Ivana Djolović

A NOTE ON MATRIX TRANSFORMATIONS AND SOME CLASSES OF OPERATORS

Abstract. The idea of this paper is to interest readers in sequence spaces and matrix transformations as the starting point for possible applications in operator theory. In this paper we give survey of the known results on the so–called classical sequence spaces - the sets ℓ_∞, c and c_0 of bounded, convergent and null sequences. We consider their basic properties, β-duals and the characterizations of matrix transformations between them. After that, we establish some results related to general linear operators from the space c into each of the classical spaces. Furthermore, we characterize the classes of compact operators between them, applying two approaches for that purpose - the Hausdorff measure of noncompactness and Sargent's results [20]. Presented results together with all the other known results about compactness, will close some gaps in the existing literature. All these results are collected in the same place and can be a useful start for further research. We also preesnted some possible ideas for further work in the area of doubly stochastic operators.

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University of Belgrade, Technical Faculty in Bor, Serbia
`idjolovic@tfbor.bg.ac.rs`

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1. Sequence spaces, matrix transformations and bounded linear operators

The theory of sequence spaces and the characterizations of matrix transformations between them play an eminent role in modern summability theory. We will see that matrix transformations between sequences spaces of a fairly general class can be considered as bounded linear operators.

1.1. Introduction into FK and BK theory. The theory of FK and BK spaces plays an important role in the characterization of matrix transformations between sequence spaces and arises from Fréchet spaces that are continuously embedded in the space ω of all complex sequences $x = (x_k)_{k=0}^{\infty}$. Hence we start with the basic notations, definitions and results.

It is well-known that ω is a Fréchet space, that is, a complete linear metric space with its metric defined by

$$d(x,y) = \sum_{k=0}^{\infty} \frac{1}{2^k} \cdot \frac{|x_k - y_k|}{1 + |x_k - y_k|}, \quad \text{for all } x, y \in \omega.$$

An FK space X is a Fréchet sequence space with continuous coordianates $P_k \colon X \to \mathbb{C}$ defined by $P_k(x) = x_k$ $(x \in X)$ for all k. Since convergence and coordinatewise convergence are equivalent in ω, and FK space is a complete linear metric sequence space with the property that convergence implies coordinatewise convergence; a BK space is normed FK space. We say that X has AK, or that X is an AK space, if $x^{[m]} = \sum_{k=0}^{m} x_k e^{(k)} \to x$ $(m \to \infty)$ for every sequence $x = (x_k)_{k=0}^{\infty} \in X$.

Let e be the sequence with $e_k = 1$ $(k = 0, 1, 2, \dots)$. By $e^{(n)}$ $(n = 0, 1 \dots)$ we denote the sequence $(e_k^{(n)})_{k=0}^{\infty}$ defined by

$$e_k^{(n)} = \begin{cases} 0 & (k \neq n) \\ 1 & (k = n). \end{cases}$$

If X and Y are normed spaces, then, as usual, $B(X, Y)$ denotes the space of all bounded linear operators $L\colon X \to Y$, which is a Banach space with the operator norm defined by $\|L\| = \{\|L(x)\| \mid \|x\| = 1\}$, whenever Y is a Banach space. We write $X^* = B(X, \mathbb{C})$ for the space of all continuous linear functionals f on X with the norm of f defined by $\|f\| = \sup\{|f(x)| \mid \|x\| = 1\}$.

Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex entries and $A_n = (a_{nk})_{k=0}^{\infty}$ denote the sequence in the n^{th} row of A. We write

$$A_n x = \sum_{k=0}^{\infty} a_{nk} x_k \text{ and } Ax = (A_n x)_{n=0}^{\infty} \text{ (provided all the series converge).}$$

If X and Y are subsets of ω, then (X, Y) denotes the class of all matrices that map X into Y, that is, $A \in (X, Y)$ if and only if the series $A_n x$ converge for all $x \in X$ and for all n, and $Ax \in Y$ for all $x \in X$.

Theorem 1.1. [10, Corollary 1.15.], [22, Corollary 4.2.3] *Let X be a Fréchet space, Y an FK space, $f\colon X \to Y$ a linear map and $P_n\colon Y \to \mathbb{C}$ $(n \in \mathbb{N}_0)$ be the n^{th} coordinate. If each map $P_n \circ f\colon X \to \mathbb{C}$ is continuous, so is $f\colon X \to Y$.*

We denote the set of all finite sequences by ϕ.

Theorem 1.2. [10, Remark 1.16.] *Let $X \supset \phi$ be an FK space. If the series $\sum_{k=0}^{\infty} a_k x_k$ converges for each $x \in X$, then the linear functional $f_a\colon X \to \mathbb{C}$ defined by*

$$f_a(x) = \sum_{k=0}^{\infty} a_k x_k \text{ for all } x \in X,$$

is continuous.

The next theorem is very important and fundamental in the theory of matrix transformations.

Theorem 1.3. [10, Theorem 1.17.], [22, Theorem 4.2.8.] *Any matrix map between FK spaces is continuous.*

Proof. Let X and Y be FK spaces, $A \in (X, Y)$ and the map $f_A\colon X \to Y$ be defined by $f_A(x) = Ax$ for all $x \in X$. By Theorem 1.2, the maps $P_n \circ f_A : X \to \mathbb{C}$ are continuous for all n; hence, by Theorem 1.1, $f_A\colon X \to Y$ is continuous. □

The following result is also of great importance in our research and frequently used; its first part is a restatement of Theorem 1.3.

Theorem 1.4. [10, Theorem 1.23], [4, Theorem 1.9.], [22, Theorem 4.2.8] (i) *Let X and Y be FK spaces. Then we have $(X, Y) \subset B(X, Y)$, that is, every $A \in (X, Y)$ defines a linear operator $L_A \in B(X, Y)$ where $L_A(x) = Ax$ $(x \in X)$.*
(ii) *Let X and Y be BK spaces and X have AK. Then $B(X, Y) \subset (X, Y)$, that is, every $L \in B(X, Y)$ can be represented by a matrix $A \in (X, Y)$ such that $L(x) = Ax$ for all $x \in X$.*

1.2. β-duals. An important role in the characterization of matrix transformations is played by β-duals. They are special case of multiplier spaces.

Definition 1.1. Let X and Y be subsets of ω. The set

$$M(X, Y) = \{a \in \omega \mid ax = (a_k x_k)_{k=0}^{\infty} \in Y \text{ for all } x \in X\}$$

is called the multiplier space of X and Y. If we denote the set of all convergent series by cs, the multiplier space $X^\beta = M(X, cs)$ is called the β-dual of X, that is,

$$X^\beta = \left\{a \in \omega \mid \sum_{k=0}^{\infty} a_k x_k \text{ converges for all } x \in X\right\}.$$

Now, it is clear that

$$A \in (X, Y) \text{ if and only if } A_n \in X^\beta \text{ for all } n \text{ and } Ax \in Y \text{ for all } x \in X.$$

The readers are referred to [9, 10, 22] for more detailed studies.

