Katica R. (Stevanović) Hedrih

THE LATEST THEORY OF BODY COLLISIONS IN ROLLING AND THE DYNAMICS OF VIBRO-IMPACT SYSTEMS THROUGH SCIENTIFIC PROJECTS OVER THREE DECADES

Abstract. The paper presents the basic elements and results of the latest theory of collision between bodies in rolling. The theory is based on the newly introduced hypothesis of conservation of the sum of the angular momentum of the motions (the sum of kinetic moments) of the bodies in rolling after the collision in relation to the collision and it defines the collision coefficient using the angular rolling velocities immediately after and immediately before the collision. Analytical expressions are derived for the outgoing angular velocities of each body immediately after the collision as a function of the axial moments of inertia of the mass of each body for corresponding instantaneous axis of rolling, the collision coefficient and the angular velocities of each body immediately before the collision. It is shown how the rolling directions of the body are determined immediately after the collision for different types of collisions. It is shown that the basis of research on projects in basic sciences, led by the author of this paper, obtained a large number of scientific results that were published in journals, one master's thesis (2010), two doctoral dissertations (1996 and 2011), two monographs, one preprint, while one monograph is in preparation for printing. Each of these titles contains the keyword "vibro-impact system". The paper provides an overview of individual works. The Master of Science thesis and both doctorates are based on the classical theory of bodies colliding in translational motion, rectilinear and curvilinear, smooth or rough, while the latest results relate to the dynamics of vibro-impact body systems in rolling along curvilinear paths in a stationary or rotating vertical plane around a vertical axis at a constant angular velocity. A new methodology for investigating the dynamics of vibro-impact systems with successive collisions between bodies in rolling using the method of phase trajectory portraits in the phase plane has been defined. The list of literature provides the most significant works of the author of this paper who is also a supervisor and a research mentor of undergraduates and doctoral students.

Mathematics Subject Classification (2010): Primary: 70-99, 74-99, 70F35, 92C10, 74L10; Secondary: 70Exx, 70KXX, 74M05, 37H20, 34D20.

Keywords: Angular rolling velocity, latest new theory of body collision in rolling, hypothesis on conservation of sum of angular momentum of body, analytical expressions for outgoing angular rolling velocities, methodology for research of vibro-impact dynamics of systems with bodies in rolling, phase portrait method.

Department of Mechanics, Mathematical Institute of Serbian Academy of Science and Arts, Belgrade, Serbia; E-mail: katicah@mi.sanu.ac.rs; khedrih@sbb.rs

Faculty of Mechanical Engineering at University of Niš, Serbia

Contents

1. Preface

The author gained the first knowledge about the dynamics of collisions from the lecture of Professor Dr. Ing. Dipl. Math.Danilo P. Rašković, as well as from his textbook Dynamics [69-71], during her studies of mechanical engineering at the Mechanical Engineering Department of the Technical Faculty in Niš. Later, the author received the first university textbook on Theoretical Mechanics published in Serbian in 1880, according to J. Weisbach, which was written by academician Ljubomir Klerić. Academician Ljubomir Klerić taught mechanics at the Belgrade Great School which later grew into the University of Belgrade and he was also the founder of the Serbian School of Mechanics and Mechanical Engineering.

Both textbooks, Rašković's and Klerić's, meticulously present the well-known classical theory of body collisions in translational motion, and are highly useful resources for the education of graduate engineers in mechanical engineering and mining, as well as other technical faculties.

Later, at the European and world congresses of mechanics, after getting acquainted with Professor Frantisek Peterka from Prague and his scientific results in the field of dynamics of vibro-impact systems [58-65], she became interested in this scientific field. She also studied Russian literature. But she did not do research in this area until she was contacted by Mr. Sc. Slavka Mitić who, under the author's supervision, completed and successfully defended her master's thesis in the scientific field of nonlinear oscillations using asymptotic methods of nonlinear mechanics by Krilov-Bogolyuboc-Mitropolski, expressing a desire for a doctoral thesis under the author's mentorship. The author of this paper and the head of the three Projects (Oscillations of the Special Elements and Systems [P.1], Basic Scientific Found of Region Niš (1981-1986), Stochastic Processes in Dynamical Systems-Applications on the Mechanical Engineering Systems [P.2], Basic Scientific Found of Region Niš (1986-1989) and Nonlinear Deterministic and Stochastic Processes with Applications in Mechanical Engineering Systems [P.3], Ministry of Science and Technology Republic of Serbia, (1990-1995)), then suggested the topic in the field of dynamics and stability of vibro-impact systems. This was at the beginning of the tenth decade of the last century, and she conducted research within the project Sub-Project 04M03A [P.5] which was led by Professor Katica (Stevanović) Hedrih. When Slavka Mitić (2. 1. 1950, Lušca Palanка, Sanski most, BIH- 20. 7. 2012, Niš) defended her doctorate [54], her mentor suggested using her doctorate to form a monograph [55], which the mentor edited and reviewed, so that the first monograph on vibro-impact systems in the Serbian language was published.

In the last two project cycles, periods 2000-2010 and 2011-2019, projects [P.4-P.9], coordinated at the Mathematical Institute of SANU and led by the author of this paper, a member of the research team was also Srdjan Jović, teaching assistant at the Faculty of Technical Sciences in Kosovska Mitrovica, along with numerous researchers from different faculties and scientific institutes in Serbia.

At that time, several scientific articles [11, 13, 20] by the author and the project leader were published, dealing with the topic of moving heavy material points along rough curved lines in a stationary or rotating vertical plane around a vertical axis at a constant angular velocity. Vibro-impact systems were the primary interest of Srdjan Jović, which the project leader proposed as a theme of the project investigation. The dynamics of vibro-impact systems of heavy bodies (heavy material points) moving translational along rough curvilinear paths and examining the change of energy and kinetic parameters in the conditions of Coulomb friction force and phase method, based on previously published articles on the movement of a heavy material point along rough curvilinear paths was proposed for research to Srdjan Jović.

The agreement was reached to first formulate the research topic of a master of science thesis, and then the topic for the preparation of a doctorate. These research topics were worked on very intensively under the mentorship of the author of this article in the period from 2009 to 2011 with constant consultations on Tuesdays and Wednesdays at MI SANU, so that the quickly achieved results formed a master's thesis and a doctorate, which Srdjan Jović quickly formulated and successfully defended before the respective committees, in which the author of this paper and the head of research on the indicated topics was the president, and the official mentor was a professor from the Faculty of Technical Sciences in Kosovska Mitrovica, because it was prescribed by law. Later, based on these results, she co-authored one preprint monograph and another monograph published under the title Dynamics of Vibro-Impact Systems [36, 46]. Before the completion of the doctorate, during the research phases, a number of co-authored papers [33-35, 46] were published in journals based on the obtained research results.

As "scientific children grow up" and "find their own ways" in some other areas of research, and with some other researchers, this project, in which Srdjan Jović achieved his titles of Magister of Science [48] and Doctor of Science [47], was left without researchers on the topic of dynamics of vibro-impact systems.

The author of this review paper, as well as the project leader, has continued research on the planned research topic for five years, since the end of the research on the topic of dynamics of vibro-impact systems, when the project cycle continued in the period 2015-2019. The continuation of the research on the topic of Dynamics of Vibro-impact Systems, in the period from 2015 to the end of the project cycle 2019, was very fruitful for the author of this paper. As none of her colleagues or researchers, to whom she suggested to research together on this topic, showed any interest (for example, a personal invitation was sent to S.J., PhD, then to B.G., PhD, and to other younger researchers), she continued independent work on this research topic. Thus, the original scientific results appeared, which were published in a series of one-author articles in prestigious scientific journals, Springer's Proceedings, proceedings of various scientific conferences, which verified significant scientific results under the following key titles: Theory of body collision in rolling; Generalized rolling pendulum along curvilinear trajectories; Rolling the ball on curved surfaces and coordinate surfaces of curvilinear coordinate systems; Phase portraits of generalized rolling pendulums; Theorems of bifurcation and triggers of coupled singularities; Methodology for studying the nonlinear dynamics of vibro-impact systems with bodies in rolling, using the phase plane method (see References [6-12], [14-19] and [24, 25]).

2. Introduction

Material (mechanical) systems whose motion is repeated in equal or different time intervals are called vibrational (oscillatory) systems, or oscillators for short. In mechanics, the notion of collision is more general than the notion of impact, so together (combined) *impact* and *collision,* we write *collision* [29, 31], for short; impact and collision are phenomena of nonlinear nature and with *alternation* and *discontinuity* of vector and scalar kinetic parameters of the system.

Systems in which vibrational (oscillatory) movements and impacts (or impacts and collisions, or only collisions) occur and appear are called *vibro-impact systems*. These are nonlinear discrete dynamical systems in which vibro-shock effects occur.

In the technical aspects of engineering practice, vibro-impact actions often occur and are widely applied [46, 72-74]. Therefore, appropriate theoretical, numerical and experimental investigations of nonlinear dynamics of vibro-impact systems are of special importance.

The first research on vibro-shock action dates back to the 1930s, and a newer wave of interest in the dynamics of vibro-impact systems emerged at the end of the last (20^{th}) century and intensified with the development of bifurcation theory [1, 4, 5, 7-15, 21, 26] and the interpretation of chaotic regimes. The most significant scientific results that improved the knowledge about the dynamics of vibro-impact systems were published in [34, 36-39, 42-46, 66-69, 72-74], out of which we single out the authentic works of Frantisek Peterka [69, 67], Dimentberg M.F. and Menyailov A.I., Foole S. and Bishop S., Lieber P. and Jensen, D., Luo G.W. and Xie J.H., Nordmark A.B., Pavlovskaia E. and Wiercigroch M. (for details see lists of References in [43-45] and in [51, 42]), Katica (Stevanović) Hedrih [11, 12, 17-19, 21-25, 29-31, 34-39, 42-49] and associates Slavka Mitić [54-57], Srđan Jović [47-49] and others; The authors of these papers used different methods to find solutions to the set tasks of the dynamics of vibro-impact systems, most often starting from the general stereo-mechanical theory of impact (collision). The latest works are based on research conducted numerically and experimentally on the basis of analytical methods.

Here we will point out the results of Serbian researchers Slavka Mitić (2. 1. 1950, Lušca Palanкa, Sanski most, BIH- 20. 7. 2012, Niš) [54-55] and Srđan Jović [47-49], who did research on projects of the Ministry of Science of the Republic of Serbia under the supervision of project leader Katica (Stevanović) Hedrih (see the list of Projects [P.1-P.9]).

The following parts of this paper present the latest authentic author results of Katica (Stevanović) Hedrih, which were obtained and published in the last five years of research within the project and topics: Phenomena of nonlinear dynamics of generalized rolling pendulums [14, 16-19] and Nonlinear dynamics and phenomena in vibro-impact systems with bodies in rolling [19-25].

According to the research program on the project Nonlinear Deterministic and Stochastic Processes with Applications in Mechanical Engineering Systems [P.3], financially supported by the Ministry of Science and Technology of the Republic of Serbia, (1990-1995) realized through Mechanical Engineering Faculty University of Nis, the research topic was “*Deterministic and stochastic processes in vibro-impact* *systems*". Slavka Mitić (2. 1. 1950, Lušca Palanка, Sanski most, BIH- 20. 7. 2012, Niš) participated as a researcher under the mentorship of the project leader Hedrih (Stevanović), KR., and on the basis of the obtained research results she formed her doctoral dissertation under the title "*Stability of deterministic and stochastic processes in vibro-impact systems*", which she successfully defended in 1994.

Later in 2006, at the suggestion of the mentor and on the basis of the content of the grant, she formed a monograph under the title "*Vibro-impact systems*"[54], edited and reviewed by her mentor Hedrih (Stevanović) K.R. This monograph was also the first monograph in the Serbian language on vibro-impact systems. This monograph used the classical theory of body collisions in translational motion.

Among the published papers based on the obtained research results on the subject of the project, on this occasion we single out the co-authored paper [56]. The paper was published in the Journal Acta Technica CSAV (Ceskoslovensk Akademie Ved) in the Czech Republic in 1997, and previously presented at the European Conference on Nonlinear oscillations - ENOC Prague 1996.

In the paper [56] written by Mitić, S. and Hedrih (Stevanović), K.R., nonlinear oscillations of the torsion oscillator with impact masses were described. This paper dealt with nonlinear oscillations of the torsion oscillator with reciprocal rigidly connected impact masses. It was assumed that two impulses occurred at one interval of the disturbing torsion moment. The asymptotic Krilow-Bogolyubov-Mitropolski method was applied, along with the stereo-mechanical impact theory for the inclusion of impact conditions, to the determination of the primary approximation of the torsion system nonlinear oscillations. Phase trajectories were drawn on the basis of the numerical results. The mathematical model of the vibro-impact system was written in the form of an autonomous nonlinear system of the first order differential equations. The integral curves and the phase trajectories were obtained by means of the Runge-Kutta method, of the Turbo-Pascal program and with the aid of the computer.

As parts of research programs of two projects [P.8] and [P.9]: "*Theoretical and Applied Mechanics of the Rigid and Solid Bodies. Mechanics of Materials*", (2006–2010), and "*Dynamics of hybrid systems with complex structures. Mechanics of materials*", (2011-2019), under the mentorship of the project leader Hedrih (Stevanović), K.R., Srdjan Jović researched the dynamics of vibro-impact systems, and based on the research results he formed a master of science thesis [48] and a doctoral dissertation [47], which he successfully defended at the Faculty of Technical Sciences in Kosovska Mitrovica. By law, an official mentor was appointed from that faculty. The titles of the master's thesis and doctorate were: "*Energy analysis of vibro-impact system dynamics*" [48], [in Serbian], Magister of Science Thesis, 2009, and "*Energy analysis of vibro-impact system dynamics with curvilinear paths and no ideal constraints*" [47], [in Serbian], Doctoral Degree Thesis, 2011. We will not show the contents of this master of science thesis and doctoral dissertation here, because the titles speak eloquently enough about their orientations. We will note that the results of a series of published works of the project leader on the dynamics of a heavy material point along rough curved lines [11, 12] and the dynamics of vibro-impact systems of heavy material points in translational motion along different rough curved lines and in successive collisions were used as basis of the advanced research (see References [34-39] by Hedrih (Stevanović), K.R. and other). Phase portraits and portraits of constant mechanical energy curves were used by Srdjan Jović (see References [47, 48]).

A large number of co-authored articles (see References [34-39]) were published, some of which were published in prestigious world-famous journals. Additionally, the obtained results related to their contents were presented at numerous prestigious scientific conferences. Based on the research results of these projects, the results of published co-authored papers [11, 34-39] as well as master's theses of sciences [48] and doctorate [47] created with three parts, one preprint [39] under the title: „*Vibro-impact system dynamics*" present an analysis of the dynamics of one class of vibro-impact systems based on oscillators along curvilinear routes and stationary non-ideal constraints, (309 pages long, 30 copies). Additionally, there was one monograph of the same name [49], which was cataloged in the National Library of Serbia, preprint (30 copies) and a published monograph (100 copies with categorization) composed of three parts, the contents of which will be presented in the continuation of this work.

The preprint [39] and the monograph [40]: „*Vibro-impact system dynamics*" contain a focused analysis of the dynamics of one class of vibro-impact systems based on oscillators with two heavy mass particles translator moving along the same rough curvilinear routes in a stationary or rotating vertical plane around a vertical axis at constant angular velocity and in successive collisions.

This book (both the preprint and the monograph form) posed an original methodology of the dynamics analysis of one class of vibro-impact systems based on the phase trajectory method. It used phase trajectory portraits of two oscillators each with one heavy mass particle translatory moving along the same curvilinear rough route in a vertical stationary or rotating plane about a vertical axis at a constant angular velocity and in successive collisions. This book was compiled as a result of selection, systematization and application of new and authentic research results in each part, which the authors attained through their research work within scientific research projects (see [P.8] project ОИ144002 (2006-2010) and [P.9] project ОИ174001 (2011-2016)). Modern information technology (commercial software tools – software package programs MathCad, MatLab, Wolfram Mathematica) was used for graphic visualization of vibro-impact system dynamics. Some of those results were previously published by the renowned scientific journals with the highest scientific reputation worldwide. Some of these are [34-39].

The book [39, 49] was divided into three parts and written in such a way that the parts could be used independently.

The abstract of the first part, authored by Hedrih (Stevanović) K.R., with the title of "*The basis of the impact and collision theory, the chosen methods of analysis of nonlinear system dynamics and the material point movement along the curved rough line*" is in the following content: This part outlines the theoretical basis for researching vibro-impact system dynamics with curvilinear translatory motion and stationary non-ideal constraints. The first chapter of this part presents the theoretical basis of the impact and collision dynamics based on the theory which was established by Isaac Newton. The second chapter outlines the basic methods upon which the methodology for researching vibro-impact system dynamics is established in the remaining chapters of this part. The third chapter of this part presents the original results of the author [11, 13, 20] on the heavy material point movement along the rough arbitrary curvilinear route as well as a circle, parabola, cycloid and ellipse. This chapter represents the theoretical basis for a choice of vibro-impact system dynamics model which is studied in the following two parts on the vibro-impact dynamics with two mass particles in translatory motions and successive collisions. The final fourth chapter of this first part presents examples which are a result of joint research, supported by Srdjan Jović [34-39].

The abstract of the second part, co-authored by Vladimir M. Raičević and Srdjan V. Jović with the title "*Analysis of the vibro-impact system dynamics based on the autonomous oscillator with curvilinear routes and stationary non-ideal bonds*" is in the following content: This part of the monograph is based on theoretical results presented in the first part [11], and shows the research results of the properties of autonomous vibro-impact dynamics with a large number (four) of approximation models of nonlinear dynamics of the real systems with one or more (two, three) degrees of freedom of vibro-impact free oscillations. The system is abstracted to the material point model which moves freely and translatory along the rough curved line, in a vertical plane, fitted with the elongation limiters of the material point movement. For the purpose of studying the dynamics of the vibro-impact system dynamic models, authors use the concepts of the impact theory presented in the first part of this monograph, nonlinear dynamics methods presented in the second chapter and the results of the material point translatory movement along the rough curved line [11] presented in the third chapter of the first part of the monograph, which are theoretical bases for studying the autonomous nonlinear dynamics models and vibro-impact dynamics phenomena. Modern information technology is also used (software tools – software package programs MathCad, CorelDraw) for graphic visualization of the vibro-impact dynamics. Most of the results presented in this part are taken from the published papers [34-39, 44, 45] of all three authors of this monograph, as well as from the PhD dissertation [48] of Srdjan Jović.

The abstract of the third part, authored by Srdjan V. Jović, with the title of "*Analysis of the vibro-impact system dynamics based on the forced oscillator with curvilinear routes and stationary non-ideal bonds*" is in the following content [47, 48]:

This part of the monograph presents the research results of the properties of forced vibro-impact dynamics with a large number (seven) of approximation models of nonlinear dynamics of real systems with one or more (two) degrees of freedom of vibro-impact forced oscillation. All study models of the system consist of one or more heavy slides moving along the curvilinear rough line, with the middle line having the shape of a curve in a vertical plane. The line is fitted with the limiters of heavy mass translatory movement. The system is abstracted to the material point model (the pellets) which move forcefully along the rough curved line, in a vertical plane, fitted with the elongation limiters of the material point translatory movement, while the material points are exposed to the effect of the outside forces. For the purpose of studying the dynamics of the vibro-impact system dynamic models, authors use the impact theory presented in the first part of this monograph, nonlinear dynamics methods presented in the second chapter and the results of material point movement along the rough curved line presented in the third chapter [111] of the first part of the monograph, which are the theoretical basis for studying the non autonomous nonlinear dynamics model and vibro-impact dynamics phenomena. Modern information technology is also used (software tools – software package programs MathCad, CorelDraw) for graphic visualization of the vibro-impact dynamics. All the results presented in this part of the monograph are the results of independent, authentic research of Srdjan Jović [44, 45] under the supervision of project leader, within the theme projects ON144002 [P8] and ON 174001 [P.8] financed by the Ministry of Science of the Republic of Serbia.

3. Central collision of two rolling balls: Theory and examples

This part of the paper focuses on central collision [15, 17, 19, 26-31] of two rolling rigid and heavy smooth balls and using elements of mathematical phenomenology and phenomenological mapping [27, 28, 66-68] to obtain corresponding post collision and outgoing angular velocities of the balls and to apply these results for investigation in vibro-impact dynamics of two rolling balls along a circular trace or curvilinear route in a stationary or rotating vertical plane. This task is fully accomplished and the obtained results are original and new. Original plans of component impact velocities and angular velocity of each of two different rolling balls in central collision and corresponding outgoing angular velocities are presented. The use of elements of mathematical phenomenology by Petrovič [66-68], especially mathematical analogy between kinetic parameters of collision of two bodies in translatory motion and collision of two rolling different size balls, new original expressions of two outgoing angular velocities for each of rolling balls after collision are defined. New hypothesis of conservation of the sum of angular momentum for instantaneous axes of rolling of two bodies in rolling before and after collision of two axisymmetric bodies is introduced [15, 17, 19, 26-28, 30-32].

Using this new and original result, vibro-impact dynamics of two rolling different heavy balls on the circle trace in a vertical plane in a period of series of successive collisions is investigated. Using a series of the elliptic integrals, new nonlinear equations for obtaining angles of balls positions at positions of collisions are defined. Branches of phase trajectories of the balls in vibro-impact dynamics are theoretically presented [15, 17, 19].

The theory of impact dynamics of systems as well as vibro-impact dynamics is an important research task nowadays. This is the reason and motivation for our research as the presentation of the theory of the collision of two rolling, rigid, homogeneous and heavy, smooth balls with different radii and different masses.

3.1. Short history

"In connection with the game of billiards .... there are various dynamic tasks, whose solutions are contained in this event. I think that people who know Theoretical mechanics, and even students of polytechnics, with interest familiarize themselves with explanations of all the original phenomenon that can be observed from the time of movement of billiard balls".

Gaspard-Gustave de Coriolis,

Mathematical theory of billiards game.
G. Coriolis (1990). *Théorie mathématique des effets du jeu de billard; suivi des deux célèbres mémoires publiés en 1832 et 1835 dans le* Journal de l'École Polytechnique*: Sur le principe des forces vives dans les mouvements relatifs des machines & Sur les équations du mouvement relatif des systèmes de corps* (Originally published by Carilian-Goeury, 1835 ed.). Éditions Jacques Gabay. ISBN 2-87647-081-0 [50-52].

In 1668, the Royal Scientific Society in London launched a call for a solution to the problem of impact and collision dynamics, and for that call, the well-known scientists Wallis (John Wallis, 1616-1703, *Mechanica sive de mote*-1688) and Huygens (Christian Huygens - *De motu corporum ex percussione*) submitted their papers. Wallis and Huygens used the results of the collision, submitted them to the Royal Society and added their generalizations. Using their work, Isaac Newton laid down the fundamental foundations of the *Theory of Impact* [30, 70, 72], which is still unsurpassed today [30]. (see Figure 1.). Even before Newton, Wallis and Huygens, there was research into the dynamics of impact. Thus, for example, collision problems were addressed by Galileo Galilei, who came to the realization that the impact force was infinitely large in relation to the pressure forces, but did not reach and learn about the relation of the impact impulses and the amount of movement. Today's knowledge of collision dynamics is not much more advanced than this collision theory, which was founded by Newton, Wallis and Huygens. In connection with this competition of the Royal Scientific Society and submitted papers, it was evident that papers contained the first set of basics of collision theory. The name also mentions Sir Christopher Michael Wren (20 October 1632 - 25 February 1723), who was also the president of the Royal Scientific Society (see Figure 2.).

The dynamics of rolling ball collisions occur in many engineering systems, and especially in the dynamics of roller bearings. Even today, no general theory of rolling ball collisions has been given. Some recent results by the author of this paper present new and original results [15, 17, 19, 26-31] in support of the classical theory of rolling ball collisions. These results are presented in the first part of this article.

In the game of billiards, collisions of rolling equal balls occur. The complexities of billiard dynamics and billiard models and the possibility of observing and noting the complex phenomena and phenomena of collision dynamics were pointed out by Coriolis (Gaspard-Gustave de Coriolis; Paris, May 21, 1792 - Paris, September 19, 1843) and to illustrate this we cite the following quotation (see References [50-52]):

"In connection with the game of billiards ...... different dynamic tasks occur, the solutions of which are contained in this event. I think that people, who know theoretical mechanics, and even students of the Polytechnic Schools, are interested in learning about the explanations of all the original phenomena, which can be observed with the movement of billiard balls."

FIGURE 1. From left to right scientists: Sir Christopher Michael Wren (20 October 1632 - 25 February 1723), John Wallis (1616-1703), Christiaan Huygens (14 April 1629 - 8 July 1695) and Gaspard-Gustave de Coriolis, (Paris, May 21, 1792 - Paris, September 19, 1843).

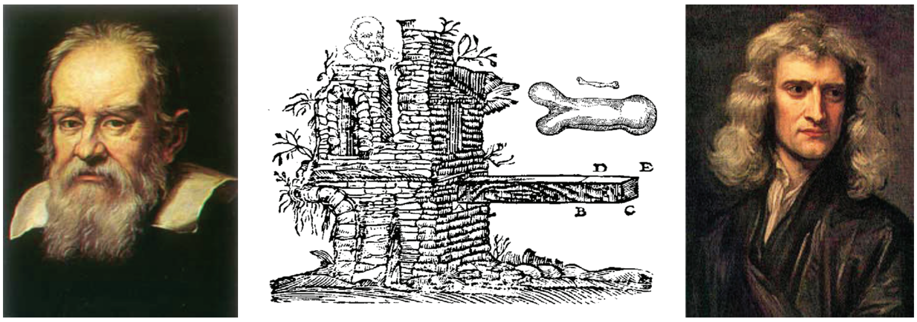


FIGURE 2. Scientists, authors of the original ideas of Theoretical and Applied Mechanics: Galileo Galilei (Paris, February 15, 1564 - Florence, January 8, 1642) and author of an authentic and significant work: *"Discorsi e dimostrazioni matematiche intorno a due nuove scienze attinenti alla meccanica e i movimenti locali" 1638* (left) and Sir Isaac Newton, (Lincolnshire, December 25, 1642 - London, March 20, 1726/7) (right) author of Basic Collision Theory and Works: Mathematical Principles of Natural Philosophy (Lat. Philosophiae Naturalis Principia Mathematica), published in 1687.

The elements of the dynamics of billiards [50-52], [30] are coupled into a complex system, whose dynamics are different from the phenomena observed in the dynamics of the system. Starting from the geometric basis for switching to the theory of impact and collisions between two or a few numbers of the balls, it is possible to see that impacts and collisions are in the center of this dynamics. Shown are the plans of translational and angular velocities of rolling of one ball before and after the impact, and also the two balls colliding. Rolling balls are the main elements in numerous mechanical engineering systems.

3.2. Collision of two bodies in translatory motion. Let us start with the largely known classical theory [70] of central collision between two bodies, with mass m_1 and m_2, in translatory motion and with translatory velocities $\vec{v}_1(t_0)$ and $\vec{v}_2(t_0)$ at the moment before collision between them. These velocities we denote as arrival, or impact or pre-impact velocities at the moment t_0 (see Figure 3.).

At this moment t_0 of the central collision start between these bodies, the contact of these two bodies is at point P, in which both bodies possess the common tangent plane–plane of contact (touch). In the theory of central collision, it is proposed that collision takes a very shorth period of time $(t_0, t_0+\tau)$, and that τ tends to zero. After this short period bodies in collision separate and are in an outgoing kinetic state by post-impact-outgoing velocities $\vec{v}_1(t_0+\tau)$ and $\vec{v}_2(t_0+\tau)$.

On the basis of hypothesis of conservation of linear momentum (impulse) of motion, the following relation is valid [70]:

$$m_1\vec{v}_1(t_0)+m_2\vec{v}_2(t_0)=m_1\vec{v}_1(t_0+\tau)+m_2\vec{v}_2(t_0+\tau) \tag{1}$$

and the coefficient of the restitution of body central collision is:

$$k=\frac{v_r(t_0+\tau)}{v_r(t_0)}=\frac{v_2(t_0+\tau)-v_1(t_0+\tau)}{v_1(t_0)-v_2(t_0)} \tag{2}$$

and presents the ratio between the difference of translatory velocities in post-collision and pre-collision kinetic states, defined by Newton's classical theory of impact.

Post-central-collision – outgoing body translator velocities are in the form [67]:

$$v_1(t_0+\tau)=v_1(t_0)-\frac{1+k}{1+\frac{m_1}{m_2}}\left(v_1(t_0)-v_2(t_0)\right) \tag{3}$$

$$v_2(t_0+\tau)=v_2(t_0)+\frac{1+k}{1+\frac{m_2}{m_1}}\left(v_1(t_0)-v_2(t_0)\right) \tag{4}$$

Impuls (linear momentum) of collision in this case is:

$$K_{Fud}=m_1\left(v_1(t_0+\tau)-v_1(t_0)\right)=-\frac{m_1m_2}{m_1+m_2}(1+k)\left(v_1(t_0)-v_2(t_0)\right) \tag{5}$$

As it is known from the classical literature [70], the coefficient of the restitution of body collision depends on the kind of collision: 1* for the pure no elastic (plastic) collision, the coefficient of restitution is equal to zero- $k=0$; 2* for the pure ideal elastic collision, the coefficient of restitution is equal to unit, $k=1$; and 3* for the arbitrary case between ideal plastic and ideal elastic collision, the coefficient of restitution is in the interval between zero and unit, $0 < k < 1$.

From the comparison between outgoing (post-collision) velocities with no elastic collision of two translatory bodies in the pre-collision state, we can point out the following conclusions:

* in the case of pure plastic collision of two bodies, $k=0$, outgoing (post-collision) velocities are equal one to other;
* in the case of ideal elastic collision of two bodies in translator motions, $k=1$, outgoing (post-collision) velocity of the body with largest pre-collision impact velocity is smaller, and outgoing (post-collision) velocity of the body with smaller pre-collision impact velocity is larger; in this case, the ideal elastic impact, $k=1$, if both pre-collision impact velocities of the bodies are of equal intensity, $v_1(t_0)=v_2(t_0)$, and opposite direction, then both outgoing velocities of the both bodies are of equal intensity, $v_1(t_0)=v_2(t_0)$, and opposite direction, independent of the body masses.
* In the case of no elastic collision between bodies, $0<k<1$, if condition $m_1v_1(t_0)+m_2v_2(t_0)=0$, or $\frac{m_1}{m_2} = -\frac{v_2(t_0)}{v_1(t_0)}$ is satisfied, outgoing (post-collision) velocities of both bodies satisfied relation: $\frac{m_1}{m_2}=-\frac{v_2(t_0+\tau)}{v_1(t_0+\tau)}$. In this case outgoing velocities are: $v_1(t_0+\tau)=-kv_1(t_0)$ and $v_2(t_0+\tau)=-kv_2(t_0)$.

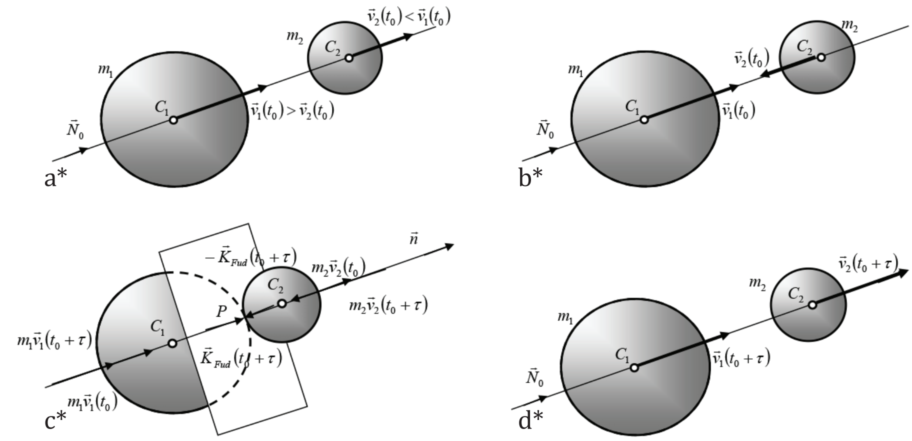


FIGURE 3. Central collision between two bodies, with mass m_1 and m_2 in translatory motion (a* and b*) and with translatory pre-impact velocities $\vec{v}_1(t_0)$ and $\vec{v}_2(t_0)$ (c*) and with outgoing post-impact velocities $\vec{v}_1(t_0+\tau)$ and $\vec{v}_2(t_0+\tau)$ (c* and d*).

3.3. Kinematics of collision of two rolling balls along a horizontal trace

3.3.1. Possible impact points at one ball for different kinds of the impacts and component velocities. Let us start with the analysis of the elements of kinematics of two rolling balls in the state of pre-collision between them. We consider two heavy smooth balls with different masses, and different radii, r_1 and r_2, each in rolling kinetic state with momentary angular velocity, $\vec{\omega}_{P,1}$ and $\vec{\omega}_{P,2}$, along corresponding straight trace of rolling, which are linear. Momentary axes of each rolling line lie in a horizontal plane and are orthogonal to the rolling trace in each moment passing through point P_2 (see Figure 4) or for first and second rolling ball through point P_2 as in point P_2 (see Figure 5. a* and b*, and also c* and d*) or Figure 6. These points P_1 and $\vec{\omega}_{P,}$are points of touch between rolling trace and corresponding rolling ball, and these points move along trace together with momentary axis of ball's rolling.

If momentary angular velocities, $\vec{\omega}_{P,1}$ and $\vec{\omega}_{P,2}$, of the rolling balls and corresponding axes of rolling the first and the second heavy ball are known, and also the radiuses of balls and mass densities of balls, then the dynamics of each ball is fully determined. Therefore, the investigation of the heavy balls dynamics is a simple task for obtaining all kinetic parameters of balls.

Let us consider possible component impact velocities in point $\mathbf{T}$ at spherical surface as a possible point of touch (contact) in a kinetic state of collision between two rolling balls. If we talk about the collision of two equal dimensions (equal radiuses) of the rolling balls all possible points $\mathbf{T}_i$, $i = 1,2,3\ldots$ of central or skew collision between balls are at a circle passing through a mass center of both balls and the balls' common tangent plane through this point of balls collision is vertical. Taking into account that trajectories of mass centers of both rolling balls are horizontal and straight lines parallel to a rolling trace, then both mass centers move translatory with velocities $\vec{v}_{C,translator}$ or $\vec{v}_{C,1,translator}$ and $\vec{v}_{C,2,translator}$ (Figure 4).

