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AN INTRODUCTION INTO FREDHOLM THEORY AND GENERALIZED DRAZIN–RIESZ INVERTIBLE OPERATORS

Abstract. After a brief introduction into classical Fredholm theory we consider Riesz operators, polynomially Riesz operators, generalized Drazin–Riesz invertible operators, as a generalization of Drazin invertible operators, as well as generalized Kato–Riesz decomposition for bounded linear operators on Banach spaces. Also, some properties of the corresponding spectra are investigated.

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1. Introduction

Fredholm operators are a generalization of invertible operators. A bounded linear operator T acting on a Banach space X is Fredholm if its kernel is finite-dimensional and its range is of finite codimension. The concept of Riesz operators is a generalization of the concept of compact as well as the concept of quasinilpotent operators. A bounded linear operator T acting on a Banach space X is said to be Riesz if every nonzero spectral point λ is an isolated point of $\sigma(T)$, it is an eigenvalue of T and the spectral projection associated with $\{\lambda\}$ is finite-dimensional. Furthermore, a bounded linear operator T is Riesz if and only if the only possible point contained in its Fredholm spectrum is zero. We say that T is polynomially

Riesz if there exists a nonzero complex polynomial p such that the image $p(T)$ is Riesz.

On the other side, Drazin invertible operators also generalize invertible operators. A bounded linear operator T acting on a Banach space X is said to be Drazin invertible if there exists $S \in L(X)$ such that $TS = ST$, $STS = S$ and $TST - T$ is nilpotent [8, 10, 25, 30]. It is well-known that $T \in L(X)$ is Drazin invertible if and only if T has finite ascent and descent [25], that is, T can be decomposed into a direct sum of an invertible operator and a nilpotent one. Moreover, $T \in L(X)$ is Drazin invertible but not invertible if and only if 0 is a pole of the resolvent of T [8].

This concept has been generalized by Koliha [26] by replacing the third condition in the previous definition with the condition that $TST - T$ is quasinilpotent. We recall that T is generalized Drazin invertible if and only if $0 \notin \operatorname{acc}\sigma(T)$, and this is also equivalent to the fact that $T = T_1 \oplus T_2$ where T_1 is invertible and T_2 is quasinilpotent [26]. If we replace the third condition in the previous definitions with condition that $TST - T$ is Riesz, we get the concept of generalized Drazin–Riesz invertible operators. We prove that $T \in L(X)$ is generalized Drazin–Riesz invertible if and only if it can be decomposed into a direct sum of an invertible operator and a Riesz operator. This is also equivalent to the fact that 0 is not an accumulation point of the Browder spectrum of T [1].

For T is said to admit a generalized Kato–Riesz decomposition if there exists a pair of T-invariant closed subspaces (M, N) such that $X = M \oplus N$, the reduction T_M is Kato and T_N is Riesz [47]. The generalized Drazin–Riesz (generalized Kato–Riesz) spectrum of T is the set of all complex λ such that $T - \lambda$ is not generalized Drazin–Riesz invertible (does not admit a generalized Kato–Riesz decomposition). We show that T is polynomially Riesz if and only if the generalized Drazin–Riesz spectrum of T is empty, and this also equivalent to the fact that the generalized Kato–Riesz spectrum of T is empty.

The second section is dedicated to some basic concepts and results of the theory of semi-Fredholm operators, while the third section concerns Riesz operators. Although these results are well-known and can be found, for instance, in [9, 18, 19, 36, 43], proofs are given in almost all cases to make the article self-contained. In the last part of the second section we also present some results from [47]. These two sections contain subjects of lectures on PhD studies at the Faculty of Science and Mathematics, University of Niš.

The forth section is devoted to polynomially Riesz operators and results presented here originate from the paper [53]. In the fifth section we consider operators which admit a generalized Kato–Riesz decomposition, in particular generalized Drazin–Riesz invertible operators and various classes of generalized Drazin–Riesz semi-Fredholm operators, while the sixth section deals with corresponding spectra. The results presented in the last two sections originate from the paper [54], and also in the forth section there is a characterization of generalized Drazin–Riesz invertible operators which has been recently proved by O. Abad and H. Zguitti [1]. For the further generalizations of the concept of Drazin invertibility we refer readers to [5, 55, 57, 58].

2. Semi-Fredholm operators

In this section we give a short introduction into Fredholm theory. For a more comprehensive survey of Fredholm theory we refer readers to [2, 3, 9, 36, 56].

Throughout this paper we use X and Y to denote infinite dimensional complex Banach spaces, although for some considerations we will state that the space is finite dimensional. Let $L(X,Y)$ be the set of all linear bounded operators and $K(X,Y)$ ($F(X,Y)$) is the set of all compact (finite rank) operators from X to Y. Let $L(X)$ be the Banach algebra of bounded linear operators acting on X. The group of all invertible operators is denoted by $L(X)^{-1}$, while $K(X)$ ($F(X)$) is the set of all compact (finite rank) operators on X. We use X' to denote the dual space of X. If $T \in L(X,Y)$, then $T' \in L(Y',X')$ is the dual (adjoint) operator of T. For $T \in L(X,Y)$, we use $\mathcal{N}(T) \subset X$ for the null space and $\mathcal{R}(T) \subset Y$ for the range of T.

Let $\mathbb{N}$ ($\mathbb{N}_0$) denote the set of all positive (non-negative) integers, and let $\mathbb{C}$ denote the set of all complex numbers. An operator $T \in L(X)$ is *nilpotent* when $T^n = 0$ for some $n \in \mathbb{N}$, while T is *quasinilpotent* if $\|T^n\|^{1/n} \to 0$, that is $T - \lambda \in L(X)^{-1}$ for all complex $\lambda \neq 0$.

For $T \in L(X)$, let $\sigma(T)$ denote the spectrum of T, $r(T)$ the spectral radius of T and $\rho(T) = \mathbb{C} \setminus \sigma(T)$ the resolvent set of T. The function $\lambda \mapsto (\lambda - T)^{-1}$, $\lambda \in \rho(T)$, is called the resolvent of T. The left spectrum of $T \in L(X)$ is denoted by $\sigma_l(T)$, while the right spectrum is denoted by $\sigma_r(T)$.

If $S \subset \mathbb{C}$, then ∂S is the boundary of S, $\operatorname{acc} S$ is the set of accumulation points of S, $\operatorname{iso} S = S \setminus \operatorname{acc} S$ and $\operatorname{int} S$ is the set of interior points of S. For $\lambda_0 \in \mathbb{C}$, the open disc, centered at λ_0 with radius ϵ in $\mathbb{C}$, is denoted by $D(\lambda_0, \epsilon)$. For $a \in X$ and $\epsilon > 0$ set $D(a,\epsilon) = \{x \in X : \|x - a\| < \epsilon\}$ and $D[a,\epsilon] = \{x \in X : \|x - a\| \leqslant \epsilon\}$. If S is a subset of $\mathbb{C}$ (or X), then the closure of S is denoted by $\overline{S}$. If S is a subset of a linear space X, we write $\operatorname{lin} S$ for the linear span of S.

2.1. Nullity and defect.

Let $A \in L(X,Y)$. Then $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, denote the null-space and the range of A. Let $\alpha(A) = \dim N(A)$ if $N(A)$ is finite dimensional, and let $\alpha(A) = \infty$ if $N(A)$ is infinite dimensional. Similarly, let $\beta(A) = \dim Y/R(A) = \operatorname{codim} R(A)$ if $Y/R(A)$ is finite dimensional, and let $\beta(A) = \infty$ if $Y/R(A)$ is infinite dimensional. The quantity $\alpha(A)$ is the *nullity*, and $\beta(A)$ is the *defect* of $A \in L(X,Y)$.

Let $M \subset X$ and $W \subset X'$. The annihilator $M^\perp$ is defined by

$$M^\perp = \{f \in X' : f(x) = 0 \text{ for all } x \in M\},$$

while the annihilator of W is the set

$$^\perp W = \{x \in X : w(x) = 0 \text{ for every } w \in W\}.$$

The set $U^\perp$ is a closed subspace of X', and $^\perp W$ is a closed subspace of X. If $A \in L(X)$, then [40, 41, 43]

$$\mathcal{R}(A)^\perp = \mathcal{N}(A'), \quad \overline{\mathcal{R}(A)} = ^\perp(\mathcal{R}(A)^\perp) = ^\perp \mathcal{N}(A'), \tag{2.1}$$

where $A' \in L(X')$ is the adjoint operator of A. Moreover, $\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A')$ is closed, and in this case

$$\mathcal{R}(A') = \mathcal{N}(A)^{\perp}, \quad \mathcal{R}(A) = {}^{\perp}\mathcal{N}(A'). \tag{2.2}$$

Lemma 2.1. *Let X_1 be a finite dimensional subspace of X, and let X_2 be a closed subspace of X. Then $X_1 + X_2$ is closed.*

Proof. Let $Q_{X_2}\colon X \mapsto X/X_2$ denote the natural quotient mapping. Since X_1 is finite dimensional, we get that $Q_{X_2}(X_1)$ is finite dimensional also. Hence, $Q_{X_2}(X_1)$ is closed in X/X_2. Since Q_{X_2} is continuous, it follows that $Q_{X_2}^{-1}(Q_{X_2}(X_1)) = X_1 + \mathcal{N}(Q_{X_2}) = X_1 + X_2$ is closed in X. □

Lemma 2.2. *If U and V are subspaces of X, such that V is closed and $V \subset U$, then*

$$\dim U/V = \dim V^{\perp}/U^{\perp}. \tag{2.3}$$

Proof. Case I. Suppose that $\dim U/V = k < \infty$. Let $\{\tilde{x}_1, \dots, \tilde{x}_k\}$ denotes a base in U/V, where $x_i \in U$, $\tilde{x}_i = x_i + V$, $i = 1, \dots, k$, and let

$$V_i = V \oplus \operatorname{lin}\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k\},\ i = 1, \dots, k.$$

Now, Lemma 2.1 implies that V_i is closed in X, and obviously $x_i \notin V_i$. From the Hahn-Banach theorem there exists some $f_i \in X'$ satisfying $f_i \in V_i^{\perp}$ and $f_i(x_i) = 1$. It is easy to see that $\tilde{f}_i = f_i + U^{\perp}$, $i = 1, \dots, k$, are linearly independent. Thus, $\dim V^{\perp}/U^{\perp} \geqslant k$.

On the other hand, if $f \in V^{\perp}$, let $f(x_i) = \alpha_i$, $i = 1, \dots, k$. Since $\{\tilde{x_1}, \dots, \tilde{x_k}\}$ is a base for U/V, it follows that for every $x \in U$, the element $\tilde{x} \equiv x + V$ has the unique representation $\tilde{x} = \lambda_1 \tilde{x}_1 + \ldots + \lambda_k \tilde{x}_k$ ($\lambda_i \in \mathbb{C}$), and consequently the representation

$$x = \lambda_1 x_1 + \ldots + \lambda_k x_k + v, \quad v \in V, \tag{2.4}$$

is unique in terms of $\lambda_1, \dots, \lambda_k$. From (2.4) we get $(f - \alpha_1 f_1 - \cdots - \alpha_k f_k)(x) = 0$ for every $x \in U$, i.e. $f - (\alpha_1 f_1 + \cdots + \alpha_k f_k) \in U^{\perp}$. Particularly, for $\tilde{f} = f + U^{\perp}$ we have $\tilde{f} = \alpha_1 \tilde{f}_1 + \cdots + \alpha_k \tilde{f}_k$. Consequently, $\dim V^{\perp}/U^{\perp} \leqslant k$.

Case II. Suppose that $\dim U/V = \infty$. Then there exits the sequence (V_n) of closed subspaces of X satisfying

$$V \subsetneqq V_1 \subsetneqq V_2 \subsetneqq \cdots \subsetneqq U.$$

From the Hahn-Banach theorem it follows that

$$V^{\perp} \supsetneqq V_1^{\perp} \supsetneqq V_2^{\perp} \supsetneqq \cdots \supsetneqq U^{\perp}$$

is satisfied. Consequently, $\dim(V^{\perp}/U^{\perp}) = \infty$. □

Lemma 2.3. *If M is a subspace of X, then*

$$\operatorname{codim} M^{\perp} = \dim M. \tag{2.5}$$

Moreover, if M is a closed subspace of X, then

$$\dim M^{\perp} = \operatorname{codim} M. \tag{2.6}$$

Proof. From Lemma 2.2, take $U = M$ and $V = \{0\}$ to obtain (2.5). Moreover, take $U = X$ and $V = M$ to obtain (2.6). $\square$

Lemma 2.4. *If $A \in L(X,Y)$ and $R(A)$ is closed, then $\alpha(A') = \beta(A)$, $\beta(A') = \alpha(A)$ and $\alpha(A) = \alpha(A'')$.*

Proof. From (2.1) and (2.6) we get

$$\alpha(A') = \dim \mathcal{N}(A') = \dim \mathcal{R}(A)^{\perp} = \operatorname{codim} \mathcal{R}(A) = \beta(A).$$

From (2.2) and (2.5) it follows

$$\beta(A') = \operatorname{codim} \mathcal{R}(A') = \operatorname{codim} \mathcal{N}(A)^{\perp} = \dim \mathcal{N}(A) = \alpha(A). \qquad \square$$

The *injectivity modulus* (*minimum modulus*) of $T \in L(X,Y)$ is defined as

$$m(T) = \inf_{\|x\|=1} \|Tx\|.$$

We immediately obtain that $\|Tx\| \geqslant m(T)\|x\|$ for every $x \in X$, and

$$m(T) = \max\{c \geqslant 0\colon \|Tx\| \geqslant c\|x\|, \quad \text{for every } x \in X\}.$$

The operator $T \in L(X,Y)$ is *bounded below* if there exists some $c > 0$ such that $c\|x\| \leqslant \|Tx\|$, for every $x \in X$. It is easy to see that T is bounded below if and only if $m(T) > 0$.

The *surjectivity modulus* of $T \in L(X,Y)$ is defined as

$$q(T) = \sup\{\alpha \geqslant 0\colon \alpha D[0,1]_Y \subset T(D[0,1]_X)\}.$$

The following results hold:

$$m(T) > 0 \iff [R(T) = \overline{R(T)} \text{ and} N(T) = \{0\}],$$
$$q(T) > 0 \iff R(T) = Y,$$
$$q(T') = m(T) \text{ and } m(T') = q(T).$$

Lemma 2.5. *Let $A \in L(X,Y)$, and let W be closed subspace of Y such that $\mathcal{R}(A) \oplus W$ is closed subspace of Y. Then $\mathcal{R}(A)$ is closed in Y.*

Proof. The space $X/\mathcal{N}(A) \times W$ is endowed with the norm

$$\|(x + \mathcal{N}(A), w)\| = \|x + \mathcal{N}(A)\| + \|w\|$$

and it is a Banach space. Define the operator $A_0\colon X/\mathcal{N}(A) \times W \to Y$ by $A_0(x + \mathcal{N}(A), w) = Ax + w, \quad x \in X,\ w \in W$. A_0 is well-defined, linear bounded operator, and $\mathcal{R}(A_0) = \mathcal{R}(A) \oplus W$. Since A_0 is injective, it is bounded below, there exists some $c > 0$ such that

$$\|Ax + w\| \geqslant c\|(x + \mathcal{N}(A), w)\| = c(\|x + \mathcal{N}(A)\| + \|w\|), \quad x \in X,\ w \in W.$$

For $w = 0$ we obtain $\|Ax\| \geqslant c\|x + \mathcal{N}(A)\|, x \in X$. The operator $\hat{A}\colon X/\mathcal{N}(A) \to Y$, which is defined as $\hat{A}(x + \mathcal{N}(A)) = A(x)$, must be bounded below. Hence, $\mathcal{R}(\hat{A})$ is closed. From $\mathcal{R}(A) = \mathcal{R}(\hat{A})$ we get that $\mathcal{R}(A)$ is closed. $\square$

Corollary 2.1. *Let $A \in L(X,Y)$ and $\beta(A) < \infty$. Then $\mathcal{R}(A)$ is closed.*

For $A \in L(X)$ set $\mathcal{R}^\infty(A) = \cap_{n=1}^\infty \mathcal{R}(A^n)$ and $\mathcal{N}^\infty(A) = \cup_{n=1}^\infty \mathcal{N}(A^n)$. Both $\mathcal{R}^\infty(A)$ and $\mathcal{N}^\infty(A)$ are linear subspaces of X. Moreover, these subspaces are A-invariant.

Lemma 2.6. *Let A be a liner operator acting on X. If there exists $m \in \mathbb{N}$ such that*

$$\mathcal{N}(A) \cap A^m(X) = N(A) \cap A^{m+k}(X), \quad \text{for all } k \in \mathbb{N}, \tag{2.7}$$

then $A(\mathcal{R}^\infty(A)) = \mathcal{R}^\infty(A)$.

Proof. From $\mathcal{R}^\infty(A) \subset A^n(X)$ it follows that $A(\mathcal{R}^\infty(A)) \subset A^{n+1}(X)$, $n = 1, 2, \dots$, so $A(\mathcal{R}^\infty(A)) \subset \cap_{n=1}^\infty A^{n+1}(X) = \mathcal{R}^\infty(A)$.

To prove the opposite inclusion, suppose that $y \in \mathcal{R}^\infty(A)$. Then for every $k \in \mathbb{N}$ there exists $x_k \in X$ such that $y = A^{m+k}(x_k)$. Let $z_k = A^m(x_1) - A^{m+k-1}(x_k)$, $k \in \mathbb{N}$. Since $z_k \in \mathcal{N}(A) \cap A^m(X)$, from (2.7) we get $z_k \in A^{m+k-1}(X)$ for every $k \in \mathbb{N}$. Consequently, $A^m(x_1) \in A^{m+k-1}(X)$ for every $k \in \mathbb{N}$, so $A^m(x_1) \in \mathcal{R}(A^\infty)$. We get $y = A^{m+1}(x_1) \in A(\mathcal{R}^\infty(A))$. ☐

Lemma 2.7. *Let A be a liner operator acting on X. If $\alpha(A) < \infty$ or $\beta(A) < \infty$, then $A(\mathcal{R}^\infty(A)) = \mathcal{R}^\infty(A)$.*

Proof. Let $\alpha(A) < \infty$. Since

$$\mathcal{N}(A) \supset \mathcal{N}(A) \cap \mathcal{R}(A) \supset \mathcal{N}(A) \cap \mathcal{R}(A^2) \supset \mathcal{N}(A) \cap \mathcal{R}(A^3) \supset \cdots$$

and since $\dim \mathcal{N}(A) < \infty$ we conclude that there exists some $m \in \mathbb{N}$ such that the statement (2.7) holds. Now applying Lemma 2.6 we get $A(\mathcal{R}^\infty(A)) = \mathcal{R}^\infty(A)$.

Suppose that $\beta(A) < \infty$. Then there exists a subspace V of X such that $\dim V < \infty$ and

$$X = V \oplus A(X) \tag{2.8}$$

Again we prove that (2.7) holds. Suppose that this statement is not true, then there exists a subsequence $(n_k)_k$ of $\mathbb{N}$ such that

$$\mathcal{N}(A) \cap A^{n_1}(X) \neq \mathcal{N}(A) \cap A^{n_1+1}(X) = \cdots = \mathcal{N}(A) \cap A^{n_2}(X) \neq \mathcal{N}(A) \cap A^{n_2+1}(X) = \cdots$$

Therefore for every $k \in \mathbb{N}$ there exists $x_k \in X$ such that $A^{n_k} x_k \in \mathcal{N}(A) \cap A^{n_k}(X)$ and $A^{n_k} x_k \notin \mathcal{N}(A) \cap A^{n_k+1}(X)$. Because of (2.8) there exist $v_k \in V$ and $w_k \in A(X)$ such that $x_k = v_k + w_k$. Since $\dim V < \infty$, the set $\{v_k : k \in \mathbb{N}\}$ is linearly dependent and so there exist $k_1, \dots, k_j \in \mathbb{N}$, $k_1 < \cdots < k_j$ and scalars $\alpha_1, \dots, \alpha_{j-1}$, $j \in \mathbb{N}$, $j \geqslant 2$, such that

$$v_{k_j} = \alpha_1 v_{k_1} + \cdots + \alpha_{j-1} v_{k_{j-1}}.$$

Therefore

$$x_{k_j} - \alpha_1 x_{k_1} - \cdots - \alpha_{j-1} x_{k_{j-1}} = w_{k_j} - \alpha_1 w_{k_1} - \cdots - \alpha_{j-1} w_{k_{j-1}}.$$

Since $A^{n_{k_j}} x_{k_1} = \cdots = A^{n_{k_j}} x_{k_{j-1}} = 0$, it follows that

$$\begin{aligned} A^{n_{k_j}} x_{k_j} &= A^{n_{k_j}} (x_{k_j} - \alpha_1 x_{k_1} - \cdots - \alpha_{j-1} x_{k_{j-1}}) \\ &= A^{n_{k_j}} (w_{k_j} - \alpha_1 w_{k_1} - \cdots - \alpha_{j-1} w_{k_{j-1}}) \in A^{n_{k_j}}(A(X)) = A^{n_{k_j}+1}(X), \end{aligned}$$

which is a contradiction.

So we again apply Lemma 2.6 to get the assertion. ☐

2.2. Semi-Fredholm operators. Sets of *upper and lower semi-Fredholm* operators from X to Y, respectively, are defined as

$$\Phi_+(X,Y) = \{T \in L(X,Y) \,:\, \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed}\},$$
$$\Phi_-(X,Y) = \{T \in L(X,Y) \,:\, \beta(T) < \infty\}.$$

For upper and lower semi-Fredholm operators the index is defined by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. If $T \in \Phi_+(X,Y) \setminus \Phi_-(X,Y)$, then $\operatorname{ind}(T) = -\infty$, and if $T \in \Phi_-(X,Y) \setminus \Phi_+(X,Y)$, then $\operatorname{ind}(T) = +\infty$. The set of semi-Fredholm operators is defined as $\Phi_\pm(X,Y) = \Phi_+(X,Y) \cup \Phi_-(X,Y)$, while the set of Fredholm operators is defined as $\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y)$. Shortly, $\Phi(X) = \Phi(X,X)$ etc.

The *semi-Fredholm domain* of $A \in L(X)$ is defined as

$$\Phi_\pm(A) = \{\lambda \in \mathbb{C} \colon A - \lambda \in \Phi_\pm(X)\}.$$

The *Fredholm domain* of $A \in L(X)$ is defined as

$$\Phi(A) = \{\lambda \in \mathbb{C} \colon A - \lambda \in \Phi(X)\}.$$

Lemma 2.8. *Let $A \in L(X,Y)$. Then $A \in \Phi_+(X,Y)$ if and only if $A' \in \Phi_-(Y',X')$. Also, $A \in \Phi_-(X,Y)$ if and only if $A' \in \Phi_+(Y',X')$. In the both cases* $\operatorname{ind}(A) = -\operatorname{ind}(A')$.

Proof. The result follows from Lemma 2.4 and Corollary 2.1. □

Now, we characterize Φ_+-operators.

Theorem 2.1. *For $A \in L(X,Y)$ the following conditions are equivalent:*

(i) *$A \in \Phi_+(X,Y)$;*
(ii) *For every bounded sequence $(x_n)_n$ in X, which does not have a convergent subsequence, it holds that $(Ax_n)_n$ does not have a convergent subsequence;*
(iii) *For every bounded but not totally bounded subset $Q \subset X$ it holds that $A(Q)$ is not totally bounded.*

Proof. (i) $\Longrightarrow$ (ii): Let $A \in \Phi_+(X,Y)$ and let $(x_n)_n$ be a sequence in X which does not have a convergent subsequence. There exists a closed subspace X_1 of X such that $X = \mathcal{N}(A) \oplus X_1$. Since A is one-one on X_1, and $A(X_1) = \mathcal{R}(A)$ is closed, we get that the reduction $A_{|X_1} \colon X_1 \to \mathcal{R}(A)$ has a bounded inverse. There exist $u_n \in \mathcal{N}(A)$ and $v_n \in X_1$ such that $x_n = u_n + v_n$, $n = 1, 2, \dots$. Suppose that the sequence $(Ax_n)_n$ has the convergent subsequence, denoted by $(Ax_{n_k})_k$. It follows that $(Av_{n_k})_k$ is the convergent sequence and consequently $A_{|X_1}^{-1}(Av_{n_k}) = v_{n_k}$ is the convergent sequence. Thus, the sequence $(v_{n_k})_k$ is bounded. Since the sequence $(x_{n_k})_k$ is bounded, it follows that $(u_{n_k})_k$ is bounded also. However, $(u_{n_k})_k$ is the sequence in the finite dimensional space $\mathcal{N}(A)$, so it contains the convergent subsequence $(u_{n_{k_j}})_j$. From $x_{n_{k_j}} = u_{n_{k_j}} + v_{n_{k_j}}$ it follows that $(x_{n_{k_j}})_j$ is the convergent sequence, which is not possible.

(ii) $\Longrightarrow$ (iii): Obvious.

(iii) $\Longrightarrow$ (i): Suppose that for every bounded set Q, which is not totally bounded, the set $A(Q)$ is not totally bounded. Let K be the closed unit ball of $\mathcal{N}(A)$. If $\mathcal{N}(A)$ is not finite dimensional, then K is not totally bounded. It follows that $A(K) = \{0\}$

is not totally bounded, and this is not true. Consequently, $\alpha(A) < \infty$. It follows that there exists a closed subspace X_1 of X such that $X = \mathcal{N}(A) \oplus X_1$. Suppose that the restriction $A_{|X_1}$ is not bounded below, i.e. $\inf_{x\in X_1,\, \|x\|=1} \|Ax\| = 0$. There exists a sequence $(x_n)_n$ in X_1 such that $\|x_n\| = 1$ and $Ax_n \to 0$ as $n \to \infty$. It follows that $\{Ax_n : n = 1, 2, \dots\}$ is totally bounded, so $\{x_n : n = 1, 2, \dots\}$ is totally bounded. There exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $x_{n_k} \to x \in X_1$ as $k \to \infty$. From $A(x_{n_k}) \to A(x) = 0$ as $k \to \infty$, we obtain $x \in X_1 \cap \mathcal{N}(A)$. It follows that $x = 0$, which is not possible since $\|x\| = 1$. We have just proved that $A_{|X_1}$ is bounded below, so $\mathcal{R}(A) = A(X_1)$ is closed. Thus, $A \in \Phi_+(X, Y)$. □

Theorem 2.2 (The index theorem)**.** *Let $A \in L(X, Y)$ and $B \in L(Y, Z)$. If A and B are upper* (*lower*) *semi-Fredholm, then BA is upper* (*lower*) *semi-Fredholm and*

$$\alpha(BA) \leqslant \alpha(A) + \alpha(B), \quad \beta(BA) \leqslant \beta(A) + \beta(B), \quad \operatorname{ind}(BA) = \operatorname{ind}(A) + \operatorname{ind}(B).$$

Proof. Let $A \in \Phi_+(X, Y)$, $B \in \Phi_+(Y, Z)$, and let Q be a bounded subset of X, but Q is not totally bounded. From Theorem 2.1 it follows that $A(Q)$ is bounded and it is not totally bounded. From the same reason $B(A(Q))$ is not totally bounded. Again from Theorem 2.1 it follows that $BA \in \Phi_+(X, Z)$.

Let X_1 be a complement of $\mathcal{N}(A)$ in X, that is

$$X = \mathcal{N}(A) \oplus X_1 \tag{2.9}$$

Then the reduction $A_{|X_1} \colon X_1 \to \mathcal{R}(A)$ has the inverse from $\mathcal{R}(A)$ to X_1, and let $W = A^{-1}_{|X_1}(\mathcal{N}(B) \cap \mathcal{R}(A))$. We shall prove that:

$$\mathcal{N}(BA) = \mathcal{N}(A) \oplus W \tag{2.10}$$

Obviously, $W \subset \mathcal{N}(BA)$ and $\mathcal{N}(A) \subset \mathcal{N}(BA)$, so $N(A) + W \subset \mathcal{N}(BA)$. Let $x \in \mathcal{N}(BA)$. From (2.9) it follows that there exist $z \in \mathcal{N}(A)$ and $x_1 \in X_1$ such that $x = z + x_1$. Since $BAx_1 = BAx = 0$, it follows that $A_{|X_1}x_1 = Ax_1 \in \mathcal{R}(A) \cap \mathcal{N}(B)$, and so $x_1 \in W$. Therefore $\mathcal{N}(BA) = \mathcal{N}(A) + W$. If $w \in \mathcal{N}(A) \cap W$, then $w \in X_1 \cap N(A) = \{0\}$. Hence, $\mathcal{N}(A) \cap W = \{0\}$. From (2.10) we obtain

$$\dim \mathcal{N}(BA) = \dim \mathcal{N}(A) + \dim(\mathcal{N}(B) \cap \mathcal{R}(A)) \tag{2.11}$$

and consequently, $\dim \mathcal{N}(BA) \leqslant \dim \mathcal{N}(A) + \dim \mathcal{N}(B)$, that is $\alpha(BA) \leqslant \alpha(A) + \alpha(B)$.

Let $Y_0 = \mathcal{R}(A) \cap \mathcal{N}(B)$. There exist subspaces Y_1 and Y_2 in Y, such that

$$\mathcal{R}(A) = Y_0 \oplus Y_1, \tag{2.12}$$

$$\mathcal{N}(B) = Y_0 \oplus Y_2. \tag{2.13}$$

Let Y_3 be the complement of $\mathcal{R}(A) \oplus Y_2$ in Y, i.e.

$$Y = \mathcal{R}(A) \oplus Y_2 \oplus Y_3. \tag{2.14}$$

We get

$$\mathcal{R}(B) = \mathcal{R}(BA) + B(Y_3). \tag{2.15}$$

From (2.12) it follows that $\mathcal{R}(BA) = B(Y_1)$. Since $(Y_1 \oplus Y_3) \cap \mathcal{N}(B) = (Y_1 \oplus Y_3) \cap (Y_0 \oplus Y_2) = \{0\}$, we get that B is one-one on $Y_1 \oplus Y_3$. Hence, $\dim B(Y_3) = \dim Y_3$, and from $Y_1 \cap Y_3 = \{0\}$ it follows that $B(Y_1) \cap B(Y_3) = \{0\}$, so $\mathcal{R}(BA) \cap B(Y_3) = \{0\}$. From (2.15) we obtain $\mathcal{R}(B) = \mathcal{R}(BA) \oplus B(Y_3)$. Consequently,

$$\beta(BA) = \beta(B) + \dim B(Y_3) = \beta(B) + \dim Y_3. \tag{2.16}$$

From (2.14) it follows that $\dim Y_3 \leqslant \beta(A)$, so from (2.16) we get $\beta(BA) \leqslant \beta(B) + \beta(A)$.

From (2.11) we obtain

$$\alpha(BA) = \alpha(A) + \dim Y_0. \tag{2.17}$$

Recall that from (2.13) it follows that the subspaces Y_0 and Y_2 are finite dimensional.

From (2.17), (2.16), (2.13) and (2.14) we get

$$\begin{aligned}\operatorname{ind}(BA) &= \alpha(BA) - \beta(BA) = \alpha(A) + \dim Y_0 - \beta(B) - \dim Y_3\\ &= \alpha(A) + (\dim Y_0 + \dim Y_2) - \beta(B) - (\dim Y_2 + \dim Y_3)\\ &= \alpha(A) + \alpha(B) - \beta(B) - \beta(A) = \operatorname{ind}(A) + \operatorname{ind}(B).\end{aligned}$$

Suppose that $A \in \Phi_-(X, Y)$ and $B \in \Phi_-(Y, Z)$. Then $A' \in \Phi_+(Y', X')$ and $B' \in \Phi_+(Z', Y')$ (Lemma 2.8), and according to the previous consideration, we have $A'B' \in \Phi_+(Z', X')$. It follows that $BA \in \Phi_-(X, Z)$. Using Lemma 2.4 we get

$$\begin{aligned}\alpha(BA) = \beta[(BA)'] = \beta(A'B') &\leqslant\\ &\leqslant \beta(A') + \beta(B') = \alpha(A) + \alpha(B).\end{aligned}$$

Analogously, $\beta(BA) \leqslant \beta(A) + \beta(B)$ and by Lemma 2.8 it follows $\operatorname{ind}(BA) = -\operatorname{ind}((BA)') = -\operatorname{ind}(A'B') = -\operatorname{ind}(A') - \operatorname{ind}(B') = \operatorname{ind}(A) + \operatorname{ind}(B)$. □

Remark 2.1. If $A\colon X \to Y$ and $B\colon Y \to Z$ are linear operators, from the proof of Theorem 2.2 it follows that

$$\alpha(BA) \leqslant \alpha(A) + \alpha(B), \quad \beta(BA) \leqslant \beta(A) + \beta(B).$$

Moreover, if $\alpha(A) < \infty$ and $\alpha(B) < \infty$ (or, $\beta(A) < \infty$ and $\beta(B) < \infty$), then

$$\operatorname{ind}(BA) = \operatorname{ind}(A) + \operatorname{ind}(B).$$

If $N(B) \subset R(A)$, from (2.11) we get $\dim N(BA) = \dim N(A) + \dim N(B)$, and so $\alpha(BA) = \alpha(A) + \alpha(B)$. From (2.13), (2.14) and (2.16) we obtain $\beta(BA) = \beta(A) + \beta(B)$.

Therefore, if A is a linear operator defined on X, then $\alpha(A^n) \leqslant n\alpha(A)$ and $\beta(A^n) \leqslant n\beta(A)$ for every $n \in \mathbb{N}$. Moreover, if $\alpha(A) < \infty$ or $\beta(A) < \infty$, then $\operatorname{ind}(A^n) = n \operatorname{ind}(A)$ for every $n \in \mathbb{N}$.

If $N(A^n) \subset R(A)$ or $N(A) \subset R(A^n)$ for every $n \in \mathbb{N}$ (these two conditions are equivalent according to Lemma 2.36), by the induction on n we prove that

$$\begin{aligned}\alpha(A^n) &= n\alpha(A),\\ \beta(A^n) &= n\beta(A).\end{aligned}$$

Corollary 2.2. (i) *Let $A \in \Phi_+(X, Y)$ and $B \in \Phi_+(Y, Z)$. If $A \notin \Phi_-(X, Y)$ or $B \notin \Phi_-(Y, Z)$, then $BA \in \Phi_+(X, Z) \setminus \Phi_-(X, Z)$.*

(ii) *Let* $A \in \Phi_-(X,Y)$ *and* $B \in \Phi_-(Y,Z)$. *If* $A \notin \Phi_+(X,Y)$ *or* $B \notin \Phi_+(Y,Z)$, *then* $BA \in \Phi_-(X,Z) \setminus \Phi_+(X,Z)$.

Proof. (i): Suppose that A and B are Φ_+-operators and at least one of them is not Fredholm. From Theorem 2.2 it follows that $BA \in \Phi_+(X,Z)$ and $\mathrm{ind}(BA) = \mathrm{ind}(B) + \mathrm{ind}(A) = -\infty$, so $BA \notin \Phi_-(X,Z)$.

(ii): It can be proved analogously. □

From Theorem 2.2 and Corollary 2.2 it follows that $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_+(X) \setminus \Phi_-(X)$ and $\Phi_-(X) \setminus \Phi_+(X)$ are semigroups in $L(X)$.

Let M and N be closed subspaces of Y. Then $J_M : M \to Y$ is the natural inclusion, and $Q_N : Y \to Y/N$ is the natural epimorphism. Obviously, $J_M \in L(M,Y)$ and $Q_N \in L(Y, Y/N)$.

Theorem 2.3. *Let* $A \in L(X,Y)$ *and* $B \in L(Y,Z)$.

(i) *If* BA *belongs to the class* Φ_+, *then* A *and* $BJ_{\mathcal{R}(A)}$ *belong to the class* Φ_+.

(ii) *If* BA *belongs to the class* Φ_-, *then* B *and* $Q_{N(B)}A$ *also belong to the class* Φ_-.

Proof. (i): Let Q be a bounded subset of X, such that Q is not totally bounded. From Theorem 2.1 it follows that $B(A(Q))$ is not totally bounded. Consequently, $A(Q)$ is not totaly bounded. From Theorem 2.1 it follows that $A \in \Phi_+(X,Y)$.

Let $(y_n)_n$ be a bounded sequence in $\mathcal{R}(A)$, which does not have a convergent subsequence. From $A \in \Phi_+(X,Y)$ it follows that there exists a closed subspace X_1 of X such that $X = \mathcal{N}(A) \oplus X_1$. Then the reduction $A_1 = A_{|X_1} : X_1 \to \mathcal{R}(A)$ has a bounded inverse from $\mathcal{R}(A)$ to X_1. There exists some $x_n \in X_1$ such that $Ax_n = y_n$, i.e. $x_n = A_1^{-1} y_n$. It follows that $(x_n)_n$ is bounded and it does not have a convergent subsequence. Since BA is a Φ_+-operator, from Theorem 2.1 it follows that $(BAx_n)_n = (By_n)_n$ does not have a convergent subsequence. Again by Theorem 2.1 we conclude that $BJ_{\mathcal{R}(A)}$ is a Φ_+-operator.

(ii): From $BA \in \Phi_-(X,Z)$ it follows that $A'B' \in \Phi_+(Z',X')$ (see Lemma 2.8). From (i) we get that B' and $A'J_{\mathcal{R}(B')} = A'J_{\mathcal{N}(B)^\perp}$ are Φ_+-operators. From (2.2) we get that $\mathcal{R}({Q_{\mathcal{N}(B)}}') = \mathcal{N}(Q_{\mathcal{N}(B)})^\perp = \mathcal{N}(B)^\perp$ and from (2.1) it follows that $\mathcal{N}(Q'_{\mathcal{N}(B)}) = \mathcal{R}(Q_{\mathcal{N}(B)})^\perp = \{0\}$. Therefore, ${Q_{\mathcal{N}(B)}}'$ is an isomorphism from $(Y/\mathcal{N}(B))'$ onto $\mathcal{N}(B)^\perp$, so ${Q_{\mathcal{N}(B)}}'$ is a Φ_+-operator. From Theorem 2.2 it follows that $(Q_{\mathcal{N}(B)}A)' = A'J_{\mathcal{N}(B)^\perp} Q'_{\mathcal{N}(B)}$ is a Φ_+-operator. According to Lemma 2.8 we get that B and $Q_{\mathcal{N}(B)}A$ are Φ_--operators. □

Theorem 2.4. *If* $A \in L(X,Y)$ *is upper (lower) semi-Fredholm and* $K \in K(X,Y)$, *then* $A + K$ *is upper (lower) semi-Fredholm.*

Proof. Let $A \in \Phi_+(X,Y)$ and $K \in K(X,Y)$. Suppose that $A + K \notin \Phi_+(X,Y)$, so there exists a bounded sequence $(x_n)_n$ in X which does not have a convergent subsequence, and $((A+K)x_n)_n$ has a convergent subsequence $((A+K)x_{n_k})_k$ (Theorem 2.1). Since $(x_{n_k})_k$ is bounded in X and $K \in K(X,Y)$, there exists a subsequence $(x_{n_{k_j}})_j$ of $(x_{n_k})_k$ such that the sequence $(Kx_{n_{k_j}})_j$ is convergent. It follows that

$(Ax_{n_{k_j}})_j$ is convergent. From Theorem 2.1 it follows that $A \notin \Phi_+(X,Y)$, which is not possible.

Let $A \in \Phi_-(X,Y)$ and $K \in K(X,Y)$. Then $A' \in \Phi_+(Y',X')$ and $K' \in K(Y',X')$. From the already proved part it follows that $A'+K' \in \Phi_+(Y',X')$, and so $A+K \in \Phi_-(X,Y)$. □

2.3. Ascent and descent. We give some assertions concerning the ascent and descent of linear operators. Almost all assertions are valid for linear spaces without norm.

Let A be a linear operator defined on a linear space X. The following inclusions are obvious:

$$\mathcal{N}(A^n) \subset \mathcal{N}(A^{n+1}), \quad (A^0 = I), \quad n = 0,1,2,\dots.$$

If $\mathcal{N}(A^n) = \mathcal{N}(A^{n+1})$ for some $n \in \{0,1,2,\dots\}$, then $\mathcal{N}(A^k) = \mathcal{N}(A^n)$ for every $k = n, n+1, \dots$.

Definition 2.1. Let A be a linear operator defined on X. If there exists some $n \in \mathbb{N}$ such that the following holds:

$$\mathcal{N}(A^n) = \mathcal{N}(A^{n+1}), \tag{2.18}$$

then A has a *finite ascent*. The smallest n such that (2.18) holds is the ascent of A, denoted by $\operatorname{asc}(A)$.

If (2.18) does not hold for any n, then the ascent of A is infinite, noted by $\operatorname{asc}(A) = \infty$.

Dually,

$$\mathcal{R}(A^n) \supset \mathcal{R}(A^{n+1}), \quad (A^0 = I), \quad n = 0,1,2,\dots,$$

If $\mathcal{R}(A^n) = \mathcal{R}(A^{n+1})$ for some $n \in \mathbb{N}$, then $\mathcal{R}(A^k) = \mathcal{R}(A^n)$ for every $k = n, n+1, \dots$.

Definition 2.2. Let A be a linear operator defined on X. If there exists some $n \in \mathbb{N}$, such that

$$\mathcal{R}(A^n) = \mathcal{R}(A^{n+1}), \tag{2.19}$$

then A has a *finite descent*. In this case the descent of A is the smallest n such that (2.19) holds, and $\operatorname{dsc}(A) = n$.

If (2.19) does not hold for any $n \in \mathbb{N}$, then A has an infinite descent, and $\operatorname{dsc}(A) = \infty$.

Lemma 2.9. *If A is a linear operator defined on X, the following conditions are equivalent:*

(i) $\operatorname{asc}(A) \leqslant m < \infty$;
(ii) $\mathcal{N}(A^n) \cap \mathcal{R}(A^m) = \{0\}$ *for every* $n \in \mathbb{N}$;
(iii) $\mathcal{N}(A^n) \cap \mathcal{R}(A^m) = \{0\}$ *for some* $n \in \mathbb{N}$;
(iv) $\mathcal{N}(A) \cap \mathcal{R}(A^m) = \{0\}$.

Proof. (i)$\Longrightarrow$(ii): If $\operatorname{asc}(A) \leqslant m < \infty$, $n \in \mathbb{N}$ and $y \in \mathcal{N}(A^n) \cap \mathcal{R}(A^m)$, then $y = A^m x$, $x \in X$ and $A^n y = 0$. Consequently $A^{m+n}x = 0$ and $x \in \mathcal{N}(A^{m+n}) = \mathcal{N}(A^m)$ and hence $y = A^m x = 0$.

(ii)$\Longrightarrow$(iii)$\Longrightarrow$(iv): Obvious.

(iv)$\Longrightarrow$(i): Suppose that $\mathcal{N}(A)\cap\mathcal{R}(A^m) = \{0\}$. If $x \in \mathcal{N}(A^{m+1})$, i.e. $A(A^m x) = 0$, then $A^m x \in \mathcal{N}(A) \cap \mathcal{R}(A^m) = \{0\}$, or $x \in \mathcal{N}(A^m)$. Therefore $\mathcal{N}(A^m) = \mathcal{N}(A^{m+1})$ and $\operatorname{asc}(A) \leqslant m$. ☐

Lemma 2.10. *If A is a linear operator defined on X, the following conditions are equivalent:*

(i) $\operatorname{dsc}(A) \leqslant m < \infty$;
(ii) *For every $n \in \mathbb{N}$, $\mathcal{R}(A^n)$ has a direct complement C_n in X, such that $C_n \subset \mathcal{N}(A^m)$;*
(iii) $\mathcal{N}(A^m) + \mathcal{R}(A^n) = X$ *for every* $n \in \mathbb{N}$;
(iv) $\mathcal{N}(A^m) + \mathcal{R}(A^n) = X$ *for some* $n \in \mathbb{N}$;
(v) $\mathcal{N}(A^m) + \mathcal{R}(A) = X$.

Proof. (i)$\Longrightarrow$(ii): Let $\operatorname{dsc}(A) \leqslant m$, $n \in \mathbb{N}$, and let C be a direct complement of $\mathcal{R}(A^n)$ in X, i.e.

$$X = C \oplus \mathcal{R}(A^n). \tag{2.20}$$

Let $\{x_i : i \in I\}$ be an algebraic base in C. Since $A^m(C) \subset A^m(X) = A^{m+n}(X)$, for every $i \in I$ there exists some $y_i \in X$ such that $A^m x_i = A^{m+n} y_i$. Let $z_i = x_i - A^n y_i$. From $A^m z_i = A^m x_i - A^{m+n} y_i = 0$ it follows that $C_n = \operatorname{lin}\{z_i : i \in I\} \subset \mathcal{N}(A^m)$. According to (2.20) for every $x \in X$ there exist scalars α_i, $i \in I$ and $y \in X$ such that

$$x = \sum \alpha_i x_i + A^n y = \sum \alpha_i (z_i + A^n y_i) + A^n y = \sum \alpha_i z_i + A^n z.$$

Hence, $X = C_n + \mathcal{R}(A^n)$. To prove that this sum is direct, suppose that $x \in C_n \cap \mathcal{R}(A^n)$. Then $x = \sum \beta_i z_i = A^n v$ and hence

$$\sum \beta_i x_i = \sum \beta_i A^n y_i + A^n v \in \mathcal{R}(A^n).$$

From (2.20) it follows that $\sum \beta_i x_i = 0$. We get $\beta_i = 0$ for every $i \in I$, and consequently $x = 0$. Therefore $X = C_n \oplus \mathcal{R}(A^n)$.

(ii)$\Longrightarrow$(iii)$\Longrightarrow$ (iv)$\Longrightarrow$(v): Obvious.

(v)$\Longrightarrow$(i): Suppose that $\mathcal{N}(A^m) + \mathcal{R}(A) = X$. Then $A^m(X) = A^m(\mathcal{R}(A)) = A^{m+1}(X)$, so $\operatorname{dsc}(A) \leqslant m$. ☐

Theorem 2.5. *Let A be a linear operator defined on X. If* $\operatorname{asc}(A) < \infty$ *and* $\operatorname{dsc}(A) < \infty$*, then* $\operatorname{asc}(A) = \operatorname{dsc}(A)$.

Proof. Let $p = \operatorname{asc}(A)$ and $q = \operatorname{dsc}(A)$. Suppose that $p \leqslant q$. Then $\mathcal{R}(A^q) \subset \mathcal{R}(A^p)$. Let $q > 0$, because otherwise the proof is finished. Let $y \in \mathcal{R}(A^p)$. From Lemma 2.10 we conclude $X = \mathcal{N}(A^q) + \mathcal{R}(A^q)$, so there exist $z \in \mathcal{N}(A^q)$ and $w \in X$ such that $y = z + A^q w$. Since $z = y - A^q w \in \mathcal{R}(A^p)$, we have $z \in \mathcal{N}(A^q) \cap \mathcal{R}(A^p)$. From Lemma 2.9 it follows that $\mathcal{N}(A^q) \cap \mathcal{R}(A^p) = \{0\}$, so $z = 0$ and therefore, $y \in \mathcal{R}(A^q)$. We have just proved that $\mathcal{R}(A^q) = \mathcal{R}(A^p)$, so $q \leqslant p$. Hence, $p = q$.