The multiplier spaces, and in particular, the β-dials of BK spaces again are BK spaces.

Theorem 1.5. [22, Theorem 4.3.15.], [10, Theorem 1.30., Corollary 1.31.] *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be BK spaces, $X \supset \phi$ and $Z = M(X, Y)$. Then, Z is a BK space with $\|\cdot\|$, defined by $\|z\| = \sup\{\|xz\|_Y \mid \|x\|_X = 1\}$ for all $z \in Z$. If X is a BK space, then X^β is also BK space with respect to $\|\cdot\|_\beta$ defined by*

$$\|a\|_\beta = \sup\left\{\sup_n \left|\sum_{k=0}^{n} a_k x_k\right| \mid \|x\|_X = 1\right\}.$$

The following result establishes a relation between the β–and continous duals of FK spaces; its first part contains Theorem 1.2.

Theorem 1.6. [22, Theorem 7.2.9.], [10, Theorem 1.34.] *Let $X \supset \phi$ be BK space. Then $X^\beta \subset X^*$, that is, there is a linear one-to-one map $T \colon X^\beta \to X^*$. If X has AK, then T is onto.*

1.3. Some classical sequence spaces and matrix transformations between them. Now we consider the sets of all bounded, convergent and null sequences, denoted by ℓ_∞, c and c_0, respectively, that is,

$$\ell_\infty = \{x \in \omega \mid \sup_k |x_k| < \infty\},$$
$$c = \{x \in \omega \mid \lim_{k\to\infty} x_k = \xi \text{ for some } \xi \in \mathbb{C}\},$$
$$c_0 = \{x \in \omega \mid \lim_{k\to\infty} x_k = 0\}.$$

These sequence spaces were the subject of research in many papers. Also, many new sequence spaces arise from various concepts of summability and the classical sequence spaces. Here we consider the sets of all bounded, convergent and null sequences and try to get the reader interested in this research area. A large number of valuable papers can be found for further reading.

We recall the definition of a Schauder basis.

Definition 1.2. A Schauder basis of a linear metric space X is a sequence $b = (b_n)_{n=0}^{\infty}$ of vectors such that for each $x \in X$ there is a unique sequence of scalars $\lambda = (\lambda_n)_{n=0}^{\infty}$ with $x = \sum_{n=0}^{\infty} \lambda_n b_n$.

Let X be BK space. Then we write

$$\|a\|^* = \|a\|_X^* = \sup \left\{ \left| \sum_{k=0}^{\infty} a_k x_k \right| \mid \|x\| = 1 \right\}$$

and observe that the expression on the left hand side exists and is finite if $a \in X^\beta$ by Theorem 1.6.

Now we list some basic known properties of the classical sequence spaces.

Remark 1.1. (i) The spaces ℓ_∞, c and c_0 are BK spaces with their natural norm $\|x\| = sup_k|x_k|$; c and c_0 are closed subspaces of ℓ_∞; c_0 has AK, every sequence $x = (x_k)_{k=0}^{\infty}$ has a unique representation

$$x = \xi e + \sum_{k=0}^{\infty} (x_k - \xi) e^{(k)} \text{ where } \xi = \lim_{k \to \infty} x_k,$$

but ℓ_∞ has no Schauder basis.
(ii) We have $c_0^\beta = c^\beta = \ell_\infty^\beta = \ell_1$ where ℓ_1 is defined by $\ell_1 = \{x \in \omega \mid \sum_{k=0}^{\infty} |x_k| < \infty\}$. By [10, Theorem 1.29.], we have $\|a\|_c^* = \|a\|_{c_0}^* = \|a\|_{\ell_\infty}^* = \|a\|_1$ for all $a \in \ell_\infty^\beta$, where $\|a\|_1 = \sum_{k=0}^{\infty} |a_k|$.

The following general results will be useful for the characterization of matrix transformations between c, c_0 and ℓ_∞.

Theorem 1.7. (i) *Let X be a BK space. Then $A \in (X, \ell_\infty)$, if and only if $\|A\|_{(X,\ell_\infty)}^* = \sup_n \|A_n\|_X^* < \infty$; moreover, if $A \in (X, Y)$ then $\|L_A\| = \|A\|_{(X,\ell_\infty)}^*$* [10, Theorem 1.23.].

(ii) *Let X and Y be FK spaces, X have AK and Y_1 be a closed FK space in Y. Then $A \in (X, Y_1)$ if and only if $A \in (X, Y)$ and $Ae^{(k)} \in Y_1$ for all k* [22, 8.3.6.].

Theorem 1.8. [22, 8.3.7.] *Let X be an FK space and $X_1 = X \oplus e$. Then $A \in (X_1, Y)$ if and only if $A \in (X, Y)$ and $Ae \in Y$.*

We do not need the following result in this paper, but it is of great importance in the theory of matrix transformations.

Theorem 1.9. [22, Theorem 8.3.9.] *Let X and Z be BK spaces with AK and $Y = Z^\beta$. Then $(X, Y) = (X^{\beta\beta}, Y)$ and*

$$A \in (X, Y) \text{ if and only if } A^T \in (Z, X^\beta).$$

The space ℓ_∞ has no Schauder basis and some classes of matrix transformations on it, such as the classes (ℓ_∞, c) and (ℓ_∞, c_0), cannot be characterized by using the FK space theory.

Theorem 1.10 (Schur)**.** [9, Theorem 6, p. 169] *We have $A \in (\ell_\infty, c)$ if and only if*

(a) $\sum_{k=0}^\infty |a_{nk}|$ *converges uniformly in n;*
(b) *there exists* $\lim_{n\to\infty} a_{nk}$ *for each fixed k.*

Corollary 1.1. *$A \in (\ell_\infty, c_0)$ if and only if*

(a) $\sum_{k=0}^\infty |a_{nk}|$ *converges uniformly in n;*
(b) $\lim_{n\to\infty} a_{nk} = 0$ *for each fixed k.*

Remark 1.2. We remark that the conditions in Corollary 1.1 can be replaced by

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} |a_{nk}| = 0 \qquad [21,\ \mathbf{21.}\ (21.1)].$$

Now, the characterizations of matrix transformations between ℓ_∞, c and c_0 and can be given.