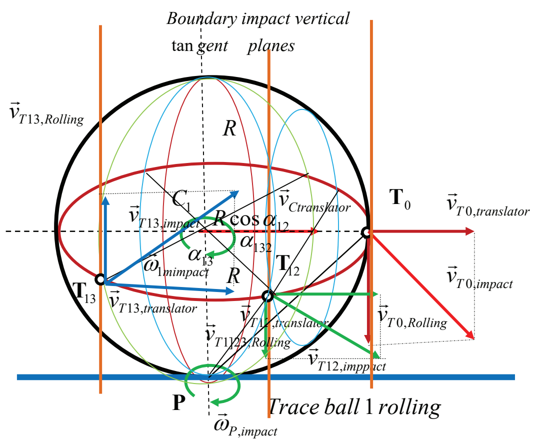


FIGURE 4. Plan of component impact velocities of impact points of a ball for the different types of collisions of two equal rolling balls

For the case of equal balls in collision (Figure 4), each impact velocity $\vec{v}_{T,impact}$ of impact at pre-collision state has two components, one horizontal equal to $\vec{v}_{T,translator} = \vec{v}_{C,translator}$ and one vertical component $\vec{v}_{\mathbf{T},rolling}$ of self rotation with angular velocity $\vec{\omega}_C = \vec{\omega}_P$ around a central axis parallel to instantaneous (momentary) axis of a ball rolling along the trail. This rolling component of impact velocity is dependent on the types of collision. If collision of balls is central with same line as a trace rolling both balls, then rolling component $\vec{v}_{T0,rolling}$ of impact velocity of point $\mathbf{T}_0$ is with maximal intensity and with intensity equal to a product between the ball radius R and the intensity of angular velocity $\omega_C = \omega_P$ of self rotation. In the case of rolling balls in skew collision between balls, impact points are at the point $\mathbf{T}_{12}$ or $\mathbf{T}_{13}$ (see Figure 4) and with angular velocities not parallel, and balls' rolling traces are with intersection, or parallel, then points $\mathbf{T}_{12}$ or $\mathbf{T}_{13}$ of the collision of rolling balls are at the distance defined by $R\cos\alpha$ to the

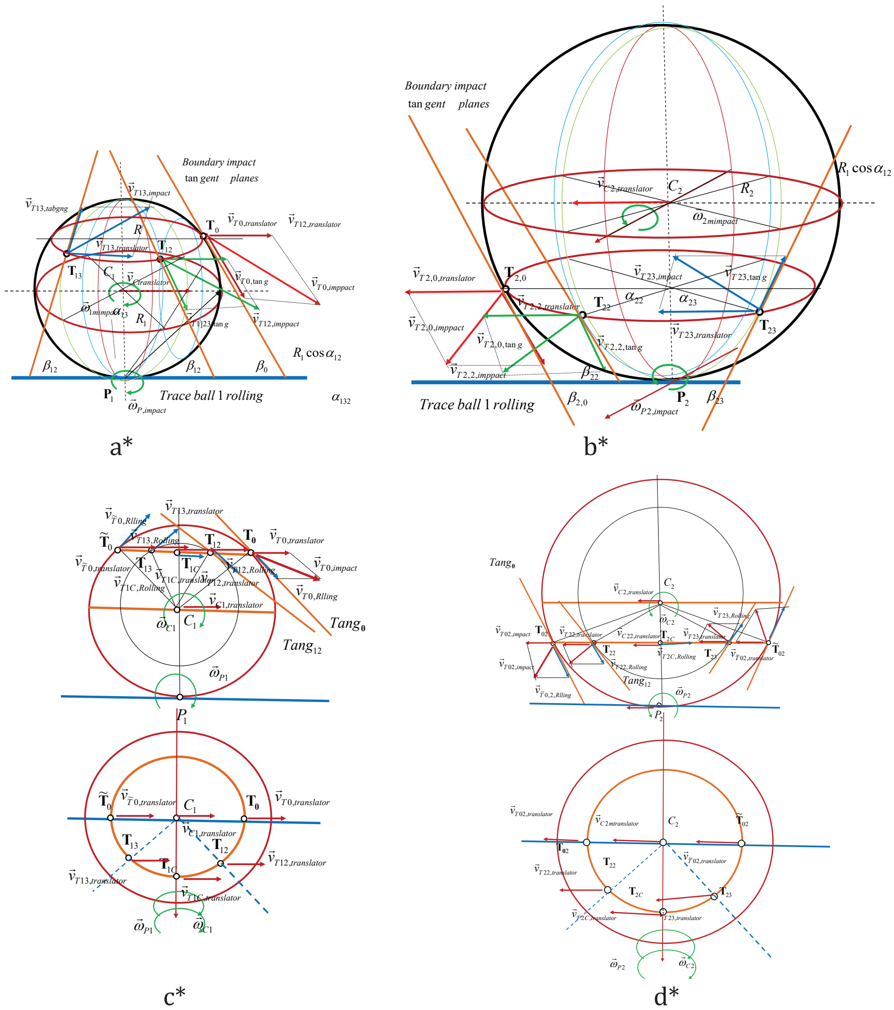


FIGURE 5. Plans of the impact velocities of possible points of collision of two rolling heavy balls with different radiuses: a* and c* for the first rolling smaller ball and b* and d* for the second biggest rolling ball.

self rotation central axis of ball, where α is the angle between trace rolling of corresponding ball and normal to the common tangent plane. Thus, the intensity of the rolling component of arrival velocity is equal to the product between the orthogonal distance $R\cos\alpha$ and intensity of angular velocity $\omega_C = \omega_P$ of self rotation. Outgoing components of the impact velocity of the impact point $\mathbf{T}_0$ in central collision of the equal rolling balls are corresponding intensities and with opposite directions in vertical and in horizontal planes, for ideal elastic collision. After analysis of the post collision motion for this case we see that it is simple, taking into consideration only that collisions appear in rolling balls with corresponding angular velocities of rolling.

3.3.2. Possible impact points at two balls for different kinds of collisions and component velocities. In the case of rolling balls with different dimensions (size) and masses and axial mass inertia moments for instantaneous axis of rolling, the kinematical plan of component velocities in collision are presented in Figures 5 and 6.

In Figure 5, the kinematical plans of the impact velocities of possible points of collision of two rolling heavy balls with different radiuses: a* and c* for the first smaller rolling ball and b* and d* for the second bigger rolling ball are presented [30].

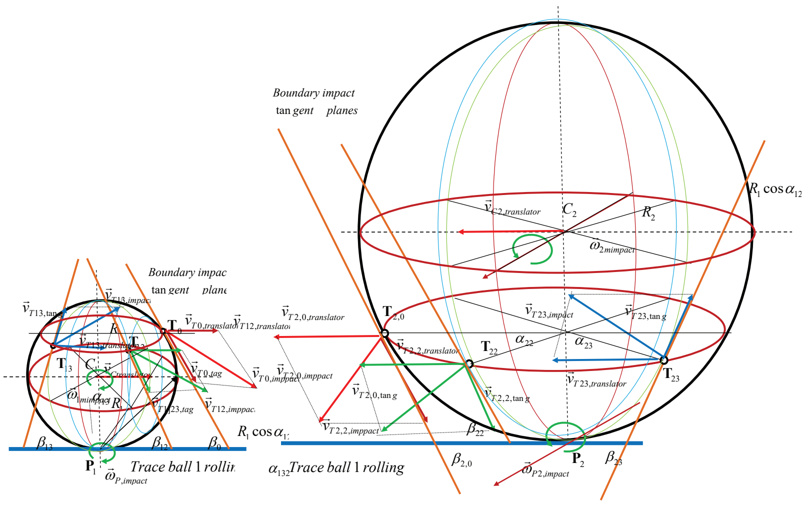


FIGURE 6. Plans of the impact velocities of possible points at corresponding circles at the same height of balls in central collision of two rolling heavy balls with different radiuses: left for the first smaller ball and right for the second bigger ball.

In Figure 6, plans of the impact velocities of possible points at corresponding circles at the same height of balls in central collision of two rolling heavy balls with different radiuses: left for the first smaller ball and right for the second bigger ball are presented.

From the listed plans of the component velocities, we can see that in the case of central collision of the rolling different dimension balls, the collision point is $\mathbf{T}_0$ at both spherical surfaces. At the smaller ball, this point $\mathbf{T}_0$ of central ball collision is higher to its mass center and at the bigger ball, this point $\mathbf{T}_0$ is lower to the mass center of the ball. The tangent plane of central collision is passing through point $\mathbf{T}_0$, and is orthogonal to the radii of both balls from point $\mathbf{T}_0$ to the corresponding ball center, C_1 and C_2.

In Figures 5, 6 and 7, the possible points of impacts at balls with different size for different types of collision with corresponding kinematic plans of velocities are presented, but this part focuses on central collisions of two rolling rigid smooth balls along the starting trace (Figures 5, 6 and 7) and circle trace (Figures 9 and 10). In Figures 5, 6 and 7, plans of the component impact and outgoing velocities at point $\mathbf{T}_0$ of central collision of two rolling heavy balls with different radiuses are presented.

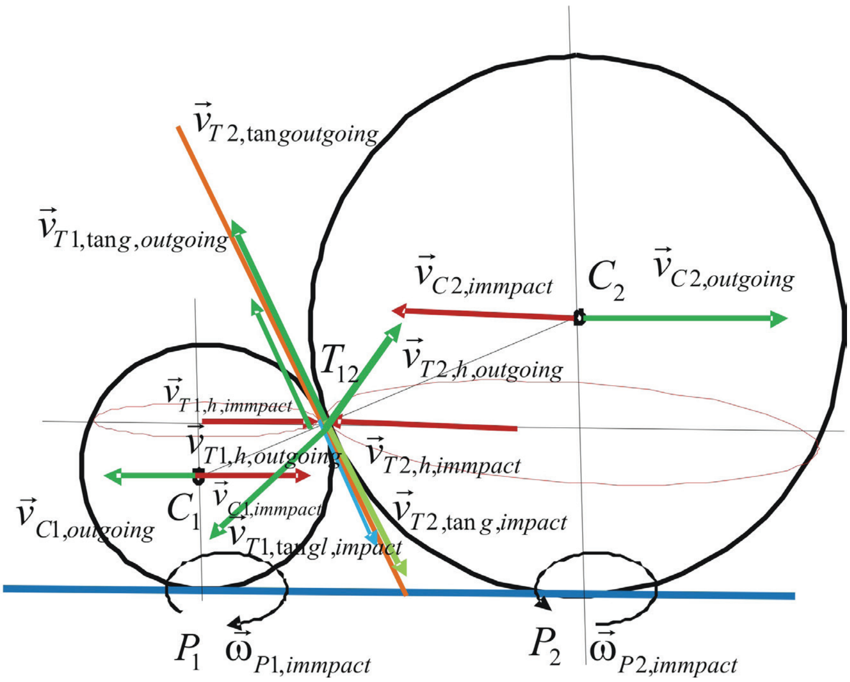


FIGURE 7. Plans of the component impact and outgoing velocities at the point of central collision of two rolling heavy balls with different radiuses

After a similar analysis of the presented kinematic plans of arrival and outgoing component velocities, as in the previous case, all conclusions from the given Figures 5, 6 and 7 directed us to a general central conclusion, that the collision of the two rolling balls is simpler to investigate in analogy with the well-known classical theory and results of kinetic, kinematic and dynamic parameters of collision between two bodies in translatory motion. See the next subchapter for detail about an elementary logical analogy between two simple motions, each with one degree of freedom. Basic

3.4. Basic assumptions of the theory of collision dynamics in non-slip rolling bodies. From the previous kinetic analysis and the conclusions we have drawn and proved, it follows that to consider the dynamics of collisions of axisymmetric rigid bodies with one plane of symmetry, which are in non-slip rolling, we must start with incoming angular velocities and rolling paths. Then, the axial moment of inertia of rolling body masses for the instantaneous axes of rolling should be included by introducing analogous assumptions, as well as for the case of analogous dynamics of impact and collisions of bodies in translational motions determined by the translational velocities and corresponding masses (see Reference [70, 30]).

The theory of collision dynamics (and in the special case of impact) is based on the following assumptions:

1* The contact time τ of two bodies in a collision is very short;
2* The impact forces $\vec{F}^{ud}$ and the corresponding impact moments $\vec{\mathfrak{M}}^{ud}$ of the forces are variable and of high intensity, of the order of magnitude $\frac{1}{\tau}$, and of short duration during the contact time τ of two bodies in the collision and during the collision they have attack points at the contact points in the collision;
3* The change of angular momentum of motion of the material two bodies in rolling for the corresponding rolling axes during the collision is finite.
4* The impulse (linear momentum) and angular momentum of ''ordinary forces'' compared to the impulse (linear momentum) and angular momentum of instantaneous collision forces are much smaller and can be neglected.

3.5. An elementary logical analogy. In the examples of the simplest dynamics of rigid bodies with one degree of freedom of movement, we will present an elementary logical analogy, which should be easily understood. Why do we begin with this article, which should be popular but at the same time contain the results of a high scientific domain?

It is well known that the most fundamental breakthroughs in science, which have become a lasting scientific heritage, are in fact elementary learning, which, in the integration of knowledge and conceptual processes, grows into complex scientific disciplines. The aim and answer of the question posed is to show that starting from the simplest dynamics of rigid bodies, translation and rolling, and then determining the elementary logical analogy among these dynamics and abstraction to the model of these dynamics, one can move to qualitative and mathematical analogies.

Therefore, by abstracting the disparate parameters of the dynamics of two real systems, one can come up with a theoretical model, a unique mathematical model with the same elements of mathematical phenomenology. We can use the knowledge of the properties of one model to convey it in the knowledge of the properties of the other, logically analogous.

Using logical, structural, qualitative and mathematical analogies [27, 28, 66-68] in both directions, we aim to obtain new original results of the theory of body-collision in rolling. We base the new results on the well-known theory of collision between bodies in translation. The unsurpassed theory of collision between bodies in translational moving has been formulated by world-renowned scientists Isaac Newton, John Wallis, and Christiaan Huygens.

Consequently, let us start with a logical analogy between the dynamics of the body systems shown in Figure 8a * and b *.

Figure 8a* shows two rigid bodies that can move along an ideally smooth horizontal surface in one direction, so that their median plane is always vertical. It is a planar, translational motion of a rigid body with one degree of freedom of movement, so we can consider it as a material point of concentrated mass in the mess center. Such a body is exposed to the effect of five constraints (links): two translations are prevented (one in the vertical direction and one in the direction perpendicular to the plane of plane motion) and three rotations around three orthogonal directions (around the direction of the body translation, around the vertical direction and around the direction perpendicular to the previous two).

The kinetic parameters of the motion translation of the rigid body model from Figure 8a* are: m_k, $k=1{,}2$, masses, $\vec{v}_{C1}$ and $\vec{v}_{C2}$ the velocities of body translation, which are the connected vectors for the mass centers of these bodies C_1 and C_1. We suppose that each of the bodies is loaded by one external force with intensity F_k, $k=1{,}2$, with a direction collinear with velocity and an attack point at the center of mass of the corresponding body affected.

On the basis of the theorem on the change of the linear momentum of motion (or quantity of motion), we construct the ordinary differential equation of translational dynamics of one and the other body in the form:

$$m_k \dot{v}_{Ck} = F_k,\ k=1{,}2 \tag{6}$$

The change in the linear momentum of body motion theorem states that this change in time equals the sum of active and reactive forces. The linear momentum of motion in the translation of a rigid body or the impulse of the translational motion of a body is the product between the mass of the body and the velocity $\vec{v}_{Ck}$, $k=1{,}2$ of the center of mass: $\vec{\mathbf{K}}_k = m_k \vec{v}_{Ck}$, $k=1{,}2$, $k=1{,}2$.

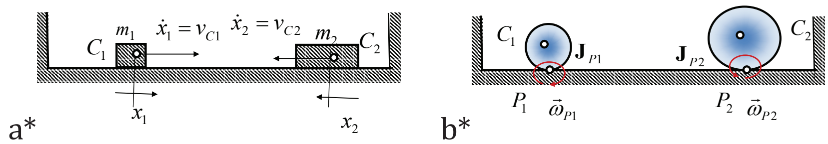


FIGURE 8. Models of the two simplest dynamics of material bodies, each with one degree of freedom of movement: a * translatory dynamics of a rigid body and b* rolling without slipping of a rigid body (with a form of a homogeneous disk, a homogeneous sphere or homogeneous with one axis and one plane of symmetry).

Figure 8 b * shows two bodies which roll without sliding straight along a linear path (guide, trace) through a circular plane contour of the body, a circular shape with a corresponding center at a point, C_1 and C_2, respectively. These rolling bodies may be spherical balls, cylinders, disks, but also other shapes, having one axis of symmetry and one plane of symmetry in which there is a contour of the shape of a circle, by which the legs are expected to roll.

Such bodies, which roll without sliding, have one degree of freedom of movement. Since each free body has six degrees of freedom of movement, this means that five motion constraints, three translation constraints and two rotations are imposed on the motion of these bodies from Figure 8b*, that is, five bonds are imposed on each of the bodies. The first limitation is that the median plane of symmetry of the body, in which the center of mass of the body, is at all times in the plane of contour of rolling. This produces the constraints of one translation perpendicular to that rolling plane, and of two rotations about two orthogonal axes in that plane of rolling. The connection with the non-slip rolling route prevented one translation in the direction of the rolling route and one translation directly on the rolling route. All these together represent five links and constraints, leaving only one degree of freedom of movement, which is rolling around the current instantaneous axis of rolling.

Therefore, we direct our further consideration to that class of bodies which roll without slipping. Figure 8b * shows two rigid bodies which roll without sliding at angular velocities $\vec{\omega}_{P1}$ and $\vec{\omega}_{P2}$, at the corresponding instantaneous axes of rolling, passing through the points, P_1 or P_2 respectively, of the contact of the bodies in rolling and the track on which they are rolling, which are directed orthogonally to the plane of the rolling. The axial moments of inertia of the masses of the body in rolling for the instantaneous axes of rolling are $\mathbf{J}_{P1}$ and $\mathbf{J}_{P2}$. Kinetic parameters of the rolling dynamics of each body are the instantaneous angular velocities of rolling $\vec{\omega}_{P1}$ and $\vec{\omega}_{P2}$, which are related to the instantaneous axes of rolling. These axes move translationally along the rolling path, and the axial moments of inertia of the mass of the body and for the corresponding instantaneous rolling axis, for the observed body class, $\mathbf{J}_{P1}$ and $\mathbf{J}_{P2}$ do not change during the rolling dynamics.

Based on the theorem on the change of angular momentum (or kinetic momentum) for the instantaneous axis of rolling without sliding, we construct the ordinary differential equation of the dynamics of rolling of one and the other body, in the form:

$$\mathbf{J}_{Pk}\dot{\omega}_{Pk} = \mathfrak{M}_{Pk}\,, \quad k = 1,2 \tag{7}$$

By comparing kinetic elements of two previous analyses of the dynamics of the bodies on two systems, one in translation and another in rolling without sliding from Figure 8a * and b *, we establish a logical, and at the same time, qualitative analogy and phenomenological mapping of the kinetic parameters of these models of two different dynamics, each with one degree of freedom. The mathematical analogy follows in the next section.

3.6. Dynamics of the central collision of two rolling balls along a horizontal trace. Let us start with the application of mathematical analogy of the classical theory of dynamics of collision to the dynamics of the collision between two rolling balls, with mass m_1 and m_2, and axial mass inertia moments $\mathbf{J}_{P1}$ and $\mathbf{J}_{P2}$ for corresponding momentary axis of the rotation in rolling along trace with pre-impact (arrival) angular velocities $\vec{\omega}_{P1,impact} = \vec{\omega}_{P1}(t_0)$ and $\vec{\omega}_{P2,impact} = \vec{\omega}_{P2}(t_0)$. Mass centers $\mathbf{C}_1$ and $\mathbf{C}_2$ of the balls move translatory with pre-impact (arrival) velocities $\vec{v}_{C1.impact} = \vec{v}_{C1}(t_0)$ and $\vec{v}_{C2.impact} = \vec{v}_{C2}(t_0)$. Angular velocities $\vec{\omega}_{P1,impact} = \vec{\omega}_{P1}(t_0)$ and $\vec{\omega}_{P2,impact} = \vec{\omega}_{P2}(t_0)$ we denote as arrival, or impact or pre-impact angular velocities at the moment t_0 (see Figures 5, 6, 7 and 8). At this moment t_0 of the collision start between these rolling balls, the contact of these two balls is at point $\mathbf{T}_{12}$, in which both balls possess common tangent plane – the plane of contact (touch). In the theory of collision, it is proposed that collision takes a very short period $(t_0, t_0 + \tau)$, and that τ tends to zero. After this short period τ bodies – two rolling balls in collision separate, outgoing by post-impact-outgoing angular velocities $\vec{\omega}_{P1,outgoing} = \vec{\omega}_{P1}(t_0 + \tau)$ and $\vec{\omega}_{P2,outgoing} = \vec{\omega}_{P2}(t_0 + \tau)$. Mass centers $\mathbf{C}_1$ and $\mathbf{C}_2$ of the balls move translatory with post-impact (outgoing) translatory velocities $\vec{v}_{C1.outgoing} = \vec{v}_{C1}(t_0 + \tau)$ and $\vec{v}_{C2.outgoing} = \vec{v}_{C2}(t_0 + \tau)$. These translatory velocities are possible to express, each by its corresponding outgoing post-collision angular velocity and radius of the corresponding ball.

Elements of mathematical phenomenology [27, 28, 66-68] and phenomenological mappings [64] between rolling balls and translatory bodies (balls), which are analogous dynamical systems with elements in impact (similar as electro-mechanical analogy between an electrical oscillator with one degree of freedom and a mechanical oscillator with one degree of freedom). Translatory motion of a body (ball) and rolling motion of a ball are analogous motions, and each with one degree of freedom. The analogies between mass and axial mass inertia moment for the rolling momentary axis and also translatory velocity and angular velocity around the momentary axis of rolling follow from the comparison of their mathematical description by ordinary differential equations of corresponding motion – translatory and rolling kinetic states. This is a visible and simple explanation presented in the previous subchapter.

Taking into account that translatory motion of two bodies in central collision is a simpler motion of two bodies, defined by corresponding inertia properties expressed by mass, m_1 and m_2, of each body, and also by corresponding translatory pre-impact velocities, $\vec{v}_1(t_0)$ and $\vec{v}_2(t_0)$ at the moment before collision and by post-impact-outgoing translatory velocities $\vec{v}_1(t_0 + \tau)$ and $\vec{v}_2(t_0 + \tau)$ it is possible to establish an analogy with the collision between two rolling balls. Explanation is in the following form.

Additionally, rolling balls along a horizontal strength trace is a simple rotation motion defined only by inertia properties in the axial ball mass inertia moments $\mathbf{J}_{P1}$ and $\mathbf{J}_{P2}$ for the corresponding momentary axis of rotation in rolling along a trace with pre-impact (arrival) angular velocities $\vec{\omega}_{P1,impact} = \vec{\omega}_{P1}(t_0)$ and $\vec{\omega}_{P2,impact} = \vec{\omega}_{P2}(t_0)$ and corresponding outgoing post-impact-outgoing angular velocities $\vec{\omega}_{P1,outgoing} = \vec{\omega}_{P1}(t_0 + \tau)$ and $\vec{\omega}_{P2,outgoing} = \vec{\omega}_{P2}(t_0 + \tau)$.

Using Petrović's theory of elements of mathematical phenomenology and phenomenological mappings [27, 28, 66-68] in parts of qualitative and mathematical analogies, we can indicate a qualitative and mathematical analogy between the system of the translatory dynamics and central collision (impact) dynamics of two bodies in translatory motion pre-impact and post impact dynamics phenomena and the system of the rolling two ball dynamics and central collision (impact) dynamics of two rolling balls in rolling motion, without slipping, pre-impact and post impact dynamics phenomena.

On the basis of these indicated qualitative and mathematical analogies, it is possible to list analogous kinetic parameters of these systems.

The axial mass inertia moments $\mathbf{J}_{P1}$ and $\mathbf{J}_{P2}$ for the corresponding momentary axis of rotation in rolling, without slipping along trace are analogous to the corresponding bodies with masses m_1 and m_2 of two bodies in collision in translatory motion.

Pre-impact (arrival) angular velocities $\vec{\omega}_{P1,impact} = \vec{\omega}_{P1}(t_0)$ and $\vec{\omega}_{P2,impact} = \vec{\omega}_{P2}(t_0)$ of the rolling balls around the corresponding momentary axis are analogous to corresponding translatory pre-impact velocities, $\vec{v}_1(t_0)$ and $\vec{v}_2(t_0)$ of two bodies at the moment before collision.

Post-impact-outgoing angular velocities $\vec{\omega}_{P1,outgoing} = \vec{\omega}_{P1}(t_0+\tau)$ and $\vec{\omega}_{P2,outgoing} = \vec{\omega}_{P2}(t_0+\tau)$ of the rolling balls are analogous to the corresponding post-impact-outgoing translatory velocities $\vec{v}_1(t_0+\tau)$ and $\vec{v}_2(t_0+\tau)$ of two bodies in translatory motion to collision.

On the basis of Petrović's theory [63-65] and qualitative and mathematical analogies considered in the previous section, it is possible to formulate the analogous *hypothesis of conservation of the sum of angular momentums* (moment of impulse for the corresponding momentary axis) of the impact dynamics of two rolling balls in pre-collision and post-collision motion; this is achieved on the basis of the hypothesis of conservation of the sum of linear momentum (impulse) (1) of the impact dynamics of two bodies in translatory motion pre-collision and post-collision, in the following relation:

$$\mathbf{J}_{P1}\vec{\omega}_{P1}(t_0) + \mathbf{J}_{P2}\vec{\omega}_{P2}(t_0) = \mathbf{J}_{P1}\vec{\omega}_{P1}(t_0+\tau) + \mathbf{J}_{P2}\vec{\omega}_{P2}(t_0+\tau) \tag{8}$$

and analogous with (2), the coefficient of the restitution of rolling balls collision is in the form:

$$k = \frac{\omega_r(t_0+\tau)}{\omega_r(t_0)} = \frac{\omega_{P2}(t_0+\tau) - \omega_{P1}(t_0+\tau)}{\omega_{P1}(t_0) - \omega_{P2}(t_0)} \tag{9}$$

as the ratio between the difference of angular velocities of rolling balls post-collision and pre-collision kinetic states.

Equation (8) is stating an important kinetic parameter of the system of two colliding rolling balls as a sum of angular momentum (sum of the moment of impulse of each of the colliding rolling balls for the corresponding momentary axis of rolling) of the colliding and rolling balls before – pre-collision and after – post-collision kinetic state of the system in analogy with and on the same level as equation (1) of the sum of linear momentum (impulse) for two colliding bodies (balls) in translatory motion, pre-collision and post-collision of bodies in translatory motion.

The restitution coefficient k expressed by (2) is determined by Newton's classical theory of impact dynamics of rigid bodies, as the ratio between the difference of the translatory velocity components after and before the impact for the case of central collision between two bodies (balls) in translatory motion after and before collision.

In the present paper the coefficient of the restitution k by expression (9) *is introduced by angular velocities after and before collision of the two rolling balls*. It is in mathematical and qualitative analogy to the basis of the theory of Elements of mathematical phenomenology and Phenomenological mappings [53-65] founded by Mihailo Petrović (Serbian scientist and one of three doctoral students of Julius Henri Poincaré) using analogous kinetic elements of translatory motion of two balls and of the rotation motion of two rolling balls, each of them with one degree of freedom.

Additionally, in analogy with the expressions (3)-(4) of post-collision – outgoing body translatory velocities, it is possible to write expressions of post-collision – outgoing rolling balls angular velocities in the following forms:

$$\omega_{P1}(t_0+\tau)=\frac{(\mathbf{J}_{P1}-k\mathbf{J}_{P2})\omega_{P1}(t_0)+(1+k)\mathbf{J}_{P2}\omega_{P2}(t_0)}{\mathbf{J}_{P1}+\mathbf{J}_{P2}}$$
$$\omega_{P1}(t_0+\tau)=\omega_{P1}(t_0)-\frac{1+k}{1+\dfrac{\mathbf{J}_{P1}}{\mathbf{J}_{P2}}}(\omega_{P1}(t_0)-\omega_{P2}(t_0)) \tag{10}$$

$$\omega_{P2}(t_0+\tau)=\frac{(\mathbf{J}_{P2}-k\mathbf{J}_{P1})\omega_{P2}(t_0)+(1+k)\mathbf{J}_{P1}\omega_{P1}(t_0)}{\mathbf{J}_{P1}+\mathbf{J}_{P2}}$$
$$\omega_{P2}(t_0+\tau)=\omega_{P2}(t_0)+\frac{1+k}{1+\dfrac{\mathbf{J}_{P2}}{\mathbf{J}_{P1}}}(\omega_{P1}(t_0)-\omega_{P2}(t_0)) \tag{11}$$

Previously obtained expressions (10) *and* (11) *of post-collision – outgoing rolling balls angular velocities are new and original results obtained on the basis of Petrović's theory of elements of mathematical phenomenology (see Reference* [63-65]*). Additionally, expression* (8) *for the hypothesis of conservation of the sum of angular momentums* (moment of impulse for corresponding momentary axis) of impact dynamics of two rolling balls pre-collision and post-collision motion is a newly introduced relation in impact dynamics as well as expression (9) for the coefficient of restitution in collision of two rolling balls with different size and in central collision. All these results are analytical and present a basis for applications in other kinds of collisions.

In analogy of expression (5) of the impulses (linear momentum) of collision two bodies in pre-collision and post-collision translatory motions, it is possible to compose analogous expressions of the moment of impulses (kinetic moment, angular momentum) of collision of two rolling balls in pre-collision and post-collision dynamics, in the following form:

$$\mathfrak{L}_{\mathfrak{M},impact}=\mathbf{J}_{P1}(\omega_{P1}(t_0+\tau)-\omega_{P1}(t_0))=-\frac{\mathbf{J}_{P1}\mathbf{J}_{P2}}{\mathbf{J}_{P1}+\mathbf{J}_{P2}}(1+k)(\omega_{P1}(t_0)-\omega_{P2}(t_0)) \tag{12}$$

As it is known from classical literature, in analogy with the coefficient of the restitution of two bodies colliding in translatory motion [70], the coefficient of collision of two rolling balls also depends on kinds of collisions: 1* for pure no elastic (plastic) collision coefficient of collision is equal to zero- $k=0$; 2* for pure ideal elastic collision coefficient of collision is equal to unit, ; and 3* for an arbitrary case between ideal plastic and ideal elastic collision coefficient of collision is in interval between zero and unique, $0<k<1$. From the comparison between outgoing (post-collision) angular velocities in no elastic collision of two rolling balls in the pre-collision state, we can point out the following conclusions:

* in the case of pure plastic collision of two rolling balls, $k=1$, outgoing (post-collision) angular velocities are equal one to another;
* in the case of ideal elastic collision of two rolling balls, $k=1$, outgoing (post-collision) angular velocities of the rolling balls are: outgoing (post-collision) angular velocity of the rolling ball with the largest pre-collision impact angular velocity is smaller, and outgoing (post-collision) angular velocity of the rolling ball with smaller pre-collision impact angular velocity is larger; in this case, ideal elastic collision without slipping, for $k=1$, if both pre-collision impact angular velocities of the rolling balls are equal, then, both outgoing (post-collision) angular velocities of the both balls are equal and independent of the balls axial inertia moments.
* In the case of no elastic collision between rolling balls, $0<k<1$, if condition $\mathbf{J}_{P1}\omega_{P1}(t_0)+\mathbf{J}_{P2}\omega_{P2}(t_0)=0$, or $\frac{\mathbf{J}_{P1}}{\mathbf{J}_{P2}}=-\frac{\omega_{P2}(t_0)}{\omega_{P1}(t_0)}$ is present, outgoing (post-collision) angular velocities of both balls achieved the condition: $\frac{\mathbf{J}_{P1}}{\mathbf{J}_{P2}}=-\frac{\omega_{P2}(t_0+\tau)}{\omega_{P1}(t_0+\tau)}$. In this case outgoing angular velocities of the rolling balls are: $\omega_{P1}(t_0+\tau)=-k\omega_{P1}(t_0)$ and $\omega_{P2}(t_0+\tau)=-k\omega_{P2}(t_0)$.

The kinetic energy of the rolling balls in the pre-collision kinetic state is in the form:

$$E_k(t_0)=\frac{1}{2}\left(\mathbf{J}_{P1}\omega_{P1}^2(t_0)+\mathbf{J}_{P2}\omega_{P2}^2(t_0)\right) \tag{13}$$

and kinetic energy of these rolling balls after collision (in the post-collision kinetic state) is:

$$E_k(t_0+\tau)=\frac{1}{2}\left(\mathbf{J}_{P1}\omega_{P1}^2(t_0+\tau)+\mathbf{J}_{P2}\omega_{P2}^2(t_0+\tau)\right) \tag{14}$$

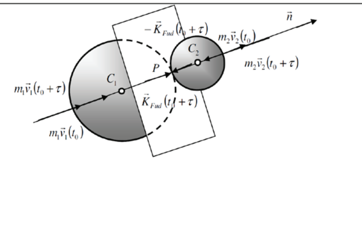
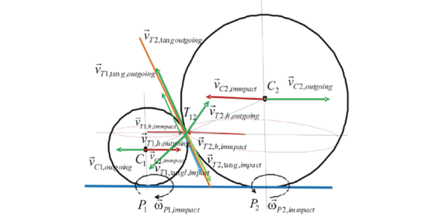
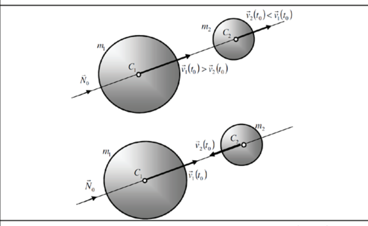
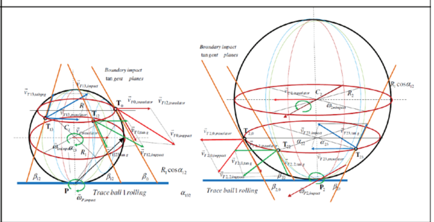
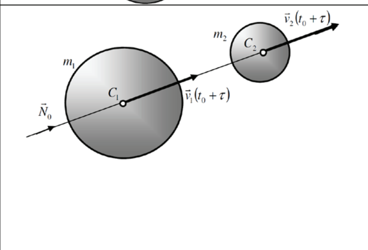
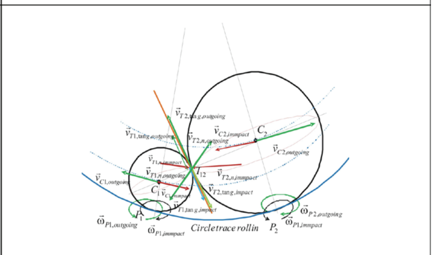
a* In the case of arbitrary coefficient of restitution, $0<k<1$, of collision, the rate of decreasing kinetic energy in comparison between the pre-collision and post-collision kinetic state of the rolling balls is equal:

$$\Delta E_k=E_k(t_0+\tau)-E_k(t_0)=-\frac{\mathbf{J}_{P1}\mathbf{J}_{P2}}{2(\mathbf{J}_{P1}+\mathbf{J}_{P2})}(1-k^2)(\omega_{P1}(t_0)-\omega_{P2}(t_0))^2 \tag{15}$$

b* For an ideal plastic collision, $k=0$, the rate of the kinetic energy decreasing in comparison between pre-collision and post-collision kinetic state of the rolling balls is equal:

$$\Delta E_{k,plast}=E_k(t_0+\tau)-E_k(t_0)=-\frac{\mathbf{J}_{P1}\mathbf{J}_{P2}}{2(\mathbf{J}_{P1}+\mathbf{J}_{P2})}(\omega_{P1}(t_0)-\omega_{P2}(t_0))^2 \tag{16}$$

Table 1. Mathematical and qualitative analogies between kinetic parameters of two system in central collision dynamics: the collision of two bodies in translatory motion and collision of two rolling balls

	Collision of two bodies in translator motion	Collision of two rolling balls
Configuration of the systems in collision state and plans f velocities and tanfent plane of bodies collisions		
		
		
Analogous theorms of conservation of linear momentum (impulse) or angular momentum	Theorm of conservation of linear momentum (impulse) in collision of two bodies in translator motion $m_1\vec{v}_1(t_0)+m_2\vec{v}_2(t_0)=m_1\vec{v}_1(t_0+\tau)+m_2\vec{v}_2(t_0+\tau)$	Theorm of conservation of angular momentum (kinetic moment) in collision of two rolling balls $\mathbf{J}_{P1}\vec{\omega}_{P1}(t_0)+\mathbf{J}_{P2}\vec{\omega}_{P2}(t_0)=\mathbf{J}_{P1}\vec{\omega}_{P1}(t_0+\tau)+\mathbf{J}_{P2}\vec{\omega}_{P2}(t_0+\tau)$
Coefficient of the restitution of two body collision	Coefficient of the restitution in collision of two bodies in translator motion $k=\dfrac{v_r(t_0+\tau)}{v_r(t_0)}=\dfrac{v_2(t_0+\tau)-v_1(t_0+\tau)}{v_1(t_0)-v_2(t_0)}$	Coefficient of the restitution in collision of two rolling balls $k=\dfrac{\omega_r(t_0+\tau)}{\omega_r(t_0)}=\dfrac{\omega_{P2}(t_0+\tau)-\omega_{P1}(t_0+\tau)}{\omega_{P1}(t_0)-\omega_{P2}(t_0)}$
Outgoing velocities of thwo bodies at post-collision moment	Outgoing velocities of the of two bodies in translator motion at post-collision moment $v_1(t_0+\tau)=v_1(t_0)-\dfrac{1+k}{1+\dfrac{m_1}{m_2}}(v_1(t_0)-v_2(t_0))$ $v_2(t_0+\tau)=v_2(t_0)+\dfrac{1+k}{1+\dfrac{m_1}{m_2}}(v_1(t_0)-v_2(t_0))$	Outgoing angular velocities of the rolling balls at post-collision moment $\omega_{P1}(t_0+\tau)=\omega_{P1}(t_0)-\dfrac{1+k}{1+\dfrac{\mathbf{J}_{P1}}{\mathbf{J}_{P2}}}(\omega_{P1}(t_0)-\omega_{P2}(t_0))$ $\omega_{P2}(t_0+\tau)=\omega_{P2}(t_0)+\dfrac{1+k}{1+\dfrac{\mathbf{J}_{P1}}{\mathbf{J}_{P2}}}(\omega_{P1}(t_0)-\omega_{P2}(t_0))$
Impuls (linear momentum) of collision	*Impuls (linear momentum) of collision of impact forces* $K_{Fud}=m_1(v_1(t_0+\tau)-v_1(t_0))=-\dfrac{m_1m_2}{m_1+m_2}(1+k)(v_1(t_0)-v_2(t_0))$	*Moment of impuls (linear momentum) of collision of impact couple (moment of impact forces)* $\mathfrak{M}_{impact}=\mathbf{J}_{P1}(\omega_{P1}(t_0+\tau)-\omega_{P1}(t_0))=-\dfrac{\mathbf{J}_{P1}\mathbf{J}_{P2}}{\mathbf{J}_{P1}+\mathbf{J}_{P2}}(1+k)(\omega_{P1}(t_0)-\omega_{P2}(t_0))$
Kinetic energy change fromprecollison to postcollision kinetic state	$\Delta E_{k,plast}=E_k(t_0+\tau)-E_k(t_0)=\dfrac{m_1m_2}{2(m_1+m_2)}(v_1(t_0)-v_2(t_0))^2$	$\Delta E_k=E_k(t_0+\tau)-E_k(t_0)=\dfrac{\mathbf{J}_{P1}\mathbf{J}_{P2}}{2(\mathbf{J}_{P1}+\mathbf{J}_{P2})}(1-k^2)(\omega_{P1}(t_0)-\omega_{P2}(t_0))^2$

c* In the case of ideal elastic collision, $k=1$, between rolling balls with no change of kinetic energy in comparison to pre-collision and post-collision kinetic states of rolling balls and is equal to zero:

$$\Delta E_{k,elast} = E_k(t_0+\tau) - E_k(t_0) = 0\,. \tag{17}$$

In this case of ideal elastic impact, low *kinetic energy conservation* is valid.

