,\quad i=1,2.$$

Now we will show that (a) implies the first statement, by checking all four Penrose equations. For the first and the second equation, it is clear. Let us check the third and fourth.

$$\begin{aligned}(III)\ D^{1/2}A_1B_1(A_1B_1)^\dagger D^{-1/2}&=D^{1/2}A_1B_1B_1^{-1}A_1^*D^{-1}D^{-1/2}\\&=D^{1/2}A_1A_1^*D^{-1}D^{-1/2},\end{aligned}$$

which is, under the premise (a), Hermitian.

$$\begin{aligned}(IV)\ (A_1B_1)^\dagger D^{-1/2}D^{1/2}A_1B_1&=B_1^{-1}A_1^*D^{-1}D^{-1/2}D^{1/2}A_1B_1\\&=B_1^{-1}A_1^*D^{-1}A_1B_1,\end{aligned}$$

For the sake of completeness, we enlist the equivalent forms for (f) and (j)

$$\begin{aligned}(f):(A_1B_1)^\dagger&=B_1^{-1}(D^{-1/2}A_1)^\dagger D^{-1/2},\\ (D^{-1/2}A_1)^\dagger D^{-1/2}A_i&=A_1^*D^{-1}A_i,\quad i=1,2;\end{aligned}$$

and

$$\begin{aligned}(j):(A_1B_1)^\dagger&=B_1^*(A_1B_1B_1^*)^\dagger,\\ (D^{1/2}A_1B_1)^\dagger D^{-1/2}A_i&=(B_1B_1^*)^{-1}A_1^*D^{-1},\quad i=1,2.\end{aligned}$$

The proof that (a)⇒(k)–(m) will be omitted here, because it will be found later, in Theorem 3.2, for more general case.

Now, it remains only part:

$(b)\Rightarrow(a)$: If we use matrix forms for the operators A and B as before in the proof, it actually remains to prove that

$$(A_1B_1)^\dagger=B_1^{-1}A_1^*D^{-1}A_1A_1^*D^{-1}\Rightarrow(A_1B_1)^\dagger=B_1^{-1}A_1^*D^{-1}.$$

Let us denote $W = A_1^* D^{-1} A_1$. For the expression $(A_1B_1)^\dagger = B_1^{-1}A_1^*D^{-1}A_1A_1^*D^{-1}$, proper Penrose equations are the following

(1) $A_1 = A_1W^2$;
(2) $W^3A_1^* = WA_1^*$;
(3) $[A_1WA_1^*, D^{-1}] = 0$;
(4) $[B_1B_1^*, W^2] = 0$.

On the other side, Penrose equations for $(A_1B_1)^\dagger = B_1^{-1}A_1^*D^{-1}$ are the following

(1) $A_1 = A_1W$;
(2) $WA_1^* = A_1^*$;
(3) $[A_1A_1^*, D^{-1}] = 0$;
(4) $[B_1B_1^*, W] = 0$,

The operator W is Hermitian, moreover-it is positive ($W = T^*T$, where $T = D^{-1/2}A_1$), hence $I + W$ is invertible, so we have

$$\begin{aligned} A_1 = A_1W^2 &\Leftrightarrow A_1(I - W^2) = 0 \Leftrightarrow A_1(I - W)(I + W) = 0 \\ &\Rightarrow A_1(I - W) = 0, \end{aligned}$$

which means $A_1 = A_1W$.

By using this fact, we proved (b)⇒(a), and therefore the proof is completed. □

The next theorem presents one possible way for generalization of some statements from the previous theorem.

Theorem 3.2. *Let $A \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$ and $B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be bounded linear operators, such that A, B and AB have closed ranges. Let m and n be arbitrary nonnegative integers. The following statements are equivalent*

(a) $(AB)^\dagger = B^\dagger A^\dagger$;
(l') $(AB)^\dagger = (B^*B)^n((AA^*)^mAB(B^*B)^n)^\dagger(AA^*)^m$ *and* $((AA^*)^mAB(B^*B)^n)^\dagger = (B(B^*B)^n)^\dagger((AA^*)^mA^*)^\dagger$;
(m') $(AB)^\dagger = B^*(BB^*)^n((A^*A)^{m+1}(BB^*)^{n+1})^\dagger(A^*A)^mA^*$ *and* $((A^*A)^{m+1}(BB^*)^{n+1})^\dagger = ((BB^*)^\dagger)^{n+1}((A^*A)^\dagger)^{m+1}$.

Proof. First, we show the existence of the operators $((A^*A)^{m+1}(BB^*)^{n+1})^\dagger$ and $((AA^*)^mAB(B^*B)^n)^\dagger$. By Lemma 1.4, if P and Q are closed-range operators, then PQ is closed-range if and only if $P^\dagger PQQ^\dagger$ is closed range. Let we put $P = (A^*A)^m$, $Q = (BB^*)^n$. They are closed-range as a powers of Hermitian closed-range operators A^*A and BB^*. Let us compute $P^\dagger PQQ^\dagger$

$$\begin{aligned} P^\dagger PQQ^\dagger &= ((A^*A)^m)^\dagger(A^*A)^m(BB^*)^n((BB^*)^n)^\dagger \\ &= ((A^*A)^\dagger)^m(A^*A)^m(BB^*)^n((BB^*)^\dagger)^n = A^\dagger ABB^\dagger, \end{aligned}$$

which is a closed-range operator because of Lemma 1.4. Thus, we proved that $(A^*A)^{m+1}(BB^*)^{n+1}$ has closed range, which implies the existence of its Moore–Penrose inverse.

Let us now put $P=(AA^*)^mA$, $Q=B(B^*B)^n$. By computing $P^\dagger PQQ^\dagger$

$$\begin{aligned}P^\dagger PQQ^\dagger &= ((AA^*)^mA)^\dagger(AA^*)^mAB(B^*B)^n(B(B^*B)^n)^\dagger\\ &= A^\dagger((A^*)^\dagger A^\dagger)^m(AA^*)^mAB(B^*B)^n(B^\dagger(B^*)^\dagger)^nB^\dagger = A^\dagger ABB^\dagger,\end{aligned}$$

we conclude using Lemma 1.4 that it is a closed range operator, which implies $(AA^*)^mAB(B^*B)^n$ has a closed range, and because of that the Moore–Penrose inverse.

We made preparations, and now the proof starts.

$$\begin{aligned}(l')\Rightarrow(a):\ (AB)^\dagger &= (B^*B)^n((AA^*)^mAB(B^*B)^n)^\dagger(AA^*)^m\\ &= (B^*B)^n(B(B^*B)^n)^\dagger((AA^*)^mA)^\dagger(AA^*)^m\\ &= (B^*B)^n((B^*B)^\dagger)^nB^\dagger A^\dagger((AA^*)^\dagger)^m(AA^*)^m = B^\dagger A^\dagger,\end{aligned}$$

where we used the following fact: if H is hermitian, then $H^2H^\dagger = H = H^\dagger H^2$

$(a)\Rightarrow(l')$: Using the decompositions

$$A=\begin{bmatrix}A_1 & A_2\\ 0 & 0\end{bmatrix},\quad B=\begin{bmatrix}B_1 & 0\\ 0 & 0\end{bmatrix},$$

the implication becomes

$$(A_1B_1)^\dagger = B_1^{-1}A_1^*D^{-1}\Rightarrow\begin{cases}(A_1B_1)^\dagger = (B_1^*B_1)^n(D^mA_1B_1(B_1^*B_1)^n)^\dagger D^m,\\ (D^mA_1B_1(B_1^*B_1)^n)^\dagger = (B_1^*B_1)^{-n}B_1^{-1}A_1^*D^{-(m+1)}.\end{cases}$$

We can easily prove that $(D^mA_1B_1(B_1^*B_1)^n)^\dagger = (B_1^*B_1)^{-n}(A_1B_1)^\dagger D^{-m}$, by immediately checking four Penrose equations under the premise $(A_1B_1)^\dagger = B_1^{-1}A_1^*D^{-1}$.

The second part is now clear

$$(D^mA_1B_1(B_1^*B_1)^n)^\dagger = (B_1B_1^*)^{-n}(A_1B_1)^\dagger D^{-m} = (B_1^*B_1)^{-l}B_1^{-1}A_1^*D^{-(m+1)},$$

so we completed this part of the proof.

$(m')\Rightarrow(a)$:

$$\begin{aligned}(AB)^\dagger &= B^*(BB^*)^n((A^*A)^{m+1}(BB^*)^{n+1})^\dagger(A^*A)^mA^*\\ &= (B^*B)^nB^*((BB^*)^\dagger)^{n+1}((A^*A)^\dagger)^{m+1}A^*(AA^*)^m\\ &= (B^*B)^nB^*(BB^*)^\dagger((BB^*)^\dagger)^n((A^*A)^\dagger)^m(A^*A)^\dagger A^*(AA^*)^m\\ &= (B^*B)^{n-1}B^*BB^\dagger((BB^*)^\dagger)^n((A^*A)^\dagger)^mA^\dagger AA^*(AA^*)^{m-1}\\ &= (B^*B)^{n-1}B^*((BB^*)^\dagger)^n((A^*A)^\dagger)^mA^*(AA^*)^{m-1}\\ &= \dots\\ &= B^*(BB^*)^\dagger(A^*A)^\dagger A^* = B^\dagger A^\dagger.\end{aligned}$$

$(a)\Rightarrow(m')$: Here we also use the following decompositions

$$A=\begin{bmatrix}A_1 & A_2\\ 0 & 0\end{bmatrix},\quad \begin{bmatrix}B_1 & 0\\ 0 & 0\end{bmatrix},$$

but in the calculation there are some steps that should be explained.

Let us denote $T = (A^*A)^{m+1}(BB^*)^{n+1}$. It is easier to compute in the following way

$$T = A^*(AA^*)^m AB(B^*B)^n B^* = \begin{bmatrix} A_1^* D^m A_1(B_1B_1^*)^{n+1} & 0 \\ A_2^* D^m A_1(B_1B_1^*)^{n+1} & 0 \end{bmatrix};$$

now,

$$T^\dagger = \begin{bmatrix} (D^{m+1/2}A_1(B_1B_1^*)^{n+1})^\dagger D^{-1/2}A_1 & (D^{m+1/2}A_1(B_1B_1^*)^{n+1})^\dagger D^{-1/2}A_2 \\ 0 & 0 \end{bmatrix}.$$

Remains to find $((A^*A)^\dagger)^{m+1}$. It can be computed on this way

$$(A^*A)^\dagger = A^\dagger(A^\dagger)^* = \begin{bmatrix} A_1^* D^{-1}A_1 & A_1^* D^{-1}A_2 \\ A_2^* D^{-1}A_1 & A_2^* D^{-1}A_2 \end{bmatrix}.$$

It is easy to prove by induction that for arbitrary nonnegative integer k

$$((A^*A)^\dagger)^k = \begin{bmatrix} A_1^* D^{-(k+1)}A_1 & A_1^* D^{-(k+1)}A_2 \\ A_2^* D^{-(k+1)}A_1 & A_2^* D^{-(k+1)}A_2 \end{bmatrix}.$$

Also, it is clear

$$(A^*A)^{k+1} = A^*(AA^*)^k A = \begin{bmatrix} A_1^* D^k A_1 & A_1^* D^k A_2 \\ A_2^* D^k A_1 & A_2^* D^k A_2 \end{bmatrix}.$$

Now, we have all necessary terms for computing (m') in the terms of A_1, A_2 and B_1. Thus, we should prove that $(A_1B_1)^\dagger = B_1^{-1}A_1^*D^{-1}$ implies

$$\begin{cases} (A_1B_1)^\dagger = B_1^*(B_1B_1^*)^n (D^{m+1/2}A_1B_1B_1^*(B_1B_1^*)^n)^\dagger D^{m+1/2}, \\ (D^{m+1/2}A_1(B_1B_1^*)^{n+1})^\dagger D^{-1/2}A_i = (B_1^*B_1)^{-(n+1)}A_1^*D^{-(m+2)}A_i, \quad i = 1, 2. \end{cases}$$

We can prove the first part is true by checking all four Penrose equations for

$$(D^{m+1/2}A_1B_1B_1^*(B_1B_1^*)^n)^\dagger = (B_1B_1^*)^{-n}(B_1^*)^{-1}(A_1B_1)^\dagger D^{-(m+1/2)},$$

under the premise $(A_1B_1)^\dagger = B_1^{-1}A_1^*D^{-1}$.

Now, the second part

$$\begin{aligned} &(D^{m+1/2}A_1(B_1B_1^*)^{n+1})^\dagger D^{-1/2}A_i \\ &\qquad = (B_1B_1^*)^{-n}(B_1^*)^{-1}(A_1B_1)^\dagger D^{-(m+1/2)}D^{-1/2}A_i \\ &\qquad = (B_1B_1^*)^{-n}(B_1^*)^{-1}B_1^{-1}A_1^*D^{-1}D^{-(m+1/2)}D^{-1/2}A_i \\ &\qquad = (B_1B_1^*)^{-(n+1)}A_1^*D^{-(m+2)}A_i. \end{aligned}$$ □

Remark 3.3. If we put $m = n = 0$ in statement (m'), it becomes (k) from the Theorem 3.1, if $m = 1$, $n = 1$ it becomes (m). Also if we put $m = n = 1$ in (l'), it becomes (l).

The next result is the immediate corollary of the Theorem 3.1 and Theorem 3.2.

Corollary 3.1. *Let $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ has a closed range. The following statements are equivalent*

(a) $(A^2)^\dagger = (A^\dagger)^2$, *namely, A is a bi-dagger*[2]*;*
(b) $((A^\dagger)^*A)^\dagger = A^\dagger A^*$*;*
(c) $(A(A^\dagger)^*)^\dagger = A^*A^\dagger$*;*
(d) $(A^2A^\dagger)^\dagger = A(A^\dagger)^2$ *and* $(A^\dagger A^2)^\dagger = (A^\dagger)^2A$*;*
(e) $(A^2)^\dagger = A^\dagger(A^\dagger A^2A^\dagger)^\dagger A^\dagger$ *and* $(A^\dagger A^2A^\dagger)^\dagger = A(A^\dagger)^2A$*;*
(f) $(A^2)^\dagger = (A^\dagger A^2)^\dagger A^\dagger$ *and* $(A^\dagger A^2)^\dagger = (A^\dagger)^2A$*;*
(g) $(A^2)^\dagger = A^\dagger(A^2A^\dagger)^\dagger$ *and* $(A^2A^\dagger)^\dagger = A(A^\dagger)^2$*;*
(h) $(A^2)^\dagger = (A^*A^2)^\dagger A^*$ *and* $(A^*A^2)^\dagger = A^\dagger(A^*A)^\dagger$*;*
(i) $(A^2)^\dagger = A^*(A^2A^*)^\dagger$ *and* $(A^2A^*)^\dagger = (AA^*)^\dagger A^\dagger$*;*
(j) $(A^2)^\dagger = A^*(A^*A^2A^*)^\dagger A^*$ *and* $(A^*A^2A^*)^\dagger = (AA^*)^\dagger(A^*A)^\dagger$*;*
(k) $(A^2)^\dagger = A^*A(AA^*A^2A^*A)^\dagger AA^*$ *and* $(AA^*A^2A^*A)^\dagger = (AA^*A)^\dagger(AA^*A)^\dagger$*;*
(l) $(A^2)^\dagger = A^*AA^*((A^*A)^2(AA^*)^2)^\dagger A^*AA^*$ *and*
$((A^*A)^2(AA^*)^2)^\dagger = ((AA^*)^2)^\dagger((A^*A)^2)^\dagger$*;*
(m) $(A^2)^\dagger = (A^*A)^n((AA^*)^mA^2(A^*A)^n)^\dagger(AA^*)^m$ *and*
$((AA^*)^mA^2(A^*A)^n)^\dagger = (A(A^*A)^n)^\dagger((AA^*)^mA^*)^\dagger$*;*
(n) $(A^2)^\dagger = A^*(AA^*)^n((A^*A)^{m+1}(AA^*)^{n+1})^\dagger(A^*A)^mA^*$ *and*
$((A^*A)^{m+1}(AA^*)^{n+1})^\dagger = ((AA^*)^\dagger)^{n+1}((A^*A)^\dagger)^{m+1}$.

For the sake of completeness, we shall repeat some results already proven in [21] as the (c)-parts of the Theorems 2.2, 2.3 and 2.4.

Theorem 3.3. *Let $A \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$ and $B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be bounded linear operators, such that A, B and AB have closed ranges. The following statements are equivalent*
(a) $(AB)^\dagger = B^\dagger A^\dagger$*;*
(b1) $AB(AB)^\dagger = ABB^\dagger A^\dagger$ *and* $(AB)^\dagger AB = B^\dagger A^\dagger AB$*;*
(b2) $A^*AB = BB^\dagger A^*AB$ *and* $ABB^* = ABB^*A^\dagger A$*;*
(b3) $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ *and* $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$*;*
(c1) $AB(AB)^\dagger A = ABB^\dagger$ *and* $A^\dagger AB = B(AB)^\dagger AB$*;*
(c2) $[A^*A, BB^\dagger] = 0$ *and* $[A^\dagger A, BB^*] = 0$*;*
(d1) $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger$ *and* $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A$*;*
(d2) $B^\dagger(ABB^\dagger)^\dagger = B^\dagger A^\dagger$ *and* $(A^\dagger AB)^\dagger = B^\dagger A^\dagger$.

The following theorem establishes the connection between the basic reverse order law $(AB)^\dagger = B^\dagger A^\dagger$ and mixed-type reverse order law $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$. This mixed-type reverse order law, thoroughly considered in the paper [15], is presented in the sixth section.

Theorem 3.4. *Let $A \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$ and $B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be such that A, B and AB have closed ranges. Then $(AB)^\dagger = B^\dagger A^\dagger$ if and only if $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$, and AB satisfies any one of the following conditions*
(a) $ABB^\dagger A^\dagger AB = AB$*;*
(b) $B^\dagger A^\dagger ABB^\dagger A^\dagger = B^\dagger A^\dagger$*;*
(c) $[A^\dagger A, BB^\dagger] = 0$*;*

[2]Matrix A is bi-dagger if $(A^\dagger)^2 = (A^2)^\dagger$, without any particular requirements, because the Moore–Penrose inverse for matrices always exists. On the other side, closed-range operator A is bi-dagger if A^2 is closed-range operator and $(A^\dagger)^2 = (A^2)^\dagger$ holds.

(d) $A^\dagger ABB^\dagger$ *is an idempotent;*
(e) $BB^\dagger A^\dagger A$ *is an idempotent;*
(f) $B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger = B^\dagger A^\dagger$*;*
(g) $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$.

Proof. Statements $(a)-(g)$ are mutually equivalent, as it is proved in [21, Theorem 2.1.]. From this result and from the statement (f), Theorem 3.1, the conclusion is easy to obtain. □

Remark that the results from this section were further studied in [30, 52–54].

4. Identities concerning the reverse order law for the Moore–Penrose inverse

As mentioned in the Introduction, some reverse order laws are actually the identities. The motivation was mainly the paper [55] dealing with complex matrix case, as well as the classic paper [9]. Those results are generalized in the paper [17], and they are presented in this section.

Throughout some proofs, we need the following auxiliary result.

Lemma 4.1. *Let $P \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$, $Q \in \mathcal{L}(\mathcal{H}_3, \mathcal{H}_4)$ and $R \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be operators such that P, Q, QP, PR have closed ranges.*
(a) *If Q is invertible, then $(P(QP)^\dagger)^\dagger = QPP^\dagger$.*
(b) *If R is invertible, then $((PR)^\dagger P)^\dagger = P^\dagger PR$.*

Proof. For (a) we verify that the operators $A = P(QP)^\dagger$ and $B = QPP^\dagger$ satisfy the Penrose equations

$$ABA = P(QP)^\dagger QPP^\dagger P(QP)^\dagger = P(QP)^\dagger QP(QP)^\dagger = P(QP)^\dagger = A,$$

$$BAB = QPP^\dagger P(QP)^\dagger QPP^\dagger = QP(QP)^\dagger QPP^\dagger = QPP^\dagger = B,$$

$$AB = P(QP)^\dagger QPP^\dagger = Q^{-1}QP(QP)^\dagger QPP^\dagger = Q^{-1}QPP^\dagger = PP^\dagger \text{ is Hermitian,}$$

$$BA = QPP^\dagger P(QP)^\dagger = QP(QP)^\dagger \text{ is Hermitian.}$$

Equation (b) is verified in a similar manner. Notice that from (a) and (b) we conclude that $P(QP)^\dagger$ and $(PR)^\dagger P$ have closed ranges. □

Now we can prove some results concerning the mixed-type reverse-order law for the Moore–Penrose inverse of a product of two and three Hilbert space operators with closed ranges.