Suppose that $q \leqslant p$ and $p > 0$. We get $\mathcal{N}(A^q) \subset \mathcal{N}(A^p)$. Let $x \in \mathcal{N}(A^p)$. From Lemma 2.10 we have $X = \mathcal{N}(A^q) + \mathcal{R}(A^p)$, so there exist $u \in \mathcal{N}(A^q)$ and $v \in X$ such that $x = u + A^p v$. Since $A^p x = 0$ and $A^p u = 0$, we have $A^{2p} v = 0$.

Consequently, $v \in \mathcal{N}(A^{2p}) = \mathcal{N}(A^p)$, so $x = u \in \mathcal{N}(A^q)$. We have just proved $\mathcal{N}(A^q) = \mathcal{N}(A^p)$, so $p \leqslant q$. Hence, $p = q$. □

If A is a linear operator acting on X and $\dim X < \infty$, then $\operatorname{asc}(A) = \operatorname{dsc}(A) < \infty$. If X is infinite dimensional, then we can not conclude $\operatorname{asc}(A) = \operatorname{dsc}(A)$.

Lemma 2.11. *Let A be a linear operator acting on X.*

(i) *If $\operatorname{asc}(A) < \infty$, then $N^\infty(A) \cap R^\infty(A) = \{0\}$.*
(ii) *If $\operatorname{dsc}(A) < \infty$, then $N^\infty(A) + R^\infty(A) = X$.*

Proof. (i): Let $m = \operatorname{asc}(A) < \infty$. Then $N^\infty(A) = \mathcal{N}(A^m)$. Since $R^\infty(A) \subset \mathcal{R}(A^m)$, we conclude that $N^\infty(A) \cap R^\infty(A) \subset \mathcal{N}(A^m) \cap \mathcal{R}(A^m)$. By Lemma 2.9, $\mathcal{N}(A^m) \cap \mathcal{R}(A^m) = \{0\}$ and hence $N^\infty(A) \cap R^\infty(A) = \{0\}$.

(ii): Let $m = \operatorname{dsc}(A) < \infty$. Then $R^\infty(A) = \mathcal{R}(A^m)$. Since $\mathcal{N}(A^m) \subset N^\infty(A)$, we conclude that $\mathcal{N}(A^m) + \mathcal{R}(A^m) \subset N^\infty(A) + R^\infty(A)$. By Lemma 2.10, $\mathcal{N}(A^m) + \mathcal{R}(A^m) = X$ and hence $N^\infty(A) + R^\infty(A) = X$. □

Remark that the condition $N^\infty(A) \cap R^\infty(A) = \{0\}$ is equivalent to the condition $\mathcal{N}(A) \cap R^\infty(A) = \{0\}$, i.e. to the fact that A is one-one on $R^\infty(A)$. Really, if $N^\infty(A) \cap R^\infty(A) = \{0\}$, because of $\mathcal{N}(A) \cap R^\infty(A) \subset N^\infty(A) \cap R^\infty(A)$ it follows $\mathcal{N}(A) \cap R^\infty(A) = \{0\}$. On the other hand, if A is one-one on $R^\infty(A)$, then A^n is also one-one on $R^\infty(A)$ for every $n \in \mathbb{N}$, that is $\mathcal{N}(A^n) \cap R^\infty(A) = \{0\}$ for every $n \in \mathbb{N}$ and hence $N^\infty(A) \cap R^\infty(A) = \{0\}$.

Corollary 2.3. *Let A be a linear operator acting on X. and* $\operatorname{asc}(A) < \infty$. *Then:*

(i) *A is one-one on $R^\infty(A)$.*
(ii) *For every subspace M of X such that $A(M) = M$, it follows that A is one-one on M.*

Proof. (i): From $\operatorname{asc}(A) < \infty$, according to Lemma 2.11 (i), it follows that $N^\infty(A) \cap R^\infty(A) = \{0\}$. Since $\mathcal{N}(A) \cap R^\infty(A) \subset N^\infty(A) \cap R^\infty(A) = \{0\}$, we obtain that $\mathcal{N}(A) \cap R^\infty(A) = \{0\}$, and so A is one-one on $R^\infty(A)$.

(ii): Suppose that M is a subspace of X such that $A(M) = M$. Then $A^n(M) = M$ for every $n \in \mathbb{N}$ and hence $M \subset \mathcal{R}(A^n)$ for every $n \in \mathbb{N}$. Thus $M \subset R^\infty(A)$ and by (i), A is one-one on M. □

Theorem 2.6. *Let A be a linear operator acting on X,* $\operatorname{asc}(A) < \infty$ *and* $\operatorname{dsc}(A) < \infty$. *Then* $\operatorname{asc}(A) = \operatorname{dsc}(A) = p$ *and*

$$X = \mathcal{R}(A^p) \oplus \mathcal{N}(A^p). \tag{2.21}$$

Moreover, $\mathcal{R}(A^p)$ and $\mathcal{N}(A^p)$ are invariant subspaces of A, the operator $A_1 = A_{|\mathcal{R}(A^p)}\colon \mathcal{R}(A^p) \to \mathcal{R}(A^p)$ is bijective, and the operator $A_2 = A_{|\mathcal{N}(A^p)}\colon \mathcal{N}(A^p) \to \mathcal{N}(A^p)$ is nilpotent. If $A \in L(X)$, then $\mathcal{R}(A^p)$ and $\mathcal{N}(A^p)$ are closed.

On the opposite, if for some $m \in \mathbb{N}$ the following holds:

$$X = \mathcal{R}(A^m) \oplus \mathcal{N}(A^m), \tag{2.22}$$

then $\operatorname{asc}(A) = \operatorname{dsc}(A) \leqslant m$.

Proof. If $\operatorname{asc}(A) = \operatorname{dsc}(A) = p < \infty$, then (2.21) follows from Lemma 2.9 and Lemma 2.10.

On the other hand, from $\operatorname{dsc}(A) = p$ it follows that $A(\mathcal{R}(A^p)) = \mathcal{R}(A^p)$ and therefore, A_1 is surjective. Since $\operatorname{asc}(A) < \infty$, A_1 is one-one by Corollary 2.3. If $x \in \mathcal{N}(A^p)$, then $A_2^p x = A^p x = 0$, so A_2 is nilpotent.

If $A \in L(X)$, then $\mathcal{N}(A^p)$ is closed and from Lemma 2.5 and (2.21) it follows that $\mathcal{R}(A^p)$ is closed.

If (2.22) holds, then from Lemma 2.9 and Lemma 2.10 we conclude $\operatorname{asc}(A) \leqslant m$ and $\operatorname{dsc}(A) \leqslant m$. According to Theorem 2.5 we have $\operatorname{asc}(A) = \operatorname{dsc}(A) \leqslant m$. □

Lemma 2.12. *If* $A \in L(X)$*, then* $\operatorname{asc}(A) = \operatorname{dsc}(A) < \infty$ *if and only if* $\operatorname{asc}(A') = \operatorname{dsc}(A') < \infty$*. Moreover,* $\operatorname{asc}(A) = \operatorname{dsc}(A) = \operatorname{asc}(A') = \operatorname{dsc}(A')$.

Proof. Let $\operatorname{asc}(A) = \operatorname{dsc}(A) = p < \infty$. Then $\mathcal{N}(A^p) = \mathcal{N}(A^{p+1})$ and $\mathcal{R}(A^p) = \mathcal{R}(A^{p+1})$. From Theorem 2.6 it follows that $\mathcal{R}(A^p)$ is closed, so according to (2.2) we get

$$\mathcal{R}((A')^p) = \mathcal{R}((A^p)') = \mathcal{N}(A^p)^{\perp} = \mathcal{N}(A^{p+1})^{\perp} = \mathcal{R}((A^{p+1})') = \mathcal{R}((A')^{p+1}).$$

From (2.1) it follows that

$$\mathcal{N}((A')^p) = \mathcal{N}((A^p)') = \mathcal{R}(A^p)^{\perp} = \mathcal{R}(A^{p+1})^{\perp} = \mathcal{N}((A^{p+1})') = \mathcal{N}((A')^{p+1}).$$

Consequently, $\operatorname{asc}(A') \leqslant p$ and $\operatorname{dsc}(A') \leqslant p$.

Suppose that $\operatorname{asc}(A') = \operatorname{dsc}(A') = q < \infty$. Then $\mathcal{N}((A')^q) = \mathcal{N}((A')^{q+1})$ and $\mathcal{R}((A')^q) = \mathcal{R}((A')^{q+1})$ is closed (Theorem 2.6). From (2.2) we get

$$\mathcal{N}(A^q) = {}^{\perp}(\mathcal{N}(A^q)^{\perp}) = {}^{\perp}(\mathcal{R}((A')^q)) = {}^{\perp}(\mathcal{R}((A')^{q+1})) = {}^{\perp}(\mathcal{N}(A^{q+1})^{\perp}) = \mathcal{N}(A^{q+1}),$$

$$\mathcal{R}(A^q) = {}^{\perp}\mathcal{N}((A')^q) = {}^{\perp}\mathcal{N}((A')^{q+1}) = \mathcal{R}(A^{q+1}).$$

It follows that $\operatorname{asc}(A) \leqslant q$ and $\operatorname{dsc}(A) \leqslant q$. Finally, $p = q$. □

Lemma 2.13. *Let* $A \in L(X)$ *and let* $\mathcal{R}(A^n)$ *be closed for* $n = 1, 2, \ldots$. *Then* $\operatorname{asc}(A) = \operatorname{dsc}(A')$ *and* $\operatorname{dsc}(A) = \operatorname{asc}(A')$.

Proof. As in the proof of Lemma 2.12, we obtain that $\operatorname{asc}(A) < \infty$ if and only if $\operatorname{dsc}(A') < \infty$, and in this case $\operatorname{asc}(A) = \operatorname{dsc}(A')$. It follows that $\operatorname{asc}(A) = \infty$ if and only if $\operatorname{dsc}(A') = \infty$. Therefore, in the both cases we have $\operatorname{asc}(A) = \operatorname{dsc}(A')$. Analogously, $\operatorname{dsc}(A) = \operatorname{asc}(A')$. □

Theorem 2.7. *Let A be a linear operator acting on X.*

(i) *If* $\operatorname{asc}(A) < \infty$*, then* $\alpha(A) \leqslant \beta(A)$.
(ii) *If* $\operatorname{dsc}(A) < \infty$*, then* $\beta(A) \leqslant \alpha(A)$.

Proof. (i): Let $p = \operatorname{asc}(A) < \infty$. If $\beta(A) = \infty$, then $\alpha(A) \leqslant \beta(A)$.

Suppose that $\beta(A) < \infty$. From Lemma 2.9 we get $\mathcal{N}(A) \cap \mathcal{R}(A^p) = \{0\}$, so $\alpha(A) \leqslant \beta(A^p) \leqslant p\beta(A) < \infty$ (Remark 2.1). Also, for every $n \geqslant p$ we have

$$n \operatorname{ind}(A) = \operatorname{ind}(A^n) = \alpha(A^n) - \beta(A^n) = \alpha(A^p) - \beta(A^n). \tag{2.23}$$

If $\operatorname{dsc}(A) < \infty$, then $\operatorname{dsc}(A) = \operatorname{asc}(A) = p$ and for every $n \geqslant p$ we get $n \operatorname{ind}(A) = \alpha(A^p) - \beta(A^p)$. Since $\alpha(A^p) - \beta(A^p)$ is constant, it follows that

$\text{ind}(A) = 0$, i.e. $\alpha(A) = \beta(A)$. If $\text{dsc}(A) = \infty$, then $\lim_{n\to\infty} \beta(A^n) = \infty$. Consequently $\alpha(A^p) - \beta(A^n) < 0$ for a suitable n. From (2.23) we obtain $\text{ind}(A) < 0$, so $\alpha(A) < \beta(A)$.

(ii): Let $q = \text{dsc}(A) < \infty$. If $\alpha(A) = \infty$, then $\beta(A) \leqslant \alpha(A)$.

Let $\alpha(A) < \infty$. From Lemma 2.10 it follows that there exists a subspace C of X such that $X = C \oplus \mathcal{R}(A)$ and $C \subset \mathcal{N}(A^q)$, so $\beta(A) = \dim C \leqslant \alpha(A^q) \leqslant q\alpha(A) < \infty$. Now, as in the proof of (i), we obtain $\beta(A) = \alpha(A)$ if $\text{asc}(A) < \infty$ holds, and $\beta(A) < \alpha(A)$ if $\text{asc}(A) = \infty$ holds. □

Corollary 2.4. *Let A be a linear operator acting on X.*

(i) *If $\text{asc}(A) < \infty$ and $\text{dsc}(A) < \infty$, then $\alpha(A) = \beta(A)$.*
(ii) *If $\alpha(A) = \beta(A) < \infty$, then $\text{asc}(A) < \infty$ if and only if $\text{dsc}(A) < \infty$.*

Proof. (i) follows from Theorem 2.7.

(ii): Let $\alpha(A) = \beta(A) < \infty$ and $p = \text{asc}(A) < \infty$. Then $\alpha(A^n) - \beta(A^n) = \text{ind}(A^n) = n\, \text{ind}(A) = 0$, $n = 0, 1, 2, \dots$ (see Remark 2.1). Hence, $\beta(A^p) = \alpha(A^p) = \alpha(A^{p+1}) = \beta(A^{p+1}) \leqslant (p+1)\beta(A) < \infty$. It follows that $\mathcal{R}(A^p) = \mathcal{R}(A^{p+1})$, i.e. $\text{dsc}(A) \leqslant p$. If $\text{dsc}(A) < \infty$, we analogously prove that $\text{asc}(A) < \infty$. □

Corollary 2.5. *Let A be a linear operator acting on X. If $\text{asc}(A) < \infty$ or $\text{dsc}(A) < \infty$, then $A(R^\infty(A)) = R^\infty(A)$.*

Proof. Suppose that $p = \text{asc}(A) < \infty$. Then by Lemma 2.9 it follows that $\mathcal{N}(A) \cap A^p(X) = \{0\}$. Since $\mathcal{N}(A) \cap A^p(X) \supset \mathcal{N}(A) \cap A^{p+k}(X)$ for every $k = 1, 2, \dots$, we have

$$\mathcal{N}(A) \cap A^p(X) = \mathcal{N}(A) \cap A^{p+k}(X) = \{0\}, \text{ for all } k \in \mathbb{N}.$$

If $q = \text{dsc}(A) < \infty$, then

$$\mathcal{N}(A) \cap A^q(X) = \mathcal{N}(A) \cap A^{q+k}(X), \text{ for all } k \in \mathbb{N}.$$

Now the assertion follows from Lemma 2.6. □

Theorem 2.8. *Let A be a linear operator acting on X. If $\alpha(A) < \infty$ or $\beta(A) < \infty$, then the following conditions are equivalent:*

(i) $\text{asc}(A) < \infty$;
(ii) $N^\infty(A) \cap R^\infty(A) = \{0\}$;
(iii) $\mathcal{N}(A) \cap R^\infty(A) = \{0\}$, *i.e. A is one-one on $R^\infty(A)$.*

Proof. The implication (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii) are proved in Lemma 2.11 (i) and Corollary 2.3.

(iii)$\Longrightarrow$(i): Suppose that $\alpha(A) < \infty$ or $\beta(A) < \infty$ and suppose that $\mathcal{N}(A) \cap R^\infty(A) = \{0\}$. Then by the proof of Theorem 2.7, there exists $m \in \mathbb{N}$ such that

$$\mathcal{N}(A) \cap A^m(X) = \mathcal{N}(A) \cap A^{m+k}(X), \text{ for all } k \in \mathbb{N}.$$

Therefore $\mathcal{N}(A) \cap A^m(X) = \mathcal{N}(A) \cap R^\infty(A) = \{0\}$. From Lemma 2.9 it follows that $\text{asc}(A) \leqslant m < \infty$. □

Theorem 2.9. *Let A be a linear operator acting on X and $\beta(A) < \infty$. Then:*

$$\operatorname{dsc}(A) < \infty \Longleftrightarrow N^\infty(A) + R^\infty(A) = X.$$

Proof. The implication ($\Longrightarrow$) is the assertion (ii) in Lemma 2.11.

Conversely, suppose that $\beta(A) < \infty$ and $N^\infty(A) + R^\infty(A) = X$. Since

$$\mathcal{R}(A) \subseteq \mathcal{N}(A) + \mathcal{R}(A) \subseteq \mathcal{N}(A^2) + \mathcal{R}(A) \subseteq \cdots \subseteq \mathcal{N}(A^n) + \mathcal{R}(A) \subseteq \dots, \quad n \in \mathbb{N},$$

and since $\mathcal{R}(A)$ is of finite codimension in X, we conclude that there exists $m \in \mathbb{N}$ such that

$$\mathcal{N}(A^m) + \mathcal{R}(A) = \mathcal{N}(A^{m+k}) + \mathcal{R}(A), \quad \text{for all k} \in \mathbb{N}.$$

Consequently, $N^\infty(A) + \mathcal{R}(A) = \mathcal{N}(A^m) + \mathcal{R}(A)$. Since $X = N^\infty(A) + R^\infty(A) \subset N^\infty(A) + \mathcal{R}(A)$, we get $\mathcal{N}(A^m) + \mathcal{R}(A) = X$. From Lemma 2.10 it follows that $\operatorname{dsc}(A) \leqslant m < \infty$. □

Theorem 2.10. *Let $K \in K(X)$. Then*

$$\operatorname{asc}(I-K) = \operatorname{dsc}(I-K) = \operatorname{asc}(I'-K') = \operatorname{dsc}(I'-K') < \infty. \tag{2.24}$$

Proof. Suppose that $\operatorname{asc}(I-K) = \infty$. There exists some $x_n \in \mathcal{N}((I-K)^n)$ such that $\|x_n\| = 1$ and $\operatorname{dist}(x_n, \mathcal{N}((I-K)^{n-1})) \geqslant 1/2$, $n = 1, 2, \dots$. Let $n > m$. Since

$$Kx_n - Kx_m = x_n - [x_m - (I-K)x_m + (I-K)x_n]$$

and $x_m - (I-K)x_m + (I-K)x_n \in \mathcal{N}((I-K)^{n-1})$, we get $\|Kx_n - Kx_m\| \geqslant 1/2$. It follows that the sequence $(Kx_n)_n$ does not have a convergent subsequence. The last statement is not possible, since $(x_n)_n$ is bounded and K is compact. It follows that $\operatorname{asc}(I-K) < \infty$.

Since K' is compact, from the already proved results, it follows that $\operatorname{asc}(I' - K') < \infty$. From Theorem 2.4 it follows that $I-K \in \Phi(X)$. Hence, $(I-K)^n \in \Phi(X)$ (Theorem 2.2), so $\mathcal{R}((I-K)^n)$ is closed for all $n = 1, 2, \dots$. From Lemma 2.13 it follows that $\operatorname{dsc}(I-K) = \operatorname{asc}(I'-K')$ and $\operatorname{asc}(I-K) = \operatorname{dsc}(I'-K')$. According to Theorem 2.5 we get (2.24). □

Theorem 2.11 (Fredholm alternative)**.** *Let $K \in K(X)$. Then*

$$\dim \mathcal{N}(I-K) = \dim \mathcal{N}(I'-K') < \infty,$$

i.e.

$$\alpha(I-K) = \beta(I-K) < \infty. \tag{2.25}$$

Consequently, $\mathcal{R}(I-K) = X$ and $\mathcal{N}(I-K) = \{0\}$ hold, or $R(I-K) \neq X$ and $\mathcal{N}(I-K) \neq \{0\}$ hold.

Proof. From Theorem 2.10 it follows that $\operatorname{asc}(I-K) = \operatorname{dsc}(I-K) < \infty$, and from Theorem 2.4 and Corollary 2.4 (i) we obtain (2.25). □

The following theorem is a corollary of the Atkinson theorem and the Fredholm alternative.

Theorem 2.12. *If $A \in \Phi(X, Y)$ and $K \in K(X, Y)$, then $A + K \in \Phi(X, Y)$ and* $\operatorname{ind}(A+K) = \operatorname{ind}(A)$.

Proof. According to Theorem 2.15 there exist $B \in \Phi(Y, X)$, $K_1 \in K(X)$ such that $BA = I + K_1$. For $K \in K(X, Y)$, according to Theorem 2.4, we get $A + K \in \Phi(X, Y)$ and

$$B(A + K) = BA + BK = I + (K_1 + BK) = I + K_2, \quad K_2 \in K(X).$$

Now, from Theorem 2.11 we have

$$\operatorname{ind}(B(A + K)) = \operatorname{ind}(I + K_2) = 0 \quad \text{and} \quad \operatorname{ind}(BA) = \operatorname{ind}(I + K_1) = 0,$$

and from Theorem 2.2 it follows that

$$\operatorname{ind}(B) + \operatorname{ind}(A + K) = 0 \quad \text{and} \quad \operatorname{ind}(B) + \operatorname{ind}(A) = 0.$$

Consequently, $\operatorname{ind}(A + K) = \operatorname{ind}(A)$. □

Corollary 2.6. *If* $A \in \Phi_+(X, Y)$ *(*$A \in \Phi_-(X, Y)$*) and* $K \in K(X, Y)$*, then* $A + K \in \Phi_+(X, Y)$ *(*$A + K \in \Phi_-(X, Y)$*) and* $\operatorname{ind}(A + K) = \operatorname{ind}(A)$*.*

Proof. If $A \in \Phi(X, Y)$, then the result follows from Theorem 2.12.

If $A \in \Phi_+(X, Y)$ and $A \notin \Phi_-(X, Y)$, from Theorem 2.4 it follows that $A + K \in \Phi_+(X, Y) \setminus \Phi_-(X, Y)$, so $\operatorname{ind}(A + K) = \operatorname{ind}(A) = -\infty$. Analogously, if $A \in \Phi_-(X, Y) \setminus \Phi_+(X, Y)$, then $A + K \in \Phi_-(X, Y) \setminus \Phi_+(X, Y)$ and $\operatorname{ind}(A + K) = \operatorname{ind}(A) = \infty$. □

2.4. Schechter's and Lebow's characterization of semi-Fredholm operators.

Φ_+ and Φ_--operators can be characterized by their compact perturbations [31, 44]. We start with the following auxiliary result.

Lemma 2.14. *Let* $A \in L(X, Y)$*. Then* $A \in \Phi_+(X, Y)$ *if and only if there exists a closed subspace* M *of* X *such that* $\operatorname{codim} M < \infty$ *and the restriction of* A *to* M*,* $A_{|M} : M \to Y$*, is bounded below.*

Proof. Let M be a closed subspace of X, $\operatorname{codim} M < \infty$ and $m(A_{|M}) > 0$. There exists a finite dimensional subspace M_1 of X such that $X = M \oplus M_1$. It follows that $\mathcal{R}(A) = A(M) + A(M_1)$. Since $A(M)$ is closed, from Lemma 2.1 it follows that $\mathcal{R}(A)$ is closed. From $\mathcal{N}(A) \cap M = \{0\}$ we get $\dim \mathcal{N}(A) \leqslant \operatorname{codim} M < \infty$. We have proved that $A \in \Phi_+(X, Y)$.

If $A \in \Phi_+(X, Y)$, then there exists closed subspace M of X such that $X = M \oplus \mathcal{N}(A)$. The restriction $A_{|M}$ is injective and $\mathcal{R}(A_{|M}) = \mathcal{R}(A)$ is closed in Y, so $A_{|M}$ is bounded below. □

Lemma 2.15. *Let* $A \notin \Phi_+(X, Y)$*. Then there exist* $x_k \in X$ *and* $x'_k \in X'$*,* $k = 1, 2, \dots,$ *such that*

$$\|x_k\| = 1,\ \|x'_k\| \leqslant 2^{k-1},\ x'_j(x_k) = \delta_{jk},\ \|Ax_k\| \leqslant 2^{1-2k},\ \textit{for all } j,\ k \in \mathbb{N}. \tag{2.26}$$

Proof. Since $A \notin \Phi_+(X, Y)$, it follows that A is not bounded below. Hence, there exists $x_1 \in X$ such that $\|x_1\| = 1$ and $\|Ax_1\| \leqslant 1/2$, and there exists $x'_1 \in X'$ such that $\|x'_1\| = 1$ and $x'_1(x_1) = 1$. Suppose that there exist x_k and x'_k, $k = 1, \dots, n-1$, such that (2.26) is satisfied. Let $N = \bigcap_{k=1}^{n-1} \mathcal{N}(x'_k)$ and $L = \operatorname{lin}\{x_1, \dots, x_{n-1}\}$. Then $X = N \oplus L$. From Lemma 2.14 it follows that $m(A_{|N}) = 0$. Hence, there exists

$x_n \in N$ such that $\|x_n\| = 1$ and $\|Ax_n\| < 2^{1-2n}$. Also, there exists $f \in X'$ such that $f(x_n) = 1$ and $\|f\| = 1$. Iyt is easy to see that x'_n, which is defined as

$$x'_n = f - \sum_{k=1}^{n-1} f(x_k)x'_k$$

satisfies the conditions $x'_n(x_k) = \delta_{nk}$, $k = 1, \dots, n$ and $\|x'_n\| \leqslant 2^{n-1}$. By the induction we obtain (2.26). □

Lemma 2.16. *Let $(a_n)_n$ be the sequence of integers, which is defined inductively as follows:*

$$a_1 = 2, \qquad a_n = 2\left(1 + \sum_1^{n-1} a_k\right), \quad n = 1, 2, \dots.$$

If $A \notin \Phi_-(X, Y)$, then there exists a sequence $(y_k)_k$ in Y, and there exists a sequence $(y'_k)_k$ in Y', such that

$$(2.27)\quad \|y_k\| \leqslant a_k,\ \|y'_k\| = 1,\ \ \|A'(y'_k)\| < \frac{1}{2^k a_k} \quad \textit{and}\ \ y'_j(y_k) = \delta_{jk},\ \ j, k = 1, 2, \dots.$$

Proof. Since $A \notin \Phi_-(X, Y)$, it follows that $A' \notin \Phi_+(Y', X')$. Hence, $m(A') = 0$, and

$$\inf_{y' \in Y',\, \|y'\|=1} \|A'y'\| = 0.$$

There exists some $y'_1 \in Y'$ such that $\|y'_1\| = 1$ and $\|A'(y'_1)\| < 1/4$. From $\|y'_1\| = 1$ it follows that there exists $y_0 \in Y$ such that $\|y_0\| = 1$ and $1/2 < |y'_1(y_0)|$. For $y_1 = y_0/y'_1(y_0)$ we have $y'_1(y_1) = 1$ and $\|y_1\| < 2$. Suppose that there exist $y_1, \dots, y_{n-1}$ and $y'_1, \dots, y'_{n-1}$ such that (2.27) holds. From Corollary 2.3 we get

$$\operatorname{codim}(\operatorname{lin}\{y_1, \dots, y_{n-1}\})^\perp = \dim(\operatorname{lin}\{y_1, \dots, y_{n-1}\}) < \infty.$$

By Lemma 2.14 it follows that the restriction of A' to $(\operatorname{lin}\{y_1, \dots, y_{n-1}\})^\perp$ is not bounded below. Because of that there exists $y'_n \in (\operatorname{lin}\{y_1, \dots, y_{n-1}\})^\perp$ such that

$$\|y'_n\| = 1 \ \text{ and } \ \|A'y'_n\| < \frac{1}{2^n a_n}.$$

There exists some $y \in Y$ such that $y'_n(y) = 1$ and $\|y\| < 2$. Let

$$y_n = y - \sum_{k=1}^{n-1} y'_k(y)y_k.$$

Then

$$\|y_n\| \leqslant \|y\|\left(1 + \sum_{k=1}^{n-1} \|y_k\|\right) \leqslant 2\left(1 + \sum_{k=1}^{n-1} a_k\right) = a_n, \quad y'_n(y_n) = 1,$$

$$y'_n(y_k) = 0, \quad y'_k(y_n) = y'_k(y) - y'_k(y) = 0, \quad k = 1, \dots, n-1.$$

By the induction, there exist sequences $(y_k)_k$ in Y, and $(y'_k)_k$ in Y' such that (2.27) holds. □

Theorem 2.13. *Let $A \in L(X, Y)$. Then $A \in \Phi_+(X, Y)$ if and only if $\alpha(A - K) < \infty$ for every $K \in K(X, Y)$.*

Proof. If $A \in \Phi_+(X,Y)$ and $K \in K(X,Y)$, from Theorem 2.4 it follows that $A-K \in \Phi_+(X,Y)$, and consequently $\alpha(A-K) < \infty$.

If $A \notin \Phi_+(X,Y)$, then from Lemma 2.15 there exist sequences $(x_k)_k$ and $(x'_k)_k$ such that (2.26) holds. Define the sequence of operators $K_n \in L(X,Y)$, $n = 1,2,\dots$, as $K_n(x) = \sum_{k=1}^n x'_k(x)A(x_k)$, $x \in X$. For $n > m$ we have

$$\|(K_n - K_m)x\| \leqslant \sum_{k=m+1}^{n} 2^{k-1}2^{1-2k}\|x\|, \quad x \in X,$$

i.e. $\|K_n - K_m\| \to 0$ as $m,\ n \to \infty$. There exists $K \in K(X,Y)$ such that $K_n \to K$ and the operator K is defined as $K(x) = \sum_{k=1}^\infty x'_k(x)A(x_k)$, $x \in X$. Obviously, $K(x_k) = A(x_k)$, $k = 1,2,\dots$. Since x_k, $k = 1,2,\dots$, are linearly independent vectors, it follows that $\alpha(A-K) = \infty$. □

Theorem 2.14. *Let $A \in L(X,Y)$. The following conditions are equivalent:*

(i) $A \in \Phi_-(X,Y)$*;*
(ii) $\dim(Y/\mathcal{R}(A-K)) < \infty$ *for every* $K \in K(X,Y)$*;*
(iii) $\dim(Y/\overline{\mathcal{R}(A-K)}) < \infty$ *for every* $K \in K(X,Y)$*.*

Proof. (i) $\Longrightarrow$ (ii): Let $A \in \Phi_-(X,Y)$ and $K \in K(X,Y)$. From Theorem 2.4 it follows that $A-K \in \Phi_-(X,Y)$, i.e. $\dim(Y/\mathcal{R}(A-K)) < \infty$.

(ii) $\Longrightarrow$ (iii): If $\dim(Y/\overline{\mathcal{R}(A-K)}) = \infty$ for some $K \in K(X,Y)$, then, because of the inclusion $\mathcal{R}(A-K) \subset \overline{\mathcal{R}(A-K)}$, it follows that $\dim(Y/\mathcal{R}(A-K)) = \operatorname{codim}\mathcal{R}(A-K) \geqslant \operatorname{codim}\overline{\mathcal{R}(A-K)} = \dim(Y/\overline{\mathcal{R}(A-K)}) = \infty$.

(iii) $\Longrightarrow$ (i): If $A \notin \Phi_-(X,Y)$, then from Lemma 2.16 it follows that there exist sequences $(y_k)_k$ in Y and $(y'_k)_k$ in Y', such that (2.27) holds. Define the sequence of operators $K_n \in L(X,Y)$, $n = 1,2,\dots$, as $[K_n(x) = \sum_{k=1}^n A'y'_k(x)y_k$, $x \in X$. Obviously, $K_n \in F(X,Y)$. For $n > m$ we have

$$\|K_n(x) - K_m(x)\| \leqslant \sum_{k=m+1}^{n} \|A'y'_k\|\|x\|\|y_k\| \leqslant (\sum_{k=m+1}^{n} \frac{1}{2^k})\|x\| \leqslant \frac{1}{2^m}\|x\|,$$

i.e. $\|K_n - K_m\| \to 0$ as $m,n \to \infty$. It follows that there exists $K \in K(X,Y)$ such that $K_n \to K$ as $n \to \infty$. It is easy to see that the operator K is defined as

$$K(x) = \sum_{k=1}^{\infty} A'y'_k(x)y_k, \quad x \in X.$$

For every $x \in X$ and every k we have:

$$y'_k(Kx) = A'y'_k(x) = y'_k(Ax).$$

It follows that $y'_k \in \mathcal{R}(A-K)^\perp = \overline{\mathcal{R}(A-K)}^\perp$. Since y'_k, $k = 1,2,\dots$, are linearly independent, we get $\dim\overline{\mathcal{R}(A-K)}^\perp = \infty$. From Corollary 2.3 it follows that $\dim(Y/\overline{\mathcal{R}(A-K)}) = \operatorname{codim}\overline{\mathcal{R}(A-K)} = \dim\overline{\mathcal{R}(A-K)}^\perp = \infty$. □

2.5. Connections with the Calkin algebra. Since the set of all compact operators $K(X)$ is a closed two-sided ideal of a Banach algebra $L(X)$, the Calkin algebra $L(X)/K(X) \equiv C(X)$ is a Banach algebra with the norm

$$\|A + K(X)\| = \inf_{K \in K(X)} \|A + K\| \equiv \|A\|_K, \quad A \in L(X).$$

Let $\pi\colon L(X) \to C(X)$ denote the natural homomorphism.

The Calkin algebra has an important role in the theory of Fredholm operators. As it can be seen from the following result, the set of all Fredholm operators in $L(X)$ is the inverse image of the set of all invertible elements in Calkin algebra. If M and N are closed subspaces of X and $X = M \oplus N$, then we use $P_{M,N}$ to denote the projection from X onto M parallel to N.

Theorem 2.15 (Atkinson). *Let $A \in L(X, Y)$. The following conditions are equivalent:*

(i) $A \in \Phi(X, Y)$;

(ii) *There exist operators $A_1,\ A_2 \in L(Y, X)$, $F_1 \in F(X)$ and $F_2 \in F(Y)$ such that*

$$A_1 A = I + F_1, \quad AA_2 = I + F_2;$$

(iii) *There exist operators $A_1,\ A_2 \in L(Y, X)$, $K_1 \in K(X)$ and $K_2 \in K(Y)$ such that*

$$A_1 A = I + K_1, \quad AA_2 = I + K_2.$$

Proof. (i)$\Longrightarrow$(ii): Let $A \in \Phi(X, Y)$. There exist a closed subspace X_1 in X, and a finite dimensional subspace Y_1 in Y, such that

$$X = \mathcal{N}(A) \oplus X_1, \quad Y = \mathcal{R}(A) \oplus Y_1.$$

The operator $A_0\colon X_1 \to \mathcal{R}(A)$, defined by $A_0 x = Ax$ for $x \in X_1$, has a bounded inverse $A_0^{-1} \in L(\mathcal{R}(A), X_1)$. Consider the projections $P_{\mathcal{R}(A),Y_1} \in L(Y)$ and $P_{\mathcal{N}(A),X_1} \in L(X)$. Let

$$A_1 = A_2 = P_{X_1,\mathcal{N}(A)} A_0^{-1} P_{\mathcal{R}(A),Y_1}.$$

Obviously, $A_1 = A_2 \in L(Y, X)$ holds, and it is easy to see that $A_1 A = P_{X_1,\mathcal{N}(A)} = I - P_{\mathcal{N}(A),X_1}$ and $AA_1 = P_{\mathcal{R}(A),Y_1} = I - P_{Y_1,\mathcal{R}(A)}$. For $F_1 = -P_{\mathcal{N}(A),X_1}$ and $F_2 = -P_{Y_1,\mathcal{R}(A)}$ we get $A_1 A = I + F_1$, $AA_1 = I + F_2$ and $F_1 \in F(X)$, $F_2 \in F(Y)$.

(ii)$\Longrightarrow$(iii): Obvious.

(iii) $\Longrightarrow$(i). Let $A_1 A = I + K_1, \quad AA_2 = I + K_2$ for $A_1,\ A_2 \in L(Y, X)$, $K_1 \in K(X)$ and $K_2 \in K(Y)$. From Theorem 2.4 it follows that $I + K_1 \in \Phi(X)$ and $I + K_2 \in \Phi(Y)$. Thus $A_1 A \in \Phi(X)$ and $AA_2 \in \Phi(Y)$, and according to Theorem 2.3 we obtain $A \in \Phi(X, Y)$. □

The next corollary follows from Atkinson theorem.

Corollary 2.7. *The operator $A \in L(X)$ is Fredholm if and only if $\pi(A)$ is invertible in the Calkin algebra $C(X)$.*

The following corollary shows that $\dim X < \infty$ if and only if $\Phi(A) = \mathbb{C}$ for an operator $A \in L(X)$. In other words:

Corollary 2.8. *Let $A \in L(X)$. Then X is infinite dimensional if and only if $\Phi(A) \neq \mathbb{C}$.*

Proof. Suppose that $\dim X = \infty$. Then $K(X)$ is a proper subspace of $L(X)$ and $C(X)$ is non-trivial. Hence, $\sigma(\pi(A)) \neq \emptyset$. Since $\Phi(A) = \mathbb{C} \setminus \sigma(\pi(A))$, it follows that $\Phi(A) \neq \mathbb{C}$.

To prove the converse implication, suppose that $\dim X < \infty$. Then for every $\lambda \in \mathbb{C}$, $\alpha(A-\lambda) < \infty$ and $\beta(A-\lambda) < \infty$, and hence, $A-\lambda$ is Fredholm. Thus, $\Phi(A) = \mathbb{C}$. □

Theorem 2.16. *Let $A \in L(X,Y)$, and $B \in L(Y,Z)$. Then:*

(i) $BA \in \Phi_+(X,Z)$ *and* $A \in \Phi(X,Y) \implies B \in \Phi_+(Y,Z)$.
(ii) $BA \in \Phi_-(X,Z)$ *and* $B \in \Phi(Y,Z) \implies A \in \Phi_-(X,Y)$.
(iii) $BA \in \Phi(X,Z) \implies (A \in \Phi(X,Y) \Longleftrightarrow B \in \Phi(Y,Z))$.

Proof. (i): Suppose that $A \in \Phi(X,Y)$. Then there exist $A_2 \in \Phi(Y,X)$ and $K_2 \in K(Y)$ such that $AA_2 = I + K_2$ (Theorem 2.15). Hence, we get $B = BAA_2 - BK_2$. Since $A_2 \in \Phi(Y,X)$ and $BA \in \Phi_+(X,Z)$, it follows that $BAA_2 \in \Phi_+(Y,Z)$ (Theorem 2.2). From $BK_2 \in K(Y,Z)$ we conclude $B \in \Phi_+(Y,Z)$ by Theorem 2.4.

(ii): Analogously.

(iii): Follows from (i), (ii) and Theorem 2.3. □

Corollary 2.9. *Let $A \in L(X,Y)$ and $B \in L(Y,Z)$. Then*

(i) $[BA \in \Phi(X,Z)$ *and* $\alpha(B) < \infty] \Longrightarrow [A \in \Phi(X,Y)$ *and* $B \in \Phi(Y,Z)]$.
(ii) $[BA \in \Phi(X,Z)$ *and* $\beta(A) < \infty] \Longrightarrow [A \in \Phi(X,Y)$ *and* $B \in \Phi(Y,Z)]$.

Proof. (i): From $\mathcal{R}(B) \supset \mathcal{R}(BA)$ it follows that $\beta(B) \leqslant \beta(BA)$, and consequently $B \in \Phi(Y,Z)$. From Theorem 2.16 (ii) we get $A \in \Phi(X,Y)$.

(ii): From $\mathcal{N}(A) \subset \mathcal{N}(BA)$ it follows that $\alpha(A) \leqslant \alpha(BA)$, so $A \in \Phi(X,Y)$. From Theorem 2.16 (iii) we get $B \in \Phi(Y,Z)$. □

The following results explain the connection between some subsets of the set of semi-Fredholm operators in $L(X)$ and right or left invertible elements in $C(X)$. We start with the following lemma.

Lemma 2.17. *Let $A \in \Phi_+(X,Y)$ and let M be closed subspace of X. Then $A(M)$ is closed in Y.*

Proof. Since $A \in \Phi_+(X,Y)$, it follows that there exists a closed subspace X_1 of X such that $X = \mathcal{N}(A) \oplus X_1$. Then the restriction $A_{|X_1}\colon X_1 \to Y$ is bounded bellow and $M = (M \cap X_1) \oplus M_1$ for some subspace M_1 of M. Since $M_1 \cap X_1 = (M_1 \cap M) \cap X_1 = M_1 \cap (M \cap X_1) = \{0\}$, it follows that $\dim M_1 \leqslant \operatorname{codim} X_1 < \infty$. Hence $A(M) = A(M \cap X_1) + A(M_1)$, $\dim A(M_1) < \infty$, and $A(M \cap X_1)$ is closed since $M \cap X_1$ is closed and $A_{|X_1}$ is bounded bellow. From Lemma 2.1 it follows that $A(M)$ is closed. □

A closed subspace M of X is said to be complemented if there is a closed subspace N of X such that $X = M \oplus N$. According to [40, Teorema 4.3.3] M is a complemented subspace of X if and only if there is a projector $P \in L(X)$ such that $\mathcal{R}(P) = M$.

Theorem 2.17. *Let $A \in L(X,Y)$. The following conditions are equivalent:*

(i) $A \in \Phi_-(X,Y)$ *and* $\mathcal{N}(A)$ *is a complemented subspace of* X*;*
(ii) *There exist* $A_1 \in L(Y,X)$ *and* $F \in F(Y)$ *such that* $AA_1 = I + F$*;*
(iii) *There exist* $A_1 \in L(Y,X)$ *and* $K \in K(Y)$ *such that* $AA_1 = I + K$.

Proof. (i)$\Longrightarrow$(ii): Suppose that $A \in \Phi_-(X,Y)$, and suppose that $\mathcal{N}(A)$ is a complemented subspace of X. Then $X = \mathcal{N}(A) \oplus X_1$, where X_1 is a closed subspace and $P_{X_1,\mathcal{N}(A)} \in L(X)$. From $A \in \Phi_-(X,Y)$ it follows that $Y = \mathcal{R}(A) \oplus Y_1$, where Y_1 is finite dimensional in Y. Now, the operator $A_0\colon X_1 \to \mathcal{R}(A)$, defined by $A_0x = Ax$ for $x \in X_1$, has the bounded inverse. We obtain that $A_1 = P_{X_1,\mathcal{N}(A)}A_0^{-1}P_{\mathcal{R}(A),Y_1} \in L(Y,X)$ and $AA_1 = P_{\mathcal{R}(A),Y_1} = I - P_{Y_1,\mathcal{R}(A)}$, i.e. $AA_1 = I + F$, where $F = -P_{Y_1,\mathcal{R}(A)} \in F(Y)$.

(ii)$\Longrightarrow$(iii): Obvious.

(iii)$\Longrightarrow$(i): Let $AA_1 = I + K$ for some $A_1 \in L(Y,X)$ and $K \in K(Y)$. Since $I + K \in \Phi(Y)$, it follows that $A_1 \in \Phi_+(Y,X)$ and $A \in \Phi_-(X,Y)$ (Theorem 2.3), and there exists a closed subspace V of Y such that $Y = \mathcal{N}(I+K) \oplus V$. From Lemma 2.17 it follows that $A_1(V)$ is closed in X.

Let $x \in \mathcal{N}(A) \cap A_1(V)$. Then $Ax = 0$ and $x = A_1y$ for $y \in V$. It follows that $AA_1y = 0$, so $(I+K)y = 0$. We get $y \in \mathcal{N}(I+K) \cap V$. It follows that $y = 0$ and $x = 0$. We shall prove that $\dim X/(\mathcal{N}(A) \oplus A_1(V)) < \infty$. If $\tilde{x}_1, \dots, \tilde{x}_n$ linearly independent in $X/(\mathcal{N}(A) \oplus A_1(V))$, $x_i \in X$, $\tilde{x}_i = x_i + \mathcal{N}(A) \oplus A_1(V)$, $i = 1, \dots n$, then $\tilde{Ax_1}, \dots, \tilde{Ax_n}$ are linearly independent in $Y/AA_1(V)$, $\tilde{Ax_i} = Ax_i + AA_1(V)$, $i = 1, \dots, n$. Since $AA_1(V) = (I+K)V = \mathcal{R}(I+K)$, we get $\dim X/(\mathcal{N}(A) \oplus A_1(V)) \leqslant \dim Y/\mathcal{R}(I+K) = \beta(I+K) < \infty$. Consequently, there exits a finite dimensional subspace M of X, such that $X = \mathcal{N}(A) \oplus (A_1(V) \oplus M)$. Since $A_1(V) \oplus M$ is closed in X (Lemma 2.1), $\mathcal{N}(A)$ is a complemented subspace of X. □

Theorem 2.18. *Let $A \in L(X,Y)$. The following conditions are equivalent:*

(i) $A \in \Phi_+(X,Y)$ *and* $\mathcal{R}(A)$ *is a complemented subspace of* Y*;*
(ii) *There exist* $A_1 \in L(Y,X)$ *and* $F \in F(X)$ *such that* $A_1A = I + F$*;*
(iii) *There exist* $A_1 \in L(Y,X)$ *and* $K \in K(X)$ *such that* $A_1A = I + K$.

Proof. (i)$\Longrightarrow$(ii): Suppose that $A \in \Phi_+(X,Y)$ and suppose that $\mathcal{R}(A)$ is a complemented subspace of Y. Then there exists a bounded projection P from Y onto $\mathcal{R}(A)$, $\dim \mathcal{N}(A) < \infty$ and there exists a closed subspace X_1 of X such that $X = \mathcal{N}(A) \oplus X_1$. The operator $A_0\colon X_1 \to \mathcal{R}(A)$, defined by $A_0x = Ax$ for $x \in X_1$ has the bounded inverse. For $A_1 = P_{X_1,\mathcal{N}(A)}A_0^{-1}P$ it is easy to see that $A_1A = P_{X_1,\mathcal{N}(A)} = I - P_{\mathcal{N}(A),X_1}$, i.e. there exist $A_1 \in L(Y,X)$ and $F = -P_{\mathcal{N}(A),X_1} \in F(X)$ such that $A_1A = I + F$.

(ii)$\Longrightarrow$(iii): Obvious.

(iii)$\Longrightarrow$(i): Let $A_1A = I + K$ for some $A_1 \in L(Y,X)$ and $K \in K(X)$. Since $I + K \in \Phi(X)$, it follows that $A \in \Phi_+(X,Y)$ and $A_1 \in \Phi_-(Y,X)$ (Theorem 2.3). In the same way as in the proof of Theorem 2.17, we conclude that there exists a closed subspace V of X such that

$$X = \mathcal{N}(I+K) \oplus V. \tag{2.28}$$

Now, $\dim Y/(\mathcal{N}(A_1)\oplus A(V))<\infty$. From (2.28) it follows that $\mathcal{R}(A)=A(\mathcal{N}(I+K))+A(V)$. Hence,

$$\mathcal{R}(A)+\mathcal{N}(A_1)=A(\mathcal{N}(I+K))+A(V)+\mathcal{N}(A_1). \tag{2.29}$$

Since $\operatorname{codim}((\mathcal{N}(A_1)\oplus A(V))<\infty$, we get that $\operatorname{codim}(\mathcal{R}(A)+\mathcal{N}(A_1))<\infty$.