Theorem 1.11. *The necessary and sufficient conditions for $A \in (X, Y)$ when $X, Y \in \{c, c_0, \ell_\infty\}$ can be read from the following table:*

To \ From	ℓ_∞	c_0	c
c	**Theorem 1.10 (a), (b)**	**2.**	**4.**
c_0	**Corollary 1.1 (a), (b)**	**3.**	**5.**
ℓ_∞	**1.**	**1.**	**1.**

where

1. (1*) *where* (1*) $\sup_n \sum_{k=0}^\infty |a_{nk}| < \infty$;
2. (1*) *and* (2*) *where* (2*) $\lim_{n\to\infty} a_{nk} = \alpha_k$ *for each k;*
3. (1*) *and* (3*) *where* (3*) $\lim_{n\to\infty} a_{nk} = 0$;
4. (1*) *and* (2*) *and* (4*) *where* (4*) $\lim_{n\to\infty} \sum_{k=0}^\infty a_{nk} = \alpha$;
5. (1*) *and* (3*) *and* (5*) *where* (5*) $\lim_{n\to\infty} \sum_{k=0}^\infty a_{nk} = 0$.

Proof. The proof is a direct consequence of the previous theorems. The condition in **1.** for the characterization of the classes $(\ell_\infty, \ell_\infty)$, (c, ℓ_∞) and (c_0, ℓ_∞) follows from Theorem 1.7 (i) and Remark 1.1 (ii). Since c_0 and c are closed subspaces of ℓ_∞, and c_0 has AK, the conditions in **3.** and **5.** for the characterization of the classes (c_0, c_0) and (c_0, c) follow from **1.** and Theorem 1.7 (ii). Finally, since $c = c_0 \oplus e$, the conditions for the characterization of the classes (c, c_0) and (c, c) in **2.** and **4.** follow from **3.** and **5.** and Theorem 1.8. □

We note that the conditions in Theorem 1.11 **4.** for the characterization of the class (c, c) are those of the famous Toeplitz theorem. The following result gives different, but equivalent conditions for the characterization of the class (c, c).

Theorem 1.12 (Kojima–Schur)**.** [9, Theorem 4, p. 166] *We have $A \in (c, c)$ if and only if*

(a) $\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$;
(b) *for each* p, *there exists* $\lim_{n\to\infty} \sum_{k=p}^{\infty} a_{nk} = a_p$.

1.4. Representation of some bounded linear operators. In the previous subsection we characterized matrix transformations between the sequence spaces ℓ_∞, c and c_0. If $X, Y \in \{\ell_\infty, c, c_0\}$, then, by Theorem 1.4 (i), every $A \in (X, Y)$ defines an operator $L_A \in B(X, Y)$ where $L_A(x) = Ax$ $(x \in X)$. Also, since c_0 is BK space with AK, by Theorem 1.4 (ii), every operator $L \in B(c_0, Y)$ can be represented by a matrix $A \in (c_0, Y)$ where $L(x) = Ax$ $(x \in c_0)$.

We are interested in the representation of the general operators $L \in B(c, Y)$ when Y is any of the spaces ℓ_∞, c or c_0. In [11, 12], the authors considered the sequence spaces c and c_0 and linear operators and matrices between them. Here add one more result to the existing ones.

Theorem 1.13. (a) *We have* $L \in B(c, \ell_\infty)$ *if and only if there exist a matrix* $A \in (c_0, \ell_\infty)$ *and a sequence* $b \in \ell_\infty$ *such that*

$$L(x) = b \cdot \lim_{k\to\infty} x_k + Ax \text{ for all } x \in c. \tag{1.1}$$

(b) *We have* $L \in B(c, c)$ *if and only if there exist a matrix* $A \in (c_0, c)$ *and a sequence* $b \in \ell_\infty$ *for which the limit*

$$\lim_{n\to\infty} \left(b_n + \sum_{k=0}^{\infty} a_{nk} \right) = \beta \text{ exists} \tag{1.2}$$

such that (1.1) *holds.*
(c) *We have* $L \in B(c, c_0)$ *if and only if there exist a matrix* $A \in (c_0, c_0)$ *and a sequence* $b \in \ell_\infty$ *with*

$$\lim_{n\to\infty} \left(b_n + \sum_{k=0}^{\infty} a_{nk} \right) = 0$$

such that (1.1) *holds.*

Proof. (a) First we assume $L \in B(c, \ell_\infty)$ and write $L_n = P_n \circ L$ for $n = 0, 1, \dots$ where each P_n is defined by $P_n(x) = x_n$ for every sequence $x = (x_k)_{k=0}^{\infty}$. Since c is a BK space, we have $L_n \in c^*$ for each n, and it follows from the well–known representation of continuous linear functionals on c [9, Theorem 8., p. 109] that

$$L_n(x) = b_n \lim_{k\to\infty} x_k + A_n x \text{ for all } x \in c,$$

$$A_n = (a_{nk})_{k=0}^{\infty} = (L_n(e^{(k)}))_{k=0}^{\infty} \in \ell_1 \text{ and } b_n = L_n(e) - \sum_{k=0}^{\infty} L_n(e^{(k)}) \text{ for all } n.$$

This yields (1.1). Also since $L(x^{(0)}) = Ax^{(0)}$ for all $x^{(0)} \in c_0$, we have $A \in (c_0, \ell_\infty) = (\ell_\infty, \ell_\infty)$ by Theorem 1.11**1.** and it follows from (1.1) that $b = L(e) - Ae \in \ell_\infty$.
Conversely, we assume that $A \in (c_0, \ell_\infty)$, $b \in \ell_\infty$ and (1.1) is satisfied. Then,

$A \in (\ell_\infty, \ell_\infty) \subset (c, \ell_\infty)$, and so $L_A \in B(c, \ell_\infty)$ where $L_A(x) = Ax$ for all $x \in c$. It follows from $b \in \ell_\infty$ and (1.1) that

$$\|L(x)\| = \sup |L_n(x)| = \sup \left|b_n \lim_{k\to\infty} x_k + A_n x\right|$$
$$\leqslant \sup \left(|b_n| \cdot \|x\| + \sum_{k=0}^{\infty} |a_{nk}| \cdot |x_k| \right) = \sup \left(|b_n| + \sum_{k=0}^{\infty} |a_{nk}| \right) \cdot \|x\|,$$

that is, $L \in B(c, \ell_\infty)$.