Theorem 4.1. *Let $A \in \mathcal{L}(\mathcal{H}_3, \mathcal{H}_4)$, $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$ and $C \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be the operators, such that A, B, C, AB, ABC have closed ranges. Then the following hold*
(a) $(AB)^\dagger = (A^\dagger AB)^\dagger(ABB^\dagger)^\dagger$*;*
(b) $(AB)^\dagger = [(A^\dagger)^* B]^\dagger (B^\dagger A^\dagger)^* [A(B^\dagger)^*]^\dagger$*;*
(c) $(ABC)^\dagger = (A^\dagger ABC)^\dagger B(ABCC^\dagger)^\dagger$*;*
(d) $(ABC)^\dagger = [(AB)^\dagger ABC]^\dagger B^\dagger [ABC(BC)^\dagger]^\dagger$*;*
(e) $(ABC)^\dagger = [(ABB^\dagger)^\dagger ABC]^\dagger B[ABC(B^\dagger BC)^\dagger]^\dagger$*;*

(f) $(ABC)^\dagger = [(A^\dagger)^* BC]^\dagger (A^\dagger)^* B (C^\dagger)^* [AB(C^\dagger)^*]^\dagger$;
(g) $(ABC)^\dagger = \{[A(B^\dagger)^*]^\dagger ABC\}^\dagger B^* B B^* \{ABC[(B^\dagger)^* C]^\dagger\}^\dagger$;
(h) $(ABC)^\dagger = \{[(AB)^\dagger]^* C\}^\dagger [(AB)^\dagger]^* B^\dagger [(BC)^\dagger]^* \{A[(BC)^\dagger]^*\}^\dagger$.

Proof. According to the Lemmas 1.3 and 1.4, it is easy to conclude that operators A, B and C have the following matrix representations with the respect to the appropriate decompositions of spaces

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1 A_1^* + A_2 A_2^*$ is invertible and positive in $\mathcal{L}(\mathcal{R}(A))$. Then

$$A^\dagger = \begin{bmatrix} A_1^* D^{-1} & 0 \\ A_2^* D^{-1} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}.$$

Moreover,

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where B_1 is invertible. Then

$$B^\dagger = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix}.$$

Finally,

$$C = \begin{bmatrix} C_1 & 0 \\ C_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix},$$

where $E = C_1^* C_1 + C_2^* C_2$ is invertible and positive in $\mathcal{L}(\mathcal{R}(C^*))$. Then

$$C^\dagger = \begin{bmatrix} E^{-1} C_1^* & E^{-1} C_2^* \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix}.$$

(a): Notice that $\mathcal{R}(A^\dagger AB) = A^\dagger A(\mathcal{R}(B)) = A^\dagger A(\mathcal{R}(BB^\dagger)) = \mathcal{R}(A^\dagger ABB^\dagger)$ is closed according to Lemma 1.4. Also, $\mathcal{R}(B^* A^*)$ is closed. Again, from Lemma 1.4 and $\mathcal{R}((ABB^\dagger)^*) = \mathcal{R}((B^*)^\dagger B^* A^*) = \mathcal{R}((B^*)^\dagger B^* A^* (A^*)^\dagger) = \mathcal{R}((A^\dagger ABB^\dagger)^*)$, it follows that $\mathcal{R}(ABB^\dagger)$ is closed. Now, using matrix forms of A and B, we have

$$ABB^\dagger = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad (ABB^\dagger)^\dagger = \begin{bmatrix} A_1^\dagger & 0 \\ 0 & 0 \end{bmatrix}, \quad A^\dagger AB = \begin{bmatrix} A_1^* D^{-1} A_1 B_1 & 0 \\ A_2^* D^{-1} A_1 B_1 & 0 \end{bmatrix},$$

$$\begin{aligned} (A^\dagger AB)^\dagger &= ((A^\dagger AB)^* (A^\dagger AB))^\dagger (A^\dagger AB)^* \\ &= \begin{bmatrix} (B_1^* A_1^* D^{-1} A_1 B_1)^\dagger B_1^* A_1^* D^{-1} A_1 & (B_1^* A_1^* D^{-1} A_1 B_1)^\dagger B_1^* A_1^* D^{-1} A_2 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Therefore, $(AB)^\dagger = (A^\dagger AB)^\dagger (ABB^\dagger)^\dagger$ is equivalent to

$$(A_1 B_1)^\dagger = (B_1^* A_1^* D^{-1} A_1 B_1)^\dagger B_1^* A_1^* D^{-1} A_1 A_1^\dagger,$$

which is further equivalent to

$$(A_1 B_1)^\dagger = (D^{-1/2} A_1 B_1)^\dagger D^{-1/2} A_1 A_1^\dagger.$$

The last equality follows by checking Penrose equations; as a sample we check the second one

$$(A_1B_1)^\dagger A_1B_1(A_1B_1)^\dagger = (D^{-1/2}A_1B_1)^\dagger D^{-1/2}A_1A_1^\dagger A_1B_1(D^{-1/2}A_1B_1)^\dagger D^{-1/2}A_1A_1^\dagger$$
$$= (D^{-1/2}A_1B_1)^\dagger D^{-1/2}A_1A_1^\dagger = (A_1B_1)^\dagger.$$

(b): Notice that $\mathcal{R}(((A^\dagger)^*B)^*) = \mathcal{R}(B^*A^\dagger) = \mathcal{R}(B^*A^*) = \mathcal{R}((AB)^*)$ is closed, so $\mathcal{R}((A^\dagger)^*B)$ is closed. Also, $\mathcal{R}(A(B^\dagger)^*) = \mathcal{R}(A((B^*B)^\dagger B^*)^*) = \mathcal{R}(AB(B^*B)^\dagger) = AB(\mathcal{R}((B^*B)^\dagger)) = AB(\mathcal{R}(B^*)) = A(\mathcal{R}(B)) = \mathcal{R}(AB)$ is closed. Again, using matrix forms of A and B, we have that $(AB)^\dagger = [(A^\dagger)^*B]^\dagger(B^\dagger A^\dagger)^*[A(B^\dagger)^*]^\dagger$ is equivalent to the following

$$(A_1B_1)^\dagger = (D^{-1}A_1B_1)^\dagger D^{-1}A_1(B_1^{-1})^*(A_1(B_1^{-1})^*)^\dagger.$$

The last equality can easily be proved by checking the Penrose equations.

(c): Note that $\mathcal{R}((A^\dagger ABC)^*) = (BC)^*(\mathcal{R}(A^\dagger A)) = (BC)^*(\mathcal{R}(A^*)) = \mathcal{R}((ABC)^*)$ is closed. Also, $\mathcal{R}(ABCC^\dagger) = AB(\mathcal{R}(CC^\dagger) = AB(\mathcal{R}(C)) = \mathcal{R}(ABC)$ is closed. Now we show that $(ABC)^\dagger = (A^\dagger ABC)^\dagger B(ABCC^\dagger)^\dagger$. First we compute factors appearing on the right side. Denote

$$T = A^\dagger ABC = \begin{bmatrix} A_1^*D^{-1}A_1B_1C_1 & 0 \\ A_2^*D^{-1}A_1B_1C_1 & 0 \end{bmatrix}.$$

Now,

$$T^\dagger = (T^*T)^\dagger T^* = \begin{bmatrix} XA_1 & XA_2 \\ 0 & 0 \end{bmatrix},$$

where $X = (C_1^*B_1^*A_1^*D^{-1}A_1B_1C_1)^\dagger C_1^*B_1^*A_1^*D^{-1}$. Let

$$S = ABCC^\dagger = \begin{bmatrix} A_1B_1C_1E^{-1}C_1^* & A_1B_1C_1E^{-1}C_2^* \\ 0 & 0 \end{bmatrix}.$$

It is easy to find

$$S^\dagger = S^*(SS^*)^\dagger = \begin{bmatrix} C_1E^{-1}C_1^*B_1^*A_1^*(A_1B_1C_1E^{-1}C_1^*B_1^*A_1^*)^\dagger & 0 \\ C_2E^{-1}C_1^*B_1^*A_1^*(A_1B_1C_1E^{-1}C_1^*B_1^*A_1^*)^\dagger & 0 \end{bmatrix}.$$

Therefore, the statement (c) is equivalent to

$$(A_1B_1C_1)^\dagger = (C_1^*B_1^*A_1^*D^{-1}A_1B_1C_1)^\dagger C_1^*B_1^*A_1^*D^{-1}A_1$$
$$\times B_1C_1E^{-1}C_1^*B_1^*A_1^*(A_1B_1C_1E^{-1}C_1^*B_1^*A_1^*)^\dagger,$$

i.e.

$$(A_1B_1C_1)^\dagger = (D^{-1/2}A_1B_1C_1)^\dagger D^{-1/2}A_1B_1C_1E^{-1/2}(A_1B_1C_1E^{-1/2})^\dagger.$$

This formula can be proved in an analogous way as in (a).

(f): Notice that $\mathcal{R}(((A^\dagger)^*BC)^*) = (BC)^*(\mathcal{R}(A^\dagger)) = (BC)^*(\mathcal{R}(A^*)) = \mathcal{R}((ABC)^*)$ is closed, so $\mathcal{R}((A^\dagger)^*BC)$ is closed. Also,

$$\mathcal{R}(AB(C^\dagger)^*) = AB(\mathcal{R}((C^\dagger)^*)) = AB(\mathcal{R}(C)) = \mathcal{R}(ABC)$$

is closed. An easy computation shows that

$$(ABC)^\dagger = [(A^\dagger)^*BC]^\dagger(A^\dagger)^*B(C^\dagger)^*[AB(C^\dagger)^*]^\dagger$$

is equivalent to

$$(A_1B_1C_1)^\dagger = (D^{-1}A_1B_1C_1)^\dagger D^{-1}A_1B_1C_1E^{-1}(A_1B_1C_1E^{-1})^\dagger.$$

This equality follows a standard argument.

So far we have proved four identities. Now we use (c), to show that (d), (e) and (g) are satisfied. Also, we use (f) to prove that (h) holds.

(d): An easy computation shows that (d) is equivalent to the following

$$(A_1B_1C_1)^\dagger = [(A_1B_1)^\dagger A_1B_1C_1]^\dagger B_1^{-1}[A_1B_1C_1(B_1C_1)^\dagger]^\dagger.$$

If we put: $A' = A_1B_1$, $B' = B_1^{-1}$, $C' = B_1C_1$, then (d) becomes already proven identity (c) for operators A', B', C'. For completeness, notice that the following operator ranges are closed

$$\begin{aligned} &\mathcal{R}(A') = \mathcal{R}(AB), && \mathcal{R}(B') = \mathcal{R}(B^*), && \mathcal{R}(C') = \mathcal{R}(BC),\\ &\mathcal{R}(A'B') = \mathcal{R}(A), && \mathcal{R}(B'C') = \mathcal{R}(C), && \mathcal{R}(A'B'C') = \mathcal{R}(ABC). \end{aligned}$$

(e): An easy computation shows that (e) is equivalent to the following

$$(A_1B_1C_1)^\dagger = [A_1^\dagger A_1B_1C_1]^\dagger B_1[A_1B_1C_1C_1^\dagger]^\dagger.$$

The last identity is proved in (c).

(g): An easy computation shows that

$$(ABC)^\dagger = \{[A(B^\dagger)^*]^\dagger ABC\}^\dagger B^*BB^*\{ABC[(B^\dagger)^*C]^\dagger\}^\dagger$$

is equivalent to the following

$$(A_1B_1C_1)^\dagger = \{[A_1(B_1^*)^{-1}]^\dagger A_1B_1C_1\}^\dagger B_1^*B_1B_1^*\{A_1B_1C_1[(B_1^*)^{-1}C_1]^\dagger\}^\dagger.$$

We put: $A'' := A_1(B_1^*)^{-1}$, $B'' := B_1^*B_1B_1^*$, $C'' := (B_1^*)^{-1}C_1$. Now we have that the following operator ranges are closed

$$\begin{gathered} \mathcal{R}(A'') = A_1(\mathcal{R}((B_1^*)^{-1})) = \mathcal{R}(AB), \quad \mathcal{R}(B'') = \mathcal{R}(B^*),\\ \mathcal{R}(C'') = \mathcal{R}(BC), \quad \mathcal{R}(A''B'') = \mathcal{R}(A_1B_1B_1^*) = \mathcal{R}(AB),\\ \mathcal{R}((B''C'')^*) = \mathcal{R}((B^*BC)^*) = C^*(\mathcal{R}(B^*B)) = \mathcal{R}((BC)^*),\\ \mathcal{R}(A''B''C'') = \mathcal{R}(ABC). \end{gathered}$$

So, conditions of identity (c) are satisfied. Hence, (g) follows from (c).

(h): An easy computation shows that (h) is equivalent to the following

$$(A_1B_1C_1)^\dagger = \{[(A_1B_1)^\dagger]^*C_1\}^\dagger[(A_1B_1)^\dagger]^*B_1^{-1}[(B_1C_1)^\dagger]^*\{A_1[(B_1C_1)^\dagger]^*\}^\dagger.$$

If we put: $A''' := A_1B_1$, $B''' := B_1^{-1}$, $C''' := B_1C_1$, then (h) becomes already proven identity (f). For the completeness, notice that the following operator ranges are closed

$$\begin{aligned} &\mathcal{R}(A''') = \mathcal{R}(AB), && \mathcal{R}(B''') = \mathcal{R}(B^*), && \mathcal{R}(C''') = \mathcal{R}(BC),\\ &\mathcal{R}(A'''B''') = \mathcal{R}(A), && \mathcal{R}(B'''C''') = \mathcal{R}(C), && \mathcal{R}(A'''B'''C''') = \mathcal{R}(ABC). \quad \square \end{aligned}$$

Remark 4.1. The existence of the Moore–Penrose inverses of various operators in the previous theorem, follows from the closedness of operator ranges $\mathcal{R}(A)$, $\mathcal{R}(B)$, $\mathcal{R}(C)$, $\mathcal{R}(AB)$, $\mathcal{R}(BC)$, $\mathcal{R}(ABC)$. This fact is important, and it is explained in detail. The same is true for the rest of the theorems.

The next two corollaries are immediate consequences of Theorem 4.1 (a).

Corollary 4.1. *Let $A \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$, $B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be operators, such that A, B, AB have closed ranges. If $A^\dagger AB = B$ and $ABB^\dagger = A$, then $(AB)^\dagger = B^\dagger A^\dagger$.*

Corollary 4.2. *If P and Q be two orthogonal projectors (i.e. $P^2 = P = P^*$ and $Q^2 = Q = Q^*$), then $(PQ)^\dagger$ is an idempotent.*

Moreover, all other corollaries from [55] are also true with some slight changes in their formulations.

If U, V are operators acting on the same space, recall that $[U, V] = UV - VU$ is the usual notation for their commutator.

Theorem 4.2. *Let $A \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$, $B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be operators, such that A, B, AB have closed ranges. Let $M \in \mathcal{L}(\mathcal{H}_3)$ and $N \in \mathcal{L}(\mathcal{H}_1)$ be positive and invertible operators. Then the weighted Moore–Penrose inverse of AB with respect to M and N satisfies the following two identities*

(a) $(AB)^\dagger_{M,N} = (A^\dagger AB)^\dagger_{I,N}(ABB^\dagger)^\dagger_{M,I}$;

(b) $(AB)^\dagger_{M,N} = [(A^\dagger_{M,I})^* B]^\dagger_{M^{-1},N}(B^\dagger_{I,N}A^\dagger_{M,I})^*[A(B^\dagger_{I,N})^*]^\dagger_{M,N^{-1}}$.

Proof. By using well-known relation $A^\dagger_{M,N} = N^{-1/2}(M^{1/2}AN^{-1/2})^\dagger M^{1/2}$, it is easy to obtain that (a) is equivalent to

$$(M^{1/2}ABN^{-1/2})^\dagger = (A^\dagger ABN^{-1/2})^\dagger (M^{1/2}ABB^\dagger)^\dagger. \tag{4.1}$$

Let $\tilde{A} = M^{1/2}A$, $\tilde{B} = BN^{-1/2}$. We prove that $(M^{-1/2}\tilde{A})^\dagger = \tilde{A}^\dagger M^{1/2}$. The last statement holds if and only if $M^{-1/2}\tilde{A}\tilde{A}^\dagger M^{1/2}$ is Hermitian, which is equivalent to $[M, \tilde{A}\tilde{A}^\dagger] = 0$. Using Lemma 1.5, the last expression is equivalent to $\mathcal{R}(M\tilde{A}) = \mathcal{R}(\tilde{A})$, which is valid, because of the invertibility of the Hermitian operator M. Analogously we prove that $(\tilde{B}N^{1/2})^\dagger = N^{-1/2}\tilde{B}^\dagger$. Now, (4.1) becomes

$$(\tilde{A}\tilde{B})^\dagger = (\tilde{A}^\dagger \tilde{A}\tilde{B})^\dagger (\tilde{A}\tilde{B}\tilde{B}^\dagger)^\dagger,$$

which is already proven identity in Theorem 4.1 (a).

Analogously, we prove statement (b). □

Theorem 4.3. *Let $A \in \mathcal{L}(\mathcal{H}_3, \mathcal{H}_4)$, $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$, $C \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be operators, such that A, B, C, AB, BC, ABC have closed ranges. Let $M \in \mathcal{L}(\mathcal{H}_4)$ and $N \in \mathcal{L}(\mathcal{H}_1)$ be positive and invertible operators. Then the weighted Moore–Penrose inverse of ABC with respect to M and N satisfies the following identities*

(a) $(ABC)^\dagger_{M,N} = (A^\dagger ABC)^\dagger_{I,N} B (ABCC^\dagger)^\dagger_{M,I}$;

(b) $(ABC)^\dagger_{M,N} = ((AB)^\dagger ABC)^\dagger_{I,N} B^\dagger (ABC(BC)^\dagger)^\dagger_{M,I}$;

(c) $(ABC)^\dagger_{M,N} = ((ABB^\dagger)^\dagger ABC)^\dagger_{I,N} B (ABC(B^\dagger BC)^\dagger)^\dagger_{M,I}$;

(d) $(ABC)^\dagger_{M,N} = [(A^\dagger_{M,I})^* BC]^\dagger_{M^{-1},N}(A^\dagger_{M,I})^* B (C^\dagger_{I,N})^* [AB(C^\dagger_{I,N})^*]^\dagger_{M,N^{-1}}$;

(e) $(ABC)^{\dagger}_{M,N} = \{[A(B^{\dagger})^{*}]^{\dagger}ABC\}^{\dagger}_{I,N}B^{*}BB^{*}\{ABC[(B^{\dagger})^{*}C]^{\dagger}\}^{\dagger}_{M,I}$;

(f) $(ABC)^{\dagger}_{M,N} = \{[(AB)^{\dagger}_{M,I}]^{*}C\}^{\dagger}_{M^{-1},N}[(AB)^{\dagger}_{M,I}]^{*}B^{\dagger}[(BC)^{\dagger}_{I,N}]^{*} \times \{A[(BC)^{\dagger}_{I,N}]^{*}\}^{\dagger}_{M,N^{-1}}$.

Proof. The proof in all cases is similar to the proof of Theorem 4.2. First, we transform all weighted Moore–Penrose inverses to the ordinary ones, then we put: $\tilde{A} = M^{1/2}A$, $\tilde{B} = B$, $\widetilde{C} = CN^{-1/2}$, and apply Lemma 1.5. After that, all cases reduce to already-proven identities from Theorem 4.1. □

Some more general identities can also be derived from previous theorems.

Theorem 4.4. *Let $A \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$, $B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be operators, such that A, B, AB have closed ranges. Let $M \in \mathcal{L}(\mathcal{H}_3)$, $N \in \mathcal{L}(\mathcal{H}_1)$ and $P \in \mathcal{L}(\mathcal{H}_2)$ be positive and invertible operators. Then the weighted Moore–Penrose inverse of AB with respect to M and N satisfies the following identity*

$$(AB)^{\dagger}_{M,N} = (A^{\dagger}_{I,P}AB)^{\dagger}_{P,N}(ABB^{\dagger}_{P,I})^{\dagger}_{M,P}.$$

Proof. The proof is similar to the proof of Theorem 4.2. First, we transform all weighted Moore–Penrose inverses to the ordinary ones, which gives

$$(M^{1/2}ABN^{-1/2})^{\dagger} = [(AP^{-1/2})^{\dagger}ABN^{-1/2}]^{\dagger}[M^{1/2}AB(P^{1/2}B)^{\dagger}]^{\dagger}.$$

If we put: $\tilde{A} = M^{1/2}AP^{-1/2}$, $\tilde{B} = P^{1/2}BN^{-1/2}$, and then apply Lemma 1.5, this statement reduces to the already-proven identity from Theorem 4.1 (a). □

The following theorem can be proven similarly.

Theorem 4.5. *Let $A \in \mathcal{L}(\mathcal{H}_3, \mathcal{H}_4)$, $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$, $C \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be operators, such that A, B, C, AB, BC, ABC have closed ranges. Let $M \in \mathcal{L}(\mathcal{H}_4)$, $N \in \mathcal{L}(\mathcal{H}_1)$, $P \in \mathcal{L}(\mathcal{H}_2)$, $Q \in \mathcal{L}(\mathcal{H}_3)$ be positive and invertible operators. Then the weighted Moore–Penrose inverse of ABC with respect to M and N satisfies the following identities*

(a) $(ABC)^{\dagger}_{M,N} = (A^{\dagger}_{I,P}ABC)^{\dagger}_{P,N}B(ABCC^{\dagger}_{Q,I})^{\dagger}_{M,Q}$;

(b) $(ABC)^{\dagger}_{M,N} = ((AB)^{\dagger}_{I,Q}ABC)^{\dagger}_{Q,N}B^{\dagger}_{P,Q}(ABC(BC)^{\dagger}_{P,I})^{\dagger}_{M,P}$;

(c) $(ABC)^{\dagger}_{M,N} = ((ABB^{\dagger}_{P,I})^{\dagger}_{M,P}ABC)^{\dagger}_{P,N}B(ABC(B^{\dagger}_{I,Q}BC)^{\dagger}_{Q,N})^{\dagger}_{M,Q}$.