In Table 1, mathematical and qualitative analogies between kinetic parameters of two systems in central collision dynamics are presented. In the left column, the kinetic parameters for collision of two bodies in translatory motion and in the right column, analogous kinetic parameters of collision of two rolling balls are presented.

Listed analytical expressions and relations (13)-(17) and conclusions a*-b*-c* of the kinetic energy decreasing in comparison, kinetic energy of two rolling balls in the pre-collision of kinetic state and the post-collision kinetic state of the balls in collision present *Carnot's theorem* generalized (*Lazare Carnot* 1753-1824, *Principes fondamentaux de l'équilibre et de mouvement* - 1803), dealing with the kinetic energy of two rolling balls in kinetic states pre- and post collision (in arrival and outgoing kinetic states): "*In the collision of two rolling balls in rolling motion for the arbitrary coefficient of the restitution,* $0<k<1$*, the loss of kinetic energy decreasing during collision is proportional to the loss of angular velocities"*.

$$\Delta E_k = E_k(t_0) - E_k(t_0+\tau) = 2E_{k,izg}(\tau) = \sum_{i=1}^{N} \mathbf{J}_P \left[\vec{\omega}_P(\tau)\right]^2 \text{“}, \tag{18}$$

Examples: Masses of the balls are $m_1 = \rho_1 \frac{4}{8} r_1^3 \pi$ and $m_2 = \rho_2 \frac{4}{8} r_2^3 \pi$, and axial mass inertia moments for momentary axis of the rolling balls are:

$$\mathbf{J}_{P1} = \mathbf{J}_{C1} + m_1 r_1^2 = \frac{2}{5} m_1 r_1^2 + m_1 r_1^2 = \frac{7}{5} m_1 r_1^2 = \frac{7}{5}\rho \frac{4}{8} r_1^5 \pi = \frac{7}{10}\rho_1 r_1^5 \pi\,;\; \mathbf{J}_{P2} = \mathbf{J}_{C2} + m_2 r_2^2 = \frac{7}{10}\rho_2 r_2^5 \pi\,.$$

For $\lambda_1 = \frac{R}{r_1}$ and $\lambda_2 = \frac{r_2}{r_1}$ ratio of the axial mass inertia moments is:

$\frac{\mathbf{J}_{P1}}{\mathbf{J}_{P2}} = \left(\frac{r_1}{r_2}\right)^5 \left(\frac{\rho_1}{\rho_2}\right)^5 = \lambda_2^5$. For a different ratio between axial mass inertia moments, balls' outgoing angular velocities around instantaneous axis at post collision kinetic state between balls are:

a* for $\lambda_2 = \left(\frac{r_2}{r_1}\right)^5 = \frac{1}{2^5} = \frac{1}{32}$; $\frac{1}{1+\frac{\mathbf{J}_{P2}}{\mathbf{J}_{P1}}} = \frac{1}{1+32} = \frac{1}{33}$

$$\omega_{P1}(t_0+\tau) = \omega_{P1}(t_0) - \frac{32}{33}(1+k)(\omega_{P1}(t_0) - \omega_{P2}(t_0))$$

$$\omega_{P2}(t_0+\tau) = \omega_{P2}(t_0) + \frac{1}{33}(1+k)(\omega_{P1}(t_0) - \omega_{P2}(t_0))$$

b* $\lambda_2 = \frac{r_2}{r_1} = \frac{1}{3^5} = \frac{1}{81\cdot 3} = \frac{1}{243}$

$$\omega_{P1}(t_0+\tau) = \omega_{P1}(t_0) - \frac{243}{244}(1+k)(\omega_{P1}(t_0) - \omega_{P2}(t_0))$$

$$\omega_{P2}(t_0+\tau) = \omega_{P2}(t_0) + \frac{1}{244}(1+k)(\omega_{P1}(t_0) - \omega_{P2}(t_0))$$

c* $\lambda_2 = \frac{r_2}{r_1} = \frac{2^5}{3^5} = \frac{2^5}{81 \cdot 3} = \frac{32}{243}$; $\frac{1}{1+\frac{\mathbf{J}_{P2}}{\mathbf{J}_{P1}}} = \frac{1}{1+\frac{243}{32}} = \frac{32}{275}$

$$\omega_{P1}(t_0+\tau) = \omega_{P1}(t_0) - \frac{243}{275}(1+k)(\omega_{P1}(t_0) - \omega_{P2}(t_0))$$

$$\omega_{P2}(t_0+\tau) = \omega_{P2}(t_0) + \frac{32}{275}(1+k)(\omega_{P1}(t_0) - \omega_{P2}(t_0))$$

3.7. Elementary approach for determination of expressions of intensity of outgoing angular velocities in the centrally centric collision of two bodies in the rolling, without sliding, immediately after the collision. We look at two axisymmetric rigid bodies with one central plane of symmetry, centrally colliding, and making contact at one point of collision, or two balls of different radii, or two disks of different radii, axial moments of inertia of masses of bodies for the corresponding instantaneous axes of rolling $\mathbf{J}_{P1}$ and $\mathbf{J}_{P2}$. These axial moments of inertia of masses do not change for the axial rolling axes in motion.

The bodies are in a non-slip rolling position at the moment t_0 and have angular velocities $\vec{\omega}_{P1}(t_0)$ and $\vec{\omega}_{P2}(t_0)$ at an instant t_0 before entering the collision configuration and are referred to as incoming (inlet or impact) angular velocities. At the moment t_0 of the start of the collision, the two bodies will touch at one contact point P where both bodies have a tangential plane in common. We assume that the collision lasts briefly over an interval $(t_0, t_0+\tau)$ of time, which lasts for a short time τ (and realistically tends to zero). After this collision of short-term contact, the bodies are separated by angular velocities $\vec{\omega}_{P1}(t_0+\tau)$ and $\vec{\omega}_{P2}(t_0+\tau)$, which we call the outgoing angular velocities. This is necessary to determine the intensities of these outgoing angular velocities, $\vec{\omega}_{P1}(t_0+\tau)$ and $\vec{\omega}_{P2}(t_0+\tau)$, and we have already determined the paths of outgoing rolling velocities and the directions of rolling and directions of those outgoing velocity, velocities immediately after the collision.

Imagine that, at the point P of contact of two bodies in a state (configuration) of collision, we have drawn a tangential plane and it's normal $\vec{n}$. This tangential plane is called the touch tangent plane, and the direction of that normal to the touch plane determines the direction of the collision. Since the centers C_1 and C_2 of mass of the bodies in collision are at this normal, and if the incoming rolling traces of the bodies are at that normal, the collision is called a centric (central) collision, and if not, the collision is skew or oblique eccentric. When the incoming angular velocities $\vec{\omega}_{P1}(t_0)$ and $\vec{\omega}_{P2}(t_0)$ of both bodies in the collision are collinear with the tangent plane, that is, direct with the direction of the collision, then it is a true (directional) collision of the rolling bodies, otherwise it is a skew collision of two rolling bodies.

3.7.1. Hypothesis of conservation of sum of angular momentum for the instantaneous axes of rolling of two bodies in rolling before and after the collision of two axisymmetric bodies. At the time of the collision, both bodies, which roll immediately before the collision, come into contact at one point, or line-derivatives. In the collision event [67], although we have made the assumption of models of rigid, axisymmetric bodies with a central plane of symmetry, during the collision they deform locally, in the local contact area. If the contact of the bodies in the collision is at the point of contact (for example, the contact of the spherical surfaces of the balls in the collision, or the rotation ellipsoids), deformation occurs in the immediate vicinity of the contact point. And this deformation lasts until the projections of the angular velocities of rotation of the body in the collision in the direction of the collision (the normal on the tangent equal to both bodies in the contact point of the collision) are equal. Additionally, the projections of the relative angular velocities of the rolling motion around the instantaneous axes of rolling of the body in the collision, one relative to the other, towards the collision direction became zero. From that moment, zero projections of relative angular velocities in the direction of the collision begin to restore the state of the body as it was before the collision until the moment when the bodies separate from each other. During this time the projection of the relative velocities of the bodies in the collision of one relative to the other begins to increase and continues until the bodies have, in the part in contact, their original shape. Then there is a moment when we consider that the bodies have practically separated and that there is a period of time after the collision. Therefore, the collision period can be divided into two parts: τ' the compression period in the tangential direction to the body at the point of contact, and τ'' the restitution period in the tangential direction to the body at the point of contact in the collision, with the total short-time duration $\tau = \tau' + \tau''$ of the collision.

Since external active forces and moments of forces of finite intensities have impulses of forces equal to zero, and couplings have kinetic moments equal to zero and, at infinitesimal intervals of time, we consider that two material rigid bodies, which roll with the incoming angular velocities and in collision, are considered as one system. Therefore, the hypothesis of the conservation of the sum of angular momentum (kinetic momentum) for the instantaneous axes of rolling of each body - the movement before and after the collision, can be applied to the dynamics of the same to in the form:

$$\mathbf{J}_{P1}\vec{\omega}_{P1}(t_0) + \mathbf{J}_{P2}\vec{\omega}_{P2}(t_0) = \mathbf{J}_{P1}\vec{\omega}_{P1}(t_0 + \tau) + \mathbf{J}_{P2}\vec{\omega}_{P2}(t_0 + \tau). \tag{8.a}$$

This hypothesis about the conservation of the sum of the angular momentum of motion by rolling in a collision of two bodies, which is analogous to the hypothesis of the conservation of the sum of the linear momentum (1) of motion of two bodies in a collision and in translational motion (see References [1, 10, 12, 24, 25, 28, 29]).

3.7.2. Coefficient of restitution or collision of two axisymmetric rolling bodies with one central plane of symmetry. When the incoming angular velocities $\vec{\omega}_{P1}(t_0)$ and $\vec{\omega}_{P2}(t_0)$ of rolling and the axial moments of inertia of mass $\mathbf{J}_{P1}$ and $\mathbf{J}_{P2}$ for the instantaneous axes of rolling of each of the bodies in a collision are known, the previous hypothesis relation (8) and (8a) of the sum of the angular momentum of motion for the instantaneous axes of rolling, before and after the collision, is not sufficient to determine two unknown outgoing angular velocities $\vec{\omega}_{P1}(t_0 + \tau)$ and $\vec{\omega}_{P2}(t_0 + \tau)$, after the collision of two bodies, which roll just before and after the collision. We need another relation, an equation, which we will set from the very properties of the body in a collision. As we have already described the collision and the contact process of two bodies, in the period of solid body compression, the angular velocity $\vec{\omega}_{P1}(t_0)$ of the first body will decrease by $\vec{\omega}_{P1}(t_0)-\vec{\omega}_{CP}$, and the second increases by $\vec{\omega}_{CP}-\vec{\omega}_{P2}(t_0)$, where $\vec{\omega}_{CP}$ the angular velocity of both bodies in the collision is at the end of the compression at the local environment of the contact point of the bodies in the collision. As both bodies are deformed in the local area around the point of joint contact in the collision, it is apparent that, during the restitution period, the deformations of the body will not be immediately lost and that the angular velocity $\vec{\omega}_{P1}(t_0)$ of the first body will decrease by $k(\vec{\omega}_{CP}-\vec{\omega}_{P2}(t_0))$ another angular velocity and the angular velocity $\vec{\omega}_{P2}(t_0)$ of the second body will increase for size $k(\vec{\omega}_{CP}-\vec{\omega}_{P2}(t_0))$, where k is the sum coefficient. Based on this analysis we can state that the outgoing angular velocities $\vec{\omega}_{P1}(t_0+\tau)$ and $\vec{\omega}_{P2}(t_0+\tau)$ of the bodies that were in the collision are outgoing

$$\vec{\omega}_{P1}(t_0+\tau)=\vec{\omega}_{P1}(t_0)-(1+k)(\vec{\omega}_{P1}(t_0)-\vec{\omega}_{e})=(1+k)\vec{\omega}_{CP}-k\vec{\omega}_{P1}(t_0), \quad (18)$$

$$\vec{\omega}_{P2}(t_0+\tau)=\vec{\omega}_{P2}(t_2)+(1+k)(\vec{\omega}_{CP}-\vec{\omega}_{P2}(t_0))=(1+k)\vec{\omega}_{CP}-k\vec{\omega}_{P2}(t_0). \quad (19)$$

Subtracting these previous relations (18) - (19) we obtain

$$\vec{\omega}_{P2}(t_0+\tau)-\vec{\omega}_{P1}(t_0+\tau)=k(\vec{\omega}_{P2}(t_0)+\vec{\omega}_{P1}(t_0)). \quad (20)$$

The ratio k of the relative angular velocities of rolling of the axisymmetric bodies after and before the collision is

$$k=\frac{\omega_r(t_0+\tau)}{\omega_r(t_0)}=\frac{\omega_{P2}(t_0+\tau)-\omega_{P1}(t_0+\tau)}{\omega_{P1}(t_0)-\omega_{P2}(t_0)} \quad (9.a)$$

and *is called the collision coefficient, or the coefficient of restitution, or the coefficient of establishment of rolling bodies in a collision.*

This coefficient k is also *newly introduced* and represents a new definition of the collision coefficient, or the coefficient of restitution or the coefficient of establishment of rolling bodies in a collision. This new definition (9a) is derived by the author of this paper.

With the introduction of this new refinement of the collision coefficient, we have generalized Newton's definition from the theory of collision between rigid bodies in translatory motion to the theory of collision between rigid bodies in rolling motion without sliding by using the difference of rolling angular velocities both after and before the collision. If a kinetic state can be defined by one angular velocity around the instantaneous axis of the rolling for each of the bodies, in attempting to define the dynamics of the collision, we implement our definition of the coefficient k of restitution over the ratio of the relative angular velocities of the rolling bodies after and before the collision.

3.7.3. Intensity of outgoing angular velocities of two bodies rolling after a collision. In order to determine the intensities of the outgoing angular velocities of the rolling of two bodies after a collision, $\vec{\omega}_{P1}(t_0+\tau)$ and $\vec{\omega}_{P2}(t_0+\tau)$, it is sufficient to eliminate from the previous relations (8a) - (9a) the unknown angular velocities of both bodies, $\vec{\omega}_{CP}$, in the collision at the end of the compression at the local environment of the point of contact of the bodies in the collision. Consequently, it is important to solve the relations by unknown outgoing angular velocities, so for the outgoing angular velocities, $\vec{\omega}_{P1}(t_0+\tau)$ and $\vec{\omega}_{P2}(t_0+\tau)$, after the collision of the balls, we get the following expressions:

$$\vec{\omega}_{P1}(t_0+\tau)=\vec{\omega}_{P1}(t_0)-\frac{1+k}{1+\frac{\mathbf{J}_{P1}}{\mathbf{J}_{P2}}}(\vec{\omega}_{P1}(t_0)-\omega_{P2}(t_0)) \tag{10.a}$$

$$\vec{\omega}_{P2}(t_0+\tau)=\omega_{\mathbf{P}1}(t_0)+\frac{1+k}{1+\frac{\mathbf{J}_{P2}}{\mathbf{J}_{P1}}}(\omega_{\mathbf{P}1}(t_0)-\omega_{\mathbf{P}2}(t_0)) \tag{11.a}$$

By determining these intensities of the outgoing angular velocities $\vec{\omega}_{P1}(t_0+\tau)$ and $\vec{\omega}_{P2}(t_0+\tau)$ of the rolling of the balls (axisymmetric bodies each with a central plane of symmetry) after the collision, we have solved the complete problem of the theory of collision of axisymmetric bodies in rolling without slipping (see References [1, 10, 12, 24, 25, 28, 29]). These expressions (8a), (9a), (10a) and (11a) are reached, also, in the form (8), (9), (10) and (11) by a logical, qualitative and mathematical analogy, starting from the theory of collisions of bodies in translational motion, as shown in the previous chapter 3.6.

4. Vibro-impact dynamics of multiple collisions of two different rolling heavy balls along a circle trace in a vertical plane

In References [37, 45], the phase trajectory portrait of the vibro-impact forced dynamics of two heavy mass particles motions along a rough circle is investigated, and also the vibro-impact of a heavy mass particle moving along a rough circle with two impact limiters was considered and studied. In References [34-39, 44-49] a series of mass particle motion along smooth or rough curvilinear lines are studied, followed by the presentation of results.

The following part examines the vibro-impact dynamics of multiple successive collisions of two rolling heavy balls along a circle trace in a vertical plane and presents the obtained results.

In Figure 9, a model of two heavy homogeneous rolling balls, with radiuses r_1 and r_2, along a circle, with radius R, in a vertical plane is presented. Let us start with the theory of dynamics of collision between these two rolling balls, with mass m_1 and m_2, and axial mass inertia moments $\mathbf{J}_{P1}$ and $\mathbf{J}_{P2}$ for the corresponding momentary axis of rotation in rolling along a curvilinear trace in the form of a circular line in a vertical plane, with pre-impact (arrival) angular velocities $\vec{\omega}_{P1,impact}=\vec{\omega}_{P1}(t_0)$ and $\vec{\omega}_{P2,impact}=\vec{\omega}_{P2}(t_0)$. Mass centers $\mathbf{C}_1$ and $\mathbf{C}_2$ of the balls move translatory along the two circles, with radius $R-r_1$ and $R-r_2$, respectively, and with pre-impact (arrival) velocities $\vec{v}_{C1.impact}=\vec{v}_{C1}(t_0)$ and $\vec{v}_{C2.impact}=\vec{v}_{C2}(t_0)$. The angular velocities $\vec{\omega}_{P1,impact}=\vec{\omega}_{P1}(t_0)$ and $\vec{\omega}_{P2,impact}=\vec{\omega}_{P2}(t_0)$ we denote as arrival, or impact or pre-collision angular velocities at the moment t_0 (see Figure 10). At this moment t_0 of the start of the collision between these rolling balls, the contact of these two balls is at point $\mathbf{T}_{12}$, in which both balls possess the common tangent plane – plane of contact (touch). In the theory of the collision, it is proposed that collision takes a very short period of time $(t_0, t_0+\tau)$, and that τ tends to zero. After this short period τ bodies - two rolling balls in collision separate, outgoing by post-collision-outgoing angular velocities $\vec{\omega}_{P1,outgoing} = \vec{\omega}_{P1}(t_0+\tau)$ and $\vec{\omega}_{P2,outgoing} = \vec{\omega}_{P2}(t_0+\tau)$. Mass centers $\mathbf{C}_1$ and $\mathbf{C}_2$ of the balls move translatory with post-collision (outgoing) translatory velocities $\vec{v}_{C1.outgoing} = \vec{v}_{C1}(t_0+\tau)$ and $\vec{v}_{C2.outgoing} = \vec{v}_{C2}(t_0+\tau)$. These translatory velocities are possible to express, each by the corresponding angular velocity and radius of the corresponding ball [26].

Taking into account that translatory motion along an ideal curvilinear line of two bodies in central collision (as the collision of two mass particles moving along a curvilinear line) is a simpler motion of two mass particles, defined by corresponding inertia properties expressed by mass, m_1 and m_2, of each body and also by a corresponding translatory pre-impact velocity, $\vec{v}_1(t_0)$ and $\vec{v}_2(t_0)$ at the moment before collision and by post-impact-outgoing translator velocities $\vec{v}_1(t_0+\tau)$ and $\vec{v}_2(t_0+\tau)$ is possible to compare with the collision of two rolling balls along a curvilinear line. Explanation is similar to the one in the case when the pre- and post-collision traces are straight lines, presented in the previous part 3.5.

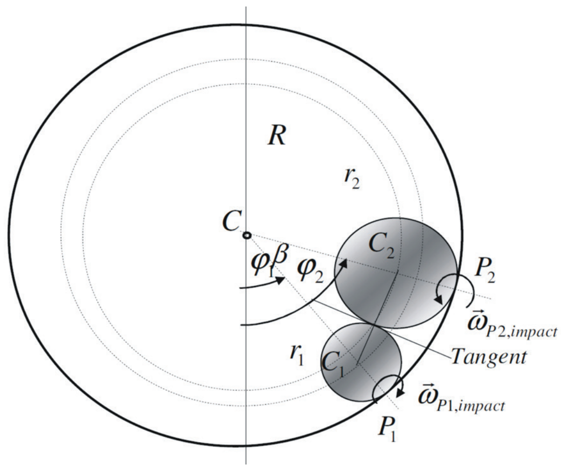


FIGURE 9. Mechanical system of collision of two heavy rolling balls along a circular trace in a vertical plane

Additionally, the rolling balls along curvilinear circle lines-traces is a simple rotation motion defined only by inertia properties in the axial mass inertia moments $\mathbf{J}_{P1}$ and $\mathbf{J}_{P2}$ for corresponding momentary axis of rotation in rolling along curvilinear circle traces with pre-impact (arrival) angular velocities $\vec{\omega}_{P1,impact} = \vec{\omega}_{P1}(t_0)$ and $\vec{\omega}_{P2,impact} = \vec{\omega}_{P2}(t_0)$ and corresponding outgoing post-impact-outgoing angular velocities $\vec{\omega}_{P1,outgoing} = \vec{\omega}_{P1}(t_0+\tau)$ and $\vec{\omega}_{P2,outgoing} = \vec{\omega}_{P2}(t_0+\tau)$. Nevertheless, for the rolling motion between two collisions we must assume that balls are in the rolling dynamics under the conservative force caused by the gravitational field, which contains the balls and the circle. But it is only necessary to take this into account during the motion of the balls between two collisions, and to obtain pre-collision angular velocities as angular velocities at end of the previous interval of rolling each of the balls in the gravitational field.

4.1. Kinetic parameters of a rolling heavy ball motion along a circle in a vertical plane.

Let us consider the rolling dynamics of one heavy smooth ball (first) along a curvilinear circle line trace in a vertical plane and in the gravitational field (rolling pendulum in References [14, 66-68]). For that reason, the kinetic and potential energies are expressed by the central angle φ_1 with respect to the circle center C_0 (see Figure 9, and Reference [26, 69,]):

$$\mathbf{E}_{\mathbf{k},1} = \frac{1}{2}\kappa_1 m_1 (R - r_1)^2 \dot{\varphi}_1^2 ; \quad \mathbf{E}_{p,1} = m_1 g h_{C1} = m_1 g (R - r_1)(1 - \cos\varphi_1) \qquad (21)$$

where the translatory velocity of the ball mass center $\vec{v}_{C1}$ and the angular velocity around the central axis $\vec{\omega}_{C1}$ and the angular velocity of the body rolling around the momentary axis $\vec{\omega}_{P1}$ are in the following relations:

$$v_{C1} = (R - r_1)\dot{\varphi}_1 = r_1 \omega_{P1} = r_1 \omega_{C1} \qquad (22)$$

$$\omega_{C1} = \omega_{P1} = \left(\frac{R}{r_1} - 1\right)\dot{\varphi}_1 = \frac{(R - r_1)}{r_1}\dot{\varphi}_1 \qquad (23)$$

Additionally, the first ball axial mass inertia moment $\mathbf{J}_{P1}$ for the instantaneous axis of rolling and the coefficient of rolling κ_1 of the first rolling ball along a circular line in a vertical plane are:

$$\mathbf{J}_{P1} = m_1\left(\frac{\mathbf{J}_{C1}}{m_1} + r_1^2\right) \quad \text{and} \quad \kappa_1 = \frac{\mathbf{J}_{C1}}{m_1 r_1^2} + 1 = \frac{i_{C1}^2}{r_1^2} + 1 \qquad (24)$$

For that reason, it is necessary to obtain the corresponding ordinary nonlinear differential equations of rolling each of balls along a curvilinear line in a vertical plane in the gravitational field.

The ordinary nonlinear differential equation of the first ball rolling along a curvilinear circular line in a vertical plane in the gravitational field is:

$$\ddot{\varphi}_1 + \frac{g}{\kappa_1 (R - r_1)} \sin\varphi_1 = 0 \qquad (25)$$

The integral of the energy of the first ball in the rolling dynamics along a circular trace in the gravitational field is:

$$\mathbf{E}_1 = \mathbf{E}_{\mathbf{k},1} + \mathbf{E}_{p,1} = \frac{1}{2}\kappa_1 m_1 (R - r_1)^2 \dot{\varphi}_1^2 + m_1 g (R - r_1)(1 - \cos\varphi_1) = C_1 = const \qquad (26)$$

and it presents an expression of total mechanical energy of the rolling ball at arbitrary moments and arbitrary positions on the circle trace. The total mechanical energy of the first rolling ball along a circular trace at the initial moment is:

$$\mathbf{E}_{1,0} = \mathbf{E}_{\mathbf{k},1,0} + \mathbf{E}_{p,1,0} = \frac{1}{2}\kappa_1 m_1 (R - r_1)^2 \dot{\varphi}_{1,0}^2 + m_1 g (R - r_1)(1 - \cos\varphi_{1,0}) = C_1 = const \quad (27)$$

where $\dot{\varphi}_{1,0} = \dot{\varphi}_1(0)$ and $\dot{\varphi}_{1,0} = \dot{\varphi}_1(0)$ are the initial values of the generalized angular coordinate and generalized angular velocity.

The first integral of the ordinary nonlinear differential equation (25) of the rolling dynamics of the first ball along a curvilinear circle line is possible to obtain from the integral of energy (26)-(27) in the following form:

$$\dot{\varphi}_1^2 + \frac{2g}{\kappa_1(R - r_1)}(1 - \cos\varphi_1) = \dot{\varphi}_{1,0}^2 + \frac{2g}{\kappa_1(R - r_1)}(1 - \cos\varphi_{1,0}) \quad (28)$$

or in the form:

$$\dot{\varphi}_1^2 = \dot{\varphi}_{1,0}^2 + \frac{2g}{\kappa_1(R - r_1)}(\cos\varphi_1 - \cos\varphi_{1,0}) \quad (29)$$

The previous non-linear equation (29) presents the equation of the phase trajectory in the phase plane $(\varphi_1, \dot{\varphi}_1)$, the curves of the constant total mechanical energy of the rolling ball between two collisions are also visible, and the total mechanical energy in this interval is constant, but depends on initial conditions, $\varphi_{1,0} = \varphi_1(0)$ and $\dot{\varphi}_{1,0} = \dot{\varphi}_1(0)$, in each of the intervals between two successive collisions. After each collision of the balls, the set of angular velocities of rolling balls are outgoing angular velocities as post-impact or post-collision angular velocities of rolling balls as the initial velocities for dynamics in the next post-collision interval of the corresponding period of rolling.

For that reason, the expression of the momentary angular velocity of rolling balls is necessary to be expressed by means of the independent generalized coordinate φ_1 in the following form:

$$\omega_{P1} = \left(\frac{R}{r_1} - 1\right)\dot{\varphi}_1 = \left(\frac{R}{r_1} - 1\right)\sqrt{\dot{\varphi}_{1,0}^2 + \frac{2g}{\kappa_1(R - r_1)}(\cos\varphi_1 - \cos\varphi_{1,0})} \quad (30)$$

The angular velocity of the first rolling ball is the function of the initial central angular velocity $\dot{\varphi}_1(0) = \dot{\varphi}_{0,1}$ in relation to the circle center C_0 and generalized coordinate $\varphi_{1,impact,1}$ of position at circle where first collision appears:

$$\omega_{P1,impact,1} = \left(\frac{R}{r_1} - 1\right)\dot{\varphi}_1(\varphi_{1,impact}) = \left(\frac{R}{r_1} - 1\right)\sqrt{\dot{\varphi}_{1,0}^2 + \frac{2g}{\kappa_1(R - r_1)}(\cos\varphi_{1,impact} - \cos\varphi_{1,0})} \quad (31)$$

and for the next impact angular velocity of each of the rolling balls depends on the outgoing angular velocity in the previous collision of the balls and the coordinate $\varphi_{1impact,1}$ of position where the next collision appears:

$$\begin{aligned} \omega_{P1,impact.2} &= \omega_{P1,outgoing,1} = \left(\frac{R}{r_1} - 1\right)\dot{\varphi}_1(\varphi_{1,impact,2}) \\ \omega_{P1,impact.2} &= \omega_{P1,outgoing,1} = \left(\frac{R}{r_1} - 1\right)\sqrt{\dot{\varphi}_{1,outgoing,1}^2 + \frac{2g}{\kappa_1(R - r_1)}(\cos\varphi_{1,impact,2} - \cos\varphi_{1,outgoing,1})} \end{aligned} \quad (32)$$

Using previous expressions for the description of the dynamics of the first rolling ball along the same circular trace in a vertical plane, for the second rolling ball, kinetic and potential energies are expressed by the central angle φ_2, and are in the following forms:

$$\mathbf{E}_{\mathbf{k},2} = \frac{1}{2}\kappa_2 m_2 (R - r_2)^2 \dot{\varphi}_2^2 \text{ and } \mathbf{E}_{p,2} = m_2 g h_{C1} = m_2 g (R - r_2)(1 - \cos\varphi_2) \quad (33)$$

where the velocity of the second ball mass center $\vec{v}_{C2}$ and the second ball angular velocity around the central axis $\vec{\omega}_{C2}$ and angular velocity about the momentary axis of rolling $\vec{\omega}_{P2}$ are in the following relations:

$$v_{C2} = (R - r_2)\dot{\varphi}_2 = r_2 \omega_{P2} = r_2 \omega_{C2} \text{ and } \omega_{C2} = \omega_{P2} = \left(\frac{R}{r_2} - 1\right)\dot{\varphi}_2 = \frac{(R - r_2)}{r_2}\dot{\varphi}_2 \quad (34)$$

and the second ball axial mass inertia moment for the instantaneous axis of rolling and the coefficient of the rolling of the second rolling ball along a circular trace in a vertical plane are in the following forms:

$$\mathbf{J}_{P2} = m_2 \left(\frac{\mathbf{J}_{\mathbf{C}2}}{m_2} + r_2^2\right) \text{ and } \kappa_2 = \frac{\mathbf{J}_{\mathbf{C}2}}{m_2 r_2^2} + 1 = \frac{i_{C2}^2}{r_2^2} + 1 \quad (35)$$

The ordinary non-linear differential equation describing the dynamics of the second ball rolling is:

$$\ddot{\varphi}_2 + \frac{g}{\kappa_2 (R - r_2)} \sin\varphi_2 = 0 \quad (36)$$

The integral of energy of the second ball rolling along the circle in the gravitational field is:

$$\mathbf{E}_2 = \mathbf{E}_{\mathbf{k},2} + \mathbf{E}_{p,2} = \frac{1}{2}\kappa_2 m_2 (R - r_2)^2 \dot{\varphi}_2^2 + m_2 g (R - r_2)(1 - \cos\varphi_2) = C_2 = const \quad (37)$$

and presents the expression of total mechanical energy of the second rolling ball at an arbitrary moment and arbitrary position on the circle trace. The total mechanical energy of the second rolling ball along the same circle trace at the initial moment is:

$$\mathbf{E}_{2,0} = \mathbf{E}_{\mathbf{k},2,0} + \mathbf{E}_{p,2,0} = \frac{1}{2}\kappa_2 m_2 (R - r_2)^2 \dot{\varphi}_{2,0}^2 + m_2 g (R - r_2)(1 - \cos\varphi_{2,0}) = C_2 = const \quad (38)$$

where $\varphi_{2,0} = \varphi_2(0)$ and $\dot{\varphi}_{2,0} = \dot{\varphi}_2(0)$ are the initial values of the generalized angular coordinate and generalized angular velocity.

The first integral of the ordinary nonlinear differential equation (36) of the rolling dynamics of the second ball along the curvilinear circle line is possible to obtain from the integral of energy (37)-(38) in the following form:

$$\dot{\varphi}_2^2 + \frac{2g}{\kappa_2 (R - r_2)}(1 - \cos\varphi_2) = \dot{\varphi}_{2,0}^2 + \frac{2g}{\kappa_2 (R - r_2)}(1 - \cos\varphi_{2,0}) \quad (39)$$

or in the form:

$$\dot{\varphi}_2^2 = \dot{\varphi}_{2,0}^2 + \frac{2g}{\kappa_2 (R - r_2)}(\cos\varphi_2 - \cos\varphi_{2,0}) \quad (40)$$

This previous nonlinear equation (40) presents the equation of the phase trajectory in a phase plane $(\varphi_2, \dot{\varphi}_2)$ and it is also visible that there are curves of constant energy of the second rolling ball between two collisions and that total mechanical energy in this interval is constant, but depends on the initial conditions $\varphi_{2,0} = \varphi_2(0)$ and $\dot{\varphi}_{2,0} = \dot{\varphi}_2(0)$ in each of the interval between two successive collisions. After each collision of the balls, angular velocities of the rolling balls are the outgoing angular velocities as post-impact angular velocities of the rolling balls as a set of initial velocities for dynamics in the next post-collision interval of the corresponding period of rolling of the balls.

For that reason, the expression of the momentary angular velocity of the rolling ball is necessary to express by means of an independent generalized angle coordinate φ_2 in the following form:

$$\omega_{P2} = \left(\frac{R}{r_2} - 1\right)\dot{\varphi}_2 = \left(\frac{R}{r_2} - 1\right)\sqrt{\dot{\varphi}_{2,0}^2 + \frac{2g}{\kappa_2(R - r_2)}\left(\cos\varphi_2 - \cos\varphi_{2,0}\right)} \qquad (41)$$

The angular velocity of the second ball rolling along an instantaneous axis is the function of the initial central angular velocity $\dot{\varphi}_2(0) = \dot{\varphi}_{0,2}$ with respect to the circle center C_0 and the angle coordinate $\varphi_{2,impact,1}$ of the position where the second collision appears:

$$\omega_{P2,impact,1} = \left(\frac{R}{r_2} - 1\right)\dot{\varphi}_2\left(\varphi_{2,impact}\right) = \left(\frac{R}{r_2} - 1\right)\sqrt{\dot{\varphi}_{2,0}^2 + \frac{2g}{\kappa_2(R - r_2)}\left(\cos\varphi_{2,impact} - \cos\varphi_{2,0}\right)} \qquad (42)$$

and for the next impact-collision angular velocity of the second ball rolling depends on the outgoing angular velocity in the previous collision of the second balls and coordinate $\varphi_{2,impact,1}$ of position where the next (second) collision appears:

$$\omega_{P2,impact.2} = \omega_{P2,outgoing,1} = \left(\frac{R}{r_2} - 1\right)\dot{\varphi}_2\left(\varphi_{2,impact,2}\right) \qquad (43)$$

$$\omega_{P2,impact.2} = \omega_{P2,outgoing,1} = \left(\frac{R}{r_2} - 1\right)\sqrt{\dot{\varphi}_{2,outgoing,1}^2 + \frac{2g}{\kappa_2(R - r_2)}\left(\cos\varphi_{2,impact,2} - \cos\varphi_{2,outgoing,1}\right)}$$

Central angle coordinates of the positions of the balls in the state of collisions are in the following relation: $\varphi_{2,impact,k} = \varphi_{1,impact,k} + \beta$, where angle depends on the geometrical parameters of the circle line radius R, and of the radiuses of both balls r_1 and r_2, and it is defined by the expression in the form (see Figure 10):

$$\beta = \arccos\frac{(R - r_1)^2 + (R - r_2)^2 - (r_1 + r_2)^2}{2(R - r_1)(R - r_2)} = \arccos\frac{\lambda_1(\lambda_1 - \lambda_2 - 1) - \lambda_2}{(\lambda_1 - \lambda_2)(\lambda_1 - 1)} \qquad (44)$$

where $\lambda_2 = \dfrac{r_2}{r_1}$ and $\lambda_2 = \dfrac{r_2}{r_1}$.