From the fact $\dim(\mathcal{N}(A_1)\cap A(\mathcal{N}(I+K)))<\infty$, and knowing that $\mathcal{N}(A_1)$ is closed, we conclude that there exists a closed subspace Z of Y, such that

$$\mathcal{N}(A_1)=(\mathcal{N}(A_1)\cap A(\mathcal{N}(I+K)))\oplus Z. \tag{2.30}$$

From (2.30) and (2.29) we get $\mathcal{R}(A)+\mathcal{N}(A_1)=\mathcal{R}(A)+Z$. We shall prove that the sum $\mathcal{R}(A)+Z$ is direct. Let $y\in\mathcal{R}(A)\cap Z$ and $y=Ax$ for $x\in X$. Since $Z\subset\mathcal{N}(A_1)$, we conclude that $0=A_1y=A_1Ax=(I+K)x$, i.e. $x\in\mathcal{N}(I+K)$ and therefore, $y\in\mathcal{N}(A_1)\cap A(\mathcal{N}(I+K))$. From $y\in Z$ and (2.30) we obtain $y=0$.

Consequently, $\operatorname{codim}(\mathcal{R}(A)\oplus Z)<\infty$, and there exists a finite dimensional subspace W of Y, such that $Y=\mathcal{R}(A)\oplus Z\oplus W$. Since $\mathcal{R}(A)$ and $Z\oplus W$ are closed, we obtain that $\mathcal{R}(A)$ is a complemented subspace of Y. □

We define the following subset of lower and upper semi-Fredholm operators in $L(X,Y)$:

$$\Phi_r(X,Y)=\{A\in\Phi_-(X,Y):\mathcal{N}(A)\text{ is a complemented subspace of }X\},$$
$$\Phi_l(X,Y)=\{A\in\Phi_+(X,Y):\mathcal{R}(A)\text{ is a complemented subspace of }Y\}.$$

Operators from the set $\Phi_r(X,Y)$ ($\Phi_l(X,Y)$) are called right (left) Fredholm operators or right (left) essentially invertible operators. Obviously, $\Phi(X,Y)=\Phi_r(X,Y)\cap\Phi_l(X,Y)$. Therefore, $A\in L(X,Y)$ is left Fredholm if $\alpha(A)<\infty$ and $\mathcal{R}(A)$ is a complemented subspace of Y, while A is right Fredholm if $\beta(A)<\infty$ and $\mathcal{N}(A)$ is a complemented subspace of X.

Corollary 2.10. *The sets $\Phi_r(X)$ and $\Phi_l(X)$ are open in $L(X)$.*

Proof. Let $C(X)_r^{-1}$ and $C(X)_l^{-1}$, respectively, denote the set of all right, and the set of all left invertible elements in $C(X)$. From Theorems 2.17 and 2.18 we conclude that $\Phi_r(X)=\pi^{-1}(C(X)_r^{-1})$ and $\Phi_l(X)=\pi^{-1}(C(X)_l^{-1})$. Since $C(X)_r^{-1}$ and $C(X)_l^{-1}$ are open in $C(X)$, and π is a continuous map, it follows that $\Phi_r(X)$ and $\Phi_l(X)$ are open sets in $L(X)$. □

An operator $T\in L(X)$ is relatively regular (or g-invertible) if there exists $S\in L(X)$ such that $TST=T$. It is well-known that T is relatively regular if and only if $\mathcal{R}(T)$ and $\mathcal{N}(T)$ are complemented subspaces of X. Therefore, $T\in L(X)$ is left Fredholm if and only if T is a relatively regular upper semi-Fredholm operator, while T is *right Fredholm* if and only if T is a relatively regular lower semi-Fredholm operator.

Corollary 2.11. *Let $A\in L(X,Y)$ and $B\in L(Y,Z)$.*

(i) *If $A\in\Phi_l(X,Y)$ and $B\in\Phi_l(Y,Z)$, then $BA\in\Phi_l(X,Z)$.*
(ii) *If $A\in\Phi_r(X,Y)$ and $B\in\Phi_r(Y,Z)$, then $BA\in\Phi_r(X,Z)$.*

Proof. (i): Let $A \in \Phi_l(X,Y)$ and $B \in \Phi_l(Y,Z)$. By Theorem 2.18 there exist $A_1 \in L(Y,X)$, $K_1 \in K(X)$, $B_1 \in L(Z,Y)$ and $K_2 \in L(Y)$ such that $A_1A = I + K_1$ and $B_1B = I + K_2$. Therefore, $A_1B_1BA = A_1(I + K_2)A = A_1A + A_1K_2A = I + K_1 + A_1K_2A$. Since $K_1 + A_1K_2A \in K(X)$, from Theorem 2.18 it follows that $BA \in \Phi_l(X,Z)$.

(ii): Analogously. □

Corollary 2.12. *Let $A \in L(X,Y)$ and $B \in L(Y,Z)$.*

(i) *If $BA \in \Phi_l(X,Z)$, then $A \in \Phi_l(X,Y)$.*
(ii) *If $BA \in \Phi_r(X,Z)$, then $B \in \Phi_r(Y,Z)$.*
(iii) *If $BA \in \Phi(X,Z)$, then $A \in \Phi_l(X,Y)$ and $B \in \Phi_r(Y,Z)$.*

Proof. (i): Let $BA \in \Phi_l(X,Z)$. By Theorem 2.18 there exist $B_1 \in L(Z,X)$ and $K \in K(X)$ such that $B_1BA = I + K$, which implies that $A \in \Phi_l(X,Y)$.

(ii): Analogously.

(iii): Follows from (i) and (ii). □

Corollary 2.13. (i) *If $A \in \Phi_l(X,Y)$ and $K \in K(X,Y)$, then $A + K \in \Phi_l(X,Y)$.*
(ii) *If $A \in \Phi_r(X,Y)$ and $K \in K(X,Y)$, then $A + K \in \Phi_r(X,Y)$.*

Proof. (i): Let $A \in \Phi_l(X,Y)$. Then there exist $A_1 \in L(Y,X)$ and $K_1 \in K(X)$ such that $A_1A = I + K_1$. Therefore, $A_1(A + K) = I + K_1 + A_1K$ and since $K_1 + A_1K \in K(X)$ we conclude that $A + K \in \Phi_l(X,Y)$.

(ii): Analogously. □

Corollary 2.14. *Let $A \in L(X,Y)$, and $B \in L(Y,Z)$. Then:*

(i) $BA \in \Phi_l(X,Z)$ *and* $A \in \Phi(X,Y) \implies B \in \Phi_l(Y,Z)$.
(ii) $BA \in \Phi_r(X,Z)$ *and* $B \in \Phi(Y,Z) \implies A \in \Phi_r(X,Y)$.

Proof. Follows from Theorem 2.15, Corollary 2.11, Corollary 2.13, analogously to the proof of Theorem 2.16. □

2.6. Semi-Weyl operators. The sets of upper semi-Weyl, lower semi-Weyl and Weyl operators from X to Y are defined as $\mathcal{W}_+(X,Y) = \{T \in \Phi_+(X,Y) : \text{ind}(T) \leqslant 0\}$, $\mathcal{W}_-(X,Y) = \{T \in \Phi_-(X,Y) : \text{ind}(T) \geqslant 0\}$ and $\mathcal{W}(X,Y) = \{T \in \Phi(X,Y) : \text{ind}(T) = 0\}$, respectively. An operator $T \in L(X,Y)$ is *left (right) Weyl* if T is left (right) Fredholm and $\text{ind}(T) \leqslant 0$ ($\text{ind}(T) \geqslant 0$) [47], [48].

Every bounded below operator is also an upper semi-Weyl operator, and every surjective operator is a lower semi-Weyl operator. We start with the following results of B. Yood [46], which show that every upper semi-Weyl operator can be obtained by a compact perturbation of a bounded below operator, and every lower semi-Weyl operator can be obtained by a compact perturbation of a surjective operator.

Lemma 2.18. *Let $A \in \mathcal{W}_+(X,Y)$. Then there exists $F \in F(X,Y)$ such that $\dim \mathcal{R}(F) = \alpha(A)$ and $A + F$ is bounded below.*

Proof. Let $\alpha(A) = n$ and let $\{x_1, x_2, \dots, x_n\}$ be a base in $\mathcal{N}(A)$. There exist $f_1, \dots, f_n \in X'$ such that $f_i(x_j) = \delta_{ij}$, $i, j = 1, \dots, n$. From $\operatorname{ind}(A) \leqslant 0$ we obtain $\beta(A) \geqslant \alpha(A)$, so $\dim \mathcal{N}(A') \geqslant n$. Let $\{g_1, \dots, g_n\}$ be a linearly independent subset of $\mathcal{N}(A')$. Then there exist $y_1, \dots, y_n \in Y$, such that $g_i(y_j) = \delta_{ij}$, $i, j = 1, \dots, n$. Define the operator $F \in L(X, Y)$ as $F(x) = \sum_{i=1}^n f_i(x) y_i$, $x \in X$. Obviously, $\dim \mathcal{R}(F) = n$ and $F \in F(X, Y)$. We prove that $A + F$ is bounded below. From Theorem 2.4 it follows that $\mathcal{R}(A+F)$ is closed. If $x_0 \in X$ and $(A+F)x_0 = 0$, then

$$Ax_0 + \sum_{i=1}^n f_i(x_0) y_i = 0 \tag{2.31}$$

$$\begin{aligned} 0 = g_j\left(Ax_0 + \sum_{i=1}^n f_i(x_0) y_i\right) &= g_j(Ax_0) + \sum_{i=1}^n f_i(x_0) g_j(y_i) \\ &= g_j(Ax_0) + f_j(x_0), \quad j = 1, \dots, n. \end{aligned}$$

Since $g_j(Ax_0) = (A' g_j) x_0 = 0$, we have $f_j(x_0) = 0$, $j = 1, \dots, n$. From (2.31) it follows that $Ax_0 = 0$, i.e. $x_0 \in \mathcal{N}(A)$. Hence, there exist scalars $\lambda_1, \dots, \lambda_n$, such that $x_0 = \lambda_1 x_1 + \dots + \lambda_n x_n$. We get $0 = f_j(x_0) = f_j(\lambda_1 x_1 + \dots + \lambda_n x_n) = \lambda_j$, i.e. $x_0 = 0$. Consequently, $A + F$ is bounded below. □

Lemma 2.19. *Let $A \in \mathcal{W}_-(X, Y)$. Then there exists $F \in F(X, Y)$ such that $\dim \mathcal{R}(F) = \beta(A)$ and $A + F$ is surjective.*

Proof. Let $\beta(A) = n$. Since $\operatorname{ind}(A) \geqslant 0$, there exist linearly independent vectors $x_1, \dots, x_n$ in $\mathcal{N}(A)$. Also, there exist functionals $f_1, \dots, f_n \in X'$, such that $f_i(x_j) = \delta_{ij}$, $i, j \in \{1, \dots, n\}$. Let $\{y_1, \dots, y_n\}$ be a base of an algebraic complement of $\mathcal{R}(A)$ in Y, and $F(x) = \sum_{i=1}^n f_i(x) y_i$. Obviously, $\dim \mathcal{R}(F) = n$ and $F \in F(X, Y)$. We prove that $A + F$ is surjective. Let $y \in Y$. Since $Y = \mathcal{R}(A) \oplus \operatorname{lin}\{y_1, \dots, y_n\}$, there exist $x_0 \in X$ and scalars $\lambda_1, \dots, \lambda_n$ such that $y = Ax_0 + \lambda_1 y_1 + \dots + \lambda_n y_n$. From $X = \cap_{i=1}^n \mathcal{N}(f_i) \oplus \operatorname{lin}\{x_1, \dots, x_n\}$ it follows that there exist $z_1 \in \cap_{i=1}^n \mathcal{N}(f_i)$ and $z_2 \in \operatorname{lin}\{x_1, \dots, x_n\}$, such that $x_0 = z_1 + z_2$. From $x_1, \dots, x_n \in \mathcal{N}(A)$ we obtain $Az_2 = 0$, so $Ax_0 = Az_1$. Since $z_1 \in \cap_{i=1}^n \mathcal{N}(f_i)$, it follows that $F(z_1) = \sum_{i=1}^n f_i(z_1) y_i = 0$. Consequently, $(A+F)z_1 = Ax_0$ and

$$(A+F)(z_1 + \lambda_1 x_1 + \dots + \lambda_n x_n) = Ax_0 + \lambda_1 y_1 + \dots + \lambda_n y_n = y. \qquad \square$$

Theorem 2.19. *Let $A \in L(X, Y)$. The following conditions are equivalent:*

(i) *$A \in \mathcal{W}_+(X, Y)$;*
(ii) *There exist $B \in L(X, Y)$ and $F \in F(X, Y)$, such that B is bounded below, $\dim \mathcal{R}(F) = \alpha(A)$ and $A = B + F$;*
(iii) *There exist $B \in L(X, Y)$ and $C \in K(X, Y)$, such that B is bounded below and $A = B + C$.*

Proof. (i)$\Longrightarrow$(ii). If $A \in \mathcal{W}_+(X, Y)$, from Lemma 2.18 it follows that there exists $F_0 \in F(X, Y)$ such that $\dim \mathcal{R}(F_0) = \alpha(A)$ and $A + F_0$ is bounded below. For $B = A + F_0$ and $F = -F_0$ we have $A = B + F$, $B \in L(X, Y)$ is bounded below, $F \in F(X, Y)$ and $\dim \mathcal{R}(F) = \alpha(A)$.

(ii)$\Longrightarrow$(iii). It is clear.

(iii)$\Longrightarrow$(i). It follows from the inclusion $\mathcal{J}(X,Y) \subset \mathcal{W}_+(X,Y)$ and Corollary 2.6. □

Analogously, with using Lemma 2.19 the following result can be proved.

Theorem 2.20. *Let $A \in L(X,Y)$. The following conditions are equivalent:*

(i) $A \in \mathcal{W}_-(X,Y)$;
(ii) *There exist $B \in L(X,Y)$ and $F \in F(X,Y)$, such that B is surjective, $\dim \mathcal{R}(F) = \beta(A)$ and $A = B + F$;*
(iii) *There exist $B \in L(X,Y)$ and $C \in K(X,Y)$, such that B is surjective and $A = B + C$.*

Theorem 2.21. *Let $A \in L(X,Y)$. The following conditions are equivalent:*

(i) $A \in \mathcal{W}(X,Y)$;
(ii) *There exist $B \in L(X,Y))$ and $F \in F(X,Y)$, such that B is an invertible operator, $\dim \mathcal{R}(F) = \alpha(A)(= \beta(A))$ and $A = B + F$;*
(iii) *There exist $B \in L(X,Y)$ and $C \in K(X,Y)$ such that B is an invertible operator and $A = B + C$.*

Proof. (i)$\Longrightarrow$(ii): If $A \in \mathcal{W}(X,Y)$, from Theorem 2.18 it follows that there exists $F_0 \in F(X,Y)$ such that $\dim \mathcal{R}(F_0) = \alpha(A)$ and $A + F_0$ is bounded below. Thus, $\alpha(A + F_0) = 0$. From Theorem 2.12 we obtain $A + F_0 \in \mathcal{W}(X,Y)$. Consequently, $\beta(A+F_0) = \alpha(A+F_0) = 0$, so $A+F_0$ is invertible. For $B = A+F_0$ and $F = -F_0$ we have $A = B+F$, B is invertible, $F \in F(X,Y)$ and $\dim \mathcal{R}(F) = \dim \mathcal{R}(F_0) = \alpha(A)$.

(ii)$\Longrightarrow$(iii): It is clear.

(iii)$\Longrightarrow$(i): It follows from Theorem 2.12 as every invertible operator is Weyl. □

2.7. Openness of the set of semi-Fredholm operators.

The following theorem provides that the set of Fredholm operators $\Phi(X,Y)$ is an open set in $L(X,Y)$.

Theorem 2.22. *Let $A \in \Phi(X,Y)$. Then there exists some $\epsilon > 0$ such that for every $B \in L(X,Y)$, if $\|B\| < \epsilon$ then $A + B \in \Phi(X,Y)$ and*

$$\operatorname{ind}(A+B) = \operatorname{ind}(A), \quad \alpha(A+B) \leqslant \alpha(A), \quad \beta(A+B) \leqslant \beta(A).$$

Proof. From Theorem 2.15 it follows that there exist $A_1 \in L(Y,X)$, $K_1 \in K(X)$ and $K_2 \in K(Y)$, such that

$$A_1 A = I + K_1, \quad AA_1 = I + K_2. \tag{2.32}$$

Let $\epsilon = \|A_1\|^{-1}$, $B \in L(X,Y)$ and $\|B\| < \epsilon$. From (2.32) it follows that

$$A_1(A+B) = (I + A_1 B) + K_1, \quad (A+B)A_1 = (I + BA_1) + K_2. \tag{2.33}$$

Since $\|A_1 B\| \leqslant \|A_1\|\|B\| < 1$, there exists $(I + A_1 B)^{-1} \in L(X)$. Analogously, there exists $(I + BA_1)^{-1} \in L(Y)$. From (2.33) we get

$$(I + A_1 B)^{-1} A_1 (A+B) = I + (I + A_1 B)^{-1} K_1, \tag{2.34}$$

$$(A+B)A_1(I + BA_1)^{-1} = I + K_2 (I + BA_1)^{-1}. \tag{2.35}$$

Since $(I + A_1B)^{-1}K_1 \in K(X)$ i $K_2(I + BA_1)^{-1} \in K(Y)$, from Theorem 2.15 we get $A + B \in \Phi(X, Y)$. From Theorem 2.2, Theorem 2.11, (2.32) and (2.34) it follows that

$$\operatorname{ind}(A_1) + \operatorname{ind}(A) = 0, \quad \operatorname{ind}(A_1) + \operatorname{ind}(A + B) = 0.$$

Consequently, $\operatorname{ind}(A + B) = \operatorname{ind}(A)$.

From (2.34) it follows that $\mathcal{N}(A + B) \subset \mathcal{N}((I + A_1B)^{-1}A_1(A + B)) = \mathcal{N}(I + (I + A_1B)^{-1}K_1)$, so $\alpha(A + B) \leqslant \alpha(I + (I + A_1B)^{-1}K_1)$. From Theorem 2.11 it follows that $\alpha(I + (I + A_1B)^{-1}K_1) = \beta(I + (I + A_1B)^{-1}K_1)$. Since $X = R(I + (I + A_1B)^{-1}K_1) + R((I + A_1B)^{-1}K_1)$, we get $\beta(I + (I + A_1B)^{-1}K_1) \leqslant \dim R((I + A_1B)^{-1}K_1) = \dim R(K_1)$. From $R(K_1) = \mathcal{N}(A)$ (see the proof of Theorem 2.15), it follows that $\alpha(A + B) \leqslant \alpha(A)$.

From (2.35) it follows that $R(A+B) \supset R(I+K_2(I+BA_1)^{-1})$ and consequently $\beta(A + B) \leqslant \beta(I + K_2(I + BA_1)^{-1}) \leqslant \dim R(K_2(I + BA_1)^{-1}) = \dim R(K_2) = \beta(A)$. □

M. Ó. Searcóid [42], L. E. Labuschagne and J. Swart [29], H. Kroh and P. Volkman [27] gave the proof of the openness of the set of proper semi-Fredholm operators in $L(X, Y)$ which does not depend on the Borsuk antipodal theorem, unlike the proof in [9] or [12].

We need the following auxiliary results [37].

Lemma 2.20. *Let X be n-dimensional. Then there exist $x_1, \dots, x_n \in X$ and $f_1, \dots, f_n \in X'$ such that $\|x_i\| = 1$, $\|f_i\| = 1$ and $f_i(x_j) = \delta_{ij}$, $i, j = 1, \dots, n$.*

Lemma 2.21. *Let M be closed subspace of X, such that $\operatorname{codim} M = m$, $m \in \mathbb{N}$. Then for every $\epsilon > 0$ there exists a projection $P \in F(X)$ such that $\mathcal{N}(P) = M$ and $\|P\| \leqslant m + \epsilon$.*

Proof. Let $Q_M : X \to X/M$ be the natural quotient mapping. Since $\dim X/M = m$, from Lemma 2.20 there exist $\tilde{x}_1, \dots, \tilde{x}_m \in X/M$ and $\tilde{f}_1, \dots, \tilde{f}_m \in (X/M)'$ such that $\|\tilde{x}_i\| = 1$, $\|\tilde{f}_i\| = 1$ and $\tilde{f}_i(\tilde{x}_j) = \delta_{ij}$, i, $j = 1, \dots, m$. Moreover, there exist elements $x_i \in X$ such that $Q_M x_i = \tilde{x}_i$ and $\|x_i\| < 1 + \epsilon/m$. Let $f_i = {Q_M}'\tilde{f}_i$, $i = 1, \dots, m$. Then $P(x) = \sum_{i=1}^m f_i(x)x_i$ is the requested projection. □

Let $\mathcal{J}(X, Y)$ denote the set of all bounded below operators, and $\mathcal{Q}(X, Y)$ denote the set of all surjective operators in $L(X, Y)$.

Lemma 2.22. *The sets $\mathcal{J}(X, Y)$ and $\mathcal{Q}(X, Y)$ are open in $L(X, Y)$.*

Proof. If $A \in \mathcal{J}(X, Y)$, then $m(A) > 0$. Let $B \in L(X, Y)$ such that $\|B\| < m(A)$. Then

$$m(A) = m(A + B - B) \leqslant m(A + B) + \|B\| < m(A + B) + m(A),$$

implying that $m(A + B) > 0$, so $A + B$ is bounded below. Consequently, $\mathcal{J}(X, Y)$ is open in $L(X, Y)$. Analogously, $\mathcal{Q}(X, Y)$ is open. □

Lemma 2.23. *Let $T \in \mathcal{J}(X,Y)$. If there exists a sequence of Fredholm operators $(T_n)_n$ of the same index k such that $T_n \to T$ $(n \to \infty)$, then* $\operatorname{ind}(T) = k$.

Proof. According to Lemma 2.22, there exists some $\epsilon > 0$ such that from $U \in L(X,Y)$ and $\|T - U\| < \epsilon$ it follows that U is bounded below. We can assume that $\|T - T_n\| < \epsilon$ for every n. Then, for every $n \in \mathbb{N}$ we have $\alpha(T_n) = 0$, so $\beta(T_n) = -\operatorname{ind}(T_n) = -k$ and $k < 0$. From Lemma 2.21 there exist projections $Q_n \in F(Y)$ such that $\mathcal{N}(Q_n) = R(T_n)$ and $\|Q_n\| \leqslant -k+\epsilon$ for every $n \in \mathbb{N}$. Let the operator $\tilde{T}_n : X \mapsto R(T_n)$ be defined as the reduction of T_n. Then $\tilde{T}_n$ is bijective and hence, $\tilde{T}_n$ has a bounded inverse defined on $R(T_n)$. Let $S_n = \tilde{T}_n^{-1}(I - Q_n)$. Obviously, $S_n \in L(Y,X)$ and

$$S_nT_n = I \quad \text{and} \quad T_nS_n = I - Q_n. \tag{2.36}$$

From (2.36), using Theorem 2.15, we conclude that $S_n \in \Phi(Y,X)$. Using Theorem 2.2 we obtain $\operatorname{ind}(S_n) = -\operatorname{ind}(T_n) = -k$. For $y \in Y$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}\|S_ny\| &\leqslant (m(T))^{-1}\|TS_ny\| = (m(T))^{-1}\|((T - T_n)S_n + (I - Q_n))y\| \\ &\leqslant (m(T))^{-1}(\|T - T_n\|\|S_n\| + \|I - Q_n\|)\|y\| \\ &\leqslant (m(T))^{-1}(\epsilon\|S_n\| + 1 - k + \epsilon)\|y\|.\end{aligned}$$

Consequently, $\|S_n\| \leqslant (m(T))^{-1}(\epsilon\|S_n\| + 1 - k + \epsilon)$, so

$$\|S_n\| \leqslant \frac{(m(T))^{-1}(1 - k + \epsilon)}{1 - \epsilon(m(T))^{-1}}.$$

We get that the sequence $(S_n)_n$ is uniformly bounded. Because of that we have $\|S_n(T-T_n)\| < 1$ for $n \geqq n_0$. Knowing that $S_nT = I+S_n(T-T_n)$, we conclude that S_nT is invertible in $L(X)$. Consequently, for $S = (S_nT)^{-1}S_n$ we have $S \in \Phi(Y)$ and $ST = I$. According Theorem 2.2 we conclude $\operatorname{ind}(S) = \operatorname{ind}((S_nT)^{-1}) + \operatorname{ind}(S_n) = \operatorname{ind}(S_n) = -k$, and from $ST = I$ we get $\operatorname{ind}(T) = -\operatorname{ind}(S) = k$. ☐

Lemma 2.24. *The sets*

$$\mathcal{J}(X,Y) \setminus \Phi_-(X,Y) = \{T \in L(X,Y) : T \textit{ is bounded below and } \operatorname{ind}(T) = -\infty\},$$
$$\mathcal{Q}(X,Y) \setminus \Phi_+(X,Y) = \{T \in L(X,Y) : T \textit{ is surjective and } \operatorname{ind}(T) = \infty\}$$

are open in $L(X,Y)$.

Proof. Let $T \in L(X,Y)$ be bounded below and $\operatorname{ind}(T) = -\infty$. Then there exists some $\epsilon > 0$ such that if $S \in L(X,Y)$ and $\|S - T\| < \epsilon$, then it follows that S is bounded below.

We prove that there exists some $\delta > 0$, such that if $S \in L(X,Y)$ and $\|S-T\| < \delta$ then it follows that S is bounded below and $\operatorname{ind}(S) = -\infty$. Suppose that the opposite holds, i.e. there exists a sequence $(T_n)_n$ in $L(X,Y)$ such that $T_n \to T$ as $n \to \infty$, $\|T - T_n\| < \epsilon$, and $\operatorname{ind}(T_n) > -\infty$ for every $n \in \mathbb{N}$. It follows that $T_n \in \Phi(X,Y)$ for every $n \in \mathbb{N}$. We prove that all T_n have the same index. For every $m,n \in \mathbb{N}$ let $b = \sup\{a \in [0,1] : \operatorname{ind}(aT_n + (1 - a)T_m) = \operatorname{ind}(T_m)\}$ and $V = bT_n + (1 - b)T_m$. Then V is bounded below (because $\|T - V\| < \epsilon$) and V is the limit of Fredholm operators each having the same index $\operatorname{ind}(T_m)$. From Lemma

2.23 it follows that $\operatorname{ind}(V) = \operatorname{ind}(T_m)$, so $b \in \{a \in [0,1] : \operatorname{ind}(aT_n + (1-a)T_m) = \operatorname{ind}(T_m)\}$. Clearly, $b \in [0,1]$. Suppose that $b < 1$. Since $V = bT_n + (1-b)T_m$ is Fredholm, from Theorem 2.22 it follows that there exists an $\epsilon_1 > 0$ such that for every $T \in L(X,Y)$, the inequality $\|T - V\| < \epsilon_1$ implies $T \in \Phi(X,Y)$ and $\operatorname{ind}(T) = \operatorname{ind}(V) = \operatorname{ind}(T_m)$. Then there exists $t \in [0,1]$ such that $b < t < b + \delta_1$ where $\delta_1 = \frac{\epsilon_1}{\|T_n\| + \|T_m\|}$ ($T_n \neq 0$ and $T_m \neq 0$ since T_n and T_m are Fredholm). For $T = tT_n + (1-t)T_m$ it follows that

$$\begin{aligned}\|T - V\| = \|tT_n + (1-t)T_m - (bT_n + (1-b)T_m)\| &\leqslant |t-b|(\|T_n\| + \|T_m\|)\\ &< \delta_1(\|T_n\| + \|T_m\|) = \epsilon_1.\end{aligned}$$

Hence T is Fredholm with $\operatorname{ind}(T) = \operatorname{ind}(T_m)$ and so, $t \in \{a \in [0,1] : \operatorname{ind}(aT_n + (1-a)T_m) = \operatorname{ind}(T_m)\}$. This contradicts to the fact that b is the supremum of the set $\{a \in [0,1] : \operatorname{ind}(aT_n + (1-a)T_m) = \operatorname{ind}(T_m)\}$. Therefore $b = 1$, i.e. $\operatorname{ind}(T_n) = \operatorname{ind}(T_m)$. Consequently, for every $n \in \mathbb{N}$ we have $\operatorname{ind}(T_n) = k$, where $k \in \mathbb{Z}$, $k < 0$. Now, we apply Lemma 2.23 to the sequence $(T_n)_n$ and T, to obtain $\operatorname{ind}(T) = k \neq -\infty$, which is not possible.

If $T \in L(X,Y)$ is surjective and $\operatorname{ind}(T) = +\infty$, then $T' \in L(Y', X')$ is bounded below (because $m(T') = q(T) > 0$), and from Lemma 2.4 it follows that $\operatorname{ind}(T') = -\operatorname{ind}(T) = -\infty$. The openness of the set $\mathcal{Q}(X,Y) \setminus \Phi_-(X,Y)$ follows from the openness of the set $\mathcal{J}(Y', X') \setminus \Phi_+(Y', X')$. □

The following theorem shows that the set of proper semi-Fredhom operators is open in $L(X,Y)$.

Theorem 2.23. *Let $A \in L(X,Y)$.*

(i) *If $A \in \Phi_+(X,Y)$ and $\beta(A) = \infty$, then there exists some $\epsilon > 0$ such that for $B \in L(X,Y)$ and $\|B\| < \epsilon$ it follows that $A + B \in \Phi_+(X,Y)$, $\alpha(A+B) \leqslant \alpha(A)$ and $\beta(A+B) = \beta(A) = \infty$.*

(ii) *If $A \in \Phi_-(X,Y)$ and $\alpha(A) = \infty$, then there exists some $\epsilon > 0$ such that for $B \in L(X,Y)$ and $\|B\| < \epsilon$ it follows that $A + B \in \Phi_-(X,Y)$, $\beta(A+B) \leqslant \beta(A)$ and $\alpha(A+B) = \alpha(A) = \infty$.*

Proof. (i): Let $A \in \Phi_+(X,Y)$ and $\beta(A) = \infty$. Then $\operatorname{ind}(A) = -\infty$. There exists a closed subspace M of X such that $X = \mathcal{N}(A) \oplus M$. It follows that the restriction $A_{|M} : M \to Y$ is bounded below and $\operatorname{ind}(A_{|M}) = -\infty$. From Lemma 2.24 we get that there exists $\epsilon > 0$ such that for every $B \in L(X,Y)$, if $\|B\| < \epsilon$ then $(A+B)|_M \in L(M,Y)$ is bounded below and $\operatorname{ind}((A+B)_{|M}) = -\infty$. By Lemma 2.14, $A + B \in \Phi_+(X,Y)$. Since $\mathcal{N}(A+B) \cap M = \{0\}$, it follows that $\alpha(A+B) \leqslant \operatorname{codim} M = \alpha(A)$. Now, $\mathcal{N}(A)$ is finite dimensional, $R(A+B) = R((A+B)_{|M}) + (A+B)(\mathcal{N}(A))$ and $\operatorname{codim} R((A+B)_{|M}) = \infty$, so $\beta(A+B) = \infty$.

(ii): Follows dually from (i). □

Theorem 2.24. *If $A \in \Phi_+(X,Y)$ (respectively $\Phi_-(X,Y)$, $Phi(X,Y)$), then there exists some $\epsilon > 0$ such that $B \in L(X,Y)$ and $\|B\| < \epsilon$ implies $A + B \in \Phi_+(X,Y)$ (respectively $\Phi_-(X,Y)$,*

$Phi(X,Y)$). Moreover,

$$\alpha(A+B) \leqslant \alpha(A), \quad \beta(A+B) \leqslant \beta(A), \quad \operatorname{ind}(A+B) = \operatorname{ind}(A).$$

Proof. Follows from Theorem 2.22 and Theorem 2.23. □

Corollary 2.15. *The sets $\Phi_+(X,Y)$, $\Phi_-(X,Y)$, $\Phi_+(X,Y) \setminus \Phi_-(X,Y)$, $\Phi_-(X,Y) \setminus \Phi_+(X,Y)$, $\Phi(X,Y)$, $\Phi_\pm(X,Y)$, $\mathcal{W}(X,Y)$, $\mathcal{W}_+(X,Y)$ and $\mathcal{W}_-(X,Y)$ are open in $L(X,Y)$. The sets $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_+(X) \setminus \Phi_-(X)$, $\Phi_-(X) \setminus \Phi_+(X)$, $\Phi(X)$, $\mathcal{W}(X)$, $\mathcal{W}_+(X)$ and $\mathcal{W}_-(X)$ are open semigroups in $L(X)$.*

For $A \in L(X)$, the sets $\Phi(A)$ and $\Phi_\pm(A)$ are open.

Proof. It follows from Theorems 2.22, 2.23, 2.2 and Corollary 2.2. □

Now we prove that the boundary of the set $\Phi(X,Y)$ (respectively, $\Phi_+(X,Y)$, $\Phi_-(X,Y)$, $\Phi_+(X,Y) \setminus \Phi_-(X,Y)$, $\Phi_-(X,Y) \setminus \Phi_+(X,Y)$, $\mathcal{W}_+(X,Y)$, $\mathcal{W}_-(X,Y)$, $\mathcal{W}(X,Y)$) is contained in the complement of the set of all semi-Fredholm operators.

Corollary 2.16. *If $A \in L(X,Y)$ belongs to the boundary of the set $\Phi(X,Y)$ (respectively $\Phi_+(X,Y)$, $\Phi_-(X,Y)$, $\Phi_+(X,Y) \setminus \Phi_-(X,Y)$, $\Phi_-(X,Y) \setminus \Phi_+(X,Y)$, $\mathcal{W}_+(X,Y)$, $\mathcal{W}_-(X,Y)$, $\mathcal{W}(X,Y)$), then A is not semi-Fredholm.*

Proof. Let $A \in \partial\Phi(X,Y)$. Since $\Phi(X,Y)$ is open, it follows that $A \notin \Phi(X,Y)$. If A is semi-Fredholm, then A belongs to the set $(\Phi_+(X,Y) \cup \Phi_-(X,Y)) \setminus \Phi(X,Y) = (\Phi_+(X,Y) \setminus \Phi_-(X,Y)) \cup (\Phi_-(X,Y) \setminus \Phi_+(X,Y))$ which is open by Theorem 2.23, and therefore, there exists a neighborhood $D(A,\epsilon)$ of A such that $D(A,\epsilon) \subset (\Phi_+(X,Y) \cup \Phi_-(X,Y)) \setminus \Phi(X,Y)$. Hence, $D(A,\epsilon) \cap \Phi(X,Y) = \emptyset$, which is impossible because of $A \in \partial\Phi(X,Y)$. The other cases can be proved similarly. □

The following lemma shows that if two semi-Fredholm operators are connected by a continuous curve contained in $\Phi_\pm(X,Y)$, then these semi-Fredholm operators have the same index.

Lemma 2.25. *Let $A,\ B \in \Phi_\pm(X,Y)$, $f\colon [0,1] \mapsto L(X,Y)$ be continuous, $f(0) = A$, $f(1) = B$ and $f([0,1]) \subset \Phi_\pm(X,Y)$. Then* $\operatorname{ind}(A) = \operatorname{ind}(B)$.

Proof. Let $S = \{\lambda \in [0,1] : \operatorname{ind}(f(\lambda)) = \operatorname{ind}(A)\}$. Clearly, $0 \in S$ and let $a = \sup S$. Then $a \in [0,1]$. Since $f(a) \in \Phi_\pm(X,Y)$, from Theorem 2.24 and continuity of f it follows that there exists $\epsilon = \epsilon(a) > 0$ such that $\operatorname{ind}(f(t)) = \operatorname{ind}(f(a))$ for each $t \in [0,1] \cap (a-\epsilon, a+\epsilon)$. Then there exists $\lambda \in S$ such that $a - \epsilon < \lambda \leqslant a$. Hence $\operatorname{ind}(f(t)) = \operatorname{ind}(A)$ for all $t \in [0,1] \cap (a-\epsilon, a+\epsilon)$, and so,

$$[0,1] \cap (a-\epsilon, a+\epsilon) \subset S. \tag{2.37}$$

Thus $a \in S$. Suppose that $a < 1$. From (2.37) it follows that there exists $\lambda \in S$ such that $\lambda > a$ which contradicts the fact that $a = \sup S$. Hence, $a = 1$ and so, $1 \in S$. Therefore, $\operatorname{ind}(f(1)) = \operatorname{ind}(A)$, i.e. $\operatorname{ind}(B) = \operatorname{ind}(A)$. □

The following assertion shows that if two points $\lambda_0, \lambda_1 \in \Phi_\pm(A)$ are connected by a continuous curve contained in $\Phi_\pm(A)$, then the operators $A - \lambda_0$ and $A - \lambda_1$ have the same index.

Lemma 2.26. *Let $A \in L(X)$, $\lambda_0, \lambda_1 \in \Phi_\pm(A)$, let $g\colon [0,1] \to \mathbb{C}$ be continuous, $g(0) = \lambda_0$, $g(1) = \lambda_1$ and $g([0,1]) \subset \Phi_\pm(A)$. Then* $\operatorname{ind}(A - \lambda_0) = \operatorname{ind}(A - \lambda_1)$.

Proof. The mapping $h\colon \mathbb{C} \to L(X)$, defined by $h(\lambda) = A - \lambda$, is continuous. Let $f = h \circ g$. Consequently, $f\colon [0,1] \to L(X)$ is continuous, $f(0) = h(g(0)) = h(\lambda_0) = A - \lambda_0$, $f(1) = h(g(1)) = h(\lambda_1) = A - \lambda_1$ and $f([0,1]) = h(g([0,1])) \subset h(\Phi_\pm(A)) \subset \Phi_\pm(X)$. From Lemma 2.25 it follows that $\operatorname{ind}(A - \lambda_0) = \operatorname{ind}(A - \lambda_1)$. □

Corollary 2.17. *The function $\lambda \mapsto \operatorname{ind}(A - \lambda)$ is constant on every connected component of the semi-Fredholm domain of $A \in L(X)$.*

Proof. The set $\Phi_\pm(A)$ is open (Theorem 2.24) and hence, its connected components, as open sets, are path connected. Now the proof follows from Lemma 2.26. □

2.8. Perturbation classes. The concept of perturbation classes was introduced by A. Lebow and M. Schecter in [31].

Suppose $\mathcal{A}$ is a complex Banach algebra with identity 1. Let $\mathcal{A}^{-1}$ denote the group of invertible elements of $\mathcal{A}$ and let $\mathcal{A}_l^{-1}$ ($\mathcal{A}_r^{-1}$) denote the semigroup of left (right) invertible elements of $\mathcal{A}$.

Let S be a subset of $\mathcal{A}$. The *perturbation class* of S, denoted by $\mathcal{P}(S)$, is the set

$$\mathcal{P}(S) = \{a \in \mathcal{A} : a + s \in S \quad \text{for every } s \in S\}$$

We assume that S satisfies the additional condition

$$\lambda S \subset S \quad \text{for every scalar } \lambda \neq 0. \tag{2.38}$$

Lemma 2.27. *Let S satisfied* (2.38). *Then $\mathcal{P}(S)$ is a subspace of $\mathcal{A}$. Moreover, if S is open in $\mathcal{A}$, then $\mathcal{P}(S)$ is closed in $\mathcal{A}$.*

Proof. Let $a,\ b \in \mathcal{P}(S)$, $s \in S$ and $\alpha \in \mathbb{C}$, $\alpha \neq 0$. Then

$$\alpha a + s = \alpha\Big(a + \frac{s}{\alpha}\Big) \in S \quad \text{and} \quad (a + b) + s = a + (b + s) \in S.$$

Consequently, $\mathcal{P}(S)$ is a subspace of $\mathcal{A}$.

Assume that S is open. Then for every $s \in S$ there exists $\delta > 0$ such that $\|c - s\| < \delta$ implies $c \in S$. Let $(x_n)_n$ be a sequence in $\mathcal{P}(S)$ converging to $x \in \mathcal{A}$. Then $\|x_n - x\| < \delta$ starting from $n \geqq n_0$. Hence $s + x - x_n \in S$, and from $x_n \in \mathcal{P}(S)$ we obtain $s + x \in S$. Consequently, $x \in \mathcal{P}(S)$. □

Lemma 2.28. *Let S_1 and S_2 be subsets of $\mathcal{A}$ satisfying the condition* (2.38). *Assume that $S_1 \subset S_2$ and that S_2 does not contain any boundary point of S_1. Then $\mathcal{P}(S_2) \subset \mathcal{P}(S_1)$.*

Proof. From the assumptions that $S_1 \subset S_2$ and that S_2 does not contain any boundary point of S_1 it follows that S_1 does not contain any of its boundary points and therefore, S_1 is an open set.

Let $s_1 \in S_1$ and $a_2 \in \mathcal{P}(S_2)$. Then

$$\alpha a_2 + s_1 = \alpha\Big(a_2 + \frac{s_1}{\alpha}\Big) \in S_2 \quad \text{for every } \alpha \in \mathbb{C},\ \alpha \neq 0.$$

Since S_1 is open, we have $\alpha a_2 + s_1 \in S_1$ for small values of $|\alpha|$. It follows that $\alpha a_2 + s_1 \in S_1$ for every α, because otherwise there would exist some α_0 such that

$\alpha_0 a_2 + s_1$ is a boundary point of S_1 which is also contained in S_2. Consequently, $a_2 + s_1 \in S_1$. □

Lemma 2.29. *If $\mathcal{A}^{-1}S \subset S$, then $\mathcal{P}(S)$ is a left ideal of $\mathcal{A}$. If $S\mathcal{A}^{-1} \subset S$, then $\mathcal{P}(S)$ is a right ideal of $\mathcal{A}$.*

Proof. Let $a \in \mathcal{A}^{-1}$, $b \in \mathcal{P}(S)$ and $s \in S$. Then $ab + s = a(b + a^{-1}s) \in S$. It follows that $ab \in \mathcal{P}(S)$. Since every element $a \in \mathcal{A}$ is a sum of two elements in $\mathcal{A}^{-1}$ ($a = (a - \lambda) + \lambda$ where $|\lambda| > \|a\|$), from Lemma 2.27 it follows that $\mathcal{P}(S)$ is a left ideal. □

Theorem 2.25. *If S is open and S satisfies $\mathcal{A}^{-1}S \subset S$ and $S\mathcal{A}^{-1} \subset S$, then $\mathcal{P}(S)$ is closed two-sided ideal.*

Proof. It follows from Lemma 2.27 and Lemma 2.29. □

The *radical* of $\mathcal{A}$ is the set

$$\text{Rad}(\mathcal{A}) = \{x \in \mathcal{A} : 1 + \mathcal{A}x \subseteq \mathcal{A}^{-1}\} = \{x \in \mathcal{A} : 1 + x\mathcal{A} \subseteq \mathcal{A}^{-1}\}. \tag{2.39}$$

The equality between the sets in (2.39) follows from the Jacobson lemma: If $x, y \in \mathcal{A}$ are arbitrary, then $1 + xy \in \mathcal{A}^{-1} \iff 1 + yx \in \mathcal{A}^{-1}$.

Lemma 2.30.

$$\begin{aligned} \text{Rad}(\mathcal{A}) &= \{x \in \mathcal{A} : r(ax) = 0 \textit{ for every } a \in \mathcal{A}\} \\ &= \{x \in \mathcal{A} : r(xa) = 0 \textit{ for every } a \in \mathcal{A}\}. \end{aligned} \tag{2.40}$$

Proof. To prove the first equality in (2.40) suppose that $x \in \text{Rad}(\mathcal{A})$ and $a \in \mathcal{A}$. Then $1 + ax \in \mathcal{A}^{-1}$ for every $a \in \mathcal{A}$. For $\lambda \in \mathbb{C}$, $\lambda \neq 0$, we have $1 - \lambda^{-1}ax \in \mathcal{A}^{-1}$. Consequently, $r(ax) = 0$.

On the other hand, let $x \in \mathcal{A}$ satisfies that $r(ax) = 0$. Hence, $-1 \notin \sigma(ax)$, i.e. $1 + ax \in \mathcal{A}^{-1}$.

The second equality in (2.40) follows from the fact that $r(ax) = r(xa)$ for every $a, x \in \mathcal{A}$. □

Lemma 2.31.

$$\text{Rad}(\mathcal{A}) = \{x \in \mathcal{A} : 1 + \mathcal{A}^{-1}x \subset \mathcal{A}^{-1}\} = \{x \in \mathcal{A} : 1 + x\mathcal{A}^{-1} \subset \mathcal{A}^{-1}\}.$$

Proof. The inclusion "$\subset$" is clear. To prove the reverse inclusion suppose that $z \in \mathcal{A}$ such that $1 + gz \in \mathcal{A}^{-1}$ for every $g \in \mathcal{A}^{-1}$, and let $a \in \mathcal{A}$. There exist $g_1, g_2 \in \mathcal{A}^{-1}$ such that $a = g_1 + g_2$. Let $c = 1 + g_1 z$. Since $c \in \mathcal{A}^{-1}$, we get

$$1 + az = 1 + g_1 z + g_2 z = c + g_2 z = c(1 + c^{-1} g_2 z) \in \mathcal{A}^{-1}.$$

Consequently, $z \in \text{Rad}(\mathcal{A})$. □

Theorem 2.26.

$$\text{Rad}(A) = \text{Ptrb}(\mathcal{A}^{-1}) = \text{Ptrb}(\mathcal{A}_l^{-1}) = \text{Ptrb}(\mathcal{A}_r^{-1}). \tag{2.41}$$

Proof. In order to prove the first equality in (2.41) suppose that $x \in \mathcal{P}(\mathcal{A}^{-1})$ and $a \in \mathcal{A}^{-1}$. Then $a^{-1} + x \in \mathcal{A}^{-1}$. Consequently, we get $1 + ax = a(a^{-1} + x) \in \mathcal{A}^{-1}$, and $x \in \operatorname{Rad}(\mathcal{A})$. On the other hand, let $x \in \operatorname{Rad}(\mathcal{A})$ and $a \in \mathcal{A}^{-1}$. Then $1 + a^{-1}x \in \mathcal{A}^{-1}$ and $a + x = a(1 + a^{-1}x) \in \mathcal{A}^{-1}$. It follows that $x \in \mathcal{P}(\mathcal{A}^{-1})$.