(b) We assume $L \in B(c, c)$. Then it follows that $L \in B(c, \ell_\infty)$ and so, by Part (a), there exist $b \in \ell_\infty$ and $A \in (c_0, \ell_\infty)$ such that (1.1) holds. It follows from (1.1) and $L(e^{(k)}) \in c$ for all k that there exist complex numbers α_k such that

$$\lim_{n\to\infty} a_{nk} = \alpha_k;$$

this and $A \in (c_0, \ell_\infty)$ imply $A \in (c_0, c)$ (see Theorem 1.11). Furthermore, $L(e) \in c$ implies that (1.2) holds.
Conversely, we assume that there exist a matrix $A \in (c_0, c)$ and a sequence $b \in \ell_\infty$ with (1.2) such that (1.1) holds. Then we have $L \in B(c, \ell_\infty)$ by Part (a). Let $x \in c$ be given. Then there are $x^{(0)} \in c_0$ and $\xi \in \mathbb{C}$ such that $x = x^{(0)} + \xi \cdot e$. Then we have for all n by (1.1)

$$L_n(x) = b_n\xi + \sum_{k=0}^{\infty} a_{nk}x_k = b_n\xi + \sum_{k=0}^{\infty} a_{nk}(x_k^{(0)} + \xi) = \xi\left(b_n + \sum_{k=0}^{\infty} a_{nk}\right) + A_n x^{(0)}.$$

Since $A \in (c_0, c)$, $\lim_{n\to\infty} A_n x^{(0)}$ exists, and we obtain by (1.2) $\lim_{n\to\infty} L_n(x) = \xi\beta + \lim_n A_n x^{(0)}$, and so $L(x) \in c$. This shows $L \in B(c, c)$.

(c) The proof of Part (c) is exactly the same as that of Part (b) with $\beta = \alpha_k = 0$ for all k. □

We close this section with the estimates of the norm of some bounded linear operators between ℓ_∞, c and c_0.

By Remark 1.1 and by Theorem 1.7, we have that if $A \in (X, Y)$ for $X, Y \in \{\ell_\infty, c, c_0\}$, then the operator L_A with $L_A(x) = Ax$ has the norm

$$\|L_A\| = \sup_n \|A_n\|_X^* = \sup_n \sum_{k=0}^{\infty} |a_{nk}|. \tag{1.3}$$

In the case of a general operator $L \in B(c, c)$, the situation is different. The following theorem describes this.

Theorem 1.14. [5, Teorema 3.19.] *Every operator $L \in B(c, c)$ can be represented by a matrix $B = (b_{nk})_{n=0,k=-1}^{\infty}$ such that the following conditions hold*

$$L(x) = \left(b_{n,-1}\xi + \sum_{k=0}^{\infty} b_{nk}x_k\right)_{n=0}^{\infty} \text{ where } \xi = \lim_{k\to\infty} x_k,$$
$$\lim_{n\to\infty} b_{nk} = \beta_k \text{ exists for each } k = 0, 1, \dots, \tag{1.4}$$

$$\lim_{n\to\infty}\sum_{k=-1}^{\infty} b_{nk} = \beta, \tag{1.5}$$

$$\|L\| = \sup_n \sum_{k=-1}^{\infty} |b_{nk}| < \infty. \tag{1.6}$$

We also have

$$\lim_{n\to\infty}(L(x))_n = \xi\cdot\beta + \sum_{k=0}^{\infty}\beta_k(x_k-\xi) = \left(\beta - \sum_{k=0}^{\infty}\beta_k\right)\xi + \sum_{k=0}^{\infty}\beta_k x_k \textit{ for all } x\in c.$$

If we use notation of Theorem 1.13, we have that if $L \in B(c,c)$, then

$$\|L\| = \sup_n\left(|b_n| + \sum_{k=0}^{\infty}|a_{nk}|\right).$$

This is the same as the norm defined in (1.6). We note the difference between the norm of the matrix operator $L_A \in (c,c)$ defined in (1.3) and the norm of the general bounded operator $L \in B(c,c)$.

2. Compact operators

Here we will give necessary and sufficient conditions for our operators to be compact. For that purpose we will use two different methods - the application of the Hausdorff measure of noncompactness and the results by Sargent in [20]. The first method is based on a result by Goldenštein, Gohberg and Markus and can be applied when the final sequence spaces have a Schauder basis. This technique has been used in many papers and different kinds of sequence spaces were treated [6–8,12–17]. When the final space is ℓ_∞ which has no Schauder basis then we apply a result by Sargent and close some existing gaps in the research.

2.1. Introduction. Let X and Y be Banach spaces and L be a linear operator from X to Y. We say that L is a compact operator if its domain is all of X and for every bounded sequence $(x_n)_{n=0}^{\infty}$ in X, the sequence $(L(x_n))_{n=0}^{\infty}$ has a convergent subsequence in Y. We denote the class of such operators by $K(X,Y)$. For further reading see [10,18,19].

Definition 2.1. [10, Definition 2.10] Let (X,d) be a metric space, Q be a bounded subset of X and $K(x,r) = \{y\in X \mid d(x,y) < r\}$. Then the Hausdorff measure of noncompactness of Q, denoted by $\chi(Q)$, is defined by

$$\chi(Q) = \inf\left\{\epsilon > 0 \mid Q \subset \bigcup_{i=1}^{n} K(x_i,\ r_i),\ x_i\in X,\ r_i<\epsilon\ (i=1,\dots,n),\ n\in\mathbb{N}_0\right\}.$$

Some properties are given below.

If Q, Q_1 and Q_2 are bounded subsets of the metric space (X,d), then we have [10, Lemma 2.11]

$$\chi(Q) = 0 \text{ if and only if } Q \text{ is a totally bounded set,}$$

$$\chi(Q) = \chi(\overline{Q}),$$

$$Q_1 \subset Q_2 \text{ implies } \chi(Q_1) \leqslant \chi(Q_2).$$

If Q, Q_1 and Q_2 are bounded subsets of the normed space X, then we have [10, Theorem 2.12]

$$\chi(Q_1 + Q_2) \leqslant \chi(Q_1) + \chi(Q_2),$$
$$\chi(Q + x) = \chi(Q) \ (x \in X),$$
$$\chi(\lambda Q) = |\lambda|\chi(Q) \text{ for all } \lambda \in \mathbb{C}.$$

Definition 2.2. [10, Definition 2.24] Let X and Y be Banach spaces and χ_1 and χ_2 be Hausdorff measures on X and Y. Then the operator $L\colon X \to Y$ is called (χ_1, χ_2)-bounded if $L(Q)$ is a bounded subset of Y for every bounded subset Q of X and there exists a positive constant K such that $\chi_2(L(Q)) \leqslant K\chi_1(Q)$ for every bounded subset Q of X. If an operator L is (χ_1, χ_2)-bounded then the number$\|L\|_{(\chi_1,\chi_2)} = \inf\{K > 0 \mid \chi_2(L(Q)) \leqslant K\chi_1(Q) \text{ for all bounded } Q \subset X\}$ is called (χ_1, χ_2)- measure of noncompactness of L. In particular, if $\chi_1 = \chi_2 = \chi$, then we write $\|L\|_{(\chi,\chi)} = \|L\|_\chi$.