4.2. Non-linear vibro-impact dynamics and phase trajectories with successive central collisions of two heavy smooth balls rolling along a circular trace in a vertical plane. Let us consider the vibro-impact dynamics of two heavy balls rolling along a circle in a vertical plane. Using the ordinary nonlinear differential equations (25) and (36), and the equations (29) and (40) of the phase trajectory of two separate rolling balls along circle line (obtained in 4.1), we can consider the dynamics of two rolling balls in vibro-impact dynamics along a circular trace in a vertical plane, taking into account that these equations are valid for the nonlinear dynamics of the both balls between two successive collisions of these balls.

4.2.1 Solution for governing nonlinear differential equations with respect to time duration of the rolling balls at a circular line. For each interval of the non-linear dynamics of balls, between two collisions, the initial conditions must take into account the position of the corresponding impact and post-collision outgoing angular velocity of the corresponding ball. We take the measure of time from zero at each next interval between two successive collisions. Also, it is necessary to obtain the time for each next collision in relation to the initial moment of motion, or from the starting interval of motion post-previous-collision.

For that reason, let us introduce the following denotations:

$$\omega_1^2 = \frac{2g}{\kappa_1 (R - r_1)}, \quad \omega_2^2 = \frac{2g}{\kappa_2 (R - r_2)}, \tag{45}$$

$$k_2^2 = \frac{2\omega_2^2}{\dot{\varphi}_{2,0}^2 + 2\omega_2^2 \sin^2 \frac{\varphi_{2,0}}{2}} = \frac{\frac{4g}{\kappa_2 (R - r_2)}}{\dot{\varphi}_{2,0}^2 + \frac{4g}{\kappa_2 (R - r_2)} \sin^2 \frac{\varphi_{2,0}}{2}}$$

$$k_2^2 = \frac{2\omega_2^2}{\dot{\varphi}_{2,0}^2 + 2\omega_2^2 \sin^2 \frac{\varphi_{2,0}}{2}} = \frac{\frac{4g}{\kappa_2 (R - r_2)}}{\dot{\varphi}_{2,0}^2 + \frac{4g}{\kappa_2 (R - r_2)} \sin^2 \frac{\varphi_{2,0}}{2}} \tag{46}$$

then equations (29) and (40) of the phase trajectory of each of the rolling heavy balls along a circular line in a vertical plane are expressed in the following forms:

$$\dot{\varphi}_1 = \sqrt{\dot{\varphi}_{1,0}^2 + \omega_1^2 (\cos \varphi_1 - \cos \varphi_{1,0})} \quad \text{and} \quad \dot{\varphi}_2 = \sqrt{\dot{\varphi}_{2,0}^2 + \omega_2^2 (\cos \varphi_2 - \cos \varphi_{2,0})} \tag{47}$$

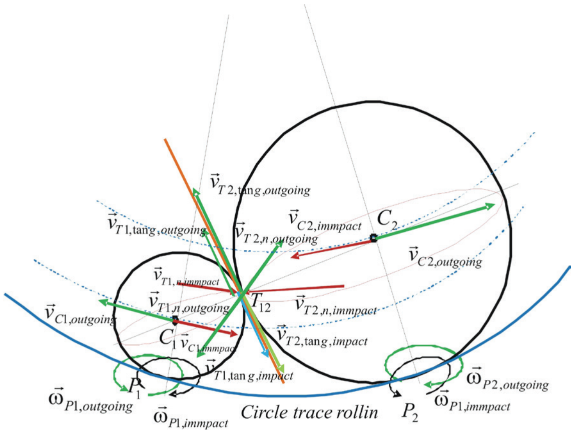


FIGURE 10. Plan of angular velocities and component velocities in pre- and post-collision of two rolling heavy balls along a circular trace in a vertical plane

The first ordinary nonlinear differential equation from (25) is possible to solve with respect to time t, and for that reason we must introduce in the first integral (29) the following trigonometric relation: $\cos\varphi_1 = 1 - 2\sin^2\frac{\varphi_1}{2}$ and after transformation, the time t of duration of rolling a ball along a circular trace between two ball positions $\varphi_{1,0} = \varphi_1(0)$ and $\varphi_1 = \varphi_1(t)$, on the circle line , is expressed by an integral in the form:

$$t = \int_{\varphi_{1,0}}^{\varphi_1} \frac{d\varphi_1}{\sqrt{\dot{\varphi}_{1,0}^2 + \omega_1^2(\cos\varphi_1 - \cos\varphi_{1,0})}} = \int_{\varphi_{1,0}}^{\varphi_1} \frac{d\varphi_1}{\sqrt{\dot{\varphi}_{1,0}^2 + 2\omega_1^2\sin^2\frac{\varphi_{1,0}}{2} - 2\omega_1^2\sin^2\frac{\varphi_1}{2}}} \tag{48}$$

or in the form:

$$t = \frac{2}{\omega_{0,1}} \int_{\varphi_{1,0}}^{\varphi_1} \frac{d\frac{\varphi_1}{2}}{\sqrt{1 - k_1^2\sin^2\frac{\varphi_1}{2}}} \tag{49}$$

The next transformation of the previous expression of the integral is by introducing relations: $u = \sin\theta = \sin\frac{\varphi_1}{2}$ and $u_{1,0} = \sin\left(\frac{1}{2}\varphi_{1,0}\right)$, that previous integral for obtaining the time t of duration of rolling a ball along a circle between two ball positions $\varphi_{1,0} = \varphi_1(0)$ and $\varphi_1 = \varphi_1(t)$, on the circle line , turns the following form:

$$t = \frac{2}{\omega_{0,1}} \int_{\sin\frac{\varphi_{1,0}}{2}}^{\sin\frac{\varphi_1}{2}} \frac{du}{\sqrt{(1-u^2)(1-k_1^2u^2)}} \tag{50}$$

The obtained integral in the expression (50) for the time t of duration of rolling a ball along a circle between two ball positions $\varphi_{1,0} = \varphi_1(0)$ and $\varphi_1 = \varphi_1(t)$ on the circular line, is a *normal elliptic integral*, known as *Legandre's elliptic integral of the first kind* (see Reference by Rašković [67] and Mitrinović, Djoković [57]).

Using the development of terms of functions in the previous integral (50) in series:

$$\left(1 - k_1^2u^2\right)^{-\frac{1}{2}} = 1 + \frac{1}{2}k_1^2u^2 + \frac{1\cdot 3}{2\cdot 4}k_1^4u^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}k_1^6u^6 + ... = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n k_1^{2n}u^{2n} \tag{51}$$

$$\left(1 - u^2\right)^{-\frac{1}{2}} = 1 + \frac{1}{2}u^2 + \frac{1\cdot 3}{2\cdot 4}u^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}u^6 + ... = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n u^{2n} \tag{52}$$

where:

$$(-1)\binom{-\frac{1}{2}}{n} = \frac{(2n-1)!!}{(2n)!!} \qquad (-1)\binom{\frac{1}{2}}{n} = -\frac{(2n-3)!!}{(2n)!!} \tag{53}$$

then it is not difficult to obtain the approximate values of the integral (50) in the following form:

$$t \approx \frac{2}{\omega_{0,1}}\left(u+\frac{1}{2\cdot 3}k_1^2u^3+\frac{1\cdot 3}{2\cdot 4\cdot 6}k_1^4u^5\right)_{\sin\frac{\varphi_{1,0}}{2}}^{\sin\frac{\varphi_1}{2}}+$$
$$+\frac{2}{\omega_{0,1}}\left(\frac{1}{2\cdot 3}u^3+\frac{1}{4\cdot 5}k_1^2u^5+\frac{1\cdot 3}{2^2\cdot 4\cdot 7}k_1^4u^7\right)_{\sin\frac{\varphi_{1,0}}{2}}^{\sin\frac{\varphi_1}{2}}+ \tag{54}$$
$$+\frac{2}{\omega_{0,1}}\left(\frac{1\cdot 3}{2\cdot 4\cdot 5}u^5+\frac{1\cdot 3}{2^2\cdot 4\cdot 7}k_1^2u^7+\frac{1}{9}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2k_1^4u^9\right)_{\sin\frac{\varphi_{1,0}}{2}}^{\sin\frac{\varphi_1}{2}}$$

The previously obtained expression (54) is an approximate value and presents the time t duration of the first ball rolling along a circular line between two ball positions $\varphi_1=\varphi_1(t)$ and $\varphi_1=\varphi_1(t)$ on the circular line, from the initial position φ_1 of the ball to the arbitrary position φ_1 on the curvilinear circle trace, where $\varphi_{0,1}$ is a coordinate angle at the initial position of the first ball initial at moment.

In the analogy with the previously obtained approximate value (54) of the time φ_1 duration of the first ball rolling from the initial position to the arbitrary position φ_1 on the curvilinear circle line, for the expression of an approximate value of the time t duration of the second ball rolling from the initial position $\varphi_{0,2}=\varphi_2(0)$ to the arbitrary position $\varphi_2(t)$ on the curvilinear circular line, we obtain:

$$t \approx \frac{2}{\omega_{0,1}}\left(u+\frac{1}{2\cdot 3}k_2^2u^3+\frac{1\cdot 3}{2\cdot 4\cdot 6}k_2^4u^5\right)_{\sin\frac{\varphi_{2,0}}{2}}^{\sin\frac{\varphi_2}{2}}+$$
$$+\frac{2}{\omega_{0,1}}\left(\frac{1}{2\cdot 3}u^3+\frac{1}{4\cdot 5}k_2^2u^5+\frac{1\cdot 3}{2^2\cdot 4\cdot 7}k_2^4u^7\right)_{\sin\frac{\varphi_{2,0}}{2}}^{\sin\frac{\varphi_2}{2}}+ \tag{55}$$
$$+\frac{2}{\omega_{0,1}}\left(\frac{1\cdot 3}{2\cdot 4\cdot 5}u^5+\frac{1\cdot 3}{2^2\cdot 4\cdot 7}k_2^2u^7+\frac{1}{9}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2k_2^4u^9\right)_{\sin\frac{\varphi_{2,0}}{2}}^{\sin\frac{\varphi_2}{2}}$$

4.2.2. Non-linear system dynamics in the interval from the initial position to the first collision of balls. For obtaining the coordinates of balls' positions in the configuration of the first collision between rolling heavy balls at a circular line in a vertical plane, it is necessary to obtain time $t_{impact,1}$ of the first collision at which both balls are in the configuration of the first collision. We propose that the mass center $C_{1,impact,1}$ of the first ball is in the position defined by the angle coordinate $\varphi_1(t_{impact,1})=\varphi_{1,impact,1}$, then the coordinate of the mass center $C_{2,impact,1}$ of the second ball is defined by the angle coordinate: $\varphi_2(t_{impacr,1})=\varphi_1(t_{ompact,1})+\beta$, where the angle β is defined by the expression (44). Using the approximate expressions (54) and (55) for time $t_{impact,1}$ duration of balls' motion from the corresponding initial positions, $\varphi_{1,0}$ and $\varphi_{2,0}$, to the positions $\varphi_{1,impact,1}$ and $\varphi_{2,impact,1}$, of the first collision between rolling balls we can write the following:

$$t_{impact,1} = t_{1,impact,1} \approx \frac{2}{\omega_{0,1}} \left(u + \frac{1}{2\cdot 3} k_1^2 u^3 + \frac{1\cdot 3}{2\cdot 4\cdot 6} k_1^4 u^5 \right)_{\sin\frac{\varphi_{1,0}}{2}}^{\sin\frac{\varphi_{1,impact,1}}{2}} +$$
$$+\frac{2}{\omega_{0,1}} \left(\frac{1}{2\cdot 3} u^3 + \frac{1}{4\cdot 5} k_1^2 u^5 + \frac{1\cdot 3}{2^2\cdot 4\cdot 7} k_1^4 u^7 \right)_{\sin\frac{\varphi_{1,0}}{2}}^{\sin\frac{\varphi_{1,impact,1}}{2}} + \tag{56}$$
$$+\frac{2}{\omega_{0,1}} \left(\frac{1\cdot 3}{2\cdot 4\cdot 5} u^5 + \frac{1\cdot 3}{2^2\cdot 4\cdot 7} k_1^2 u^7 + \frac{1}{9}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k_1^4 u^9 \right)_{\sin\frac{\varphi_{1,0}}{2}}^{\sin\frac{\varphi_{1,impact,1}}{2}}$$

$$t_{impact,1} = t_{2,impact,1} \approx \frac{2}{\omega_{0,2}} \left(u + \frac{1}{2\cdot 3} k_2^2 u^3 + \frac{1\cdot 3}{2\cdot 4\cdot 6} k_2^4 u^5 \right)_{\sin\frac{\varphi_{2,0}}{2}}^{\sin\frac{\varphi_{1,impact,1}+\beta}{2}} +$$
$$+\frac{2}{\omega_{0,2}} \left(\frac{1}{2\cdot 3} u^3 + \frac{1}{4\cdot 5} k_2^2 u^5 + \frac{1\cdot 3}{2^2\cdot 4\cdot 7} k_2^4 u^7 \right)_{\sin\frac{\varphi_{2,0}}{2}}^{\sin\frac{\varphi_{1,impact,1}+\beta}{2}} + \tag{57}$$
$$+\frac{2}{\omega_{0,1}} \left(\frac{1\cdot 3}{2\cdot 4\cdot 5} u^5 + \frac{1\cdot 3}{2^2\cdot 4\cdot 7} k_1^2 u^7 + \frac{1}{9}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k_1^4 u^9 \right)_{\sin\frac{\varphi_{2,0}}{2}}^{\sin\frac{\varphi_{1,impact,1}+\beta}{2}}$$

Taking into account that both balls start from the initial positions, $\varphi_{1,0}$ and $\varphi_{2,0}$, we must arrive at the configuration of the first position of the first collision, defined by the coordinates $\varphi_{1,impact,1}$ and $\varphi_{2,impact,1}$, show that the expressions (56) and (57) are equal, first to one another, and then as *result is a nonlinear transcendent equation with respect to the unknown angle coordinate* $\varphi_1(t_{impact,1}) = \varphi_{1,impact,1}$ of the mass center position of the first ball at the position of the first collision between balls. This task of finding the first real root of this transcendental equation is not possible to solve analytically and it is necessary to use certain numerical methods and commercial software tools. In this paper, we deal with ideas and analytical approaches to the defined task of vibro-impact dynamics. We propose that we have the first real root of this transcendental equation obtained numerically.

Furthermore, we suppose that we have the angle coordinate $\varphi_1(t_{impact,1}) = \varphi_{1,impact,1}$ of the mass center $C_{1,impact,1}$ of the first ball at the position of the first collision between balls, and also the angle coordinate $\varphi_2(t_{impact,1}) = \varphi_1(t_{ompact,1}) + \beta$ of the mass center $C_{2,impact,1}$ of the second ball at the position of the first collision between balls, then it is possible to compose the pre-first collision impact angular velocities $\omega_{P1,impact,1}$ and $\omega_{P2,impact,1}$ of the both heavy rolling balls using the expressions (32) and (43), in the following forms:

$$\omega_{P1,impact,1} = (\lambda_1 - 1)\dot{\varphi}_{1.impact,1} = (\lambda_1 - 1)\sqrt{\dot{\varphi}_{1,0}^2 + \omega_1^2\left(\cos\varphi_{1,impact,1} - \cos\varphi_{1,0}\right)} \tag{58}$$

$$\omega_{P2,impact,1} = (\lambda_2 - 1)\dot{\varphi}_{2,impact,1} = (\lambda_2 - 1)\sqrt{\dot{\varphi}_{2,0}^2 + \omega_2^2\left[\cos(\varphi_{1,impact,1} + \beta) - \cos\varphi_{2,0}\right]} \qquad (59)$$

For obtaining the post-first-collision outgoing angular velocities $\omega_{P1}(t_{u,1} + \tau) = \omega_{P1,outgoing,1}$ and $\omega_{P2}(t_0 + \tau) = \omega_{P2,outgoing,1}$ of the rolling balls along a circular line, at the same position of the first collision between balls, determined by the generalized coordinates $\varphi_{1,impact,1}$ and $\varphi_{2,impact,1}$, we use the expressions (10) and (11) and we obtain the following expressions:

$$\omega_{P1}(t_{u,1} + \tau) = \omega_{P1,outgoing,1} = \omega_{P1,impact,1}(t_{u,1}) - \frac{1+k}{1 + \frac{\mathbf{J}_{P1}}{\mathbf{J}_{P2}}}\left(\omega_{P1,impact,1}(t_{u,1}) - \omega_{P2,impact,1}(t_{u,1})\right) \qquad (60)$$

$$\omega_{P2}(t_0 + \tau) = \omega_{P2,outgoing,1} = \omega_{P2,impact,1}(t_{u,1}) + \frac{1+k}{1 + \frac{\mathbf{J}_{P2}}{\mathbf{J}_{P1}}}\left(\omega_{P1,impact,1}(t_{u,1}) - \omega_{P2,impact,1}(t_{u,1})\right) \qquad (61)$$

or in developed forms:

$$\begin{aligned}\omega_{P1}(t_0 + \tau) = \omega_{P1,outgoing,1} &= (\lambda_1 - 1)\sqrt{\dot{\varphi}_{1,0}^2 + \omega_1^2(\cos\varphi_{1,impact,1} - \cos\varphi_{1,0})} - \\ &- \frac{1+k}{1 + \frac{\mathbf{J}_{P1}}{\mathbf{J}_{P2}}}\left\{(\lambda_1 - 1)\sqrt{\dot{\varphi}_{1,0}^2 + \omega_1^2(\cos\varphi_{1,impact,1} - \cos\varphi_{1,0})} - \right. \\ &\left. -(\lambda_2 - 1)\sqrt{\dot{\varphi}_{2,0}^2 + \omega_2^2\left[\cos(\varphi_{1,impact,1} + \beta) - \cos\varphi_{2,0}\right]}\right\} \end{aligned} \qquad (62)$$

$$\begin{aligned}\omega_{P2}(t_0 + \tau) = \omega_{P2,outgoing,1} &= (\lambda_2 - 1)\sqrt{\dot{\varphi}_{2,0}^2 + \omega_2^2\left[\cos(\varphi_{1,impact,1} + \beta) - \cos\varphi_{2,0}\right]} + \\ &+ \frac{1+k}{1 + \frac{\mathbf{J}_{P2}}{\mathbf{J}_{P1}}}\left\{(\lambda_1 - 1)\sqrt{\dot{\varphi}_{1,0}^2 + \omega_1^2(\cos\varphi_{1,impact,1} - \cos\varphi_{1,0})} - \right. \\ &\left. -(\lambda_2 - 1)\sqrt{\dot{\varphi}_{2,0}^2 + \omega_2^2\left[\cos(\varphi_{1,impact,1} + \beta) - \cos\varphi_{2,0}\right]}\right\} \end{aligned} \qquad (63)$$

For obtaining the kinetic parameters of the rolling balls in the form of pre-collision and post-collision angular velocities and the angle coordinate of the ball position at a series of successive collisions between balls we must use the approach similar to the one presented in this part.

In case of dealing with numerical data, a discussion is possible about the directions of outgoing angular velocities of corresponding rolling balls, depending on the relation between the intensities and directions of the arrival at the pre-collision angular velocities and the position of the collision between balls. It is possible to have various cases, so that after the considered collision, balls departures are in opposite directions or in the same direction depending on the listed kinetic parameters. But this is a task with numerical analysis.

4.2.3 Non-linear system dynamics in the interval from the position of the first collision to the second collision of the balls. The next period of the motion of the rolling balls, between the first and the second collisions of balls, is starting with the measures of time interval with zero, and the initial conditions are equal to the outgoing kinetic parameters at the post-first-collision state of rolling balls:

* for the first rolling ball the initial coordinates are $\varphi_{1,impact,1} = \varphi_{1,outgoing,1}$ and the initial angular velocity of the first ball mass center with respect to the circular trace center is

$$\dot{\varphi}_1\left(t_{impact,1}\right) = \dot{\varphi}_{1,outgoing,1} = \frac{\omega_{P1,outgoing,1}}{\left(\lambda_1 - 1\right)}$$

and the equation of the phase trajectory branch of the first ball dynamics in the interval between the first and the second collision is:

$$\varphi_2\left(t_{impact,1}\right) = \varphi_{2,impact,1} = \varphi_1\left(t_{impact,1}\right) + \beta = \varphi_{1,impact,1} + \beta \tag{64}$$

and

* for the second rolling ball the initial coordinate is

$\varphi_2\left(t_{impact,1}\right) = \varphi_{2,impact,1} = \varphi_1\left(t_{impact,1}\right) + \beta = \varphi_{1,impact,1} + \beta$ and the initial angular velocity is $\dot{\varphi}_2\left(t_{impact,1}\right) = \dot{\varphi}_{2,outgoing,1} = \dfrac{\omega_{P2,outgoing,1}}{\left(\lambda_2 - 1\right)}$ of the second ball mass center with respect to the circular trace center, and the equation of the phase trajectory branch of the second ball dynamics in the interval between the first and the second collision is:

$$\dot{\varphi}_2 = \sqrt{\dot{\varphi}^2_{2,outgoing,1} + \omega^2_2\left[\cos\varphi_2 - \cos\left(\varphi_{1,u,1} + \beta\right)\right]} \tag{65}$$

For obtaining the time $t_{impact,2} = t_{1,impact,2} = t_{2,impact,2}$ of the second collision between rolling balls and the duration between the first and the second collision between balls, it is necessary to use the previous approach, and on the basis of this write the following integrals:

$$t_{1,impact,2} = \int\limits_{\varphi_{1,impact,1}}^{\varphi_{1,impact,2}} \frac{d\varphi_1}{\sqrt{\dot{\varphi}^2_{1,outgoing,1} + \omega^2_1\left(\cos\varphi_1 - \cos\varphi_{1,impact,1}\right)}} \tag{66}$$

$$t_{2,impact2} = \int\limits_{\varphi_{1,mpact,1}+\beta}^{\varphi_{1,impact,2}+\beta} \frac{d\varphi_2}{\sqrt{\dot{\varphi}^2_{2,outgoing,1} + \omega^2_2\left[\cos\varphi_2 - \cos\left(\varphi_{1,impact,1} + \beta\right)\right]}} \tag{67}$$

On the basis of the previous explanation in an analogy with the approximate expressions (56) and (57), for time $t_{impact,2} = t_{1,impact,2} = t_{2,impact,2}$ of duration of the intervals between the first and the second collisions of the rolling balls dynamics, it is possible to write:

* $t_{1,impact,2}$ for the first ball

$$t_{impact,2} = t_{1,impact,2} \approx \frac{2}{\omega_{0,1}}\left(u + \frac{1}{2\cdot 3}k_1^2 u^3 + \frac{1\cdot 3}{2\cdot 4\cdot 6}k_1^4 u^5 \right)_{\sin\frac{\varphi_{1,impact,1}}{2}}^{\sin\frac{\varphi_{1,impact,2}}{2}} +$$
$$+\frac{2}{\omega_{0,1}}\left(\frac{1}{2\cdot 3}u^3 + \frac{1}{4\cdot 5}k_1^2 u^5 + \frac{1\cdot 3}{2^2\cdot 4\cdot 7}k_1^4 u^7 \right)_{\sin\frac{\varphi_{1,impact,1}}{2}}^{\sin\frac{\varphi_{1,impact,2}}{2}} + \tag{68}$$
$$+\frac{2}{\omega_{0,1}}\left(\frac{1\cdot 3}{2\cdot 4\cdot 5}u^5 + \frac{1\cdot 3}{2^2\cdot 4\cdot 7}k_1^2 u^7 + \frac{1}{9}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k_1^4 u^9 \right)_{\sin\frac{\varphi_{1,impact,1}}{2}}^{\sin\frac{\varphi_{1,impact,2}}{2}}$$

* $t_{2,impact,2}$ for the second ball

$$t_{impact,2} = t_{2,impact,2} \approx \frac{2}{\omega_{0,2}}\left(u + \frac{1}{2\cdot 3}k_2^2 u^3 + \frac{1\cdot 3}{2\cdot 4\cdot 6}k_2^4 u^5 \right)_{\sin\frac{\varphi_{2,o,pact,1}}{2}}^{\sin\frac{\varphi_{1,impact,2}+\beta}{2}} +$$
$$+\frac{2}{\omega_{0,2}}\left(\frac{1}{2\cdot 3}u^3 + \frac{1}{4\cdot 5}k_2^2 u^5 + \frac{1\cdot 3}{2^2\cdot 4\cdot 7}k_2^4 u^7 \right)_{\sin\frac{\varphi_{2,o,pact,1}}{2}}^{\sin\frac{\varphi_{1,impact,2}+\beta}{2}} + \tag{69}$$
$$+\frac{2}{\omega_{0,1}}\left(\frac{1\cdot 3}{2\cdot 4\cdot 5}u^5 + \frac{1\cdot 3}{2^2\cdot 4\cdot 7}k_1^2 u^7 + \frac{1}{9}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k_1^4 u^9 \right)_{\sin\frac{\varphi_{2,o,pact,1}}{2}}^{\sin\frac{\varphi_{1,impact,2}+\beta}{2}}$$

Taking into account that both balls, starting from the position of the first collision, which is now the initial position of both rolling balls along a circular trace, in the interval between the first and the second collisions, we must arrive at the configuration of the position of the second collision for equal time $t_{impact,2} = t_{1,impact,2} = t_{2,impact,2}$.

We show that the expressions (68) and (69) are equal first to one another, and as a result there is a non-linear transcendental equation with respect to the unknown angle coordinate $\varphi_1\left(t_{impact,2}\right) = \varphi_{1,impact,2}$ of the mass center $C_{1,impact,2}$ of the first ball at the position of the second collision between balls. The task is to find the first real root of this transcendental equation, which is not solvable analytically and it is necessary to use some numerical methods as well as some commercial software tools. In this paper we deal with the ideas and the analytical approach to the defined task of vibro-impact dynamics. We propose that we have the first real root of this transcendental equation obtained numerically.

Suppose that we have the angle coordinate $\varphi_1\left(t_{impact,2}\right) = \varphi_{1,impact,2}$ of the mass center $C_{1,impact,2}$ of the first ball at the position of the second collision between balls, and also the angle coordinate $\varphi_2\left(t_{impact,1}\right) = \varphi_1\left(t_{impact,1}\right) + \beta$ of the mass center $C_{2,impact,2}$ of the second ball at the position of the second collision between balls, then it is possible to compose pre-second-collision impact angular velocities $\omega_{P1,impact,2}$ and $\omega_{P2,impact,2}$ of the heavy rolling of balls around the corresponding instantaneous axis, using the expressions (32) and (43), in the following forms:

$$\omega_{P1,u,2} = \omega_{P1,impact,2} = (\lambda_1 - 1)\dot{\varphi}_{1.impact,2}$$
$$\omega_{P1,u,2} = (\lambda_1 - 1)\sqrt{\dot{\varphi}^2_{1,impact,1} + \omega^2_1(\cos\varphi_{1,impact,2} - \cos\varphi_{1,immpact,1})} \tag{70}$$

$$\omega_{P2,u,2} = \omega_{P2,impact,2} = (\lambda_2 - 1)\dot{\varphi}_{2,impact,2}$$
$$\omega_{P2,u,2} = (\lambda_2 - 1)\sqrt{\dot{\varphi}^2_{2,impact,1} + \omega^2_2[\cos(\varphi_{1,impact,2} + \beta) - \cos(\varphi_{1,impact,1} + \beta)]} \tag{71}$$

For obtaining the post-second-collision outgoing angular velocities $\omega_{P1}(t_{impact,2} + \tau) = \omega_{P1,outgoing,2}$ and $\omega_{P2}(t_{impact,2} + \tau) = \omega_{P2,outgoing,2,1}$ of the rolling balls along a circular line, we use the expressions (10) and (11) and we obtain the following expressions:

$$\omega_{P1}(t_{impact,2} + \tau) = \omega_{P1,outgoing,2} = \omega_{P1,impact,2}(t_{u,2}) - \frac{1+k}{1 + \frac{\mathbf{J}_{P1}}{\mathbf{J}_{P2}}}(\omega_{P1,impact,2}(t_{u,2}) - \omega_{P2,impact,2}(t_{u,2})) \tag{72}$$

$$\omega_{P2}(t_{impact,2} + \tau) = \omega_{P2,outgoing,2} = \omega_{P2,impact,2}(t_{u,2}) + \frac{1+k}{1 + \frac{\mathbf{J}_{P2}}{\mathbf{J}_{P1}}}(\omega_{P1,impact,2}(t_{u,2}) - \omega_{P2,impact,2}(t_{u,2})) \tag{73}$$

4.2.4 Non-linear system dynamics in the interval from the position of *n*-th to *n+1*-th collision of balls. Based on the previous consideration of the present series of successive collisions between balls, it is possible to make a generalization of the expressions for kinetic parameters between two successful collisions of the balls.

The next period of the motion of two rolling balls, after n -th collision, $n \geq 2$, between balls, with the measures of time interval $t_{impact,(n+1)} = t_{1,impact,(n+1)} = t_{,2impact,(n+1)}$ starting with zero, and the initial conditions equal to outgoing kinetic parameters at the position of n -th -collision state of the rolling balls:

* for the first rolling ball the initial angle coordinate is $\varphi_{1,impact,n} = \varphi_{1,outgoing,n}$, $n \geq 2$ and the initial angular velocity is

$$\dot{\varphi}_1(t_{impact,n}) = \dot{\varphi}_{1,outgoing,n} = \frac{\omega_{P1,outgoing,m}}{(\lambda_1 - 1)}$$

and the equation of phase trajectory branch of the first ball dynamics between n -th and $n+1$ -th $n \geq 2$ collisions of the rolling balls is:

$$\dot{\varphi}_1 = \sqrt{\dot{\varphi}^2_{1,outgoing,n} + \omega^2_1(\cos\varphi_1 - \cos\varphi_{1,impact,n})},$$

between n -th and $n+1$ -th $n \geq 2$ (74)

and

* for the second rolling ball, the initial angle coordinate is $\varphi_{2,impact,n} = \varphi_{1,impact,n} + \beta$ and the initial angular velocity between n -th and $n \geq 2$ -th $n \geq 2$ is

$$\dot{\varphi}_2\left(t_{impact,n}\right)=\dot{\varphi}_{2,outgoing,n}=\frac{\omega_{P2,outgoing,n}}{\left(\lambda_2-1\right)}$$

and the equation of the phase trajectory branch of the second ball dynamics after n-th collision and in the interval between n-th and $n+1$-th $n\geq 2$ collisions of the second rolling balls is:

$$\dot{\varphi}_2=\sqrt{\dot{\varphi}_{2,outgoing,n}^2+\omega_2^2\left[\cos\varphi_2-\cos\left(\varphi_{1,impact,n}+\beta\right)\right]},$$

between n-th and $n+1$-th $n\geq 2$ (75)

For obtaining the time $t_{impact,(n+1)}=t_{1,impact,(n+1)}=t_{2,impact,(n+1)}$ of the ($n+1$)-th collisions between rolling balls, it is necessary to use the previous approach, and based on this, write the following integrals:

$$t_{impact,(n+1)}=t_{1,impact,(n+1)}=\int_{\varphi_{1,u,n}}^{\varphi_{1,u,(n+1)}}\frac{d\varphi_1}{\sqrt{\dot{\varphi}_{1,outgoing,n}^2+\omega_1^2\left(\cos\varphi_1-\cos\varphi_{1,impact,n}\right)}} \quad (76)$$

$$t_{impact,(n+1)}=t_{2,impact,(n+1)}=\int_{\varphi_{1,u,n}+\beta}^{\varphi_{1,u,(n+1)}}\frac{d\varphi_2}{\sqrt{\dot{\varphi}_{2,outgoing,n}^2+\omega_2^2\left[\cos\varphi_2-\cos\left(\varphi_{1,impact,n}+\beta\right)\right]}} \quad (77)$$

Based on the explanation in part 4.2.2, and an analogy with approximate expressions (56) and (57), for the interval $t_{impact,(n+1)}=t_{1,impact,(n+1)}=t_{,2impact,(n+1)}$ between n-th and $n+1$-th $n\geq 2$ collisions of the rolling balls it is possible to write:

$$\begin{aligned} t_{inpact,(n+1)}=t_{1,inpact,(n+1)}\approx&\frac{2}{\omega_{0,1}}\left(u+\frac{1}{2\cdot 3}k_1^2u^3+\frac{1\cdot 3}{2\cdot 4\cdot 6}k_1^4u^5\right)_{\sin\frac{\varphi_{1,impact,n}}{2}}^{\sin\frac{\varphi_{1,ompact,(n+1)}}{2}}+\\ &+\frac{2}{\omega_{0,1}}\left(\frac{1}{2\cdot 3}u^3+\frac{1}{4\cdot 5}k_1^2u^5+\frac{1\cdot 3}{2^2\cdot 4\cdot 7}k_1^4u^7\right)_{\sin\frac{\varphi_{1,impact,n}}{2}}^{\sin\frac{\varphi_{1,ompact,(n+1)}}{2}}+\\ &+\frac{2}{\omega_{0,1}}\left(\frac{1\cdot 3}{2\cdot 4\cdot 5}u^5+\frac{1\cdot 3}{2^2\cdot 4\cdot 7}k_1^2u^7+\frac{1}{9}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2k_1^4u^9\right)_{\sin\frac{\varphi_{1,impact,n}}{2}}^{\sin\frac{\varphi_{1,ompact,(n+1)}}{2}} \end{aligned} \quad (78)$$

$$\begin{aligned} t_{impact,(n+1)}=t_{2,inpact,(n+1)}\approx&\frac{2}{\omega_{0,2}}\left(u+\frac{1}{2\cdot 3}k_2^2u^3+\frac{1\cdot 3}{2\cdot 4\cdot 6}k_2^4u^5\right)_{\sin\frac{\varphi_{2,impact,n}}{2}}^{\sin\frac{\varphi_{1,ompact,(n+1)}+\beta}{2}}+\\ &+\frac{2}{\omega_{0,2}}\left(\frac{1}{2\cdot 3}u^3+\frac{1}{4\cdot 5}k_2^2u^5+\frac{1\cdot 3}{2^2\cdot 4\cdot 7}k_2^4u^7\right)_{\sin\frac{\varphi_{2,impact,n}}{2}}^{\sin\frac{\varphi_{1,ompact,(n+1)}+\beta}{2}}+\\ &+\frac{2}{\omega_{0,1}}\left(\frac{1\cdot 3}{2\cdot 4\cdot 5}u^5+\frac{1\cdot 3}{2^2\cdot 4\cdot 7}k_1^2u^7+\frac{1}{9}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2k_1^4u^9\right)_{\sin\frac{\varphi_{2,impact,n}}{2}}^{\sin\frac{\varphi_{1,ompact,(n+1)}+\beta}{2}} \end{aligned} \quad (79)$$

Taking into account that both balls, starting from the position of n-th, $n \geq 2$ collision, now the initial position of the rolling balls along a circle, in the interval between n-th and $n+1$-th, $n \geq 2$ collisions, we must arrive at the configuration of the position of $n+1$-th, $n \geq 2$ collision, and show the expressions (78) and (79) are equal, first to one another. As a result, there is a non-linear transcendental equation with respect to the unknown angle coordinate $\varphi_1\left(t_{impact,(n+1)}\right) = \varphi_{1,impact,(n+1)}$ of the mass center $C_{1.impact,(n+1)}$ of the first ball at the position of $n+1$-th, $n \geq 2$ collision between balls. The task is to find the first real root of this transcendental equation and it is not analytically solvable. It is necessary to use some numerical methods as well as some commercial software tools, as we explained in the previous parts.