Boundary points of the set $\mathcal{A}^{-1}$ are two-sided divisors of zero, and hence they are neither left invertible nor right invertible. From Lemma 2.28 it follows that $\mathcal{P}(\mathcal{A}_l^{-1}) \subset \mathcal{P}(\mathcal{A}^{-1})$ and $\mathcal{P}(\mathcal{A}_r^{-1}) \subset \mathcal{P}(\mathcal{A}^{-1})$. We prove that $\operatorname{Rad}(\mathcal{A}) \subset \mathcal{P}(\mathcal{A}_l^{-1})$. Let $x \in \operatorname{Rad}(\mathcal{A})$ and let $a \in \mathcal{A}_l^{-1}$. Then $b(a + x) = 1 + bx \in \mathcal{A}^{-1}$, where $ba = 1$. We get $(1 + bx)^{-1}b(a + x) = 1$, and consequently $a + x \in \mathcal{A}_l^{-1}$, i.e. $x \in \mathcal{P}(\mathcal{A}_l^{-1})$. Analogously we prove $\operatorname{Rad}(\mathcal{A}) \subset \mathcal{P}(\mathcal{A}_r^{-1})$. □

Corollary 2.18. *Rad($\mathcal{A}$) is closed two-sided ideal of $\mathcal{A}$.*

Proof. It follows from Theorem 2.26 and Theorem 2.25. □

The quasinilpotents of $\mathcal{A}$ form the set $\operatorname{QN}(\mathcal{A}) = \{d \in \mathcal{A}\colon \|d^n\|^{1/n} \to 0\}$.

Lemma 2.32. *If $\mathcal{A}$ is a Banach algebra, then the radical is the largest right, and the largest left ideal of $\mathcal{A}$ contained in the quasinilpotents.*

Proof. Suppose that I is a right ideal of $\mathcal{A}$ and $I \subset \operatorname{QN}(\mathcal{A})$. From Lemma 2.30 it follows that $r(x) = 0$ for every $x \in \operatorname{Rad}(\mathcal{A})$. Thus $\operatorname{Rad}(\mathcal{A}) \subset \operatorname{QN}(\mathcal{A})$. Let $x \in I$ and $a \in \mathcal{A}$. Since I is a right ideal of $\mathcal{A}$, we have $xa \in I$ and since $I \subset \operatorname{QN}(\mathcal{A})$, we get $r(xa) = 0$. It follows that $x \in \operatorname{Rad}(\mathcal{A})$. □

The set of *inessential* operators, denoted by $I(X)$, is the set of all operators $A \in L(X)$ satisfying $\pi(A) \in \operatorname{Rad}(C(X))$, i.e., $I(X) = \pi^{-1}(\operatorname{Rad}(C(X)))$.

Lemma 2.33. *$I(X)$ is a closed two-sided ideal of $L(X)$, and*

$$\begin{aligned} I(X) &= \{E \in L(X) : I + EA \in \Phi(X) \textit{ for every } A \in L(X)\} \\ &= \{E \in L(X) : I + AE \in \Phi(X) \textit{ for every } A \in L(X)\} \\ &= \{E \in L(X) : I + EA \in \Phi(X) \textit{ for every } A \in \Phi(X)\} \\ &= \{E \in L(X) : I + AE \in \Phi(X) \textit{ for every } A \in \Phi(X)\}. \end{aligned}$$

Proof. It follows from Corollary 2.18, Lemma 2.30, Corollary 2.31 and Corollary 2.7. □

Lemma 2.34. *$I(X) = \mathcal{P}(\Phi(X)) = \mathcal{P}(\Phi_l(X)) = \mathcal{P}(\Phi_r(X))$.*

Proof. From (2.41) and Corollary 2.7 it follows that

$$\begin{aligned} I(X) &= \pi^{-1}(\operatorname{Rad}(C(X))) = \pi^{-1}(\mathcal{P}(C(X)^{-1})) \\ &= \mathcal{P}(\pi^{-1}(C(X)^{-1})) = \mathcal{P}(\Phi(X)). \end{aligned}$$

Since $\Phi_l(X) = \pi^{-1}(C(X)_l^{-1})$ and $\Phi_r(X) = \pi^{-1}(C(X)_r^{-1})$ (Corollary 2.10), we prove analogously that $I(X) = \mathcal{P}(\Phi_l(X)) = \mathcal{P}(\Phi_r(X))$. □

Corollary 2.19.

$$\mathcal{P}(\Phi_\pm(X)) = \mathcal{P}(\Phi_+(X)) \cap \mathcal{P}(\Phi_-(X)),$$

$$K(X) \subset \begin{matrix} \mathcal{P}(\Phi_+(X)) \\ \mathcal{P}(\Phi_-(X)) \end{matrix} \subset \mathcal{P}(\Phi(X)).$$

Proof. Obviously, $\mathcal{P}(\Phi_+(X)) \cap \mathcal{P}(\Phi_-(X)) \subset \mathcal{P}(\Phi_\pm(X))$. Since the set $\Phi_+(X)$ is open, $\Phi_+(X) \subset \Phi_\pm(X)$ and $\Phi_\pm(X)$ does not contain boundary points of $\Phi_+(X)$ (Corollary 2.16), from Lemma 2.28 it follows that $\mathcal{P}(\Phi_\pm(X)) \subset \mathcal{P}(\Phi_+(X))$. The inclusions

$$\mathcal{P}(\Phi_\pm(X)) \subset \mathcal{P}(\Phi_-(X)), \qquad \mathcal{P}(\Phi_+(X)) \subset \mathcal{P}(\Phi(X)),$$
$$\mathcal{P}(\Phi_-(X)) \subset \mathcal{P}(\Phi(X))$$

can be proved analogously.

The inclusions $K(X)) \subset \mathcal{P}(\Phi_+(X))$ and $K(X) \subset \mathcal{P}(\Phi_-(X))$ follow from Theorem 2.4. □

We mention that there exists a Banach space X such that $K(X) \neq \mathcal{P}(\Phi_-(X))$, $K(X') \neq \mathcal{P}(\Phi_+(X'))$ as well as $\mathcal{P}(\Phi_+(X)) \neq \mathcal{P}(\Phi(X))$, $\mathcal{P}(\Phi_-(X')) \neq \mathcal{P}(\Phi(X'))$ and $\mathcal{P}(\Phi_+(X')) \neq \mathcal{P}(\Phi_-(X'))$ [9, Example 5.6.10].

Lemma 2.35. *Let $A \in \Phi_+(X)$ (respectively, $\Phi_-(X)$, $\Phi(X)$) and $B \in \mathcal{P}(\Phi_+(X))$ (respectively, $\mathcal{P}(\Phi_-(X))$, $\mathcal{P}(\Phi(X))$). Then* $\operatorname{ind}(A+B) = \operatorname{ind}(A)$.

Proof. From Lemma 2.27 it follows that $\lambda B \in \mathcal{P}(\Phi_+(X))$ for every $\lambda \in [0,1]$. Consequently, $A + \lambda B \in \Phi_+(X)$. From Lemma 2.25 it follows that $\operatorname{ind}(A+B) = \operatorname{ind}(A)$. □

Theorem 2.27. $\mathcal{P}(\mathcal{W}(X)) = \mathcal{P}(\Phi(X))$.

Proof. From Lemma 2.35 (or also from Lemma 2.28 and Corollary 2.16) it follows that $\mathcal{P}(\Phi(X)) \subset \mathcal{P}(\mathcal{W}(X))$.

In order to prove the reverse inclusion, it is enough to prove that $I + BE \in \Phi(X)$ for every $B \in L(X)$ and $E \in \mathcal{P}(\mathcal{W}(X))$ (see Lemma 2.34 and Lemma 2.33). For given B there exist invertible operators A_1 and A_2 such that $B = A_1 + A_2$ ($A_1 = \lambda + B$ for big enough $|\lambda|$ and $A_2 = -\lambda I$). It follows that $A_i,\ A_i^{-1} \in \Phi(X)$ and $\operatorname{ind}(A_i) = 0$, $\operatorname{ind}(A_i^{-1}) = 0$, $i = 1, 2$.

Since

$$A_2^{-1}(I + BE) = A_2^{-1}(I + A_1E) + E = A_2^{-1}A_1(A_1^{-1} + E) + E,$$

and $E \in \mathcal{P}(\mathcal{W}(X))$, according to Theorem 2.2, we get $A_2^{-1}(I + BE) \in \mathcal{W}(X)$. As $A_2^{-1} \in \Phi(X)$, from Theorem 2.16 (iii) we conclude $I + BE \in \Phi(X)$. □

2.9. Kato operators.

For a linear operator A acting on X we set

$$k(A) = \dim(\mathcal{N}(A)/(\mathcal{N}(A) \cap \mathcal{R}^\infty(A)).$$

Lemma 2.36. *For a linear operator A acting on X the following conditions are equivalent:*

(i) $\mathcal{N}(A) \subset \mathcal{R}(A^m)$ *for every* $m \in \mathbb{N}$.
(ii) $\mathcal{N}(A^n) \subset \mathcal{R}(A^m)$ *for every* $n \in \mathbb{N}$ *and every* $m \in \mathbb{N}$.
(iii) $\mathcal{N}(A^n) \subset \mathcal{R}(A)$ *for every* $n \in \mathbb{N}$.
(iv) $\mathcal{N}(A) \subset \mathcal{R}^\infty(A)$.

(v) $\mathcal{N}^\infty(A) \subset \mathcal{R}(A)$.
(vi) $\mathcal{N}^\infty(A) \subset \mathcal{R}^\infty(A)$.
(vii) $k(A) = 0$.

Proof. (i) $\Longrightarrow$ (ii): We use the induction on n. If $n = 1$ then the statement obviously holds. Suppose that the statements holds for n, and we prove it for $n+1$. Let $x \in \mathcal{N}(A^{n+1})$. From $A^n(Ax) = 0$ it follows that $Ax \in \mathcal{N}(A^n)$. According to the inductive assumption, we have $Ax = A^{m+1}x_1$ for some $x_1 \in X$. Hence, $x - A^m(x_1) \in \mathcal{N}(A)$, and from (i) it follows that $x - A^m x_1 \in \mathcal{R}(A^m)$. Thus, $x \in \mathcal{R}(A^m)$.

(ii)$\Longrightarrow$(i): Obvious.

(iii) $\Longrightarrow$ (ii): We use the induction on m. If $m = 1$ then the statement holds. Suppose that the statement holds for m and we prove it for $m+1$. Let $x \in \mathcal{N}(A^n)$. By the inductive assumption we have $x \in \mathcal{R}(A^m)$, and there exists some $x_1 \in X$ such that $x = A^m x_1$. From $0 = A^n x = A^{n+m}x_1$ we get $x_1 \in \mathcal{N}(A^{n+m})$, and from (iii) we obtain $x_1 \in \mathcal{R}(A)$. Consequently, $x \in \mathcal{R}(A^{m+1})$.

(ii)$\Longrightarrow$(iii): Obvious.

The equivalences (i)$\Longleftrightarrow$(iv)$\Longleftrightarrow$(vii), (iii)$\Longleftrightarrow$(v), (ii)$\Longleftrightarrow$(vi) are obvious. $\square$

Definition 2.3. An operator $T \in L(X)$ is *Kato* if $\mathcal{R}(T)$ is closed and T satisfies one of the equivalent conditions of Lemma 2.36.

Lemma 2.37. *Let $A \in L(X)$. If $R(A^n)$ is closed for every $n \in \mathbb{N}$, then*

$$k(A) = k(A'). \tag{2.42}$$

Proof. According to [14, Theorem 3.7] we have that

$$k(A) = \dim(\mathcal{R}(A) + \mathcal{N}^\infty(A))/\mathcal{R}(A). \tag{2.43}$$

Now according to (2.1), (2.2), (2.3) it follows that

$$\begin{aligned} k(A') &= \dim \mathcal{N}(A')/(\mathcal{N}(A') \cap \mathcal{R}^\infty(A')) = \dim \mathcal{R}(A)^\perp/(\mathcal{R}(A)^\perp \cap \cap_n \mathcal{N}(A^n)^\perp) \\ &= \dim \mathcal{R}(A)^\perp/(\mathcal{R}(A) \cup \cup_n \mathcal{N}(A^n))^\perp = \dim \mathcal{R}(A)^\perp/(\mathcal{R}(A) + \mathcal{N}^\infty(A))^\perp \\ &= \dim(\mathcal{R}(A) + \mathcal{N}^\infty(A))/\mathcal{R}(A), \end{aligned}$$

which together with (2.43) gives (2.42). $\square$

Corollary 2.20. *Let $A \in L(X)$. Then A is Kato if and only if A' is Kato.*

Proof. It follows from Lemma 2.37 and the fact that $\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A')$ is closed. $\square$

Corollary 2.21. *Let $A \in L(X)$. Then:*

(i) *A is Kato and* $\operatorname{asc}(A) < \infty$ *if and only if A is bounded below.*
(ii) *A is Kato and* $\operatorname{dsc}(A) < \infty$ *if and only if A is surjective.*

Proof. (i) Suppose that A is Kato and $\operatorname{asc}(A) < \infty$. Then $k(A) = 0$ and $R(A)$ is closed. From Lemma 2.9 it follows that $\mathcal{N}(A) \cap \mathcal{R}^\infty(A) = \{0\}$, which implies $k(A) = \alpha(A)$. Thus $\alpha(A) = 0$, and we conclude that A is bounded below.

The reverse implication is obvious.

The assertion (ii) can be proved by duality. $\square$

2.10. The punctured neighbourhood theorem. The punctured neighbourhood theorem shows that the function $\lambda \mapsto \alpha(A-\lambda)$ (respectively, $\lambda \mapsto \beta(A-\lambda)$) has the constant value in a punctured neighborhood of $\lambda_0 \in \Phi_\pm(A)$, and this value differs from $\alpha(A-\lambda_0)$ (respectively, $\beta(A-\lambda_0)$) for the number $k(A-\lambda_0)$ (see also [9, Theorem 3.2.10]).

Theorem 2.28. *Let $A \in L(X)$ be semi-Fredholm. Then $k(A) < \infty$ and there exists $\epsilon > 0$ such that $A - \lambda$ is semi-Fredholm and*

$$\alpha(A-\lambda) = \alpha(A) - k(A), \quad \beta(A-\lambda) = \beta(A) - k(A), \quad \operatorname{ind}(A-\lambda) = \operatorname{ind}(A),$$

for every $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \epsilon$.

Proof. From the definition of $k(A)$ it follows that $k(A) \leqslant \alpha(A)$. From (2.43) it follows that $k(A) \leqslant \beta(A)$. Consequently, $k(A) \leqslant \min(\alpha(A), \beta(A))$, and if A is semi-Fredholm, then $k(A) < \infty$. Suppose that $A \in \Phi_+(X)$. From Theorem 2.24 it follows that there exists some $\epsilon_1 > 0$, such that if $|\lambda| < \epsilon_1$ then $A - \lambda \in \Phi_+(X)$, $\alpha(A-\lambda) \leqslant \alpha(A)$, $\beta(A-\lambda) \leqslant \beta(A)$ and $\operatorname{ind}(A-\lambda) = \operatorname{ind}(A)$.

Let $x \in \mathcal{N}(A-\lambda)$, i.e. $Ax = \lambda x$. It follows that $A^2x = \lambda Ax = \lambda^2 x$ and finally $A^n x = \lambda^n x$, $n = 1, 2, \ldots$. If $\lambda \neq 0$, then

$$\mathcal{N}(A-\lambda) \subset \bigcap_{n=1}^{\infty} \mathcal{R}(A^n) = \mathcal{R}(A^\infty). \tag{2.44}$$

From $A \in \Phi_+(X)$ it follows $A^n \in \Phi_+(X)$ by Lemma 2.2 and hence $\mathcal{R}(A^n)$ is closed for every $n \in \mathbb{N}$. Hence, $\mathcal{R}^\infty(A)$ is closed in X. Denote $X_1 = \mathcal{R}^\infty(A)$. From Lemma 2.7 it follows that $A(X_1) = X_1$. Let A_1 be the reduction of A on X_1. Since $A_1 \in \Phi(X_1)$, according to Theorem 2.24 there exists some $\epsilon_2 > 0$, such that if $0 < |\lambda| < \epsilon_2$ then $A_1 - \lambda \in \Phi(X_1)$, $\alpha(A_1 - \lambda) \leqslant \alpha(A_1)$, $\beta(A_1 - \lambda) \leqslant \beta(A_1)$, $\operatorname{ind}(A_1 - \lambda) = \operatorname{ind}(A_1)$. Since $\beta(A_1) = 0$, it follows that $\beta(A_1 - \lambda) = 0$. From (2.44) we have

$$\alpha(A-\lambda) = \alpha(A_1 - \lambda) = \operatorname{ind}(A_1 - \lambda) = \operatorname{ind}(A_1) = \alpha(A_1).$$

From $\alpha(A_1) = \dim(\mathcal{N}(A) \cap \mathcal{R}^\infty(A)) = \dim \mathcal{N}(A) - \dim \mathcal{N}(A)/(\mathcal{N}(A) \cap \mathcal{R}^\infty(A)) = \alpha(A) - k(A)$, we get that $\alpha(A-\lambda)$ has a constant value $\alpha(A) - k(A)$ in a neighborhood $0 < |\lambda| < \epsilon_2$. Let $\epsilon = \min(\epsilon_1, \epsilon_2)$. For $0 < |\lambda| < \epsilon$ we have $\operatorname{ind}(A-\lambda) = \operatorname{ind}(A)$ and $\alpha(A-\lambda) = \alpha(A) - k(A)$, so $\beta(A-\lambda)$ has a constant value $\beta(A) - k(A)$.

If $A \in \Phi_-(X)$, then $A' \in \Phi_+(X')$ by Proposition 2.8, and from the previously proved part it follows that there exists some $\epsilon > 0$, such that if $0 < |\lambda| < \epsilon$ then $\alpha(A' - \lambda)$ has the constant value $\alpha(A') - k(A')$, $\beta(A' - \lambda)$ has the constant value $\beta(A') - k(A')$ and $\operatorname{ind}(A' - \lambda) = \operatorname{ind}(A')$. Since $A^n \in \Phi_-(X)$, it follows that $\mathcal{R}(A^n)$ is closed for every $n \in \mathbb{N}$, and by (2.42) we have $k(A') = k(A)$. Now, $\operatorname{ind}(A-\lambda) = -\operatorname{ind}(A'-\lambda) = -\operatorname{ind}(A') = \operatorname{ind}(A)$, and $\beta(A-\lambda) = \alpha(A'-\lambda) = \alpha(A') - k(A') = \beta(A) - k(A)$. Analogously, it can be proved $\alpha(A-\lambda) = \alpha(A) - k(A)$. □

The approximate point spectrum of $T \in L(X)$ is defined as $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not bounded below}\}$ and the surjective spectrum is defined as $\sigma_{su}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not surjective}\}$. We set $\rho_{ap}(T) = \mathbb{C} \setminus \sigma_{ap}(T)$ and $\rho_{su}(T) = \mathbb{C} \setminus \sigma_{su}(T)$.

Corollary 2.22. *Let $A \in \Phi_+(X)$. The following conditions are equivalent*

(i) $\operatorname{asc}(A) < \infty$;
(ii) $0 \notin \operatorname{acc} \sigma_{ap}(A)$;
(iii) $0 \notin \operatorname{int} \sigma_{ap}(A)$.

Proof. (i)$\Longrightarrow$(ii) Suppose that $\operatorname{asc}(A) < \infty$. From Lemma 2.11 (i) it follows that $\mathcal{N}(A) \cap \mathcal{R}^\infty(A) = \{0\}$, and so $k(A) = \alpha(A)$. Now, by Theorem 2.28, there exists $\epsilon > 0$ such that for every λ, $0 < |\lambda| < \epsilon$ it follows that $A - \lambda \in \Phi_+(X)$ and $\alpha(A - \lambda) = \alpha(A) - k(A) = 0$, and so $A - \lambda$ is bounded below.

(ii)$\Longrightarrow$(iii) It is clear.

(iii)$\Longrightarrow$(i) Suppose that $0 \notin \operatorname{int} \sigma_{ap}(A)$. If $0 \notin \sigma_{ap}(A)$, then A is bounded below and $\operatorname{asc}(A) = 0$. Now assume that $0 \in \sigma_{ap}(A)$. From $A \in \Phi_+(X)$ according to Theorem 2.28 it follows that there exists $\epsilon > 0$ such that for every $\lambda \in \mathbb{C}$,

$$0 < |\lambda| < \epsilon \Longrightarrow \alpha(A - \lambda) = \alpha(A) - k(A). \tag{2.45}$$

From $0 \notin \operatorname{int} \sigma_{ap}(A)$ and $0 \in \sigma_{ap}(A)$ it follows that there exists $\mu \in \mathbb{C}$ such that $0 < |\mu| < \epsilon$ and $A - \mu$ is bounded bellow, and so $\alpha(A - \mu) = 0$. From (2.45) it follows that $\alpha(A) = k(A)$ and hence $N(A) \cap R^\infty(A) = \{0\}$. Now from Theorem 2.8 it follows that $\operatorname{asc}(A) < \infty$. □

Corollary 2.23. *Let $A \in \Phi_-(X)$. The following conditions are equivalent*

(i) $\operatorname{dsc}(A) < \infty$;
(ii) $0 \notin \operatorname{acc} \sigma_{su}(A)$;
(iii) $0 \notin \operatorname{int} \sigma_{su}(A)$.

Proof. It follows from Corollary 2.22 by duality. □

2.11. Semi-Browder operators.

An operator $T \in L(X)$ is upper semi-Browder if it is upper semi-Fredholm of finite ascent, and T is lower semi-Browder if it is lower semi-Fredholm of finite descent. Let $\mathcal{B}_+(X)$ ($\mathcal{B}_-(X)$) denote the set of all upper (lower) semi-Browder operators. The set of Browder operators is defined as $\mathcal{B}(X) = \mathcal{B}_+(X) \cap \mathcal{B}_-(X)$. If $T \in L(X)$ is upper (lower) semi-Browder and relatively regular, then we say that T is left (right) Browder [47], [48]. Therefore, $T \in L(X)$ is left Browder if and only if T is left Fredholm of finite ascent, and T is right Browder if and only if it is right Fredholm of finite descent. The set of all left (right) Browder operators is denoted by $\mathcal{B}_l(X)$ ($\mathcal{B}_r(X)$).

From Theorem 2.7 it follows that $T \in \mathcal{B}_+(X)$ if and only if $T \in \mathcal{W}_+(X)$ and $\operatorname{asc}(T) < \infty$, and $T \in \mathcal{B}_-(X)$ if and only if $T \in \mathcal{W}_-(X)$ and $\operatorname{dsc}(T) < \infty$. Also, T is left Browder if and only if $T \in \mathcal{W}_l(X)$ and $\operatorname{asc}(T) < \infty$, and T is right Browder if and only if $T \in \mathcal{W}_r(X)$ and $\operatorname{dsc}(T) < \infty$.

Lemma 2.38. *Let $A \in L(X)$. Then $A \in \mathcal{B}_+(X)$ if and only if $A' \in \mathcal{B}_-(X')$; moreover, $A \in \mathcal{B}_-(X)$ if and only if $A' \in \mathcal{B}_+(X')$.*

Proof. It follows from Lemma 2.13 and Lemma 2.8. □

Theorem 2.29. *Let $A,\ B \in L(X)$ and $AB = BA$. Then*

(i) $A,\ B \in \mathcal{B}_+(X) \Longleftrightarrow AB \in \mathcal{B}_+(X)$;

(ii) $A,\, B \in \mathcal{B}_-(X) \Longleftrightarrow AB \in \mathcal{B}_-(X)$.

Proof. (ii): Suppose that $A,\, B \in \mathcal{B}_-(X)$. According to Theorem 2.2 it follows that $AB \in \Phi_-(X)$. Let $p = \max(\mathrm{dsc}(A), \mathrm{dsc}(B))$. Then $A^p(X) = A^{p+1}(X)$ and $B^p(X) = B^{p+1}(X)$. Hence,

$$\begin{aligned}(AB)^p(X) &= A^pB^p(X) = A^pB^{p+1}(X) = B^{p+1}A^p(X)\\ &= B^{p+1}A^{p+1}(X) = (AB)^{p+1}(X).\end{aligned}$$

It follows that $\mathrm{dsc}(AB) < \infty$, so $AB \in \mathcal{B}_-(X)$.

On the other hand, suppose that $AB \in \mathcal{B}_-(X)$. From $AB = BA \in \Phi_-(X)$, using Theorem 2.3 (ii), it follows that $A, B \in \Phi_-(X)$. If $p = \mathrm{dsc}(AB) < \infty$, then $R^\infty(AB) = (AB)^p(X)$, and since $(AB)^p \in \Phi_-(X)$, we get $\mathrm{codim} R^\infty(AB) < \infty$. For every $n \in \mathbb{N}$ we have

$$A^n(X) \supset A^nB^n(X) = (AB)^n(X) \supset R^\infty(AB),$$

so $\mathrm{dsc}(A) < \infty$. Analogously, $\mathrm{dsc}(B) < \infty$ and $A, B \in \mathcal{B}_-(X)$.

(i) follows from (ii) and Lemma 2.38. □

If M is a subspace of X such that $T(M) \subset M$, $T \in L(X)$, it is said that M is T-*invariant*. We define $T_M \colon M \to M$ as $T_Mx = Tx$, $x \in M$. If M and N are two closed T-invariant subspaces of X such that $X = M \oplus N$, we say that T is *completely reduced* by the pair (M, N) and it is denoted by $(M, N) \in \mathrm{Red}(T)$. In this case we write $T = T_M \oplus T_N$ and say that T is a *direct sum* of T_M and T_N.

We need the following auxiliary assertions.

Lemma 2.39. *Let $T \in L(X)$ and $(M, N) \in \mathrm{Red}(T)$. The following statements hold:*

(i) $\mathrm{asc}(T) < \infty$ *if and only if* $\mathrm{asc}(T_M) < \infty$ *and* $\mathrm{asc}(T_N) < \infty$*;*

(ii) $\mathrm{dsc}(T) < \infty$ *if and only if* $\mathrm{dsc}(T_M) < \infty$ *and* $\mathrm{dsc}(T_N) < \infty$.

Proof. Since $\mathcal{N}(T^n) = \mathcal{N}(T_M^n) \oplus \mathcal{N}(T_N^n)$, for every $n \in \mathbb{N}$, we conclude that $\mathrm{asc}(T) < \infty$ if and only if $\mathrm{asc}(T_M) < \infty$ and $\mathrm{asc}(T_N) < \infty$ and in that case $\mathrm{asc}(T) = \max\{\mathrm{asc}(T_M), \mathrm{asc}(T_N)\}$. Similarly, as $\mathcal{R}(T^n) = \mathcal{R}(T_M^n) \oplus \mathcal{R}(T_N^n)$, $n \in \mathbb{N}$, we get that $\mathrm{dsc}(T) < \infty$ if and only if $\mathrm{dsc}(T_M) < \infty$ and $\mathrm{dsc}(T_N) < \infty$, with $\mathrm{dsc}(T) = \max\{\mathrm{dsc}(T_M), \mathrm{dsc}(T_N)\}$. □

Let $T \in L(X)$ and $\epsilon > 0$. We shall write

$$\mathrm{comm}_\epsilon^{-1}(T) = \{S \in L(X)^{-1} : S \text{ commutes with } T,\ \|S\| < \epsilon\}.$$

Theorem 2.30. *Let $T \in L(X)$. The following conditions are equivalent:*

(i) *T is upper semi-Browder;*

(ii) *$T \in \Phi_+(X)$ and there exists $\epsilon > 0$ such that for every $S \in \mathrm{comm}_\epsilon^{-1}(T)$ it follows that $T - S$ is bounded below;*

(iii) *$T \in \Phi_+(X)$ and $0 \notin \mathrm{acc}\, \sigma_{ap}(T)$;*

(iv) *$T \in \Phi_+(X)$ and $0 \notin \mathrm{int}\, \sigma_{ap}(T)$;*

(v) *There exists $(M, N) \in \mathrm{Red}(T)$ such that T_M is bounded below, T_N is nilpotent and $\dim N < \infty$;*

(vi) *There exists a projector* $P \in F(X)$ *which commutes with* T *such that* TP *is nilpotent operator and* $T + P$ *is bounded below;*
(vii) *There exist* $B \in F(X)$ *and* $A \in L(X)$ *such that* A *is bounded below,* $AB = BA$ *and* $T = A + B$*;*
(viii) *There exist* $B \in K(X)$ *and* $A \in L(X)$ *such that* A *is bounded below,* $AB = BA$ *and* $T = A + B$*.*

Proof. (i)$\Longrightarrow$(ii): Suppose that $T \in L(X)$ is upper semi-Browder. Then $T \in \Phi_+(X)$ and Theorem 2.2 provides that $\mathcal{R}(T^n)$ is closed for every $n \in \mathbb{N}$. According to Theorem 2.24 there exists some $\epsilon_1 > 0$ such that if $B \in L(X)$ and $\|B\| < \epsilon_1$, then $T - B \in \Phi_+(X)$. Let $X_1 = \mathcal{R}^\infty(T)$. X_1 is a Banach space and $T(X_1) = X_1$ by Lemma 2.7 (or Corollary 2.5). For the operator $T_1 \colon X_1 \to X_1$ induced by T we have that $\alpha(T_1) < \infty$ and $\beta(T_1) = 0$. From $T_1 \in \Phi(X_1)$, again by Theorem 2.24, it follows that there exists some $\epsilon_2 > 0$, such that for $B \in L(X_1)$, $\|B\| < \epsilon_2$ implies $T_1 - B \in \Phi(X_1)$, $\alpha(T_1 - B) \leqq \alpha(T_1)$, $\beta(T_1 - B) \leqq \beta(T_1)$, $\operatorname{ind}(T_1 - B) = \operatorname{ind}(T_1)$. Set $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ and let $S \in \operatorname{comm}_\epsilon^{-1}(T)$. Since S commutes with T, it follows that $S(\mathcal{R}(T^n)) \subset \mathcal{R}(T^n)$ for every $n \in \mathbb{N}$, and therefore, $S(X_1) = S(\cap_{n=1}^\infty \mathcal{R}(T^n)) = \cap_{n=1}^\infty S(\mathcal{R}(T^n)) \subset \cap_{n=1}^\infty \mathcal{R}(T^n) = X_1$. Let $S_1 \colon X_1 \to X_1$ be the operator induced by operator S. Then $\|S_1\| < \epsilon$ and $\beta(T_1) = 0$, and so $\beta(T_1 - S_1) = 0$.

We prove that

$$\mathcal{N}(T - S) \subset \mathcal{R}^\infty(T). \tag{2.46}$$

Let $x \in \mathcal{N}(T - S)$. Then $Tx = Sx$ and $T^2x = T(Tx) = T(Sx) = S(Tx) = S^2x$. By induction we conclude $T^n x = S^n x$ for every $n \in \mathbb{N}_0$.

Since S commutes with T, it follows that S^{-1} commutes with T. Fix $n_0 \in \mathbb{N}$. From $T^{n_0}x = S^{n_0}x = SS^{n_0-1}x$ it follows that $S^{-1}T^{n_0}x = S^{n_0-1}x$, and now from the fact that S^{-1} commutes with T we get $T^{n_0}S^{-1}x = S^{n_0-1}x$. Continuing this method we obtain $T^{n_0}(S^{-1})^{n_0}x = x$. Hence $x \in \mathcal{R}(T^{n_0})$, and since n_0 was arbitrary, we get $x \in \mathcal{R}^\infty(T)$.

From (2.46) we conclude that $\mathcal{N}(T - S) = \mathcal{N}(T_1 - S_1)$. Cosequently,

$$\alpha(T - S) = \alpha(T_1 - S_1) = i(T_1 - S_1) = i(T_1) = \alpha(T_1).$$

From Lemma 2.11 (i) it follows that $\mathcal{N}(T) \cap \mathcal{R}^\infty(T) = \{0\}$ and hence, $\alpha(T_1) = 0$. Therefore, $\alpha(T - S) = 0$. Since $T - S \in \Phi_+(X)$, we have that $T - S$ has closed range, and hence $T - S$ is bounded below.

(ii)$\Longrightarrow$(iii): It is obvious.

The equivalences (i)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv) follow from Corollary 2.22.

(i)$\Longrightarrow$(v): Suppose that T is upper semi-Browder. From [36, Theorem 16.21] it follows that there exists $(M, N) \in \operatorname{Red}(T)$ such that T_M is Kato, T_N is nilpotent and $\dim N < \infty$. Since $\operatorname{asc}(T) < \infty$, Lemma 2.39 (i) provides that $\operatorname{asc}(T_M) < \infty$. Now from Corollary 2.21 (i) we obtain that T_M is bounded below.

(v)$\Longrightarrow$(vi): Suppose that there exists $(M, N) \in \operatorname{Red}(T)$ such that T_M is bounded below, T_N is nilpotent and $\dim N < \infty$. Let P be a projector such that $\mathcal{R}(P) = N$ and $\mathcal{N}(P) = M$. Clearly, $P \in F(X)$. For $x \in X$ there exist $u \in M$ and $v \in N$ such that $x = u + v$. As $TPx = Tv$ and $PTx = P(Tu + Tv) = Tv$, we conclude that P commutes with T. Since T_N is nilpotent, there is an $n \in \mathbb{N}$ such that

$T_N^n = 0$. From $TP = 0 \oplus T_N$ we conclude that $(TP)^n = 0 \oplus T_N^n = 0$, and so TP is nilpotent. The fact that T_N is nilpotent provides that $T_N + I$ is invertible, and hence $\mathcal{N}(T + P) = \mathcal{N}(T_M) \oplus \mathcal{N}(T_N + I) = \{0\}$ and $\mathcal{R}(T + P) = \mathcal{R}(T_M) \oplus \mathcal{R}(T_N + I) = \mathcal{R}(T_M) \oplus N$. Since $\mathcal{R}(T_M)$ is closed and $\dim N < \infty$ we get that $\mathcal{R}(T+P)$ is closed. Consequently, $T + P$ is bounded below.

(vi)$\Longrightarrow$(vii): Let there exists a projector $P \in F(X)$ which commutes with T such that TP is nilpotent and $T + P$ is bounded below. For $A = T + P$ and $B = -P$ we have that $B \in F(X)$, A is bounded below and B commutes with A.

(vii)$\Longrightarrow$(viii): It is clear.

(viii)$\Longrightarrow$(i): Let there exist $B \in K(X)$ and $A \in L(X)$ such that A is bounded below, $AB = BA$ and $T = A + B$. Then asc $A < \infty$ and $A \in \Phi_+(X)$, and hence $A + \lambda B \in \Phi_+(X)$ for $\lambda \in [0, 1]$ according to Theorem 2.4. Since B commutes with A, from [13, Theorem 3] it follows that the function $\lambda \to \overline{\mathcal{N}^\infty(A + \lambda B)} \cap \mathcal{R}^\infty(A + \lambda B)$ is locally constant on the set $[0, 1]$ and therefore this function is constant on $[0, 1]$. As $\mathrm{asc}(A) < \infty$, from Theorem 2.8 it follows that $\overline{\mathcal{N}^\infty(A)} \cap \mathcal{R}^\infty(A) = \mathcal{N}^\infty(A) \cap \mathcal{R}^\infty(A) = \{0\}$ and hence, $\overline{\mathcal{N}^\infty(A + B)} \cap \mathcal{R}^\infty(A + B) = \{0\}$. It implies $\mathcal{N}^\infty(A + B) \cap \mathcal{R}^\infty(A + B) = \{0\}$, and again according to Theorem 2.8 we get $\mathrm{asc}\,(A + B) < \infty$. Therefore, $T = A + B \in \mathcal{B}_+(X)$. □

Theorem 2.31. *Let $T \in L(X)$. The following conditions are equivalent:*

(i) *T is lower semi-Browder;*
(ii) *$T \in \Phi_-(X)$ and there exists $\epsilon > 0$ such that for every $S \in \mathrm{comm}_\epsilon^{-1}(T)$ it follows that $T - S$ is surjective;*
(iii) *$T \in \Phi_-(X)$ and $0 \notin \mathrm{acc}\,\sigma_{su}(T)$;*
(iv) *$T \in \Phi_-(X)$ and $0 \notin \mathrm{int}\,\sigma_{su}(T)$;*
(v) *There exists $(M, N) \in \mathrm{Red}(T)$ such that T_M surjective, T_N is nilpotent and $\dim N < \infty$;*
(vi) *There exists a projector $P \in F(X)$ which commutes with T such that TP is nilpotent operator and $T + P$ is surjective;*
(vii) *There exist $B \in F(X)$ and $A \in L(X)$ such that A is surjective, $AB = BA$ and $T = A + B$;*
(viii) *There exist $B \in K(X)$ and $A \in L(X)$ such that A is surjective, $AB = BA$ and $T = A + B$.*

Proof. The implication (i)$\Longrightarrow$(ii) can be got from the implication (i)$\Longrightarrow$(ii) in Theorem 2.30 by taking adjoints (see Lemma 2.38).

(ii)$\Longrightarrow$(iii): It is obvious.

The equivalences (i)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv) follow from Corollary 2.23.

(i)$\Longrightarrow$(v): It follows from Lemma 2.39 (ii) and Corollary 2.21 (ii), similarly to the proof of the implication (i)$\Longrightarrow$(v) in Theorem 2.30.

(v)$\Longrightarrow$(vi)$\Longrightarrow$(vii)$\Longrightarrow$(viii): Similarly as in the proof of Theorem 2.30.

(viii)$\Longrightarrow$(i): It can be got by taking adjoints, and using Lemma 2.38, Lemma 2.13 and the implication (viii)$\Longrightarrow$(i) in Theorem 2.30 □

Theorem 2.32. *Let $T \in L(X)$. The following conditions are equivalent:*

(i) *T is Browder;*

(ii) *$T \in \Phi(X)$ and there exists $\epsilon > 0$ such that for every $S \in \mathrm{comm}_\epsilon^{-1}(T)$ it follows that $T - S$ is invertible;*
(iii) *$T \in \Phi(X)$ and $0 \notin \operatorname{acc} \sigma(T)$;*
(iv) *$T \in \Phi(X)$ and $0 \notin \operatorname{int} \sigma(T)$;*
(v) *There exists $(M, N) \in \mathrm{Red}(T)$ such that T_M invertible, T_N is nilpotent and $\dim N < \infty$;*
(vi) *There exists a projector $P \in F(X)$ which commutes with T such that TP is nilpotent operator and $T + P$ is invertible;*
(vii) *There exist $B \in F(X)$ and $A \in L(X)$ such that A is invertible, $AB = BA$ and $T = A + B$;*
(viii) *There exist $B \in K(X)$ and $A \in L(X)$ such that A is invertible, $AB = BA$ and $T = A + B$.*

Proof. The equivalences (i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv) and the implication (viii)$\Longrightarrow$(i) follows from Theorems 2.30 and 2.31. The implications (i)$\Longrightarrow$(v)$\Longrightarrow$(vi)$\Longrightarrow$(vii)$\Longrightarrow$(viii) can be proved similarly to the proof of the appropriate implications in Theorem 2.30. □

In order to characterize left and right Browder operators we need the following lemma.

Lemma 2.40. *Let $A \in L(X)$ and let X be a direct sum of closed subspaces X_1 and X_2 which are A-invariant. If $A_1 = A_{|X_1} \colon X_1 \to X_1$ and $A_2 = A_{|X_2} \colon X_2 \to X_2$, then the operator A is g-invertible if and only if A_1 and A_2 are g-invertible.*

Proof. The operator A has the following matrix form with respect to the decomposition $X = X_1 \oplus X_2$:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

Suppose that A is g-invertible. Then there exists $B \in L(X)$ such that $ABA = A$, and B has the following matrix form:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

and we get

$$\begin{bmatrix} A_1 B_{11} A_1 & A_1 B_{12} A_2 \\ A_2 B_{21} A_1 & A_2 B_{22} A_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

which implies $A_1 B_{11} A_1 = A_1$ and $A_2 B_{22} A_2 = A_2$. Thus, A_1 and A_2 are g-invertible operators.

Conversely, suppose that $A_1 \in L(X_1)$ and $A_2 \in L(X_2)$ are g-invertible operators. Then there exist $B_1 \in L(X_1)$ and $B_2 \in L(X_2)$ such that $A_1 B_1 A_1 = A_1$ and $A_2 B_2 A_2 = A_2$. Let

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}.$$

Then we have $B \in L(X)$ and $ABA = A$, so A is a g-invertible operator. □

Recall that an operator $T \in L(X)$ is left (right) invertible if and only if T is relatively regular bounded below (surjective) operator. Using the openness of the sets $\Phi_l(X)$ and $\Phi_r(X)$ (Corollary 2.10), the stability of these sets under compact perturbations (Corollary 2.13) and Lemma 2.40, similarly to the proof of Theorem 2.30 and Theorem 2.31, we get the following characterizations of left (right) Browder operators.

Theorem 2.33. *Let $T \in L(X)$. The following conditions are equivalent:*

- (i) *T is left Browder;*
- (ii) *$T \in \Phi_l(X)$ and there exists $\epsilon > 0$ such that for every $S \in \mathrm{comm}_\epsilon^{-1}(T)$ it follows that $T - S$ is left invertible;*
- (iii) *$T \in \Phi_l(X)$ and $0 \notin \mathrm{acc}\,\sigma_l(T)$;*
- (iv) *$T \in \Phi_l(X)$ and $0 \notin \mathrm{int}\,\sigma_l(T)$;*
- (v) *$T \in \Phi_l(X)$ and $0 \notin \mathrm{acc}\,\sigma_{ap}(T)$;*
- (vi) *$T \in \Phi_l(X)$ and $0 \notin \mathrm{int}\,\sigma_{ap}(T)$;*
- (vii) *There exists $(M, N) \in \mathrm{Red}(T)$ such that T_M is left invertible, T_N is nilpotent and $\dim N < \infty$;*
- (viii) *There exists a projector $P \in F(X)$ which commutes with T such that TP is nilpotent operator and $T + P$ is left invertible;*
- (ix) *There exist $B \in F(X)$ and $A \in L(X)$ such that A is left invertible, $AB = BA$ and $T = A + B$;*
- (x) *There exist $B \in K(X)$ and $A \in L(X)$ such that A is left invertible, $AB = BA$ and $T = A + B$.*

Theorem 2.34. *Let $T \in L(X)$. The following conditions are equivalent:*

- (i) *T is right Browder;*
- (ii) *$T \in \Phi_r(X)$ and there exists $\epsilon > 0$ such that for every $S \in \mathrm{comm}_\epsilon^{-1}(T)$ it follows that $T - S$ is right invertible;*
- (iii) *$T \in \Phi_r(X)$ and $0 \notin \mathrm{acc}\,\sigma_r(T)$;*
- (iv) *$T \in \Phi_r(X)$ and $0 \notin \mathrm{int}\,\sigma_r(T)$;*
- (v) *$T \in \Phi_r(X)$ and $0 \notin \mathrm{acc}\,\sigma_{su}(T)$;*
- (vi) *$T \in \Phi_r(X)$ and $0 \notin \mathrm{int}\,\sigma_{su}(T)$;*
- (vii) *There exists $(M, N) \in \mathrm{Red}(T)$ such that T_M is right invertible, T_N is nilpotent and $\dim N < \infty$;*
- (viii) *There exists a projector $P \in F(X)$ which commutes with T such that TP is nilpotent operator and $T + P$ is right invertible;*
- (ix) *There exist $B \in F(X)$ and $A \in L(X)$ such that A is right invertible, $AB = BA$ and $T = A + B$;*
- (x) *There exist $B \in K(X)$ and $A \in L(X)$ such that A is right invertible, $AB = BA$ and $T = A + B$.*

The following two examples show that the commutativity condition in Theorems 2.30, 2.31, 2.32, 2.33, 2.34 is essential.

Example 2.1. Let $X = \ell_1 = \big\{(\xi_1, \xi_2, \dots) : \xi_i \in \mathbb{C},\ \sum_1^\infty |\xi_i| < \infty\big\}$. For $x = (\xi_1, \xi_2, \dots)$ and $y = (\eta_1, \eta_2, \dots)$ define $Ax = y$ as $\eta_1 = \xi_2$, $\eta_{2n} = \xi_{2n+2}$ and

$\eta_{2n+1} = \xi_{2n-1}$, $n = 1, 2, \dots$, so $Ax = (\xi_2, \xi_4, \xi_1, \xi_6, \xi_3, \xi_8, \dots)$. Obviously, A is an isometric isomorphism on X. Define K as $Kx = y$, where $\eta_1 = \xi_2$, $\eta_i = 0$, $i \neq 1$. Clearly, $K \in K(X)$. Let $T = A - K$, and let $x_n = (\xi_i^n)_i$ be defined as

$$\xi_{2n}^n = 1, \quad \xi_i^n = 0, \ i \neq 2n.$$

By the induction on n we prove that $T^n(x_n) = 0$ and $T^{n-1}(x_n) \neq 0$, $n \in \mathbb{N}$. The statement is satisfied for $n = 1$, so we suppose that it is valid for some m, $m \geqslant 1$. Then

$$T^{m+1}(x_{m+1}) = T^m T(x_{m+1}) = T^m(x_m) = 0,$$
$$T^m(x_{m+1}) = T^{m-1} T(x_{m+1}) = T^{m-1}(x_m) \neq 0.$$

We get that $\operatorname{asc}(T) = \infty$. Since $T \in \mathcal{W}(X)$ (Theorem 2.12), from Corollary 2.4 (ii) it follows that $\operatorname{dsc}(T) = \infty$.

Example 2.2. Let $X = \ell_1 = \{(\dots, \xi_{-1}, \xi_0, \xi_1, \dots) : \xi_i \in \mathbb{C}, \ \sum_{-\infty}^{\infty} |\xi_i| < \infty\}$, and let $A, K \in L(X)$ be defined as follows: for $x = (\dots, \xi_{-1}, \xi_0, \xi_1, \dots) \in X$, we have $A(x) = y$, where $y = (\eta_n)$ is given by $\eta_n = \xi_{n-1}$, $n = 0, \pm 1, \pm 2, \dots$, and

$$(Kx)_n = \begin{cases} 0, & \text{for } n \neq 1, \\ -\xi_0, & \text{for } n = 1. \end{cases}$$

Then A is an isometric isomorphism on X, K is finite rank (so it is compact also), $AK \neq KA$, and $\operatorname{asc}(A + K) = \operatorname{dsc}(A + K) = \infty$.

Corresponding to the classes $\Phi_+(X)$, $\Phi_-(X)$, $\Phi(X)$, $\mathcal{W}_+(X)$, $\mathcal{W}_-(X)$, $\mathcal{W}(X)$, $\mathcal{W}_l(X)$, $\mathcal{W}_r(X)$, $\mathcal{B}_+(X)$, $\mathcal{B}_-(X)$, $\mathcal{B}(X)$, $\mathcal{B}_l(X)$, $\mathcal{B}_r(X)$, we have the following spectra and resolvent sets of $T \in L(X)$: the upper Fredholm spectrum $\sigma_{\Phi_+}(T)$ of T is defined by $\sigma_{\Phi_+}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_+(X)\}$ and the upper Fredholm resolvent set is $\rho_{\Phi_+}(T) = \mathbb{C} \setminus \sigma_{\Phi_+}(T)$, and similarly for the lower Fredholm (resp. Fredholm, upper (lower) Weyl, Weyl, left (right) Weyl, upper (lower) Browder, Browder, left (right) Browder) spectrum and resolvent set.