Now, we return to one classic matrix identity from [9]. Our next theorem shows that the result from [9] holds for bounded linear Hilbert space operators.

Theorem 4.6. *Let $A \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$, $B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be operators such that A, B, AB have closed ranges. Then $(AB)^{\dagger} = (A^{\dagger}AB)^{\dagger}(AB(A^{\dagger}AB)^{\dagger})^{\dagger}$.*

Proof. Using a method described in Theorem 4.1 (and decompositions and matrix forms for A and B) we conclude that $(AB)^{\dagger} = (A^{\dagger}AB)^{\dagger}(AB(A^{\dagger}AB)^{\dagger})^{\dagger}$ is equivalent to the following (D is positive and invertible as in Lemma 1.3)

$$(A_1B_1)^{\dagger} = (D^{-1/2}A_1B_1)^{\dagger}(A_1B_1(D^{-1/2}A_1B_1)^{\dagger})^{\dagger},$$

which is, by Lemma 4.1, further equivalent to

$$(A_1B_1)^{\dagger} = (D^{-1/2}A_1B_1)^{\dagger}D^{-1/2}A_1B_1(A_1B_1)^{\dagger}.$$

We check directly all four Penrose equations, so we have the proof. □

We mention that results from this section are further investigated e.g. in [8].

5. Hartwig's triple reverse order law revisited

The classical result of Hartwig [26] deals with the triple reverse order law of the form

$$(ABC)^\dagger = C^\dagger B^\dagger A^\dagger, \tag{5.1}$$

where A, B, C are complex matrices of appropriate dimensions. Hartwig established several equivalent conditions such that (5.1) holds, offering a very general proof of the main result. However, one implication in [26] is not valid in infinite dimensional Hilbert spaces, and thus we find it interesting to extend Hartwig's proof in this direction. The results presented in this section were published in 2014. in [18].

Lemma 5.1. *Let $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$ be closed range operator and let P_M be orthogonal projection from $\mathcal{K}$ to closed subspace $\mathcal{R}(M) \subset \mathcal{R}(A)$. Then A^*P_MA has a closed range.*

Proof. According to Lemma 1.2 and Lemma 1.3, operators A and P_M have the following forms

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(M) \\ \mathcal{N}(M) \end{bmatrix},$$
$$P_M = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(M) \\ \mathcal{N}(M) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(M) \\ \mathcal{N}(M) \end{bmatrix}.$$

It is obvious that $A^*P_MA = (P_MA)^*P_MA$, and by using Lemma 1.1 it is enough to prove that P_MA is a closed range operator. From the form of P_MA

$$P_MA = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(M) \\ \mathcal{N}(M) \end{bmatrix},$$

we have $\mathcal{R}(P_MA) = \mathcal{R}(A_1) = A_1(\mathcal{R}(A^*))$, which is closed because A_1 is onto. Indeed, let us suppose A_1 is not onto; this means there is some $y \in \mathcal{R}(M)\backslash\mathcal{R}(A_1)$. Because of $\mathcal{R}(M) \subset \mathcal{R}(A)$, there is some $x \in \mathcal{R}(A^*)$ such that $y = A_1x + A_2x$, provided that $A_2x \neq 0$. Therefore, $\mathcal{R}(M) \ni y - A_1x = A_2x \in \mathcal{N}(M)$, and sum $\mathcal{R}(M) \oplus \mathcal{N}(M)$ is direct, so $A_2x = 0$, which is contradiction. Therefore, A_1 is onto. □

Let $A \in \mathcal{L}(\mathcal{H}_3,\mathcal{H}_4)$, $B \in \mathcal{L}(\mathcal{H}_2,\mathcal{H}_3)$ and $C \in \mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$ be bounded linear operators with closed ranges. We use notations in the same way as in [26]

$$M = ABC, \quad X = C^\dagger B^\dagger A^\dagger, \quad E = A^\dagger A, \quad F = CC^\dagger, \quad P = EBF, \quad Q = FB^\dagger E.$$

Recall that $K \in \mathcal{L}(\mathcal{H})$ is EP, if K has a closed range and $KK^\dagger = K^\dagger K$.

The main result is the following theorem.

Theorem 5.1. *Let A, B, C be closed-range operators such that ABC also has a closed range. The following statements are equivalent*

(a) $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$;
(b) $PQP = P$, $QPQ = Q$, *and both* A^*APQ, $QPCC^*$ *are Hermitian;*
(c) $PQP = P$, $QPQ = Q$, *and both* A^*APQ, $QPCC^*$ *are EP;*
(d) $PQP = P$, $\mathcal{R}(A^*AP) = \mathcal{R}(Q^*)$, $\mathcal{R}(CC^*P^*) = \mathcal{R}(Q)$;
(e) $(PQ)^2 = PQ$, $\mathcal{R}(A^*AP) = \mathcal{R}(Q^*)$, $\mathcal{R}(CC^*P^*) = \mathcal{R}(Q)$.

Remark 5.1. In [26] Hartwig made the following remark

> The results of Theorem 1 can be extended to a regular ring $\mathcal{R}$, with involution $(\cdot)^*$ and unit 1 for which $ab = 1 \Rightarrow ba = 1$ and $a^*a = 0 \Rightarrow a = 0$ hold.

Since for bounded linear operators on infinite dimensional Hilbert spaces the implication $AB = I \Rightarrow BA = I$ does not hold, we find it important to finish this proof in more general settings.

Proof. The proof given by Hartwig stays valid for (a)⇔(b), (b)⇒(c), (c)⇒(d) and (d)⇒(e). The only case which does not hold in general is actually the implication (e)⇒(b), which involves properties of the matrix rank. Thus, this part of the proof is not applicable to operators on infinite dimensional Hilbert space.

To complete the proof, we will prove (e)⇒(a) in a different way, using properties of operator matrices.

Using Lemma 1.2 we conclude that the operator C has the following matrix form

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{bmatrix},$$

where C_1 is invertible. Then

$$C^\dagger = \begin{bmatrix} C_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(C^*) \\ \mathcal{N}(C) \end{bmatrix}.$$

From Lemma 1.3 it follows that the operator B has the following matrix form

$$B = \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(C) \\ \mathcal{N}(C^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where $G = B_1B_1^* + B_2B_2^*$ is invertible and positive in $\mathcal{L}(\mathcal{R}(B))$. Then

$$B^\dagger = \begin{bmatrix} B_1^*G^{-1} & 0 \\ B_2^*G^{-1} & 0 \end{bmatrix}.$$

From Lemma 1.3 it also follows that the operator A has the following matrix form

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1A_1^* + A_2A_2^*$ is invertible and positive in $\mathcal{L}(\mathcal{R}(A))$. Then

$$A^\dagger = \begin{bmatrix} A_1^*D^{-1} & 0 \\ A_2^*D^{-1} & 0 \end{bmatrix}.$$

Let us find the expressions for the operators M, X, E, F, P and Q. It is easy to find that

$$M = ABC = \begin{bmatrix} A_1B_1C_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_1 = A_1B_1C_1;$$

$$X = C^\dagger B^\dagger A^\dagger = \begin{bmatrix} C_1^{-1}B_1^*G^{-1}A_1^*D^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$X_1 = C_1^{-1}B_1^*G^{-1}A_1^*D^{-1};$$

$$E = A^\dagger A = \begin{bmatrix} A_1^*D^{-1}A_1 & A_1^*D^{-1}A_2 \\ A_2^*D^{-1}A_1 & A_2^*D^{-1}A_2 \end{bmatrix}; \quad F = CC^\dagger = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix};$$

$$P = EBF = \begin{bmatrix} A_1^*D^{-1}A_1B_1 & 0 \\ A_2^*D^{-1}A_1B_1 & 0 \end{bmatrix} = \begin{bmatrix} A_1^*D^{-1}M_1C_1^{-1} & 0 \\ A_2^*D^{-1}M_1C_1^{-1} & 0 \end{bmatrix};$$

$$Q = FB^\dagger E = \begin{bmatrix} B_1^*G^{-1}A_1^*D^{-1}A_1 & B_1^*G^{-1}A_1^*D^{-1}A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} C_1X_1A_1 & C_1X_1A_2 \\ 0 & 0 \end{bmatrix}.$$

It will be convenient to compute here matrix forms for some expressions appearing in the rest of the proof

$$PQ = \begin{bmatrix} A_1^*D^{-1}M_1X_1A_1 & A_1^*D^{-1}M_1X_1A_2 \\ A_2^*D^{-1}M_1X_1A_1 & A_2^*D^{-1}M_1X_1A_2 \end{bmatrix};$$

$$QP = \begin{bmatrix} C_1X_1M_1C_1^{-1} & 0 \\ 0 & 0 \end{bmatrix};$$

$$A^*AP = \begin{bmatrix} A_1^*M_1C_1^{-1} & 0 \\ A_2^*M_1C_1^{-1} & 0 \end{bmatrix};$$

$$CC^*P^* = \begin{bmatrix} C_1M_1^*D^{-1}A_1 & C_1M_1^*D^{-1}A_2 \\ 0 & 0 \end{bmatrix};$$

$$(PQ)^2 = \begin{bmatrix} A_1^*D^{-1}M_1X_1M_1X_1A_1 & A_1^*D^{-1}M_1X_1M_1X_1A_2 \\ A_2^*D^{-1}M_1X_1M_1X_1A_1 & A_2^*D^{-1}M_1X_1M_1X_1A_2 \end{bmatrix}.$$

Now, we will find equivalent expressions for the conditions (a) and (e) in the terms of the components of the operators A, B and C.

(a) : This is equivalent to $(A_1B_1C_1)^\dagger = C_1^{-1}B_1^*G^{-1}A_1^*D^{-1}$, or $M_1^\dagger = X_1$.

(e): This is equivalent to the following three expressions

(e.1) $\Leftrightarrow A_i^*D^{-1}(M_1X_1)^2A_j = A_i^*D^{-1}M_1X_1A_j$, for all $i, j \in \{1, 2\}$;

(e.2) $\Leftrightarrow \mathcal{R}\left(\begin{bmatrix} A_1^*M_1C_1^{-1} & 0 \\ A_2^*M_1C_1^{-1} & 0 \end{bmatrix}\right) = \mathcal{R}\left(\begin{bmatrix} A_1^*X_1^*C_1^* & 0 \\ A_2^*X_1^*C_1^* & 0 \end{bmatrix}\right)$;

(e.3) $\Leftrightarrow \mathcal{R}\left(\begin{bmatrix} C_1M_1^*D^{-1}A_1 & C_1M_1^*D^{-1}A_2 \\ 0 & 0 \end{bmatrix}\right) = \mathcal{R}\left(\begin{bmatrix} C_1X_1A_1 & C_1X_1A_2 \\ 0 & 0 \end{bmatrix}\right)$.

Recall that we prove the implication (e)$\Rightarrow$(a).

Now, if we premultiply (e.1) by A_i, and use summation over $i = 1, 2$ we yield $(M_1X_1)^2A_j = M_1X_1A_j$, for $j = 1, 2$. If we now postmultiply the last expression by A_j^* and add them, we have $(M_1X_1)^2 = M_1X_1$. Therefore

$$(\text{e.1}) \Rightarrow (M_1X_1)^2 = M_1X_1.$$

On the other hand, (e.2) is equivalent to: $\mathcal{R}(A_i^*M_1C_1^{-1}) = \mathcal{R}(A_i^*X_1^*C_1^*)$, $i = 1, 2$. Again, if A_i acts on both sides, and we add them, we obtain $\mathcal{R}(M_1C_1^{-1}) = \mathcal{R}(X_1^*C_1^*)$. Hence, we have $\mathcal{R}(M_1) = \mathcal{R}(X_1^*)$, which implies $M_1M_1^\dagger = X_1^\dagger X_1$. Therefore,

$$(\text{e.2}) \Rightarrow M_1M_1^\dagger = X_1^\dagger X_1.$$

Let us now write (e.3) as

$$\mathcal{N}\left(\begin{bmatrix} A_1^*D^{-1}M_1C_1^* & 0 \\ A_2^*D^{-1}M_1C_1^* & 0 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} A_1^*X_1^*C_1^* & 0 \\ A_2^*X_1^*C_1^* & 0 \end{bmatrix}\right).$$

Notice that

$$\mathcal{N}\left(\begin{bmatrix} A_1^*D^{-1}M_1C_1^* & 0 \\ A_2^*D^{-1}M_1C_1^* & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} : \begin{bmatrix} A_1^*D^{-1}M_1C_1^* & 0 \\ A_2^*D^{-1}M_1C_1^* & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\},$$

and we conclude

$$\mathcal{N}\left(\begin{bmatrix} A_1^*D^{-1}M_1C_1^* & 0 \\ A_2^*D^{-1}M_1C_1^* & 0 \end{bmatrix}\right) = \big(\mathcal{N}(A_1^*D^{-1}M_1C_1^*) \cap \mathcal{N}(A_2^*D^{-1}M_1C_1^*)\big) \oplus \mathcal{N}(C^*),$$

which is further equal (easy to see) to $\mathcal{N}(M_1C_1^*) \oplus \mathcal{N}(C^*)$.

With a little effort, we find

$$\begin{aligned} \mathcal{N}\left(\begin{bmatrix} A_1^*X_1^*C_1^* & 0 \\ A_2^*X_1^*C_1^* & 0 \end{bmatrix}\right) &= \big(\mathcal{N}(A_1^*X_1^*C_1^*) \cap \mathcal{N}(A_2^*X_1^*C_1^*)\big) \oplus \mathcal{N}(C^*) \\ &= \mathcal{N}(X_1^*C_1^*) \oplus \mathcal{N}(C^*). \end{aligned}$$

Hence, condition (e.3) implies $\mathcal{N}(M_1C_1^*) = \mathcal{N}(X_1^*C_1^*)$, which is the same as $\mathcal{R}(C_1M_1^*) = \mathcal{R}(C_1X_1)$, or $\mathcal{R}(M_1^*) = \mathcal{R}(X_1)$, or even further: $M_1^\dagger M_1 = X_1X_1^\dagger$. Since we intend to prove (e)⇒(a), it is enough to prove the following implication

$$\big\{(M_1X_1)^2 = M_1X_1, \quad M_1M_1^\dagger = X_1^\dagger X_1, \quad M_1^\dagger M_1 = X_1X_1^\dagger\big\} \Rightarrow M_1^\dagger = X_1.$$

The following completes the proof

$$\begin{aligned} M_1 &= M_1X_1X_1^\dagger = M_1X_1X_1^\dagger X_1X_1^\dagger = M_1X_1M_1M_1^\dagger X_1^\dagger \\ &= M_1X_1M_1X_1X_1^\dagger M_1^\dagger X_1^\dagger = M_1X_1X_1^\dagger M_1^\dagger X_1^\dagger \\ &= M_1M_1^\dagger X_1^\dagger = X_1^\dagger X_1X_1^\dagger = X_1^\dagger. \end{aligned}$$

For the sake of completeness, we remark that operators A^*APQ and $QPCC^*$ from part (c) of our Theorem have closed ranges. It immediately follows from Lemma 5.1 because

$$A^*APQ = A^*MM^\dagger A = A^*P_{\mathcal{R}(M)}A, \; QPCC^* = CM^\dagger MC^* = CP_{\mathcal{R}(M)}C^*. \quad \square$$

Remark that the results from this section were further studied e.g. in [31, 34, 40, 53, 64, 65].

6. Mixed-type reverse order law $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$ and its equivalencies

Many necessary and sufficient condition for $(AB)^\dagger = B^\dagger A^\dagger$ to hold were given in the literature. In the paper of Tian [48], one can find the following important relation

$$(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow (AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger \wedge (A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A.$$

Therefore, it is necessary to seek various equivalent conditions for the expression

$$(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$$

to satisfy. Results presented in this section are from the paper [15] of Dinčić, Djordjević and Mosić, and it represents the generalization of results from [51] to infinite dimensional settings.

We need the following two auxiliary results, which are proven in [33] in the setting of C^*-algebras.

Let $\mathcal{A}$ be a unital C^*–algebra with the unit 1, and let us denote by $\mathcal{P}(\mathcal{A})$ the set of all projections, i.e. $\mathcal{P}(\mathcal{A}) = \{p \in \mathcal{A} : p^2 = p = p^*\}$.

Lemma 6.1. [33] *Let $p, q \in \mathcal{P}(\mathcal{A})$. The following statements are equivalent*

(a) *pq is Moore–Penrose invertible;*
(b) *qp is Moore–Penrose invertible;*
(c) *$(1-p)(1-q)$ is Moore–Penrose invertible;*
(d) *$(1-q)(1-p)$ is Moore–Penrose invertible.*

Lemma 6.2. [33] *Let $p, q \in \mathcal{P}(\mathcal{A})$. If pq is Moore–Penrose invertible, then*

$$(qp)^\dagger = pq - p[(1-p)(1-q)]^\dagger q.$$

We shall use these results in the case of $\mathcal{A} = \mathcal{L}(\mathcal{H})$.

Our main result presents 27 equivalent conditions to the mixed-type reverse order law we are considering here.

Theorem 6.1. *Let $A \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$ and $B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be operators such that A, B and AB have closed ranges. The following statements are equivalent*

(a1) $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$;
(a2) $(AB)^\dagger = B^*(A^*ABB^*)^\dagger A^*$;
(a3) $(AB)^\dagger = B^\dagger A^\dagger - B^\dagger((I-BB^\dagger)(I-A^\dagger A))^\dagger A^\dagger$;
(b1) $((A^\dagger)^*B)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^*$;
(b2) $((A^\dagger)^*B)^\dagger = B^*((A^*A)^\dagger BB^*)^\dagger A^\dagger$;
(b3) $((A^\dagger)^*B)^\dagger = B^\dagger A^* - B^\dagger((I-BB^\dagger)(I-A^\dagger A))^\dagger A^*$;
(c1) $(A(B^\dagger)^*)^\dagger = B^*(A^\dagger ABB^\dagger)^\dagger A^\dagger$;
(c2) $(A(B^\dagger)^*)^\dagger = B^\dagger(A^*A(BB^*)^\dagger)^\dagger A^*$;
(c3) $(A(B^\dagger)^*)^\dagger = B^*A^\dagger - B^*((I-BB^\dagger)(I-A^\dagger A))^\dagger A^\dagger$;
(d1) $(B^\dagger A^\dagger)^\dagger = A(BB^\dagger A^\dagger A)^\dagger B$;
(d2) $(B^\dagger A^\dagger)^\dagger = (A^\dagger)^*((BB^*)^\dagger(A^*A)^\dagger)^\dagger(B^\dagger)^*$;
(d3) $(B^\dagger A^\dagger)^\dagger = AB - A((I-A^\dagger A)(I-BB^\dagger))^\dagger B$;
(e1) $(A^\dagger AB)^\dagger A^\dagger = B^\dagger(ABB^\dagger)^\dagger$;

(e2) $(A^\dagger AB)^\dagger A^* = B^\dagger((A^\dagger)^* BB^\dagger)^\dagger$;
(e3) $(A^\dagger A(B^\dagger)^*)^\dagger A^\dagger = B^*(ABB^\dagger)^\dagger$;
(e4) $(BB^\dagger A^\dagger)^\dagger B = A(B^\dagger A^\dagger A)^\dagger$;
(e5) $(A^* AB)^\dagger A^* = B^*(ABB^*)^\dagger$;
(e6) $((A^* A)^\dagger B)^\dagger A^\dagger = B^*((A^\dagger)^* BB^*)^\dagger$;
(e7) $(A^* A(B^\dagger)^*)^\dagger A^* = B^\dagger(A(BB^*)^\dagger)^\dagger$;
(e8) $B^\dagger((A^*)^\dagger(BB^*)^\dagger)^\dagger = ((A^* A)^\dagger(B^*)^\dagger)^\dagger A^\dagger$;
(e9) $(AA^* ABB^* B)^\dagger = B^\dagger(A^* ABB^*)^\dagger A^\dagger$;
(f1) $(A^\dagger AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger$ *and* $(ABB^\dagger)^\dagger = (A^\dagger ABB^\dagger)^\dagger A^\dagger$;
(f2) $(A^\dagger AB)^\dagger = B^*(A^\dagger ABB^*)^\dagger$ *and* $(ABB^\dagger)^\dagger = (A^* ABB^\dagger)^\dagger A^*$;
(f3) $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A - B^\dagger((I - BB^\dagger)(I - A^\dagger A))^\dagger A^\dagger A$ *and* $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger - BB^\dagger((I - BB^\dagger)(I - A^\dagger A))^\dagger A^\dagger$;
(g1) $\mathcal{R}((AB)^\dagger) = \mathcal{R}(B^\dagger(A^\dagger ABB^\dagger)A^\dagger)$ *and* $\mathcal{R}(((AB)^\dagger)^*) = \mathcal{R}((B^\dagger(A^\dagger ABB^\dagger)A^\dagger)^*)$;
(g2) $\mathcal{R}((AB)^\dagger) = \mathcal{R}(B^\dagger A^\dagger)$ *and* $\mathcal{R}((B^* A^*)^\dagger) = \mathcal{R}((A^*)^\dagger(B^*)^\dagger)$;
(g3) $\mathcal{R}(AA^* AB) = \mathcal{R}(AB)$ *and* $\mathcal{R}(B^* B(AB)^*) = \mathcal{R}((AB)^*)$.