The next result is most useful for our characterizations of compact operators.

Theorem 2.1. [10, Theorem 2.25] *Let X and Y be Banach spaces, $L \in B(X, Y)$, $S_X = \{x \in X \mid \|x\| = 1\}$ and $\bar{B}_X = \{x \in X \mid \|x\| \leqslant 1\}$ denote the unit sphere and closed unit ball in X. Then the Hausdorff measure of noncompactness of L is given by $\|L\|_\chi = \chi(L(\bar{B}_X))) = \chi(L(S_X))$.*

The following properties are also interesting and useful for us. If X and Y are Banach spaces and $L \in B(X, Y)$ then

$$L \text{ is a compact if and only if } \|L\|_\chi = 0 \ \ [10, \text{Corollary 2.26 (2.58)}],$$
$$\|L\|_\chi \leqslant \|L\| \ \ [10, \text{Corollary 2.26 (2.59)}].$$

The fundamental result for the application of the Hausdorff measure of noncompactness is the next theorem.

Theorem 2.2 (Goldenštein, Gohberg, Markus)**.** [10, Theorem 2.23] *Let X be a Banach space with a Schauder basis $(e_1, e_2, \dots)$, Q be a bounded subset of X, and $P_n\colon X \to X$ be the projector onto the linear span of $\{e_1, e_2, \dots, e_n\}$. Then we have*

$$\frac{1}{a}\limsup_{n\to\infty}\Big(\sup_{x\in Q}\|(I - P_n)(x)\|\Big) \leqslant \chi(Q) \leqslant \limsup_{n\to\infty}\Big(\sup_{x\in Q}\|(I - P_n)(x)\|\Big),$$

where $a = \limsup_{n\to\infty}\|I - P_n\|$.

In particular, if $X = c$, then $a = 2$ in the previous theorem.

Theorem 2.3. [19, Theorem 2.8.], [10, Theorem 2.15] *Let Q be a bounded subset of the normed space X, where ℓ_p $(1 \leqslant p < \infty)$ or c_0. If $P_n\colon X \to X$ is the operator defined by $P_n(x) = (x_0, x_1, \dots, x_n, 0, 0 \dots)$ for $x = (x_k)_{k=0}^\infty \in X$, then*

$$\chi(Q) = \lim_{n\to\infty}\Big(\sup_{x\in Q}\|(I - P_n)(x)\|\Big).$$

2.2. The Hausdorff measure of noncompactness and operators between ℓ_∞, c and c_0. After having introduced the necessary notations and general results, we consider the classes of compact operators between the spaces of bounded, convergent and null sequences.

We start with an estimate for the Hausdorff measure of noncompactness of a bounded linear operator from an arbitrary BK space with AK into c.

Theorem 2.4. [6, Theorem 3.4] *Let X be a BK space with AK. Then every operator $L \in B(X,c)$ can be represented by an infinite complex matrix $A = (a_{nk})_{n,k=0}^\infty$ such that $(L(x))_n = A_n x = \sum_{k=0}^\infty a_{nk}x_k$ for all n and all $x \in X$. The Hausdorff measure of noncompactness of L satisfies*

$$\frac{1}{2} \cdot \limsup_{r\to\infty} \left(\sup_{n\geqslant r} \|A_n - \alpha\|_X^*\right) \leqslant \|L\|_\chi \leqslant \limsup_{r\to\infty} \left(\sup_{n\geqslant r} \|A_n - \alpha\|_X^*\right) \tag{2.1}$$

where

$$\alpha_k = \lim_{k\to\infty} a_{nk} \text{ for every } k \text{ and } \alpha = (\alpha_k)_{k=0}^\infty. \tag{2.2}$$

Proof. We write $\|\cdot\|^* = \|\cdot\|_X^*$ and $\|A\| = \|A\|_{(X,\ell_\infty)}^*$, for short. The first part is by Theorem 1.4 (ii). We also observe that obviously $a \leqslant 2$. Furthermore $A \in (X,c)$ implies $\|A\| = \|A\|_{(X,\ell_\infty)}^* = \sup_n \|A_n\|^* < \infty$ by Theorem 1.7 (i). Since X has AK, we have $e^{(k)} \in X$, hence $Ae^{(k)} \in c$ for all k, that is, the limits α_k in (2.2) exist for all k. Now we show

$$\alpha \in X^\beta. \tag{2.3}$$

Let $x \in X$ be given. Since X has AK, there is a positive constant K such that $\|x^{[m]}\| \leqslant K\|x\|$ for all $m \in \mathbb{N}_0$, and it follows that

$$\left|\sum_{k=0}^m a_{nk}x_k\right| = \left|A_n x^{[m]}\right| \leqslant K\,\|A_n\|^*\,\|x\| \leqslant K\,\|A\|^*\,\|x\| \text{ for all } m \text{ and all } n,$$

hence by (2.2)

$$\left|\sum_{k=0}^m \alpha_k x_k\right| = \lim_{n\to\infty} \left|\sum_{k=0}^m a_{nk}x_k\right| \leqslant K\,\|A\|^*\,\|x\| \text{ for all } m.$$

Therefore $(\alpha_k x_k)_{k=0}^\infty \in bs$, and since $x \in X$ was arbitrary, we conclude $\alpha \in X^\gamma$. Since X has AK, we have $\alpha \in X^\gamma = X^\beta$ [22, Theorem 7.2.7], so (2.3) holds. Also $\alpha \in X^\beta$ implies $\|\alpha\|^* < \infty$ by [22, Theorem 7.2.9]. Now we show

$$\lim_{n\to\infty} A_n x = \sum_{k=0}^\infty \alpha_k x_k \text{ for all } x \in X. \tag{2.4}$$

Let $x \in X$ and $\varepsilon > 0$ be given. Since X has AK, there is a non–negative integer k_0 such that

$$\|x - x^{[k_0]}\| < \frac{\varepsilon}{2(\|A\|^* + \|\alpha\|^* + 1)}. \tag{2.5}$$

Also it follows from (2.2) that there is a non–negative integer n_0 such that

$$\left|\sum_{k=0}^{k_0}(a_{nk}-\alpha_k)x_k\right| < \frac{\varepsilon}{2} \text{ for all } n \geqslant n_0. \tag{2.6}$$