From the equivalencies (i)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv) in Theorem 2.32 it follows that the Browder spectrum can be described in the following way:

$$\sigma_{\mathcal{B}}(A) = \sigma_\Phi(A) \cup \operatorname{acc} \sigma(A) = \sigma_\Phi(A) \cup \operatorname{int} \sigma(A). \tag{2.47}$$

Also, from Theorems 2.30, 2.31, 2.33, 2.34 it follows that

$$\sigma_{\mathcal{B}_+}(A) = \sigma_{\Phi_+}(A) \cup \operatorname{acc} \sigma_{ap}(A) = \sigma_{\Phi_+}(A) \cup \operatorname{int} \sigma_{ap}(A),$$
$$\sigma_{\mathcal{B}_-}(A) = \sigma_{\Phi_-}(A) \cup \operatorname{acc} \sigma_{su}(A) = \sigma_{\Phi_-}(A) \cup \operatorname{int} \sigma_{su}(A)$$

and

$$\begin{aligned} \sigma_{\mathcal{B}_l}(A) &= \sigma_{\Phi_l}(A) \cup \operatorname{acc} \sigma_l(A) = \sigma_{\Phi_l}(A) \cup \operatorname{int} \sigma_l(A), \\ &= \sigma_{\Phi_l}(A) \cup \operatorname{acc} \sigma_{ap}(A) = \sigma_{\Phi_l}(A) \cup \operatorname{int} \sigma_{ap}(A), \end{aligned} \tag{2.48}$$

$$\begin{aligned} \sigma_{\mathcal{B}_r}(A) &= \sigma_{\Phi_r}(A) \cup \operatorname{acc} \sigma_r(A) = \sigma_{\Phi_r}(A) \cup \operatorname{int} \sigma_r(A), \\ &= \sigma_{\Phi_r}(A) \cup \operatorname{acc} \sigma_{su}(A) = \sigma_{\Phi_r}(A) \cup \operatorname{int} \sigma_{su}(A). \end{aligned} \tag{2.49}$$

3. Riesz operators

Riesz operators are a generalization of compact operators, as well as quasinilpotent operators.

Let $A \in L(X)$. If λ_0 is an isolated point of the spectrum u $\sigma(A)$, then λ_0 is and isolated singular point of the resolvent function of A. Hence, there exists the Laurent series of the resolvent in powers of $\lambda - \lambda_0$:

$$(\lambda - A)^{-1} = \sum_{n=-\infty}^{\infty} (\lambda - \lambda_0)^n B_n,$$

which is convergent in a punctured neighborhood of λ_0. The coefficients $B_n \in L(X)$ are given as

$$B_n = \frac{1}{2\pi i} \int_c \frac{(\lambda - A)^{-1}}{(\lambda - \lambda_0)^{n+1}}\, d\lambda, \quad n = 0, \pm 1, \pm 2, \dots, \tag{3.1}$$

where c is a circle around λ_0, separating this point from the rest of the spectrum, and oriented counter-clockwise. Particularly, B_{-1} is the *spectral projection corresponding to the point* λ_0 *and to the operator* A. We use $P_{\lambda_0}(A) = B_{-1}$.

The isolated spectral point λ_0 is *the pole of the resolvent of* A *of the order* p, if $B_{-p} \neq 0$ and $B_{-n} = 0$ for $n > p$.

The coefficients B_{-n}, $n = 1, 2, \dots$, of the Laurent series of the resolvent of A in a punctured neighborhood of λ_0 satisfies the following:

$$B_{-n} = (A - \lambda_0)^{n-1} B_{-1} = (A - \lambda_0)^{n-1} P_{\lambda_0}(A), \quad n = 1, 2, \dots. \tag{3.2}$$

The subspaces $X_1 = \mathcal{N}(P_{\lambda_0}(A))$ and $X_2 = \mathcal{R}(P_{\lambda_0}(A))$ are closed and invariant under A and $X = X_1 \oplus X_2$. If we write A_i for the reduction of A onto X_i, $i = 1, 2$, then $\sigma(A_1) = \sigma(A) \setminus \{\lambda_0\}$ and $\sigma(A_2) = \{\lambda_0\}$. Since $A_1 - \lambda_0$ is invertible, it follows that $(A_1 - \lambda_0)^n$ is bijective on $X_1 = \mathcal{N}(P_\lambda(A))$ for all $n \in \mathbb{N}$, and therefore, $\mathcal{N}((A_1 - \lambda_0)^n) = \{0\}$ and $\mathcal{R}((A_1 - \lambda_0)^n) = X_1$ for all $n \in \mathbb{N}$. Hence,

$$\begin{aligned} \mathcal{N}((A - \lambda_0)^n) &= \mathcal{N}((A_1 - \lambda_0)^n) \oplus \mathcal{N}((A_2 - \lambda_0)^n) \\ &= \mathcal{N}((A_2 - \lambda_0)^n) \subset X_2 = \mathcal{R}(P_{\lambda_0}(A)) \end{aligned} \tag{3.3}$$

$$\begin{aligned} \mathcal{R}((A - \lambda_0)^n) &= \mathcal{R}((A_1 - \lambda_0)^n) \oplus \mathcal{R}((A_2 - \lambda_0)^n) \\ &= X_1 \oplus \mathcal{R}((A_2 - \lambda_0)^n) \supset X_1 = \mathcal{N}(P_{\lambda_0}(A)), \end{aligned} \tag{3.4}$$

for all $n \in \mathbb{N}$. It implies that

$$\begin{aligned} \mathcal{N}^\infty(A - \lambda_0) &= \bigcup_{n=1}^{\infty} \mathcal{N}((A - \lambda_0)^n) \subset \mathcal{R}(P_{\lambda_0}(A)), \\ \mathcal{R}^\infty(A - \lambda_0) &= \bigcap_{n=1}^{\infty} \mathcal{R}((A - \lambda_0)^n) \supset \mathcal{N}(P_{\lambda_0}(A)). \end{aligned} \tag{3.5}$$

Lemma 3.1. *Let $A \in L(X)$ and let X be a direct sum of closed subspaces X_1 and X_2, which are invariant for A, and let A_i be the reduction of A onto X_i, $i = 1, 2$. If A_1 is bijective and A_2 is nilpotent, i.e. there exists $p \in \mathbb{N}$ such that $A_2^{p-1} \neq 0 = A_2^p$, then:*

(i) *$\lambda = 0$ is an isolated point of $\sigma(A)$.*
(ii) *$\lambda = 0$ is the pole of the resolvent of A of the order p.*
(iii) *The spectral projection of A corresponding to the point $\lambda = 0$ has the range equal to X_2, and the null space equal to X_1.*

Proof. (i): Since A_1 is invertible, we have $0 \in \rho(A_1)$. Hence, there exists $\epsilon > 0$ such that for $|\lambda| < \epsilon$ we have $\lambda \in \rho(A_1)$. Since A_2 is nilpotent, it follows that $\sigma(A_2) = \{0\}$, and for every $\lambda \neq 0$ we have $\lambda \in \rho(A_2)$. Since $\sigma(A) = \sigma(A_1) \cup \sigma(A_2)$, we obtain $0 \in \sigma(A)$. Consequently, if $0 < |\lambda| < \epsilon$ then $\lambda \in \rho(A_1) \cap \rho(A_2) = \rho(A)$, so 0 is an isolated point of $\sigma(A)$.

(ii): The decomposition

$$X = X_1 \oplus X_2 \tag{3.6}$$

completely reduce $(\lambda - A)^{-1}$ for every $\lambda \in \rho(A)$, and the reduction of $(\lambda - A)^{-1}$ onto X_i is equal to $(\lambda - A_i)^{-1}$, $i = 1, 2$. Hence, from (3.1) it follows that the decomposition (3.6) completely reduce every B_n, $n = 0, \pm 1, \pm 2, \dots$, where we consider the Laurent series in a punctured neighborhood of zero:

$$(\lambda - A)^{-1} = \sum_{n=-\infty}^{\infty} B_n \lambda^n.$$

Let $B_n^{(i)}$ denote the reduction of B_n on X_i, $i = 1, 2$. Then

$$(\lambda - A_i)^{-1} = \sum_{n=-\infty}^{\infty} B_n^{(i)} \lambda^n.$$

Since $0 \in \rho(A_1)$, the function $\lambda \mapsto (\lambda - A_1)^{-1}$ is analytic in a neighborhood of 0, and $B_{-n}^{(1)} = 0$ for every $n = 1, 2, \dots$. Since $\sigma(A_2) = \{0\}$, for $\lambda \neq 0$, it follows that

$$(\lambda - A_2)^{-1} = \sum_{n=1}^{\infty} \lambda^{-n} A_2^{n-1} = \sum_{n=1}^{p} \lambda^{-n} A_2^{n-1}.$$

Since the Laurent series is unique, it follows that $B_{-n}^{(2)} = A_2^{n-1}$ for $1 \leqslant n \leqslant p$, and $B_{-n}^{(2)} = 0$ for $n > p$.

Since there exists a projection $P \in L(X)$ such that $\mathcal{R}(P) = X_2$ and $\mathcal{N}(P) = X_1$, for $n = 1, 2, \dots$ we have

$$B_{-n} = B_{-n}(I - P) + B_{-n}P = B_{-n}^{(1)}(I - P) + B_{-n}^{(2)}P = 0 + B_{-n}^{(2)}P. \tag{3.7}$$

Consequently, $B_{-n} = 0$ for $n > p$, and $B_{-p} \neq 0$. We conclude that $\lambda = 0$ is the pole of the resolvent of A, and the order of this pole is p.

(iii): Spectral projection corresponding to $\lambda = 0$ is the operator B_{-1}. From (3.7) it follows that $B_{-1} = B_{-1}^{(2)}P = A_2^0 P = P$. □

Lemma 3.2. *Let $A \in L(X)$, $\operatorname{asc}(A) < \infty$, $\operatorname{dsc}(A) < \infty$ and $\operatorname{asc}(A) = \operatorname{dsc}(A) = p \neq 0$. Then:*

(i) *$X = \mathcal{R}(A^p) \oplus \mathcal{N}(A^p)$.*

(ii) *$\mathcal{R}(A^p)$ and $\mathcal{N}(A^p)$ are closed invariant subspaces for A, i.e. the decomposition* (i) *completely reduces A.*
(iii) *The operator A is bijective from $\mathcal{R}(A^p)$ onto $\mathcal{R}(A^p)$, and the reduction of A onto $\mathcal{N}(A^p)$ is nilpotent.*
(iv) *$\lambda = 0$ is an isolated point of the set $\sigma(A)$.*
(v) *$\lambda = 0$ is the pole of the resolvent of A, and the order of this pole is p.*
(vi) *Let P_0 be the spectral projection of A corresponding to the point $\lambda = 0$. Then $\mathcal{R}(P_0) = \mathcal{N}(A^p)$ and $\mathcal{N}(P_0) = \mathcal{R}(A^p)$.*

On the other hand, if $\lambda = 0$ is the pole of the resolvent of A, and the order of this pole is p, then $\operatorname{asc}(A) = \operatorname{dsc}(A) = p$.

Proof. ($\Longrightarrow$): Follows from Theorem 2.6 and Lemma 3.1.

($\Longleftarrow$): Let $\lambda = 0$ be the pole of the resolvent of A, and the order of this pole is p. Then there exists $\epsilon > 0$ such that

$$(\lambda - A)^{-1} = \sum_{n=-p}^{\infty} \lambda^n B_n, \quad \text{for } 0 < |\lambda| < \epsilon.$$

Let P_0 be the spectral projection of A corresponding to the point 0. From (3.2) we get

$$0 \neq B_{-p} = A^{p-1}B_{-1} = A^{p-1}P_0, \tag{3.8}$$

$$0 = B_{-(p+1)} = A^p B_{-1} = A^p P_0. \tag{3.9}$$

From (3.9) it follows that $\mathcal{R}(P_0) \subset \mathcal{N}(A^p)$. According to (3.5), we have $\mathcal{N}(A^{p+1}) \subset \mathcal{R}(P_0)$. Consequently, $\mathcal{N}(A^{p+1}) = \mathcal{R}(P_0) = \mathcal{N}(A^p)$. If $\mathcal{N}(A^{p-1}) = \mathcal{N}(A^p) = \mathcal{R}(P_0)$, then $A^{p-1}P_0 = 0$, which is not possible because of (3.8). Therefore, $\operatorname{asc}(A) = p$.

The operators A and P_0 mutually commutes, so from (3.8) and (3.9) we obtain

$$P_0 A^{p-1} \neq 0, \tag{3.10}$$

$$P_0 A^p = 0. \tag{3.11}$$

From (3.11) it follows that $\mathcal{R}(A^p) \subset \mathcal{N}(P_0)$, while from (3.4) we get $\mathcal{N}(P_0) \subset \mathcal{R}(A^{p+1})$. Therefore, $\mathcal{R}(A^p) = \mathcal{R}(A^{p+1})$ and $\operatorname{dsc}(A) \leqslant p$. If $\mathcal{R}(A^{p-1}) = \mathcal{R}(A^p) = \mathcal{N}(P_0)$, then $P_0 A^{p-1} = 0$, which is not possible because of (3.10). Hence, $\operatorname{dsc}(A) = p$. □

Definition 3.1. Let $A \in L(X)$. The point $\lambda_0 \in \sigma(A)$ is the *Riesz point* of A, if the two following conditions hold:

(i) λ_0 is an isolated point of $\sigma(A)$.
(ii) X is a direct sum of a closed subspace $F(\lambda_0)$ and a finite dimensional subspace $N(\lambda_0)$, the subspaces $F(\lambda_0)$ and $N(\lambda_0)$ are invariant for A, the reduction of $A - \lambda_0$ onto $F(\lambda_0)$ is invertible, and the reduction of $A - \lambda_0$ onto $N(\lambda_0)$ is nilpotent.

The set of Riesz points of A is denoted by $p_{00}(A)$.

The following theorem shows that if $\dim \mathcal{R}(P_{\lambda_0}(A)) < \infty$, then λ_0 is a pole of the resolvent of A, and in this case the pole λ_0 is the pole of the finite algebraic multiplicity.

Theorem 3.1. *Let $A \in L(X)$ and $\lambda_0 \in \mathbb{C}$. The following conditions are equivalent:*

(i) $\lambda_0 \in \sigma(A)$ *and there exists* $F \in F(X)$*, such that* $AF = FA$ *and* $\lambda_0 \in \rho(A+F)$.
(ii) $\lambda_0 \in \sigma(A)$ *and there exists* $K \in K(X)$*, such that* $AK = KA$ *and* $\lambda_0 \in \rho(A+K)$.
(iii) $A - \lambda_0 \in \Phi(X)$ *and* $0 < \operatorname{asc}(A - \lambda_0) = \operatorname{dsc}(A - \lambda_0) < \infty$*, i.e.* $A - \lambda_0$ *is singular and Browder.*
(iv) λ_0 *is an isolated point of* $\sigma(A)$ *and* $A - \lambda_0$ *is semi-Fredholm.*
(v) λ_0 *is an isolated point of* $\sigma(A)$ *and* $A - \lambda_0 \in \Phi(X)$.
(vi) λ_0 *is an isolated point of* $\sigma(A)$ *and* $A - \lambda_0 \in \mathcal{W}(X)$.
(vii) λ_0 *is an isolated point of* $\sigma(A)$ *and it is an eigenvalue of* A *of the finite algebraic multiplicity.*
(viii) λ_0 *is an isolated point of* $\sigma(A)$ *of the finite algebraic multiplicity.*
(ix) λ_0 *is a pole of the resolvent of* A *of the finite algebraic multiplicity.*
(x) λ_0 *is a Riesz point of* A.
(xi) $\lambda_0 \in \sigma(A) \cap \Phi^R(A)$.

Proof. (i)$\Longleftrightarrow$(ii) $\Longleftrightarrow$(iii): Follows from Theorem 2.32.

(iii)$\Longrightarrow$(iv): Suppose that $A - \lambda_0 \in \Phi(X)$ and $0 < \operatorname{asc}(A - \lambda_0) = \operatorname{dsc}(A - \lambda_0) < \infty$. Using Lemma 3.2 we get that λ_0 is an isolated point of $\sigma(A)$.

(iv)$\Longrightarrow$(v)$\Longrightarrow$(vi): If λ_0 is isolated point of $\sigma(A)$ and if $A - \lambda_0$ is semi-Fredholm, from Theorem 2.28 it follows that $A - \lambda_0$ is Weyl.

(vi)$\Longrightarrow$(vii): Let λ_0 be an isolated point of $\sigma(A)$, $A - \lambda_0 \in \mathcal{W}(X)$ and let $P_{\lambda_0}(A)$ be the spectral projection of A corresponding to λ_0. For $X_1 = \mathcal{N}(P_{\lambda_0}(A))$ and $X_2 = \mathcal{R}(P_{\lambda_0}(A))$ we have $X = X_1 \oplus X_2$. Closed subspaces X_1 and X_2 are invariant for A, $\sigma(A_1) = \sigma(A) \setminus \{\lambda_0\}$, and $\sigma(A_2) = \{\lambda_0\}$ where A_i is the reduction of A onto X_i, $i = 1, 2$.

From (3.3) it follows that $\mathcal{N}(A_2 - \lambda_0) = \mathcal{N}(A - \lambda_0)$, so $\alpha(A_2 - \lambda_0) < \infty$.

By (3.4) we have

$$\mathcal{R}(A - \lambda_0) = X_1 \oplus \mathcal{R}(A_2 - \lambda_0).$$

Let

$$X_2 = \mathcal{R}(A_2 - \lambda_0) \oplus Y. \tag{3.12}$$

Then

$$X = X_1 \oplus X_2 = X_1 \oplus \mathcal{R}(A_2 - \lambda_0) \oplus Y = \mathcal{R}(A - \lambda_0) \oplus Y. \tag{3.13}$$

From (3.12) and (3.13) we obtain $\beta(A_2 - \lambda_0) = \dim Y = \beta(A - \lambda_0) < \infty$. Consequently, $\lambda_0 \in \Phi(A_2)$. Since $\sigma(A_2) = \{\lambda_0\}$, we get $\Phi(A_2) = \mathbb{C}$. From Corollary 2.8 it follows that $\dim X_2 < \infty$, so λ_0 is an isolated point of $\sigma(A)$, and its multiplicity is finite. From $A - \lambda_0 \in \mathcal{W}(X)$ and $\lambda_0 \in \sigma(A)$ we get $\alpha(A - \lambda_0) > 0$, so λ_0 is an eigenvalue of A.

(vii)$\Longrightarrow$(viii): Obvious.

(viii)$\Longrightarrow$(iii): Let λ_0 be an isolated point of $\sigma(A)$ with $\dim \mathcal{R}(P_{\lambda_0}(A)) < \infty$. According to (3.3), $\mathcal{N}((A-\lambda_0)^n) \subset \mathcal{R}(P_{\lambda_0}(A))$ and since $\dim \mathcal{R}(P_{\lambda_0}(A)) < \infty$, we conclude that $\alpha(A-\lambda_0) < \infty$ and $\operatorname{asc}(A-\lambda_0) < \infty$. From (3.4) we have that

$$\mathcal{N}(P_{\lambda_0}(A)) \subset \mathcal{R}((A-\lambda_0)^n)$$

and therefore $\operatorname{codim} \mathcal{R}((A-\lambda_0)^n) \leqslant \operatorname{codim} \mathcal{N}(P_{\lambda_0}(A)) = \dim \mathcal{R}(P_{\lambda_0}(A)) < \infty$ for every $n \in \mathbb{N}$. Consequently, $\beta(A-\lambda_0) < \infty$ and $\operatorname{dsc}(A-\lambda_0) < \infty$.

(viii)$\Longrightarrow$(ix): Suppose that λ_0 is an isolated point of $\sigma(A)$, and let $\dim \mathcal{R}(P_{\lambda_0}(A)) < \infty$. As in the proof for (viii)$\Longrightarrow$(iii) we conclude that $\operatorname{asc}(A-\lambda_0) < \infty$ and $\operatorname{dsc}(A-\lambda_0) < \infty$. From Lemma 3.2 it follows that λ is a pole of the resolvent of A.

(ix)$\Longrightarrow$(viii): Obvious.

(ix)$\Longrightarrow$(x): Follows from Lemma 3.2.

(x)$\Longrightarrow$(ix): Follows from Lemma 3.1.

(v)$\Longrightarrow$(xi): Suppose that λ_0 is an isolated point of $\sigma(A)$, and let $A-\lambda_0 \in \Phi(X)$. There exists $\epsilon > 0$ such that $0 < |\mu - \lambda_0| < \epsilon$ implies $\mu \in \rho(A)$. It follows that $D(\lambda_0, \epsilon) \subset \Phi_\pm(A)$ and $D(\lambda_0, \epsilon) \cap \rho(A) \neq \emptyset$. The set $D(\lambda_0, \epsilon)$ is connected, so there exists the component K of $\Phi_\pm(A)$ such that $D(\lambda_0, \epsilon) \subset K$. Since $K \subset \Phi^R(A)$, we get $\lambda_0 \in \Phi^R(A)$.

(xi)$\Longrightarrow$(vi): Let $\lambda_0 \in \Phi^R(A) \cap \sigma(A)$. There exists the component K of the set $\Phi_\pm(A)$ such that $\lambda_0 \in K$ and $K \cap \rho(A) \neq \emptyset$. From Corollary 2.17 it follows that $A - \lambda_0 \in \mathcal{W}(X)$. Let $\lambda' \in K \cap \rho(A)$. The set K is path connected, so there exists a curve Γ in K starting at λ_0, and ending at λ'. Since $\lambda' \in \rho(A)$, it follows that there exists $\epsilon(\lambda') > 0$, such that $\mu \in D(\lambda', \epsilon(\lambda'))$ implies $\mu \in \rho(A)$, i.e. $\alpha(A-\mu) = \beta(A-\mu) = 0$. Let $\nu \in \Gamma \setminus \{\lambda'\}$. Then $A - \nu$ is semi-Fredholm and from Theorem 2.28 it follows that there exists $\epsilon(\nu) > 0$ such that if $0 < |\mu - \nu| < \epsilon(\nu)$, then $\alpha(A-\mu)$ and $\beta(A-\mu)$ have constant values. The set Γ is compact, so there exists a finite set of points $\nu_1, \nu_2, \ldots, \nu_n \in \Gamma$ such that $\nu_1 = \lambda_0$, $\nu_n = \lambda'$, $\Gamma \subset \cup_{i=1}^n D(\nu_i, \epsilon(\nu_i))$ and $D(\nu_i, \epsilon(\nu_i)) \cap D(\nu_{i+1}, \epsilon(\nu_{i+1})) \neq \emptyset$ for $i = 1, 2, \ldots, n-1$. Since $D(\nu_{n-1}, \epsilon(\nu_{n-1}))$ and $D(\nu_n, \epsilon(\nu_n)) = D(\lambda', \epsilon(\lambda'))$ have nonempty intersection, we conclude that $\alpha(A-\mu)$ and $\beta(A-\mu)$ have the constant value 0 in the set $0 < |\mu - \nu_{n-1}| < \epsilon(\nu_{n-1})$. We continue this method and conclude that if $0 < |\mu - \lambda_0| < \epsilon(\lambda_0)$, then $\alpha(A-\mu) = 0$ and $\beta(A-\mu) = 0$, so $\mu \in \rho(A)$. Consequently, λ_0 is an isolated point of $\sigma(A)$. $\square$

Remark 3.1. If λ_0 is an eigenvalue of $A \in L(X)$, then $\dim \mathcal{N}(A-\lambda_0)$, is the *geometric multiplicity* of λ_0. Moreover, if λ_0 is an isolated point of $\sigma(A)$, then from the inclusion (3.5) it follows that $\alpha(A-\lambda_0) \leqslant \dim \mathcal{R}(P_{\lambda_0}(A))$, so the geometric multiplicity of an eigenvalue λ_0 is not greater then its algebraic multiplicity.

Let λ_0 is a Riesz point of A and the subspaces $F(\lambda_0)$ and $\mathcal{N}(\lambda_0)$ be the same as in Definition 3.1. From Lemma 3.1 and Lemma 3.2 it follows that

$$N(\lambda_0) = \mathcal{R}(P_{\lambda_0}(A)) = \mathcal{N}((A-\lambda_0)^p), \quad F(\lambda_0) = \mathcal{N}(P_{\lambda_0}(A)) = \mathcal{R}((A-\lambda_0)^p),$$

where p is the order of the pole of the resolvent of A at the point λ_0, i.e. $p = \operatorname{asc}(A-\lambda_0) = \operatorname{dsc}(A-\lambda_0)$. From

$$\{0\} \neq \mathcal{N}(A-\lambda_0) \neq \mathcal{N}((A-\lambda_0)^2) \neq \ldots \neq \mathcal{N}((A-\lambda_0)^{p-1}) \neq \mathcal{N}((A-\lambda_0)^p)$$

we conclude that $\mathcal{N}((A-\lambda_0)^p)$ contains at least p linearly independent vectors, so

$$p \leqslant \dim \mathcal{N}((A-\lambda_0)^p) = \dim \mathcal{R}(P_{\lambda_0}(A)) < \infty.$$

We get that the order of the pole of the resolvent of A at the Riesz point λ_0 is not greater then the algebraic multiplicity of λ_0.

Since for the Browder spectrum of $A \in L(X)$ it holds

$$\sigma_{\mathcal{B}}(A) = \{\lambda \in \mathbb{C} : A-\lambda \notin \mathcal{B}(X)\} = \sigma(A) \setminus \{\lambda \in \sigma(A) : A-\lambda \in \mathcal{B}(X)\},$$

according to the equivalency (iii)$\Longleftrightarrow$(viii) in Theorem 3.1, we get that the Browder spectrum of A consists of all points for $\sigma(A)$, except those points $\lambda \in \sigma(A)$ for which the corresponding spectral projection $P_\lambda(A)$ is a finite rank operator, i.e.

$$\sigma_{\mathcal{B}}(A) = \sigma(A) \setminus \{\lambda \in \sigma(A) : \lambda \text{ is an isolated point of } \sigma(A) \text{ and } \dim \mathcal{R}(P_\lambda(A)) < \infty\}.$$

From the equivalency (iii)$\Longleftrightarrow$(x) in Theorem 3.1 it follows that

$$\sigma_{\mathcal{B}}(A) = \sigma(A) \setminus \{\lambda \in \sigma(A) : \lambda \text{ is the Riesz point of } A\} = \sigma(A) \setminus p_{00}(A), \tag{3.14}$$

while from the equivalency (iii)$\Longleftrightarrow$(vii) we obtain that

$$\sigma_{\mathcal{B}}(A) = \sigma(A) \setminus \{\lambda \in \sigma(A) : \lambda \text{ is an isolated eigenvalue of finite algebraic multiplicity}\}.$$

Notice that the set $\sigma(A) \setminus \sigma_{\mathcal{B}}(A)$ is at most countable (it consists of isolated points).

Definition 3.2. $A \in L(X)$ is *Riesz* if every nonzero point of $\sigma(A)$ is a Riesz point of A.

The set of all Riesz operators in $L(X)$ is denoted by $R(X)$.

Corollary 3.1. *An operator $A \in L(X)$ is Riesz if and only if every nonzero point of $\sigma(A)$ is a pole of the finite algebraic multiplicity.*

Proof. Follows from the equivalence (x)$\Longleftrightarrow$(ix) in Theorem 3.1. □

Corollary 3.2. *An operator $A \in L(X)$ is Riesz if and only if every nonzero point of $\sigma(A)$ is isolated in $\sigma(A)$, and its algebraic multiplicity is finite.*

Proof. Follows from the equivalence (x)$\Longleftrightarrow$(viii) in Theorem 3.1. □

Theorem 3.2. *Let $A \in L(X)$. The following statements are equivalent:*

(i) *A is Riesz.*
(ii) $\{\lambda \in \mathbb{C} : A-\lambda \in \mathcal{B}(X)\} = \mathbb{C} \setminus \{0\}$.
(iii) $\{\lambda \in \mathbb{C} : A-\lambda \in \mathcal{W}(X)\} = \mathbb{C} \setminus \{0\}$.
(iv) $\Phi(A) = \mathbb{C} \setminus \{0\}$.
(v) $\{\lambda \in \mathbb{C} : A-\lambda \in \mathcal{B}_+(X)\} = \mathbb{C} \setminus \{0\}$.
(vi) $\{\lambda \in \mathbb{C} : A-\lambda \in \mathcal{B}_-(X)\} = \mathbb{C} \setminus \{0\}$.
(vii) $\{\lambda \in \mathbb{C} : A-\lambda \in \mathcal{W}_+(X)\} = \mathbb{C} \setminus \{0\}$.
(viii) $\{\lambda \in \mathbb{C} : A-\lambda \in \mathcal{W}_-(X)\} = \mathbb{C} \setminus \{0\}$.
(ix) $\{\lambda \in \mathbb{C} : A-\lambda \in \Phi_+(X)\} = \mathbb{C} \setminus \{0\}$.

(x) $\{\lambda \in \mathbb{C} : A - \lambda \in \Phi_-(X)\} = \mathbb{C} \setminus \{0\}$.
(xi) $\Phi_\pm(A) = \mathbb{C} \setminus \{0\}$.

Proof. (i)$\Longrightarrow$(ii): Let $A \in R(X)$ and let $\lambda \neq 0$. If $\lambda \in \sigma(A)$, then λ is a Riez point of A. By Theorem 3.1, $A-\lambda$ is Browder. If $\lambda \in \rho(A)$, then obviously, $A-\lambda$ is Browder. Therefore, $\mathbb{C} \setminus \{0\} \subset \{\lambda \in \mathbb{C} : A - \lambda \in \mathcal{B}(X)\}$. Since the space X is infinite dimensional and $\{\lambda \in \mathbb{C} : A - \lambda \in \mathcal{B}(X)\} \subset \Phi(A)$, from Corollary 2.8 it follows that $\{\lambda \in \mathbb{C} : A - \lambda \in \mathcal{B}(X)\} \neq \mathbb{C}$. Therefore, $\{\lambda \in \mathbb{C} : A - \lambda \in \mathcal{B}(X)\} = \mathbb{C} \setminus \{0\}$.

(ii)$\Longrightarrow$(iv): Follows from the inclusion $\{\lambda \in \mathbb{C} : A - \lambda \in \mathcal{B}(X)\} \subset \Phi(A)$ and Corollary 2.8.

(iv)$\Longrightarrow$(i): Suppose that $\Phi(A) = \mathbb{C} \setminus \{0\}$. We prove that every nonzero point of $\sigma(A)$ is a Riesz point of A, which means that A is Riesz.

Let $\lambda \in \sigma(A)$ and $\lambda \neq 0$. Since $\Phi(A)$ is equal to $\mathbb{C} \setminus \{0\}$, it is a connected subset of $\Phi_\pm(A)$ which intersects $\rho(A)$ and therefore $\Phi(A) \subset \Phi^R(A)$. Hence, from $\lambda \neq 0$ it follows that $\lambda \in \Phi(A)$ and consequently, $\lambda \in \Phi^R(A)$. Therefore, $\lambda \in \sigma(A) \cap \Phi^R(A)$ and so, λ is a Riesz point of A by Theorem 3.1.

The equivalences (iii)$\Longleftrightarrow$(iv)$\Longleftrightarrow$(vii)$\Longleftrightarrow$(viii)$\Longleftrightarrow$(ix)$\Longleftrightarrow$(x)$\Longleftrightarrow$(xi) follow from Corollary 2.17.

The implications (ii)$\Longrightarrow$(v)$\Longrightarrow$(ix) and (ii)$\Longrightarrow$(vi)$\Longrightarrow$(x) are clear. □

Remark 3.2. For infinite dimensional space X, according to Theorem 3.2, if $A \in R(X)$, then $0 \notin \Phi_\pm(A)$, i.e. A is not semi-Fredholm operator. In other words, the set of Riesz operators does not intersect the set of semi-Fredholm operators. Clearly, if X is finite dimensional, then $L(X) = R(X) = \Phi(X) = \Phi_+(X) = \Phi_-(X)$ and all sets on the left side of the equalities in Theorem 3.2 are equal to $\mathbb{C}$.

So, for infinite-dimensional space X and $A \in L(X)$, according to Theorem 3.2, the following equivalence holds:

$$A \in R(X) \Longleftrightarrow \theta(A) = \{0\}, \tag{3.15}$$

where θ is one of σ_Φ, σ_{Φ_+}, σ_{Φ_-}, $\sigma_{\mathcal{W}}$, $\sigma_{\mathcal{W}_+}$, $\sigma_{\mathcal{W}_-}$, $\sigma_{\mathcal{B}}$, $\sigma_{\mathcal{B}_+}$, $\sigma_{\mathcal{B}_-}$.

Hence for a bounded linear operator A acting on arbitrary Banach space (finite dimensional or infinite dimensional), it follows that A is Riesz if and only if $\theta(A) \subset \{0\}$, where θ is as above.

Corollary 3.3. *An operator $A \in L(X)$ is Riesz if and only if $\pi(A)$ is quasinilpotent in the Calkin algebra $C(X)$.*

Proof. From Theorem 3.2 and Corollary 2.7. □

Corollary 3.4. *Every compact operator $A \in L(X)$ is Riesz.*

Proof. Follows from Corollary 3.3. □

Corollary 3.5. *Every quasinilpotent operator $A \in L(X)$ is Riesz.*

Proof. Let $A \in L(X)$ is quasinilpotent. Since $\sigma(\pi(A)) \subset \sigma(A)$, we get $\sigma(\pi(A)) = \{0\}$, i.e. $\pi(A)$ is quasinilpotent in $C(X)$. From Corollary 3.3 it follows that A is Riesz. □

The following example shows that there exists a quasinilpotent operator which is not compact.

Example 3.1. Let X^2 be a Banach space endowed with the norm $\|(x,y)\| = \|x\| + \|y\|$, and let $T \in L(X^2)$ be defined as $T(x,y) = (y,0)$. It is easy to see that T is not compact. Since $T^2 = 0$, we get that T is quasinilpotent.

Corollary 3.6. *The sum of a compact operator and a quasinilpotent operator is Riesz.*

Proof. Let $K \in K(X)$ and $A \in L(X)$ is quasinilpotent. Since $\pi(A+K) = \pi(A)$ is quasinilpotent in $C(X)$, it follows that $A + K \in R(X)$. □

The opposite question: can every Riesz operator be represented as a sum of a compact and a quasinilpotent operator, is less trivial. Precisely, this is still an open question if X is a Banach space. If X is a Hilbert space, T. T.West [45] gave the affirmative solution to this problem.

Corollary 3.7. *If A is Riesz, then $0 \in \sigma(A)$. Moreover, $\sigma(A)$ is at most countable and $0 \in \sigma(A)$ is the only one possible point of accumulation.*

Proof. Let $A \in R(X)$. From Theorem 3.2 it follows that $\Phi(A) = \mathbb{C} \setminus \{0\}$. Since $\rho(A) \subset \Phi(A)$, we get $0 \in \sigma(A)$. Every nonzero point of $\sigma(A)$ is a Riesz point of A, so it is an isolated point of $\sigma(A)$ (Theorem 3.1). It follows that $\sigma(A)$ is at most countable and 0 is the only possible point of accumulation of $\sigma(A)$. □

Corollary 3.8. *Let $A \in R(X)$, $\lambda \neq 0$ and $\lambda \in \sigma(A)$. Then λ is an eigenvalue of A and λ is a pole of a resolvent of A. The spectral projection $P_\lambda(A)$, which corresponds to λ, is a finite rank operator, and $\dim \mathcal{N}(A-\lambda) \leqslant \dim \mathcal{R}(P_\lambda(A))$.*

Moreover, let $p(\lambda)$ be the order of the pole of the resolvent of A at λ. Then

$$\operatorname{asc}(A-\lambda) = \operatorname{dsc}(A-\lambda) = p(\lambda),$$

$$\mathcal{R}(P_\lambda(A)) = \mathcal{N}((A-\lambda)^{p(\lambda)}) \quad \text{and} \quad \mathcal{N}(P_\lambda(A)) = \mathcal{R}((A-\lambda)^{p(\lambda)}),$$

$$1 \leqslant p(\lambda) \leqslant \dim \mathcal{R}(P_\lambda(A)).$$

Proof. Every nonzero point of $\sigma(A)$ is a Riesz point of A, so the statements follow from Remark 3.1. □

Corollary 3.9. *Let $A \in R(X)$. Then the only possible cases are as follows:*

(i) $\sigma(A) = \{0\}$,

(ii) $\sigma(A) = \{0, \lambda_1, \dots, \lambda_n\}$, *where for* $1 \leqslant k \leqslant n$, $\lambda_k \neq 0$, *every* λ_k *is an eigenvalue of* A *and* $\dim \mathcal{N}(A-\lambda_k) < \infty$,

(iii) $\sigma(A) = \{0, \lambda_1, \lambda_2, \dots\}$, *where for every* $k \geqslant 1$, $\lambda_k \neq 0$, *every* λ_k *is an eigenvalue of* A, $\dim \mathcal{N}(A-\lambda_k) < \infty$, *and* $\lim_{k\to\infty} \lambda_k = 0$.

Proof. Follows from Corollary 3.7 and Corollary 3.8. □

Corollary 3.10. *The operator $A \in L(X)$ is Riesz if and only if $A' \in L(X')$ is Riesz.*

Proof. Since $\Phi(A) = \Phi(A')$ (Proposition 2.8), the statement follows from Theorem 3.2. $\square$

Theorem 3.3. *Let $A \in L(X)$ be Riesz and let $0 \neq \lambda \in \sigma(A)$. Then*

$$\dim \mathcal{N}((A-\lambda)^n) = \dim \mathcal{N}((A'-\lambda)^n), \quad \textit{for every } n \in \mathbb{N}. \tag{3.16}$$

Moreover, the ranges of the spectral projections $P_\lambda(A)$ and $P_\lambda(A')$ have the same dimension.

Proof. Let $A \in R(X)$ and $0 \neq \lambda \in \sigma(A)$. Then λ is a Riesz point of A. According to Theorem 3.1 it follows that $A-\lambda \in \mathcal{W}(X)$, so $(A-\lambda)^n \in \mathcal{W}(X)$ for every $n \in \mathbb{N}$, by Theorem 2.2. Thus $\alpha((A-\lambda)^n) = \beta((A-\lambda)^n) < \infty$ and $\mathcal{R}((A-\lambda)^n)$ is closed. From that, using Proposition 2.4, we get (3.16).

Let p be the order of the pole of A at λ. From Lemma 3.2 and Lemma 2.12 it follows that

$$\operatorname{asc}(A-\lambda) = \operatorname{dsc}(A-\lambda) = \operatorname{asc}(A'-\lambda) = \operatorname{dsc}(A'-\lambda) = p,$$
$$\mathcal{R}(P_\lambda(A)) = \mathcal{N}((A-\lambda)^p) \quad \text{and} \quad \mathcal{R}(P_\lambda(A')) = \mathcal{N}((A'-\lambda)^p).$$

Now, from (3.16) we get $\dim \mathcal{R}(P_\lambda(A)) = \dim \mathcal{R}(P_\lambda(A'))$. $\square$

Corollary 3.11. *$I(X)$ is the greatest ideal contained in $R(X)$.*

Proof. Since $\operatorname{Rad}(C(X)) \subset \operatorname{QN}(C(X))$ it follows that $I(X) = \pi^{-1}(\operatorname{Rad}(C(X))) \subset \pi^{-1}(\operatorname{QN}(C(X))) = R(X)$. Let J be a left (right) ideal and $J \subset R(X)$. Since π is onto, $\pi(J)$ is a left (right) ideal in the Calkin algebra $C(X)$. By Corollary 3.3, $\pi(J)$ is contained in the set of all quasinilpotent elements of $C(X)$. Using Lemma 2.32 we find $\pi(J) \subset \operatorname{Rad}\,(C(X))$, so $J \subset \pi^{-1}(\pi(J)) \subset \pi^{-1}(\operatorname{Rad}(C(X))) = I(X)$. $\square$

The following theorem shows that the ideal of all compact operators or the ideal of all finite rank operators in the Atkinson theorem can be replaced with any nonzero two-sided ideal contained in the set of Riesz operators.

Theorem 3.4. *For $A \in L(X)$ the following conditions are equivalent:*

(i) $A \in \Phi(X)$.

(ii) *There exist operators $A_1,\ A_2 \in L(X)$, $J_1 \in I(X)$ and $J_2 \in I(Y)$ such that*

$$A_1 A = I + J_1, \quad AA_2 = I + J_2.$$

Proof. (i)$\Longrightarrow$(ii): Follows from Theorem 2.15 and and the fact that $F(X) \subset I(X)$.

(ii)$\Longrightarrow$(i): Suppose that there exist operators $A_1,\ A_2 \in L(X)$, $J_1 \in I(X)$ and $J_2 \in I(X)$ such that $A_1A = I + J_1$, $\quad AA_2 = I + J_2$. Since $I(X) = P(\Phi(X))$, it follows that $I + J_1 \in \Phi(X)$ and $I + J_2 \in \Phi(X)$, i.e. $A_1A \in \Phi(X)$ and $AA_2 \in \Phi(X)$. By Theorem 2.3 we get $A \in \Phi(X)$. $\square$

Let π_I be the natural mapping from $L(X)$ onto the quotient algebra $L(X)/I(X)$. The next corollary shows that the set of all Fredholm operators in $L(X)$ is the inverse image of the set of all invertible elements in the quotient algebra $L(X)/I(X)$.

Corollary 3.12. *The operator $A \in L(X)$ is Fredholm if and only if $\pi_J(A)$ is invertible in the quotient algebra $L(X)/J(X)$.*

Proof. Follows from Theorem 3.4. □

Corollary 3.13. *Let $A \in L(X)$. Then $\sigma(\pi_I(A)) = \mathbb{C} \setminus \Phi(A)$.*

Proof. Follows from Corollary 3.12. □

Corollary 3.14. *$A \in L(X)$ is Riesz if and only if $\pi_I(A)$ is quasinilpotent in $L(X)/I(X)$.*

Proof. From Theorem 3.2 and Corollary 3.13 it follows that A is Riesz if and only if $\sigma(\pi_J(A)) = \{0\}$. □

Theorem 3.5. (i) *If $A \in R(X)$ and $\lambda \in \mathbb{C}$, then $\lambda A \in R(X)$.*
(ii) *If $A \in R(X)$, $B \in L(X)$ and $AB - BA \in I(X)$, then $AB, BA \in R(X)$.*
(iii) *If A, $B \in R(X)$ and $AB - BA \in I(X)$, then $A + B \in R(X)$.*
(iv) *If $A \in L(X)$ is Riesz, f is a complex analytic function in a neighborhood of $\sigma(A)$ and $f(0) = 0$, then $f(A)$ is Riesz.*

Proof. (ii): Let $A \in R(X)$ and let π_I be the quotient map from $L(X)$ onto $L(X)/I(X)$. By Corollary 3.14, $r(\pi_I(A)) = 0$. From $AB - BA \in I(X)$ it follows that $\pi_I(A)\pi_I(B) = \pi_I(B)\pi_I(A)$ and hence

$$r(\pi_I(AB)) = r(\pi_I(A)\pi_I(B)) \leqslant r(\pi_I(A))r(\pi_I(B)) = 0.$$

Thus $\pi_I(AB)$ is quasinilpotent in $L(X)/I(X)$, so $AB \in R(X)$.

(iv): Since f is a complex analytic function in a neighborhood of $\sigma(A)$ and $f(0) = 0$, then there exists a complex analytic function h in a neighborhood of $\sigma(A)$ such that $f(\lambda) = \lambda h(\lambda)$. Hence $f(A) = Ah(A) = h(A)A$, and since A is Riesz, from (ii) it follows that $f(A)$ is Riesz. □

The sum and the product of two Riesz operators in not necessarily a Riesz operator.

Example 3.2. Let X^2 and T be defined as in Example 3.5, and let $S \in L(X^2)$ be defined as $S(x, y) = (0, x)$. Since $T^2 = S^2 = 0$, we get that T and S are Riesz operators, but the operator $(S + T)(x, y) = (y, x)$ is invertible so it is not Riesz. Further, $ST(x, y) = (0, y)$ and $TS(x, y) = (x, 0)$. Since $\dim \mathcal{N}(I - ST) = \dim \mathcal{N}(TS) = \dim X = \infty$, we get $I - ST \notin \Phi(X^2)$, i.e. $1 \notin \Phi(ST)$. From Theorem 3.2 it follows that ST is not Riesz. Analogously, $TS \notin R(X^2)$.

We need the following result.

Lemma 3.3. *Let $(a_n)_n$ be a sequence of quasinilpotent elements of a Banach algebra $\mathcal{A}$. If there exists $a \in \mathcal{A}$ such that $\lim_{n\to\infty} a_n = a$ and $a_n a = a a_n$, $n = 1, 2, \ldots$, then a is quasinilpotent also.*

Proof. Since $a - a_n$ commutes with a_n, $n \in \mathbb{N}$, we get

$$r(a) = r(a - a_n + a_n) \leqslant r(a - a_n) + r(a_n) = r(a - a_n) \leqslant \|a - a_n\|.$$

Letting $n \to \infty$ gives $r(a) = 0$. □

Theorem 3.6. *Let $(A_n)_n$ be a sequence of Riesz operators in $L(X)$. Suppose that $\lim_{n\to\infty} A_n = A$ and $A_n A - A A_n \in I(X)$ for every $n \in \mathbb{N}$. Then A is Riesz.*

Proof. Follows from Lemma 3.3 and Corollary 3.14. □

In general, $R(X)$ is not closed in $L(X)$.

Theorem 3.7. *([9, Lemma 3.5.1]) Let $A \in L(X)$ be Riesz, let M be a closed subspace of X, such that M is invariant for A. Then the reduction $A_M \colon M \to M$ is Riesz.*

Proof. We prove that for every $\lambda \in \rho(A)$ the equality $(\lambda - A)M = M$ holds. Since $(\lambda - A)M \subset M$ obviously holds, we prove $M \subset (\lambda - A)M$, $\lambda \in \rho(A)$. To show this, it is sufficient to prove $(\lambda - A)^{-1}M \subset M$, $\lambda \in \rho(A)$. Let $|\lambda| > r(A)$. Then $(\lambda - A)^{-1} = \sum_{n=0}^{\infty} \lambda^{-(n+1)} A^n$. Since M is closed and M is invariant for A, we conclude that $(\lambda - A)^{-1}M \subset M$. Let $x \in M$ and $x^* \in M^{\perp}$. Since $x^*((\lambda - A)^{-1}x) = 0$ for every λ such that $|\lambda| > r(A)$, and $\rho(A)$ is connected (because A is Riesz), we conclude that the analytic function $\lambda \mapsto x^*((\lambda - A)^{-1}x)$ is equal to zero in the domain $\rho(A)$, i.e. $x^*((\lambda - A)^{-1}x) = 0$ for every $\lambda \in \rho(A)$. It follows that $(\lambda - A)^{-1}x \in {}^{\perp}(M^{\perp}) = M$ (since M is closed, from [43, Chapter III, Lemma 3.2] it follows that ${}^{\perp}(M^{\perp}) = M$). Consequently, $(\lambda - A)^{-1}M \subset M$ for every $\lambda \in \rho(A)$, so $(\lambda - A)M = M$, for every $\lambda \in \rho(A)$. Hence, from $\lambda \in \rho(A)$ we get $\lambda \in \rho(A_M)$, or

$$\sigma(A_M) \subset \sigma(A). \tag{3.17}$$

For $\lambda \in \rho(A)$ we have $(\lambda - A_M)^{-1} = (\lambda - A)^{-1}_{|M}$. Let $\lambda_0 \in \sigma(A_M)$ and $\lambda_0 \neq 0$. From (3.17) we conclude $\lambda_0 \in \sigma(A)$, so λ_0 is an isolated point of $\sigma(A)$, so it is isolated in $\sigma(A_M)$. Let $P_M \in L(M)$ be the spectral projection corresponding to λ_0 and A_M, and let $P_0 \in L(X)$ be the spectral projection corresponding to λ_0 and A. We obtain

$$M = \frac{1}{2\pi i}\int_c (\lambda - A_M)^{-1} d\lambda = \frac{1}{2\pi i}\int_c (\lambda - A)^{-1}_{|M} d\lambda = \left[\frac{1}{2\pi i}\int_c (\lambda - A)^{-1} d\lambda\right]_{|M} = P_{0|M}$$

where c is the circle $|\lambda - \lambda_0| = r$, and r is small enough. Since A is Riesz, we get that P_0 is finite rank (Corollary 3.2). Consequently, P_M is finite rank. From Corollary 3.2 it follows that A_M is Riesz. □

4. Polynomially Riesz operators

We shall write $H +_{\text{comm}} K = \{c + d : (c, d) \in H \times K \;\; cd = dc\}$ for the commuting sum and $H \cdot_{\text{comm}} K = \{cd : (c, d) \in H \times K \;\; cd = dc\}$ for the commuting product of subsets $H, K \subseteq A$.