Proof. The existence of various terms appearing in the statements of the theorem follows mainly from the Lemma 1.4, and from some properties of the kernel and range of operators (see Proposition 1.1). The existence of the Moore–Penrose inverse of the products like $(I - BB^\dagger)(I - A^\dagger A)$ follows from Lemma 6.1.

Using Lemma 1.2, we conclude that the operator B has the following matrix form

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where B_1 is invertible. Then

$$B^\dagger = \begin{bmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix}.$$

From Lemma 1.3 also follows that the operator A has the following matrix form

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1 A_1^* + A_2 A_2^*$ is invertible and positive in $\mathcal{L}(\mathcal{R}(A))$. Then

$$A^\dagger = \begin{bmatrix} A_1^* D^{-1} & 0 \\ A_2^* D^{-1} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}.$$

First we find an equivalent form for the statement (a1). We have

$$S = A^\dagger ABB^\dagger = \begin{bmatrix} A_1^* D^{-1} A_1 & 0 \\ A_2^* D^{-1} A_1 & 0 \end{bmatrix},$$

and consequently

$$S^\dagger = (S^* S)^\dagger S^* = \begin{bmatrix} (A_1^* D^{-1} A_1)^\dagger A_1^* D^{-1} A_1 & (A_1^* D^{-1} A_1)^\dagger A_1^* D^{-1} A_2 \\ 0 & 0 \end{bmatrix}.$$

It follows that
$$B^\dagger S^\dagger A^\dagger = \begin{bmatrix} B_1^{-1}(A_1^* D^{-1} A_1)^\dagger A_1^* D^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$
Therefore,
$$(AB)^\dagger = B^\dagger (A^\dagger A B B^\dagger)^\dagger B^\dagger$$
is equivalent to
$$(A_1 B_1)^\dagger = B_1^{-1}(A_1^* D^{-1} A_1)^\dagger A_1^* D^{-1} = B_1^{-1}(D^{-1/2} A_1)^\dagger D^{-1/2}.$$
By checking the Penrose equations, the last formula holds if and only if
$$[B_1 B_1^*, (D^{-1/2}A_1)^\dagger D^{-1/2} A_1] = 0 \quad \text{and} \quad [D, D^{-1/2} A_1 (D^{-1/2} A_1)^\dagger] = 0. \tag{6.1}$$

Hence, the statement (a1) is equivalent to (6.1).

Let us now find some more equivalent statements to the condition (a1). Using Lemma 1.5, we get that (6.1) is equivalent to
$$\mathcal{R}(DA_1) = \mathcal{R}(A_1) \quad \text{and} \quad \mathcal{R}(B_1 B_1^* A_1^*) = \mathcal{R}(A_1^*).$$
or
$$\mathcal{R}(DA_1) = \mathcal{R}(A_1) \quad \text{and} \quad \mathcal{N}(A_1 B_1 B_1^*) = \mathcal{N}(A_1),$$
If we apply Lemma 1.6, for $X = A_1 B_1$, $C = B_1^{-1}$, $B = D^{-1/2}$, the equality
$$(A_1 B_1)^\dagger = B_1^{-1}(D^{-1/2} A_1)^\dagger D^{-1/2}$$
is equivalent to
$$\mathcal{R}(D^{-1} A_1 B_1) = \mathcal{R}(A_1 B_1) \quad \text{and} \quad \mathcal{N}(A_1 B_1 (B_1^* B_1)^{-1}) = \mathcal{N}(A_1 B_1),$$
or
$$\mathcal{R}(D^{-1} A_1 B_1) = \mathcal{R}(A_1 B_1) \quad \text{and} \quad \mathcal{R}((B_1^* B_1)^{-1}(A_1 B_1)^*) = \mathcal{R}((A_1 B_1)^*).$$

Now, we find an equivalent statement to (g3). Conditions
$$\mathcal{R}(AA^*AB) = \mathcal{R}(AB) \quad \text{and} \quad \mathcal{R}(B^*B(AB)^*) = \mathcal{R}((AB)^*)$$
are equivalent to
$$\mathcal{R}(DA_1 B_1) = \mathcal{R}(A_1 B_1) \quad \text{and} \quad \mathcal{R}(B_1^* B_1 (A_1 B_1)^*) = \mathcal{R}((A_1 B_1)^*)$$
which is equivalent to (6.1). Hence, (g3) is equivalent to (a1).

Analogously, the equivalencies: (b1)⇔(g3),(c1)⇔(g3) and (d1)⇔(g3) can be proved.

Let us now prove, for example, (c2)⇔(g3). Using above notations, and
$$T = A^*A(BB^*)^\dagger = \begin{bmatrix} A_1^* A_1 (B_1 B_1^*)^{-1} & 0 \\ A_2^* A_1 (B_1 B_1^*)^{-1} & 0 \end{bmatrix},$$
it is easy to see that
$$\begin{aligned} T^\dagger &= (T^*T)^\dagger T^* \\ &= \begin{bmatrix} (D^{1/2} A_1 (B_1 B_1^*)^{-1})^\dagger D^{-1/2} A_1 & (D^{1/2} A_1 (B_1 B_1^*)^{-1})^\dagger D^{-1/2} A_2 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$
Now, $(A(B^\dagger)^*)^\dagger = B^\dagger (A^*A(BB^*)^\dagger)^\dagger A^*$ if and only if

$$(A_1(B_1^*)^{-1})^\dagger = B_1^{-1}(D^{1/2}A_1(B_1B_1^*)^{-1})^\dagger D^{1/2}.$$

Applying Lemma 1.6, for $X = A_1(B_1^*)^{-1}$, $C = B_1^{-1}$, $B = D^{1/2}$, the last equality is equivalent to

$$\mathcal{R}(DA_1(B_1^*)^{-1}) = \mathcal{R}(A_1(B_1^*)^{-1}) \text{ and } \mathcal{N}(A_1(B_1^*)^{-1}B_1^{-1}(B_1^*)^{-1}) = \mathcal{N}(A_1(B_1^*)^{-1}),$$

i.e.

$$\mathcal{R}(DA_1B_1) = \mathcal{R}(A_1B_1) \quad \text{and} \quad \mathcal{R}(B_1^{-1}A_1^*) = \mathcal{R}((A_1B_1)^*),$$

so we have just proved that (c2) is equivalent to (g3).

Analogously, we prove the equivalencies (a2)$\Leftrightarrow$(g3),(b2)$\Leftrightarrow$(g3) and (d2)$\Leftrightarrow$(g3).

In proving equivalencies including e-statements, there are no other techniques besides those we have already shown in the previous part of the proof.

The table of proper statements is given below as some kind of summary overview, and also for the sake of completeness

(a1) $(A_1B_1)^\dagger = B_1^{-1}(D^{-1/2}A_1)^\dagger D^{-1/2}$;
(a2) $(A_1B_1)^\dagger = B_1^*(D^{1/2}A_1B_1B_1^*)^\dagger D^{1/2}$;
(b1) $(D^{-1}A_1B_1)^\dagger = B_1^{-1}(D^{-1/2}A_1)^\dagger D^{1/2}$;
(b2) $(D^{-1}A_1B_1)^\dagger = B_1^*(D^{-3/2}A_1B_1B_1^*)^\dagger D^{-1/2}$;
(c1) $(A_1(B_1^*)^{-1})^\dagger = B_1^*(D^{-1/2}A_1)^\dagger D^{-1/2}$;
(c2) $(A_1(B_1^*)^{-1})^\dagger = B_1^{-1}(D^{1/2}A_1(B_1B_1^*)^{-1})^\dagger D^{1/2}$;
(d1) $(B_1^{-1}A_1^*D^{-1})^\dagger = D^{1/2}(A_1^*D^{-1/2})^\dagger B_1$;
(d2) $(B_1^{-1}A_1^*D^{-1})^\dagger = D^{-1/2}((B_1B_1^*)^{-1}A_1^*D^{-3/2})^\dagger(B_1^*)^{-1}$;
(e1) $(D^{-1/2}A_1B_1)^\dagger D^{-1/2} = B_1^{-1}A_1^\dagger$;
(e2) $(D^{-1/2}A_1B_1)^\dagger D^{-1/2} = B_1^{-1}(D^{-1}A_1)^\dagger D^{-1}$;
(e3) $(D^{-1/2}A_1(B_1^*)^{-1})^\dagger = B_1^*A_1^\dagger D^{1/2}$;
(e4) $(B_1^{-1}A_1^*D^{-1/2})^\dagger = D^{-1/2}(A_1^*D^{-1})^\dagger B_1$;
(e5) $(D^{1/2}A_1B_1)^\dagger = B_1^*(A_1B_1B_1^*)^\dagger D^{-1/2}$;
(e6) $(D^{-1}A_1B_1B_1^*)^\dagger = (B_1^*)^{-1}(D^{-3/2}A_1B_1)^\dagger D^{-1/2}$;
(e7) $(D^{1/2}A_1(B_1^*)^{-1})^\dagger = B_1^{-1}(A_1(B_1B_1^*)^{-1})^\dagger D^{-1/2}$;
(e8) $(D^{-1}A_1(B_1B_1^*)^{-1})^\dagger = B_1(D^{-3/2}A_1(B_1^*)^{-1})^\dagger D^{-1/2}$;
(e9) $(DA_1B_1B_1^*B_1)^\dagger = B_1^{-1}(D^{1/2}A_1B_1B_1^*)^\dagger D^{-1/2}$.

Each of those statements is equivalent to

$$\mathcal{R}(D^\alpha A_1B_1) = \mathcal{R}(A_1B_1) \quad \text{and} \quad \mathcal{N}(A_1B_1(B_1^*B_1)^\beta) = \mathcal{N}(A_1B_1),$$

for some $\alpha, \beta \in \{-1, 1\}$. More precisely, we have:

α	β	statement
1	1	a2, d1, e3, e6
1	-1	b1, c2, e1, e8
-1	1	b2, c1, e4, e5
-1	-1	a1, d2, e2, e7, e9

Using Lemma 1.5, we have

$$\mathcal{R}(D^\alpha A_1B_1) = \mathcal{R}(A_1B_1) \Leftrightarrow [D^\alpha, A_1B_1(A_1B_1)^\dagger] = 0 \Leftrightarrow [D, A_1B_1(A_1B_1)^\dagger] = 0,$$

and

$$\begin{aligned}\mathcal{N}(A_1B_1(B_1^*B_1)^\beta) = \mathcal{N}(A_1B_1) &\Leftrightarrow \mathcal{R}((B_1^*B_1)^\beta(A_1B_1)^*) = \mathcal{R}((A_1B_1)^*)\\ &\Leftrightarrow [(B_1^*B_1)^\beta, (A_1B_1)^*((A_1B_1)^*)^\dagger] = 0\\ &\Leftrightarrow [(B_1^*B_1)^\beta, (A_1B_1)^\dagger A_1B_1] = 0\\ &\Leftrightarrow [B_1^*B_1, (A_1B_1)^\dagger A_1B_1] = 0,\end{aligned}$$

which means that each statement mentioned in the table above is equivalent to (g3).

Now, we prove the equivalencies $(x3) \Leftrightarrow (x1)$, where $x \in \{a, b, c, d, f\}$. First, we prove (a3)⇔(a1)

$$(a3) \Leftrightarrow (AB)^\dagger = B^\dagger A^\dagger - B^\dagger[(I - BB^\dagger)(I - A^\dagger A)]^\dagger A^\dagger.$$

Using Lemma 6.2, for $P = BB^\dagger$ and $Q = A^\dagger A$, we have

$$(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A - BB^\dagger[(I - BB^\dagger)(I - A^\dagger A)]^\dagger A^\dagger A. \tag{6.2}$$

If we premultiply this expression by $B^\dagger$ and postmultiply it by $A^\dagger$, we obtain

$$B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger = B^\dagger A^\dagger - B^\dagger[(I - BB^\dagger)(I - A^\dagger A)]^\dagger A^\dagger = (AB)^\dagger,$$

and we have the proof.

Analogously, way we can prove that (b3)⇔(b1) and (c3)⇔(c1); the part (d3)⇔ (d1) is very similar-the difference is in taking $Q = BB^\dagger$ and $P = A^\dagger A$.

Let us now prove (f3)⇔(f1)

$$(f3.1) \Leftrightarrow (A^\dagger AB)^\dagger = B^\dagger A^\dagger A - B^\dagger((I - BB^\dagger)(I - A^\dagger A))^\dagger A^\dagger A.$$

If we premultiply (6.2) by $B^\dagger$, we have

$$B^\dagger(A^\dagger ABB^\dagger)^\dagger = B^\dagger A^\dagger A - B^\dagger((I - BB^\dagger)(I - A^\dagger A))^\dagger A^\dagger A = (A^\dagger AB)^\dagger,$$

i.e. part (f1.1). Also,

$$(f3.2) \Leftrightarrow (ABB^\dagger)^\dagger = BB^\dagger A^\dagger - BB^\dagger((I - BB^\dagger)(I - A^\dagger A))^\dagger A^\dagger.$$

If we postmultiply (6.2) by $A^\dagger$, we have

$$(A^\dagger ABB^\dagger)^\dagger A^\dagger = BB^\dagger A^\dagger - BB^\dagger((I - BB^\dagger)(I - A^\dagger A))^\dagger A^\dagger = (ABB^\dagger)^\dagger,$$

i.e. part (f1.2). We have finished this part of the proof.

Let us now see what are the equivalent of statements (f1) and (f2).

A simple computation shows that (f1) is equivalent to the following two statements

$$(D^{-1/2}A_1B_1)^\dagger D^{-1/2}A_i = B_1^{-1}(D^{-1/2}A_1)^\dagger D^{-1/2}A_i, \quad i = 1, 2; \tag{6.3}$$

$$A_1^\dagger = (D^{-1/2}A_1)^\dagger D^{-1/2}. \tag{6.4}$$

Suppose that (f1) holds; if we substitute (6.4) in (6.3), then postmultiply each of modified equations (6.3) by A_i^*, and add them, we get

$$(D^{-1/2}A_1B_1)^\dagger = B_1^{-1}A_1^\dagger D^{1/2},$$

which holds if and only if

$$[D, A_1A_1^\dagger] = 0 \text{ and } [B_1B_1^*, A_1^\dagger A_1] = 0,$$

which is, by Lemma 1.5, equivalent to

$$\mathcal{R}(DA_1) = \mathcal{R}(A_1) \quad \text{and} \quad \mathcal{R}(B_1B_1^*A_1^*) = \mathcal{R}(A_1^*),$$

i.e. we get the statement (a1). It is not difficult to see that the reverse implication also holds.

An easy computation shows that (f2) is equivalent to the following two statements

$$(D^{-1/2}A_1B_1)^\dagger D^{-1/2}A_i = B_1^*(D^{-1/2}A_1B_1B_1^*)^\dagger D^{-1/2}A_i, \quad i = 1, 2; \tag{6.5}$$

$$A_1^\dagger = (D^{-1/2}A_1)^\dagger D^{-1/2}. \tag{6.6}$$

Suppose that (f2) holds; if we postmultiply each equation of (6.5) by A_i^*, and add them, we obtain $(D^{-1/2}A_1B_1)^\dagger = B_1^*(D^{-1/2}A_1B_1B_1^*)^\dagger$, which holds, by Lemma 1.6, if and only if $\mathcal{N}(A_1B_1B_1^*B_1) = \mathcal{N}(A_1B_1)$. As in the previous part of the proof, (6.6) is equivalent to $\mathcal{R}(DA_1) = \mathcal{R}(A_1)$. So, we have the part (f2)⇒(a1). The reverse implication can easily be obtained.

Let us now see what are the equivalent statements of (g1) and (g2).

First, (g1)

$$\begin{aligned}
&\mathcal{R}(B^\dagger(A^\dagger ABB^\dagger)A^\dagger) = \mathcal{R}((AB)^\dagger) = \mathcal{R}((AB)^*)\\
&\qquad \Leftrightarrow \mathcal{R}(B_1^*A_1^*) = \mathcal{R}(B_1^{-1}(D^{-1/2}A_1)^\dagger D^{-1/2}) = \mathcal{R}(B_1^{-1}(D^{-1/2}A_1)^\dagger)\\
&\qquad \Leftrightarrow B_1\mathcal{R}(B_1^*A_1^*) = \mathcal{R}(B_1B_1^*A_1^*) = \mathcal{R}((D^{-1/2}A_1)^\dagger) = \mathcal{R}((D^{-1/2}A_1)^*) = \mathcal{R}(A_1^*),
\end{aligned}$$

so we actually have $\mathcal{R}(B_1B_1^*A_1^*) = \mathcal{R}(A_1^*)$. The second condition: $\mathcal{R}(((AB)^\dagger)^*) = \mathcal{R}((B^\dagger(A^\dagger ABB^\dagger)A^\dagger)^*)$ becomes

$$\begin{aligned}
&\mathcal{N}(B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger) = \mathcal{N}((AB)^\dagger) = \mathcal{N}((AB)^*)\\
&\Leftrightarrow \mathcal{N}(A_1^*) = \mathcal{N}(B_1^*A_1^*) = \mathcal{N}(B_1^{-1}(D^{-1/2}A_1)^\dagger D^{-1/2}) = \mathcal{N}((D^{-1/2}A_1)^\dagger D^{-1/2})\\
&\Leftrightarrow \mathcal{R}(A_1) = \mathcal{R}(D^{-1/2}(A_1^*D^{-1/2})^\dagger)\\
&\Leftrightarrow D^{1/2}\mathcal{R}(A_1) = \mathcal{R}(D^{1/2}A_1) = \mathcal{R}((A_1^*D^{-1/2})^\dagger) = \mathcal{R}((A_1^*D^{-1/2})^*) = \mathcal{R}(D^{-1/2}A_1)
\end{aligned}$$

so we have: $\mathcal{R}(DA_1) = \mathcal{R}(A_1)$. Those two things are equivalent to the (a1), so we have just proved (g1)⇔(a1).

Now, (g2)

$$\begin{aligned}
&\mathcal{R}(B^\dagger A^\dagger) = \mathcal{R}((AB)^\dagger) = \mathcal{R}((AB)^*)\\
&\qquad \Leftrightarrow \mathcal{R}(B_1^*A_1^*) = \mathcal{R}(B_1^*A_1^*D^{-1}) = \mathcal{R}(B_1^{-1}A_1^*)\\
&\qquad \Leftrightarrow B_1\mathcal{R}(B_1^*A_1^*) = \mathcal{R}(B_1B_1^*A_1^*) = \mathcal{R}(A_1^*),
\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}((B^*A^*)^\dagger) &= \mathcal{R}((A^*)^\dagger(B^*)^\dagger)\\ &\Leftrightarrow \mathcal{N}((AB)^\dagger) = \mathcal{N}(B^\dagger A^\dagger) = \mathcal{N}((AB)^*)\\ &\Leftrightarrow \mathcal{N}(B_1^*A_1^*) = \mathcal{N}(B_1^{-1}A_1^*D^{-1})\\ &\Leftrightarrow \mathcal{N}(A_1^*) = \mathcal{N}(A_1^*D^{-1})\\ &\Leftrightarrow \mathcal{R}(A_1) = \mathcal{R}(D^{-1}A_1),\end{aligned}$$

which together are equivalent to (a1), so we have just proved (g2)⇔(a1). □

Now we formulate analogous result for the weighted Moore–Penrose inverse.