Let $n \geqslant n_0$ be given. Then it follows from (2.5) and (2.6) that

$$\begin{aligned}\left|A_n x - \sum_{k=0}^{\infty}\alpha_k x_k\right| &\leqslant \left|\sum_{k=0}^{k_0}(a_{nk}-\alpha_k)x_k\right| + \left|\sum_{k=k_0+1}^{\infty}(a_{nk}-\alpha_k)x_k\right| \\ &< \frac{\varepsilon}{2} + \|A_n-\alpha\|^*\,\|x-x^{[k_0]}\| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Since $x \in X$ was arbitrary, (2.4) follows. Now we show (2.1). Let $y=(y_n)_{n=0}^\infty \in c$ be given. Then the sequence y has a unique representation $y = \eta\cdot e + \sum_{n=0}^\infty (y_n - \eta)e^{(n)}$ with $\eta = \lim_{n\to\infty} y_n$, and we obtain $(I-P_r)(y) = \sum_{n=r+1}^\infty (y_n-\eta)e^{(n)}$ for all $r = -1, 0, 1, \ldots$. Writing $y_n = A_n x$ $(n = 0, 1, \ldots)$ and $B = (b_{nk})_{n,k=0}^\infty$ for the matrix with $b_{nk} = a_{nk} - \alpha_k$ for all n and k, we obtain by (2.4)

$$\|(I-P_r)(Ax)\| = \sup_{n\geqslant r+1}|y_n-\eta| = \sup_{n\geqslant r+1}\left|A_n x - \sum_{k=0}^\infty \alpha_k x_k\right| = \sup_{n\geqslant r+1}|B_n x|,$$

whence $\sup_{x\in S_X}\|(I-P_r)(Ax)\| = \sup_{n\geqslant r+1}\|B_n\|^*$ for all r. Now the inequalities in (2.1) follow from Theorems 2.1 and 2.2. □

If we put $X = c_0$, which is a BK space with AK, in the previous theorem, we obtain the next result.

Corollary 2.1. *If $L \in B(c_0, c)$, then*

$$\frac{1}{2}\cdot\limsup_{r\to\infty}\left(\sup_{n\geqslant r}\sum_{k=0}^\infty |a_{nk}-\alpha_k|\right) \leqslant \|L\|_\chi \leqslant \limsup_{r\to\infty}\left(\sup_{n\geqslant r}\sum_{k=0}^\infty |a_{nk}-\alpha_k|\right),$$

where $A \in (c_0, c)$ is the matrix that represents L by Theorem 1.4 (ii)*, and L is compact if and only if*

$$\lim_{r\to\infty}\left(\sup_{n\geqslant r}\sum_{k=0}^\infty |a_{nk}-\alpha_k|\right)=0.$$

The previous theorem does not cover the case $X = c$, but the next one does.

Theorem 2.5. [5, Theorem 3.21], [12, Theorem 1] *Let $L \in B(c, c)$. Then we have*

$$\begin{aligned}&\frac{1}{2}\limsup_{n\to\infty}\left(\left|b_{n,-1}-\beta+\sum_{k=0}^\infty \beta_k\right| + \sum_{k=0}^\infty |b_{nk}-\beta_k|\right)\\ &\qquad\leqslant \|L\|_\chi \leqslant \limsup_{n\to\infty}\left(\left|b_{n,-1}-\beta+\sum_{k=0}^\infty \beta_k\right| + \sum_{k=0}^\infty |b_{nk}-\beta_k|\right),\end{aligned}$$

where $b_{nk} = b_k^{(n)} = L_n(e^k)$ for $n = 0, 1, \ldots$; $k \geqslant 0$ and $b_{-1}^{(n)} = L_n(e) - \sum_{k=0}^\infty L_n(e^{(k)})$; β and β_k $(k = 0, 1, \ldots)$ are given in (1.5) *and* (1.4).

Using the fact that $L \in B(c,c)$ is compact if and only if $\|L\|_\chi = 0$, we obtain the following characterization for compact operators in $B(c,c)$ from Theorem 2.5.

Corollary 2.2. *If $L \in B(c,c)$, then L is compact if and only if*

$$\lim_{n\to\infty}\left(\left|b_{n,-1} - \beta + \sum_{k=0}^{\infty}\beta_k\right| + \sum_{k=0}^{\infty}|b_{nk} - \beta_k|\right) = 0.$$

Now, instead of the general bounded operators on c, we consider the matrix operators L_A where $A \in (c,c)$. If we put $b_{n,-1} = 0$ for $n = 0,1,\dots$ and $b_{nk} = a_{nk}$ for $n,k = 0,1,\dots$ in Theorem 2.5, we get the next corollary.

Corollary 2.3. [5, Corollary 3.22] *Let $A \in (c,c)$ and $L_A(x) = Ax$ for each $x \in c$. Then we have*

$$\frac{1}{2}\limsup_{n\to\infty}\left(\left|\sum_{k=0}^{\infty}\alpha_k - \alpha\right| + \sum_{k=0}^{\infty}|a_{nk} - \alpha_k|\right) \leqslant \|L_A\|_\chi \leqslant \limsup_{n\to\infty}\left(\left|\sum_{k=0}^{\infty}\alpha_k - \alpha\right| + \sum_{k=0}^{\infty}|a_{nk} - \alpha_k|\right),$$

where $\alpha_k = \lim_{n\to\infty} a_{nk}$ for $k = 0,1,\dots$ and $\alpha = \lim_{n\to\infty}\sum_{k=0}^{\infty} a_{nk}$, and A is compact if and only if

$$\lim_{n\to\infty}\left(\left|\sum_{k=0}^{\infty}\alpha_k - \alpha\right| + \sum_{k=0}^{\infty}|a_{nk} - \alpha_k|\right) = 0.$$

Theorem 2.6. *If $A \in (c,c_0)$, or $A \in (c_0,c_0)$, then*

$$\|L_A\|_\chi = \limsup_{r\to\infty}\left(\sup_{n\geqslant r}\sum_{k=0}^{\infty}|a_{nk}|\right)$$

and A is compact if and only if

$$\lim_{r\to\infty}\left(\sup_{n\geqslant r}\sum_{k=0}^{\infty}|a_{nk}|\right) = 0.$$

Proof. In the proof, we use the same technique as in the proofs in many papers with the final space c_0. We write $K = \{x \in X \mid \|x\| \leqslant 1\}$, where $X \in \{c, c_0\}$. We have that

$$\|L_A\|_\chi = \chi(AK) = \lim_{r\to\infty}\left[\sup_{x\in K}\|(I-P_r)(Ax)\|\right]$$

where $P_r\colon c_0 \to c_0$ $(r = 0,1,\dots)$ is the projector such that $P_r(x) = (x_0, x_1, \dots, x_r, 0, 0\dots)$ for $x = (x_k)_k \in c_0$. It is known that $\|I - P_r\| = 1$ for all r. Let $A_{(r)} = (\bar{a}_{nk})_{n,k}$ be the infinite matrix with

$$\bar{a}_{nk} = \begin{cases} 0 & \text{if } 0 \leqslant n \leqslant r \\ a_{nk} & \text{if } r < n \end{cases}.$$

Now, we have that $A_{(r)} \in (X, c_0)$ and

$$\sup_{x\in K} \|(I - P_r)(Ax)\| = \|L_{A_{(r)}}\| = \limsup_{r\to\infty} \left(\sup_{n\geqslant r} \sum_{k=0}^{\infty} |a_{nk}| \right). \qquad \square$$

Finally, let the final space Y be ℓ_∞.