We say that $S \subseteq A$ is a *commutative ideal* if $S +_{\text{comm}} S \subseteq S$, $A \cdot_{\text{comm}} S \subseteq S$.

We shall write $\text{Poly} = \mathbf{C}[z]$ for the algebra of complex polynomials. If $S \subseteq A$ is an arbitrary set we shall write that $a \in \text{Poly}^{-1}(S)$ if there exists a nonzero complex polynomial $p(z)$ such that $p(a) \in S$. If $S \subseteq A$ is a commutative ideal, the set

$$\mathcal{P}_a^S = \{p \in \text{Poly} : p(a) \in S\}$$

of polynomials p for which $p(a) \in S$ will be an ideal of the algebra Poly. Since the natural numbers are well ordered there will be a unique polynomial p of minimal degree with leading coefficient 1 contained in $\mathcal{P}_a^S$ which we call the minimal polynomial of a; we shall write $p = m_a \equiv m_a^S$. Then $\mathcal{P}_a^S$ is generated by $p = m_a$, i.e. $\mathcal{P}_a^S = m_a \cdot \mathrm{Poly}$.

According to Theorem 3.5 ((ii), (iii)), we conclude that the set of Riesz operators $R(X)$ is a commutative ideal in the algebra $L(X)$.

We shall say that an operator $A \in L(X)$ is *polynomially Riesz* and write $A \in \mathrm{Poly}^{-1}R(X)$ if there exists a nonzero complex polynomial $p(z)$ such that $p(A) \in R(X)$.

Lemma 4.1. *If $A, B \in L(X)$ commute, A is Fredholm and AB is Riesz, then B is Riesz.*

Proof. Suppose that $AB = BA \in R(X)$ and $A \in \Phi(X)$. According to Corollary 2.7, $\pi(A)$ is invertible in the Calkin algebra $C(X)$, while $\pi(A)\pi(B) = \pi(AB)$ is quasinilpotent in $C(X)$ (Corollary 3.3) and $(\pi(A))^{-1}$ commute with $\pi(B)$ and hence also with $\pi(A)\pi(B)$. It follows that $\pi(B) = (\pi(A))^{-1}\pi(A)\pi(B)$ is quasinilpotent in $C(X)$, and so, according to Corollary 3.3, we conclude that $B \in R(X)$. □

We remark that the assertion of the previous lemma also holds if the condition of commutativity of operators A and B is replaced by a weaker condition that $AB - BA$ belongs to the perturbation class of the set of Fredholm operators $\mathrm{Ptrb}(\Phi(X))$.

Theorem 4.1. *Let $A \in L(X)$. Then $A \in \mathrm{Poly}^{-1}R(X)$ if and only if $\sigma_{\mathcal{B}}(A)$ is finite and in that case $\sigma_{\mathcal{B}}(A) = m_A^{-1}(0)$ where m_A is the minimal polynomial of A.*

Proof. Suppose that $A \in \mathrm{Poly}^{-1}R(X)$. Then $m_A(A) \in R(X)$ and from (3.15) and [15] it follows that $m_A(\sigma_{\mathcal{B}}(A)) = \sigma_{\mathcal{B}}(m_A(A)) = \{0\}$, and therefore,

$$\sigma_{\mathcal{B}}(A) \subset m_A^{-1}(0). \tag{4.1}$$

To prove the opposite suppose that $\sigma_{\mathcal{B}}(A)$ is finite and let $\sigma_{\mathcal{B}}(A) = \{\lambda_1, \dots, \lambda_n\}$. For $p(z) = (z - \lambda_1) \cdot \dots \cdot (z - \lambda_n)$ we have $\{0\} = p(\sigma_{\mathcal{B}}(A)) = \sigma_{\mathcal{B}}(p(A))$, and so, $p(A) \in R(X)$ by (3.15).

Let $A \in \mathrm{Poly}^{-1}R(X)$ and let λ be a zero of the minimal polynomial m_A. Then $m_A(z) = (z - \lambda)q(z)$ and therefore, $m_A(A) = (A - \lambda)q(A) = q(A)(A - \lambda) \in R(X)$. We show that $\lambda \in \sigma_\Phi(A)$. If $\lambda \notin \sigma_\Phi(A)$, then $A - \lambda$ is Fredholm, and from Lemma 4.1 it follows that $q(A) \in R(X)$ which contradicts the fact that the polynomial m_A is minimal. Therefore, $m_A^{-1}(0) \subset \sigma_\Phi(A)$, which together with (4.1) gives $\sigma_\Phi(A) = \sigma_{\mathcal{B}}(A) = m_A^{-1}(0)$. This completes the proof. □

The *connected hull* of a compact subset K of the complex plane $\mathbb{C}$, denoted by ηK, is the complement of the unbounded component of $\mathbb{C} \setminus K$ [18, Definition 7.10.1]. Given a compact subset K of the plane, a hole of K is a bounded component of $\mathbb{C} \setminus K$, and so a hole of K is a component of $\eta K \setminus K$.

We recall that, for compact subsets $H, K \subset \mathbb{C}$, the following implication holds [18, Theorem 7.10.3]:

$$\partial H \subset K \subset H \Longrightarrow \partial H \subset \partial K \subset K \subset H \subset \eta K = \eta H. \tag{4.2}$$

Evidently, if $K \subseteq \mathbb{C}$ is at most countable, then $\eta K = K$. Therefore, for compact subsets $H, K \subseteq \mathbb{C}$, if $\eta K = \eta H$, then H is at most countable if and only if K is at most countable, and in that case $H = K$.

An operator $A \in L(X)$ is called *essentially Kato* if $\mathcal{R}(A)$ is closed and $k(A) = \dim \mathcal{N}(A)/(\mathcal{N}(A) \cap \mathcal{R}(A^\infty)) < \infty$ [36, Definition 21.4, Theorem 21.3]; [39, Theorem 2.1]. Clearly, every semi-Fredholm operator is an essentially Kato operator. The corresponding spectrum is $\sigma_{Ke}(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not essentially Kato}\}$. In [39] this spectrum was investigated and called Browder's essential generalized spectrum of A.

Racall the following result for $A \in L(X)$ [36, Theorem 21.11]:

$$\partial \sigma_\Phi(A) \subset \sigma_{Ke}(A) \subset \sigma_\Phi(A).$$

Also we recall the following result.

Theorem 4.2. [47, Theorem 10] *If $T \in L(X)$, then for each $* = +, -, l, r$ there are inclusions*

$$\partial \sigma_{\mathcal{B}}(T) \subset \partial \sigma_{\mathcal{B}_*}(T) \subset \partial \sigma_{\mathcal{W}_*}(T) \subset \partial \sigma_{\Phi_*}(T) \subset \sigma_{\mathcal{W}_*}(T) \subset \sigma_{\mathcal{B}_*}(T) \subset \sigma_{\mathcal{B}}(T).$$

Theorem 4.3. *Let $A \in L(X)$. If θ is one of σ_{Ke}, σ_Φ, σ_{Φ_+}, σ_{Φ_-}, σ_{Φ_l}, σ_{Φ_r}, $\sigma_{\mathcal{W}}$, $\sigma_{\mathcal{W}_+}$, $\sigma_{\mathcal{W}_-}$, $\sigma_{\mathcal{W}_l}$, $\sigma_{\mathcal{W}_r}$, $\sigma_{\mathcal{B}}$, $\sigma_{\mathcal{B}_+}$, $\sigma_{\mathcal{B}_-}$, $\sigma_{\mathcal{B}_l}$, $\sigma_{\mathcal{B}_r}$, then*

$$A \in \mathrm{Poly}^{-1}R(X) \iff \theta(A) \text{ is finite},$$

and in that case $\theta(A) = m_A^{-1}(0)$ where m_A is the minimal polynomial of A.

Proof. From Theorem 4.2 and (4.2) it follows that if one of the mentioned spectra is finite, then any other of them is also finite and they are equal. Now the rest of the assertion follows from Theorem 4.1. □

Corollary 4.1. *If $A \in \mathrm{Poly}^{-1}R(X)$, then for all $\lambda \in m_A^{-1}(0)$, $A - \lambda$ is not essentially Kato, and for all $\lambda \notin m_A^{-1}(0)$, $A - \lambda$ is Browder.*

Proof. Follows from Theorem 4.3. □

Corollary 4.2. *If $A \in \mathrm{Poly}^{-1}R(X)$, then $\sigma(A)$ is at most countable.*

Proof. From (2.47) it follows that the set $\sigma(A) \setminus \sigma_{\mathcal{B}}(A)$ consists of the isolated points of $\sigma(A)$ and consequently, it is at most countable. Since, by Theorem 4.1, $\sigma_{\mathcal{B}}(A)$ is finite, it follows that $\sigma(A)$ is at most countable. □

We recall the following concept introduced by A. Jeribi and N. Moalla in [20, Definition 1.2],:

Definition 4.1. An operator $A \in L(X)$ is called *generalized Riesz* if there exists E a finite subset of $\mathbb{C}$ such that

(1) For all $\lambda \in \mathbb{C} \setminus E$, $A - \lambda$ is a Fredholm operator on X.
(2) For all $\lambda \in \mathbb{C} \setminus E$, $A - \lambda$ has finite ascent and finite descent.
(3) All $\lambda \in \sigma(A) \setminus E$ are eigenvalues of finite multiplicity, and have no accumulation point except possibly points of E.

Theorem 4.4. *Let $A \in L(X)$. Then A is generalized Riesz if and only if A has the finite Browder spectrum.*

Proof. Suppose that A is generalized Riesz. Then there exists a finite subset E of $\mathbb{C}$ such that for every $\lambda \in \mathbb{C} \setminus E$, $A - \lambda$ is a Fredholm operator on X with finite ascent and finite descent, i.e. $A - \lambda$ is Browder. Therefore, $\sigma_{\mathcal{B}}(A) \subset E$ and hence, $\sigma_{\mathcal{B}}(A)$ is a finite subset of $\mathbb{C}$.

To prove the converse, suppose that $\sigma_{\mathcal{B}}(A)$ is a finite subset of $\mathbb{C}$. For all $\lambda \in \mathbb{C} \setminus \sigma_{\mathcal{B}}(A)$, $A - \lambda$ is Browder. Let $\lambda \in \sigma(A) \setminus \sigma_{\mathcal{B}}(A)$. Then $A - \lambda$ is a Weyl operator (Proposition 2.7) and $\mathrm{asc}(A - \lambda) = \mathrm{dsc}(A - \lambda) = p < \infty$. From $\mathrm{ind}(A - \lambda) = 0$ it follows that $\alpha(A - \lambda) = \beta(A - \lambda)$ and since $\lambda \in \sigma(A)$ we get $\alpha(A - \lambda) > 0$, i.e. λ is an eigenvalue of A. Since $\mathcal{N}^\infty(A - \lambda) = \mathcal{N}((A - \lambda)^p)$ and $(A - \lambda)^p \in \Phi(X)$ (Theorem 2.2) we get $\dim \mathcal{N}^\infty(A - \lambda) < \infty$. From (2.47) it follows that all $\lambda \in \sigma(A) \setminus \sigma_{\mathcal{B}}(A)$ have no accumulation point except possibly points of $\sigma_{\mathcal{B}}(A)$. Therefore, A is generalized Riesz with $E = \sigma_{\mathcal{B}}(A)$. □

Corollary 4.3. *Let $A \in L(X)$. Then A is generalized Riesz if and only if A is polynomially Riesz.*

Proof. Follows from Theorem 4.1 and Theorem 4.4. □

We recall the following result which was proved by H. Baklouti [6, Theorem 2.1]:

Theorem 4.5. *Let $A \in L(X)$ and let p be a non-zero complex polynomial. If μ be in $\sigma(p(A)) \setminus \beta_{ess}(p(A))$, then μ is an eigenvalue of $p(A)$ and*

$$\mathcal{N}^\infty(\mu - p(A)) = \bigoplus_{\substack{\lambda \in \sigma(A) \\ p(\lambda) = \mu}} \mathcal{N}^\infty(\lambda - A).$$

In particular, $\mathrm{mult}(p(A), \mu) = \sum_{\substack{\lambda \in \sigma(A) \\ p(\lambda) = \mu}} \mathrm{mult}(A, \lambda)$.

The following two results extend in [6, Corollary 2.1] and [20, Proposition 3.1].

Theorem 4.6. *Let $A \in \mathrm{Poly}^{-1} R(X)$ and let $\mu \in \sigma(m_A(A)) \setminus \{0\}$. Then*

$$\mathcal{N}^\infty(\mu - m_A(A)) = \bigoplus_{\substack{\lambda \in \sigma(A) \\ m_A(\lambda) = \mu}} \mathcal{N}^\infty(\lambda - A),$$

$$\mathrm{mult}(m_A(A), \mu) = \sum_{\substack{\lambda \in \sigma(A) \\ m_A(\lambda) = \mu}} \mathrm{mult}(A, \lambda).$$

Proof. From $m_A(A) \in R(X)$ it follows that $\sigma_{\mathcal{B}}(m_A(A)) = \{0\}$ and the assertion follows from Theorem 4.5. □

Theorem 4.7. *Let $A \in L(X)$. Then $A \in \mathrm{Poly}^{-1} R(X)$ if and only if $A' \in \mathrm{Poly}^{-1} R(X')$ and the minimal polynomial of A is equal to the minimal polynomial of A'.*

Moreover, if $A \in \mathrm{Poly}^{-1} R(X)$, then for every $\lambda \in \sigma(A) \setminus m_A^{-1}(0)$,

$$\mathrm{mult}(A, \lambda) = \mathrm{mult}(A', \lambda).$$

Proof. For a nonzero complex polynomial $p(z)$, from Corollary 3.10 it follows that $p(A) \in R(X)$ if and only if $p(A') = p(A)' \in R(X')$. Therefore, $A \in \mathrm{Poly}^{-1}R(X)$ if and only if $A' \in \mathrm{Poly}^{-1}R(X')$ and $m_A = m_{A'}$.

Let $A \in \mathrm{Poly}^{-1}R(X)$ and let $\lambda \in \sigma(A) \setminus m_A^{-1}(0)$. From Theorem 4.1 it follows that $\lambda \in \sigma(A) \setminus \sigma_{\mathcal{B}}(A)$, and hence $A - \lambda$ is Browder and λ is an eigenvalue of A. Consequently, $\mathcal{N}^\infty(A - \lambda) = \mathcal{N}((A - \lambda)^p)$ and $\mathcal{N}^\infty(A' - \lambda) = \mathcal{N}((A' - \lambda)^p)$ where $p = \mathrm{asc}(A - \lambda) = \mathrm{asc}(A' - \lambda)$. Since $A - \lambda$ is Weyl, it follows that $(A - \lambda)^p$ is Weyl (Theorem 2.2). Hence and according to Proposition 2.4, it follows

$$\begin{aligned}\mathrm{mult}(A, \lambda) &= \dim \mathcal{N}((A - \lambda)^p) = \alpha((A - \lambda)^p) = \beta((A - \lambda)^p)\\ &= \alpha((A' - \lambda)^p) = \dim \mathcal{N}((A' - \lambda)^p) = \mathrm{mult}(A', \lambda).\end{aligned}$$

□

Gilfeather [11, Theorem 1] proved that if A is a polynomially compact operator on a Banach space, then every λ zero of the minimal polynomial of A is the limit of the eigenvalues of A or else there exists a closed infinite dimensional A-invariant subspace X_λ such that the reduction of $A - \lambda$ on X_λ is quasinilpotent. We shall show that polynomially Riesz operators have the same property. In order to prove this we need the following simple assertion.

Lemma 4.2. *Let $A \in L(X)$ and let $X = X_1 \oplus \cdots \oplus X_n$ where X_i is a closed A-invariant subspace of X and $A = A_1 \oplus \cdots \oplus A_n$ where A_i is the reduction of A on X_i, $i = 1, \ldots, n$. Then*

$$\sigma_\Phi(A) = \sigma_\Phi(A_1) \cup \cdots \cup \sigma_\Phi(A_n), \tag{4.3}$$

and A is Riesz if and only if $A_1, \ldots, A_n$ are Riesz.

Proof. For $\lambda \in \mathbb{C}$, from

$$\begin{aligned}\mathcal{N}(A - \lambda) &= \mathcal{N}(A_1 - \lambda) \oplus \cdots \oplus \mathcal{N}(A_n - \lambda),\\ \mathcal{R}(A - \lambda) &= \mathcal{R}(A_1 - \lambda) \oplus \cdots \oplus \mathcal{R}(A_n - \lambda)\end{aligned}$$

it follows that $A - \lambda$ is Fredholm if and only if $A_i - \lambda$ is Fredholm, $i = 1, \ldots, n$. This implies (4.3) and therefore, $\sigma_\Phi(A) = \{0\}$ if and only if $\sigma_\Phi(A_i) = \{0\}$, $i = 1, \ldots, n$, which means that A is Riesz if and only if $A_1, \ldots, A_n$ are Riesz. □

Theorem 4.8. *Let $A \in \mathrm{Poly}^{-1}R(X)$. Then each $\lambda \in m_A^{-1}(0)$ is either the limits of of the eigenvalues of A or else there exists a closed A-invariant subspace X_λ of X which is infinite dimensional and $\sigma(A_\lambda) = \{\lambda\}$ where A_λ is the reduction of A on X_λ.*

Proof. From Theorem 4.1 and Theorem 4.4 it follows that each $\lambda \in m_A^{-1}(0)$ is either the limits of of the eigenvalues of A or else λ is an isolated point of $\sigma(A)$. Let $\lambda \in m_A^{-1}(0)$ and let λ be an isolated point of $\sigma(A)$. Then there exist open sets Ω_1 and Ω_2 which boundaries are simple closed rectifiable curves and such that $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \emptyset$, $\lambda \in \Omega_1$, the closure of Ω_1 contains no other point of $\sigma(A)$ and $\sigma(A) \subset \Omega_1 \cup \Omega_2$. Using the spectral projection $P_\lambda = \frac{1}{2\pi i} \int_{\partial\Omega_1} (zI - A)^{-1} dz$ we have $X = X_\lambda \oplus X_\mu$, where $X_\lambda = P_\lambda X$ and $X_\mu = (I - P_\lambda)X$ are closed A-invariant subspaces of X, $A = A_\lambda \oplus A_\mu$, where A_λ (A_μ) is the reduction of A on X_λ (X_μ), and also $\sigma(A_\lambda) = \{\lambda\}$ and $\sigma(A_\mu) = \sigma(A) \setminus \{\lambda\}$. From $m_A(A) \in R(X)$ it follows that $m_A(A_\mu)$ is Riesz by Theorem 3.7. Since $(A_\mu - \lambda)^{-1}$ commutes with $m_A(A_\mu)$, by Theorem 3.5 (ii), we get that $(A_\mu - \lambda)^{-1} m_A(A_\mu)$ is Riesz, that is there exists a polynomial q which degree is less than the degree of the polynomial m_A such that $q(A_\mu)$ is Riesz. We shall prove that $\dim X_\lambda = \infty$. Suppose the opposite that $\dim X_\lambda < \infty$. Then $q(A_\lambda)$ is compact and hence it is Riesz. Since $q(A) = q(A_\lambda) \oplus q(A_\mu)$, from Lemma 4.2 it follows that $q(A)$ is Riesz which contradicts the fact that the polynomial m_A is minimal. Thus X_λ is infinite dimensional and the proof is complete. □

Gilfeather [11, Theorem 1] described the structure of polynomially compact operators proving that every polynomially compact operator on a Banach space is the finite direct sum of translates of operators which have property that the finite power of the operators is compact. The structure of polynomially Riesz operators on Hilbert spaces was described by Y. M. Han, S. H. Lee and W. Y. Lee [16, Lemma 3]: every polynomially Riesz operators on a Hilbert space is the finite direct sum of translates of Riesz operators. This assertion holds also for polynomially Riesz operators on Banach spaces:

Theorem 4.9. *If $A \in \text{Poly}^{-1}R(X)$ and $m_A^{-1}(0) = \{\lambda_1, \dots, \lambda_n\}$, then the Banach space X is decomposed into the direct sum $X = X_1 \oplus \cdots \oplus X_n$ where X_i is closed A-invariant subspace of X, and $A = A_1 \oplus \cdots \oplus A_n$ where A_i is the reduction of A on X_i and $A_i - \lambda_i$ is Riesz, $i = 1, \dots, n$.*

Proof. Let $A \in \text{Poly}^{-1}R(X)$ and $m_A^{-1}(0) = \{\lambda_1, \dots, \lambda_n\}$. There exist open sets $\Omega_1, \dots, \Omega_n$ such that $\lambda_i \in \Omega_i$, $\partial\Omega_i$ is a rectifiable simple closed curve, $i = 1, \dots, n$, $\sigma(A) \subset \cup_{i=1}^n \Omega_i$, $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ for $i \neq j$ and if λ_i is an isolated point of $\sigma(A)$, then $\sigma(A) \cap \Omega_i = \{\lambda_i\}$. For the spectral projections $P_i = \int_{\partial\Omega_i} (zI - A)^{-1} dz$ and $X_i = P_i X$ we have $X = X_1 \oplus \cdots \oplus X_n$ and X_i is a closed A-invariant subspace of X, $i = 1, \dots n$. Also $A = A_1 \oplus \cdots \oplus A_n$ where A_i is the reduction of A on X_i and $\sigma(A_i) = \sigma(A) \cap \Omega_i$. By Theorem 4.3, $\sigma_\Phi(A) = \{\lambda_1, \dots, \lambda_n\}$ and according to (4.3) we conclude $\sigma_\Phi(A_i) = \{\lambda_i\}$, $i = 1, \dots, n$. Hence $\sigma_\Phi(A_i - \lambda_i) = \{0\}$, that is $A_i - \lambda_i$ is Riesz, $i = 1, \dots, n$. □

Gilfeather [11, Theorem 2] described also the structure of polynomially compact normal operators on a Hilbert space. In the following theorem we extend Gilfeather's result on polynomially Riesz normal operators.

Theorem 4.10. *Let H be a Hilbert space and let $A \in L(H)$ be normal and polynomially Riesz. Let $m_A^{-1}(0) = \{\lambda_1, \dots, \lambda_n\}$. Then for each $i \in \{1, \dots, n\}$, $\lambda \in m_A^{-1}(0)$ is either the limits of of the eigenvalues of A or else it is an isolated eigenvalues with the infinite dimensional eigenspace.*

Then H is decomposed into the orthogonal direct sum $H = H_1 \oplus \cdots \oplus H_n$ and $A = A_1 \oplus \cdots \oplus A_n$ where A_i is the reduction of A on H_i, and $A_i - \lambda_i$ is compact, $i = 1, \dots, n$. Moreover, if λ_i is an isolated point of $\sigma(A)$, then $A_i = \lambda_i I$.

Proof. Applying the functional calculus as in the proof of Theorem 4.9, we get that the spectral projections P_i, $i = 1, \dots, n$, are orthogonal and $P_i P_j = 0$, $i \neq j$. Therefore, $H_i = P_i X$ are closed A-invariant subspaces of H such that $H_i \perp H_j$, $i \neq j$, and hence H is decomposed into the orthogonal direct sum $H = H_1 \oplus \cdots \oplus H_n$. If A_i is the reduction of A on H_i, then from Theorem 4.9 we have that $A_i - \lambda_i$ is Riesz. Let $\pi_i \colon L(H_i) \to C(H_i)$ denote the natural homomorphism where $C(H_i)$ is the Calkin algebra over H_i, $C(H_i) = L(H_i)/K(H_i)$, $i = 1, \dots, n$. Since $C(H_i)$ is a C^* algebra and since $\pi_i(A_i - \lambda_i)$ is normal and quasinilpotent, it follows that $\pi_i(A_i - \lambda_i) = 0$, that is $A_i - \lambda_i$ is compact.

From Theorem 4.8 it follows that each $\lambda_i \in m_A^{-1}(0)$ is either the limits of of the eigenvalues of A or else it is an isolated point of $\sigma(A)$ in which case $\sigma(A_i) = \{\lambda_i\}$ and H_i is infinite dimensional. Since $\sigma(A_i - \lambda_i) = \{0\}$ and $A_i - \lambda_i$ is normal, it follows that $A_i - \lambda_i = 0$, i.e. $A_i = \lambda_i I$. As H_i is infinite dimensional, λ_i is an eigenvalue with the infinite dimensional eigenspace. □

If H is a Hilbert space, an operator $A \in L(H)$ is called hyponormal if $\|A^*x\| \leqslant \|Ax\|$ for all $x \in H$, that is $A^*A - AA^* \geqslant 0$. Gilfeather proved for $A \in L(H)$ [11, Proposition 4]:

$$A \text{ is hyponormal}, sigma(A) \text{ is countable} \implies A \text{ is normal.} \tag{4.4}$$

The conclusion of (4.4) extends to *paranormal* operators, i.e. a paranormal operator $A \in L(H)$ with countable spectrum is normal [32, 35], where A is paranormal if $||Ax||^2 \leqq ||A^2x||$ for all unit vectors $x \in H$. Evidently, $A \in L(H)$ hyponormal implies A paranormal. The following corollary is an extension of Corollary 2 in [11] to polynomially Riesz paranormal operators.

Corollary 4.4. *Every polynomially Riesz paranormal operator in $L(H)$ is normal.*

Proof. If $A \in \mathrm{Poly}^{-1}\mathrm{R(H)}$ and $m_A^{-1}(0) = \{\lambda_1, \dots, \lambda_n\}$, then it follows from Theorem 2.7 that $H = \bigoplus_{i=1}^n H_i$, where each H_i is a closed A-invariant subspace of H, and $A = \bigoplus_{i=1}^n A_i$. Here each $A_i - \lambda_i$ is Riesz, the operator A_i has at best a countable spectrum (with λ_i as its only possible limit point) and all points of $\sigma(A_i)$ other than the point λ_i are eigenvalues of the operator. Recall that the restriction of a paranormal operator to an invariant subspace is paranormal and (as noted above) a paranormal operator with countable spectrum is normal. Hence each A_i, and consequently A, is normal. □

The following theorem is an extension of Theorem 2.6 in [20] and Theorem 5.2, Chapter V in [43].

Theorem 4.11. *Let $B \in L(X)$. Then the following conditions are equivalent:*

(4.11.1) *B is Browder.*

(4.11.2) *There exist $n \in \mathbb{N}$, $T \in L(X)$ and $A \in K(X)$ such that*

$$TB^n = B^nT = I - A.$$

(4.11.3) *There exist $n \in \mathbb{N}$, $T \in L(X)$, $A \in \mathrm{Poly}^{-1}K(X)$ and $\lambda \in \mathbb{C}$ such that $m_A(\lambda) \neq 0$ and*

$$TB^n = B^nT = \lambda - A.$$

(4.11.4) *There exist* $n \in \mathbb{N}$, $T \in L(X)$, $A \in \text{Poly}^{-1}R(X)$ *and* $\lambda \in \mathbb{C}$ *such that* $m_A(\lambda) \neq 0$ *and*

$$TB^n = B^nT = \lambda - A. \tag{4.5}$$

Proof. (4.11.1) $\Longrightarrow$ (4.11.2): Follows from [43, Theorem 5.2, p. 123–124].

(4.11.2) $\Longrightarrow$ (4.11.3): Suppose that there exist $n \in \mathbb{N}$, $T \in L(X)$ and $A \in K(X)$ such that $TB^n = B^nT = I - A$. Then $A \in \text{Poly}^{-1}K(X)$ with $m_A(z) = z$ and hence $m_A(1) \neq 0$. Thus, the statement (4.11.3) holds for $\lambda = 1$.

(4.11.3) $\Longrightarrow$ (4.11.4): Follows from the inclusion $K(X) \subset R(X)$.

(4.11.4) $\Longrightarrow$ (4.11.1): If (4.11.4) holds, then from Corollary 4.1 it follows that $\lambda - A$ is Browder and from (4.5), according to [18, Theorem 7.10.2], we get B^n is Browder. Again by [18, Theorem 7.10.2], we conclude that B is Browder. □

If $A \in L(X,Y)$, $B \in L(Y,X)$ and $\lambda \in \mathbb{C}$, $\lambda \neq 0$, it is well know that [8, Chapter 5.1, Lemma 7]

$$BA - \lambda \text{ is left (right) invertible} \Longleftrightarrow AB - \lambda \text{ is left (right) invertible,} \tag{4.6}$$

$$BA - \lambda \text{ is left (right) Fredholm} \Longleftrightarrow AB - \lambda \text{ is left (right) Fredholm.} \tag{4.7}$$

Lemma 4.3. *Let* $A \in L(X,Y)$, $B \in L(Y,X)$ *and let* $\lambda \in \mathbb{C}$, $\lambda \neq 0$. *Then*

$$BA - \lambda \textit{ is left (right) Browder} \Longleftrightarrow AB - \lambda \textit{ is left (right) Browder.} \tag{4.8}$$

Proof. For $T \in L(X)$ we have (see (2.48))

$$\sigma_{\mathcal{B}_l}(T) = \sigma_{\Phi_l}(T) \cup \operatorname{acc} \sigma_l(T) \tag{4.9}$$

From (4.6) it follows that

$$\sigma_l(BA) \cup \{0\} = \sigma_l(AB) \cup \{0\}, \tag{4.10}$$

while (4.7) implies

$$\sigma_{\Phi_l}(BA) \cup \{0\} = \sigma_{\Phi_l}(AB) \cup \{0\}. \tag{4.11}$$

From (4.10) it follows that

$$\operatorname{acc} \sigma_l(BA) = \operatorname{acc} \sigma_l(AB). \tag{4.12}$$

Now from (4.9), (4.11) and (4.12) we conclude

$$\sigma_{\mathcal{B}_l}(BA) \cup \{0\} = \sigma_{\mathcal{B}_l}(AB) \cup \{0\}, \tag{4.13}$$

and similarly (see (2.49))

$$\sigma_{\mathcal{B}_r}(BA) \cup \{0\} = \sigma_{\mathcal{B}_r}(AB) \cup \{0\}. \tag{4.14}$$

The equivalence (4.8) follows from (4.13) and (4.14). □

If $A \in L(X,Y)$ and $B \in L(Y,X)$, from (4.13) and (4.14) it follows

$$\sigma_{\mathcal{B}}(BA) \cup \{0\} = \sigma_{\mathcal{B}}(AB) \cup \{0\}. \tag{4.15}$$

A. Jeribi and N. Moalla [20, Proposition 3.3] proved that if BA is polynomially compact, then AB and BA are generalized Riesz operators. We improve on this:

Theorem 4.12. *Let $A \in L(X,Y)$ and $B \in L(Y,X)$. Then*

$$BA \in \mathrm{Poly}^{-1}R(X) \Longleftrightarrow AB \in \mathrm{Poly}^{-1}R(Y) \tag{4.16}$$

and in that case

$$m_{BA}^{-1}(0) \cup \{0\} = m_{AB}^{-1}(0) \cup \{0\}, \tag{4.17}$$

$$\sigma(BA) \setminus (m_{BA}^{-1}(0) \cup \{0\}) = \sigma(AB) \setminus (m_{AB}^{-1}(0) \cup \{0\}). \tag{4.18}$$

Proof. From Theorem 4.1 and (4.15) we get (4.16) and (4.17). (4.18) follows from (4.6) and (4.17). □

We remark that if $A \in L(X,Y)$, $B \in L(Y,X)$ and $BA \in \mathrm{Poly}^{-1}R(X)$, then for all $\lambda \in \sigma(BA) \setminus (m_{BA}^{-1}(0) \cup \{0\})$ it follows

$$\mathrm{mult}(BA, \lambda) = \mathrm{mult}(AB, \lambda). \tag{4.19}$$

For a proof of the equality (4.19) we refer the reader to the proof of Proposition 3.3 in [20].

For more about polynomially Riesz operators and perturbations by polynomially Riesz operators, as well as polynomially Riesz elements of a Banach algebra, we refer the reader to [49–52].

5. Generalized Kato–Riesz decomposition and generalized Drazin–Riesz invertible operators

5.1. Generalized Drazin–Riesz invertible and generalized Drazin–Riesz semi-Fredholm operators.

In this section we shall use the following notations:

$\mathbf{R_1} = \mathcal{J}(X)$	$\mathbf{R_2} = \mathcal{Q}(X)$	$\mathbf{R_3} = L(X)^{-1}$
$\mathbf{R_4} = \mathcal{B}_+(X)$	$\mathbf{R_5} = \mathcal{B}_-(X)$	$\mathbf{R_6} = \mathcal{B}(X)$
$\mathbf{R_7} = \Phi_+(X)$	$\mathbf{R_8} = \Phi_-(X)$	$\mathbf{R_9} = \Phi(X)$
$\mathbf{R_{10}} = \mathcal{W}_+(X)$	$\mathbf{R_{11}} = \mathcal{W}_-(X)$	$\mathbf{R_{12}} = \mathcal{W}(X)$

We need the following auxiliary assertions.

Lemma 5.1. *Let $X = X_1 \oplus X_2 \cdots \oplus X_n$ where $X_1,\ X_2, \dots, X_n$ are closed subspaces of X and let M_i be a closed subset of X_i, $i = 1, \dots, n$. Then the set $M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is closed.*

Proof. Consider Banach space $X_1 \times X_2 \times \cdots \times X_n$ provided with the canonical norm $\|(x_1, \dots, x_n)\| = \sum_{i=1}^n \|x_i\|$, $x_i \in X_i$, $i = 1, \dots, n$. Then the map $f\colon X_1 \times \cdots \times X_n \to X_1 \oplus \cdots \oplus X_n = X$ defined by $f((x_1, \dots, x_n)) = x_1 + \cdots + x_n$, $x_i \in X_i$, $i = 1, \dots, n$, is a homeomorphism. Since $M_1 \times M_2 \times \cdots \times M_n$ is closed in $X_1 \times X_2 \times \cdots \times X_n$, it follows that $f(M_1 \times M_2 \times \cdots \times M_n) = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is closed. □

Lemma 5.2. *Let $T \in L(X)$ and $(M,N) \in \mathrm{Red}(T)$. The following statements hold:*

(i) *$T \in \mathbf{R}_i$ if and only if $T_M \in \mathbf{R}_i$ and $T_N \in \mathbf{R}_i$, $1 \leqq i \leqq 9$, and in that case $\mathrm{ind}(T) = \mathrm{ind}(T_M) + \mathrm{ind}(T_N)$;*

(ii) *If $T_M \in \mathbf{R}_i$ and $T_N \in \mathbf{R}_i$, then $T \in \mathbf{R}_i$, $10 \leqq i \leqq 12$.*

(iii) *If $T \in \mathbf{R}_i$ and T_N is Weyl, then $T_M \in \mathbf{R}_i$, $10 \leqq i \leqq 12$.*

Proof. (i): From the equalities

$$\mathcal{N}(T) = \mathcal{N}(T_M) \oplus \mathcal{N}(T_N) \quad \mathcal{R}(T) = \mathcal{R}(T_M) \oplus \mathcal{R}(T_N) \tag{5.1}$$

it follows that $\alpha(T) = \alpha(T_M) + \alpha(T_N)$ and $\beta(T) = \beta(T_M) + \beta(T_N)$. It implies that $\alpha(T) < \infty$ if and only if $\alpha(T_M) < \infty$ and $\alpha(T_N) < \infty$, and also, $\beta(T) < \infty$ if and only if $\beta(T_M) < \infty$ and $\beta(T_N) < \infty$. Also, $R(T)$ is closed if and only if $R(T_M)$ and $R(T_N)$ are closed. Indeed, if $\mathcal{R}(T_M)$ and $\mathcal{R}(T_N)$ are closed, then from Lemma 5.1 and (5.1) it follows that $\mathcal{R}(T)$ is closed, and it is easy to prove that if $\mathcal{R}(T)$ is closed, then $\mathcal{R}(T_M)$ and $\mathcal{R}(T_N)$ are closed. Therefore T is bounded below (surjective, upper semi-Fredholm, lower semi-Fredholm) if and only if T_M and T_N are bounded below (surjective, upper semi-Fredholm, lower semi-Fredholm), and in that case $\mathrm{ind}(T) = \alpha(T) - \beta(T) = (\alpha(T_M) + \alpha(T_N)) - (\beta(T_M) + \beta(T_N)) = \mathrm{ind}(T_M) + \mathrm{ind}(T_M)$.

Now from Lemma 2.39 we conclude that T is upper (lower) semi-Browder if and only if T_M and T_N are upper (lower) semi-Browder.

(ii): Follows from (i).

(iii): Suppose that $T \in \mathcal{W}_+(X)$ and that T_N is Weyl. Then $\mathrm{ind}(T_N) = 0$ and according to (i) it follows that $T_M \in \Phi_+(X)$ and $\mathrm{ind}(T_M) = \mathrm{ind}(T_M) + \mathrm{ind}(T_N) = \mathrm{ind}(T) \leqslant 0$. Thus T_M is upper semi-Weyl.

The cases $i = 11$ and $i = 12$ can be proved similarly. □

For $T \in L(X)$ and $n \in \mathbb{N}_0$ we set

$$\alpha_n(T) = \dim \mathcal{N}(T^{n+1})/\mathcal{N}(T^n) \quad \text{and} \quad \beta_n(T) = \dim \mathcal{R}(T^n)/\mathcal{R}(T^{n+1}).$$

From [23, Lemmas 3.1 and 3.2] it follows that $\alpha_n(T) = \dim(\mathcal{N}(T) \cap \mathcal{R}(T^n))$ and $\beta_n(T) = \mathrm{codim}(\mathcal{R}(T) + \mathcal{N}(T^n))$.

For each $n \in \mathbb{N}_0$, T induced a linear transformation from the vector space $\mathcal{R}(T^n)/\mathcal{R}(T^{n+1})$ to the space $\mathcal{R}(T^{n+1})/\mathcal{R}(T^{n+2})$ and $k_n(T)$ denotes the dimension of the null space of the induced map. We recall that [14]

$$k_n(T) = \dim(\mathcal{R}(T^n) \cap \mathcal{N}(T))/(\mathcal{R}(T^{n+1}) \cap \mathcal{N}(T)),$$
$$k_n(T) = \dim(\mathcal{R}(T) + \mathcal{N}(T^{n+1}))/(\mathcal{R}(T) + \mathcal{N}(T^n)).$$

This implies that $k_n(T) = \alpha_n(T) - \alpha_{n+1}(T)$ whenever$\alpha_{n+1}(T) < \infty$, and $k_n(T) = \beta_n(T) - \beta_{n+1}(T)$ whenever $\beta_{n+1}(T) < \infty$. If there is $d \in \mathbb{N}_0$ for which $k_n(T) = 0$ for $n \geqslant d$, then T is said to have uniform descent for $n \geqslant d$. An operator $T \in L(X)$ has uniform descent for $n \geqslant d$ if and only if the sequence of subspaces $(\mathcal{R}(T^n) \cap \mathcal{N}(T))$ is constant for $n \geqslant d$, which is also equivalent to the fact that the sequence of subspaces $(\mathcal{R}(T) + \mathcal{N}(T^n))$ is constant for $n \geqslant d$ [14, Theorem 3.1].

For $T \in L(X)$ and every $d \in \mathbb{N}_0$, the operator range topology on $\mathcal{R}(T^d)$ is defined by the norm $\|\cdot\|_d$ such that for every $y \in \mathcal{R}(T^d)$,

$$\|y\|_d = \inf\{\|x\| : x \in X, \quad y = T^d x\}.$$

For $T \in L(X)$ if there is $d \in \mathbb{N}_0$ for which T has uniform descent for $n \geqslant d$ and if $\mathcal{R}(T^n)$ is closed in the operator range topology of $\mathcal{R}(T^d)$ for $n \geqslant d$, then we say that T has *topological uniform descent for* (TUD for brevity) $n \geqslant d$ [14].

We recall that an operator $T \in L(X)$ is Kato if and only if T has TUD for $n \geqslant 0$ [3, Corollary 1.83 (i)]

Lemma 5.3. *If $T \in L(X)$ is Kato, then there exists a positive constant $\delta > 0$, such that*

(i) *$T - \lambda$ is Kato for all $|\lambda| < \delta$;*
(ii) *$\alpha(T - \lambda) = \alpha(T)$ for all $|\lambda| < \delta$;*
(iii) *$\beta(T - \lambda) = \beta(T)$ for all $|\lambda| < \delta$.*

Proof. Let T be Kato. Then T has TUD for $n \geqslant 0$. Now by using [14, Theorem 4.7] we obtain that there exists a $\delta > 0$ such that for every $\lambda \in \mathbb{C}$, $0 < |\lambda| < \delta$ implies that: $T - \lambda$ is Kato, and

$$\alpha_n(T-\lambda) = \alpha_0(T) = \alpha(T), \quad \beta_n(T-\lambda) = \beta_0(T) = \beta(T), \text{ for every } n \in \mathbb{N}_0,$$

and so

$$\alpha(T-\lambda) = \alpha_0(T-\lambda) = \alpha(T), \quad \beta(T-\lambda) = \beta_0(T-\lambda) = \beta(T).$$

Hence (1), (2) and (3) hold for all $|\lambda| < \delta$. □

The following proposition shows that every Kato operator T has a property that if 0 is not an interior point of $\sigma_{\mathbf{R}_i}(T)$, then 0 can not be even a boundary point of $\sigma_{\mathbf{R}_i}(T)$, that is, $0 \in \rho_{\mathbf{R}_i}(T)$, $1 \leqslant i \leqslant 12$.

Proposition 5.1. *Let $T \in L(X)$. Then the following implications hold:*

(i) *If T is Kato and 0 is an accumulation point of $\rho_{ap}(T)$ (resp. $\rho_{su}(T)$, $\rho(T)$), then T is bounded below (resp. surjective, invertible);*
(ii) *If T is Kato and 0 is an accumulation point of $\rho_{\mathcal{B}_+}(T)$ (resp. $\rho_{\mathcal{B}_-}(T)$, $\rho_{\mathcal{B}}(T)$), then T is bounded below (resp. surjective, invertible);*
(iii) *If T is Kato and 0 is an accumulation point of $\rho_{\Phi_+}(T)$ (resp. $\rho_{\Phi_-}(T)$, $\rho_{\Phi}(T)$), then T is upper semi-Fredholm (resp. lower semi-Fredholm, Fredholm);*
(iv) *If T is Kato and 0 is an accumulation point of $\rho_{\mathcal{W}_+}(T)$ (resp. $\rho_{\mathcal{W}_-}(T)$, $\rho_{\mathcal{W}}(T)$), then T is upper semi-Weyl (resp. lower semi-Weyl, Weyl).*

Proof. (i): Suppose that T is Kato. According to Lemma 5.3 there exists $\delta > 0$ such that for all $|\lambda| < \delta$ it holds that

$$T - \lambda \text{ is Kato,} \tag{5.2}$$

$$\alpha(T-\lambda) = \alpha(T) \quad \text{and} \quad \beta(T-\lambda) = \beta(T). \tag{5.3}$$

Suppose now that $0 \in \operatorname{acc} \rho_{ap}(T)$ ($0 \in \operatorname{acc} \rho_{su}(T)$). Then there exists μ such that $0 < |\mu| < \delta$ and $T - \mu$ is bounded below (surjective), and so from (5.3) it follows that $\alpha(T) = \alpha(T - \mu) = 0$ ($\beta(T) = \beta(T - \mu) = 0$). Consequently, T is bounded below (surjective).

(ii): Let T be Kato and let $0 \in \operatorname{acc} \rho_{\mathcal{B}_+}(T)$ ($0 \in \operatorname{acc} \rho_{\mathcal{B}_-}(T)$). Then there exists $\delta > 0$ such that (5.2) and (5.3) hold for all $|\lambda| < \delta$. From $0 \in \operatorname{acc} \rho_{\mathcal{B}_+}(T)$ ($0 \in \operatorname{acc} \rho_{\mathcal{B}_-}(T)$) it follows that there exists μ such that $0 < |\mu| < \delta$ and $T - \mu$ is upper semi-Browder (lower semi-Browder). Since $T - \mu$ is Kato, from Corollary 2.21 it follows that $T - \mu$ is bounded below (surjective). Now, as in the previous part of the proof, we can conclude that T is bounded below (surjective).

The implications (iii) and (iv) follow also from Lemma 5.3. □

An operator $T \in L(X)$ is said to admit a *generalized Kato decomposition*, abbreviated as GKD, if there exists a pair $(M, N) \in \text{Red}(T)$ such that T_M is Kato and T_N is quasinilpotent. A relevant case is obtained if we assume that T_N is nilpotent. In this case T is said to be of *Kato type.* If T admits a GKD (M, N) such that N is finite-dimensional, then T_N is nilpotent, since every quasinilpotent operator on a finite dimensional space is nilpotent, and hence T is essentially Kato according to [36, Theorem 21.3]. For $T \in L(X)$, *the Kato spectrum*, *the Kato type spectrum* and *the generalized Kato spectrum* is defined by

$\sigma_K(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Kato}\}$,

$\sigma_{Kt}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not of Kato type}\}$,

$\sigma_{gK}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ does not admit a GKD}\}$, respectively.

Definition 5.1. An operator $T \in L(X)$ is said to admit a *Kato–Riesz decomposition*, abbreviated as GKRD, if there exists a pair $(M, N) \in \text{Red}(T)$ such that T_M is Kato and T_N is Riesz.