Theorem 6.2. *Let $A \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$ and $B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be operators such that A, B and AB have closed ranges. Suppose $M \in \mathcal{L}(\mathcal{H}_3)$ and $\mathcal{N} \in \mathcal{L}(\mathcal{H}_1)$ are positive definite invertible operators. The following statements are equivalent*

(a1) $(AB)^\dagger_{M,N} = B^\dagger_{I,N}(A^\dagger_{M,I}ABB^\dagger_{I,N})^\dagger A^\dagger_{M,I}$;
(a2) $(AB)^\dagger_{M,N} = N^{-1}B^*(A^*MABN^{-1}B^*)^\dagger A^*M$;
(a3) $(AB)^\dagger_{M,N} = B^\dagger_{I,N}A^\dagger_{M,I} - B^\dagger_{I,N}((I-BB^\dagger_{I,N})(I-A^\dagger_{M,I}A))^\dagger A^\dagger_{M,I}$;
(b1) $((A^*)^\dagger_{I,M^{-1}}B)^\dagger_{M^{-1},N} = B^\dagger_{I,N}(A^\dagger_{M,I}ABB^\dagger_{I,N})^\dagger A^*$;
(b2) $((A^*)^\dagger_{I,M^{-1}}B)^\dagger_{M^{-1},N} = N^{-1}B^*((A^*MA)^\dagger(BN^{-1}B^*))^\dagger A^\dagger_{M,I}M^{-1}$;
(b3) $((A^*)^\dagger_{I,M^{-1}}B)^\dagger_{M^{-1},N} = B^\dagger_{I,N}A^* - B^\dagger_{I,N}((I-BB^\dagger_{I,N})(I-A^\dagger_{M,I}A))^\dagger A^*$;
(c1) $(A(B^*)^\dagger_{N^{-1},I})^\dagger_{M,N^{-1}} = B^*(A^\dagger_{M,I}ABB^\dagger_{I,N})^\dagger A^\dagger_{M,I}$;
(c2) $(A(B^*)^\dagger_{N^{-1},I})^\dagger_{M,N^{-1}} = NB^\dagger_{I,N}((A^*MA)(BN^{-1}B^*)^\dagger)^\dagger A^*M$;
(c3) $(A(B^*)^\dagger_{N^{-1},I})^\dagger_{M,N^{-1}} = B^*A^\dagger_{M,I} - B^*((I-BB^\dagger_{I,N})(I-A^\dagger_{M,I}A))^\dagger A^\dagger_{M,I}$;
(d1) $(B^\dagger_{I,N}A^\dagger_{M,I})^\dagger_{N,M} = A(BB^\dagger_{I,N}A^\dagger_{M,I}A)^\dagger B$;
(d2) $(B^\dagger_{I,N}A^\dagger_{M,I})^\dagger_{N,M} = M^{-1}(A^*)^\dagger_{I,M^{-1}}((BN^{-1}B^*)^\dagger(A^*MA)^\dagger)^\dagger(B^*)^\dagger_{N^{-1},I}N$;
(d3) $(B^\dagger_{I,N}A^\dagger_{M,I})^\dagger_{N,M} = AB - A((I-A^\dagger_{M,I}A)(I-BB^\dagger_{I,N}))^\dagger B$;
(e1) $(A^\dagger_{M,I}AB)^\dagger_{I,N}A^\dagger_{M,I} = B^\dagger_{I,N}(ABB^\dagger_{I,N})^\dagger_{M,I}$;
(e2) $(A^\dagger_{M,I}AB)^\dagger_{I,N}A^* = B^\dagger_{I,N}((A^*)^\dagger_{I,M^{-1}}BB^\dagger_{I,N})^\dagger_{M^{-1},I}$;
(e3) $(A^\dagger_{M,I}A(B^*)^\dagger_{N^{-1},I})^\dagger_{I,N^{-1}}A^\dagger_{M,I} = B^*(ABB^\dagger_{I,N})^\dagger_{M,I}$;
(e4) $(BB^\dagger_{I,N}A^\dagger_{M,I})^\dagger_{I,M}B = A(B^\dagger_{I,N}A^\dagger_{M,I}A)^\dagger_{N,I}$;
(e5) $N(A^*MAB)^\dagger_{I,N}A^*M = B^*(ABN^{-1}B^*)^\dagger_{M,I}$;
(e6) $N((A^*MA)^\dagger B)^\dagger_{I,N}A^\dagger_{M,I} = B^*((A^*)^\dagger_{I,M^{-1}}BN^{-1}B^*)^\dagger_{M^{-1},I}M$;
(e7) $(A^*MA(B^*)^\dagger_{N^{-1},I})^\dagger_{I,N^{-1}}A^*M = NB^\dagger_{I,N}(A(BN^{-1}B^*)^\dagger)^\dagger_{M,I}$;
(e8) $NB^\dagger_{I,N}((A^*)^\dagger_{I,M^{-1}}(BN^{-1}B^*)^\dagger)^\dagger_{M^{-1},I}M = ((A^*MA)^\dagger(B^*)^\dagger_{N^{-1},I})^\dagger_{I,N^{-1}}A^\dagger_{M,I}$
(e9) $(AA^*MABN^{-1}B^*B)^\dagger_{M,N} = B^\dagger_{I,N}(A^*MABN^{-1}B^*)^\dagger A^\dagger_{M,I}$;
(f1) $(A^\dagger_{M,I}AB)^\dagger_{I,N} = B^\dagger_{I,N}(A^\dagger_{M,I}ABB^\dagger_{I,N})^\dagger$ *and*
$(ABB^\dagger_{I,N})^\dagger_{M,I} = (A^\dagger_{M,I}ABB^\dagger_{I,N})^\dagger A^\dagger_{M,I}$;

(f2) $(A^\dagger_{M,I}AB)^\dagger_{I,N} = N^{-1}B^*(A^\dagger_{M,I}ABN^{-1}B^*)^\dagger$ *and*
$(ABB^\dagger_{I,N})^\dagger_{M,I} = (A^*MABB^\dagger_{I,N})^\dagger A^*M$;

(f3) $(A^\dagger_{M,I}AB)^\dagger_{I,N} = B^\dagger_{I,N}A^\dagger_{M,I}A - B^\dagger_{I,N}((I-BB^\dagger_{I,N})(I-A^\dagger_{M,I}A))^\dagger A^\dagger_{M,I}A$ *and*
$(ABB^\dagger_{I,N})^\dagger_{M,I} = BB^\dagger_{I,N}A^\dagger_{M,I} - BB^\dagger_{I,N}((I-BB^\dagger_{I,N})(I-A^\dagger_{M,I}A))^\dagger A^\dagger_{M,I}$;

(g1) $\mathcal{R}((AB)^\dagger_{M,N}) = \mathcal{R}(B^\dagger_{I,N}(A^\dagger_{M,I}ABB^\dagger_{I,N})^\dagger A^\dagger_{M,I})$ *and*
$\mathcal{R}(((AB)^\dagger_{M,N})^*) = \mathcal{R}((B^\dagger_{I,N}(A^\dagger_{M,I}ABB^\dagger_{I,N})^\dagger A^\dagger_{M,I})^*)$;

(g2) $\mathcal{R}((AB)^\dagger_{M,N}) = \mathcal{R}(B^\dagger_{I,N}A^\dagger_{M,I})$ *and*
$\mathcal{R}((B^*A^*)^\dagger_{N^{-1},M^{-1}}) = \mathcal{R}((A^*)^\dagger_{I,M^{-1}}(B^*)^\dagger_{N^{-1},I})$;

(g3) $\mathcal{R}(AA^*MAB) = \mathcal{R}(AB)$ *and* $\mathcal{R}((ABN^{-1}B^*B)^*) = \mathcal{R}((AB)^*)$.

Proof. Using the basic relation between ordinary and weighted Moore–Penrose inverse $A^\dagger_{M,N} = N^{-1/2}(M^{1/2}AN^{-1/2})^\dagger M^{1/2}$, and the substitutions $\tilde{A} = M^{1/2}A$, $\tilde{B} = BN^{-1/2}$, all statements from this theorem reduce to the statements of the already-proven Theorem 6.1. For example, let us prove (e6)⇔(g2). For (e6) we have the following chain of equivalencies

$$
\begin{aligned}
&N((A^*MA)^\dagger B)^\dagger_{I,N}A^\dagger_{M,I} = B^*((A^*)^\dagger_{I,M^{-1}}BN^{-1}B^*)^\dagger_{M^{-1},I}M\\
&\quad\Leftrightarrow N^{1/2}((A^*MA)^\dagger BN^{-1/2})^\dagger(M^{1/2}A)^\dagger M^{1/2} = B^*((A^*M^{-1/2})^\dagger BN^{-1}B^*)^\dagger M^{1/2}\\
&\quad\Leftrightarrow ((\tilde{A}^*\tilde{A})^\dagger\tilde{B})^\dagger\tilde{A}^\dagger = \tilde{B}^*((\tilde{A}^*)^\dagger\tilde{B}\tilde{B}^*)^\dagger,
\end{aligned}
$$

which is actually (e6) from Theorem 6.1.

On the other side, (g2) becomes

$$
\begin{aligned}
(g2.1) &\Leftrightarrow \mathcal{R}((AB)^\dagger_{M,N}) = \mathcal{R}(B^\dagger_{I,N}A^\dagger_{M,I})\\
&\Leftrightarrow \mathcal{R}(N^{-1/2}(M^{1/2}ABN^{-1/2})^\dagger M^{1/2}) = \mathcal{R}(N^{-1/2}(BN^{-1/2})^\dagger(M^{1/2}A)^\dagger M^{1/2})\\
&\Leftrightarrow \mathcal{R}(N^{-1/2}(\tilde{A}\tilde{B})^\dagger M^{1/2}) = \mathcal{R}(N^{-1/2}\tilde{B}^\dagger\tilde{A}^\dagger M^{1/2})\\
&\Leftrightarrow \mathcal{R}(N^{-1/2}(\tilde{A}\tilde{B})^\dagger) = \mathcal{R}(N^{-1/2}\tilde{B}^\dagger\tilde{A}^\dagger)\\
&\Leftrightarrow \mathcal{R}((\tilde{A}\tilde{B})^\dagger) = \mathcal{R}(\tilde{B}^\dagger\tilde{A}^\dagger),
\end{aligned}
$$

and

$$
\begin{aligned}
(g2.2) &\Leftrightarrow \mathcal{R}((B^*A^*)^\dagger_{N^{-1},M^{-1}}) = \mathcal{R}((A^*)^\dagger_{I,M^{-1}}(B^*)^\dagger_{N^{-1},I})\\
&\Leftrightarrow \mathcal{R}(M^{1/2}(N^{-1/2}B^*A^*M^{1/2})^\dagger N^{-1/2})\\
&\qquad = \mathcal{R}(M^{1/2}(A^*M^{1/2})^\dagger(N^{-1/2}B^*)^\dagger N^{-1/2})\\
&\Leftrightarrow \mathcal{R}(M^{1/2}(\tilde{B}^*\tilde{A}^*)^\dagger N^{-1/2}) = \mathcal{R}(M^{1/2}(\tilde{A}^*)^\dagger(\tilde{B}^*)^\dagger N^{-1/2})\\
&\Leftrightarrow \mathcal{R}(M^{1/2}(\tilde{B}^*\tilde{A}^*)^\dagger) = \mathcal{R}(M^{1/2}(\tilde{A}^*)^\dagger(\tilde{B}^*)^\dagger)\\
&\Leftrightarrow \mathcal{R}((\tilde{B}^*\tilde{A}^*)^\dagger) = \mathcal{R}((\tilde{A}^*)^\dagger(\tilde{B}^*)^\dagger),
\end{aligned}
$$

which means we have (g2) from Theorem 6.1. Since we have Theorem 6.1 already proven, the proof of this theorem follows immediately. □

We remark that results presented in this section are further generalized in [38] for elements of C^*- algebras and in [39] for elements from a ring with an involution [53, 54].

7. Mixed-type reverse order law, ternary powers and functional calculus

This section mainly consists of the results from a recent paper [13]. This paper is directly motivated by [14], whose results generalized those from Tian's paper [47] from complex matrix case to infinite dimensional Hilbert spaces settings, using operator matrices. In this paper further significant generalizations are done by using the ternary powers and ternary polynomials of bounded operators between different Hilbert spaces and the Borel functional calculus for bounded Hermitian operators.

7.1. Ternary powers and ternary polynomials. The definition of the ternary powers and ternary polynomials used in the paper is slightly different from one presented in [27, page 167]. We also go one step further by defining negative ternary powers.

Definition 7.1. Let $\mathcal{A}$ be a ternary algebra in the sense of Hestenes [27]. For $A \in \mathcal{A}$ and $k \in \mathbb{N}_0$, the element $A^{(k)}$ defined recursively as

$$A^{(0)} = A, \quad A^{(k)} = A^{(k-1)}A^*A = AA^*A^{(k-1)}, \tag{7.1}$$

is called k-th *ternary power* of A. To any algebraic polynomial

$$p(\lambda) = p_0 + p_1\lambda + \cdots + p_n\lambda^n$$

corresponds the unique element

$$t_p(A) = p_0A^{(0)} + p_1A^{(1)} + \cdots + p_nA^{(n)}, \tag{7.2}$$

which originates from $p(\lambda)$ when one replace λ^k, $k = \overline{0, n}$, by the k-th ternary power $A^{(k)}$ of A. The function of A, defined by (7.2) is *ternary polynomial* in A. For any element A the class of all elements given by ternary polynomials of A is denoted by $\mathcal{T}(A)$.

We are interested in the case when $\mathcal{A} = \mathbb{C}^{m\times n}$ or $\mathcal{A} = \mathcal{L}(\mathcal{H}, \mathcal{K})$.

Remark 7.1. In this section only generalized inverses we are dealing with are Moore–Penrose and weighted Moore–Penrose inverses, so the expression $A^{(k)}$ means k-th ternary power, **not** the generalized inverse of A which satisfies the k-th Penrose equation ($k = \overline{1, 4}$).

The next definition is a more operative form of the Definition 7.1.

Definition 7.2. Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $k \in \mathbb{N}_0$. The k-th *ternary power* of A is given by $A^{(k)} = (AA^*)^kA = A(A^*A)^k$.

Remark 7.2. Note that the relation (7.1) can be written as

$$A^{(0)} = A, \ A^{(k)} = A(A^*)^{(k-1)}A, \quad k \in \mathbb{N},$$

and also in a more symmetric form as

$$A^{(0)} = A, \quad A^{(1)} = AA^*A, \quad A^{(k+2)} = AA^*A^{(k)}A^*A, \qquad k \in \mathbb{N}.$$

Moreover, it is easy to prove that

$$A^{(k)} = (AA^*)^p A^{(k-p-q)} (A^*A)^q, \quad p, q \in \mathbb{N},\ p+q \leqslant k. \tag{7.3}$$

Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ be a closed-range operator. It is not hard to see that the set $\mathcal{T}(A) = \{t_p(A) : p \in \mathbb{C}[\lambda]\}$ is a unital algebra (unit is $A^{(0)} \equiv A$) with the product (denoted by $\circ$) defined by

$$t_p(A) \circ t_q(A) := t_p(A) A^\dagger t_q(A). \tag{7.4}$$

According to the Definition 7.2, the ternary polynomial (7.2) can be rewritten in two equivalent ways as

$$t_p(A) = \sum_{k=0}^{n} p_k A^{(k)} = \begin{cases} A \sum_{k=0}^{n} p_k (A^*A)^k = Ap(A^*A), \\ \sum_{k=0}^{n} p_k (AA^*)^k \, A = p(AA^*)A. \end{cases}$$

Now (7.4) becomes

$$t_p(A) \circ t_q(A) = t_p(A) A^\dagger t_q(A) = A \cdot (pq)(A^*A) = (pq)(AA^*) \cdot A = t_{p \cdot q}(A).$$

Particularly, $A^{(k)} \circ A^{(\ell)} = A^{(\ell)} \circ A^{(k)} = A^{(k+\ell)}$. The mapping $\gamma_A \colon \mathbb{C}[\lambda] \to \mathcal{T}(A) \subset \mathcal{L}(\mathcal{H}, \mathcal{K})$, which makes the correspondence between λ^k and the ternary power $A^{(k)}$ for $k \in \mathbb{N}_0$ is linear homomorphism (in the sense $\gamma_A(p \cdot q) = \gamma_A(p) \circ \gamma_A(q)$ for any $p, q \in \mathbb{C}[\lambda]$) such that $\gamma_A(1) = A^{(0)} = A$ and $\gamma_A(t) = A^{(1)}$, and it gives the ternary polynomial calculus. For example, for some polynomial $p(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 \in \mathbb{C}[\lambda]$ we have

$$\begin{aligned} \gamma_A(p) &= \gamma_A(a_0 + a_1\lambda + a_2\lambda^2) = a_0\gamma_A(1) + a_1\gamma_A(\lambda) + a_2\gamma_A(\lambda \cdot \lambda) \\ &= a_0\gamma_A(1) + a_1\gamma_A(\lambda) + a_2(\gamma_A(\lambda) \circ \gamma_A(\lambda)) \\ &= a_0A^{(0)} + a_1A^{(1)} + a_2(A^{(1)} \circ A^{(1)}) = a_0A^{(0)} + a_1A^{(1)} + a_2A^{(2)} = t_p(A). \end{aligned}$$

We will define negative ternary powers of A as follows.

Definition 7.3. Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ be a closed-range operator and $k \in \mathbb{N}$. The k-th *negative ternary power* of A is given by $A^{(-k)} := ((AA^*)^\dagger)^k A = A((A^*A)^\dagger)^k$.

According to the Lemma 1.1, the closedness of the range of A is enough for the existence of $(AA^*)^\dagger$ and $(A^*A)^\dagger$, so the definition is correct.

We should remark that the inductive definition of the ternary power given by (7.1) actually holds for all integers k. Because of the following chain of equivalencies

$$A^{(k)} = A^{(k-1)}A^*A \Leftrightarrow A^{(k)}A^\dagger = A^{(k-1)}A^* \Leftrightarrow A^{(k)}(A^*A)^\dagger = A^{(k-1)},$$

one can derive another useful relation between subsequent ternary powers

$$A^{(k-1)} = A^{(k)}(A^*A)^\dagger = (AA^*)^\dagger A^{(k)}, \quad k \in \mathbb{Z}. \tag{7.5}$$

It is clear that relation (7.1) is more suitable for generating positive ternary powers, while the relation (7.5) is more suitable for negative ones.

Remark 7.3. Note that the relation (7.5) can be written in a more symmetric form $A^{(k)} = (AA^*)^\dagger A^{(k+2)}(A^*A)^\dagger$. It is easy to prove that

$$A^{(k)} = ((AA^*)^\dagger)^p A^{(k+p+q)}((A^*A)^\dagger)^q, \quad p, q \in \mathbb{N}. \tag{7.6}$$

The formulae (7.3) and (7.6) can be further combined and generalized.

Example 7.1. We enlist some ternary powers of the given operator $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$

$$\ldots, \quad A^{(-2)} = (A^*)^\dagger A^\dagger (A^*)^\dagger, \quad A^{(-1)} = (A^\dagger)^*, \quad A^{(0)} = A, \quad A^{(1)} = AA^*A, \quad \ldots$$

Recall that for negative ternary powers we must ensure that operator A has a closed range.

Theorem 7.1. *For closed-range operator $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $k \in \mathbb{Z}$ the operator $T^{(k)}$ has closed range.*

Proof. We use the mathematical induction method. For $k = 0$ operator $T^{(0)} = T$ has a closed range by the statement of the Theorem. For $k = 1$ we have $T^{(1)} = T(T^*T)$ and the Lemma 1.4 gives $T^\dagger TT^*T(T^*T)^\dagger = T^*(T^*)^\dagger = (T^\dagger T)^* = T^\dagger T$, and since $\mathcal{R}(T^\dagger T) = \mathcal{R}(T^*)$ is closed subspace, the range of $T^{(1)}$ is closed. Suppose that the range of $T^{(k)}$ is closed, and let us prove the closedness of the range of $T^{(k+1)}$. By the Lemma 1.4, for $T^{(k+1)} = (TT^*)T^{(k)}$, one have

$$(TT^*)^\dagger TT^*T^{(k)}(T^{(k)})^\dagger = (T^*)^\dagger T^*T^{(k)}(T^{(k)})^\dagger = TT^\dagger T^{(k)}(T^{(k)})^\dagger = T^{(k)}(T^{(k)})^\dagger,$$

which, by the induction hypothesis, has the closed range (because $\mathcal{R}(BB^\dagger) = \mathcal{R}(B)$ for some closed-range bounded linear operator B).

We now prove the Theorem for negative ternary powers $T^{(-k)} = ((TT^*)^\dagger)^k T$, again by Lemma 1.4 and Lemma 1.2

$$\begin{aligned}(((TT^*)^\dagger)^k)^\dagger((TT^*)^\dagger)^k TT^\dagger &= (TT^*)^k((TT^*)^\dagger)^k TT^*(TT^*)^\dagger \\ &= TT^*(TT^*)^\dagger TT^*(TT^*)^\dagger = TT^*(TT^*)^\dagger = TT^\dagger,\end{aligned}$$

which is a closed-range operator. Therefore, we completed the proof. □

Remark 7.4. We emphasize that for some closed-range operator $T \in \mathcal{L}(\mathcal{H})$ it may happen that for some $k \in \mathbb{N}$ the operator T^k does not have a closed range (see e.g. [6, p. 123]).

Some properties of ternary powers are collected in the following theorem.