Theorem 2.7. *Let X be any of spaces c, c_0 or ℓ_∞. If $A \in (X, \ell_\infty)$, then*

$$0 \leqslant \|L_A\|_\chi \leqslant \limsup_{r\to\infty} \left(\sup_{n\geqslant r} \sum_{k=0}^{\infty} |a_{nk}| \right)$$

and A is compact if

$$\lim_{r\to\infty} \left(\sup_{n\geqslant r} \sum_{k=0}^{\infty} |a_{nk}| \right) = 0.$$

Proof. We define the projector $P_r \colon \ell_\infty \to \ell_\infty$ $(r = 0, 1, \dots)$ by
$P_r(x) = (x_0, x_1, \dots, x_r, 0, 0 \dots)$ for $x = (x_k)_k \in \ell_\infty$.
Since $AK \subset P_r(AK) + (I - P_r)(AK)$, applying the properties of χ, we obtain

$$\begin{aligned} \chi(AK) &\leqslant \chi(P_r(AK)) + \chi((I - P_r)(AK)) \\ &= \chi((I - P_r)(AK)) \leqslant \sup_{x\in K} \|(I - P_r)Ax\| = \|L_{A_{(r)}}\|. \end{aligned}$$

Now, the conclusion is clear. $\square$

As we have seen, in the cases where the final space is ℓ_∞, we have only been able to give sufficient conditions for the compactness of an operator. Hence, the characterization of compact matrix operators in the class (X, ℓ_∞) is not complete yet.

A characterization of compact operators on ℓ_∞ can be found in [1, Lemma 4.1(a)].

We will use a result by Sargent [20] and complete the characterization, where it is possible.

2.3. A Result by Sargent in the Case $Y = \ell_\infty$.

In this subsection we will "improve" the results related to the compactness of operators in the classes (X, ℓ_∞) where X is one of the spaces c, c_0 or ℓ_∞. This will be achieved by applying Sargent's results.

As usual, let A_i denote the sequence of elements in the i-th row of the matrix A and $A_i^{(n)}$ denote the sequence whose first n coordinates coincide with those of A_i.

Necessary and sufficient conditions for compactness of matrix operator L_A associated with matrix A from the class (X, ℓ_∞) for $X \in \{c, c_0, \ell_\infty\}$ can be found in [20, p. 85] and we list them below.

Theorem 2.8. (i) [20, p. 85, (b)] *Let $A \in (\ell_\infty, \ell_\infty)$. Then L_A is compact if and only if*

$$\sup_i \|A_i\|_1 < \infty, \tag{2.7}$$

$$\lim_{n\to\infty} \left(\sup_i \|A_i - A_i^{(n)}\|_1 \right) = 0. \tag{2.8}$$

(ii) [20, p. 85, (f)] *Let* $A \in (c_0, \ell_\infty)$. *Then* L_A *is compact if and only if conditions* (2.7) *and* (2.8) *hold.*

(iii) [20, p. 85, (g)] *Let* $A \in (c, \ell_\infty)$. *Then* L_A *is compact if and only if condition* (2.8) *holds.*

As we know $(c, \ell_\infty) = (c_0, \ell_\infty) = (\ell_\infty, \ell_\infty)$ and the condition **(1.)** from Theorem 1.11 is actually the condition (2.7). Hence, if we suppose that $A \in (c, \ell_\infty) = (c_0, \ell_\infty) = (\ell_\infty, \ell_\infty)$, the condition (2.7) is redundant and the theorem can be stated in the following way.

Theorem 2.9. *Let* X *be one of the spaces* c, c_0 *or* ℓ_∞ *and* $A \in (X, \ell_\infty)$. *Then* L_A *is compact if and only if*

$$\lim_{n\to\infty}\left(\sup_i \sum_{k=n+1}^{\infty} |a_{ik}|\right) = 0.$$

Thus we have given necessary and sufficient conditions for the compactness of matrix operators in the mentioned classes, even when the final space has no Schauder basis, that is, in our case ℓ_∞.

3. Matrix transformations and doubly stochastic operators

In this section we will extend the theory of matrix transformations and infinite matrices to doubly stochastic operators considering the classical sequence spaces mentioned above. The idea is to make the relation with the research from [3]. Maybe this can be the start point to extend the results and connect them with the results from [2]. First, let us recall that a square matrix is said to be doubly stochastic if its elements are non-negative and all row sums and column sums are equal to one [2]. In the case of infinite dimensional matrix, we say that $A = (a_{nk})_{n,k=0}^{\infty}$ is doubly stochastic if its entries are non negative and all row and column sums are one, that is:

$$\sum_{n=0}^{\infty} a_{nk} = 1 \text{ for all } k \text{ and } \sum_{k=0}^{\infty} a_{nk} = 1 \text{ for all } n. \tag{3.1}$$

Also, recall the next definition.

Definition 3.1. [3] An operator $D_0 \colon \ell_1 \to \ell_1$ is called a doubly stochastic operator on ℓ_1 if it is positive, i.e. $D_0 f \geqslant 0$ for each non-negative $f \in \ell_1$, and

$$\forall n \in \mathbb{N}, \sum_{m=1}^{\infty} D_0 e_n(m) = 1, \quad \forall m \in \mathbb{N}, \sum_{n=1}^{\infty} D_0 e_n(m) = 1,$$

where $e_n \in \ell_\infty$ denotes the sequence $e_n(j) = 0$ for all $j \neq n$ and $e_n(n) = 0$. (Here we cite the original notation from the definition but it is clear that e_n is the same as $e^{(n)}$ used above).

The set ℓ_1, the set of absolute convergent series is one more classical sequence space and well known result which will be useful in further lines is: $\ell_\infty^\beta = c^\beta = c_0^\beta = \ell_1$. Also, let us recall that $A \in (\ell_1, \ell_1)$ if and only if $\sup_k \sum_{n=0}^{\infty} |a_{nk}| < \infty$ [10,18,21].