Suppose that we have the necessary angle coordinate $\varphi_1\left(t_{impact,(n+1)}\right) = \varphi_{1,impact,(n+1)}$ of the mass center $C_{1.impact,(n+1)}$ of the first ball at the position of $n+1$-th, $n \geq 2$ collision between balls, and also the angle coordinate $\varphi_2\left(t_{impact,(n+1)}\right) = \varphi_1\left(t_{impact,(n+1)}\right) + \beta$ of the mass center $C_{2,impact,(n+1)}$ of the second ball at the position of $n+1$ collision between balls, then it is possible to compose pre-($n+1$)-th-collision impact angular velocities $\omega_{P1,impact,(n+1)}$ and $\omega_{P2,impact,(n+1)}$ of the rolling of heavy balls, using the expressions (32) and (43), in the following forms:

$$\begin{aligned} \omega_{P1,impact,(n+1)} &= (\lambda_1 - 1)\dot{\varphi}_{1.impact,(n+1)} \\ \omega_{P1,impact,(n+1)} &= (\lambda_1 - 1)\sqrt{\dot{\varphi}^2_{1,impact,n} + \omega^2_1\left(\cos\varphi_{1,impact,(n+1)} - \cos\varphi_{1,impact,n}\right)} \end{aligned} \tag{80}$$

$$\begin{aligned} \omega_{P2,impact,(n+1)} &= (\lambda_2 - 1)\dot{\varphi}_{2,u,(n+1)} \\ \omega_{P2,impact,(n+1)} &= (\lambda_2 - 1)\sqrt{\dot{\varphi}^2_{2,impact,n} + \omega^2_2\left[\cos\left(\varphi_{1,impact,(n+1)} + \beta\right) - \cos\left(\varphi_{1,impact,n} + \beta\right)\right]} \end{aligned} \tag{81}$$

For obtaining post-($n+1$)-th, $n \geq 2$-central collision outgoing angular velocities $\omega_{P1}\left(t_{impact,(n+1)} + \tau\right) = \omega_{P1,outgoing,(n+1)}$ and $\omega_{P2}\left(t_{impact,(n+1)} + \tau\right) = \omega_{P2,outgoing,(n+1)1}$ around the corresponding instantaneous axis of each of the rolling balls along a circular line, we use the expressions (10)-(11) and we obtain the following expressions:

$$\begin{aligned} \omega_{P1}\left(t_{u,(n+1)} + \tau\right) &= \omega_{P1,outgoing,(n+1)} \\ \omega_{P1}\left(t_{u,(n+1)} + \tau\right) &= \omega_{P1,impact,(n+1)}\left(t_{u,(n+1)}\right) - \frac{1+k}{1+\frac{\mathbf{J}_{P1}}{\mathbf{J}_{P2}}}\left[\omega_{P1,impact,(n+1)}\left(t_{u,(n+1)}\right) - \omega_{P2,impact,(n+1)}\left(t_{u,(n+1)}\right)\right] \end{aligned} \tag{82}$$

$$\begin{aligned} \omega_{P2}\left(t_{u,(n+1)} + \tau\right) &= \omega_{P2,outgoing,(n+1)} \\ \omega_{P2}\left(t_{u,(n+1)} + \tau\right) &= \omega_{P2,impact,2}\left(t_{u,(n+1)}\right) + \frac{1+k}{1+\frac{\mathbf{J}_{P2}}{\mathbf{J}_{P1}}}\left[\omega_{P1,impact,(n+1)}\left(t_{u,(n+1)}\right) - \omega_{P2,impact,(n+1)}\left(t_{u,(n+1)}\right)\right] \end{aligned} \tag{83}$$

The next period of the motion of the rolling balls, after ($n+1$)-th collision, $n \geq 2$ between balls, with the measures of time $t_{impact,(n+2)}$ interval starting with zero, and initial conditions equal to the outgoing kinetic parameters at post-($n+1$)-th -collision state of the rolling balls:

* for the first rolling ball, the initial coordinate is $\varphi_{1,impact,(n+1)}$, $n \geq 2$ and the initial angular velocity around the circular line center C_0 is $\dot{\varphi}_1\left(t_{impact,(n+1)}\right) = \dot{\varphi}_{1,outgoing,(n+1)} = \dfrac{\omega_{P1,outgoing,(n+1)}}{(\lambda_1 - 1)}$

and the equation of the phase trajectory branch of the first ball dynamics between $n+1$-th and $n+2$-th, $n \geq 2$ is:

$$\dot{\varphi}_1 = \sqrt{\dot{\varphi}^2_{1,outgoing,(n+1)} + \omega^2_1 \left(\cos\varphi_1 - \cos\varphi_{1,impact,(n+1)}\right)},$$

$$n+1\text{-th and } n+2\text{-th, } n \geq 2 \tag{84}$$

and

* for the second rolling ball, the initial coordinate is $\varphi_{2,impact,(n+1)} = \varphi_{1,impact,(n+1)} + \beta$ and the initial angular velocity between $n+1$ -th and $n+2$-th, $n \geq 2$ central collisions is $\dot{\varphi}_2\left(t_{impact,(n+1)}\right) = \dot{\varphi}_{2,outgoing,(n+1)} = \dfrac{\omega_{P2,outgoing,(n+1)}}{\left(\lambda_2 - 1\right)}$, and the equation of the phase trajectory branch of the second ball dynamics after ($n+1$)-th collision, $n \geq 2$, and between $n+1$-th and $n+2$-th, $n \geq 2$ collisions of the balls is:

$$\dot{\varphi}_2 = \sqrt{\dot{\varphi}^2_{2,outgoing,(n+1)} + \omega^2_2 \left[\cos\varphi_2 - \cos\left(\varphi_{1,impact,(n+1)} + \beta\right)\right]},$$

$$\text{between } n+1\text{-th and } n+2\text{-th, } n \geq 2. \tag{85}$$

4.2.5. Sketch of the phase trajectory branches of the rolling ball dynamics between successive central collisions of two rolling heavy balls along a circular trace in a vertical plane. In Figure 11, phase trajectory portraits of vibro-impact dynamics of two rolling balls along a curvilinear circular line in a vertical plane with successive two first collisions are presented: (upper) for the second rolling ball and (lower) for the first rolling ball non-linear dynamics. In Figure 12, plans of the configurations of the rolling balls in vibro-impact dynamics of rolling heavy balls along a curvilinear circular line in a vertical plane with successive first two collisions are presented.

Let us explain how to obtain phase trajectories of vibro-impact nonlinear dynamics of two rolling balls along a circle starting from the initial conditions defined by the corresponding initial position and the initial angular velocity of the rolling balls: $\left(\varphi_{1,0}, \vec{\omega}_{1,0}\right)$ and $\left(\varphi_{2,0}, \vec{\omega}_{2,0}\right)$. Balls in this configuration are presented in Figure 12 (upper and left). In Figure 11, phase portraits for different initial conditions are presented: for the first ball (lower) portrait in the phase plane $\left(\varphi_1, \dot{\varphi}_1\right)$ and for the second ball (upper) portrait in the phase plane $\left(\varphi_2, \dot{\varphi}_2\right)$, for the cases of both single balls rolling along a circular line. We can see that on the phase portraits there are three types of phase trajectories visible. The closed phase trajectory corresponds to oscillatory motions with the constant total mechanical energy of the nonlinear oscillation dynamics. Open trajectories correspond to progressive balls rolling along a circular line in one direction. Trajectories with cross sections passing through unstable saddle type singular points are separatrices and homoclinic trajectories. Saddle points at the phase portrait correspond to the upper ball position on the circle line and present no stable equilibrium position. A stable center type singular point corresponds to a lower position of the ball at the circle line, and presents a stable equilibrium position of the ball at circle line.

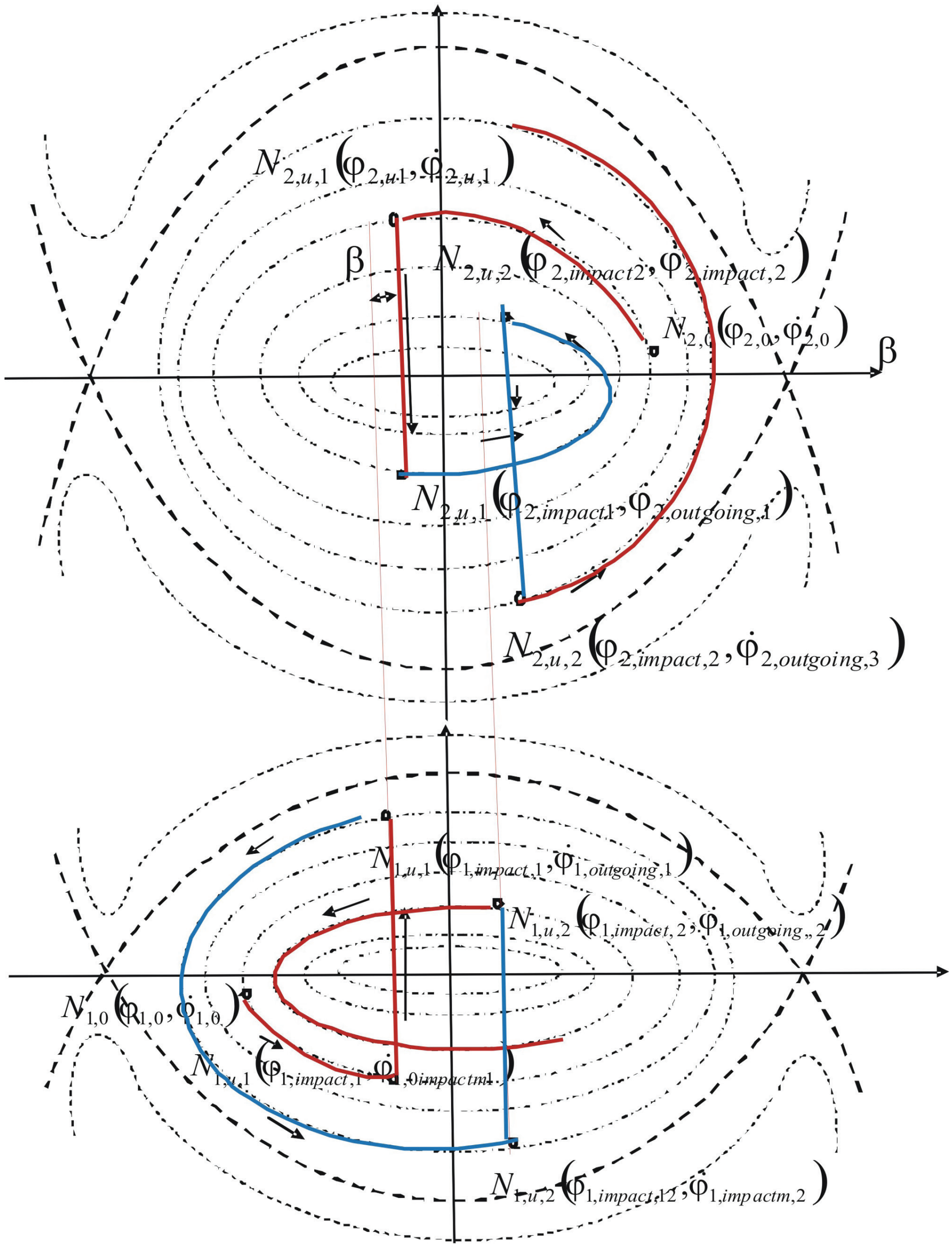


FIGURE 11. Phase trajectory portraits of vibro-impact dynamics of rolling balls along a curvilinear circular line in a vertical plane with two first successive collisions: (upper) for the second rolling ball and (lower) for the first rolling ball

Using these phase portraits for a single ball rolling along a circular line, we start to construct phase trajectory branches for the vibro-impact dynamics of each of the rolling balls at a circular line, starting by the corresponding initial position and the initial angular velocity. In the phase portrait, see Figure 11, starting kinetic states of the balls are presented by phase representative points: $N_{1.0}(\varphi_{1,0},\dot{\varphi}_{1,0})$ and $N_{2,0}(\varphi_{2,0},\dot{\varphi}_{2,0})$, taking into account $\dot{\varphi}_{1,0}=\frac{\omega_{1,0}}{\lambda_1-1}$ and $\dot{\varphi}_{2,0}=\frac{\omega_{2,0}}{\lambda_2-1}$. The interval of the balls non-linear dynamics is along the corresponding phase trajectory of the single ball non-linear dynamics between points from $N_{1.0}(\varphi_{1,0},\dot{\varphi}_{1,0})$ to $N_{1.impact,1}(\varphi_{1,impact,1},\dot{\varphi}_{1,impact,1})$, for the first rolling ball and from $N_{2,0}(\varphi_{2,0},\dot{\varphi}_{2,0})$ to $N_{2,impact,1}(\varphi_{2,impact},\dot{\varphi}_{2,impact,1})$ for the second rolling ball non-linear dynamics.

Taking into account $\dot{\varphi}_{1,ompact,1}=\frac{\omega_{1,impact,1}}{\lambda_1-1}$ and $\dot{\varphi}_{2,impact,1}=\frac{\omega_{2,impact,1}}{\lambda_2-1}$, the equation of the corresponding branch of phase trajectory for the first and second rolling ball along a circular line is defined, respectively, by (29) and (40). Phase representative points $N_{1.impact,1}(\varphi_{1,impact,1},\dot{\varphi}_{1,impact,1})$ and $N_{2,impact,1}(\varphi_{2,impact},\dot{\varphi}_{2,impact,1})$, in phase portraits in Figure 11, correspond to the pre-first-collision state, and phase representative points $N_{1.outgoing,1}(\varphi_{1,impact,1},\dot{\varphi}_{1,outgoing,1})$ and $N_{2,outgoing,1}(\varphi_{2,impact},\dot{\varphi}_{2,outgoingm1})$ correspond to the post-first-collision kinetic state of the rolling balls along a circular line. From these representative points at the phase portrait a jump in velocity for each of the ball dynamics appears and this jump is a jump from one to another phase trajectory depending on the outgoing angular velocity for each of the rolling balls defined by the expressions (62) and (63) or (64) and (65).

The jump on one phase trajectory branch to another branch of another trajectory appears between the following representative points: from $N_{1.impact,1}(\varphi_{1,impact,1},\dot{\varphi}_{1,impact,1})$ to $N_{1.outgoing,1}(\varphi_{1,impact,1},\dot{\varphi}_{1,outgoing,1})$ for the first ball and from $N_{2,impact,1}(\varphi_{2,impact,1},\dot{\varphi}_{2,impact,1})$ to $N_{2,outgoing,1}(\varphi_{2,impact,1},\dot{\varphi}_{2,outgoing,1})$ for the second ball. This is created by the change of the angular velocities of rolling balls pre- and post- collision kinetic state at the same position and caused by the change of angular velocity directions of both rolling balls after the collision between them. The next corresponding branch of the corresponding phase trajectory of the first ball and the one for the second ball are defined by the expressions (70) and (71) respectively. These new branches are defined and bounded by the pairs of the following representative points: for the first ball rolling from the representative point $N_{1.outgoing,1}(\varphi_{1,impact,1},\dot{\varphi}_{1,outgoing,1})$ to $N_{1.impact,2}(\varphi_{1,impact,2},\dot{\varphi}_{1,impact,2})$, and for the second ball rolling from the representative point $N_{2,outgoing,1}(\varphi_{2,impact,1},\dot{\varphi}_{2,outgoing,1})$ to $N_{2,impact,2}(\varphi_{2,impact,2},\dot{\varphi}_{2,outgoing,2})$, respectively.

Next jumps appear from the representative points $N_{1.impact,2}(\varphi_{1,impact,2},\dot{\varphi}_{1,impact,2})$ and $N_{2,impact,2}(\varphi_{2,impact,2},\dot{\varphi}_{2,outgoing,2})$ to the representative points $N_{1.outgoing,2}(\varphi_{1,impact,2},\dot{\varphi}_{1,outgoing,2})$ and $N_{2,outgoing,2}(\varphi_{2,impact,2},\dot{\varphi}_{2,outgoing,2})$, respectively.

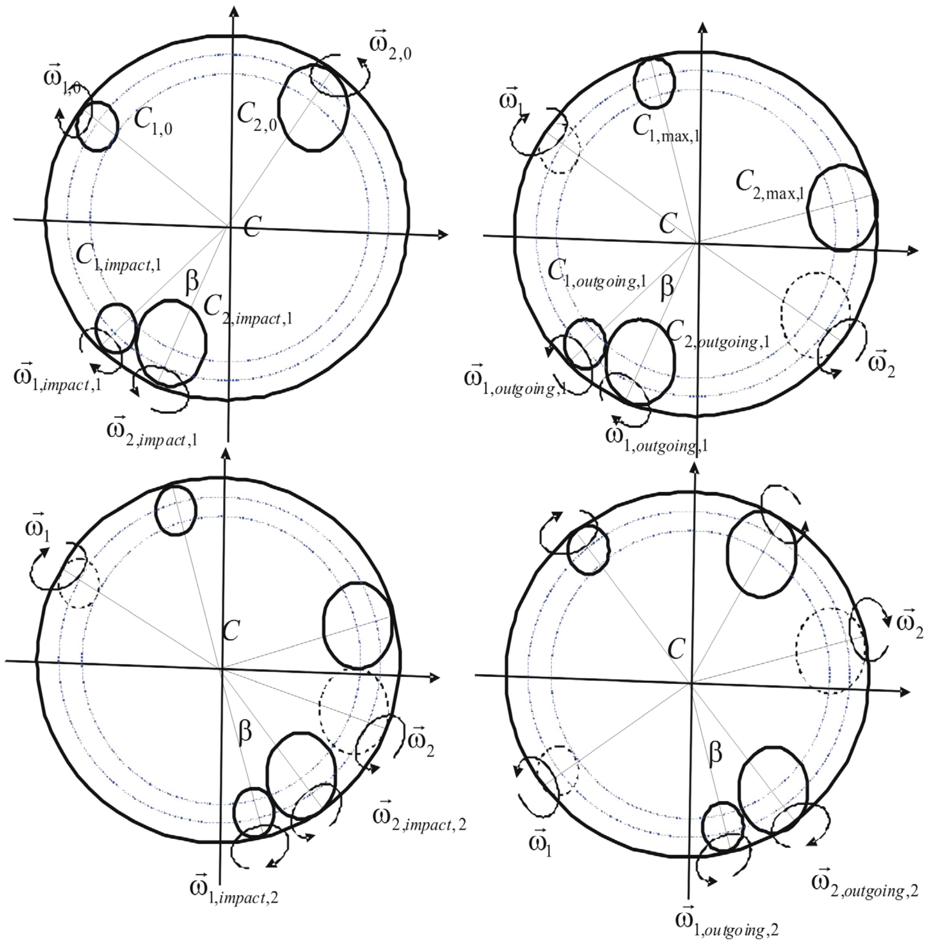


FIGURE 12. Plan of configurations of rolling balls in vibro-impact dynamics of rolling heavy balls along a curvilinear circular line in a vertical plane with successive first two collisions

The next branches of trajectories are defined by the expressions (80) and (81), bounded by the representative points: for the first rolling ball from $N_{1.outgoing,n}\left(\varphi_{1,impact,n},\dot{\varphi}_{1,outgoing,n}\right)$ to $N_{1.impact,(n+1)}\left(\varphi_{1,impact,(n+1)},\dot{\varphi}_{1,impact,(n+1)}\right)$, and for the second ball from $N_{2,outgoing,n}\left(\varphi_{2,impact,n},\dot{\varphi}_{2,outgoing,n}\right)$ to $N_{2,impact,(n+1)}\left(\varphi_{2,impact,(n+1)},\dot{\varphi}_{2,outgoing,(n+1)}\right)$, respectively.

Next jumps appear from the points $N_{1.impact,(n+1)}\left(\varphi_{1,impact,(n+1)},\dot{\varphi}_{1,impact,(n+1)}\right)$ and $N_{2,impact,(n+1)}\left(\varphi_{2,impact,(n+1)},\dot{\varphi}_{2,outgoing,(n+1)}\right)$ to the points $N_{1.outgoing,(n+1)}\left(\varphi_{1,impact,(n+1)},\dot{\varphi}_{1,outgoing,(n+1)}\right)$ and $N_{2,outgoing,(n+1)}\left(\varphi_{2,impact,(n+1)},\dot{\varphi}_{2,outgoing,(n+1)}\right)$, respectively, for $n = 2,3,4,5,6,\ldots$.

4.2.6. Energy analysis of the vibro-impact non-linear dynamics with successive central collisions of two rolling heavy balls along a circular trace in a vertical plane. From the phase portraits of the rolling heavy balls along a circular line in a vertical plane it is possible to conclude that the nonlinear dynamics of both balls between impacts is conservative with constant total mechanical energies for each of the rolling balls. Jumps of the representative point in the corresponding ball phase portrait in pre- and post- collision caused the change of total mechanical energy of each ball, from upper to lower total mechanical energy for one and opposite for another ball. If impacts are ideally elastic and the sum total of mechanical energies of both balls are constant, if there is no ideal elastic collision, this sum of total mechanical energy decreases and after numerous successive collisions tends to zero. Conversely, in this case of no ideal elastic collisions a series of jumps appear from one to another phase trajectory branch.

In case of ideal elastic collisions between rolling balls in the vibro-impact dynamics of the whole system with constant mechanical energy and the change of mechanical energy between balls in each of the collisions appears. The vibro-impact dynamics continued in an infinite period and with infinite numbers of collisions. In the case of no ideal elastic collisions between rolling balls in the vibro-impact dynamics of the whole system with no constant mechanical energy, the energy dissipation appears in each collision and the change of mechanical energy appears between balls in each collision. Then the vibro-impact dynamics continued in a finite period and with finite numbers of collisions up to the rest of the system after finite numbers of the collisions.

Taking into account that non-linear dynamics of the single heavy ball rolling along a circle in a vertical plane is in conservative motion, and that for each ball energy integrals are presented in the forms: (26)-(28) for the first rolling ball and (36)-(38)-(39) for the second rolling ball along a circular line and that each branch of the phase trajectories in phase portraits between two successive collisions also present the corresponding branch of the curves of the constant system total mechanical energy for each of a single ball motion, it is possible to make some conclusions concerning the vibro-impact dynamics of the two rolling balls. In each collision between two rolling balls, the rolling ball with a large angular velocity of the ball after collision is smaller and its total mechanical energy obtains a jump from the upper level to the lower level, and the rolling ball with smaller angular velocity after collision obtains a larger angular velocity and its total mechanical energy obtains a jump from the lower to the upper level. The jumps of total mechanical energy of each ball appear there after each collision of the balls.

Concluding remarks. In concluding remarks, it is necessary to point out the importance of Petrović's theory of the Elements of mathematical phenomenology and Phenomenological Mapping [58-60] for obtaining the original results of the kinetic parameters of two rolling balls in the central collision when both balls roll along a straight trace as well as along a curvilinear trace in a vertical plane, on the basis of analogy with kinetic parameters of the central collision between two bodies in translatory motion. In Table 1, on the basis of mathematical and qualitative analogies between the kinetic parameters of two system central collision dynamics, the corresponding analogous kinetic parameters of the central collision of two bodies in translatory motion are presented and the central collision of two rolling balls.

Also, the kinetic parameters of the collision between two rolling balls presented in this paper are used to present the vibro-impact dynamics of two rolling heavy balls along a curvilinear circular line in a vertical plane. For the vibro-impact dynamics a sketch of phase trajectory is presented in Figure 11.

At the end, it is useful to conclude that the obtained kinetic parameters of the central collision of two rolling balls are possible to use in a study of the skew collision of two rolling balls that roll along two straight line traces with the intersection as well as parallel at a distance smaller than the sum of the balls' radiuses.

The aim of this part is not to present an overview about the generalization of all results in the area of the collision of two rolling balls with different properties of balls and collisions. The part is focused on the central collision of two rolling rigid and heavy smooth balls and using the elements of mathematical phenomenology and phenomenological mapping to obtain the corresponding new expressions for the post-collision and the outgoing angular velocity of each ball and applied these results for the investigation of the vibro-impact dynamics of two rolling balls along a circular trace. This task is analytically solved in full and the obtained analytical results are original and new. Also, these results can be fundamental for the next development and investigation of the special class of collision of the rigid and/or deformable balls and also in application in different areas of engineering systems with coupled rotations (in rolling bearings, rolling vibro-impact dampers - mechanisms for the dynamic absorption of torsional vibrations, or other).

5. Generalized rolling pendulum along a curvilinear trace: Phase portrait, singular points and total mechanical energy surface

5.1. Kinetic parameters of a rolling heavy ball motion along three circle arches in a vertical plane. This part of the paper contains a description of a generalized rolling pendulum along a curvilinear trace consisting of three circle arches in a vertical plane. Sets of three non-linear differential equations of dynamics of the described generalized rolling pendulum along each of three circle arches are presented. A set of three equations of each of three phase trajectory branches which correspond to the dynamics of the described generalized rolling pendulum along each of three circle arches is derived. A phase portrait, a set of singular points and total mechanical energy surface are graphically presented for a particular case of geometrical parameters of the system (for details see Reference [14]).

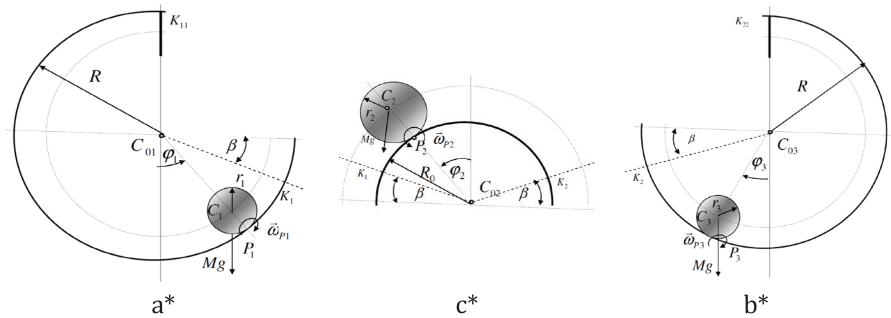


FIGURE 13. Decomposition of the rolling ball dynamics along the curvilinear line rolling trace consisting of three circle arches

To begin with, it is necessary to define a mechanical model of generalized rolling pendulum. A rigid body with one axis of symmetry and a plane of symmetry, which can roll along a curvilinear line with one or more minimums in a vertical plane, is a rolling pendulum, in our definition. In this part of the paper, non-linear dynamics of a rolling pendulum along a curvilinear line, as a rolling trace, consisting of three circle arches is investigated. In Figure 13, the decomposition of the curvilinear rolling trace into three separated circle arches with different radiuses is presented. The system of the ordinary nonlinear differential equations of a heavy ball rolling along a rolling trace into three separated circle arches with different radiuses is in the forms:

$$\ddot{\varphi}_1 + \frac{g}{\kappa_1 (R - r_1)} \sin \varphi_1 = 0 \text{, for } -\pi \le \varphi_1 \le \frac{\pi}{2} - \beta \tag{85}$$

$$\ddot{\varphi}_2 - \frac{g}{\kappa_2 (R_0 + r_3)} \sin \varphi_2 = 0 \text{, for } -\left(\frac{\pi}{2} - \beta\right) \le \varphi_2 \le \frac{\pi}{2} - \beta \tag{86}$$

$$\ddot{\varphi}_3 + \frac{g}{\kappa_3 (R - r_3)} \sin \varphi_3 = 0 \text{, for } -\pi \le \varphi_1 \le \frac{\pi}{2} - \beta \tag{87}$$

The system of the first integral of the previously listed ordinary nonlinear differential equations (85)-(87) of a heavy ball rolling along a rolling trace into three separated circle arches with different radiuses are in the forms:

$$\dot{\varphi}_1^2 = \dot{\varphi}_{1,0}^2 + \frac{2g}{\kappa_1 (R - r_1)} \left(\cos \varphi_1 - \cos \varphi_{1,0}\right) \text{, for } -\pi \le \varphi_1 \le \frac{\pi}{2} - \beta \tag{88}$$

$$\dot{\varphi}_2^2 = \dot{\varphi}_{2,0}^2 - \frac{2g}{\kappa_2 (R + r_2)} \left(\cos \varphi_2 - \cos \varphi_{2,0}\right) \text{, for} -\left(\frac{\pi}{2} - \beta\right) \le \varphi_2 \le \frac{\pi}{2} - \beta \tag{89}$$

$$\dot{\varphi}_3^2 = \dot{\varphi}_{3,0}^2 + \frac{2g}{\kappa_3 (R - r_3)} \left(\cos \varphi_3 - \cos \varphi_{3,0}\right) \text{, for } -\pi \le \varphi_1 \le \frac{\pi}{2} - \beta \tag{90}$$

and present the equations of the branches of phase trajectory portraits.

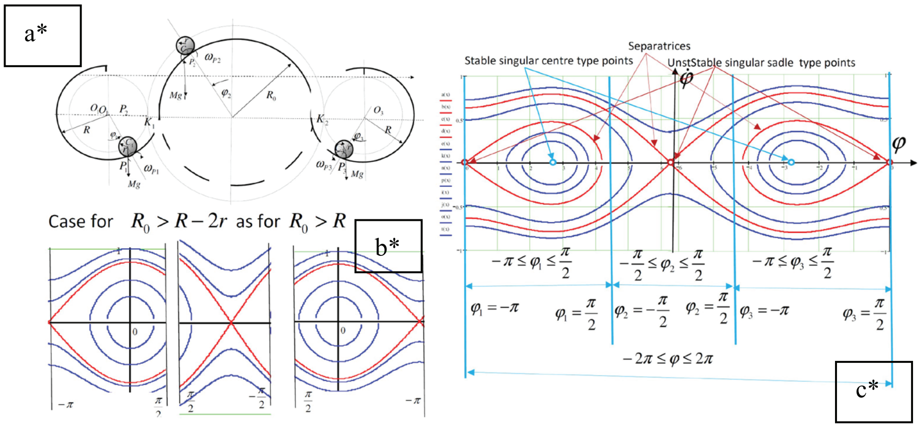
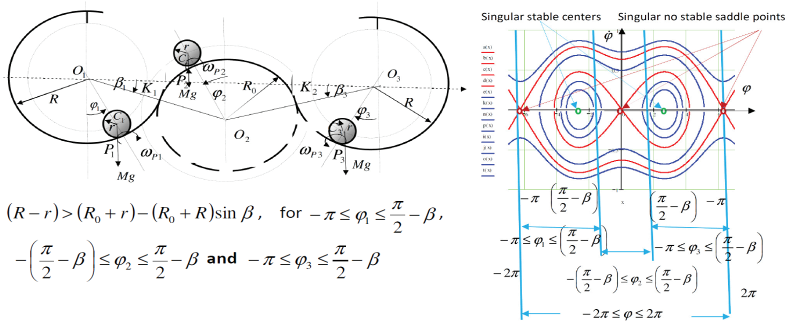


FIGURE 14. A rolling ball along a curvilinear line consisting of three circle arches, each with a central angle of $3\frac{\pi}{2}$ or π or $3\frac{\pi}{2}$, successively, and in the singular case for $R_0 > R - 2r$. a* Mechanical model of "the generalized rolling pendulum" along a curvilinear rolling trace. b* Three parts of a phase portrait which correspond to non-linear dynamics of a rolling ball along each of three circle arches as rolling traces. c* The complete phase portrait of the rolling dynamics of a ball along a curvilinear line-trace consisting of three circle arches with central angles $3\frac{\pi}{2}$ or π or $3\frac{\pi}{2}$, successively, in the case for $R_0 > R - 2r$ and with a half of two triggers of coupled each of two singular points and a homoclinic orbit in the form of half of number "eight" with one cross section in one non stable saddle type singular point and with the second type of homoclinic phase trajectory with a cross section in a non-stable saddle type singular point and containing a stable center type singular point.

5.1.1. Particular case for: $\beta = 0$ and $R_0 > R - 2r$, (see Figure 14.a*). For that case, the set of the nonlinear equations of phase trajectory branches of nonlinear dynamics of rolling balls along three circle arches of the rolling trace is in the form (88)-(90) for $\beta = 0$.

Using the previous set of equations and changing the initial conditions, and taking into account the conditions of continuity in the common posits between the first and the second circle arches as well as the second and the third circle arches of the rolling trace and that at the left end of the first circle arch and at the right end of the third circle arch, limiters are positioned, we can obtain a set of three particular parts of the phase portraits for a rolling ball along each of three circle arches of a rolling trace presented in Figure 13b*. Using the obtained set of three particular phase portraits, which preset decomposition of the complete phase trajectory portrait for the considered case of a rolling ball along a curvilinear trace composed of three circle arches for $\beta = 0$ and $R_0 > R - 2r$, (see Figure 14b*). In the results of composition it is visible that the complete phase trajectory portrait (in phase plane $(\varphi_1, \dot{\varphi}_1)$-$(\varphi_2, \dot{\varphi}_2)$-$(\varphi_3, \dot{\varphi}_3)$), of a rolling ball along a curvilinear trace composed of three circle arches for $\beta = 0$ and $R_0 > R - 2r$, is presented in Figure 14c*.



b*

FIGURE 15. A rolling ball along a curvilinear line consisting of three circle arches, each with a central angle of $3\frac{\pi}{2} - \beta$ or $\pi - 2\beta$ or $3\frac{\pi}{2} - \beta$, successively, and in the case for $(R-r) > (R_0 + r) - (R_0 + R)\sin\beta$, for $-\pi \le \varphi_1 \le \frac{\pi}{2} - \beta$, $-\left(\frac{\pi}{2} - \beta\right) \le \varphi_2 \le \frac{\pi}{2} - \beta$ and $-\pi \le \varphi_3 \le \frac{\pi}{2} - \beta$. a* Mechanical model of "the generalized rolling pendulum" along a curvilinear rolling trace. b* Complete phase portrait of rolling dynamics of a ball along curvilinear line-trace consisting of three circle arches in the case for $(R-r) > (R_0 + r) - (R_0 + R)\sin\beta$ and with a trigger of coupled each of three singular points and a homoclinic orbit in the form of number "eight" with one cross section in one non-stable saddle type singular point and with the second type of the homoclinic phase trajectory with two cross-sections in two non-stable saddle type singular points.

5.1.2. Particular case for: $\beta \neq 0$ and $(R - r) > (R_0 + r) - (R_0 + R)\sin\beta$, (see Figure 15a*). For that case, the set of the nonlinear equations of phase trajectory branches of nonlinear dynamics of rolling balls along three circle arches of a rolling trace is in the following form (93)-(95) for $\beta \neq 0$. For that case, the complete phase trajectory portrait in the phase plane $(\varphi_1, \dot{\varphi}_1)$-$(\varphi_2, \dot{\varphi}_2)$-$(\varphi_3, \dot{\varphi}_3)$, is in the form presented in Figure 15b*.

5.2. Total mechanical energy surface of generalized rolling pendulum.

For a graphical presentation of the total mechanical energy surface of a generalized rolling pendulum, we take into consideration a particular case for: $\beta \neq 0$ and $(R-r) > (R_0 + r) - (R_0 + R)\sin\beta$, (see Figure 16a* and 4a*). For that case, the set of the nonlinear function of the total mechanical energy surface in the phase space $(\mathbf{E}_1, \varphi_1, \dot{\varphi}_1)$-$(\mathbf{E}_2, \varphi_2, \dot{\varphi}_2)$-$(\mathbf{E}_3, \varphi_3, \dot{\varphi}_3)$ is defined by (88)-(90).