Theorem 7.2. *Let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ (we request that A is with a closed range when dealing with the Moore–Penrose inverses and negative ternary powers!) and $k \in \mathbb{Z}$. Then we have*

(1) $(A^*)^{(k)} = (A^{(k)})^*$, $AA^\dagger A^{(k)} = A^{(k)}A^\dagger A = A^{(k)}$;
(2) $A^{(k)}A^* = (AA^*)^{k+1}$, $A^*A^{(k)} = (A^*A)^{k+1}$, $k \in \mathbb{N}_0$;
(3) $A^{(-k)}A^* = ((AA^*)^\dagger)^{k-1}$, $A^*A^{(-k)} = ((A^*A)^\dagger)^{k-1}$, $k \in \mathbb{N}_0$;
(4) $A^*A^{(k)}A^* = (A^*)^{(k+1)}$, $A^*A^{(-k)}A^* = (A^*)^{(-k+1)}$, $k \in \mathbb{N}_0$;
(5) $A^{(k)}A^\dagger = (AA^*)^k$, $A^\dagger A^{(k)} = (A^*A)^k$, *for* $k \in \mathbb{N}_0$;
(6) $A^{(-k)}A^\dagger = ((AA^*)^\dagger)^k$, $A^\dagger A^{(-k)} = ((A^*A)^\dagger)^k$, *for* $k \in \mathbb{N}_0$;

(7) $(A^*)^{(k)}A^{(k)} = (A^*A)^{2k+1}$, $(A^*)^{(-k)}A^{(-k)} = ((A^*A)^{2k-1})^\dagger$, $k \in \mathbb{N}_0$;
$A^{(k)}(A^*)^{(k)} = (AA^*)^{2k+1}$, $A^{(-k)}(A^*)^{(-k)} = ((AA^*)^{2k-1})^\dagger$, $k \in \mathbb{N}_0$;
(8) $A^{(k)}(A^{(k)})^*A = A^{(2k+1)}$, $(A^{(k)})^*A^{(k)}A^* = (A^{(2k+1)})^*$;
(9) $A^{(k)}(A^{(k)})^\dagger = AA^\dagger$, $(A^{(k)})^\dagger A^{(k)} = A^\dagger A$;
(10) $(A^{(k)})^\dagger = (A^\dagger)^{(k)}$;
A is invertible, then $(A^{(k)})^{-1} = (A^{-1})^{(k)} = A^{-1}(AA^*)^{-k} = (A^*A)^{-k}A^{-1}$;
(11) $A^{(k)} \circ A^{(\ell)} = A^{(k+\ell)}$ *for any* $\ell \in \mathbb{Z}$. *Particularly,* $A^{(k)} \circ A^{(-k)} = A^{(0)} = A$.
(12) $A^{(-k)} = ((A^*)^\dagger)^{(k-1)}$, $A^{(k)} = ((A^*)^\dagger)^{(-(k+1))}$.
(13) $(A^{(m)})^{(n)} = A^{(m+n+2mn)} = (A^{(n)})^{(m)}$, $m, n \in \mathbb{Z}$. *Particularly,* $(A^{(m)})^{(n)} = A^{(0)} \Leftrightarrow m = n = 0 \vee m = n = -1$;
(14) $A^{(k)} = (A^{(0)})^{(k)}$, $A^{(-k)} = (A^{(-1)})^{(k-1)}$, $(A^{(-m)})^{(-n)} = (A^{(m-1)})^{(n-1)}$.

Proof. 1. Obvious from the definition.

2. $A^{(k)}A^* = (AA^*)^kAA^* = (AA^*)^{(k+1)}$, the second part can be proved on analogous way.
3. $A^{(-k)}A^* = ((AA^*)^\dagger)^kAA^* = (AA^*)^{(k-1)}$, the second part can be proved on analogous way.
4. By using definition and part 3, we have $A^*A^{(k)}A^* = A^*(AA^*)^k = (A^*)^{(k)}$ and $A^*A^{(-k)}A^* = A^*(AA^*)^{(k-1)} = (A^*)^{(k-1)}$
5. By part 2 and Lemma 1.2, $A^{(k)}A^\dagger = A^{(k)}A^*(AA^*)^\dagger = (AA^*)^{k+1}AA^* = (AA^*)^k$; the second part can be proved on analogous way.
6. $A^{(-k)}A^\dagger = ((AA^*)^\dagger)^kAA^\dagger = ((AA^*)^\dagger)^kAA^*(AA^*)^\dagger = ((AA^*)^\dagger)^k$, the second part can be proved in an analogous way.
7. By the definitions of ternary powers,

$$(A^*)^{(k)}A^{(k)} = (A^*A)^kA^*A(A^*A)^k = (A^*A)^{2k+1},$$
$$(A^*)^{(-k)}A^{(-k)} = ((A^*A)^\dagger)^kA^*A((A^*A)^\dagger)^k = ((A^*A)^\dagger)^{2k-1}.$$

The second part follows when we replace A with A^*.

8. From part 7: $A^{(k)}(A^{(k)})^*A = (AA^*)^{(2k+1)}A = A^{(2k+1)}$; for positive and $A^{(-k)}(A^{(-k)})^*A = ((AA^*)^\dagger)^{(2k-1)}A = A^{(-(2k-1))} = A^{(-2k+1)}$ for negative k; the second part is just a conjugate transpose of the first.
9. Note that, by the Theorem 7.1, there exist $(A^{(k)})^\dagger$. By part 7

$$A^{(k)}(A^{(k)})^\dagger = A^{(k)}(A^{(k)})^*(A^{(k)}(A^{(k)})^*)^\dagger = (AA^*)^{2k+1}((AA^*)^{2k+1})^\dagger$$
$$= (AA^*)^{2k+1}((AA^*)^\dagger)^{2k+1} = AA^*(AA^*)^\dagger = AA^\dagger;$$

$$(A^{(-k)})^\dagger A^{(-k)} = ((A^{(-k)})^*A^{(-k)})^\dagger(A^{(-k)})^*A^{(-k)}$$
$$= (((A^*A)^\dagger)^{2k-1})^\dagger((A^*A)^\dagger)^{2k-1} = (A^*A)^{2k-1}((A^*A)^\dagger)^{2k-1}$$
$$= A^*A(A^*A)^\dagger = A^*(A^\dagger)^* = A^\dagger A;$$

The relations $A^{(-k)}(A^{(-k)})^\dagger$ and $(A^{(k)})^\dagger A^{(k)}$ can be proven analogously.

10. We prove positive and negative ternary powers cases separately, by using Lemma 1.2.

$$\begin{aligned}(A^{(k)})^\dagger &= ((A^{(k)})^*A^{(k)})^\dagger (A^{(k)})^* = ((A^*)^{(k)}A^{(k)})^\dagger (A^*)^{(k)}\\ &= ((A^*A)^{2k+1})^\dagger (A^*A)^k A^* = ((A^*A)^\dagger)^{2k+1}(A^*A)^k A^*\\ &= ((A^*A)^\dagger)^{k+1}A^* = ((A^*A)^\dagger)^k (A^*A)^\dagger A^*\\ &= ((A^*A)^\dagger)^k A^\dagger = (A^\dagger (A^\dagger)^*)^k A^\dagger = (A^\dagger)^{(k)},\ k\in\mathbb{N}_0;\end{aligned}$$

$$\begin{aligned}(A^{(-k)})^\dagger &= ((A^{(-k)})^*A^{(-k)})^\dagger (A^{(-k)})^* = ((A^*)^{(-k)}A^{(-k)})^\dagger (A^*)^{(-k)}\\ &= (((A^*A)^\dagger)^{2k-1})^\dagger ((A^*A)^\dagger)^k A^* = (A^*A)^{2k-1}((A^*A)^\dagger)^k A^*\\ &= (A^*A)^{k-1}A^*AA^\dagger = (A^*A)^k A^\dagger = (A^\dagger)^{(-k)},\ k\in\mathbb{N}_0.\end{aligned}$$

We used: $(A^\dagger)^{(-k)} = \left((A^\dagger (A^\dagger)^*)^\dagger\right)^k A^\dagger = (((A^*A)^\dagger)^\dagger)^k A^\dagger = (A^*A)^k A^\dagger$.

11. We have $A^{(k)}\circ A^{(\ell)} = (AA^*)^k AA^\dagger A(A^*A)^\ell = (AA^*)^k (AA^*)^\ell A = A^{(k+\ell)}$, for $k,\ell\in\mathbb{N}_0$. Also

$$\begin{aligned}A^{(k)}\circ A^{(-\ell)} &= (AA^*)^k AA^\dagger A((A^*A)^\dagger)^\ell = (AA^*)^k A((A^*A)^\dagger)^\ell\\ &= (AA^*)^k((AA^*)^\dagger)^\ell A = \begin{cases}(AA^*)^{k-\ell}A, & k\geqslant \ell,\\ ((AA^*)^\dagger)^{\ell-k}A, & k\leqslant \ell,\end{cases} = A^{(k-\ell)}.\end{aligned}$$

$$A^{(-k)}\circ A^{(-\ell)} = ((AA^*)^\dagger)^k AA^\dagger A((A^*A)^\dagger)^\ell = ((AA^*)^\dagger)^{k+\ell}A = A^{(-k-\ell)},$$

where we used the fact $A(A^*A)^\dagger = (A^*)^\dagger = (AA^*)^\dagger A$.

12.

$$\begin{aligned}A^{(-k)} &= ((AA^*)^\dagger)^k AA^*(A^*)^\dagger = ((AA^*)^\dagger)^{k-1}(A^*)^\dagger\\ &= ((A^*)^\dagger A^\dagger)^{k-1}(A^*)^\dagger = ((A^*)^\dagger)^{(k-1)}.\end{aligned}$$

$$\begin{aligned}((A^*)^\dagger)^{(-k-1)} &= (((A^*)^\dagger A^\dagger)^\dagger)^{k+1}(A^*)^\dagger = (AA^*)^{k+1}(AA^*)^\dagger A\\ &= (AA^*)^k A = A^{(k)}.\end{aligned}$$

13. We prove separately cases for positive and negative ternary powers; here $m,n\in\mathbb{N}_0$. Part 7 is often used, and part 12 also.

$$\begin{aligned}(A^{(m)})^{(n)} &= (A^{(m)}(A^*)^{(m)})^n A^{(m)} = \left((AA^*)^{2m+1}\right)^n (AA^*)^m A\\ &= (AA^*)^{(2m+1)n+m}A = A^{(m+n+2mn)};\end{aligned}$$

$$\begin{aligned}(A^{(m)})^{(-n)} &= ((A^{(m)}(A^*)^{(m)})^\dagger)^n A^{(m)} = \left(((AA^*)^{2m+1})^\dagger\right)^n A^{(m)}\\ &= ((AA^*)^\dagger)^{(2m+1)n}(AA^*)^m A = ((AA^*)^\dagger)^{(2mn+n-m)}A\\ &= A^{(-(-m+n+2mn))} = A^{(m+(-n)+2m(-n))};\end{aligned}$$

$$\begin{aligned}(A^{(-m)})^{(n)} &= \left(A^{(-m)}(A^*)^{(-m)}\right)^n A^{(-m)}\\ &= \left(((AA^*)^{2m-1})^\dagger\right)^n ((AA^*)^\dagger)^m A = ((AA^*)^\dagger)^{2mn-n+m} A\\ &= A^{(-m+n-2mn)} = A^{((-m)+n+2(-m)n)};\end{aligned}$$

$$\begin{aligned}(A^{(-m)})^{(-n)} &= (((A^{(-m)})^*)^\dagger)^{(n-1)} = (((((A^*)^\dagger)^{(m-1)})^*)^\dagger)^{(n-1)}\\ &= (A^{(m-1)})^{(n-1)} = A^{2(m-1)(n-1)+m-1+n-1}\\ &= A^{2mn-m-n} = A^{(-m)+(-n)+2(-m)(-n)};\end{aligned}$$

14. It follows from 13. □

Remark 7.5. Another importance of ternary powers is the fact that the Moore–Penrose inverse can be expressed via some ternary polynomial. For the complex matrix case we have: $A^\dagger = t_p(A^*)$, for more details please see [3, page 250], and the references therein. For the closed-range Hilbert space operator T we have (Euler–Knopp method!)

$$T^\dagger = \lim_{n\to\infty} S_n(T), \quad S_n(T) = \sum_{k=0}^{n} \alpha(I-\alpha T^*T)^k T^*, \qquad 0<\alpha<||T||^2,$$

so $T^\dagger$ is actually the strong limit of ternary polynomials in T^*; see e.g. [25, pp. 64–65], or [6, p. 42].

If the operator matrix for A is given by Lemma 1.2, for all $k \in \mathbb{N}_0$ we have

$$A^{(k)} = \begin{bmatrix} A_1^{(k)} & 0\\ 0 & 0\end{bmatrix} : \begin{bmatrix}\mathcal{R}(A^*)\\ \mathcal{N}(A)\end{bmatrix} \to \begin{bmatrix}\mathcal{R}(A)\\ \mathcal{N}(A^*)\end{bmatrix}, \tag{7.7}$$

where $A_1^{(k)}$ is invertible. Then

$$(A^{(k)})^\dagger = \begin{bmatrix} (A_1^{(k)})^{-1} & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix} (A_1^{-1})^{(k)} & 0\\ 0 & 0\end{bmatrix}. \tag{7.8}$$

For some ternary polynomial we have

$$t_p(A) = \begin{bmatrix} t_p(A_1) & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix} p(A_1A_1^*)A_1 & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix} A_1 p(A_1^*A_1) & 0\\ 0 & 0\end{bmatrix}.$$

Note that by using the operator matrices (7.7) and (7.8) one can easily prove the Theorem 7.2 under the assumption that A has a closed range. The reason why it is done in an algebraic way, without using the operator matrices, is to provide the possibility for generalization to rings with involutions and C^*-algebras.

The relation between ternary powers and partial isometries is natural, and it is presented in the next proposition.

Proposition 7.1. *Let $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$ be a close-range operator. Then A is partial isometry iff $A^{(k)} = A^{(k+1)}$ for some (all) $k \in \mathbb{Z}$.*

Proof. If $A^\dagger = A^*$, then

$$A^{(k)} = A^{(k+1)}(A^*A)^\dagger = A^{(k+1)}A^\dagger(A^\dagger)^* = A^{(k+1)}A^\dagger A = A^{(k+1)}.$$

Conversely,

$$0 = A^{(k+1)} - A^{(k)} = (AA^*)^k(AA^*A - A) \Rightarrow 0 = A^{(1)} - A^{(0)},$$

i.e. $A = AA^*A$, which means A is partial isometry. □

7.2. The mixed-type ROLs for ternary powers. Recall that throughout the rest of the section $\mathcal{H}_k,\ k = \overline{1,4}$, denote arbitrary Hilbert spaces, and $A_k \in \mathcal{L}(\mathcal{H}_{k+1}, \mathcal{H}_k)$, $k = 1, 2, 3$, denote bounded linear operators. Also, let $M = A_1A_2A_3$.

Theorem 7.3. *Let the operators A_1, A_3, M and $(A_1^{(k)})^\dagger M(A_3^{(\ell)})^\dagger$, $k, \ell \in \mathbb{N}_0$, have closed ranges. Then the following statements are equivalent*

(a) $M^\dagger = (A_3^{(\ell)})^\dagger((A_1^{(k)})^\dagger M(A_3^{(\ell)})^\dagger)^\dagger(A_1^{(k)})^\dagger$;

(b) $\mathcal{R}(A_1^{(k)}(A_1^{(k)})^*M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^{(\ell)})^*A_3^{(\ell)}M^*) = \mathcal{R}(M^*)$.

Proof. According to Theorem 7.1, closedness of the ranges of A_1 and A_3 implies closedness of the range of $A_1^{(k)}$ and $A_3^{(\ell)}$ for any $k, \ell \in \mathbb{Z}$.

Suppose, by using Lemma 1.2, that the operators A_1 and A_3 have the following matrix forms

$$A_k = \begin{bmatrix} A_{k1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A_k^*) \\ \mathcal{N}(A_k) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A_k) \\ \mathcal{N}(A_k^*) \end{bmatrix}, \quad k = 1 \text{ or } 3,$$

where A_{11} and A_{31} are invertible. Then

$$A_k^\dagger = \begin{bmatrix} A_{k1}^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A_k) \\ \mathcal{N}(A_k^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A_k^*) \\ \mathcal{N}(A_k) \end{bmatrix}, \quad k = 1 \text{ or } 3.$$

According to such space decompositions, it follows that operator A_2 has the following matrix form for some bounded linear operators $A_{21}\colon \mathcal{R}(A_3) \to \mathcal{R}(A_1^*)$, $A_{22}\colon \mathcal{N}(A_3^*) \to \mathcal{R}(A_1^*)$, $A_{23}\colon \mathcal{R}(A_3) \to \mathcal{N}(A_1)$ and $A_{24}\colon \mathcal{N}(A_3^*) \to \mathcal{N}(A_1)$

$$A_2 = \begin{bmatrix} A_{21} & A_{22} \\ A_{23} & A_{24} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A_3) \\ \mathcal{N}(A_3^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A_1^*) \\ \mathcal{N}(A_1) \end{bmatrix}.$$

Also, we use the notation $M_1 = A_{11}A_{21}A_{31}$, so the matrix form of M is

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Now, (a) is equivalent to

$$M_1^\dagger = (A_{31}^{-1})^{(\ell)}((A_{11}^{-1})^{(k)}M_1(A_{31}^{-1})^{(\ell)})^\dagger(A_{11}^{-1})^{(k)}.$$

By checking the Penrose equations, we see this is equivalent to

$$(A_{11}^{-1})^{(k)}M_1M_1^\dagger A_{11}^{(k)}, \quad A_{31}^{(\ell)}M_1^\dagger M_1(A_{31}^{-1})^{(\ell)}$$

to be Hermitian, which means

$$[A_{11}^{(k)}(A_{11}^{(k)})^*, M_1M_1^\dagger] = 0, \quad [(A_{31}^{(\ell)})^*A_{31}^{(\ell)}, M_1^\dagger M_1] = 0.$$

On the other hand, conditions (b) are equivalent to

$$\mathcal{R}(A_{11}^{(k)}(A_{11}^{(k)})^*M_1) = \mathcal{R}(M_1), \quad \mathcal{R}((A_{31}^{(\ell)})^*A_{31}^{(\ell)}M_1^*) = \mathcal{R}(M_1^*).$$

By Lemma 1.5, we have the proof. □

Remark 7.6. The condition b) can be written in more condensed form as

$$\mathcal{R}(A_1^{(2k+1)}A_2A_3) = \mathcal{R}(A_1A_2A_3), \quad \mathcal{R}((A_3^*)^{(2\ell+1)}A_2^*A_1^*) = \mathcal{R}(A_3^*A_2^*A_1^*).$$

Corollary 7.1. *Suppose $A \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$, $B \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and AB are closed-range operators. The following statements are equivalent*

a) $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$,
b) $\mathcal{R}(AA^*AB) = \mathcal{R}(AB) \wedge \mathcal{R}(B^*BB^*A^*) = \mathcal{R}(B^*A^*)$,
c) $(AB)^\dagger = (A^\dagger AB)^\dagger A^\dagger \wedge (AB)^\dagger = B^\dagger(ABB^\dagger)^\dagger$.

Proof. a)⇔b): It follows from the Theorem 7.3 when we put $A_2 = I$ and $k = \ell = 0$.
b)⇔c): It follows from the Theorem 7.3 when we put $A_3 = I$ for the first part, $A_1 = I$ for the second part, and $k = \ell = 0$. Remark that this result is a part of [21, Theorem 2.6]. □

As a corollaries of Theorem 7.3 one can obtain following two corollaries for $k = \ell = 0$ and $k = \ell = 1$, respectively.

Corollary 7.2. [14, Theorem 2.1]*Let A_1, A_3, M, $A_1^\dagger MA_3^\dagger$ have closed ranges. Then the following statements are equivalent*

(a) $M^\dagger = A_3^\dagger(A_1^\dagger MA_3^\dagger)^\dagger A_1^\dagger$;
(b) $\mathcal{R}(A_1A_1^*M) = \mathcal{R}(M)$ *and* $\mathcal{R}(A_3^*A_3M^*) = \mathcal{R}(M^*)$.

Corollary 7.3. [14, Theorem 2.5] *Let A_1, A_3, M, $(A_1A_1^*A_1)^\dagger M(A_3A_3^*A_3)^\dagger$ have closed ranges. Then the following statements are equivalent.*

(a) $M^\dagger = (A_3A_3^*A_3)^\dagger((A_1A_1^*A_1)^\dagger M(A_3A_3^*A_3)^\dagger)^\dagger(A_1A_1^*A_1)^\dagger$;
(b) $\mathcal{R}((A_1A_1^*)^3M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^3M^*) = \mathcal{R}(M^*)$.

Proposition 7.2. *Under the assumptions of Corollary 7.3, with $\mathcal{H}_1 = \mathcal{H}_2$, the following statements are equivalent (k is a non-negative integer)*

(a) $M^\dagger = (A_3A_3^*A_3)^\dagger[((A_1A_1^*A_1)^\dagger)^kM(A_3A_3^*A_3)^\dagger]^\dagger((A_1A_1^*A_1)^\dagger)^k$,
(a) $\mathcal{R}((A_1A_1^*)^{3k}M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^3M^*) = \mathcal{R}(M^*)$.

Proposition 7.3. *Under the conditions of Corollary 7.3, with $\mathcal{H}_3 = \mathcal{H}_4$, the following statements are equivalent (ℓ is a non-negative integer)*

(a) $M^\dagger = ((A_3A_3^*A_3)^\dagger)^\ell[(A_1A_1^*A_1)^\dagger M((A_3A_3^*A_3)^\dagger)^\ell]^\dagger(A_1A_1^*A_1)^\dagger$,
(b) $\mathcal{R}((A_1A_1^*)^3M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^{3\ell}M^*) = \mathcal{R}(M^*)$.

Remark 7.7. The previous two propositions were incorrectly stated as Propositions 2.3 and 2.4 in [14]. Additional hypothesis $\mathcal{H}_1 = \mathcal{H}_2$ for the first, and $\mathcal{H}_3 = \mathcal{H}_4$ for the second one, should be added.

Theorem 7.4. *Let the operators M and $(A_1^{(k)})^*M(A_3^{(\ell)})^*$, $k, \ell \in \mathbb{N}_0$, have closed ranges. Then the following statements are equivalent*

(a) $M^\dagger = (A_3^{(\ell)})^* \left((A_1^{(k)})^* M (A_3^{(\ell)})^*\right)^\dagger (A_1^{(k)})^*$;

(b) $\mathcal{R}(A_1^{(k)}(A_1^{(k)})^* M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^{(\ell)})^* A_3^{(\ell)} M^*) = \mathcal{R}(M^*)$.