For $T \in L(X)$ *the generalized Kato–Riesz spectrum* is defined by

$$\sigma_{gKR}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ does not admit a GKRD}\}.$$

Clearly, $\sigma_{gKR}(T) \subset \sigma_{gK}(T) \subset \sigma_{Kt}(T) \subset \sigma_{eK}(T) \subset \sigma_K(T) \subset \sigma_{ap}(T) \cap \sigma_{su}(T)$.

Proposition 5.2. *Let $T \in L(X)$ and $(M, N) \in Red(T)$. Then T admits a GKRD(M, N) if and only if T' admits a GKRD$(N^\perp, M^\perp)$.*

Proof. Suppose that T admits a GKRD(M, N). Then $(M, N) \in Red(T)$, T_M is Kato, T_N is Riesz. Let P_M be the projection of X onto M along N. Then P'_M is also a projector, $\mathcal{N}(P'_M) = \mathcal{R}(P_M)^\perp = M^\perp$ and $\mathcal{R}(P'_M) = \mathcal{N}(P_M)^\perp = N^\perp$ since $\mathcal{R}(P_M)$ is closed. Consequently, $X' = \mathcal{R}(P'_M) \oplus \mathcal{N}(P'_M) = N^\perp \oplus M^\perp$ and since subspaces $N^\perp$ and $M^\perp$ are invariant under T', it follows that $(N^\perp, M^\perp) \in \text{Red}(T')$. For $P_N = I - P_M$ we have that $(M, N) \in Red(TP_N)$, $TP_N = P_N T$ and since $TP_N = (TP_N)_M \oplus (TP_N)_N = 0 \oplus T_N$, according to Lemma 4.2, it follows that TP_N is Riesz. Consequently, $T'P'_N = P'_N T'$ is Riesz, $(N^\perp, M^\perp) \in Red(T'P'_N)$ and since $\mathcal{R}(P'_N) = \mathcal{N}(P_N)^\perp = M^\perp$, we conclude that $(T'P'_N)_{M^\perp} = T'_{M^\perp}$ is Riesz according to Lemma 4.2.

In order to show that $T'_{N^\perp}$ is Kato we need to prove that $\mathcal{R}(T'_{N^\perp}) = T'(N^\perp)$ is closed and $\mathcal{N}((T'_{N^\perp})^n) \subset \mathcal{R}(T'_{N^\perp}) = T'(N^\perp)$ for every $n \in \mathbb{N}$. Now we proceed as in the proof of [2, Theorem 1.43].

Since T_M is Kato, we have that $TP_M(X) = T(M)$ is closed, and hence, $(TP_M)'(X')$ is closed. From $TP_M = P_M T$ it follows that $T'P'_M = P'_M T'$ and hence $(TP_M)'(X') = T'P'_M(X') = T'(N^\perp)$ is closed.

For every $n \in \mathbb{N}$ we have

$$\mathcal{N}((T'_{N^\perp})^n) = \mathcal{N}((T')^n) \cap N^\perp = \mathcal{R}(T^n)^\perp \cap N^\perp = (\mathcal{R}(T^n) + N)^\perp.$$

Since T_M is Kato, we have

$$\mathcal{N}(TP_M) = \mathcal{N}(T_M) + N \subset T^n(M) + N \subset \mathcal{R}(T^n) + N,$$

and therefore,

$$\mathcal{N}((T'_{N^\perp})^n) = (\mathcal{R}(T^n) + N)^\perp \subset (\mathcal{N}(TP_M))^\perp = \mathcal{R}((TP_M)') = \mathcal{R}(T'P'_M)$$

$$= T'(P'_M(X')) = T'(N^\perp) = \mathcal{R}(T'_{N^\perp}).$$

Consequently, $T'_{N^\perp}$ is Kato and $(N^\perp, M^\perp)$ is a GKRD for T'.

Let $(M,N) \in Red(T)$ and let T' admit a GKRD$(N^\perp, M^\perp)$. Then $T'_{N^\perp}$ is Kato and $T'_{M^\perp}$ is Riesz. As $(N^\perp, M^\perp) \in Red(T'P'_N)$, then $T'P'_N = (T'P'_N)_{N^\perp} \oplus (T'P'_N)_{M^\perp} = 0 \oplus T'_{M^\perp}$, and according to Lemma 4.2 we obtain $T'P'_N$ is Riesz, which implies TP_N is Riesz. Since $TP_N = 0 \oplus T_N$, from Lemma 4.2 we get that T_N is Riesz. Let $P_M = I - P_N$. Since $T'_{N^\perp}$ is Kato, it follows that $\mathcal{R}((T'_{N^\perp})^n)$ is closed and

$$\mathcal{N}((T'_{N^\perp})^n) \subset \mathcal{R}(T'_{N^\perp}), \quad \text{for every} \quad n \in \mathbb{N}. \tag{5.4}$$

From $(N^\perp, M^\perp) \in Red(T'P'_M)$, we have $(T'P'_M)^n = (T'P'_M)^n_{N^\perp} \oplus (T'P'_N)^n_{M^\perp} = {T'}^n_{N^\perp} \oplus 0$ and

$$\mathcal{R}((T'P'_M)^n) = \mathcal{R}((T'_{N^\perp})^n). \tag{5.5}$$

So $\mathcal{R}((T'P'_M)^n)$ is closed which implies that $\mathcal{R}((TP_M)^n) = \mathcal{R}(T^n_M)$ is closed. As

$$\begin{aligned}\mathcal{N}((T'_{N^\perp})^n) &= \mathcal{N}((T')^n) \cap N^\perp = \mathcal{R}(T^n)^\perp \cap N^\perp = (\mathcal{R}(T^n) + N)^\perp \\ &= (\mathcal{R}(T^n_M) + \mathcal{R}(T^n_N) + N)^\perp = (\mathcal{R}(T^n_M) + N)^\perp,\end{aligned}$$

from (5.4) and (5.5) we obtain

$$(\mathcal{R}(T^n_M) + N)^\perp \subset \mathcal{R}((TP_M)') = \mathcal{N}(TP_M)^\perp, \quad \text{for every} \quad n \in \mathbb{N},$$

which implies

$$\mathcal{N}(TP_M) = {}^\perp(\mathcal{N}(TP_M)^\perp) \subset {}^\perp((\mathcal{R}(T^n_M) + N)^\perp) \quad \text{for every} \quad n \in \mathbb{N}. \tag{5.6}$$

From Lemma 5.1 it follows that $\mathcal{R}(T^n_M) + N$ is closed and therefore, according to [43, Chapter III, Lemma 3.2], we get ${}^\perp((\mathcal{R}(T^n_M) + N)^\perp) = \mathcal{R}(T^n_M) + N$, $n \in \mathbb{N}$. Now from (5.6) we obtain

$$\mathcal{N}(T_M) + N \subset \mathcal{R}(T^n_M) + N \quad \text{for every} \quad n \in \mathbb{N}.$$

It implies $\mathcal{N}(T_M) \subset \mathcal{R}(T^n_M)$ for every $n \in \mathbb{N}$ and we can conclude that T_M is Kato. □

An operator $T \in L(X)$ is said to have the single-valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP at λ_0 for breviety) if for every open disc $\mathcal{D}_{\lambda_0}$ centerd at λ_0 the only analitic function $f\colon \mathcal{D}_{\lambda_0} \to X$ satisfying $(T-\lambda)f(\lambda) = 0$ for all $\lambda \in \mathcal{D}_{\lambda_0}$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$. We denote by $\mathcal{S}(T)$ the open set of $\lambda \in \mathbb{C}$ where T fails to have SVEP at λ.

Evidently, $T \in L(X)$ has the SVEP at every point of the resolvent set $\rho(T)$. Moreover, from the identity theorem for analytic function and $\sigma(T) = \sigma(T')$, where $T' \in L(X')$ is the adjoint operator of T, it follows that T and T' have the SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T and T' have the SVEP at every isolated point of the spectrum. Hence, there is implication

$$\sigma(T) \text{ does not cluster at } \lambda_0 \Longrightarrow T \text{ and } T' \text{ have the SVEP at } \lambda_0. \tag{5.7}$$

Moreover, from the indentity theorem for analytic functions we have (see [4], p. 182):

(5.8) $\sigma_{ap}(T)$ does not cluster at $\lambda_0 \Longrightarrow T$ has the SVEP at λ_0

and

(5.9) $\sigma_{su}(T)$ does not cluster at $\lambda_0 \Longrightarrow T'$ has the SVEP at λ_0.

An operator $T \in L(X)$ is said to be Drazin invertible if there exists $S \in L(X)$ such that $TS = ST$, $STS = S$ and $TST - T$ is nilpotent. This concept has been generalized by Koliha [26] by replacing the third condition in this definition with the condition that $TST - T$ is quasinilpotent. Recall that T is generalized Drazin invertible if and only if $0 \notin \operatorname{acc}\sigma(T)$, and this is also equivalent to the fact that $T = T_1 \oplus T_2$ where T_1 is invertible and T_2 is quasinilpotent. If we replace the third condition in the previous definitions by condition that $TST - T$ is Riesz, we get the concept of generalized Drazin–Riesz invertible operators.

Definition 5.2. An operator $T \in L(X)$ is *generalized Drazin–Riesz invertible* if there exists $S \in L(X)$ such that

$$TS = ST, \quad STS = S, \quad TST - T \quad \text{is Riesz.}$$

The generalized Drazin–Riesz spectrum of $T \in L(X)$ is defined by $\sigma_{gDR}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not generalized Drazin–Riesz invertible}\}$.

Clearly, every Riesz operator is generalized Drazin–Riesz invertible. The set of generalized Drazin–Riesz invertible operators is denoted by $L(X)^{DR}$. Harte introduced the concept of a quasipolar element in a Banach algebra [18, Definition 7.5.2]: an element a of a Banach algebra A is *quasipolar* if there is an idempotent $q \in A$ commuting with a such that $a(1-q)$ is quasinilpotent and $q \in (Aa) \cap (aA)$. Koliha [26] proved that an element is quasipolar if and only if it is generalized Drazin invertible. We shall say that $T \in L(X)$ is *Riesz-quasipolar* if there exists a bounded projection Q satisfying

(5.10) $TQ = QT$, $T(I-Q)$ is Riesz, $Q \in (L(X)T) \cap (TL(X))$.

For $T \in L(X)$, a subset σ of $\sigma(T)$ is called a *spectral set* of T if it is both open and closed in the relative topology of $\sigma(T)$.

In the following theorem we show that an operator $T \in L(X)$ is Riesz-quasipolar if and only if T is generalized Drazin–Riesz invertible and moreover, it is also equivalent to the fact that T admits a GKRD and T and T' have the SVEP at 0.

Theorem 5.1. *Let $T \in L(X)$. The following conditions are equivalent:*

(i) *There exists $(M, N) \in \operatorname{Red}(T)$ such that T_M is invertible and T_N is Riesz;*
(ii) *T admits a GKRD and $0 \notin \operatorname{int}\sigma(T)$.*
(iii) *T admits a GKRD and T and T' have the SVEP at 0;*
(iv) *T is generalized Drazin–Riesz invertible;*
(v) *T is Riesz-quasipolar;*
(vi) *There exists a bounded projection P on X which commutes with T such that $T + P$ is Browder and TP is Riesz;*
(vii) *There exists $(M, N) \in \operatorname{Red}(T)$ such that T_M is Browder and T_N is Riesz;*

(viii) T *admits a GKRD and* $0 \notin \operatorname{acc}\sigma_{\mathcal{B}}(T)$*;*
(ix) T *admits a GKRD and* $0 \notin \operatorname{int}\sigma_{\mathcal{B}}(T)$*;*
(x) $0 \notin \operatorname{acc}\sigma_{\mathcal{B}}(T)$*.*

Proof. (i)$\Longrightarrow$(ii): Suppose that there exists $(M,N) \in \operatorname{Red}(T)$ such that T_M is invertible and T_N is Riesz. Then T_M is Kato and hence, T admits a GKRD (M,N). Since T_M is invertible, $0 \in \rho_{ap}(T_M)$ and there exists $\epsilon > 0$ such that $D(0,\epsilon) \subset \rho(T_M)$. As T_N is Riesz, it follows that $0 \in \operatorname{acc}\rho(T_N)$. Consequently, $0 \in \operatorname{acc}(\rho(T_M) \cap \rho(T_N)) = \operatorname{acc}\rho(T)$ and so, $0 \notin \operatorname{int}\sigma(T)$.

(ii)$\Longrightarrow$(i): Suppose that T admits a GKRD and $0 \notin \operatorname{int}\sigma(T)$. Then there exists $(M,N) \in \operatorname{Red}(T)$ such that T_M is Kato and T_N is Riesz and $0 \in \operatorname{acc}\,\rho(T)$. According to Lema 5.2 (i), it follows that $0 \in \operatorname{acc}\,\rho(T_M)$. From Proposition 5.1 (i) it follows that T_M is invertible.

(ii)$\Longrightarrow$(iii): Follows from the fact that $\sigma(T) = \sigma(T')$ and the identity theorem for analytic functions.

(iii)$\Longrightarrow$(ii): Suppose that T admits a GKRD and T and T' have the SVEP at 0. Then there exists $(M,N) \in \operatorname{Red}(T)$ such that T_M is Kato and T_N is Riesz. Since the SVEP at 0 of T is inherited by the reductions on every closed invariant subspaces, we get that T_M has the SVEP at 0. According to [2, Theorem 2.49] it follows that T_M is bounded below and so there exists $\epsilon_1 > 0$ for which $T_M - \lambda$ is bounded below for every $|\lambda| < \epsilon_1$, that is $D(0,\epsilon_1) \subset \rho_{ap}(T_M)$. Since T_N is Riesz, then $D(0,\epsilon_1) \setminus C_1 \subset \rho_{ap}(T_N)$ where C_1 is most countable set of Riesz points of T_N. Consequently, $D(0,\epsilon_1) \setminus C_1 \subset \rho_{ap}(T_M) \cap \rho_{ap}(T_N) = \rho_{ap}(T)$. From Proposition 5.2 it follows that T' admits the GKRD $(N^\perp, M^\perp)$ and since T' has the SVEP at 0, according to already proved we conclude that there exists $\epsilon_2 > 0$ such that $D(0,\epsilon_2) \setminus C_2 \subset \rho_{ap}(T') = \rho_{su}(T)$ where C_2 is most countable set of Riesz points of $T'_{M^\perp}$. Let $\epsilon = \min\{\epsilon_1,\epsilon_2\}$ and $C = C_1 \cup C_2$. Than C is most countable and $D(0,\epsilon) \setminus C \subset \rho_{ap}(T) \cap \rho_{su}(T) = \rho(T)$. Consequently, $0 \notin \operatorname{int}\sigma(T)$.

(i)$\Longrightarrow$(iv): Suppose that there exists $(M,N) \in \operatorname{Red}(T)$ such that T_M is invertible and T_N is Riesz. Let $S = T_M^{-1} \oplus 0$, i.e.

$$S = \begin{bmatrix} T_M^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} M \\ N \end{bmatrix} \to \begin{bmatrix} M \\ N \end{bmatrix}$$

and let $x \in X$. Then $x = m+n$, where $m \in M$ and $n \in N$, and $TSx = TS(m+n) = TT_M^{-1}m = m$ and $STx = ST(m+n) = S(T_Mm + T_Nn) = T_M^{-1}T_Mm = m$. Thus $TS = ST$. Further, $STSx = STS(m+n) = Sm = S(m+n) = Sx$ and hence, $STS = S$. From

$$\begin{aligned} T - T^2S &= \begin{bmatrix} T_M & 0 \\ 0 & T_N \end{bmatrix} - \begin{bmatrix} T_M^2 & 0 \\ 0 & T_N^2 \end{bmatrix} \cdot \begin{bmatrix} T_M^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} T_M & 0 \\ 0 & T_N \end{bmatrix} - \begin{bmatrix} T_M & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & T_N \end{bmatrix}, \end{aligned}$$

according to Lemma 4.2, it follows that $T - T^2S$ is Riesz.

(iv)$\Longrightarrow$(v): Suppose that T is generalized Drazin–Riesz invertible. Then there exists $S \in L(X)$ such that $ST = TS$, $STS = S$ and $T - T^2S$ is Riesz. Let

$Q = TS$. Then Q is a bounded projection which commutes with T, $Q = TS = ST \in (L(X)T) \cap (TL(X))$ and $T(I-Q) = T(I-TS) = T - T^2S$ is Riesz.

(v)$\Longrightarrow$(vi): Suppose that T is Riesz-quasipolar. Then there exists a bounded projection Q satisfying (5.10). Let $P = I - Q$. Then $TP = PT$ and TP is Riesz. From $I - P = Q \in (L(X)T) \cap (TL(X))$ it follows that there exist $U, V \in L(X)$ such that $I - P = UT = TV$. Then

$$(T+P)(UTV+P) = (UTV+P)(T+P) = I + TP. \tag{5.11}$$

Since TP is Riesz, from Theorem 3.2 it follows that $I + TP$ is Browder. Now from (5.11), according to Theorem 2.29, we obtain that $T + P$ is Browder.

(vi)$\Longrightarrow$(vii): Suppose that there exists a projection $P \in L(X)$ that commutes with T such that $T+P$ is Browder and TP is Riesz. For $M = \mathcal{N}(P)$ and $N = \mathcal{R}(P)$ we have that $(M, N) \in \text{Red}(T)$ and $T_N = (TP)_N$ is Riesz by Lemma 4.2. From Lemma 5.2(i) it follows that $T_M = (T+P)_M$ is Browder.

(vii)$\Longrightarrow$(viii): Let $(M, N) \in \text{Red}(T)$ and $T = T_M \oplus T_N$, where T_M is Browder and T_N is Riesz. Then $0 \in \rho_{\mathcal{B}}(T_M)$ and there exists $\epsilon > 0$ such that $D(0, \epsilon) \subset \rho_{\mathcal{B}}(T_M)$. Since T_N is Riesz, $\sigma_{\mathcal{B}}(T_N) \subset \{0\}$ by Theorem 3.2 and hence, $D(0,\epsilon) \setminus \{0\} \subset \rho_{\mathcal{B}}(T_M) \cap \rho_{\mathcal{B}}(T_N)$. According to Lemma 5.2(i) $\rho_{\mathcal{B}}(T_M) \cap \rho_{\mathcal{B}}(T_N) = \rho_{\mathcal{B}}(T)$ and so, $D(0,\epsilon) \setminus \{0\} \subset \rho_{\mathcal{B}}(T)$. Therefore, $0 \notin \operatorname{acc} \sigma_{\mathcal{B}}(T)$.

From [36, Theorem 16.21] it follows that there exist two closed T-invariant subspaces M_1 and M_2 such that $M = M_1 \oplus M_2$, M_2 is finite dimensional, T_{M_1} is Kato and T_{M_2} is nilpotent. Hence $X = M_1 \oplus (M_2 \oplus N)$ and $M_2 \oplus N$ is closed. From Lemma 4.2 it follows that $T_{M_2 \oplus N} = T_{M_2} \oplus T_N$ is Riesz and thus T admits the GKRD $(M_1, M_2 \oplus N)$.

(viii)$\Longrightarrow$(ix), (viii)$\Longrightarrow$(x): It is obvious.

(ix)$\Longrightarrow$(i): Suppose that T admits a GKRD and $0 \notin \operatorname{int} \sigma_{\mathcal{B}}(T)$. Then there exists $(M, N) \in \text{Red}(T)$ such that T_M is Kato and T_N is Riesz. Since $0 \in \operatorname{acc} \rho_{\mathcal{B}}(T)$, from Lema 5.2(i) we get that $0 \in \operatorname{acc} \rho_{\mathcal{B}}(T_M)$, which according to Proposition 5.1(ii) implies that T_M is invertible.

(x)$\Longrightarrow$(iv): Suppose that $0 \notin \operatorname{acc} \sigma_{\mathcal{B}}(T)$. There are two cases:

1. If $0 \notin \operatorname{acc} \sigma(T)$, then T is generalized Drazin invertible, and hence T is generalized Drazin–Riesz invertible.
2. If $0 \in \operatorname{acc} \sigma(T)$, then $0 \in \operatorname{acc} \sigma(T) \setminus \operatorname{acc} \sigma_{\mathcal{B}}(T) \subset \sigma_{\mathcal{B}}(T) \setminus \operatorname{acc} \sigma_{\mathcal{B}}(T) = \operatorname{iso} \sigma_{\mathcal{B}}(T)$.

Since $\sigma(T)$ is the disjoint union of $\sigma_{\mathcal{B}}(T)$ and the set of Riesz points $p_{00}(T)$ according to (3.14), and since $0 \in \operatorname{acc} \sigma(T)$ and $0 \in \operatorname{iso} \sigma_{\mathcal{B}}(T)$, we conclude that $0 \in \operatorname{acc} p_{00}(T)$. Hence there is a sequence (λ_n) of mutually different Riesz points which converges to 0, where $|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant \dots$. There is $n_0 \in \mathbb{N}$ such that for $n \in \mathbb{N}$, $n > n_0$ implies that $0 < |\lambda_n| < 1$. Then $\sigma = \{0, \lambda_{n_0+1}, \lambda_{n_0+2}, \dots\}$ is a spectral set. Let P_σ be the spectral projection of T corresponding to σ. From [28, Teorem 9.8.10] it follows that $(\mathcal{R}(P_\sigma), \mathcal{N}(P_\sigma)) \in Red(T)$, $\sigma(T_{\mathcal{R}(P_\sigma)}) = \sigma$ and $\sigma(T_{\mathcal{N}(P_\sigma)}) = \sigma(T) \setminus \sigma$. Since the spectral radius $r(T_{\mathcal{R}(P_\sigma)}) = \sup\{|\lambda_{n_0+1}|, |\lambda_{n_0+2}|, \dots\} = |\lambda_{n_0+1}| < 1$, it follows that $T_{\mathcal{R}(P_\sigma)} - I_{\mathcal{R}(P_\sigma)}$ is invertible, and since $0 \notin \sigma(T_{\mathcal{N}(P_\sigma)})$, we have that $T_{\mathcal{N}(P_\sigma)}$ is invertible. Now from $T - P_\sigma = (T_{\mathcal{R}(P_\sigma)} - I_{\mathcal{R}(P_\sigma)}) \oplus T_{\mathcal{N}(P_\sigma)}$ we conclude that $T - P_\sigma$ is invertible.

We shall prove that $T^{D,\sigma} = (T - P_\sigma)^{-1}(I - P_\sigma)$ is a generalized Drazin–Riesz inverse for T.

Since T commutes with P_σ, it follows that T commutes with $T^{D,\sigma}$. As

$$TT^{D,\sigma} = T(T - P_\sigma)^{-1}(I - P_\sigma) = (T - P_\sigma)(T - P_\sigma)^{-1}(I - P_\sigma) = I - P_\sigma,$$

we obtain that

$$T^{D,\sigma}TT^{D,\sigma} = T^{D,\sigma}(I - P_\sigma) = T^{D,\sigma},$$
$$T - TT^{D,\sigma}T = T - (I - P_\sigma)T = P_\sigma T = TP_\sigma.$$

We have that TP_σ is completely reduced by the pair $(R(P_\sigma), N(P_\sigma))$ and

$$TP_\sigma = T_{R(P_\sigma)} \oplus 0. \tag{5.12}$$

Since $T - \lambda_k$ is Fredholm, because of $T = T_{R(P_\sigma)} \oplus T_{N(P_\sigma)}$, it follows that $T_{R(P_\sigma)} - \lambda_k$ is Fredholm, for every $k \in \mathbb{N}$. From (5.12) it follows that

$$TP_\sigma - \lambda_k = (T_{R(P_\sigma)} - \lambda_k) \oplus (-\lambda_k I_{\mathcal{N}(P_\sigma)}),$$

and so $TP_\sigma - \lambda_k$ is Fredholm, for every $k \in \mathbb{N}$. As $\sigma(TP_\sigma) = \sigma$, it follows that $\sigma_\Phi(TP_\sigma) \subset \{0\}$, and therefore TP_σ is Riesz. Consequently, $T^{D,\sigma}$ is a generalized Drazin–Riesz inverse for T. □

The equivalence of the conditions (i)-(ix) in Theorem 5.1 was proved in the paper [54], while the equivalence (x)$\Longleftrightarrow$ (iv) was proved by O. Abad and H. Zguitti in [1]. Evidently, every Riesz operator and every Browder operator is generalized Drazin–Riesz invertible operator.

The following theorem shows a connection between polynomially Riesz operators and generalized Drazin–Riesz invertible operators.

Theorem 5.2. *Let $T \in L(X)$. If T is polynomially Riesz, then $T - \lambda$ is generalized Drazin–Riesz invertible for every $\lambda \in \mathbb{C}$.*

Proof. Let T be polynomially Riesz and $m_T^{-1}(0) = \{\lambda_1, \dots, \lambda_n\}$. According to Theorem 4.1 we have that $\sigma_{\mathcal{B}}(T) = m_T^{-1}(0)$ where m_T is the minimal polynomial of T. It implies that $T - \lambda$ is Browder and hence, generalized Drazin–Riesz invertible for every $\lambda \notin m_T^{-1}(0)$. According to Theorem 4.9, X is decomposed into the direct sum $X = X_1 \oplus \cdots \oplus X_n$ where X_i is closed T-invariant subspace of X, $T = T_1 \oplus \cdots \oplus T_n$ where T_i is the reduction of T on X_i and $T_i - \lambda_i$ is Riesz, $i = 1, \dots, n$. Since $T_i - \lambda_i$ is Riesz, it follows that $\sigma_{\mathcal{B}}(T_i - \lambda_i) \subset \{0\}$ and hence, $\sigma_{\mathcal{B}}(T_i) \subset \{\lambda_i\}$, $i = 1, \dots, n$. It implies that $T_i - \lambda_j$ is Browder for $i \neq j$, $i, j \in \{1, \dots, n\}$.

According to Lemma 5.1 it follows that $\{0\} \oplus X_2 \oplus \cdots \oplus X_n = X_2 \oplus \cdots \oplus X_n$ is closed in X. Consider the decomposition

$$T - \lambda_1 = (T_1 - \lambda_1) \oplus (T_2 - \lambda_1) \oplus \cdots \oplus (T_n - \lambda_1).$$

Since $(X_1, X_2 \oplus \cdots \oplus X_n) \in \mathrm{Red}(T)$, $(T - \lambda_1)_{X_1} = T_1 - \lambda_1$ is Riesz, and since $(T - \lambda_1)_{X_2 \oplus \cdots \oplus X_n} = (T_2 - \lambda_1) \oplus \cdots \oplus (T_n - \lambda_1)$ is Browder as a direct sum of Browder operators $T_2 - \lambda_1, \dots, T_n - \lambda_1$ (Lema 5.2 (i)), it follows that $T - \lambda_1$ is generalized Drazin–Riesz invertible. In that way we can prove that $T - \lambda_i$ is generalized Drazin–Riesz invertible for every $i \in \{1, \dots, n\}$. Consequently, $T - \lambda$ is generalized Drazin–Riesz invertible for every $\lambda \in \mathbb{C}$. □

Let $T \in L(X)$ be generalized Drazin–Riesz invertible operator T, that is, there exists $(M, N) \in \operatorname{Red}(T)$ such that T_M is invertible and T_N is Riesz. For the operator T_N we shall say that it is a Riesz part of T.

A Riesz operator with infinite spectrum has the property that the sequence of its Riesz points converges to 0. In the following proposition we show that a generalized Drazin–Riesz invertible operator T, which Riesz part T_N has infinite spectrum, has the same property: there exists a sequence of nonzero Riesz points of T which converges to 0. Moreover, it holds the converse: if T admits a GKRD and there exists a sequence of nonzero Riesz points of T which converges to 0, then T is generalized Drazin–Riesz operator which Riesz part has infinite spectrum.

Proposition 5.3. *Let $T \in L(X)$. The following statements are equivalent:*

(i) $T = T_M \oplus T_N$ *where T_M is invertible and T_N is Riesz with infinite spectrum;*
(ii) T *admits a GKRD and there exists a sequence of nonzero Riesz points of T which converges to* 0.

Proof. (i)$\Longrightarrow$(ii): Suppose that $T = T_M \oplus T_N$ where T_M is invertible and T_N is Riesz with infinite spectrum. Then T admits a GKRD(M, N) and $\sigma(T_N) = \{0, \mu_1, \mu_2, \dots\}$ where μ_n, $n \in \mathbb{N}$, are nonzero Riesz points of T_N and

$$\lim_{n \to \infty} \mu_n = 0. \tag{5.13}$$

According to Theorem 5.1 we have that $0 \notin \operatorname{acc} \sigma_{\mathcal{B}}(T)$, i.e. there exists $\epsilon > 0$ such that $\mu \notin \sigma_{\mathcal{B}}(T)$ for $0 < |\mu| < \epsilon$. From (5.13) it follows that there exists $n_0 \in \mathbb{N}$ such that $0 < |\mu_n| < \epsilon$ for $n \geqslant n_0$. Hence $\mu_n \in \sigma(T) \setminus \sigma_{\mathcal{B}}(T)$ for all $n \geqslant n_0$ and, since the set $\sigma(T) \setminus \sigma_{\mathcal{B}}(T)$ is exactly the set of the Riesz points of T, we see that $(\mu_n)_{n=n_0}^{\infty}$ is the sequence of nonzero Riesz points of T which converges to 0.

(ii)$\Longrightarrow$(i): Suppose that $T = T_M \oplus T_N$ where T_M is Kato, T_N is Riesz and let (λ_n) is the sequence of nonzero Riesz points of T such that $0 = \lim_{n \to \infty} \lambda_n$. Since $\lambda_n \in \rho_{\mathcal{B}}(T)$ for all $n \in \mathbb{N}$, it follows that $0 \in \operatorname{acc} \rho_{\mathcal{B}}(T)$. As in the proof of Theorem 5.1 we conclude that T_M is invertible. Thus there exists an $\epsilon > 0$ such that $D(0, \epsilon) \subset \rho(T_M)$ and there exists $n_0 \in \mathbb{N}$ such that $\lambda_n \in D(0, \epsilon)$ for all $n \geqslant n_0$. Consequently, $\lambda_n \notin \sigma(T_M)$ for all $n \geqslant n_0$ and since $\lambda_n \in \sigma(T) = \sigma(T_M) \cup \sigma(T_N)$, it follows that $\lambda_n \in \sigma(T_N)$ for all $n \geqslant n_0$. Therefore, the spectrum of T_N is infinite. □

Corollary 5.1. *Let $T \in L(X)$ be generalized Drazin–Riesz invertible and let $0 \in \operatorname{acc} \sigma(T)$. Then there exists a sequence of nonzero Riesz points of T which converges to* 0.

Proof. According to Theorem 5.1 it follows that $T = T_M \oplus T_N$ with T_M invertible and T_N Riesz. Since $0 \in \operatorname{acc} \sigma(T)$, it follows that $0 \in \operatorname{acc} \sigma_N(T)$ and so, $\sigma_N(T)$ is infinite. Applying Proposition 5.3 we obtain that there exists a sequence of nonzero Riesz points of T which converges to 0. □

In the following text operators which are a direct sum of a Riesz operator and a bounded below (resp. surjective, upper (lower) semi-Fredholm, upper (lower) semi-Weyl, upper (lower) semi-Browder) operator are characterized. These operators generalize the class of generalized Drazin invertible operators and also the class of generalized Drazin–Riesz invertible operators, and hence we shall call them *generalized Drazin-Riesz bounded below (resp. generalized Drazin–Riesz surjective, generalized Drazin-Riesz upper (lower) semi-Fredholm, generalized Drazin–Riesz Fredholm, etc) operators*, and we shall use the following notations:

$$gDR\mathbf{R}_i(X) = \{T \in L(X) : T = T_1 \oplus T_2,\ T_1 \in \mathbf{R}_i,\ T_2 \text{ is Riesz}\}, \quad 1 \leqslant i \leqslant 12.$$

Using Proposition 5.1 ((i), (ii)), similarly to the proof of Theorem 5.1, the following theorems can be proved.

Theorem 5.3. *Let $T \in L(X)$. The following conditions are equivalent:*

- (i) *There exists $(M, N) \in \mathrm{Red}(T)$ such that T_M is bounded below and T_N is Riesz, that is $T \in gDR\mathcal{M}(X)$;*
- (ii) *T admits a GKRD and $0 \notin \mathrm{int}\,\sigma_{ap}(T)$;*
- (iii) *T admits a GKRD and T has the SVEP at 0;*
- (iv) *There exists $(M, N) \in \mathrm{Red}(T)$ such that T_M is upper semi-Browder and T_N is Riesz, that is $T \in gDR\mathcal{B}_+(X)$;*
- (v) *T admits a GKRD and $0 \notin \mathrm{acc}\,\sigma_{\mathcal{B}_+}(T)$;*
- (vi) *T admits a GKRD and $0 \notin \mathrm{int}\,\sigma_{\mathcal{B}_+}(T)$;*
- (vii) *There exists a bounded projection P on X which commutes with T such that $T + P$ is uper semi-Browder and TP is Riesz.*

Theorem 5.4. *Let $T \in L(X)$. The following conditions are equivalent:*

- (i) *There exists $(M, N) \in \mathrm{Red}(T)$ such that T_M is surjective and T_N is Riesz, that is $T \in gDR\mathcal{Q}(X)$;*
- (ii) *T admits a GKRD and $0 \notin \mathrm{int}\,\sigma_{su}(T)$;*
- (iii) *T admits a GKRD and T' has the SVEP at 0;*
- (iv) *There exists $(M, N) \in \mathrm{Red}(T)$ such that T_M is lower semi-Browder and T_N is Riesz, that is $T \in gDR\mathcal{B}_-(X)$;*
- (v) *T admits a GKRD and $0 \notin \mathrm{acc}\,\sigma_{\mathcal{B}_-}(T)$;*
- (vi) *T admits a GKRD and $0 \notin \mathrm{int}\,\sigma_{\mathcal{B}_-}(T)$;*
- (vii) *There exists a bounded projection P on X which commutes with T such that $T + P$ is lower semi-Browder and TP is Riesz.*

The condition that $0 \notin \mathrm{int}\,\sigma_{ap}(T)$ ($0 \notin \mathrm{int}\,\sigma_{su}(T)$) in the statement (ii) in Theorem 5.3 (Theorem 5.4) can not be replaced with the stronger condition that $0 \notin \mathrm{acc}\,\sigma_{ap}(T)$ ($0 \notin \mathrm{acc}\,\sigma_{su}(T)$). The example which shows that is a Riesz operator with infinite spectrum. Namely, if $T \in L(X)$ is Riesz with infinite spectrum, then obviously T is generalized Drazin–Riesz invertible, but $0 \in \mathrm{acc}\,\sigma_{ap}(T) = \mathrm{acc}\,\sigma_{su}(T)$ and $0 \notin \mathrm{int}\,\sigma_{ap}(T) = \mathrm{int}\,\sigma_{su}(T)$.

P. Aiena and E. Rosas proved that if $\lambda_0 - T$ is of Kato type, then the implications (5.7), (5.8) and (5.9) can be reversed [4]. Q. Jiang and H. Zhong [21] showed that if $\lambda_0 - T$ admits a GKD, then the following statements are equivalent:

(i) T (T') has the SVEP at λ_0;
(ii) $\sigma_{ap}(T)$ ($\sigma_{su}(T)$) does not cluster at λ_0;
(iii) λ_0 is not an interior point of $\sigma_{ap}(T)$ ($\sigma_{su}(T)$),

that is, the implications (5.7), (5.8) and (5.9) can be also reversed in the case that $\lambda_0 - T$ admits a GKD. In the following corollary this result is extended to the case of operators which admit a GKRD.

Corollary 5.2. *Let $T \in L(X)$. If $\lambda_0 - T$ admits a GKRD, then the following statements are equivalent:*
(i) T (T') has the SVEP at λ_0*;*
(ii) λ_0 *is not an interior point of* $\sigma_{ap}(T)$ ($\sigma_{su}(T)$)*;*
(iii) $\sigma_{\mathcal{B}_+}(T)$ ($\sigma_{\mathcal{B}_-}(T)$) *does not cluster at* λ_0*;*
(iv) λ_0 *is not an interior point of* $\sigma_{\mathcal{B}_+}(T)$ ($\sigma_{\mathcal{B}_-}(T)$)*.*

Proof. Follows from the equivalences (ii)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(v)$\Longleftrightarrow$(vi) in Theorem 5.3 (Theorem 5.4). □

A Riesz operator T with the infinite spectrum is an example of an operator which admits GKRD and has the SVEP, but the spectra $\sigma_{ap}(T) = \sigma_{su}(T) = \sigma(T)$ cluster at 0. So, if $\lambda_0 - T$ admits a GKRD, then the statement that T (T') has the SVEP at λ_0 is not in general equivalent to the statement that $\sigma_{ap}(T)$ ($\sigma_{su}(T)$) does not cluster at λ_0.

The previous results can be extended to the cases of essential spectra.

Theorem 5.5. *Let $T \in L(X)$ and $7 \leqslant i \leqslant 12$. The following conditions are equivalent:*
(i) *There exists $(M, N) \in \text{Red}(T)$ such that $T_M \in \mathbf{R}_i$ and T_N is Riesz, that is $T \in gDR\mathbf{R_i}(X)$;*
(ii) *T admits a GKRD and $0 \notin \operatorname{acc} \sigma_{\mathbf{R}_i}(T)$;*
(iii) *T admits a GKRD and $0 \notin \operatorname{int} \sigma_{\mathbf{R}_i}(T)$;*
(iv) *There exists a bounded projection P on X which commutes with T such that $T + P \in \mathbf{R}_i$ and TP is Riesz.*

Proof. (i)$\Longrightarrow$(ii) Suppose that there exists $(M, N) \in \text{Red}(T)$ such that $T_M \in \mathbf{R}_i$ and T_N is Riesz. As in the proof of the implication (vii)$\Longrightarrow$(viii) in Theorem 5.1 it follows that T admits a GKRD.

Since $\mathbf{R}_i$ is open, from $T_M \in \mathbf{R}_i$ it follows that there exists $\epsilon > 0$ such that $D(0, \epsilon) \subset \rho_{\mathbf{R}_i}(T_M)$. Since T_N is Riesz, according to Theorem 3.2, it follows that $\sigma_{\mathbf{R}_i}(T_N) \subset \{0\}$ and hence, $D(0, \epsilon) \setminus \{0\} \subset \rho_{\mathbf{R}_i}(T_M) \cap \rho_{\mathbf{R}_i}(T_N)$. According to Lemma 5.2(i),(ii), $\rho_{\mathbf{R}_i}(T_M) \cap \rho_{\mathbf{R}_i}(T_N) \subset \rho_{\mathbf{R}_i}(T)$ and so, $D(0, \epsilon) \setminus \{0\} \subset \rho_{\mathbf{R}_i}(T)$. Therefore, $0 \notin \operatorname{acc} \sigma_{\mathbf{R}_i}(T)$.

(ii)$\Longrightarrow$(iii) Obvious.

(iii)$\Longrightarrow$(i) Suppose that T admits a GKRD and $0 \notin \operatorname{int} \sigma_{\Phi_+}(T)$. Then there exists $(M, N) \in \text{Red}(T)$ such that T_M is Kato and T_N is Riesz and $0 \in \operatorname{acc} \rho_{\Phi_+}(T)$. According to Lema 5.2(i), it follows that $0 \in \operatorname{acc} \rho_{\Phi_+}(T_M)$. From Proposition 5.1(iii) it follows that T_M is upper semi-Fredholm. The cases $i = 8$ and $i = 9$ can be proved similarly.

Suppose that T admits a GKRD and $0 \in \text{acc}\ \rho_{\mathcal{W}_+}(T)$. Then there exists $(M,N) \in \text{Red}(T)$ such that T_M is Kato and T_N is Riesz. We show that $0 \in \text{acc}\ \rho_{\mathcal{W}_+}(T_M)$. Let $\epsilon > 0$. From $0 \in \text{acc}\ \rho_{\mathcal{W}_+}(T)$ it follows that there exists $\lambda \in \mathbb{C}$ such that $0 < |\lambda| < \epsilon$ and $T - \lambda \in \mathcal{W}_+(X)$. As T_N is Riesz, $T_N - \lambda$ is Fredholm of index zero, and so, according to Lema 5.2(iii), we conclude that $T_M - \lambda \in \mathcal{W}_+(M)$, that is $\lambda \in \rho_{\mathcal{W}_+}(T_M)$. Therefore, $0 \in \text{acc}\ \rho_{\mathcal{W}_+}(T_M)$ and from Proposition 5.1(iv) it follows that T_M is upper semi-Weyl, and so $T \in gDR\mathcal{W}_+(X)$. The cases $i = 11$ and $i = 12$ can be proved similarly.

(i)$\Longrightarrow$(iv): Suppose that there exists $(M,N) \in \text{Red}(T)$ such that $T_M \in \mathbf{R}_i$ and T_N is Riesz. Let $P \in L(X)$ be a projection such that $\mathcal{N}(P) = M$ and $\mathcal{R}(P) = N$. Then $TP = PT$ and since $TP = (TP)_M \oplus (TP)_N = 0 \oplus T_N$ and T_N is Riesz, from Lemma 4.2 it follows that TP is Riesz. Also from the fact that T_N is Riesz it follows that $\sigma_{\mathbf{R}_i}(T_N) \subset \{0\}$ and so, $(T+P)_N = T_N + I_N \in \mathbf{R}_i$, where I_N is identity on N. Since $(T+P)_M = T_M \in \mathbf{R}_i$, we have that $T + P \in \mathbf{R}_i$ by Lemma 5.2(i),(ii).

(iv)$\Longrightarrow$(i): Suppose that there exists a projection $P \in L(X)$ that commutes with T such that $T + P \in \mathbf{R}_i$ and TP is Riesz. For $M = \mathcal{N}(P)$ and $N = \mathcal{R}(P)$ we have that $(M,N) \in \text{Red}(T)$ and $T_N = (TP)_N$ is Riesz. For $i \in \{7,8,9\}$ from Lemma 5.2(i) it follows that $T_M = (T+P)_M \in \mathbf{R}_i$. Suppose that $i \in \{10,11,12\}$. Since T_N is Riesz, it follows that $T_N + I_N$ is Weyl and so, from $T+P = (T+P)_M \oplus (T+P)_N = T_M \oplus (T_N + I_N)$, according to Lemma 5.2(iii), it follows that $T_M \in \mathbf{R}_i$. □

From the equivalence (ii)$\Longleftrightarrow$(iii) in Theorems 5.5 it follows that if $\lambda_0 - T$ admits a GKRD, then λ_0 is not an interior point of $\sigma_{\mathbf{R}}(T)$ if and only if $\sigma_{\mathbf{R}}(T)$ does not cluster at λ_0 where $\mathbf{R}$ is one of $\Phi_+, \Phi_-, \Phi, \mathcal{W}_+, \mathcal{W}_-, \mathcal{W}$.

Corollary 5.3. *Let $T \in L(X)$ and let $0 \in \partial\sigma_{\mathbf{R}_i}(T)$, $1 \leqq i \leqq 12$. Then T admits a generalized Kato–Riesz decomposition if and only if T belongs to $gDR\mathbf{R}_i(X)$.*

Proof. Follows from the equivalences (i)$\Longleftrightarrow$(ii) in Theorems 5.1, 5.3, 5.5 and the equivalence (i)$\Longleftrightarrow$(iii) in Theorem 5.5. □

Theorem 5.6. *Let $T \in L(X)$ and let f be a complex analytic function in a neighborhood of $\sigma(T)$. If $T \in gDR\mathbf{R}_i(X)$ and $f^{-1}(0) \cap \sigma_{\mathbf{R_i}}(T) = \{0\}$, then $f(T) \in gDR\mathbf{R}_i(X)$, $1 \leqslant i \leqslant 12$.*

Proof. We shall prove the assertion for the case $i = 4$ and $i = 10$. Suppose that $T \in gDR\mathbf{R}_4(X)$. Then, according to Theorem 5.1, T is generalized Drazin–Riesz bounded below and there exists $(M,N) \in \text{Red}(T)$ such that T_M is bounded below and T_N is Riesz. The pair (M,N) completely reduces $(\lambda - T)^{-1}$ for every $\lambda \in \rho(T)$ and so, $f(T) = \frac{1}{2\pi i}\int_\gamma f(\lambda)(\lambda - T)^{-1} d\lambda$, where γ is a contour surrounding $\sigma(T)$ and which lies in the domain of f, is also reduced by the pair (M,N), $f(T)_M = f(T_M)$ and $f(T)_N = f(T_N)$, and consequently $f(T) = f(T_M) \oplus f(T_N)$.

Suppose also that $f^{-1}(0) \cap \sigma_{\mathcal{B}_+}(T) = \{0\}$. Using the fact that $0 \notin \sigma_{\mathcal{B}_+}(T_M) \subset \sigma_{\mathcal{B}_+}(T)$, we obtain $0 \notin f(\sigma_{\mathcal{B}_+}(T_M))$. According to the spectral mapping theorem it follows $0 \notin f(\sigma_{\mathcal{B}_+}(T_M)) = \sigma_{\mathcal{B}_+}(f(T_M))$ [38, Theorem 3.4], so $f(T_M)$ is upper semi-Browder. Since $f(0) = 0$, it follows that $f(T_N)$ is Riesz by Theorem 3.5(iv).

Consequently, $f(T) \in gDR\mathbf{R}_4(X)$, i.e. $f(T)$ is generalized Drazin–Riesz bounded below.

The cases for $i = 1, 2, 3, 5, 6, 7, 8, 9$ can be proved similarly.

Suppose that T is generalized Drazin–Riesz upper semi-Weyl and

$$f^{-1}(0) \cap \sigma_{\mathcal{W}_+}(T) = \{0\}.$$

Then there exists $(M, N) \in \mathrm{Red}(T)$ such that T_M is upper semi-Weyl and T_N is Riesz. As above we conclude that $f(T) = f(T_M) \oplus f(T_N)$ and $f(T_N)$ is Riesz. From $0 \notin \sigma_{\mathcal{W}_+}(T_M) \subset \sigma_{\mathcal{W}_+}(T)$, we obtain $0 \notin f(\sigma_{\mathcal{W}_+}(T_M))$. Since $\sigma_{\mathcal{W}_+}(f(T_M)) \subset f(\sigma_{\mathcal{W}_+}(T_M))$ [38, Theorem 3.3], it follows that $0 \notin \sigma_{\mathcal{W}_+}(f(T_M))$, and so $f(T_M)$ is upper semi-Weyl. Consequently, $f(T)$ is generalized Drazin–Riesz upper semi-Weyl.