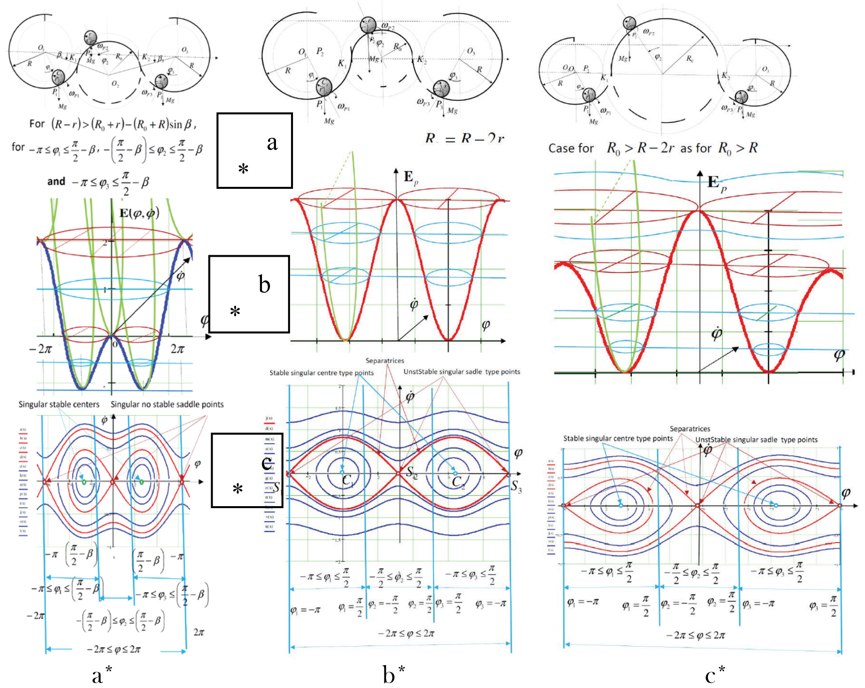


FIGURE 16. A generalized rolling pendulum with a rolling ball along a curvilinear trace consisting of three circle arches each with the central angle of $3\frac{\pi}{2}-\beta$ or $\pi-2\beta$ or $3\frac{\pi}{2}-\beta$, successively, and in the case of $(R-r)>(R_0+r)-(R_0+R)\sin\beta$, for $-\pi\le\varphi_1\le\frac{\pi}{2}-\beta$, $-\left(\frac{\pi}{2}-\beta\right)\le\varphi_2\le\frac{\pi}{2}-\beta$ and $-\pi\le\varphi_3\le\frac{\pi}{2}-\beta$. a* The mechanical model of "the generalized rolling pendulum" along a curvilinear rolling trace. b* Surface of total mechanical energy of the rolling dynamics of a ball along a curvilinear line consisting of circle arches with central angles of $3\frac{\pi}{2}-\beta$ or $\pi-2\beta$ or $3\frac{\pi}{2}-\beta$, successively, in the singular case for $(R-r)>(R_0+r)-(R_0+R)\sin\beta$ and with three maximum values of the total mechanical energy, two same local maximum values of total mechanical energy and the smallest maximum value between the previous, which correspond to three non-stable saddle type singular points and two minimum of total mechanical energy values correspond to two stable centre type singular points; c* Complete phase portrait of rolling dynamics of a ball along curvilinear line-trace consisting of three circle arches in the case for $(R-r)>(R_0+r)-(R_0+R)\sin\beta$ and with the triggers of coupled each of three singular points and homoclinic orbit in the form of number "eight" with one cross section in one non-stable saddle type singular point and with the second type of homoclinic phase trajectory with two cross-sections in two non-stable saddle type singular points.

In Figure 16a*, for a particular case for: $\beta \neq 0$ and $(R-r) > (R_0+r) - (R_0+R)\sin\beta$, the graphical presentation of the total mechanical energy surface in the phase space $(\mathbf{E}_1, \varphi_1, \dot{\varphi}_1)$-$(\mathbf{E}_2, \varphi_2, \dot{\varphi}_2)$-$(\mathbf{E}_3, \varphi_3, \dot{\varphi}_3)$ of nonlinear dynamics of a generalized rolling pendulum is presented.

In the same Figure 16, the mechanical model (a*) of the generalized rolling pendulum is presented for that case and also the corresponding complete phase trajectory portrait (c*) is presented. The paper starts with a description of a generalized rolling pendulum (see Reference [14]) along a curvilinear trace consisting of three circle arches in a vertical plane. The rolling body of a generalized rolling pendulum is a rigid body with an axis of symmetry and one plane of symmetry with a cross-section in a plane of symmetry in the form of a circle. Sets of three non-linear differential equations and a set of three equations of each of three phase trajectory branches which correspond to the dynamics of the described generalized rolling pendulum along each of three circle arches are derived. The phase portrait, a set of singular points and the total mechanical energy surface are graphically presented for particular cases of geometrical parameters of the system.

It is possible to use the presented analytical and graphical presentation for a heavy mass particle moving along a curvilinear trace consisting of three circle arches, introducing a nonlinear differential equations and other analytical expressions that the coefficient of rolling is equal to unique, and that the radius of a rolling ball is equal to zero (for detail see References [11-14]).

The presented analytical and graphical elements in the previous parts are the basis of the methodology for the investigation of the vibro-impact dynamics of a system with two rolling bodies in successive collisions (see References [10, 12, 15, 17, 24, 26, 29, 30, 31]). For obtaining the outgoing angular velocity of each rolling body after each collision in a series of successive collisions we can use the analytical expressions presented in References [15, 26, 30] from the extended classical theory of impact by kinematics and dynamics of collision between two rolling bodies founded by Hedrih (Stevanović) R. K.

5.3. Analytical generalization of a nonlinear dynamics description of a rolling heavy thin disk along a curvilinear trace in a rotating vertical plane around a vertical axis at a constant angular velocity. The nonlinear differential equation of dynamics of a heavy thin disk rolling, without slipping, along a general curvilinear trace, in a rotating vertical plane, around the vertical axis with the constant angular velocity, is derived. The first integral of this nonlinear differential equation is determined. The first integral presents the nonlinear equation of the phase trajectory in a phase plane of rolling, without slipping, a heavy thin disk along a general curvilinear trace, in a rotating vertical plane, around the vertical axis with the constant angular velocity. A theorem about bifurcation and triggers of coupled singularities is formulated. A qualitative analysis of the stability of singular points and relative equilibrium positions on a trace of a rolling body is presented.

The characteristic equation of dynamics of the generalized rolling pendulum, along a trajectory in a rotating vertical plane at a constant angular velocity around vertical axis is presented (for details see References [6, 7, 26]).

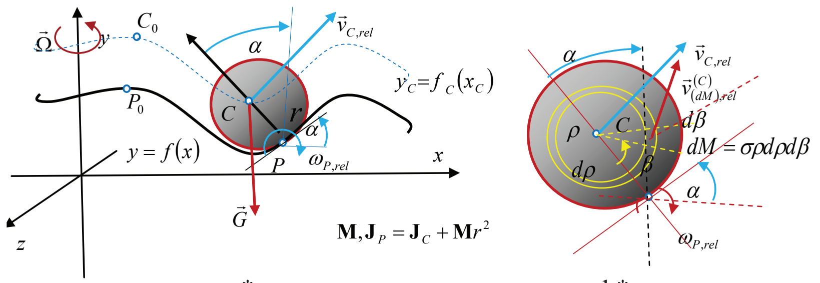


FIGURE 17. Geometric parameters of the rolling of heavy rigid disks on a rotating curvilinear trace in a vertical plane around a vertical axis

Suppose there is a curvilinear trace determined by $y = f(x)$, so that the curvature radius of each of its concave arches is larger than the radius of the contour of the disk circle in the plane of symmetry, by which the disk rolls, without slipping, along the curvilinear trace, rotating, around the vertical axis with the constant angular velocity Ω, in the rotating vertical plane (see Refs. [6, 7, 26]). The rolling body, without slipping, rotating around the vertical axis with the constant angular velocity Ω, has a degree of freedom of movement along the curvilinear trace, because it has five constraints. For an independent generalized coordinate, we select the abscise coordinate x, in the rotating vertical plane of the coordinate system, by which we will express the angular velocity $\omega_P(x,\dot{x},\Omega)$ of the instantaneous relative rotation around the current instantaneous axis of the relative rolling of the disk along a curvilinear trace in a rotating vertical plane with the constant angular velocity Ω around the vertical axis. $\omega_P(x,\dot{x},\Omega) = \frac{1}{r} v_{Crel}(x,\dot{x},\Omega)$. The expression of kinetic energy E_k of a disk in relative rolling along a curvilinear trace $y = f(x)$ in a vertically rotating plane around the vertical axis by a constant angular velocity Ω, determines the integral of the kinetic energy $dE_{k(dM)} = \frac{1}{2}\left[v_{(d)}\right]^2 dM$ of the elementary mass dM of the disk. (see right subfigure b* in Figure 17).

5.3.1. Nonlinear differential equation of a rolling disk along a rotating curvilinear line and the equation of phase trajectory. The nonlinear differential equation of the rolling motion of a heavy thin rigid disk, without slipping with the radius r, along the curvilinear trace of the form $y = f(x)$, in a rotating vertical plane around the vertical axis with the constant angular velocity Ω, (and where the coefficient of rolling $\kappa = \frac{\mathbf{J}_P}{Mr^2} = \frac{\mathbf{i}_P^2}{r^2} = \frac{\mathbf{i}_C^2}{r^2} + 1 = \kappa$), is (for details see Refs. [6, 7, 26]):

$$\ddot{x} + \frac{1}{2}\dot{x}^2 \frac{F'(x,r)}{F(x,r)} - \frac{r^2}{2}\Omega^2 \frac{\mathbf{J}'_z(x,M,r)}{\mathbf{J}_P F(x,r)} + \frac{g}{\kappa}\frac{f'(x)}{F(x,r)} = 0 \qquad (91)$$

In order to get the first integral of the preceding nonlinear differential equation (91), we introduce $u = \dot{x}^2$ as a change of the variable coordinate (see Reference [6, 7, 26]). The first integral of the rolling motion differential equation of a rigid disk, without slipping, along the rotating curvilinear trace of the form $y = f(x)$, around the vertical axis with the constant angular velocity Ω, is:

$$[\dot{x}(x)] = \pm\sqrt{[\dot{x}_0(x_0)]^2 \frac{F(x_0,r)}{F(x,r)} + \Omega^2 \frac{r^2[\mathbf{J}_z(x,M,r) - \mathbf{J}_z(x_0,M,r)]}{\mathbf{J}_P F(x,r)} + \frac{2g}{\kappa F(x,r)}[f_C(x_0 r) - f_C(x,r)]} \quad (92)$$

If we introduce the coefficient of disk rolling, without slipping, in the form $\kappa = \frac{\mathbf{J}_P}{Mr^2} = \frac{\mathbf{i}_P^2}{r^2} = \frac{\mathbf{i}_C^2}{r^2} + 1 = \kappa$, which for a thin disk is $\kappa = \frac{3}{2}$, then the nonlinear ordinary differential equation of rolling without sliding a heavy rigid thin disk along a curvilinear line in the rotating vertical plane around the vertical axis at the constant angular velocity $\dot{\vartheta} = \Omega$, is in the following form:

$$\ddot{x}F(x,r) + \frac{1}{2}\dot{x}^2 F'(x,r) - \frac{r^2}{2}\Omega^2 \frac{\mathbf{J}'_z(x,M,r)}{\mathbf{J}_P} + \frac{g}{\kappa} f'_C(x) = 0 \quad (93)$$

where $f_C(x)$ and $F(x,r)$ are expressed by the following expressions (for details see References [6, 7, 26]):

$$F(x,r) = \left\langle 1 + [f'(x)]^2 \right\rangle \left\{ 1 - \frac{rf''(x)}{\left[1 + [f'(x)]^2\right]^{\frac{3}{2}}} \right\}^2 \quad (94)$$

$$f_C(x) = y + r\frac{1}{\sqrt{1 + [f'(x)]^2}} \quad (95)$$

In the previous papers of the author (for details see References [6, 7, 26]), the main attention was paid to a more detailed analysis of the characteristic equation of the dynamics of the generalized rolling pendulum, along the trajectory in a rotating vertical plane at a constant angular velocity around the vertical axis, which was performed in the form:

$$f'(x)\left\{ 1 - r\frac{f''(x)}{\left[1 + [f'(x)]^2\right]^{\frac{3}{2}}} \right\} - \frac{2\kappa}{3g}\Omega^2 \left\langle x - \frac{rf'(x)}{\sqrt{1 + [f'(x)]^2}} \right\rangle \left\langle 1 - \frac{rf''(x)}{\left[1 + [f'(x)]^2\right]\sqrt{1 + [f'(x)]^2}} \right\rangle = 0 \quad (96)$$

and in which: $y = f(x)$ in general, or in particular cases $y = f(x) = kx^2(x^2 - a^2)^2$ or $y = f(x) = kx^2(x^2 - a^2)^2(x^2 - b^2)$ or $f(x) = -kx^2(x^2 - a^2)\left[c^4 - (x^2 - b^2)^2\right]$ is the equation of the curvilinear path, where b, b, c and k are the known constants, and with the following relation $a < b$, κ, $\kappa = 1 + \frac{\mathbf{i}_C^2}{\mathbf{r}^2}$ the rolling coefficient, Ω the radius of the circle of the body of the pendulum by which the pendulum rolls along curvilinear paths, Ω the angular velocity of rotation of the vertical plane about the vertical axis, and in which there is the curvilinear rolling route of the generalized rolling pendulum.

We can draw a conclusion in the form of the following theorem: *The kinetic energy of a thin disk rolling along a curvilinear trace in a vertical rotating plane around a vertical axis, with a constant angular velocity* Ω*, consists of the kinetic energy of the rotation of the rigid disk around the vertical axis at angular velocity* Ω*, and the kinetic energy of relative rolling disk along the curvilinear trace with the angular rolling velocity* $\omega_P(x,\dot{x},\Omega)=\frac{1}{r}v_{Crel}(x,\dot{x},\Omega)$ *along a curvilinear rotating trace:* $E_k=\frac{1}{2}\mathbf{J}_z\Omega^2+\frac{1}{2}\mathbf{J}_P[\omega_{P,rel}]^2$.

The theorem on bifurcation and on the trigger of coupled singularities in the nonlinear dynamics of generalized rolling pendulums along curvilinear routes in a rotating vertical plane around a vertical axis at a constant angular velocity Ω:

Let us present the curved line, given with $f(x)=f(-x)$*, for which the following is valid* $f(x)=f(-x)$*, and which has at the points for extreme values* $EX_s(x_s,y_s=f(x_s))$ *for* $f'(x_s)=0$*, the minimums* $C_s(x_s,y_s=f(x_s))$ *for* $f'(x_s)=0$, $f''(x_s)>0$*, and the maximums* $S_s(x_s,y_s=f(x_s))$ *for* $f''(x_s)<0$, $f''(x_s)<0$*, the curvilinear route, along which a heavy homogeneous thin disk of the radius* $r>0$ *rolls without slipping and let it be located in the Earth's gravitational field, in the vertical plane which rotates around the vertical axis, at a constant angular velocity* $\Omega>0$*. The characteristic equation for determining the singular points, as well as the position of the relative equilibrium of the disk on the curvilinear path, in the vertical rotating plane around the vertical axis at a constant angular velocity* $\Omega>0$*, is of the form:*

$$h(x)=f'(x)\left\{1-r\frac{f''(x)}{\left[1+[f'(x)]^2\right]^{\frac{3}{2}}}\right\}-\frac{2\kappa}{3g}\Omega^2\left\langle x-\frac{rf'(x)}{\sqrt{1+[f'(x)]^2}}\right\rangle\left\langle 1-\frac{rf''(x)}{\left[1+[f'(x)]^2\right]\sqrt{1+[f'(x)]^2}}\right\rangle=0 \quad (97)$$

in which it is $\kappa=\frac{\mathbf{J}_P}{Mr^2}=\frac{\mathbf{i}_P^2}{r^2}=\frac{\mathbf{i}_C^2}{r^2}+1=\kappa$ *, that is* $\kappa=\frac{3}{2}$*, the rolling coefficient of the disk, because* $\mathbf{J}_{Cz}=\sigma\frac{r^4}{4}\pi=M\frac{r^2}{4}$ *and* $\mathbf{J}_P=\mathbf{J}_C+Mr^2=\frac{3}{2}Mr^2$ *, and* g *is the acceleration of the Earth that is heavier. Around each extremum of the curvilinear trajectory, which is the minimum defined by* $C_s(x_s,y_s=f(x_s))$ *for* $f''(x_s)>0$, $f''(x_s)>0$*, in the nonlinear dynamics of the thin disk rolling, bifurcations and triggers of coupled singularities appear, and around each extremum, which is a maximum defined with* $S_s(x_s,y_s=f(x_s))$ *for* $f'(x_s)=0$, $f''(x_s)<0$*, there are neither bifurcation nor triggers of coupled singularities* (for details see References [6, 7, 26]).

6. Vibro-impact dynamics of two rolling heavy thin disks along a rotating curvilinear line and energy analysis

In this part of the paper, a construction of the phase trajectory portraits of a generalized rolling pendulum along a rotating curvilinear line is presented. The generalized rolling pendulum containing a rolling thin heavy disk rotates along the curvilinear line consisting of three circle arches, in a rotating vertical plane at a constant angular velocity around a vertical eccentric/central axis. Depending on the system parameters, different possible forms of the phase portraits appear with different structures of the sets of singular points and forms of phase trajectories. A trigger of coupled singular points and homoclinic orbit in the form of deformed number eight appears. A mathematical analogy [67-69] between nonlinear differential equations of the considered generalized rolling pendulum and motion of the heavy mass particle along the same form of the curvilinear line, in a rotating vertical plane around the vertical axis at a constant angular velocity, is pointed out. On the basis of the obtained different possible phase trajectory portraits, a non-linear phenomenon in vibro-impact dynamics of two rolling thin disks on a rotating curvilinear line around the vertical axis at a constant angular velocity, is investigated. Energy transfer between rolling disks in each of the series of successive collisions is analyzed and presented on relative mechanical energy portraits for the dynamics of each of the rolling disks in collision (see Reference [24]).

6.1. Vibro-impact system description. The vibro-impact system (see Figure 18) contains two generalized rolling pendulums, each in the form of a thin heavy disk, which are in rolling motions, without slipping, along a rotating curvilinear line trace with the constant angular velocity Ω around a vertical axis. The curvilinear line trace is symmetric and consists of three circle arches, two with the same radius value and one with the different radius.

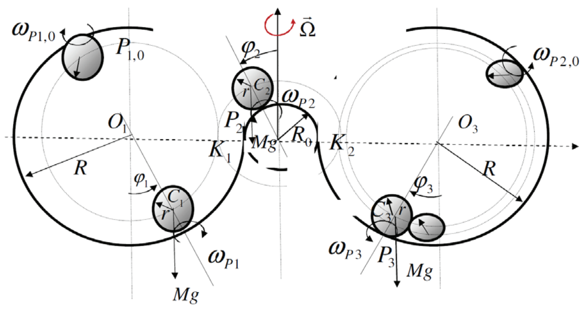


FIGURE 18. Vibro-impact system, containing two generalized rolling pendulums each in the form of a thin heavy disk, which are in rolling motions, without slipping, along a rotating curvilinear line trace with the constant angular velocity Ω around the vertical axis of a curvilinear line axial symmetry

Figure 19 represents one combination of many possible parts of phase trajectory portraits finalized into a complex phase trajectory portrait of a dynamics of a relative rolling heavy thin disk along a rotating curvilinear trace around a vertical axis of its symmetry at a constant angular velocity, with the notation of the corresponding set of singular points, and two triggers of coupled singular points and two homoclinic trajectory orbits in a form of two deformed figure eight shapes.

It is necessary to point out, that depending on the relations between geometrical parameters and values of angular velocity of the rotation of a curvilinear rolling trace, there is the exit of the component phase trajectory portraits of the dynamics of a rolling thin heavy disk along the component circle arches of a curvilinear trace rotating around eccentric/centric axes. For graphical presentations in this paper, there are more complicated cases of the component phase trajectory portraits with triggers of coupled singular points and homoclinic trajectory in a figure eight form, as the results of bifurcation with change of bifurcation parameter.

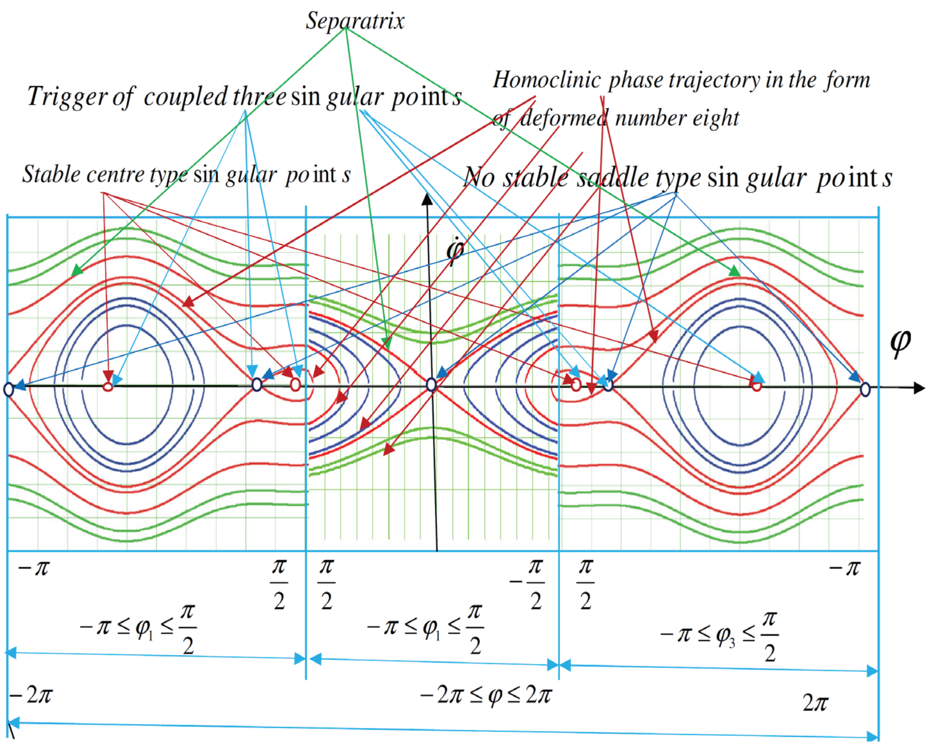


FIGURE 19. One combination of many possible parts of phase trajectory portraits finalized into a complex phase trajectory portrait of a dynamics of a relative rolling heavy thin disk along a rotating curvilinear trace around a vertical axis of its symmetry at a constant angular velocity, with the notation of the corresponding set of singular points, and two triggers of coupled singular points and homoclinic orbits in a form of two deformed figure eight shapes

Additionally, it can be concluded that there are numerous combinations for the continuation between parts of phase trajectories, from different parts of phase trajectory portraits, in the construction of the complex phase trajectory portrait for the dynamics of the relative rolling of a heavy thin disk along a curvilinear trace rotating around a vertical axis of its symmetry, with constant angular velocity.

Depending on the initial conditions of the dynamics of rolling a heavy thin disk along a rotating curvilinear trace, and the obtained in results of this dynamics of the form of the phase trajectory presented in Figure 19, along which the representative point in phase plane is moving, it is possible to evaluate and conclude about the character of disk rolling, periodic or twice-periodic or progressive rolling up to a limiter of rolling along the corresponding circle arch.

6.2. The vibro-impact dynamics with the central collision of two disks which are rolling along a rotating curvilinear track.

It is possible to determine next the impact angular velocity of relative rolling of a disk along a curvilinear trace at the position of the configuration of the first collision. For that task, we use the following relations:

a* first collision appears at the first circle arch trace impact rolling angular velocities are:

$$\omega_{P1,rel,impact,1} = \left(\frac{R-r}{r}\right)\dot{\varphi}_{1,1,impact,1} \quad \text{and} \quad \omega_{P2,rel,impact,1} = \left(\frac{R-r}{r}\right)\dot{\varphi}_{2,1,impact,1} \tag{98}$$

where $\dot{\varphi}_{1,1,impact,1}$ and $\dot{\varphi}_{2,1,impact,1}$ are angular velocities, determined by the equation of phase trajectory of the rolling disk along the rotating first circle arch around an eccentric vertical axis, for the angle coordinate of the first and the second disk in position of the first collisions in position at the first circle arch (for details see Reference [24]):

$$\begin{aligned} \dot{\varphi}_{1,1,impact,1} &= \sqrt{\dot{\varphi}_{1,0}^2 + 2\frac{\Omega^2}{\kappa}\left\langle \lambda\left(\cos\varphi_{1,1,impact,1} - \cos\varphi_{1,0}\right) - \frac{1}{4}\left(\cos 2\varphi_{1,1,impact,1} - \cos 2\varphi_{1,0}\right)\right\rangle - 2\frac{\Omega^2}{\kappa}\varepsilon\left(\sin\varphi_{1,1,impact,1} - \sin\varphi_{1,0}\right)} \\ \dot{\varphi}_{2,1,impact,1} &= \sqrt{\dot{\varphi}_{1,0}^2 + 2\frac{\Omega^2}{\kappa}\left\langle \lambda\left(\cos\varphi_{2,1,impact,1} - \cos\varphi_{2,2,continuity,2}\right) - \frac{1}{4}\left(\cos 2\varphi_{2,1,impact,1} - \cos 2\varphi_{2,2,continuity,2}\right)\right\rangle - 2\frac{\Omega^2}{\kappa}\varepsilon\left(\sin\varphi_{2,1,impact,1} - \sin\varphi_{2,2,continuity,2}\right)} \end{aligned} \tag{99}$$

b* first collision appears at the second circle arch trace impact rolling angular velocities are:

$$\omega_{P1,rel,impact,1} = \left(\frac{R_0+r}{r}\right)\dot{\varphi}_{1,2,impact,1} \quad \text{and} \quad \omega_{P2,rel,impact,1} = \left(\frac{R_0+r}{r}\right)\dot{\varphi}_{2,2,impact,1} \tag{100}$$

where $\dot{\varphi}_{1,1,impact,1}$ and $\dot{\varphi}_{2,1,impact,1}$ are angular velocities, determined by the equation of phase trajectory of the rolling disk along a rotating first circle arch around an eccentric vertical axis, for the angle coordinate of the first and the second disk in the position of the first collisions in the position at the first circle arch (for details see Reference [24]):

$$\begin{aligned} \dot{\varphi}_{1,2,impact,1} &= \sqrt{\dot{\varphi}_{1,0}^2 + 2\frac{\Omega^2}{\kappa}\left\langle \lambda\left(\cos\varphi_{1,2,impact,1} - \cos\varphi_{1,ootinuity,1}\right) - \frac{1}{4}\left(\cos 2\varphi_{1,2,impact,1} - \cos 2\varphi_{1,ootinuity,1}\right)\right\rangle} \\ \dot{\varphi}_{2,2,impact,1} &= \sqrt{\dot{\varphi}_{1,0}^2 + 2\frac{\Omega^2}{\kappa}\left\langle \lambda\left(\cos\varphi_{2,2,impact,1} - \cos\varphi_{2,2,continuity,2}\right) - \frac{1}{4}\left(\cos 2\varphi_{2,2,impact,1} - \cos 2\varphi_{2,2,continuity,2}\right)\right\rangle} \end{aligned} \tag{101}$$

c* first collision appears at the first circle arch trace impact rolling angular velocities are:

$$\omega_{P1,rel,impact,1} = \left(\frac{R-r}{r}\right)\dot{\varphi}_{1,3,impact,1} \text{ and } \omega_{P2,rel,impact,1} = \left(\frac{R-r}{r}\right)\dot{\varphi}_{2,2,impact,1} \qquad (102)$$

where $\dot{\varphi}_{1,1,impact,1}$ and $\dot{\varphi}_{2,1,impact,1}$ are angular velocities, determined by the equation (18) of phase trajectory of the rolling disk along the rotating first circle arch around eccentric vertical axis, for the angle coordinate of the first and the second disk in position of the first collisions in position at the first circle arch (for details see Reference [24]):

$$\dot{\varphi}_{1,3,impact,1} = \sqrt{\dot{\varphi}_{1,0}^2 + 2\frac{\Omega^2}{\kappa}\left\langle \lambda\left(\cos\varphi_{1,3,impact,1} - \cos\varphi_{1,3,continuity,2}\right) - \frac{1}{4}\left(\cos 2\varphi_{1,3,impact,1} - \cos 2\varphi_{1,0}\right)\right\rangle + 2\frac{\Omega^2}{\kappa}\varepsilon\left(\sin\varphi_{1,3,impact,1} - \sin\varphi_{1,3,continuity,2}\right)}$$

$$\dot{\varphi}_{2,3,impact,1} = \sqrt{\dot{\varphi}_{1,0}^2 + 2\frac{\Omega^2}{\kappa}\left\langle \lambda\left(\cos\varphi_{2,3,impact,1} - \cos\varphi_{3,0}\right) - \frac{1}{4}\left(\cos 2\varphi_{2,3,impact,1} - \cos 2\varphi_{3,0}\right)\right\rangle + 2\frac{\Omega^2}{\kappa}\varepsilon\left(\sin\varphi_{2,3,impact,1} - \sin\varphi_{3,0}\right)} \qquad (103)$$

Let us start with the theory of dynamics of a central collision between two rolling disks, with mass m_1 and m_2, and axial mass inertia moments $\mathbf{J}_{P2}$ and $\mathbf{J}_{P2}$ for the corresponding momentary axis of the relative rolling along a rotating curvilinear trace with pre-impact (arrival) relative angular velocities $\vec{\omega}_{P1,impact} = \vec{\omega}_{P1}(t_0)$ and $\vec{\omega}_{P2,impact} = \vec{\omega}_{P2}(t_0)$. The mass centers $\mathbf{C}_1$ and $\mathbf{C}_2$ of the disks are moving in a relative translational move with pre-impact (arrival) relative velocities $\vec{v}_{C1.impact} = \vec{v}_{C1}(t_0)$ and $\vec{v}_{C2.impact} = \vec{v}_{C2}(t_0)$. Relative angular velocities $\vec{\omega}_{P1,impact} = \vec{\omega}_{P1}(t_0)$ and $\vec{\omega}_{P2,impact} = \vec{\omega}_{P2}(t_0)$, we denote as arrival, or impact or pre-impact relative angular velocities at the moment t_0. At this moment t_0 of the start of the collision between these relative rolling disks, the contact of these two disks is at point $\mathbf{T}_{12}$, in which both of the disks possess a common tangent plane – plane of contact (touch). In the theory of collision, it is proposed that collision takes a very short period of time $(t_0, t_0 + \tau)$, and that τ tends to zero. After this short period τ, bodies - two relative rolling disks in collision separate and outgo with post-impact-outgoing relative angular velocities $\vec{\omega}_{P1,outgoing} = \vec{\omega}_{P1}(t_0 + \tau)$ and $\vec{\omega}_{P2,outgoing} = \vec{\omega}_{P2}(t_0 + \tau)$. The mass centers $\mathbf{C}_1$ and $\mathbf{C}_2$ of the disks perform relative translational motion with post-impact (outgoing) translatory velocities $\vec{v}_{C1.outgoing} = \vec{v}_{C1}(t_0 + \tau)$ and $\vec{v}_{C2.outgoing} = \vec{v}_{C2}(t_0 + \tau)$. These relative translational velocities could be expressed by the corresponding relative angular velocity and the radius of the corresponding disk.

Using Hedrih' s expressions for outgoing angular velocities (chapter 3, parts 3.6 and 3.7), in the expressions (10) and (11), after the first collision, one can write the expressions for the outgoing angular velocities of the rolling disks after the first collision, for the first $\omega_{P1,rel,outgoing,1}$ and for the second $\omega_{P2,rel,outgoing,1}$.

Taking into account that the collision kinetic state appears and disappears during a very short time period, tends to zero, $\tau \to 0$, and that in the classical theory the hypothesis that collision is a quasi-static process is introduced, and also taking into account only the change of the rolling disk angular velocities, but not changing the disks positions, everything presented in chapters 3, (parts 3.6 and 3.7) 4 and 5 of this paper has been proved and is valid for the relative rolling disks in collision positioned on the curvilinear line (for details see Reference [24]).

For obtaining the first branches of phase trajectories of the first and the second disk, the equation of phase trajectory with the initial condition of the corresponding rolling disk, for starting the investigation of vibro-impact dynamics is used. One can take into account the corresponding eccentricity of the vertical axis depending on the disk rolling along the corresponding one, two or three circle arches of a rotating curvilinear trace. These initial conditions are $\varphi_{1,0}$ and $\dot{\varphi}_{1,0}$ for the first disk and $\varphi_{2,0}$ and $\dot{\varphi}_{2,0}$ for the second disk, which define the corresponding initial relative positions and initial relative angular velocity of the corresponding disk center C_1 and C_2 in relation to a rotating curvilinear trace and a circle arches center O. By using these initial conditions, we define the expressions for the pre-first collision impact (arrival) derivative of the first generalized coordinate $\dot{\varphi}_{1,impact,1}$ and the second generalized coordinate $\varphi_{2,impact,1}$ when disks are in the positions of the first collision, $\varphi_{1,impact,1}$ and $\varphi_{2,impact,1}$. Then we use these expressions for obtaining the post-first collision outgoing relative angular velocities $\omega_{P2,rel,impact,1} = \omega_{P2,rel,impact,1}$ and $\omega_{P2,outgoing,1} = \omega_{P2,rel,outgoing,1}$ and the corresponding $\dot{\varphi}_{1,outgoing,1}$ and $\dot{\varphi}_{2,outgoing,1}$, which present the initial conditions for the second branches of the rolling disks between the first and the second collision: $\varphi_{1,outgoing,1} = \varphi_{1,impact,1}$, $\dot{\varphi}_{1,outgoing,1}$, $\varphi_{2,outgoing,1} = \varphi_{2,impact,1}$ and $\dot{\varphi}_{2,outgoing,1}$.

For i-th branch of phase trajectories of the rolling disks relative dynamics along the rotating curvilinear line between $(i-1)$-th and i-th collisions, the initial conditions are:
$\varphi_{1,outgoing,(i-1)} = \varphi_{1,impact,(i-1)}$ and $\dot{\varphi}_{1,outgoing,(i-1)}$, and, $\varphi_{2,outgoing,(i-1)} = \varphi_{2,impact,(i-1)}$ and $\dot{\varphi}_{2,outgoing,(i-1)}$, obtained by the expressions for $(i-1)$ phase trajectory branch. For the pre i-th collision impact (arrival) derivative of the first generalized $\dot{\varphi}_{1,impact,i}$ and the second generalized coordinate $\dot{\varphi}_{2,impact,1}$ when disks are in the positions of the i-th collision, $\varphi_{1,impact,i}$ and $\varphi_{2,impact,i}$ are defined by the corresponding phase trajectory branches. For obtaining the expressions of the post i-th collision relative angular velocities $\dot{\varphi}_{1,outgoing,i}$ and $\dot{\varphi}_{2,outgoing,i}$, expressions for the corresponding phase trajectory branches are used. We can conclude that this algorithm contains successive applications of expressions for corresponding phase trajectory branches with a combination of expressions of corresponding outgoing angular velocities.

The application of expressions for the corresponding phase trajectory branches and for the corresponding outgoing angular velocities is clear and it is easy to obtain all the necessary pre i-th collision and post i-th collision kinetic parameters, arrival and outgoing relative angular velocities of both disks, if we know the position of each of the successive collisions for both rolling disks. However, the main problem is the lack of solvability of the transcendental equation analytically. This task is still possible to solve numerically and obtain coordinates of both disks' positions for each at i-th collision, if collision exists. This problem is solvable numerically, but requires consideration in each step of the existence of the next collision between disks.

In Figure 20, phase trajectory branches in phase portraits of two rolling heavy thin disks for relative motion in the interval between the initial condition configuration and configurations of the pre-first-collision and post-first-collision between two rolling disks with vibro-impact dynamics on a rotating curvilinear trace with the constant angular velocity around a vertical central axis of its symmetry, for bifurcation parameters $\lambda_i < 1$, $i = 1{,}2$, are presented (for details see Reference [24]).