Proof. Suppose, by using Lemma 1.2, that the operators A_1 and A_3 have the following matrix forms (remark that $\overline{W}$ means the closure of some subspace W)

$$A_k = \begin{bmatrix} A_{k1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{\mathcal{R}(A_k^*)} \\ \mathcal{N}(A_k) \end{bmatrix} \to \begin{bmatrix} \overline{\mathcal{R}(A_k)} \\ \mathcal{N}(A_k^*) \end{bmatrix}, \quad k = 1 \text{ or } 3,$$

where A_{11} and A_{31} are invertible.

According to such space decompositions, it follows that operator A_2 has the following matrix form

$$A_2 = \begin{bmatrix} A_{21} & A_{22} \\ A_{23} & A_{24} \end{bmatrix} : \begin{bmatrix} \overline{\mathcal{R}(A_3)} \\ \mathcal{N}(A_3^*) \end{bmatrix} \to \begin{bmatrix} \overline{\mathcal{R}(A_1^*)} \\ \mathcal{N}(A_1) \end{bmatrix}.$$

Also, we use the notation $M_1 = A_{11}A_{21}A_{31}$, so the matrix form of M is

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Now, (a) is equivalent to the following

$$M_1^\dagger = (A_{31}^*)^{(\ell)}((A_{11}^*)^{(k)} M_1 (A_{31}^*)^{(\ell)})^\dagger (A_{11}^*)^{(k)}.$$

By checking the Penrose equations, we see this is equivalent to

$$(A_{11}^*)^{(k)} M_1 M_1^\dagger ((A_{11}^*)^{(k)})^{-1}, \quad ((A_{31}^*)^{(\ell)})^{-1} M_1^\dagger M_1 (A_{31}^*)^{(\ell)}$$

to be Hermitian, which means

$$[A_{11}^{(k)}(A_{11}^{(k)})^*, M_1 M_1^\dagger] = 0, \quad [(A_{31}^{(\ell)})^* A_{31}^{(\ell)}, M_1^\dagger M_1] = 0.$$

On the other hand, conditions (b) are equivalent to

$$\mathcal{R}(A_{11}^{(k)}(A_{11}^{(k)})^* M_1) = \mathcal{R}(M_1), \quad \mathcal{R}((A_{31}^{(\ell)})^* A_{31}^{(\ell)} M_1^*) = \mathcal{R}(M_1^*).$$

By Lemma 1.5, we have the proof. □

Remark 7.8. The statement b) in both theorems is the very same one, therefore statements a) are equivalent. Note the difference in the requirements for the range closedness in those two theorems.

As a corollary, we can obtain the following well-known result.

Corollary 7.4. *Let* $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ *be closed-range operator,* $M \in \mathcal{L}(\mathcal{K})$ *and* $N \in \mathcal{L}(\mathcal{H})$ *Hermitian positive definite operators. Then we have*

$$A^\dagger = A_{M,N}^\dagger \Leftrightarrow \mathcal{R}(MA) = \mathcal{R}(A) \wedge \mathcal{R}(N^{-1}A^*) = \mathcal{R}(A^*).$$

Proof. If we put $A_1 := M^{1/2}$, $A_2 := M^{-1/2}AN^{1/2}$, $A_3 := N^{-1/2}$ and $k = \ell = 0$ in the Theorem 7.4, we have the proof. □

As a corollaries of Theorem 7.4 one can obtain following two corollaries for $k = \ell = 0$ and $k = \ell = 1$, respectively.

Corollary 7.5. [14, Theorem 2.2]*Let $A_1, A_3, M, A_1^*MA_3^*$ have closed ranges. Then the following statements are equivalent*

(a) $M^\dagger = A_3^*(A_1^*MA_3^*)^\dagger A_1^*$;
(b) $\mathcal{R}(A_1A_1^*M) = \mathcal{R}(M)$ *and* $\mathcal{R}(A_3^*A_3M^*) = \mathcal{R}(M^*)$.

Corollary 7.6. [14, Theorem 2.6] *Let A_1, A_3, M, $(A_1A_1^*A_1)^*M(A_3A_3^*A_3)^*$ have closed ranges. Then the following statements are equivalent*

(a) $M^\dagger = (A_3A_3^*A_3)^*((A_1A_1^*A_1)^*M(A_3A_3^*A_3)^*)^\dagger(A_1A_1^*A_1)^*$;
(b) $\mathcal{R}((A_1^*A_1A_1^*)^2M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^3M^*) = \mathcal{R}(M^*)$.

Proposition 7.4. *Under the assumptions of Corollary 7.6, with $\mathcal{H}_3 = \mathcal{H}_4$, the following statements are equivalent (k is a non-negative integer)*

(a) $M^\dagger = (A_3A_3^*A_3)^*[((A_1A_1^*A_1)^*)^kM(A_3A_3^*A_3)^*]^\dagger((A_1A_1^*A_1)^*)^k$,
(b) $\mathcal{R}((A_1A_1^*)^{3k}M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^3M^*) = \mathcal{R}(M^*)$.

Proposition 7.5. *Under the conditions of Corollary 7.6, with $\mathcal{H}_1 = \mathcal{H}_2$, the following statements are equivalent (ℓ is a non-negative integer)*

(a) $M^\dagger = ((A_3A_3^*A_3)^*)^\ell[(A_1A_1^*A_1)^*M((A_3A_3^*A_3)^*)^\ell]^\dagger(A_1A_1^*A_1)^*$,
(b) $\mathcal{R}((A_1A_1^*)^3M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^{3\ell}M^*) = \mathcal{R}(M^*)$.

Remark 7.9. The previous two propositions were incorrectly stated as Propositions 2.5 and 2.6 in [14]. Additional hypothesis $\mathcal{H}_3 = \mathcal{H}_4$ for the first, and $\mathcal{H}_1 = \mathcal{H}_2$ for the second one, should be added.

Remark 7.10. Because of part 12. from Theorem 7.2, we have $(A^{(-k)})^* = (A^\dagger)^{(k-1)}$ and $(A^{(-k)})^\dagger = (A^*)^{(k-1)}$, so two previous theorems, 7.3 and 7.4, are actually equivalent under the condition that operator A is closed-range.

7.3. The mixed-type ROLs for bounded self-adjoint operators and Borel functions. Recall that function $f\colon X \to Y$ between the topological spaces X and Y is Borel if $f^{-1}(A)$ is Borel set for any open set $A \subset Y$. Particularly, every continuous mapping is a Borel function; see e.g. [42] for further properties.

The Borel functional calculus for self-adjoint operators is a well-known topic in the operator theory, see e.g. [41]. Recall that any self-adjoint operator has the unique Borel functional calculus.

Theorem 7.5. *Let f and g be two bounded complex-valued Borel functions on the real line such that*

$$(\forall\lambda \in \sigma(A_{11}A_{11}^*)) \quad f(\lambda) \neq 0, \qquad (\forall\lambda \in \sigma(A_{31}^*A_{31})) \quad g(\lambda) \neq 0, \tag{7.9}$$

where $A_{11} = A_1|_{\overline{\mathcal{R}(A_1^*)}}\colon \overline{\mathcal{R}(A_1^*)} \to \mathcal{R}(A_1)$, $A_{31} = A_3|_{\overline{\mathcal{R}(A_3^*)}}\colon \overline{\mathcal{R}(A_3^*)} \to \mathcal{R}(A_3)$. Suppose that the operators A_1, A_3, M and $f(A_1A_1^*)Mg(A_3^*A_3)$ have closed ranges. Then the following statements are equivalent

(a) $M^\dagger = g(A_3^*A_3)(f(A_1A_1^*)Mg(A_3^*A_3))^\dagger f(A_1A_1^*)$;
(b) $\mathcal{R}(f(A_1A_1^*)^*f(A_1A_1^*)M) = \mathcal{R}(M)$ *and* $\mathcal{R}(g(A_3^*A_3)g(A_3^*A_3)^*M^*) = \mathcal{R}(M^*)$.

Proof. Remark that by the Borel functional calculus for the Hermitian operators the operators

$$f(A_1A_1^*) = \begin{bmatrix} f(A_{11}A_{11}^*) & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad g(A_3^*A_3) = \begin{bmatrix} g(A_{31}^*A_{31}) & 0 \\ 0 & 0 \end{bmatrix}$$

are well defined. Also, note that $\sigma(A_{11}A_{11}^*) \subset (0, ||A_1||^2)$ and $\sigma(A_{31}^*A_{31}) \subset (0, ||A_3||^2)$ because $A_{11}A_{11}^*$ and $A_{31}^*A_{31}$ are positive operators.

Suppose, by using Lemma 1.2, that the operators A_1 and A_3 have the following matrix forms

$$A_k = \begin{bmatrix} A_{k1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{\mathcal{R}(A_k^*)} \\ \mathcal{N}(A_k) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A_k) \\ \mathcal{N}(A_k^*) \end{bmatrix}, \quad k = 1 \text{ or } 3,$$

where A_{11} and A_{31} are invertible. According to such space decompositions, it follows that operator A_2 has the following matrix form

$$A_2 = \begin{bmatrix} A_{21} & A_{22} \\ A_{23} & A_{24} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A_3) \\ \mathcal{N}(A_3^*) \end{bmatrix} \to \begin{bmatrix} \overline{\mathcal{R}(A_1^*)} \\ \mathcal{N}(A_1) \end{bmatrix}.$$

Also, we use the notation $M_1 = A_{11}A_{21}A_{31}$, so the matrix form of M is

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Now, (a) is equivalent to the following

$$M_1^\dagger = g(A_{31}^*A_{31})(f(A_{11}A_{11}^*)M_1g(A_{31}^*A_{31}))^\dagger f(A_{11}A_{11}^*).$$

Note that condition (7.9) ensures the invertibility of the operators $f(A_{11}A_{11}^*)$ and $g(A_{31}^*A_{31})$, according to the spectral mapping theorem (there holds only inclusion, not the equality). Indeed, $\sigma(g(A_{31}^*A_{31})) \subset g(\sigma(A_{31}^*A_{31}))$ and $0 \notin g(\sigma(A_{31}^*A_{31}))$ imply $0 \notin \sigma(g(A_{31}^*A_{31}))$. The analogous fact holds for $f(A_{11}A_{11}^*)$.

Direct calculations show this is equivalent to the following

$$f(A_{11}A_{11}^*)M_1M_1^\dagger f(A_{11}A_{11}^*)^{-1},\ g(A_{31}^*A_{31})^{-1}M_1^\dagger M_1 g(A_{31}^*A_{31})$$

should be Hermitian, which is equivalent to

$$[f(A_{11}A_{11}^*)^*f(A_{11}A_{11}^*), M_1M_1^\dagger] = 0,\ [g(A_{31}^*A_{31})g(A_{31}^*A_{31})^*, M_1^\dagger M_1] = 0.$$

On the other hand, the conditions (b) are equivalent to conjunction

$$\mathcal{R}(f(A_{11}A_{11}^*)^*f(A_{11}A_{11}^*)M_1) = \mathcal{R}(M_1),$$
$$\mathcal{R}(g(A_{31}^*A_{31})g(A_{31}^*A_{31})^*M_1^*) = \mathcal{R}(M_1^*).$$

By Lemma 1.5, we have the proof. □

As a corollaries of Theorem 4.1 we can obtain Theorem 2.4 (for $f(x) = g(x) = x$), Proposition 2.2 (for $f(x) = x^k,\ g(x) = x^\ell$), Theorem 2.8 (for $f(x) = g(x) = x^2$) and Proposition 2.8 (for $f(x) = x^{2k},\ g(x) = x^{2\ell}$) from [14].

Corollary 7.7. *Let A_1, A_3, M, $A_1A_1^*MA_3^*A_3$ have closed ranges. Then the following statements are equivalent*

(a) $M^\dagger = A_3^*A_3(A_1A_1^*MA_3^*A_3)^\dagger A_1A_1^*$;

(b) $\mathcal{R}((A_1A_1^*)^2M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^2M^*) = \mathcal{R}(M^*)$.

Corollary 7.8. *Under the assumptions of Corollary 7.7, the following statements are equivalent (k and ℓ are non-negative integers)*

(a) $M^\dagger = (A_3^*A_3)^\ell[(A_1A_1^*)^k M(A_3^*A_3)^\ell]^\dagger(A_1A_1^*)^k$,
(b) $\mathcal{R}((A_1A_1^*)^{2k}M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^{2\ell}M^*) = \mathcal{R}(M^*)$.

Corollary 7.9. *Let A_1, A_3, M, $(A_1A_1^*)^2M(A_3^*A_3)^2$ have closed ranges. Then the following statements are equivalent*

(a) $M^\dagger = (A_3^*A_3)^2((A_1A_1^*)^2M(A_3^*A_3)^2)^\dagger(A_1A_1^*)^2$;
(b) $\mathcal{R}((A_1A_1^*)^4M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^4M^*) = \mathcal{R}(M^*)$.

Corollary 7.10. *Under the assumptions of the Corollary 7.9, the following statements are equivalent (k and ℓ are non-negative integers)*

(a) $M^\dagger = ((A_3^*A_3)^*)^{2\ell}(((A_1A_1^*)^*)^{2k}M((A_3^*A_3)^*)^{2\ell})^\dagger((A_1A_1^*)^*)^{2k}$,
(b) $\mathcal{R}((A_1A_1^*)^{4k}M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^{4\ell}M^*) = \mathcal{R}(M^*)$.

Theorem 7.6. *Let f and g be two bounded complex-valued Borel functions on the real line such that*

$$(\forall\lambda \in \sigma((A_{11}A_{11}^*)^{-1})) \quad f(\lambda) \neq 0, \qquad (\forall\lambda \in \sigma((A_{31}^*A_{31})^{-1})) \quad g(\lambda) \neq 0,$$

where $A_{11} = A_1|_{\mathcal{R}(A_1^*)}: \mathcal{R}(A_1^*) \to \mathcal{R}(A_1)$, $A_{31} = A_3|_{\mathcal{R}(A_3^*)}: \mathcal{R}(A_3^*) \to \mathcal{R}(A_3)$. Suppose that the operators A_1, A_3, M and $f((A_1A_1^*)^\dagger)Mg((A_3^*A_3)^\dagger)$ have closed ranges. Then the following statements are equivalent

(a) $M^\dagger = g((A_3^*A_3)^\dagger)(f((A_1A_1^*)^\dagger)Mg((A_3^*A_3)^\dagger))^\dagger f((A_1A_1^*)^\dagger)$;
(b) $\mathcal{R}(f(A_1A_1^*)^*f(A_1A_1^*)M) = \mathcal{R}(M)$ *and* $\mathcal{R}(g(A_3^*A_3)g(A_3^*A_3)^*M^*) = \mathcal{R}(M^*)$

The proof of this theorem is very similar to that of Theorem 7.5.

As the corollaries of Theorem 7.6 one can obtain Theorem 2.3 (for $f(x) = g(x) = x$), Proposition 2.1 (for $f(x) = x^k$, $g(x) = x^\ell$), Theorem 2.7 (for $f(x) = g(x) = x^2$) and Proposition 2.7 (for $f(x) = x^{2k}$, $g(x) = x^{2\ell}$) from [14].

Corollary 7.11. *Let $A_1, A_3, M, (A_1A_1^*)^\dagger M(A_3^*A_3)^\dagger$ have closed ranges. Then the following statements are equivalent*

(a) $M^\dagger = (A_3^*A_3)^\dagger[(A_1A_1^*)^\dagger M(A_3^*A_3)^\dagger]^\dagger(A_1A_1^*)^\dagger$;
(b) $\mathcal{R}((A_1A_1^*)^2M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^2M^*) = \mathcal{R}(M^*)$.

Proposition 7.6. *Under the assumptions of Proposition 7.11, the following statements are equivalent (k and ℓ are non-negative integers)*

(a) $M^\dagger = ((A_3^*A_3)^\dagger)^\ell[((A_1A_1^*)^\dagger)^k M((A_3^*A_3)^\dagger)^\ell]^\dagger((A_1A_1^*)^\dagger)^k$;
(b) $\mathcal{R}((A_1A_1^*)^{2k}M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^{2\ell}M^*) = \mathcal{R}(M^*)$.

Corollary 7.12. *Let A_1, A_3, M, $((A_1A_1^*)^2)^\dagger M((A_3^*A_3)^2)^\dagger$ have closed ranges. Then the following statements are equivalent*

(a) $M^\dagger = ((A_3^*A_3)^\dagger)^2[((A_1A_1^*)^2)^\dagger M(A_3^*A_3)^2)^\dagger]^\dagger((A_1A_1^*)^\dagger)^2$;
(b) $\mathcal{R}((A_1A_1^*)^4M) = \mathcal{R}(M)$ *and* $\mathcal{R}((D_3)^4M^*) = \mathcal{R}(M^*)$.

Proposition 7.7. *Under the assumptions of Corollary 7.12, the following statements are equivalent (k and ℓ are non-negative integers)*

(a) $M^\dagger = ((A_3^*A_3)^\dagger)^{2\ell}[((A_1A_1^*)^{2k})^\dagger M(A_3^*A_3)^{2\ell})^\dagger]^\dagger((A_1A_1^*)^\dagger)^{2k}$,
(b) $\mathcal{R}((A_1A_1^*)^{4k}M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^*A_3)^{4\ell}M^*) = \mathcal{R}(M^*)$.

7.3.1. Some equivalencies. In what follows we present some equivalencies established between the ROLs considered in the previous section. The results are unpublished, and they are significant generalizations of those from [47] and [14].

Theorem 7.7. *The following statements are equivalent (provided that we apply the Moore–Penrose inverse to closed range operators and $k, \ell \in \mathbb{N}_0$)*

(a) $M^\dagger = (A_3^{(\ell)})^\dagger((A_1^{(k)})^\dagger M(A_3^{(\ell)})^\dagger)^\dagger(A_1^{(k)})^\dagger$;

(b) $M^\dagger = (A_3^{(\ell)})^*((A_1^{(k)})^* M(A_3^{(\ell)})^*)^\dagger(A_1^{(k)})^*$;

(c) $(A_3^{(\ell)})^\dagger((A_1^{(k)})^\dagger M(A_3^{(\ell)})^\dagger)^\dagger(A_1^{(k)})^\dagger$
$= ((A_3^*A_3)^\dagger)^{2\ell+1}(((A_1A_1^*)^\dagger)^{2k+1}M((A_3^*A_3)^\dagger)^{2\ell+1})^\dagger((A_1A_1^*)^\dagger)^{2k+1}$;

(d) $(A_3^{(\ell)})^*((A_1^{(k)})^\dagger M(A_3^{(\ell)})^\dagger)^\dagger(A_1^{(k)})^* = (A_3^*A_3)^{2\ell+1}M^\dagger(A_1A_1^*)^{2k+1}$;

(e) $(A_3^{(\ell)})^*((A_1^{(k)})^* M(A_3^{(\ell)})^*)^\dagger(A_1^{(k)})^*$
$= (A_3^*A_3)^{2\ell+1}((A_1A_1^*)^{2k+1}M(A_3^*A_3)^{2\ell+1})^\dagger(A_1A_1^*)^{2k+1}$;

(f) $\mathcal{R}(A_1^{(k)}(A_1^{(k)})^*M) = \mathcal{R}(M)$ *and* $\mathcal{R}((A_3^{(\ell)})^*A_3^{(\ell)}M^*) = \mathcal{R}(M^*)$.

Proof. Part (a)⇔(f): follows from Theorem 7.3, while (b)⇔(f) follows from Theorem 7.4.