Let $A = (a_{nk})_{n,k=0}^{\infty}$ be infinite doubly stochastic matrix.

That means that $a_{nk} \geqslant 0$ for all n, k and (3.1) holds. Further, it is clear that $\sup_k \sum_{n=0}^{\infty} |a_{nk}| < \infty$ and we can conclude that *every infinite doubly stochastic matrix* actually represents matrix transformation from the class (ℓ_1, ℓ_1) and by Theorem 1.4 defines bounded linear operator L_A.

Is this operator doubly stochastic operator on ℓ_1 in the sense of definition (3.1)?

It is clear that for all non-negative $x \in \ell_1$ we have that $Ax = L_A x$ and $L_A x = (A_n x)_n = (\sum_{k=0}^{\infty} a_{nk} x_k)_n$ is non-negative. Since

$$L_A e^{(n)} = A e^{(n)} = (A_i e^{(n)})_i = \left(\sum_{k=0}^{\infty} a_{ik} e_k^{(n)} \right)_i = (a_{in})_i = (a_{1n}, a_{2n}, a_{3n}, \dots)$$

it is clear that

$$\sum_m L_A e^{(n)}(m) = \sum_m (Ae^{(n)})_m = \sum_m a_{mn} = 1,$$
$$\sum_n L_A e^{(n)}(m) = \sum_n (Ae^{(n)})_m = \sum_n a_{mn} = 1$$

where we write $L_A e^{(n)}(m)$ and $(Ae^{(n)})_m$ for appropriate m-th coordinates.

Hence, every doubly stochastic matrix defines doubly stochastic operator on ℓ_1 in the sense of Definition 3.1. The next question is: does the opposite holds, that is, if we have doubly stochastic operator on ℓ_1 in the sense of Definition 3.1, can we find infinite doubly stochastic matrix such that they coincide on the space ℓ_1?

Since the sequence space ℓ_1 is AK space, it is clear by Theorem 1.4 that $B(\ell_1, \ell_1) \subseteq (\ell_1, \ell_1)$. It remains to show that the appropriate infinite matrix determined with the doubly stochastic operator is doubly stochastic matrix.

Let L_A be doubly stochastic operator on ℓ_1. That means that $L_A x \geqslant 0$ for all $x \geqslant 0$, $x \in \ell_1$ ($x \geqslant 0$ if and only if $x_k \geqslant 0$ for all k). Also, we know by Theorem 1.4 that we can find infinite matrix A such that $L_A x = Ax$ for all $x \in \ell_1$. What is the form of the entries of A and is A doubly stochastic matrix?

Since ℓ_1 is AK space, using Schauder basis, every $x \in \ell_1$ can be represented as $x = \sum_{k=0}^{\infty} x_k e^{(k)}$. Now, from

$$L_A x = \sum_{k=0}^{\infty} x_k L_A(e^{(k)}) = \left(\sum_{k=0}^{\infty} x_k L_A(e^{(k)}(n)) \right)_n = Ax = \left(\sum_{k=0}^{\infty} a_{nk} x_k \right)_n$$

follows that $A = (a_{nk})_{n,k=0}^{\infty}$ is matrix with entries $a_{nk} = L_A(e^{(k)}(n))$. L_A is doubly stochastic operator and the following hold:

$$\sum_{k=0}^{\infty} L_A(e^{(k)}(n)) = \sum_{k=0}^{\infty} a_{nk} = 1 \quad \text{and} \quad \sum_{n=0}^{\infty} L_A(e^{(k)}(n)) = \sum_{n=0}^{\infty} a_{nk} = 1.$$

This means that the matrix A is doubly stochastic matrix from the class (ℓ_1, ℓ_1).

As in [3], but this time using the theory of matrix transformations, we have shown that every doubly stochastic matrix on ℓ_1 defines doubly stochastic operator and the opposite holds, too. The only restriction was the sign of the entries of matrix A!

We continue with the application of the theory of matrix transformations in this area but on the space ℓ_∞. We have the big difference because the space ℓ_∞ is not AK space and important inclusion from Theorem 1.4 does not hold - it is true that $(\ell_\infty, \ell_\infty) \subset B(\ell_\infty, \ell_\infty)$ but the opposite does not hold. Because of that we will use some properties of β-duals and the characterization of certain class of matrix transformations. Since we consider BK spaces, definition [2, Definition 1.2] (for $I = \mathbb{N}$) related to adjoint operator can not be used here.

Let us introduce new definition which can be useful. We will consider the transpose matrix instead of the adjoint operator. It is not the same but here this approach can give nice relation.

Definition 3.2. A bounded linear operator $L_A \colon \ell_\infty \to \ell_\infty$ determined with infinite matrix A is called doubly stochastic operator if operator $L_B \colon \ell_1 \to \ell_1$ is doubly stochastic operator where $B = A^T$.

Let $A = (a_{nk})_{n,k=0}^\infty$ be arbitrary infinite doubly stochastic matrix. It is clear that $A \in (\ell_\infty, \ell_\infty)$ ($A \in (\ell_\infty, \ell_\infty)$ if and only if $\sup_n \sum_{k=0}^\infty |a_{nk}| < \infty$) determines bounded linear matrix operator L_A such that $Ax = L_A x$ for all $x \in \ell_\infty$. Is L_A doubly stochastic operator?

We put $X = Y = \ell_1$ and $Z = \ell_\infty$ and apply Theorem 1.9. We obtain that

$$A \in (\ell_1, \ell_1) \text{ if and only if } A^T \in (\ell_\infty, \ell_1^\beta) = (\ell_\infty, \ell_\infty),$$

that is

$$A \in (\ell_\infty, \ell_\infty) \text{ if and only if } A^T \in (\ell_1, \ell_1).$$

Put $B = A^T$. According to the results related to ℓ_1, we conclude that L_B is doubly stochastic operator on ℓ_1 and further, by our new definition, L_A is doubly stochastic operator on ℓ_∞.

Now, let us suppose that we have doubly stochastic operator L_A on ℓ_∞ determined with infinite matrix $A \in (\ell_\infty, \ell_\infty)$. It is clear that for $B = A^T$ operator $L_B \colon \ell_1 \to \ell_1$ is doubly stochastic operator. Then, B is doubly stochastic matrix and A also.

But, here we must emphasize that the assumption about determination of operator with matrix is important. Actually, since ℓ_∞ is not AK space, the inclusion $B(\ell_\infty, \ell_\infty) \subset (\ell_\infty, \ell_\infty)$ does not hold generally and we are not sure that every bounded linear operator on ℓ_∞, even doubly stochastic, can be represented with infinite matrix. Hence we must give the assumption that we consider matrix operator on ℓ_∞.

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