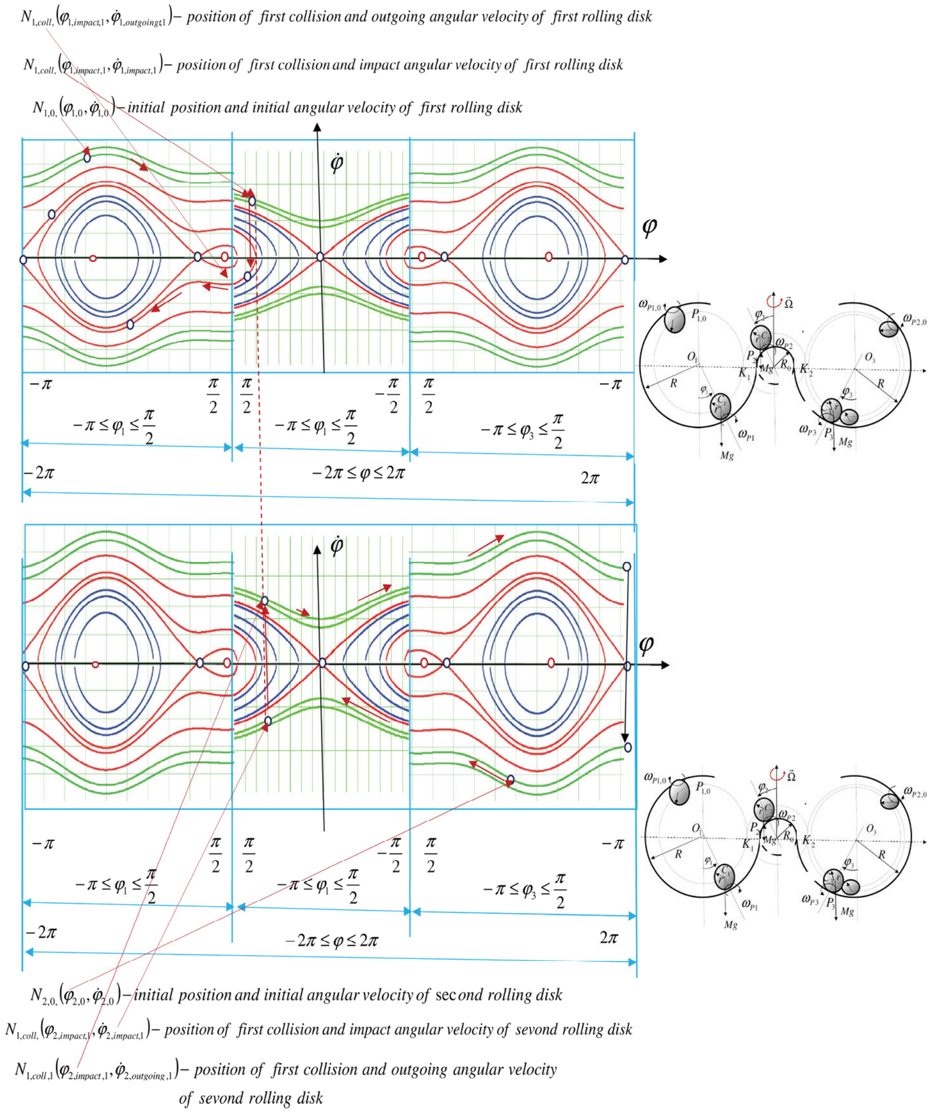


FIGURE 20. Phase trajectory branches in phase portraits of two rolling heavy thin disks for relative motion in the interval between the initial condition configuration and configurations of the pre-first-collision and post-first-collision between two rolling disks with vibro-impact dynamics on a rotating curvilinear trace with the constant angular velocity around the vertical central axis of its symmetry and for bifurcation parameters $\lambda_i < 1$, $i = 1{,}2$.

The basic results presented in this part, published in the Reference [24], are the constructions of two-phase portraits, each of the corresponding nonlinear dynamics of each of two rolling different heavy thin disks in successive central collisions along a rotating complex curvilinear trace, as well as the determination of the kinetic parameters before and after each successive central collision between rolling disks. New Hedrih's expressions published in References [12, 15, 17, 26, 27, 28, 30] and presented in the third section (parts 3.6 and 3.7)
of this paper, related to the outgoing angular velocity after central collision are applied for determining each of the outgoing angular velocities of the rolling disks after each successive collision. Results include determining the transformed elliptic integrals for the determination of time and position of each of the successive collisions. Moreover, the energy analysis with corresponding energy jumps between disks is presented on the comparative corresponding constant energy curve portraits.

Additionally, the aim of the paper is the presentation of advanced analytical results in the development of principal methodology based on new results in the extension of classical theory of collision with kinematics and dynamics of the collision of two rolling disks along a rotating curvilinear rolling trace, for the investigation of vibro-impact dynamics using phase trajectory portraits. This is an analytical approach which is sufficient for a qualitative analysis of nonlinear and vibro-impact dynamics.

The results are presented theoretically with the necessary analytical expressions and explanations, as well as with numerous graphical presentations of the different phase portraits of the dynamics of rolling heavy thin disks. This methodology could be applied for the investigation of the numerous engineering vibro-impact system dynamics.

The full methodology is useful for investigation, not only for considering the vibro-impact system dynamics; it is a useful methodology for the application to other similar vibro-impact system dynamics of two or more rolling disks without slipping along an arbitrary curvilinear line stationary or rotating around a vertical axis or a skew positioned axis. This methodology could be applied for the investigation of numerous engineering vibro-impact system dynamics.

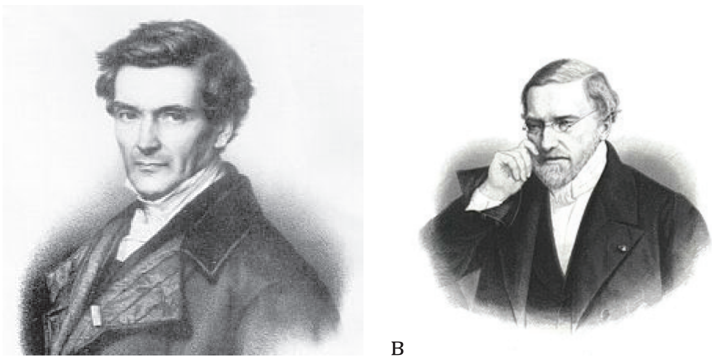
7. Mechanics of billiards – Geometry and kinematics

For an introduction to the content of billiards mechanics, which includes content on the geometry, kinematics and dynamics of playing billiards, it is most illustrative to rely on a few Coriolis' sentences, which we have adopted as the motto of this paper. Obviously, many authors believe that these sentences point to the complexity of geometry, kinematics and dynamics of billiards; because the analysis, which we will expose here, reveals that the dynamics of billiards includes many phenomena of the dynamics of real systems. Our presentation will be based on our results, but also on the comparisons with the results achieved today by other authors, both mechanics and mathematicians. Additionally, it is evident that our results are original.

Multiple keywords and concepts of kinematics and dynamics, such as: ball dynamics as a rigid body, non-slip rolling, collision, alternation of velocity direction, impact velocity, outgoing velocity, ball rolling path, center of gravity trajectory, central and skew (oblique) collision, force impulse, kinetic energy, impact and collision, collision of two spheres, collision of three spheres, impulse forces, linear momentum of motion, angular momentum of motion, speak about the complexity of dynamics of the system of billiards elements. This is evident when we observe only the movement of one billiards ball; when more balls are involved, then we have a really complex system, hybrid structures and dynamic configurations of billiards.

Let us start with the names of famous scientists who contributed to the knowledge of certain aspects of the dynamics of billiards.

As we were quoting a few sentences from Gaspard-Gustavo de Coriolis at the outset, the following part provides biographical information about this scientist whose work was essential in the field of mechanics. He was a mathematician, a mechanical engineer, and a scientist. He was best known for his work on the Coriolis acceleration and Coriolis force. Coriolis was the first to coin the term "work" for the product of force and distance. In 1829, Coriolis published a textbook, "*Calcul de l'Effet des Machines"*, which presented mechanics in a way that could easily be applied in industry. During this period, the true term for kinetic energy $\mathbf{E_k} = \frac{1}{2}mv^2$ was established, as well as its relation to mechanical work (see References [50, 51]).



A B

FIGURE 21. Gaspard-Gustave de Coriolis or Gustave Coriolis (Paris, May 21, 1792 - Paris, September 19, 1843) and Jean-Victor Poncelet (July 1, 1788 - December 22, 1867)

Coriolis explored the possibilities of generalizing kinetic energy and work on rotating systems and, as a result, produced the work "*Sur les équations du mouvement relatif des systèmes de corps"*, presented at the French Academy of Sciences (1832). Coriolis wrote the work "*Sur les équations du mouvement relatif des systèmes de corps"*, 1835. In the 20th century, the terms "*Coriolis acceleration"* and "*Coriolis force"* appeared on systems with the coupled transverse rotational motion and relative curvilinear motion.

Beginning with the works of George Birkoff, billiards systems have become a popular topic of study, drawing on a variety of fundamentals, beginning with the ergodic Mors's theory, KAM theory, and others. Also, the dynamics of billiards systems is interesting because it occurs quite naturally in a number of tasks of mechanics and physics: in the dynamics of vibro-impact systems, diffraction of short waves, dynamics of ball bearings, etc.

The basis of the dynamics of billiards is the theory of dynamics of systems with one-sided constraints. There are essentially various models of the theory of impact. The one-sided bond imposed on the system can be replaced by the field of conservative and dissipative forces, and then, the coefficients of elasticity and dissipation by some assumption to the aspirations of infinity, as Kozlov writes in [55]. It can then be shown that the movement of such a "*released"* system with fixed initial data, at each finite interval of time, tends to move with an impact.

Finally, it is necessary to emphasize again that in the approach to investigating the properties of the billiards game phenomenon, it is necessary to establish the basic models of billiards. These models, under the same keyword "*billiards"*, distinguish between defined tasks: geometry, kinematics, and the dynamics of billiards, or the totality of all these tasks. For example, starting tasks are about the properties of mathematical billiards. When approached by mathematicians, then it remains in the domain of billiards geometry and the elements are the geometric point and its possible open or closed polygonal paths, that is, polygons inscribed in a certain area by a unilaterally bounded closed contour line. Periodic trajectories are possible, depending on the initial position and the initial direction of the trajectory of the geometric point. Therefore, the basic determinations are the lengths and angles that determine the directions of the path of the geometric point. Furthermore, it works with lengths and angles, and the units of measurement are meters and degrees or radians. This is a rough approximation of the real billiards system and does not take into account the time at which the geometric point is moved along the trajectory. If the basic determination of time in addition to lengths and angles is included, then it moves into the field of kinematics, so kinetic parameters, elements of translation velocity and angular velocity of rolling are included.

If we stay only on the mathematical model that we enrich with basic determination over time, with a unit in seconds, then only the kinematic element of the geometric point translation rate, with the unit of meter per second, is included. A mass associated with a geometric point can be added to this model, so we have a model of gross abstraction of a real billiard by a material point that has mass, velocity, and its motion in time is observed. The mass has a unit in pounds. We have already included this in the model in addition to geometry and kinematics and dynamics, and with that we open the questions of determining the impulses of motion, kinetic energy and forces under which the dynamics are realized, including impacts at unilateral holding bonds and collisions between material points. However, for better abstraction of the billiard system to the model of billiard dynamics, it is not a satisfactory model neither with a geometric point nor with a material point, but with a rolling ball which has its mass, a certain mass distribution, defined by the axial moment of inertia of the masses for the axis of rolling of the ball, and has a certain instantaneous angular velocity of rolling about the instantaneous axis of rolling, measured in units of radian per second, or the velocity of translation of the center of mass in units of meters per second, and the angular velocity of rotation about the central eigen axis of self-rotation measured in units of radians per second. It means that if a billiards model is formed as an abstraction of a real model of billiards with one or more balls rolling, then the elements of the billiards events are studied as the elements of geometry, kinematics and dynamics of billiards.

Thus, there is a body of a certain shape and a line along which it rolls, so the definitions of length are used, so we are in the domain of geometry, and when we include the analysis of translational velocities of the center of mass and the angular velocity of rotation of the central self-rotation axis, then we are in the domain of kinematics. If we now include masses and axial moments of inertia of masses and impulses of motion and kinetic moment (angular momentum), kinetic energy and impulse forces, we completed the task of kinetics of billiards or the dynamics of the system of billiards. In the following chapters, we will first define the models of billiards and then analyze the elements of geometry, kinematics and dynamics of billiards within the limits of contemporary knowledge of the scientific literature in this field, as well as the original results of the authors of this chapter.

7.1. Billiard geometry in a nutshell. In this section, we also highlight the contributions of the French scientist, engineer and mathematician, Jean-Victor Poncelet (July 1, 1788 - December 22, 1867), who was known for the following works: "*Traité des propriétés projectives des figures"* (1822) and "*Introduction à la mécanique industrielle* (1829)". And here, we will point out a number of his theorems in geometry, which are of great importance for the investigation of the geometry of billiards and for determining the periodic paths of the rolling of billiards balls in elliptic billiards, as well as other forms of contours in billiards (see Reference [2]).

It is certainly important to find out the properties of ideal mathematical billiards. Today, many researches on this topic are based on a series of basic theorems of Jean-Victor Poncelet, so here we will list the definitions of some of them (see Reference [1]).

Theorem 1. (Poncelet Theorem): Consider two conical sections (conics) C and D, which lie in the plane. Suppose that a polygon is inscribed in a conical section C, and described around a conical section (conics) D. Then there is an infinite number of such polygons inscribed in one conical section, and described around the other conical section, all of which have an equal number of sides. Moreover, every point of the conical intersection C is the subject of such a broken line.

Theorem 2. (Poncelet Theorem): Suppose that there is such a point on the conical section (conic) Γ that the polygon $\mathbf{T}_0\mathbf{T}_1\mathbf{T}_2\mathbf{T}_3\mathbf{T}_4\ldots\mathbf{T}_{n-2}\mathbf{T}_0\mathbf{T}_{n-1}\mathbf{T}_0$ with the number of sides $n \geq 3$ is inscribed in the conic section (conic) Γ and described around the conic section Γ_u and that the right Γ_u does not tangent the conic section (conic) Γ_u for $i = 2,3,3,\ldots,n-2,n-1$. Then for arbitrary $n \geq 3$ there is some polygon $\mathbf{N}_0\mathbf{N}_1\mathbf{N}_2\mathbf{N}_3\mathbf{N}_4\ldots\mathbf{N}_{n-2}\mathbf{N}_0\mathbf{N}_{n-1}\mathbf{N}_0$ with sides inscribed in a conical section Γ_u and described around a conical section Γ_u.

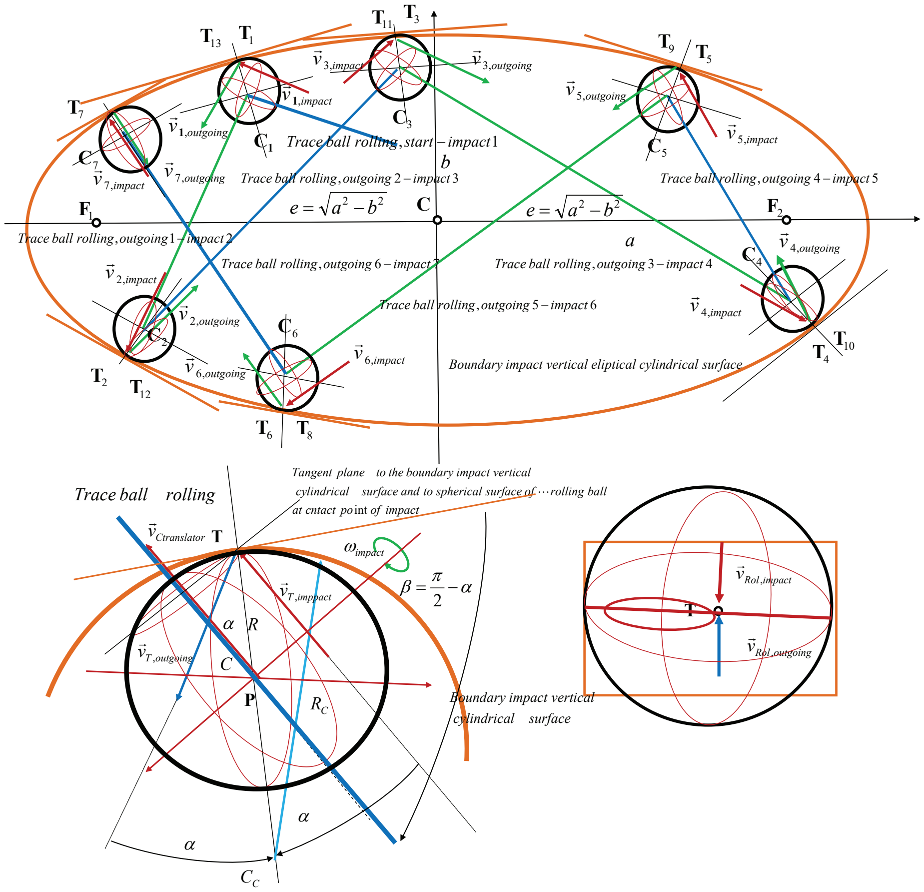


FIGURE 22. Line-trace of the rolling motion of the ball along the horizontal plane of the elliptical billiards with the plan of horizontal components of the incoming impact velocities and the outgoing velocities of the contact point after the impact of the ball into a contour elliptical cylindrical surface with vertical derivatives (up); Detail of the plan of the velocity of one ball in the configuration of the skew impact (down)

THEOREM 3. (Poncelet area theorem): Suppose that C and C are two areas in space (on the surface). If there is some closed polygon, inscribed in the area C and described around the area D, it means that there are infinitely many polygons such as that one. In addition, each point of the area C appears as the theme of such a polygon and all polygons have an equal number of sides.

One of the numerous Poncelet theorems is the Comprehensive Generalized Poncelet Theorem, but we don't preset it, because in our opinion the previous three listed theorems are a good illustration of the content and aims of these series of Poncelet theorems for applications in the geometry of models of billiards as an abstraction of real billiards into mathematical billiards, as pre geometrical objects without reality (for more details see mathematical references [2]).

Figure 22 shows the mechanical model as a good abstraction of real elliptical billiards. The track of the ball rolling along the horizontal plane of the elliptical billiards with the plan of horizontal components of the incoming impact velocities and outgoing velocities after the impact of the ball into a contour of elliptical cylindrical surface with vertical derivatives, as well as a detail plan of the velocity of one ball skew impacts.

7.2. Elements of billiards kinematics. In Figure 22, we have presented a model of an elliptical billiard with one rolling ball, which rolls on a horizontal surface bounded by a single one-side contour elliptical-cylindrical surface with vertical derivatives. The billiards balls are of the same dimensions and radius R, which is not negligible with respect to the semi-axes of ellipse in the horizontal flat base of the elliptical billiards, which we denote by a and b, which, according to the semi-axes, has an eccentricity of focus $e=\sqrt{a^2-b^2}$. With this in mind, the model of elliptical billiards in Figure 22 cannot be reduced to mathematical billiards by reducing the billiard sphere to a material or geometric point. Her line-trace of a billiard ball rolling, which is a broken polygonal line, open or closed, is a polygon, and its path can be determined purely geometrically using Poncelet theorems, or other results in the field of mathematical billiard models from the literature, which remain in the domain of pure geometry. It is necessary to first determine the geometric location of the points of ball centers of the impacts in relation to the contour of the elliptic area of billiards, and to determine the paths of rolling the ball centers in the new contour.

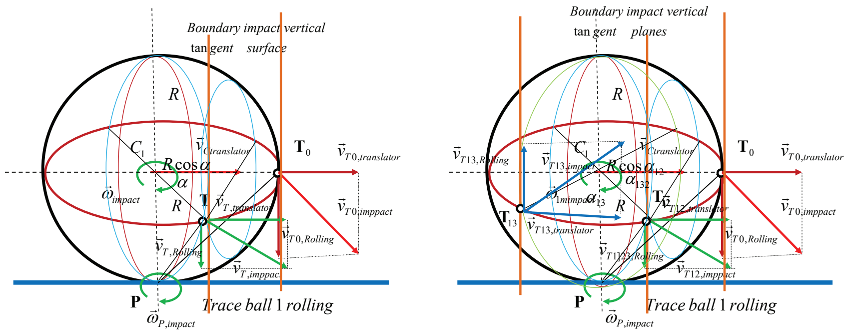


FIGURE 23. Plan of the velocity of impact of a point on a large circle of a sphere, which rolls along a straight line in a horizontal plane

In the observed model of the elliptical billiard from the case in Figure 22, we can see that the parts of the track of the roll of the billiard ball in arriving at the point of impact (the ball and contour), and leaving it after being hit by the contour elliptical-cylindrical surface are parallel to the corresponding horizontal velocity component just before the impact and by the corresponding horizontal component of the outgoing velocity of the point of impact of the ball after impact. The translational velocities of the center of mass of the sphere, before and after impact in the contour surface, are equal to the horizontal components of the incoming velocity before impact and the outgoing velocity after the collision, the point at which the sphere impact to the contour elliptical-cylindrical surface (see Reference [30]).

We will, now, turn to the representation of the kinematic elements of each of the possible impacts of a ball in rolling, to the contour surface, as well as to the kinematic elements of different cases of mutual collision of two balls.

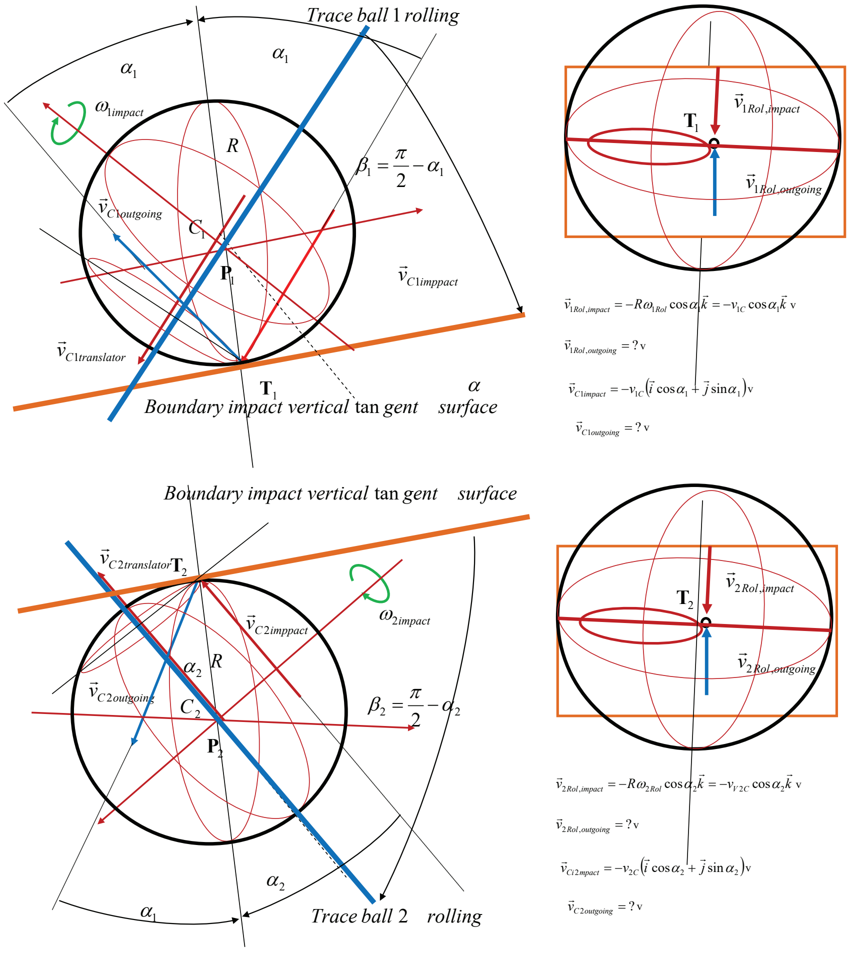


FIGURE 24. Decomposed system of two equal billiard balls in configuration of an oblique (skew) collision. Plan of the incoming and outgoing velocities of the collision contact points of each of the two balls at their oblique collision.

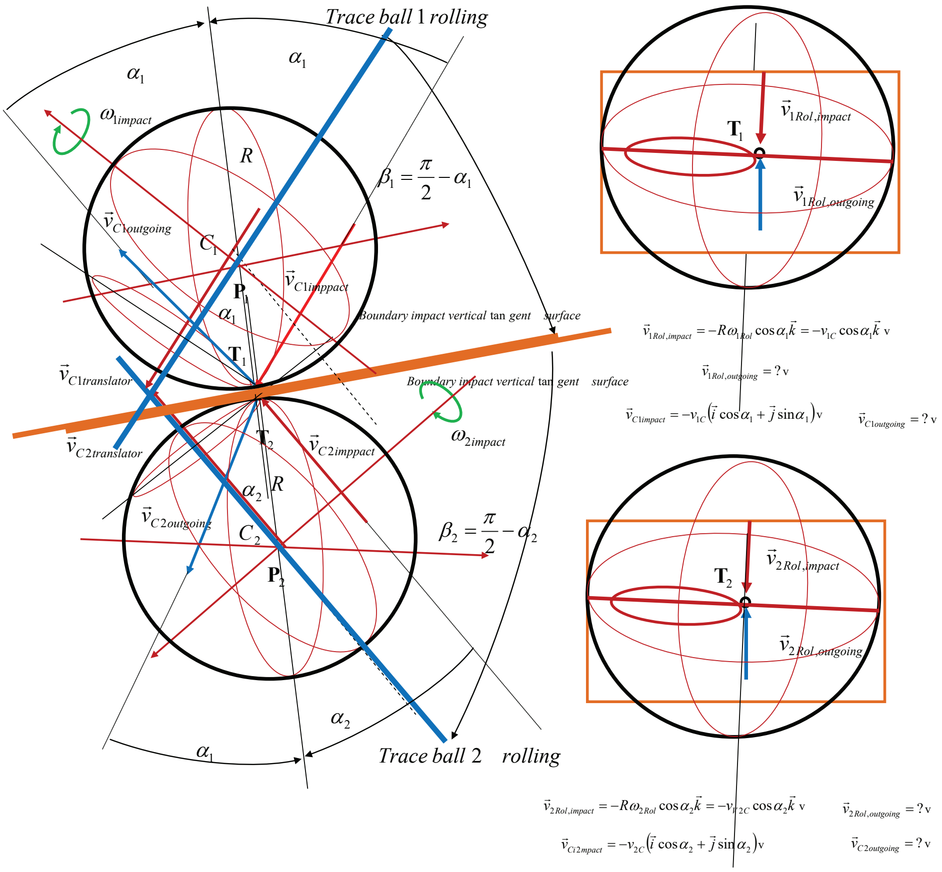


FIGURE 25. A system of two equal billiards balls that roll on rolling tracks in an oblique (skew) collision configuration. The plan of the incoming and outgoing velocities of the collision points of each of the two balls at their oblique collision

7.2.1. An oblique collision of two billiard balls in a non-slip roll. If the rolling traces of two billiard balls on rolling without slipping intersect and the balls simultaneously reach the position so that they collide, then such a collision is said to be an oblique collision of two balls. Figures 24 and 25 show the plans of the component incoming $\vec{v}_{C1,impact}$ and $\vec{v}_{C2,impact}$, as well as outgoing $\vec{v}_{C1,outgoing}$ and $\vec{v}_{C2,outgoing}$ velocities of the ball center mass $\mathbf{C}_1$ and $\mathbf{C}_2$ of an oblique collision of two billiards balls rotating horizontally relative to a common vertical tangent plane at a common point of collision.

Their velocities $\vec{v}_{C1,impact}$ and $\vec{v}_{C2,impact}$ of the centers of mass $\mathbf{C}_1$ and $\mathbf{C}_2$ are non-collinear and we are using the collision model decomposition to subsystems presented in Figure 24. Figure 25 shows a plan of the component incoming $\vec{v}_{C1,impact}$ and $\vec{v}_{C2,impact}$ and of the component outgoing $\vec{v}_{C1,outgoing}$ and $\vec{v}_{C2,outgoing}$ velocities of an oblique collision of two billiards balls rolling along a horizontal plane, relative to a common vertical tangent plane at a common point of oblique collision when their translational velocities of their centers $\mathbf{C}_2$ and $\mathbf{C}_2$ of ball mass are noncollinear, using a complex system model .

We ask how many such positions in the collision configuration can be insulated and how to define using the distance of the centers of mass of the two balls. Here is the conclusion of the theorem.

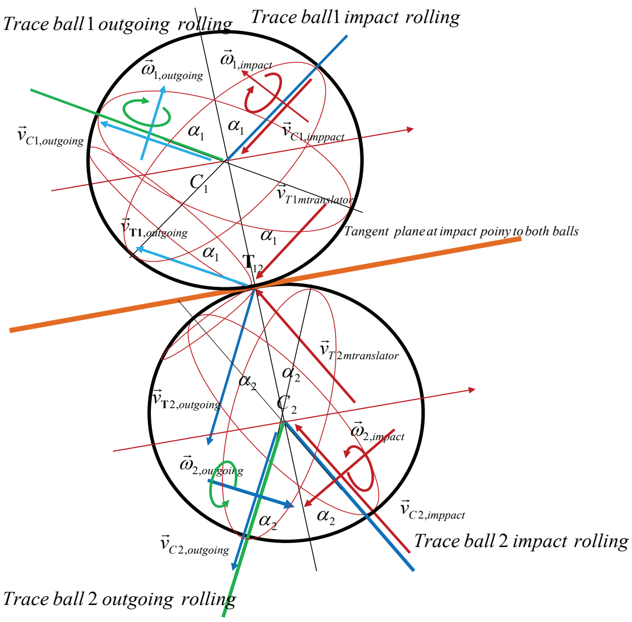


FIGURE 26. Collision of two rolling balls and the plan of incoming and outgoing angular velocities of the rolling balls with their rolling paths before and after an oblique collision

7.2.2. The theorem of the feasibility of the collision of two billiards balls. *Two billiard balls of equal radius, in rolling without slipping, and rolling along the intersecting tracks, can reach the configuration of the collision, and the realization of the same, only if their centers* $\mathbf{C}_2$ *and* $\mathbf{C}_2$ *of mass are located at the corresponding distance* $\overline{\mathbf{C}_1\mathbf{C}_2} = 2R$ *at points, which are at the translational velocities of their centers of mass* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *are directed to the point of intersection of their rolling paths. There are infinitely many such configurations of the collision feasibility of two balls. A collision is possible before both balls pass the point of intersection of the tracks, and also on condition that one has not reached the position of the section of the tracks and that the other ball has passed through that section of the tracks so that the intensities of the speeds allow the first round to cross the position of cross-sections of the tracks.*

We shall now give an analysis and explanation of the plan of the component velocities $\vec{v}_{\mathbf{T}1,hor,outgoing}$ and $\vec{v}_{\mathbf{T}1,hor,outgoing}$, as well as $\vec{v}_{\mathbf{T}1,Rol,outgoing}$ and $\vec{v}_{\mathbf{T}2,Rol,outgoing}$ of the points $\mathbf{T}_1$ and $\mathbf{T}_2$ of the balls in which the balls collide, using Figures 26 and

27 respectively. If there was a collision through the points $\mathbf{T}_1$ and $\mathbf{T}_2$ at the collision point $\mathbf{T}_1 \equiv \mathbf{T}_2 \equiv \mathbf{T}_{12}$, then the two balls set up an imaginary common tangent plane, which is tangent to both sphere surfaces at the point of their collision $\mathbf{T}_1 \equiv \mathbf{T}_2 \equiv \mathbf{T}_{12}$. Let us set the coordinate system so that the coordinate axis x is in their common tangent plane through the collision point, and the axis y in the normal direction to that tangent plane and passes through both centers of the mass $\mathbf{C}_2$ *and* $\mathbf{C}_2$ of the billiards balls in the collision configuration, while the axis z is in the vertical direction in the tangent plane in which the tracks of rolling balls lie and are pulled through the collision point of billiards balls (for details see Reference [30]).

The unit vectors of orientation of the direction and focusing of the incoming ball rolling paths, just before the oblique collision, are defined by the unit orientation vectors:

$$\vec{n}_{1,impact} = -\left(\vec{i}\sin\alpha_1 + \vec{j}\cos\alpha_1\right), \text{ and } \quad \vec{n}_{2,impact} = -\vec{i}\sin\alpha_2 + \vec{j}\cos\alpha_2 . \tag{104}$$

in which the angles α_1 and α_2 of the rolling path of the ball are locked by the direction of the normal to the tangent plane to the sphere surfaces of the balls at the point of their collision, that is, by the direction of the line passing through the centers $\mathbf{C}_1$ and $\mathbf{C}_2$ of mass and both balls.

The components of the incoming velocities of the balls in rolling before the collision are:

The translation velocities $\vec{v}_{C1,impact}$ and $\vec{v}_{C2,impact}$ of the centers of mass $\mathbf{C}_1$ and $\mathbf{C}_2$, or of the balls just before their collision, are:

$$\vec{v}_{C1,impact} = v_{C1}\vec{n}_{1,impact} = -v_{C1}\left(\vec{i}\sin\alpha_1 + \vec{j}\cos\alpha_1\right),$$

and

$$\vec{v}_{C2,impact} = v_{C2}\vec{n}_{2,impact} = v_{C2}\left(-\vec{i}\sin\alpha_2 + \vec{j}\cos\alpha_2\right) \tag{105}$$

The instantaneous angular velocities $\vec{\omega}_{P1,impact}$ and $\vec{\omega}_{P2,impact}$ of rolling due to the rolling of balls on the corresponding incoming traces immediately before their collision are:

$$\vec{\omega}_{1,impact} = \vec{\omega}_{P1,impact} = \omega_{P1,impact}\left(\vec{i}\cos\alpha_1 + \vec{j}\sin\alpha_1\right) = \frac{v_{C1,impact}}{R}\left(\vec{i}\cos\alpha_1 + \vec{j}\sin\alpha_1\right)$$

$$\vec{\omega}_{2,impact} = \vec{\omega}_{P2,impact} = \omega_{P2,impact}\left(\vec{i}\cos\alpha_2 - \vec{j}\sin\alpha_2\right) = \frac{v_{C2,impact}}{R}\left(\vec{i}\cos\alpha_2 - \vec{j}\sin\alpha_2\right) \tag{106}$$

The horizontal components $\vec{v}_{\mathbf{T}1,hor,impact}$ and $\vec{v}_{\mathbf{T}2,hor,impact}$ of the velocities of the points $\mathbf{T}_1$ and $\mathbf{T}_2$ that the balls collide which are equal to the velocities of the corresponding centers $\mathbf{C}_2$, and $\mathbf{C}_2$ of mass, of the corresponding balls:

$$\vec{v}_{\mathbf{T}1,hor,impact} = \vec{v}_{C1,impact} = v_{C1}\vec{n}_{1,impact} = -v_{C1}\left(\vec{i}\sin\alpha_1 + \vec{j}\cos\alpha_1\right)$$

and

$$\vec{v}_{\mathbf{T}2,hor,impact} = \vec{v}_{C2,impact} = v_{C2}\vec{n}_{2,impact} = v_{C2}\left(-\vec{i}\sin\alpha_2 + \vec{j}\cos\alpha_2\right) \tag{107}$$

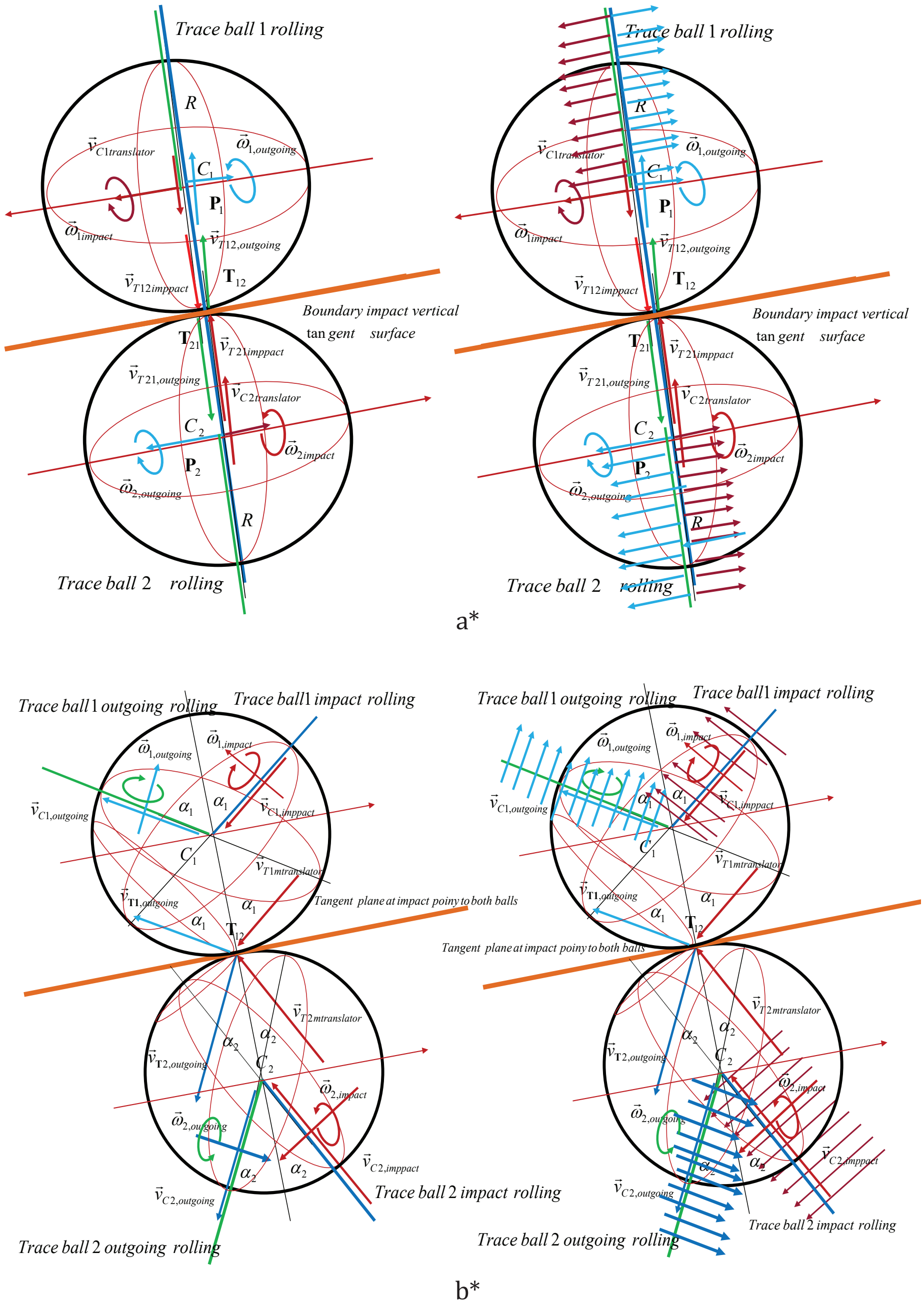


FIGURE 27. Partial analogies between the central and oblique collisions of two rolling balls: the opening of the traces of ball rolling in the oblique (skew) collision (b *) with respect to the central collision of the balls (a *).