Suppose, by using Lemma 1.2, that the operators A_1 and A_3 have the following matrix forms (remark that $\overline{W}$ means the closure of some subspace W)

$$A_k = \begin{bmatrix} A_{k1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \overline{\mathcal{R}(A_k^*)} \\ \mathcal{N}(A_k) \end{bmatrix} \to \begin{bmatrix} \overline{\mathcal{R}(A_k)} \\ \mathcal{N}(A_k^*) \end{bmatrix}, \quad k = 1 \text{ or } 3,$$

where A_{11} and A_{31} are invertible. According to such space decompositions, it follows that operator A_2 has the following matrix form

$$A_2 = \begin{bmatrix} A_{21} & A_{22} \\ A_{23} & A_{24} \end{bmatrix} : \begin{bmatrix} \overline{\mathcal{R}(A_3)} \\ \mathcal{N}(A_3^*) \end{bmatrix} \to \begin{bmatrix} \overline{\mathcal{R}(A_1^*)} \\ \mathcal{N}(A_1) \end{bmatrix}.$$

Also, we use the notation $M_1 = A_{11}A_{21}A_{31}$, so the matrix form of M is

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then

$$A_1^{(k)} = \begin{bmatrix} A_{11}^{(k)} & 0 \\ 0 & 0 \end{bmatrix}, \quad A_3^{(\ell)} = \begin{bmatrix} A_{31}^{(\ell)} & 0 \\ 0 & 0 \end{bmatrix},$$

Now we can express those statements in a more convenient form

$$(c) \Leftrightarrow (A_{31}^{(\ell)})^\dagger((A_{11}^{(k)})^\dagger M_1(A_{31}^{(\ell)})^\dagger)^\dagger(A_{11}^{(k)})^\dagger$$
$$= ((A_{31}^*A_{31})^\dagger)^{2\ell+1}(((A_{11}A_{11}^*)^\dagger)^{2k+1}M_1((A_{31}^*A_{31})^\dagger)^{2\ell+1})^\dagger((A_{11}A_{11}^*)^\dagger)^{2k+1},$$

$$(d) \Leftrightarrow (A_{31}^{(\ell)})^*((A_{11}^{(k)})^{-1}M_1(A_{31}^{(\ell)})^{-1})^\dagger(A_{11}^{(k)})^* = (A_{31}^*A_{31})^{2\ell+1}M_1^\dagger(A_{11}A_{11}^*)^{2k+1};$$

$$(e) \Leftrightarrow (A_{31}^{(\ell)})^*((A_{11}^{(k)})^*M_1(A_{31}^{(\ell)})^*)^\dagger(A_{11}^{(k)})^*$$
$$= (A_{31}^*A_{31})^{2\ell+1}((A_{11}A_{11}^*)^{2k+1}M_1(A_{31}^*A_{31})^{2\ell+1})^\dagger(A_{11}A_{11}^*)^{2k+1}$$

By checking the Penrose equations (and using Lemma 1.5) we conclude that all three statements are equivalent to

$$\mathcal{R}(A_{11}^{(k)}(A_{11}^{(k)})^*M_1) = \mathcal{R}(M_1) \wedge \mathcal{R}((A_{31}^{(\ell)})^*A_{31}^{(\ell)}M_1^*) = \mathcal{R}(M_1^*),$$

i.e. to (f).

Remark that during the proof, an obvious fact

$$\mathcal{R}(PQ) = \mathcal{R}(SQ) \Leftrightarrow \mathcal{R}(P) = \mathcal{R}(S)$$

if Q is invertible, is used. □

Theorem 7.8. *The following statements are equivalent (provided that we apply the Moore–Penrose inverse to closed range operators)*

(a) $M^\dagger = g((A_3^*A_3)^\dagger)(f((A_1A_1^*)^\dagger)Mg((A_3^*A_3)^\dagger))^\dagger f((A_1A_1^*)^\dagger)$;
(b) $M^\dagger = g(A_3^*A_3)(f(A_1A_1^*)Mg(A_3^*A_3))^\dagger f(A_1A_1^*)$;
(c) $g((A_3^*A_3)^\dagger)A_3^*(A_1^*f((A_1A_1^*)^\dagger)Mg((A_3^*A_3)^\dagger)A_3^*)^\dagger A_1^*f((A_1A_1^*)^\dagger)$
$= A_3^*(A_1^*MA_3^*)^\dagger A_1^*$;
(d) $\mathcal{R}(f(A_1A_1^*)^*f(A_1A_1^*)M) = \mathcal{R}(M) \wedge \mathcal{R}(g(A_3^*A_3)g(A_3^*A_3)^*M^*) = \mathcal{R}(M^*)$.

Proof. From Theorems 7.6 and 7.5 it follows that (a)⇔(b)⇔(d). Using the method described in those two theorems, we easily conclude that (we abbreviate $G := g((A_{31}^*A_{31})^{-1})$ and $F := f((A_{11}A_{11}^*)^{-1})$)

$$(c) \Leftrightarrow GA_{31}^*(A_{11}^*FM_1GA_{31}^*)^\dagger A_{11}^*F = A_{31}^*(A_{11}^*M_1A_{31}^*)^\dagger A_{11}^*,$$

i.e.

$$(A_{11}^*FM_1GA_{31}^*)^\dagger = (A_{31}^*)^{-1}G^{-1}A_{31}^*(A_{11}^*M_1A_{31}^*)^\dagger A_{11}^*F^{-1}(A_{11}^*)^{-1}.$$

Note that F and G are invertible, as in the proof of Theorem 7.6. Now, the third Penrose equation becomes

$$\begin{aligned} &A_{11}^*FM_1GA_{31}^*(A_{31}^*)^{-1}G^{-1}A_{31}^*(A_{11}^*M_1A_{31}^*)^\dagger A_{11}^*F^{-1}(A_{11}^*)^{-1} \\ &\quad = A_{11}^*FM_1A_{31}^*(A_{11}^*M_1A_{31}^*)^\dagger A_{11}^*F^{-1}(A_{11}^*)^{-1} \\ &\quad = A_{11}^*F(A_{11}^*)^{-1}A_{11}^*M_1A_{31}^*(A_{11}^*M_1A_{31}^*)^\dagger A_{11}^*F^{-1}(A_{11}^*)^{-1} \\ &\quad = A_{11}^{-1}(F^{-1})^*A_{11}A_{11}^*M_1A_{31}^*(A_{11}^*M_1A_{31}^*)^\dagger A_{11}^{-1}F^*A_{11}, \end{aligned}$$

which is, by Lemma 1.5, equivalent to

$$\mathcal{R}(A_{11}^{-1}F^*A_{11}A_{11}^*F(A_{11}^*)^{-1}A_{11}^*M_1A_{31}^*) = \mathcal{R}(A_{11}^*M_1A_{31}^*).$$

By using the fact that $F = f((A_{11}A_{11}^*)^{-1})$ commutes with $A_{11}A_{11}^*$, we easily find that

$$\mathcal{R}(F^*FM_1) = \mathcal{R}(M_1).$$

By using the fourth Penrose equation, we obtain the other range equality. Therefore, we conclude that (c)⇔(d). □

Theorem 7.9. *The following statements are equivalent (provided that we apply the Moore–Penrose inverse to closed range operators)*

(a) $M^\dagger = (A_3^{(\ell)})^\dagger((A_1^{(k)})^\dagger M(A_3^{(\ell)})^\dagger)^\dagger(A_1^{(k)})^\dagger$;
(b) $M^\dagger = (A_3^{(\ell)})^*((A_1^{(k)})^*M(A_3^{(\ell)})^*)^\dagger(A_1^{(k)})^*$;

(c) $A_3^\dagger(A_1^\dagger MA_3^\dagger)^\dagger A_1^\dagger = A_3^\dagger A_3^{(\ell)}(A_1^{(k)}A_1^\dagger MA_3^\dagger A_3^{(\ell)})^\dagger A_1^{(k)}A_1^\dagger$;
(d) $(A_1^\dagger MA_3^\dagger)^\dagger = A_3^{(\ell)}(A_1^{(k)}A_1^\dagger MA_3^\dagger A_3^{(\ell)})^\dagger A_1^{(k)}$;
(e) $\mathcal{R}(A_1^{(k)}(A_1^{(k)})^*M) = \mathcal{R}(M) \wedge \mathcal{R}((A_3^{(\ell)})^*A_3^{(\ell)}M^*) = \mathcal{R}(M^*)$.

Proof. From Theorems 7.5 and 7.6 it follows that (a)⇔(b)⇔(e). Since A_{11} and A_{31} are invertible, we have (c)⇔(d). Let us now prove part d). Using the method described in those two theorems, we easily conclude that

$$(A_{31}^{-1})^{(\ell)}(A_{11}^{-1}M_1A_{31}^{-1})^\dagger(A_{11}^{-1})^{(k)} = (A_{11}^{(k)}A_{11}^{-1}M_1A_{31}^{-1}A_{31}^{(\ell)})^\dagger.$$

From the third Penrose equation we have

$$\begin{aligned} A_{11}^{(k)}A_{11}^{-1}M_1A_{31}^{-1}A_{31}^{(\ell)}(A_{31}^{-1})^{(\ell)}(A_{11}^{-1}M_1A_{31}^{-1})^\dagger(A_{11}^{-1})^{(k)} \\ = A_{11}^{(k)}A_{11}^{-1}M_1A_{31}^{-1}(A_{11}^{-1}M_1A_{31}^{-1})^\dagger(A_{11}^{-1})^{(k)} \\ = (A_{11}^{(k)}A_{11}^{-1}M_1A_{31}^{-1}(A_{11}^{-1}M_1A_{31}^{-1})^\dagger(A_{11}^{-1})^{(k)})^* \\ = ((A_{11}^*)^{-1})^{(k)}A_{11}^{-1}M_1A_{31}^{-1}(A_{11}^{-1}M_1A_{31}^{-1})^\dagger(A_{11}^*)^{(k)}, \end{aligned}$$

which is, by Lemma 1.5, equivalent to

$$\mathcal{R}((A_{11}^*)^{(k)}A_{11}^{(k)}A_{11}^{-1}M_1A_{31}^{-1}) = \mathcal{R}(A_{11}^{-1}M_1A_{31}^{-1}).$$

Since

$$\begin{aligned} (A_{11}^*)^{(k)}A_{11}^{(k)}A_{11}^{-1} &= (A_{11}^*A_{11})^{2k+1}A_{11}^{-1} = A_{11}^{-1}A_{11}A_{11}^*(A_{11}A_{11}^*)^{2k} \\ &= A_{11}^{-1}(A_{11}A_{11}^*)^{2k} = A_{11}^{-1}A_{11}^{(k)}(A_{11}^*)^{(k)}, \end{aligned}$$

we have

$$\mathcal{R}(A_{11}^{-1}A_{11}^{(k)}(A_{11}^*)^{(k)}M_1A_{31}^{-1}) = \mathcal{R}(A_{11}^{-1}M_1A_{31}^{-1}),$$

from where it follows $\mathcal{R}(A_{11}^{(k)}(A_{11}^*)^{(k)}M_1) = \mathcal{R}(M_1)$.

In a similar way, one can prove the other range equality. □

Theorem 7.10. *The following statements are equivalent* (*provided that we apply the Moore–Penrose inverse to closed range operators*)

(a) $M^\dagger = g((A_3^*A_3)^\dagger)(f((A_1A_1^*)^\dagger)M(g(A_3^*A_3)^\dagger))^\dagger f((A_1A_1^*)^\dagger)$;
(b) $M^\dagger = g(A_3^*A_3)(f(A_1A_1^*)Mg(A_3^*A_3))^\dagger f(A_1A_1^*)$;
(c) $\mathcal{R}(f(A_1A_1^*)^*f(A_1A_1^*)M) = \mathcal{R}(M) \wedge \mathcal{R}(g(A_3^*A_3)g(A_3^*A_3)^*M^*) = \mathcal{R}(M^*)$.

Proof. From Theorems 7.5 and 7.6 it follows that (a)⇔(b)⇔(c). □

Remark that those results are further investigated in e.g. [43, 54, 63].

Acknowledgement. The authors are supported by the Ministry of Education, Science and Technological Development, Republic of Serbia, grant No. 451-03-68/2022-14/200124.

References

[1] E. Arghiriade, *Sur les matrices qui sont permutables avec leur inverse généralisée*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **35** (1963), 244–251.

[2] O. M. Baksalary, G. Trenkler, *The Moore–Penrose inverse: a hundred years on a frontline of physics research*, Eur. Phys. J. H **46** (2021), Article nr. 9.

[3] A. Ben-Israel, T. N. E. Greville, *Generalized Inverses: Theory and Applications*, 2nd Ed., Springer, New York, 2003.

[4] R. H. Bouldin, *The pseudo-inverse of a product*, SIAM J. Appl. Math. **24**(4) (1973), 489–495.

[5] R. H. Bouldin, *Generalized inverses and factorizations*, in: S. La Vern Campbell, ed., *Recent applications of generalized inverses*, Pitman Ser. Res. Notes in Math. **66** (1982), 233–248.

[6] S. R. Caradus, *Generalized inverses and operator theory*, Queen's Pap. Pure Appl. Math., Queen's University, Kingston, Ontario, 1978.

[7] S. L. Campbell, C. D. Meyer, *Generalized inverses of linear transformations*, Classics Appl. Math. **56**, SIAM, 2009.

[8] N. Castro-González, R. E. Hartwig, *Perturbation results and the forward order law for the Moore–Penrose inverse of a product*, Electron. J. Linear Algebra **34** (2018), 514–525.

[9] R. E. Cline, *Note on the generalized inverse of the product of matrices*, SIAM Rev. **6**(1) (1964), 57–58.

[10] T. Damm, H. K. Wimmer, *A cancellation property of the Moore–Penrose inverse of triple products*, J. Aust. Math. Soc. **86** (2009), 33–44.

[11] A. R. De Pierro, M. Wei, *Reverse order law for reflexive generalized inverses of products of matrices*, Linear Algebra Appl. **277** (1998), 299–311.

[12] N. Č. Dinčić, *Extending the Moore–Penrose inverse*, Filomat **30**(2) (2016), 419–428

[13] N. Č. Dinčić, *Mixed-type reverse order law, ternary powers and functional calculus*, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **114**(10) (2020)

[14] N. Č. Dinčić, D. S. Djordjević, *Mixed-type reverse order law for the products of three operators*, Linear Algebra Appl. **435**(11) (2011), 2658–2673.

[15] N. Č. Dinčić, D. S. Djordjević, D. Mosić, *Mixed-type reverse order law and its equivalencies*, Stud. Math. **204** (2011), 123–136.

[16] N. Č. Dinčić, D. S. Djordjević, *Basic reverse order law and its equivalencies*, Aequationes Math. **85**(3) (2013), 505–517.

[17] N. Č. Dinčić, D. S. Djordjević, *Identities concerning the reverse order law for the Moore–Penrose inverse*, Appl. Math. Comput. **220** (2013), 439–445.

[18] N. Č. Dinčić, D. S. Djordjević, *Hartwig's triple reverse order law revisited*, Linear Multilinear Algebra, **62**(7) (2014), 918–924.

[19] D. S. Djordjević, *Unified approach to the reverse order rule for generalized inverses*, Acta Sci. Math. (Szeged) **67** (2001), 761–776.

[20] D. S. Djordjević, *Further results on the reverse order law for generalized inverses*, SIAM J. Matrix Anal. Appl. **29**(4) (2007), 1242–1246.

[21] D. S. Djordjević, N. Č. Dinčić, *Reverse order law for the Moore–Penrose inverse*, J. Math. Anal. Appl. **361**(1) (2010), 252–261.

[22] D. S. Djordjević, V. Rakočević, *Lectures on Generalized Inverses*, Faculty of Sciences and Mathematics, University of Niš, 2008.

[23] A. M. Galperin, Z. Waksman, *On pseudoinverse of operator products*, Linear Algebra Appl. **33** (1980), 123–131.

[24] T. N. E. Greville, *Note on the generalized inverse of a matrix product*, SIAM Rev. **8**(4) (1966), 518–521.

[25] C. W. Groetsch, *Generalized Inverses of Linear Operators: Representation and Approximation*, Marcel Dekker, 1977.

[26] R. E. Hartwig, *The reverse order law revisited*, Linear Algebra Appl. **76** (1986), 241–246.

[27] M. R. Hestenes, *A ternary algebra with applications to matrices and linear transformations*, Arch. Ration. Mech. Anal. **11**(1) (1962), 138–194.

[28] S. Izumino, *The product of operators with closed range and an extension of the reverse order law*, Tôhoku Math. J. **34** (1982), 43–52.
[29] J. J. Koliha, D. S. Djordjević, D. S. Cvetković–Ilić, *Moore–Penrose inverse in rings with involution*, Linear Algebra Appl. **426** (2007), 371–381.
[30] A. Korporal, G. Regensburger, *On the product of projectors and generalized inverses*, Linear Multilinear Algebra **62**(12) (2014), 1567–1582.
[31] D. Krishnaswamy, V. Vijayaselvi, *Hartwig's triple reverse order law for the core inverse in C^*-algebras*, J. Math. Comput. Sci. **11** (2021), 7717–7727
[32] S. H. Kulkarni, M. T. Nair, G. Ramesh, *Some properties of unbounded operators with closed range*, Proc. Indian Acad. Sci. (Math. Sci.) **118**(4) (2008), 613–625.
[33] Y. Li, *The Moore–Penrose inverses of products and differences of projections in a C^*-algebra*, Linear Algebra Appl. **428** (2008), 1169–1177.
[34] J. Milošević, *Hartwig's triple reverse order law in C^*-algebras*, Filomat **32**(12) (2018), 4229–4232.
[35] D. Mosić, *Generalized Inverses*, Univerzitet u Nišu, Prirodno-matematički fakultet, Niš, 2018.
[36] D. Mosić, N. Č. Dinčić, *Reverse order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ in rings with involution*, Filomat **28**(9) (2014), 1791–1815.
[37] D. Mosić, D. S. Djordjević, *Reverse order law for the Moore–Penrose inverse in C^*-algebras*, Electron. J. Linear Algebra **22** (2011), 92–111.
[38] D. Mosić, D. S. Djordjević, *Reverse order law in C^*-algebras*, Appl. Math. Comput. **218**(7) (2011), 3934–3941.
[39] D. Mosić, D. S. Djordjević, *Mixed-type reverse order laws for generalized inverses in rings with involution*, Publ. Math. Debrecen **82**(3–4) (2013), 641–650.
[40] J. Nikolov Radenković, *Reverse order law for generalized inverses of multiple operator product*, Linear Multilinear Algebra **64**(7) (2016), 1266–1282.
[41] M. Reed, B. Simon, *Methods of Modern Mathematical Physics Vol I: Functional Analysis*, Academic Press, 1980.
[42] W. Rudin, *Real and Complex Analysis*, 3$^{\text{rd}}$ edition, McGraw-Hill, 1987.
[43] K. S. Stojanović, D. Mosić, *Generalization of the Moore–Penrose inverse*, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **114**(4) (2020), paper No. 196, 16 p.
[44] W. Sun, Y. Wei, *Inverse order rule for weighted generalized inverse*, SIAM J. Matrix Anal. Appl. **19**(3) (1998), 772–775.
[45] H. Tian, *On the reverse order laws $(AB)^D = B^D A^D$*, J. Math. Res. Expo. **19** (1999), 355–358.
[46] Y. Tian, *Using rank formulas to characterize equalities for Moore–Penrose inverse of a matrix product*, Appl. Math. Comput. **147** (2004), 581–600.
[47] Y. Tian, *Some mixed-type reverse-order laws for the Moore–Penrose inverse of a triple matrix product*, Rocky Mt. J. Math. **37**(4) (2007), 1327–1347.
[48] Y. Tian, *The equivalence between $(AB)^\dagger = B^\dagger A^\dagger$ and other mixed-type reverse-order laws*, Int. J. Math. Educ. Sci. Technol. **37**(3) (2007), 331–339.
[49] Y. Tian, *The Moore–Penrose inverse of a triple matrix product*, Math. Pract. Theory **1** (1992), 64–70. (in Chinese)
[50] Y. Tian, *Reverse order laws for the generalized inverses of multiple matrix products*, Linear Algebra Appl. **211** (1994), 85–100.
[51] Y. Tian, *The reverse-order law $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$ and its equivalent equalities*, J. Math. Kyoto. Univ. **45**(4) (2005), 841–850.
[52] Y. Tian, *Equalities and inequalities for ranks of products of generalized inverses of two matrices and their applications*, J. Inequal. Appl. **182** (2016), 51 p.
[53] Y. Tian, *Miscellaneous reverse order laws for generalized inverses of matrix products with applications*, Adv. Oper. Theory **5**(4) (2020), 1889–1942.
[54] Y. Tian, *Two groups of mixed reverse order laws for generalized inverses of two and three matrix products*, Comput. Appl. Math. **39**(3) (2020), paper No. 181, 15 p.

[55] Y. Tian, S. Cheng, *Some identities for Moore–Penrose inverses of matrix products*, Linear Multilinear Algebra **52**(6) (2004), 405–420.
[56] G. Wang, *The reverse order law for the Drazin inverses of multiple matrix products*, Linear Algebra Appl. **348** (2002), 265–272.
[57] G. Wang, Y. Wei, S. Qiao, *Generalized Inverses: Theory and Computations*, Science Press, Beijing–New York, 2018.
[58] Y. Wei, *Equivalent conditions for generalized inverse of products*, Linear Algebra Appl. **266** (1997), 347–363.
[59] M. Wei, *Reverse order laws for generalized inverses of multiple matrix products*, Linear Algebra Appl. **293** (1999), 273–288.
[60] M. Wei, W. Guo, *Reverse order laws for least squares g-inverses and minimum norm g-inverses of products of two matrices*, Linear Algebra Appl. **342** (2002), 117–132.
[61] H. J. Werner, *G-inverses of matrix products*, in: S. Schach, G. Trenkler, eds., *Data Analysis and Statistical Inference*, Verlag Josef Eul, 1992, 531–546.
[62] H. J. Werner, *When is B^-A^- a generalized inverse of AB?*, Linear Algebra Appl. **210** (1994), 255–263.
[63] L. Wang, S. S. Zhang, X. X. Zhang, J. L. Chen, *Mixed-type reverse order law for Moore–Penrose inverse of products of three elements in ring with involution*, Filomat **28**(10) (2014), 1997–2008.
[64] Z. Xiong, Y. Qin, *Triple reverse order law for Moore–Penrose inverse of operator product*, J. Comput. Anal. Appl. **23**(8) (2017), 1347–1358.
[65] Q. Xu, C. Song, G. Wang, *Multiplicative perturbations of matrices and the generalized triple reverse order law for the Moore–Penrose inverse*, Linear Algebra Appl. **530** (2017), 366–383.