Similarly for $i = 11, 12$. □

Proposition 5.4. *Let $T \in L(X)$ and let f be a complex analytic function in a neighborhood of $\sigma(T)$ such that $f^{-1}(0) \cap \mathrm{acc}\,\sigma(T) = \emptyset$. Then $f(T) = A + K$, where $A \in L(X)$ is generalized Dazin-Riesz Fredholm and $K \in K(X)$.*

Proof. Since $\sigma(\pi(T)) \subset \sigma(T)$, f is analytic in a neighborhood of $\sigma(\pi(T))$ and we have $f(\pi(T)) = \pi(f(T))$, where $\pi\colon L(X) \to C(X)$ denotes the natural homomorphism from $L(X)$ into the Calkin algebra $C(X)$. Then

$$\mathrm{acc}\ \sigma(\pi(f(T)) = \mathrm{acc}\ \sigma(f(\pi(T)) \subset f(\mathrm{acc}\ \sigma(\pi(T)) \subset f(\mathrm{acc}\ \sigma(T))$$

according to [17, Theorem 2]. By the assumption it follows $0 \notin \mathrm{acc}\ \sigma(\pi(f(T))$, i.e. $\pi(f(T))$ is generalized Drazin invertible. Now, we apply [7, Theorem 3.11], which completes the proof. □

Corollary 5.4. *Let $T \in L(X)$ have finite spectrum and let f be a complex analytic function in a neighborhood of $\sigma(T)$. Then $f(T) = A + K$, where $A \in L(X)$ is generalized Dazin-Riesz Fredholm and $K \in K(X)$.*

Proof. Since $\mathrm{acc}\ \sigma(T) = \emptyset$, so the condition $f^{-1}(0) \cap \mathrm{acc}\,\sigma(T) = \emptyset$ is automatically satisfied, and apply Proposition 5.4. □

Corollary 5.5. *Let $T \in L(X)$ be polynomially Riesz and let f be a complex analytic function in a neighborhood of $\sigma(T)$ such that $f^{-1}(0) \cap m_T^{-1}(0) = \emptyset$ where m_T is the minimal polynomial of T. Then $f(T) = A + K$, where $A \in L(X)$ is generalized Dazin-Riesz Fredholm and $K \in K(X)$.*

Proof. Notice that if $T \in L(X)$ is polynomially Riesz, then $\mathrm{acc}\ \sigma(T) \subset \sigma_{\mathcal{B}}(T) = m_T^{-1}(0)$, so $f^{-1}(0) \cap \mathrm{acc}\,\sigma(T) \subset f^{-1}(0) \cap m_T^{-1}(0) = \emptyset$ and we apply Proposition 5.4. □

5.2. Spectra. For $T \in L(X)$ we define the spectra with respect to the sets $gDR\mathbf{R}_i(X)$:

$$\sigma_{gDR\mathbf{R}_i}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin gDR\mathbf{R}_i(X)\}, \ \ 1 \leqslant i \leqslant 12.$$

Instead of $\sigma_{gDRL(X)^{-1}}(T)$ we shall write simpler $\sigma_{gDR}(T)$. From Theorems 5.3, 5.4 and 5.1 it follows that

$$\begin{aligned}\sigma_{gDR\mathcal{J}}(T) &= \sigma_{gKR}(T)\cup \operatorname{int}\sigma_{ap}(T) = \sigma_{gKR}(T)\cup \operatorname{int}\sigma_{\mathcal{B}_+}(T)\\ &= \sigma_{gKR}(T)\cup \operatorname{acc}\sigma_{\mathcal{B}_+}(T)) = \sigma_{gKR}(T)\cup \mathcal{S}(T),\end{aligned}\tag{5.14}$$

$$\begin{aligned}\sigma_{gDR\mathcal{Q}}(T) &= \sigma_{gKR}(T)\cup \operatorname{int}\sigma_{su}(T) = \sigma_{gKR}(T)\cup \operatorname{int}\sigma_{\mathcal{B}_-}(T)\\ &= \sigma_{gKR}(T)\cup \operatorname{acc}\sigma_{\mathcal{B}_-}(T) = \sigma_{gKR}(T)\cup \mathcal{S}(T')\end{aligned}\tag{5.15}$$

$$\begin{aligned}\sigma_{gDR}(T) &= \sigma_{gKR}(T)\cup \operatorname{int}\sigma(T) = \sigma_{gKR}(T)\cup \operatorname{int}\sigma_{\mathcal{B}}(T)\\ &= \operatorname{acc}\sigma_{\mathcal{B}}(T) = \sigma_{gKR}(T)\cup \mathcal{S}(T)\cup \mathcal{S}(T').\end{aligned}\tag{5.16}$$

From Theorem 5.5 it follows that

$$\begin{aligned}\sigma_{gD R\mathbf{R}_i}(T) &= \sigma_{gKR}(T)\cup \operatorname{acc}\, \sigma_{\mathbf{R}_i}(T)\\ &= \sigma_{gKR}(T)\cup \operatorname{int}\sigma_{\mathbf{R}_i}(T),\ 7\leqq i\leqq 12.\end{aligned}\tag{5.17}$$

Clearly,

$$\sigma_{gKR}(T)\subset \begin{matrix}\sigma_{gDR\Phi_+}(T)\subset \sigma_{gDR\mathcal{W}_+}(T)\subset \sigma_{gDR\mathcal{J}}(T)\subset \sigma_{ap}(T)\\ \sigma_{gDR\Phi_-}(T)\subset \sigma_{gDR\mathcal{W}_-}(T)\subset \sigma_{gDR\mathcal{Q}}(T)\subset \sigma_{su}(T),\end{matrix}\tag{5.18}$$

$$\sigma_{gKR}(T)\subset \begin{matrix}\sigma_{gDR\Phi_+}(T)\subset \sigma_{gDR\mathcal{W}_+}(T)\subset \sigma_{gDR\mathcal{J}}(T)\\ \sigma_{gDR\Phi_-}(T)\subset \sigma_{gDR\mathcal{W}_-}(T)\subset \sigma_{gDR\mathcal{Q}}(T)\end{matrix}\subset \sigma_{gDR}(T)\tag{5.19}$$

$$\sigma_{gKR}(T)\subset \sigma_{gDR\Phi}(T)\subset \sigma_{gDR\mathcal{W}}(T)\subset \sigma_{gDR}(T).\tag{5.20}$$

Remark 5.1. We remark that

$$\Phi_+(X)\setminus \mathcal{W}_+(X)\subset gDR\Phi_+(X)\setminus gDR\mathcal{W}_+(X),$$

$$\Phi_-(X)\setminus \mathcal{W}_-(X)\subset gDR\Phi_-(X)\setminus gDR\mathcal{W}_-(X),\tag{5.21}$$

$$\Phi(X)\setminus \mathcal{W}(X)\subset gDR\Phi(X)\setminus gDR\mathcal{W}(X).\tag{5.22}$$

Indeed, as the index is locally constant, the set $\Phi_+(X)\setminus\mathcal{W}_+(X) = \{T\in\Phi(X) : \operatorname{ind}(T) > 0\}$ is open, which implies that the set $\sigma_{\mathcal{W}_+}(T)\setminus\sigma_{\Phi_+}(T) = \rho_{\Phi_+}(T)\setminus \rho_{\mathcal{W}_+}(T)$ is open for every $T\in L(X)$. Suppose that $T\in\Phi_+(X)\setminus\mathcal{W}_+(X)$. Then $T\in gDR\Phi_+(X)$ and $0\in\sigma_{\mathcal{W}_+}(T)\setminus\sigma_{\Phi_+}(T)$. There exists $\epsilon>0$ such that $D(0,\epsilon)\subset \sigma_{\mathcal{W}_+}(T)\setminus\sigma_{\Phi_+}(T)$. Hence, $0\in \operatorname{int}\sigma_{\mathcal{W}_+}(T)$ and $T\notin gDR\mathcal{W}_+(X)$ according to Theorem 5.5. Similarly for (5.21) and (5.22).

The following example shows that the inclusions $\sigma_{gDR\Phi_+}(T)\subset\sigma_{gDR\mathcal{W}_+}(T)$, $\sigma_{gDR\Phi_-}(T)\subset\sigma_{gDR\mathcal{W}_-}(T)$ and $\sigma_{gDR\Phi}(T)\subset\sigma_{gDR\mathcal{W}}(T)$ can be proper.

Example 5.1. If X is one of $c_0(\mathbb{N}), c(\mathbb{N}), \ell_\infty(\mathbb{N}), \ell_p(\mathbb{N})$, $p\geqslant 1$, the forward and the backward unilateral shifts U and V on X are Fredholm, $\operatorname{ind}(U)=-1$ and $\operatorname{ind}(V)=1$. Therefore, $U\in\Phi_-(X)\setminus\mathcal{W}_-(X)$ and $V\in\Phi_+(X)\setminus\mathcal{W}_+(X)$, and also $U,V\in\Phi(X)\setminus\mathcal{W}(X)$. Hence, according to Remark 5.1, $U\in gDR\Phi_-(X)\setminus gDR\mathcal{W}_-(X)$, $V\in gDR\Phi_+(X)\setminus gDR\mathcal{W}_+(X)$ and $U,V\in gDR\Phi(X)\setminus gDR\mathcal{W}(X)$. This implies that $0\in\sigma_{gDR\mathcal{W}_-}(U)\setminus\sigma_{gDR\Phi_-}(U)$, $0\in\sigma_{gDR\mathcal{W}_+}(V)\setminus\sigma_{gDR\Phi_+}(V)$ and $0\in\sigma_{gDR\mathcal{W}}(U)\setminus\sigma_{gDR\Phi}(U)$.

The following example shows that the inclusions $\sigma_{gDR\mathcal{W}_+}(T)\subset\sigma_{\mathbf{gDR}\mathcal{J}}(T)$ and $\sigma_{gDR\mathcal{W}_-}(T)\subset\sigma_{\mathbf{gDR}\mathcal{Q}}(T)$ can be proper. Set $\mathbb{D}=\{\lambda\in\mathbb{C} : |\lambda|\leqslant 1\}$.

Example 5.2. If X is one of $c_0(\mathbb{N}), c(\mathbb{N}), \ell_\infty(\mathbb{N}), \ell_p(\mathbb{N})$, $p \geqslant 1$, let $T = U \oplus V$ where U and V are forward and the backward unilateral shifts, respectively. Then, according to Lemma 5.2(i), T is Fredholm and $\operatorname{ind}(T) = \operatorname{ind}(U) + \operatorname{ind}(V) = 0$. Thus T is Weyl and hence, T is generalized Drazin–Riesz Weyl. Since $\sigma_{ap}(U) = \sigma_{su}(V) = \partial\mathbb{D}$ and $\sigma_{su}(U) = \sigma_{ap}(V) = \mathbb{D}$, it follows that $\sigma_{ap}(T) = \sigma_{ap}(U) \cup \sigma_{ap}(V) = \mathbb{D}$ and $\sigma_{su}(T) = \sigma_{su}(U) \cup \sigma_{su}(V) = \mathbb{D}$. Therefore, $0 \in \operatorname{int}\sigma_{ap}(T)$ and $0 \in \operatorname{int}\sigma_{su}(T)$ and from Theorems 5.3 and 5.4 it follows that T is neither generalized Drazin–Riesz bounded below nor generalized Drazin–Riesz surjective and so, $0 \in \sigma_{\mathbf{gDR}\mathcal{J}}(T) \setminus \sigma_{gDR\mathcal{W}_+}(T)$ and $0 \in \sigma_{\mathbf{gDR}\mathcal{Q}}(T) \setminus \sigma_{gDR\mathcal{W}_-}(T)$.

We need the following result.

Proposition 5.5. [34, p. 143] *Let $T \in L(X)$ and $(M, N) \in \operatorname{Red}(T)$. Then T is essentially Kato if and only if T_M and T_N are essentially Kato.*

Theorem 5.7. *Let $T \in L(X)$ and let T admits a GKRD(M, N). Then there exists $\epsilon > 0$ such that $T - \lambda$ is essentially Kato for each λ such that $0 < |\lambda| < \epsilon$.*

Proof. If $M = \{0\}$, then T is Riesz and hence $T - \lambda$ is Fredholm for all $\lambda \neq 0$. From [36, Theorem 16.21] it follows that $T - \lambda$ is essentially Kato for all $\lambda \neq 0$.

Suppose that $M \neq \{0\}$. From [2, Theorem 1.31] it follows that $|\lambda| < \gamma(T_M)$, $T_M - \lambda$ is Kato. Since T_N is Riesz, then $T_N - \lambda$ is esentially Kato for all $\lambda \neq 0$. Let $\epsilon = \gamma(T_M)$. From Proposition 5.5 it follows that $T - \lambda$ is essentially Kato for each λ such that $0 < |\lambda| < \epsilon$. □

Corollary 5.6. *Let $T \in L(X)$. Then $\sigma_{gKR}(T)$ is compact and the sets $\sigma_{eK} \setminus \sigma_{gKR}(T)$, $\sigma_{Kt}(T) \setminus \sigma_{gKR}(T)$ and $\sigma_{gK}(T) \setminus \sigma_{gKR}(T)$ consist of at most countably many points.*

Proof. From Theorem 5.7 it follows that $\sigma_{gKR}(T)$ is closed and since $\sigma_{gKR}(T) \subset \sigma(T)$ it follows that $\sigma_{gKR}(T)$ is bounded. Thus, $\sigma_{gKR}(T)$ is compact.

Let $\lambda_0 \in \sigma_{eK}(T) \setminus \sigma_{gKR}(T)$. Then $T - \lambda_0$ admits a GKRD and according to Theorem 5.7 there exists $\epsilon > 0$ such that $T - \lambda$ is essentially Kato for each λ such that $0 < |\lambda - \lambda_0| < \epsilon$. This means that $\lambda_0 \in \operatorname{iso}\sigma_{eK}(T)$. Therefore $\sigma_{eK}(T) \setminus \sigma_{gKR}(T) \subset \operatorname{iso}\sigma_{eK}(T)$ and since $\sigma_{gK}(T) \subset \sigma_{Kt}(T) \subset \sigma_{eK}(T)$, it follows that $\sigma_{Kt}(T) \setminus \sigma_{gKR}(T) \subset \operatorname{iso}\sigma_{eK}(T)$ and $\sigma_{gK}(T) \setminus \sigma_{gKR}(T) \subset \operatorname{iso}\sigma_{eK}(T)$, which implies that $\sigma_{eK} \setminus \sigma_{gKR}(T)$, $\sigma_{Kt}(T) \setminus \sigma_{gKR}(T)$ and $\sigma_{gK}(T) \setminus \sigma_{gKR}(T)$ are at most countable. □

Proposition 5.6. *For $T \in L(X)$ the following statements hold:*

(i) $\sigma_{gDR\mathbf{R}_i}(T) \subset \sigma_{\mathbf{R}_i}(T) \subset \sigma(T)$, $1 \leqq i \leqq 12$;
(ii) $\sigma_{gDR\mathbf{R}_i}(T)$ *is a compact subset of* $\mathbb{C}$, $1 \leqq i \leqq 12$;
(iii) $\sigma_{\mathbf{R}_i}(T) \setminus \sigma_{gDR\mathbf{R}_i}(T)$ *consists of at most countably many isolated points,* $4 \leqslant i \leqslant 12$.

Proof. (i): It is obvious.

(ii): It suffices to prove that the complement of $\sigma_{gDR\mathbf{R}_i}(T)$ is open. If $\lambda_0 \notin \sigma_{gDR\mathcal{J}}(T)$, then $T - \lambda_0 \in gDR\mathcal{J}(X)$ and by Theorem 5.3 there exists $\epsilon > 0$ such that $T - \lambda_0 - \lambda \in \mathcal{B}_+(X) \subset gDR\mathcal{B}_+(X) = gDR\mathcal{J}(X)$ for $0 < |\lambda| < \epsilon$.

Thus, $T - \lambda_0 - \lambda \in gDR\mathcal{J}(X)$ for each λ such that $|\lambda| < \epsilon$, that is $D(\lambda_0, \epsilon) \subset \mathbb{C} \setminus \sigma_{gDR\mathcal{J}}(T)$. Therefore, $\mathbb{C} \setminus \sigma_{gDR\mathcal{J}}(T)$ is open. For the other cases we apply similar consideration.

(iii): If $\lambda \in \sigma_{\mathcal{B}_+}(T) \setminus \sigma_{gDR\mathcal{B}_+}(T)$, then $\lambda \in \sigma_{\mathcal{B}_+}(T)$ and $T - \lambda \in gDR\mathcal{B}_+(X)$. From Theorem 5.3 we obtain that $\lambda \in \mathrm{iso}\, \sigma_{\mathcal{B}_+}(T)$ and thus, $\sigma_{\mathcal{B}_+}(T) \setminus \sigma_{gDR\mathcal{B}_+}(T)$ consists of at most countably many isolated points. Similarly for the other cases when $5 \leqslant i \leqslant 12$. □

Corollary 5.7. *Let $T \in L(X)$ and $i \in \{1, \dots, 12\}$. If T is an operator for which*

$$\sigma_{\mathbf{R}_i}(T) = \partial\sigma_{\mathbf{R}_i}(T), \tag{5.23}$$

then $\sigma_{gKR}(T) = \sigma_{gDR\mathbf{R}_i}(T)$. In particular, if $\sigma(T) = \partial\sigma(T)$, then $\sigma_{gKR}(T) = \sigma_{gDR}(T)$.

Proof. From (5.23) it follows that $\mathrm{int}\, \sigma_{\mathbf{R}_i}(T) = \emptyset$ and so, according to (5.14), (5.15), (5.16), (5.17), we get $\sigma_{gDR\mathbf{R}_i}(T) = \sigma_{gKR}(T)$. □

From Corollary 5.7 it follows that if $\sigma(T)$ is at most countable or contained in a line, then $\sigma_{gKR}(T) = \sigma_{gDR}(T)$. Every self-adjoint, as well as, unitary operator on Hilbert space have the spectrum contained in a line. The spectrum of a polynomially Riesz operator [53] or polynomially meromorphic operator [24] is at most countable.

Proof. Follows from the equivalences (ii)$\Longleftrightarrow$(vi) in Theorems 5.3 and 5.4, and the equivalence (ii)$\Longleftrightarrow$(ix) in Theorem 5.1. □

Corollary 5.8. *Let $T \in L(X)$ have the SVEP. Then all accumulation points of $\sigma_{\mathcal{B}_+}(T)$ belong to $\sigma_{gKR}(T)$.*

Proof. Follows from the equivalence (iii)$\Longleftrightarrow$(v) of Theorem 5.3. □

Corollary 5.9. *Suppose that for $T \in L(X)$, T' has the SVEP. Then all accumulation points of $\sigma_{\mathcal{B}_-}(T)$ belong to $\sigma_{gKR}(T)$.*

Proof. Follows from the equivalence (iii)$\Longleftrightarrow$(v) of Theorem 5.4. □

Corollary 5.10. *Suppose that both T and T' have the SVEP. Then all accumulation points of $\sigma_{\mathcal{B}}(T)$ belong to $\sigma_{gKR}(T)$.*

Proof. Follows from the equivalence (iii)$\Longleftrightarrow$(viii) of Theorem 5.1. □

Corollary 5.11. *Let T be unilateral weighted right shift operator on $\ell_p(\mathbb{N})$, $1 \leqslant p < \infty$, with weight (ω_n), and let $c(T) = \lim_{n\to\infty} \inf(\omega_1 \cdot \dots \cdot \omega_n)^{1/n} = 0$. Then $\sigma_{gKR}(T) = \sigma_{gDR\mathbf{R}_i}(T) = \sigma(T) = \overline{D(0, r(T))}$.*

Proof. According to [2, Corollary 3.118] it follows that $\sigma_{ap}(T) = \sigma_{\mathcal{B}_+}(T) = \sigma(T) = \overline{D(0, r(T))}$ and T has the SVEP. Since every $\lambda \in \sigma(T)$ is an accumulation point of $\sigma_{\mathcal{B}_+}(T)$, according to Corollary 5.8 it follows that $\sigma_{gKR}(T) = \sigma_{gDR\mathbf{R}_i}(T) = \sigma(T) = \overline{D(0, r(T))}$. □

Theorem 5.8. *Let $T \in L(X)$ and $4 \leqslant i \leqslant 12$. Then the following implication holds:*

$$0 \in \partial\sigma_{\mathbf{R}_i}(T) \text{ and } T \text{ admits a GKRD} \Longrightarrow 0 \in \operatorname{iso}\sigma_{\mathbf{R}_i}(T).$$

Moreover,

$$0 \in \partial\sigma_{ap}(T) \text{ and } T \text{ admits a GKRD} \Longrightarrow 0 \notin \operatorname{acc}\sigma_{\mathcal{B}_+}(T);$$

$$0 \in \partial\sigma_{su}(T) \text{ and } T \text{ admits a GKRD} \Longrightarrow 0 \notin \operatorname{acc}\sigma_{\mathcal{B}_-}(T); \tag{5.24}$$

$$0 \in \partial\sigma(T) \text{ and } T \text{ admits a GKRD} \Longrightarrow 0 \notin \operatorname{acc}\sigma_{\mathcal{B}}(T). \tag{5.25}$$

Proof. Let $0 \in \partial\sigma_{\mathcal{B}_+}(T)$ and let T admit a GKRD. Then $0 \in \sigma_{\mathcal{B}_+}(T)$ and $0 \notin \operatorname{int}\sigma_{\mathcal{B}_+}(T)$. From the equivalence (v)$\Longleftrightarrow$(vi) in Theorem 5.3, it follows that $0 \notin \operatorname{acc}\sigma_{\mathcal{B}_+}(T)$. Since $0 \in \sigma_{\mathcal{B}_+}(T)$, it means that $0 \in \operatorname{iso}\sigma_{\mathcal{B}_+}(T)$. The remaining cases ($5 \leqslant i \leqslant 12$) can be proved in a similar way.

Let $0 \in \partial\sigma_{ap}(T)$ and let T admit a GKRD. Then $0 \in \sigma_{ap}(T)$ and $0 \notin \operatorname{int}\sigma_{ap}(T)$. From the equivalence (ii)$\Longleftrightarrow$(v) in Theorem 5.3, it follows that $0 \notin \operatorname{acc}\sigma_{\mathcal{B}_+}(T)$. Similarly for (5.24) and (5.25). ☐

Theorem 5.9. *Let $T \in L(X)$. Then the following inclusions hold:*

$$\partial\sigma_{\mathbf{R}_i}(T) \cap \operatorname{acc}\sigma_{\mathbf{R}_i}(T) \subset \sigma_{gKR}(T), \quad 4 \leqslant i \leqslant 12. \tag{5.26}$$

Moreover,

$$\partial\sigma_{ap}(T) \cap \operatorname{acc}\sigma_{\mathcal{B}_+}(T) \subset \sigma_{gKR}(T);$$

$$\partial\sigma_{su}(T) \cap \operatorname{acc}\sigma_{\mathcal{B}_-}(T) \subset \sigma_{gKR}(T);$$

$$\partial\sigma(T) \cap \operatorname{acc}\sigma_{\mathcal{B}}(T) \subset \sigma_{gKR}(T). \tag{5.27}$$

Proof. Follows from Theorem 5.8. ☐

It follows an example of a operator which does not admit a GKRD.

Example 5.3. Let X be an infinite dimensional Banach space and let $\tilde{X} = \oplus_{n=1}^{\infty} X_i$ where $X_i = X$, $i \in \mathbb{N}$. Let $A = \oplus_{n=1}^{\infty}(\frac{1}{n}I)$ where I is identity on X. Then $A \in L(\tilde{X})$ and $\sigma(A) = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. The operator $A-(1/n)$ has finite ascent and descent and hence $1/n$ is a pole of the resolvent of A, but $\alpha(A-(1/n)) = \beta(A-(1/n)) = +\infty$ and so, $1/n$ is a pole of the infinite algebraic multiplicity and $1/n$ belongs to the Fredholm spectrum of A. Consequently, A is meromorphic and Fredholm spectrum of A is equal to the spectrum of A. Therefore, $\sigma_{\mathcal{B}}(A) = \sigma_{\mathcal{W}}(A) = \sigma_{\Phi}(A) = \sigma(A)$. From (5.27) it follows that $\{0\} = \partial\sigma(A) \cap \operatorname{acc}\sigma_{\mathcal{B}}(A) \subset \sigma_{gKR}(A)$. Since $\sigma_{gKR}(A) \subset \sigma_{gK}(A) \subset \sigma_{gD}(A) = \operatorname{acc}\sigma(A) = \{0\}$, we get $\sigma_{gKR}(A) = \{0\}$ and hence, A does not admit a GKRD. Also, we remark that $\sigma_{\Phi_+}(A) = \sigma_{\Phi_-}(A) = \sigma_{\mathcal{W}_+}(A) = \sigma_{\mathcal{W}_-}(A) = \sigma_{\mathcal{B}_+}(A) = \sigma_{\mathcal{B}_-}(A) = \sigma_{ap}(A) = \sigma_{su}(A) = \sigma(A)$ and $0 \notin \operatorname{int}\sigma(T)$. This means that for $T \in L(X)$ the condition that $0 \notin \operatorname{int}\sigma_{\mathbf{R}_i}(T)$, $i \in \{1, \dots, 12\}$ is not sufficient for T to admit a GKRD. Therefore, the condition that the operator admits a GKRD in the statements (ii) and (ix) of Theorem 5.1, as well as in the statements (ii) and (vi) of Theorems 5.3 and 5.4, and also, in the statements (iii) of Theorem 5.5, can not be ommited.

Theorem 5.10. *Let $T \in L(X)$. Then*

$$\begin{array}{ccccccccc}
 & & \partial\,\sigma_{gDR\mathcal{M}}(T) & \subset & \partial\,\sigma_{gDR\mathcal{W}_+}(T) & \subset & \partial\,\sigma_{gDR\Phi_+}(T) & & \\
 & \subset & & \subset & & \subset & & \subset & \\
\partial\,\sigma_{gDR}(T) & \subset & \partial\,\sigma_{gDR\mathcal{W}}(T) & \subset & \partial\,\sigma_{gDR\Phi}(T) & \subset & & & \partial\,\sigma_{gKR}(T) \\
 & \subset & & \subset & & \subset & & \subset & \\
 & & \partial\,\sigma_{gDR\mathcal{Q}}(T) & \subset & \partial\,\sigma_{gDR\mathcal{W}_-}(T) & \subset & \partial\,\sigma_{gDR\Phi_-} & &
\end{array}$$

$$\partial\sigma_{gDR\Phi}(T) \subset \partial\sigma_{gDR\Phi_+}(T),\ \partial\sigma_{gDR\Phi}(T) \subset \partial\sigma_{gDR\Phi_-}(T),$$
$$\partial\sigma_{gDR\mathcal{W}}(T) \subset \partial\sigma_{gDR\mathcal{W}_+}(T),\ \partial\sigma_{gDR\mathcal{W}}(T) \subset \partial\sigma_{gDR\mathcal{W}_-}(T),$$

and

$$\begin{aligned}\eta\sigma_{gKR}(T) &= \eta\sigma_{gDR\Phi_+}(T) = \eta\sigma_{gDR\mathcal{W}_+}(T) = \eta\sigma_{gDR\mathcal{J}}(T) \\ &= \eta\sigma_{gDR\Phi_-}(T) = \eta\sigma_{gDR\mathcal{W}_-}(T) = \eta\sigma_{gDR\mathcal{Q}}(T) \\ &= \eta\sigma_{gDR\Phi}(T) = \eta\sigma_{gDR\mathcal{W}}(T) = \eta\sigma_{gDR}(T).\end{aligned} \tag{5.28}$$

Proof. According to (4.2) and the inclusions (5.19) and (5.20) it is sufficient to prove the inclusions $\partial\sigma_{gDR\mathbf{R_i}}(T) \subset \sigma_{gKR}(T)$, $i \in \{1, 2, 3, 7, 8, 9, 10, 11, 12\}$.

Suppose that $\lambda_0 \in \partial\sigma_{gDR\mathbf{R_i}}(T)$. Since $\sigma_{gDR\mathbf{R_i}}(T)$ is closed, it follows that

$$\lambda_0 \in \sigma_{gDR\mathbf{R_i}}(T) = \sigma_{gKR}(T) \cup \operatorname{int}\sigma_{\mathbf{R_i}}(T). \tag{5.29}$$

We shall prove that

$$\lambda_0 \notin \operatorname{int}\sigma_{\mathbf{R_i}}(T). \tag{5.30}$$

Suppose that $\lambda_0 \in \operatorname{int}\sigma_{\mathbf{R_i}}(T)$. Then there exists an $\epsilon > 0$ such that $D(\lambda_0, \epsilon) \subset \sigma_{\mathbf{R_i}}(T)$. This implies that $D(\lambda_0, \epsilon) \subset \operatorname{int}\sigma_{\mathbf{R_i}}(T)$ and hence, $D(\lambda_0, \epsilon) \subset \sigma_{gDR\mathbf{R_i}}(T)$, which contradicts to the fact that $\lambda_0 \in \partial\sigma_{gDR\mathbf{R_i}}(T)$. Now from (5.29) and (5.30), it follows that $\lambda_0 \in \sigma_{gKR}(T)$. □

From (5.28) it follows that $\sigma_{gKR}(T)$ is at most countable if and only if $\sigma_{gDR\mathbf{R}_i}(T)$ is at most countable for arbitrary $i \in \{1, \dots, 12\}$, and in that case $\sigma_{gKR}(T) = \sigma_{gDR\mathbf{R}_i}(T)$ for all $i \in \{1, \dots, 12\}$. Also, from (5.28) it follows that $\sigma_{gKR}(T) = \emptyset$ if and only if $\sigma_{gDR\mathbf{R}_i}(T) = \emptyset$ where i is one of $1, \dots, 12$. Moreover, the following theorem holds:

Theorem 5.11. *Let $T \in L(X)$. The following statements are equivalent:*

(i) $\sigma_{gKR}(T) = \emptyset$;
(ii) $\sigma_{gDR}(T) = \emptyset$;
(iii) *T is polynomially Riesz;*
(iv) *$\sigma_{\mathcal{B}}(T)$ is a finite set;*

Proof. The equivalence (i)⟺(ii) follows from (5.28).

The equivalence (iii)⟺(iv) has been proved in Theorem 4.1.

(ii)⟺(iv): According to (5.16) we have that $\sigma_{gDR}(T) = \emptyset$ if and only if $\operatorname{acc}\sigma_{\mathcal{B}}(T) = \emptyset$, which is equivalent to the fact that $\sigma_{\mathcal{B}}(T)$ is a finite set. □

Remark 5.2. The implication (iii)⟹(ii) follows also from Theorem 5.2.

The inclusion $\sigma_{gKR}(T) \subset \sigma_{gK}(T)$ can be proper as it can be seen on the example of a Riesz operator T with infinite spectrum. Namely, according to Theorem 5.11 it follows that $\sigma_{gKR}(T) = \emptyset$, while $\sigma_{gK}(T) = \{0\}$. Moreover, if T is polynomially Riesz with infinite spectrum, then $\sigma_{gKR}(T) = \emptyset$, while from [22, p. 700] we have that $\sigma_{gK}(T) = \sigma_{gD}(T) = \operatorname{acc}\sigma(T) \neq \emptyset$. □

The generalized Drazin–Riesz resolvent set of $T \in L(X)$ is defined by $\rho_{gKR}(T) = \mathbb{C} \setminus \sigma_{gKR}(T)$.

Corollary 5.12. *Let $T \in L(X)$ and let $\rho_{gKR}(T)$ has only one component. Then $\sigma_{gKR}(T) = \sigma_{gDR}(T)$.*

Proof. Since $\rho_{gKR}(T)$ has only one component, it follows that $\sigma_{gKR}(T)$ has no holes, and so $\sigma_{gKR}(T) = \eta\sigma_{gKR}(T)$. From (5.28) it follows that $\sigma_{gDR}(T) \supset \sigma_{gKR}(T) = \eta\sigma_{gKR}(T) = \eta\sigma_{gDR}(T) \supset \sigma_{gDR}(T)$ and hence, $\sigma_{gDR}(T) = \sigma_{gKR}(T)$. □

Theorem 5.12. *Let $T \in L(X)$ and $4 \leqslant i \leqslant 12$. If $\partial\sigma_{\mathbf{R}_i}(T) \subset \operatorname{acc}\sigma_{\mathbf{R}_i}(T)$, then*

$$\partial\sigma_{\mathbf{R}_i}(T) \subset \sigma_{gKR}(T) \subset \sigma_{gK}(T) \subset \sigma_{Kt}(T) \subset \sigma_{eK}(T) \subset \sigma_{\mathbf{R}_i}(T), \tag{5.31}$$

$$\partial\sigma_{\mathbf{R}_i}(T) \subset \sigma_{gKR}(T) \subset \sigma_{gDR\mathbf{R}_i}(T) \subset \sigma_{\mathbf{R}_i}(T), \tag{5.32}$$

$$\eta\sigma_{\mathbf{R}_i}(T) = \eta\sigma_{gKR}(T) = \eta\sigma_{gK}(T) = \eta\sigma_{Kt}(T) = \eta\sigma_{eK}(T) = \eta\sigma_{gDR\mathbf{R}_i}(T). \tag{5.33}$$

If $1 \leqslant i \leqslant 3$, then $\partial\sigma_{\mathbf{R}_i}(T) \subset \operatorname{acc}\sigma_{\mathbf{R}_{i+3}}(T)$ implies

$$\partial\sigma_{\mathbf{R}_i}(T) \subset \sigma_{gKR}(T) \subset \sigma_{gK}(T) \subset \sigma_{Kt}(T) \subset \sigma_{eK}(T) \subset \sigma_{K}(T) \subset \sigma_{\mathbf{R}_i}(T)$$

$$\begin{aligned}\eta\sigma_{\mathbf{R}_i}(T) &= \eta\sigma_{gKR}(T) = \eta\sigma_{gK}(T) = \eta\sigma_{Kt}(T)\\ &= \eta\sigma_{eK}(T) = \eta\sigma_{K}(T) = \eta\sigma_{gDR\mathbf{R}_i}(T).\end{aligned}$$

Proof. From $\partial\sigma_{\mathbf{R}_i}(T) \subset \operatorname{acc}\sigma_{\mathbf{R}_i}(T)$ it follows that $\partial\sigma_{\mathbf{R}_i}(T) \cap \operatorname{acc}\sigma_{\mathbf{R}_i}(T) = \partial\sigma_{\mathbf{R}_i}(T)$, and so from (5.26) it follows that $\partial\sigma_{\mathbf{R}_i}(T) \subset \sigma_{gKR}(T)$. (5.33) follows from (5.31), (5.32) and (4.2).

For $1 \leqslant i \leqslant 3$, remark that $\sigma_K(T) \subset \sigma_{\mathbf{R}_i}(T)$ and we can proceed analogously as above. □

The Goldberg spectrum of $T \in L(X)$ is defined by

$$\sigma_{ec}(T) = \{\lambda \in \mathbb{C} : \mathcal{R}(T-\lambda) \text{ is not closed}\}.$$

Obviously, $\sigma_{ec}(T) \subset \sigma_{R_i}(T)$ for all $i \in \{1, \dots, 12\}$.

Theorem 5.13. *Let $T \in L(X)$ and $4 \leqslant i \leqslant 12$. If*

$$\sigma_{\mathbf{R}_i}(T) = \partial\sigma_{\mathbf{R}_i}(T) = \operatorname{acc}\sigma_{\mathbf{R}_i}(T),$$

then

$$\sigma_{ec}(T) \subset \sigma_{gKR}(T) = \sigma_{gK}(T) = \sigma_{Kt}(T) = \sigma_{eK}(T) = \sigma_{\mathbf{R}_i}(T) = \sigma_{gDR\mathbf{R}_i}(T).$$

For $1 \leqslant i \leqslant 3$, $\sigma_{\mathbf{R}_i}(T) = \partial\sigma_{\mathbf{R}_i}(T) \subset \operatorname{acc}\sigma_{\mathbf{R}_{i+3}}(T)$, implies

$$\begin{aligned}\sigma_{ec}(T) \subset \sigma_{gKR}(T) &= \sigma_{gK}(T) = \sigma_{Kt}(T) = \sigma_{eK}(T)\\ &= \sigma_{K}(T) = \sigma_{\mathbf{R}_i}(T) = \sigma_{gDR\mathbf{R}_i}(T).\end{aligned}$$

Proof. Suppose that $\sigma_{ap}(T) = \partial\sigma_{ap}(T)$ and that every $\lambda \in \sigma_{ap}(T)$ is an accumulation point of $\sigma_{\mathcal{B}_+}(T)$. From Theorem 5.12 it follows that

$$\sigma_{ap}(T) = \partial\sigma_{ap}(T) \subset \sigma_{gKR}(T) \subset \sigma_{gK}(T) \subset \sigma_{Kt}(T) \subset \sigma_{eK}(T) \subset \sigma_K(T) \subset \sigma_{ap}(T),$$
$$\sigma_{ap}(T) = \partial\sigma_{ap}(T) \subset \sigma_{gKR}(T) \subset \sigma_{gDR\mathcal{J}}(T) \subset \sigma_{ap}(T),$$

and so $\sigma_{ec}(T) \subset \sigma_{ap}(T) = \sigma_{gKR}(T) = \sigma_{gK}(T) = \sigma_{Kt}(T) = \sigma_{eK}(T) = \sigma_K(T) = \sigma_{gDR\mathcal{J}}(T)$.

The other cases ($i = 2, \dots, 12$) can be proved similarly. □

We remark that if $K \subset \mathbb{C}$ is compact, then for $\lambda \in \partial K$ there is equivalence:

$$\lambda \in \operatorname{acc} K \Longleftrightarrow \lambda \in \operatorname{acc} \partial K. \tag{5.34}$$

Theorem 5.14. *Let $T \in L(X)$ be an operator for which $\sigma_{ap}(T) = \partial\sigma(T)$ and every $\lambda \in \partial\sigma(T)$ is not isolated in $\sigma(T)$. Then $\sigma_{ec}(T) \subset \sigma_{ap}(T) = \sigma_{gKR}(T) = \sigma_{gK}(T) = \sigma_{Kt}(T) = \sigma_{eK}(T) = \sigma_K(T) = \sigma_{gDR\mathcal{J}}(T)$.*

Proof. From $\sigma_{ap}(T) = \partial\sigma(T)$, since $\partial\sigma(T) \subset \partial\sigma_{ap}(T) \subset \sigma_{ap}(T)$ it follows that $\sigma_{ap}(T) = \partial\sigma_{ap}(T)$, while from (5.34) it follows that every $\lambda \in \partial\sigma(T)$ is not isolated in $\partial\sigma(T)$. Therefore, every $\lambda \in \partial\sigma(T)$ is not isolated in $\sigma_{ap}(T)$ and hence, $\sigma_{ap}(T) \subset \operatorname{acc}\sigma_{ap}(T)$. Since $\operatorname{acc}\sigma_{ap}(T) \subset \sigma_{\mathcal{B}_+}(T) \subset \sigma_{ap}(T)$ [36, Corollary 20.20], we get that $\sigma_{ap}(T) = \operatorname{acc}\sigma_{ap}(T) = \sigma_{\mathcal{B}_+}(T)$. Hence $\operatorname{acc}\sigma_{\mathcal{B}_+}(T) = \operatorname{acc}\sigma_{ap}(T) = \sigma_{ap}(T) = \partial\sigma_{ap}(T)$. Now from Theorem 5.13 we get $\sigma_{ec}(T) \subset \sigma_{ap}(T) = \sigma_{gKR}(T) = \sigma_{gK}(T) = \sigma_{Kt}(T) = \sigma_{eK}(T) = \sigma_K(T) = \sigma_{gDR\mathcal{J}}(T)$. □

Theorem 5.15. *Let $T \in L(X)$ be an operator for which $\sigma_{su}(T) = \partial\sigma(T)$ and every $\lambda \in \partial\sigma(T)$ is not isolated in $\sigma(T)$. Then $\sigma_{ec}(T) \subset \sigma_{su}(T) = \sigma_{gKR}(T) = \sigma_{gK}(T) = \sigma_{Kt}(T) = \sigma_{eK}(T) = \sigma_K(T) = \sigma_{gDR\mathcal{Q}}(T)$.*

Proof. Follows from the inclusions $\partial\sigma(T) \subset \partial\sigma_{su}(T) \subset \sigma_{su}(T)$, (5.34) and Theorem 5.13, analogously to the proof of Theorem 5.14. □

Example 5.4. If X is one of $c_0(\mathbb{Z})$ and $\ell_p(\mathbb{Z})$, $p \geqslant 1$, then for the forward and backward bilateral shifts $W_1,\ W_2 \in L(X)$ there are equalities

$$\sigma_{ap}(W_1) = \sigma_{su}(W_1) = \sigma(W_1) = \partial\mathbb{D}, \tag{5.35}$$
$$\sigma_{ap}(W_2) = \sigma_{su}(W_2) = \sigma(W_2) = \partial\mathbb{D}. \tag{5.36}$$

For every $i = 1, \dots, 12$, from Theorem 5.13 (or Theorems 5.14 and 5.15), (5.35), (5.36) and (5.16) it follows that

$$\sigma_{gKR}(W_1) = \sigma_{gDR\mathbf{R}_i}(W_1) = \sigma_{gDR}(W_1) = \sigma(W_1) = \partial\mathbb{D},$$
$$\sigma_{gKR}(W_2) = \sigma_{gDR\mathbf{R}_i}(W_2) = \sigma_{gDR}(W_2) = \sigma(W_2) = \partial\mathbb{D}.$$

It follows also from Corollary 5.7 and the inclusions in (5.19).

Example 5.5. For each $X \in \{c_0(\mathbb{N}), c(\mathbb{N}), \ell_\infty(\mathbb{N}), \ell_p(\mathbb{N})\}$, $p \geqslant 1$, and the forward and backward unilateral shifts $U, V \in L(X)$ there are equalities $\sigma(U) = \sigma(V) = \mathbb{D}$, $\sigma_{ap}(U) = \sigma_{su}(V) = \partial\mathbb{D}$, and hence, $\sigma_{ap}(U) = \partial\sigma(U) \subset \operatorname{acc}\sigma(U)$ and $\sigma_{su}(V) = \partial\sigma(V) \subset \operatorname{acc}\sigma(V)$. From Theorems 5.14 and 5.15, and (5.18) we get $\sigma_{gKR}(U) = \sigma_{gK}(U) = \sigma_K(U) = \sigma_{ap}(U) = \sigma_{gDR\mathcal{J}}(U) = \sigma_{gDR\mathcal{W}_+}(U) = \sigma_{gDR\Phi_+}(U) = \partial\mathbb{D}$

and $\sigma_{gKR}(V) = \sigma_{gK}(V) = \sigma_K(V) = \sigma_{su}(V) = \sigma_{gDR\mathcal{Q}}(V) = \sigma_{gDR\mathcal{W}_-}(V) = \sigma_{gDR\Phi_-}(V) = \partial\mathbb{D}$. Since $\sigma_{su}(U) = \sigma_{ap}(V) = \mathbb{D}$, from (5.17) sledi $\sigma_{gDR\mathcal{J}}(V) = \sigma_{gDR\mathcal{Q}}(U) = \mathbb{D}$, and hence $\sigma_{gDR}(U) = \sigma_{gDR}(V) = \mathbb{D}$.

Since $\sigma_\Phi(U) = \sigma_\Phi(V) = \partial\mathbb{D}$ [50, Theorem 4.2], from (5.17) we obtain that $\sigma_{gDR\Phi}(U) = \sigma_{gDR\Phi_-}(U) = \partial\mathbb{D}$ and $\sigma_{gDR\Phi}(V) = \sigma_{gDR\Phi_+}(V) = \partial\mathbb{D}$. From $\sigma_\Phi(V) = \partial\mathbb{D}$, $\sigma_{ap}(V) = \mathbb{D}$ and $\sigma_{su}(V) = \partial\mathbb{D}$, we conclude that for $|\lambda| < 1$ it holds that $V - \lambda I$ is Fredholm with positive index and so, $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma_{\mathcal{W}_+}(V) \subset \sigma_{\mathcal{W}}(V) \subset \mathbb{D}$, which implies that $\sigma_{\mathcal{W}_+}(V) = \sigma_{\mathcal{W}}(V) = \mathbb{D}$. Similarly, from $\sigma_\Phi(U) = \partial\mathbb{D}$, $\sigma_{su}(U) = \mathbb{D}$ and $\sigma_{ap}(U) = \partial\mathbb{D}$, we have that $\sigma_{\mathcal{W}_-}(U) = \sigma_{\mathcal{W}}(U) = \mathbb{D}$. Now again from (5.17) it follows that $\sigma_{gDR\mathcal{W}_+}(V) = \sigma_{gDR\mathcal{W}}(V) = \mathbb{D}$, $\sigma_{gDR\mathcal{W}_-}(U) = \sigma_{gDR\mathcal{W}}(U) = \mathbb{D}$.

Remark that the forward unilateral shift U is non-invertible isometry. In [4], p. 187, it is noticed that every non-invertible isometry T has the property that $\sigma(T) = \mathbb{D}$ and $\sigma_{ap}(T) = \partial\mathbb{D}$, and hence $\sigma_{ap}(T) = \partial\sigma(T)$ and every $\lambda \in \partial\sigma(T)$ is not isolated in $\sigma(T)$. Therefore, according to Theorem 5.14, for arbitrary non-invertible isometry T we get that $\sigma_{gKR}(T) = \sigma_{gDR\Phi_+}(T) = \sigma_{gDR\mathcal{W}_+}(T) = \sigma_{gDR\mathcal{J}}(T) = \sigma_{ap}(T) = \partial\mathbb{D}$.

Example 5.6. For the *Cesáro operator* C_p defined on the classical Hardy space $H_p(\mathbf{D})$, $\mathbf{D}$ the open unit disc and $1 < p < \infty$, by

$$(C_p f)(\lambda) = \frac{1}{\lambda}\int_0^\lambda \frac{f(\mu)}{1-\mu}d\mu, \quad \text{for all } f \in H_p(\mathbf{D}) \text{ and } \lambda \in \mathbf{D},$$

it is known that its spectrum is the closed disc Γ_p centered at $p/2$ with radius $p/2$ and $\sigma_{Kt}(C_p) = \sigma_{ap}(C_p) = \partial\Gamma_p$ [33], [4], and also $\sigma_\Phi(C_p) = \partial\Gamma_p$ [33], [4]. According to Theorem 5.14 we get that $\sigma_{gKR}(C_p) = \sigma_{gDR\Phi_+}(C_p) = \sigma_{gDR\mathcal{W}_+}(C_p) = \sigma_{gDR\mathcal{J}}(C_p) = \sigma_{ap}(C_p) = \partial\Gamma_p$, and since $\operatorname{int}\sigma_\Phi(C_p) = \operatorname{int}\sigma_{\Phi_-}(C_p) = \emptyset$, according to (5.17) we have that $\sigma_{gDR\Phi}(C_p) = \sigma_{gDR\Phi_-}(C_p) = \sigma_{gKR}(C_p) = \partial\Gamma_p$. From $\sigma(C_p) = \Gamma_p$ and $\sigma_{ap}(C_p) = \partial\Gamma_p$ it follows that and $\sigma_{su}(C_p) = \Gamma_p$ which together with $\sigma_\Phi(C_p) = \partial\Gamma_p$ implies that $\sigma_{\mathcal{W}_-}(C_p) = \sigma_{\mathcal{W}}(C_p) = \Gamma_p$. According to (5.17) we conclude that $\sigma_{gDR\mathcal{W}_-}(C_p) = \sigma_{gDR\mathcal{W}}(C_p) = \sigma_{gD\mathcal{J}\mathcal{Q}}(C_p) = \sigma_{gD\mathcal{J}}(C_p) = \Gamma_p$.

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