Vertical components $\vec{v}_{\mathbf{T}1,Rol,impact}$ and $\vec{v}_{\mathbf{T}2,Rol,impact}$ of velocities of the contact points $\mathbf{T}_1$ and $\mathbf{T}_2$ of the balls that collide with the balls are the velocities due to the rotation of the balls along the horizontal track by the incoming angular velocities and are defined as:

$$\vec{v}_{\mathbf{T}1,Rol,impact} = -R\omega_{\mathbf{P}1,Rol}\vec{k} = -v_{\mathbf{C}1,impact}\vec{k}$$

and

$$\vec{v}_{\mathbf{T}2,Rol,impact} = -R\omega_{\mathbf{P}2,Rol}\vec{k} = -v_{\mathbf{C}2,impact}\vec{k} \quad (108)$$

The horizontal components $\vec{v}_{\mathbf{T}1,hor,ougoing}$ and $\vec{v}_{\mathbf{T}2,hor,outgoing}$ of the velocities of contact points $\mathbf{T}_1$ and $\mathbf{T}_2$, with which each of the balls leaves after the mutual collision are:

$$\vec{v}_{\mathbf{T}1,hor,ougoing} = \vec{v}_{C1,outgoing} = v_{C1}\vec{n}_{1,outgoing} = v_{C1}\left(-\vec{i}\sin\alpha_1 + \vec{j}\cos\alpha_1\right)$$

and

$$\vec{v}_{\mathbf{T}2,hor,outgoing} = \vec{v}_{C2,outgoing} = v_{C2}\vec{n}_{2,outgoing} = -v_{C2}\left(\vec{i}\sin\alpha_2 + \vec{j}\cos\alpha_2\right) \quad (109)$$

The outgoing velocities $\vec{v}_{C1,outgoing}$ and $\vec{v}_{C2,outgoing}$ of translation of the centers $\mathbf{C}_1$ and $\mathbf{C}_2$ of mass, or the corresponding sphere, immediately after the mutual collision of the billiards balls are equal to the corresponding horizontal component $\vec{v}_{\mathbf{T}1,hor,ougoing}$ and $\vec{v}_{\mathbf{T}2,hor,outgoing}$ of the outgoing velocity of the corresponding contact points $\mathbf{T}_1$ and $\mathbf{T}_2$ sphere after the collision:

$$\vec{v}_{C1,outgoing} = v_{C1}\vec{n}_{1,outgoing} = \vec{v}_{\mathbf{T}1,hor,ougoing} = v_{\mathbf{T}1,hor,ougoing}\left(-\vec{i}\sin\alpha_1 + \vec{j}\cos\alpha_1\right)$$

and

$$\vec{v}_{C2,outgoing} = v_{C2}\vec{n}_{2,outgoing} = \vec{v}_{\mathbf{T}2,hor,outgoing} = -v_{\mathbf{T}2,hor,outgoing}\left(\vec{i}\sin\alpha_2 + \vec{j}\cos\alpha_2\right) \quad (110)$$

The instantaneous angular velocities $\vec{\omega}_{\mathbf{P}1,outgoing}$ and $\vec{\omega}_{\mathbf{P}1o,utgoing}$ of the rolling and due to the outgoing rolling of each of the balls per track immediately after the collision are:

$$\vec{\omega}_{1,outgoing} = \vec{\omega}_{\mathbf{P}1,outgoing} = \omega_{P1,outgoing}\left(\vec{i}\cos\alpha_1 + \vec{j}\sin\alpha_1\right) = \frac{v_{\mathbf{T}1,hor,outgoing}}{R}\left(\vec{i}\cos\alpha_1 + \vec{j}\sin\alpha_1\right)$$

$$\vec{\omega}_{2,outgoing} = \vec{\omega}_{\mathbf{P}2,outgoing} = \omega_{P2,outgoing}\left(-\vec{i}\cos\alpha_2 + \vec{j}\sin\alpha_2\right)$$

$$= \frac{v_{\mathbf{T}2,hor,outgoing}}{R}\left(-\vec{i}\cos\alpha_2 + \vec{j}\sin\alpha_2\right) \quad (111)$$

The vertical component $\vec{v}_{\mathbf{T}1,Rol,outgoig}$ and $\vec{v}_{\mathbf{T}2,Rol,outgoig}$ of the departure-outgoing velocities of the contact points $\mathbf{T}_1$ and $\mathbf{T}_2$ of balls, by which each of the balls leaves after each other's oblique collision, is the cause of the outgoing rolling of each of the balls along the horizontal track at angular velocity, $\vec{\omega}_{P1,outgoing}$ respectively $\vec{\omega}_{P2,outgoing}$, and are determined by the following expressions:

$$\vec{v}_{\mathbf{T}1,Rol,outgoing} = R\omega_{\mathbf{P}1,outgoing}\cos\alpha_1\vec{k} = v_{\mathbf{T}1,impact}\cos\alpha_1\vec{k}$$

$$\vec{v}_{\mathbf{T}2,Rol,outgoing} = R\omega_{\mathbf{P}2,outgoing}\cos\alpha_2\vec{k} = v_{\mathbf{T}2,impact}\cos\alpha_2\vec{k} \qquad (112)$$

The unit vectors of orientation of the direction and focusing of the outgoing traces of the outgoing roll of the balls in the outlet after the collision are:

$$\vec{n}_{1,outgoing} = -\vec{i}\sin\alpha_1 + \vec{j}\cos\alpha_1 ,$$

respectively

$$\vec{n}_{2,outgoing} = -\vec{i}\sin\alpha_2 + \vec{j}\cos\alpha_2 \qquad (113)$$

For next details see Reference [30].

Conclusion-Theorem: *If the incoming paths of the balls in the rolling equal angles with the common tangent plane to the balls at the contact points of collision of the balls, then the horizontal components of the outgoing velocities of those points close the same angles with that tangent plane. Also, the outgoing paths of one or the other ball, as well as the translational velocities of the centers of mass of the ball, close the same angles with this common tangent plane of collision.*

7.2.3. Billiard-Concluding considerations. Using the analysis of the elements of geometry, kinematics and dynamics of impact (see Figure 24.) and the collision of balls (see Figures 25, 26 and 27) given in the previous parts of this paper, it is possible to draw some conclusions and generalizations.

Here, we can imagine that the route-traces of the rolling of both balls, by which the balls are rolling "*corrected*" into straight lines and that each ball is rolling along the corresponding "*corrected straight route*" until the configuration which is impacted by an appropriate incoming angular velocity, and from the configuration of the collision by the outgoing angular velocities along *"broken traces with a break",* as if a central collision had occurred, but we should take into account that at the point of collision on the route a "*fracture*" of the route occurs and that the angular velocity discontinuity occurs with the alternative of the angular velocity direction. The same goes for the other ball. This practically means that there is *a jump in the intensity of the angular velocity of rolling in the kinetic state of the collision*, and that the "*breaking*" *of the rolling path changes and the direction of the instantaneous angular velocity of the roll-out after the collision changes*. The "fracture of the rolling path" depends on the angle at which the rolling path comes from the normal tangent plane to the balls at the point of impact.

This consideration takes us back to the central collision of the two balls. Therefore, the task comes down to a central collision of two balls from which we need to determine the intensities of the translational outgoing velocities and the outgoing angular velocities of rolling about the current axis, or around it parallel to its own central axis of each of the balls; then the "*track legs expand*" as corner angles with the apex in the point of collision configuration and set at angles with respect to the normal of the tangent plane at the collision-contact points of the rolling trace of balls, depending on the angles that the incoming ball rolling paths close to that normal. This leads us to the conclusion that it is also possible to use a central collision to determine the intensities of the angular velocity of rolling and to change the angular momentum of motion, and in the case of oblique collision of rolling balls of different radii and masses use appropriate analogies [27, 28, 66-68]. It should be emphasized that it is necessary to determine the outgoing rolling routes after a collision taking into account the incoming routes.

In the dynamics of billiards, there is a simultaneous collision of several balls, so the problem of determining the directions of the components of the outgoing velocities after the collision is not difficult to determine, but even today the question of determining the intensity of the individual components of the outgoing angular velocities of rolling formed in the collision of several balls is open.

8. Concluding Remarks

The elements of geometry, kinematics and dynamics of rolling homogeneous balls along curvilinear lines are defined (see Reference [30]). The complete Hedrih's theory (see References [15, 19, 24, 25, 27, 28, 30]) of the impact and collision of heavy rolling balls, through geometry, kinematics and dynamics of rolling balls, is defined (see References [15, 19, 24, 25, 27, 28, 30]).

A new definition of the coefficient of restitution (collision) was introduced, starting from the hypothesis of the conservation of the sum of angular momentum of the balls in rolling, for instant rolling axes, after the collision in relation to the time before collision of the bodies. The expressions for the outgoing angular velocities of the ball rolling after the collision have been derived and their rolling paths after the impact or collision have been determined and various possible anchors have been shown.

The difference between the content of the term billiards used in mathematical works of many mathematicians, as well as the research that remains in the field of geometry is pointed out. Our theory of ball rolling and collision is based on the examples of the abstraction of real rolling systems of heavy homogeneous billiards to a mechanical model.

Based on both of the new Hedrih's results (see References [15, 19, 24, 25, 27, 28, 30]), the theory of collision between rolling bodies and dynamics of generalized rolling pendulums (see References [6, 7, 14, 16, 18, 24, 26, 27, 28, 30, 31]) in successive collisions, and the use of phase trajectory method, a new methodology of vibro-impact dynamics investigation is founded and presented through a number of applications in mechanical system dynamics.

We must point out again that the elements of geometry, kinematics and dynamics of rolling homogeneous balls along curvilinear lines are defined (see Reference accepted for ICTAM 2020+1 and Reference [30] accepted and published in EURODYN 2020 Proceedings). The complete theory of the impact and collision of heavy rolling balls, through geometry, kinematics and dynamics of rolling balls, is defined by Hedrih (Stevanović).

A new definition of the coefficient of restitution (collision) was introduced and expressed by angular velocities of the rolling about instantaneous axis of each body after and before collision. Starting from the hypothesis of the conservation of the sum of angular momentum of the balls in rolling, for corresponding instant rolling axes, after the collision in relation to the before collision of the bodies, theory of collision obtained the basic foundation. This hypothesis is the main foundation of the new theory of collision between bodies in rolling, different from the known hypothesis of the conservation of the sum of linear momentum in the classical theory of collision between bodies in translatory motion before and after collision.

The new expressions for the outgoing angular velocities of the ball rolling after the collision have been derived and determined on the basis of the newly introduced coefficient of restitution and the hypothesis of the conservation of the sum of angular momentum. Also, the rolling paths after the impact or collision of each rolling ball have been determined and various possible anchors have been shown. The difference between the content of the term billiards used in mathematical works [2] of many mathematicians, as well as the research that remains in the field of geometry is pointed out. These results come down to the task of inscribing open or closed polygonal lines in some restricted areas, and annals are with tasks in optics, exploring the path of the light beam which is reflected off mirrors at the boundaries defined by the regions. They are based on a series of Poncelet theorems in geometry and do not reach the dynamics of the real billiards systems.

Our theory of ball rolling and collision is based on the examples of the abstraction of real rolling systems of heavy homogeneous billiards to the dynamics of a mechanical model.

Construction of the phase trajectory portraits of a generalized rolling pendulum along a rotating curvilinear line is presented in the following References [8-33]. The generalized rolling pendulum containing a rolling thin heavy disk rotates along the curvilinear line consisting of three circle arches, with constant angular velocity around a vertical eccentric/central axis. Depending on system parameters, different possible forms of the phase portraits appear with different structures of the sets of singular points and forms of phase trajectories. Trigger of coupled singular points and a homoclinic orbit in the form of a deformed figure eight appears (see References [6-14]). A mathematical analogy (see References [3, 27, 28, 66-68]) between nonlinear differential equations of the considered generalized rolling pendulum and motion of the heavy mass particle along a rotating curvilinear line points out the same forms. On the basis of the obtained different possible phase trajectory portraits, non-linear phenomena in vibro-impact dynamics of two rolling thin disks on a rotating curvilinear line is investigated. Energy transfer between rolling disks in each of the series of successive collisions is analyzed and presented on relative mechanical energy portraits for the dynamics of each of the rolling disks in collision (see Reference [14, 19, 21, 22, 23, 27, 28, 29, 32]).

By using the phase plane method, the non-linear phenomena in the dynamics of vibro-impact system containing two rolling bodies along a rotating circle, or along a different stationary curvilinear line is investigated and the results are presented in author's References. The newly established Hedrih's theory of the collision between two rolling bodies was published by the author in References [15, 30] and applied in References [6-32]. Two generalized rolling pendulums [6, 7, 10-32] along the same curvilinear trace are the main sub-systems of each of previously investigated nonlinear dynamics of the vibro-impact systems. In previously listed published papers, energy analysis is done and the successive energy jumps, between two rolling bodies in successive collisions, are indicated.

In a series of the co-authored papers (see References [32-39]), the research attention is focused to nonlinear phenomena and energy analysis in the dynamics of the vibro-impact systems containing two heavy mass particles moving along curvilinear rough lines with Amontons-Coulomb's frictions and in successive collisions.

In this review paper, the research attention is focused and based on the newly obtained Hedrih's results, the theory of collision between rolling bodies and the dynamics of generalized rolling pendulums in successive collisions, and the use of phase trajectory method; a new methodology of vibro-impact dynamics investigation is founded and presented through a number of previously published applications in mechanical vibro-impact system dynamics as well as the research results of vibro-impact dynamics in general through six Projects (P4-P9) and in the period of five project cycles in the period 1991-2019.

References

[1] A.A. Andronov, A. A. Vitt, S. Haykin, *Teoriya kolebaniy* (*Theory of oscillations*), Nauka, Moskva, 1981.

[2] В. Драгович, М. Раднович, *Интегрируемые бильярды, квадрики и многомерные поризмы Понселе*, Издательство РХД, Ижевск, 2010.

[3] K.R. (Stevanović) Hedrih, I. Kosenko, P. Krasilnikov, P.D. Spanos (eds.), *Elements of mathematical phenomenology and phenomenological mapping*, International Journal of Non-linear Mechanics **73** (2015), https://www.sciencedirect.com/journal/international-journal-of-non-linear-mechanics/vol/73.

[4] J. Guckenheimer, Ph. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Fields*, Springer-Verlag, 1983.

[5] J. Gerard, J. Daniel, *Elementary Stability and BIfurcation Theory*, Springer Verlag, 1980.

[6] K.R. (Stevanović) Hedrih, *Rolling Heavy Ball Over the Surface with Arbitrary Shape in Real Rn3 Space*, In: J. Awrejcewicz (ed.), *Perspectives in Dynamical Systems I: Mechatronics and Life Sciences. DSTA 2019*, Springer Proceedings in Mathematics & Statistics **362**, Springer, Cham. https://doi.org/10.1007/978-3-030-77306-9_22.

[7] K.R. (Stevanović) Hedrih, *Bifurcation and triggers of coupled singularities in the dynamics of generalized rolling pendulums*, In: W. Lacarbonara et al. (eds.), *Advances in Nonlinear Dynamics*, NODYCON Conference Proceedings Series (2022), 361-371.

[8] K.R. (Stevanović) Hedrih: *Trigger of Coupled Singularities,* In: J. Awrejcewicz, J. Grabski, J. Nowakowski (eds.), *Dynamical Systems-Theory and Applications*, Łódź, Poland, 51-78, 2001.

[9] K.R. (Stevanović) Hedrih, *A Trigger of Coupled Singularities,* Meccanica **39**(3) (2004), 295-314. DOI: 10.1023/B:MECC.0000022994.81090.5f.

[10] K.R. (Stevanović) Hedrih, *Existence of trigger of coupled singularities in nonlinear dynamics of mechanical system with coupled rotations*, Scientific Technical Review **62**(1) (2012), 48-56.

[11] K.R. (Stevanović) Hedrih, *Vibrations of a Heavy Mass Particle Moving along a Rough Line with Friction of Coulomb Type*, International Journal of Nonlinear Sciences and Numerical Simulation **11**(3) (2010), 203-210.

[12] K.R. (Stevanović) Hedrih, *Vibro-impact dynamics in systems with trigger of coupled three singular points: Collision of two rolling bodies.* In: J.M. Floryan (ed.), *Contributions to the Foundations of Multidisciplinary Research in Mechanics* **1** (2017), 212-213, (ICTAM 2016), Montreal, Canada, 2016, ISBN 978-0-660-05459-9.

[13] K.R. (Stevanović) Hedrih, *Nonlinear Dynamics of a Heavy Material Particle along a Circle Which Rotates and Optimal Control*, Solid Mechanics and its Applications **122** (2003), 37-45.

[14] K.R. (Stevanović) Hedrih, *Generalized rolling pendulum along curvilinear trace: Phase portrait, singular points and total mechanical energy surface*, In: A. Prokopenya, A. Gil-Swiderska (eds.), *Computer Algebra Systems in Teaching and Resea*rch **VI** (2017), 204-216, CASTR'2016, Siedlce University of Natural Sciences and Humanities, Siedlce, Poland.

[15] K.R. (Stevanović) Hedrih, *Central collision of two rolling balls: theory and examples*, Advances in Theoretical and Applied Mechanics **10**(1) (2017), 33-79. https://doi.org/10.12988/atam.2017.765.

[16] K.R. (Stevanović) Hedrih, *Rolling heavy disk along rotating circle with constant angular velocity,* In: A. N. Prokopenya, M. Jakubiak (eds.), *Computer Algebra Systems in Teaching and Research* **V** (2015), 293-304, CASTR'2015, Siedlce University of Natural Sciences and Humanities, Siedlce, Poland. ISBN: 978-83-7051-779-3.

[17] K.R. (Stevanović) Hedrih, *On-linear phenomena in vibro-impact dynamics: Central collision and energy jumps between two rolling bodies*, Nonlinear Dynamics **91**(3) (2018), 1885–1907. DOI: 10.1007/s11071-017-3988-x.

[18] K.R. (Stevanović) Hedrih, *Rolling heavy ball over the sphere in real Rn3 space***,** Nonlinear Dynamics **97** (2019), 63–82. https://doi.org/10.1007/s11071-019-04947-1.

[19] K.R. (Stevanović) Hedrih, *Vibro-impact dynamics of two rolling heavy thin disks along rotate curvilinear line and energy analysis*, Nonlinear Dynamics, **98**(4) (2019), 2551-2579.

[20] K.R. (Stevanović) Hedrih, *Discontinuity of kinetic parameter properties in nonlinear dynamics of mechanical systems*, In: *Proceedings of the 9th Brazilian Conference on Dynamics Control and their Applications*, Serra Negra, SP, Brazil, 8-40, 2010.

[21] K.R. (Stevanović) Hedrih, *Nonlinear Dynamics of the Gyrorotor in the Field with Turbulent Damping*, Nauka Tehnika Bezbednost **1** (2001), 109-127.

[22] K.R. (Stevanović) Hedrih, *Energy analysis in a nonlinear hybrid system containing linear and nonlinear subsystems coupled by hereditary element*, Nonlinear Dynamics **51**(1-2) (2008), 127-140. DOI: 10.1007/s11071-007-9197-2.

[23] K.R. (Stevanović) Hedrih, *Energy and Nonlinear Dynamics of Hybrid Systems* In: A. Luo, J. Machado, D. Baleanu (eds.), *Dynamical Systems and Methods*, Springer, 29-83, 2012. DOI: 10.1007/978-1-4614-0454-5_2.

[24] K.R. (Stevanović) Hedrih, *Vibro-impact dynamics of two rolling balls along curvilinear trace*, Procedia Engineering **199** (2017), 663-668.

[25] K.R. (Stevanović) Hedrih, *Dynamics of Impacts and Collisions of the Rolling Balls,* In: *Dynamical Systems: Theoretical and Experimental Analysis,* Springer Proceedings in Mathematics & Statistics **182** (2017), 157-168.

[26] K.R. (Stevanović) Hedrih, *Dynamics of a rolling heavy thin disk along rotate curvilinear trace on vertical plane about vertical axis*, Proceedings, of the 7th International Congress of the Serbian Society of Mechanics, Sremski Karlovci, Serbia, 96-97, 2019.

[27] K.R. (Stevanović) Hedrih, *Elements of mathematical phenomenology: I. Mathematical and qualitative analogies*, Труды МАИ **84**(42) (2015), 1-42.

[28] K.R. (Stevanović) Hedrih, *Elements of mathematical phenomenology: II. Phenomenological approximate mappings*, Труды МАИ **84**(29) (2015), 1-29.

[29] K.R. (Stevanović) Hedrih, *Energy analysis and methodology of impact dynamics investigation of system with rolling bodies (Енергетичний аналіз та методологія дослідження динаміки віброударної системи з тілами кочення*), In: *Proceedings 14th International Symposium of Ukrainian Mechanical Engineers*, Lviv**,** 51-52, 2019.

[30] K.R. (Stevanović) Hedrih: *The theory of body collision in rolling through geometry, kinematics and dynamics of billiards*, In: M. Papadrakakis, M. Fragiadakis, C. Papadimitriou (eds.), *EURODYN 2020*, XI International Conference on Structural Dynamics, Athens, Greece, 2020.

[31] K.R. (Stevanović) Hedrih, *Rolling pendulums along curvilinear traces and surfaces: Collinisin theory and dynamics of vibro-impact systems*, XXV International Congress of Theoretical and Applied Mechanics (ICTAM), Milano, Italy, 2021.

[32] K.R. (Stevanović) Hedrih, *Energy analysis in the nonlinear hybrid system containing linear and nonlinear subsystem coupled by hereditary element,* Nonlinear Dynamics **51**(1) (2007), 127-140. DOI: 10.1007/s11071-007-9197-2.

[33] K.R. (Stevanović) Hedrih, G. Janevski, *Nonlinear dynamics of a gyro-disc-rotor and structural dependence of a phase portrait on the initial conditions*, In: *Proceedings of Dynamics of Machines Institute of Thermomechanics*, Czech Committee of the European Mechanics Society, Prague, 81-88, 2000.

[34] K.R. (Stevanović) Hedrih, S. Jović, *Energy of the vibro-impact systems with Coulomb's type frictions*, ESMC Lisbon 2009, Minisymosium MS-24 Kinetics, Control and Vibrorheology KINCONVIB, 7-11 Septembar 43-47, 2009.

[35] K.R. (Stevanović) Hedrih, S. Jović, *Energy of the vibroimpact system dynamics: Heavy material particle oscillations along rough circle with one side impact limit*, 6th IFAC International workshop on Knowledge and Technology Transfer into developing countries: automation and infrastructure, Ohrid Republic of Macedonia, Septembar 26-29, 2009.

[36] K.R. (Stevanović) Hedrih, S. Jović, *Vibroimpact system dynamics: Heavy material particle oscillations along rough circle with one side impact limit*, In: J. Awrejcewicz, M. Kazmierczak, P. Olejnik, J. Mrozowski (eds.), *Proceedings of the 10th Conference on Dynamical Systems - Theory and Applications, DSTA 2009*, Łódź, Poland. 213-220, 2009.

[37] K.R. (Stevanović) Hedrih, S. Jović, *Models of Technological Processes on the Basis of Vibro-impact Dynamics*, Scientific Technical Review **59**(2) (2009), 51-72.

[38] K.R. (Stevanović) Hedrih, V. Raičević, S. Jović, *Phase Trajectory Portrait of the Vibro-impact Forced Dynamics of Two Heavy Mass Particles Motions along Rough Circle*, Communications in Nonlinear Science and Numerical Simulations 16(12) (2011), 4745-4755. DOI: 10.1016/j.cnsns.2011.05.027.

[39] K.R. (Stevanović) Hedrih, V. Raičević, S. Jović, *Vibro-impact system dynamics– Analysis of the dynamics of one class of vibro-impact systems based on oscillators with curvilinear routes and stationary non-ideal bonds*, 2015.

[40] K.R. (Stevanović) Hedrih, M. D. Mikić, *Rolling a heavy ball over a curvilinear path of specific shapes in the real plane*, 2020.

[41] K.R. (Stevanović) Hedrih, M.D. Mikić, *Rolling a heavy ball over parabola and her generalisations in real plane*, In: I. Atanasovska, A. Hedrih, M. Cajić (eds.), *Booklet of Abstracts of the Symposium Nonlinear Dynamics - Scientific work of Prof. Dr Katica* (*Stevanović*) *Hedrih*, Mathematical Institute of the Serbian Academy of Sciences and Arts, Belgrade, 119-120, 2019.

[42] K.R. (Stevanović) Hedrih, Sl. Mitić, *Stationary and nonstationary resonant state, jumps and chaotic attraction in the nonlinear vibrations in continuous systems*, IUTAM Symposium "Nonlinearity and Chaos in Engineering Dynamics", University College London, 1993.

[43] K.R. (Stevanović) Hedrih, Sl. Mitić, R. Pavlović, P. Kozić, *Mnogochastotnie vinuzhdenie kolebaniya tonkoy uprugoy oboloschki s nachalynimi nepravilynostyami,* (*Analiticheskiy analiz*) (*Multi-frequency forced vibrations of thin elastic shell with initial imperfections* (*Analytical Analysis*)), Theoretical and Applied Mechanics (Beograd) **12** (1986), 41-58.

[44] K.R. (Stevanović) Hedrih, V. Raičević, S. Jović, *Vibro-impact of a Heavy Mass Particle Moving along a Rough Circle with Two Impact Limiters in forced regime*, International Journal of Nonlinear Sciences & Numerical Simulation **10**(11) (2009), 1713-1720.

[45] K.R. (Stevanović) Hedrih, V. Raičević, S. Jović, *Vibro-impact of a Heavy Mass Particle Moving along a Rough Circle with Two Impact Limiters*, International Journal of Nonlinear Sciences and Numerical Simulation **11**(3) (2010), 211-224.

[46] A. Ismail, W.J. Stronge, *Viscoplastic analysis for direct impact of sports balls*, International Journal of Non-Linear Mechanics **47**(4) (2012), 16-21.

[47] S. Jović*, Energijska analiza dinamike vibroudarnih sistema sa krivolinijskim putanjama i neidealnim vezama* (*Energy analysis of vibroimpact system dynamics with curvilinear paths and no ideal constraints*), PhD Thesis, Faculty of Technical Sciences in Kosovska Mitrovica, University of Priština, 2011. (in Serbian)

[48] S. Jović, *Energijska analiza dinamike vibroudarnih sistema* (*Energy analysis of vibroimpact system dynamics*), Magister Thesis, Faculty of Technical Sciences in Kosovska Mitrovica, University of Priština, 2009. (in Serbian)

[49] S. Jović, K. R. (Stevanović) Hedrih, V. Raičević, *Vibro-impact system dynamics – Analysis of the dynamics of one class of vibro-impact systems based on oscillators with curvilinear routes and stationary non-ideal bonds*, Faculty of Technical Sciences in Kosovska Mitrovica, University of Priština, 2015.

[50] G. Coriolis, *Théorie mathématique des effets du jeu de billard,* Carilian-Goeury, Paris. 1835.

[51] G. Coriolis, *Sur le principe des forces vives dans les mouvements relatifs des machines,* J. De l'Ecole royale polytechnique **13** (1832), 268–302.

[52] G.G. Coriolis, *Sur les équations du mouvement relatif des systèmes de corps,* J. De l'Ecole royale polytechnique **15** (1835), 144–154.

[53] В.В. Козлов, Д.В. Трещев, *Биллиарды. Генетическое введение в динамику систем с ударами*, Издателство Московского университета, Москва, 1991.

[54] Sl. Mitić: *Stabilnost determinističkih i stohastičkih procesa u vibroudarnim sistemima* (*Stability of desterministic and stochastic processes in vibroimpact systems*), PhD Thesis, Faculty of Mechanical Engineering in Niš, 1994. (in Serbian)

[55] Sl. Mitić: *Vibroudarni sistemi*, Univerzitet u Nišu, Fakultet zaštite na radu, Niš, 2006.

[56] Sl. Mitić, K.R. (Stevanović) Hedrih, *Nonlinear oscillations of the torsion oscillator with impact masses*, Acta Technica CSAV **42**(2) (1997), 213-226.

[57] D.S. Mitrinović, D. Ž. Djoković, *Special functions* (*Specijalne funkcije*)*,* Gradjevinska knjiga, Beograd, 1964.

[58] F. Peterka, *Laws of Impact Motion of Mechanical Systems with one Degree of Freedom, Part I - Theoretical Analysis of n-Multiple (1/n) - Impact Motions*, Acta Technica CSAV **4** (1974), 462-473.

[59] F. Peterka, *Laws of Impact No 4, Motion of Mechanical Systems with one Degree of Freedom: Part II - Resalts of Analogue Computer Modelling of the Motion*, Acta Technica CSAV **5** (1974), 569-580.

[60] F. Peterka, *Laws of Impact Motion of Mechanical Systems with one Degree of Freedom: Part IV - Analytical solution of the 2/n-Impact Motions and its Stability*, Acta Technica CSAV **6** (1981), 747-758.

[61] F. Peterka, *Dynamics of Mechanical Systems with Impacts and Dry Friction,* EUROMECH-2nd ENOC, Prague, 33-44, 1996.

[62] F. Peterka, *Dynamics of the Impact Oscillator*, In: *IUTAM 97*, Cluver Ac. Publ., 283-292, 1997.

[63] F. Peterka, *More Detail Wiew on the Dynamics of the Impact Damper*, Facta Universitaties, Series Mechanics, Automatic Control and Robotics, 2003.

[64] F. Peterka, *Bifurcations and transition phenomena in an impact oscillator*, Chaos, Solitons and Fractals **7**(10) (1996), 1635-1647.

[65] F. Peterka, J. Vacik, *Laws of Impact Motion of Mechanical Systems with one Degree of Freedom: Part III - Statistic Characteristics of Beat Motions*, Acta Technica CSAV **2** (1981), 161-184.

[66] M. Petrović, *Elementi matematičke fenomenologije* (*Elements of mathematical phenomenology*)*,* Srpska kraljevska akademija, Beograd, 1911.

[67] M. Petrović, *Fenomenološko preslikavanje* (*Phenomenological mapp*), Srpska kraljevska akademija, Beograd, 1933.

[68] M. Petrović, *Mecanismes communs aux phenomenes disparates*, Paris, 1921.

[69] D.P. Rašković, *Teorija oscilacija* (*Theory of oscillatins*)*,* Naučna knjiga, First Edition 1952, Second Edition, 1965.

[70] D.P. Rašković, *Mehanika III – Dinamika* (*Mechanics III-Dynamics*), Fourth Edition, Naučna knjiga, 1972.

[71] D.P. Rašković, *Analitička mehanika*, Mašinski fakultet Kragujevac, 1974.

[72] W.J. Stronge, *Impact Mechanics*, Cambridge University Press, Cambridge, 2000.

[73] W.J. Stronge, R. James, B. Ravani, *Oblique impact with friction and tangential compliance*, Philos, Trans. R. Soc. A Math. Phys. Eng. Sci. 359 (2001) 2447–2465.

[74] D.E. Stewart, *Rigid body dynamics with friction and impact*, SIAM Rev. **42**(1) (2000), 3–39. http://dx.doi.org/10.1137/S0036144599360110.

Appendix. Projects

[P.1] Project: *Oscillations of the Special Elements and Systems*, Basic Scientific Found of Region Niš (1981-1986). Project leader: Katica (Stevanović) Hedrih, Some research results included in two Magister of sciences theses of P. Kozić and R. Pavlović. Mechanical Engineering Faculty, University of Niš.

[P.2] Project: *Stochastic Processes in Dynamical Systems-Applications on the Mechanical Engineering Systems*, Basic Scientific Found of Region Niš (1986-1989). Project leader: Katica (Stevanović) Hedrih, Some research results included in Magister of sciences theses of Sl. Mitić and in two doctoral dissertations of P. Kozić and R. Pavlović. Mechanical Engineering Faculty, University of Niš.

[P.3] Project: *Nonlinear Deterministic and Stochastic Processes with Applications in Mechanical Engineering Systems*, Ministry of Science and Technology, Republic of Serbia, (1990-1995). Project leader: Katica (Stevanović) Hedrih, Some research results included in two Magister of sciences theses of Blagoj Pavlov and Aleksandar Filipovski and in a doctoral dissertation of Sl. Mitić. Mechanical Engineering Faculty, University of Niš.

[P.4] Sub-Projects: 5.1: *Stress and Strain State of the Deformable Bodies* and 5.2: *Vector Interpretation of Body Kinetic Parameters*, as a part of Project: *Actual Problems on Mechanics*, (1990-1995), Themes Leader dr Katica (Stevanović) Hedrih; Project Leader prof. dr Mane Šašić. Ministry of Sciences, Technology and Development of Republic Serbia. Some research results included in three Magister of sciences theses of Ljubiša Perić, Dragan Jovanović and Snežana Mitić, Mathematical Institute SANU.

[P.5] Sub-Project: 04M03A *Actual Problems of Mechanics and Applications*, (1996–2000), Mathematical Institure SASA. Project Leader: dr Katica (Stevanović) Hedrih. (between researchers at Sub-Project: 04M03A are Dragović Vladimir, Jovanović Božidar, Gajić Borislav, Milena Radnović). Sub-Project 04M03A is a part of Project 4M *Methods and Models Theoretical, Industrial and Applied Mathematics.* Project Leader prof. dr Gradimir Milovanović, Ministry of Sciences, Technology and Development of Republic Serbia. Mathematica Institute SANU. http://www.mi.sanu.ac.rs/projects/scientific_p04m03/sproject_p04m03a.htm

[P.6] Project 1616: *Real Problems On Mechanics"* (2002–2005) , Project Leader: Katica (Stevanović) Hedrih. http://www.mi.sanu.ac.rs/projects/1616e.htm.

[P.7] Project ON1828: Dynamics and Control of active Structures (2001-2005), Project Leader: Katica (Stevanović) Hedrih. Basic Science-Mathematics and Mechanics, Ministry of Sciences, Technology and Development of Republic Serbia. Some research results included in two doctoral dissertations of Ljubiša Perić and Dragan Jovanović. Mechanical Engineering Faculty University of Niš.

[P.8] Project 144002: *Theoretical and Applied Mechanics of the Rigid and Solid Bodies. Mechanics of Materials*, (2006–2010). Project Leader: Katica (Stevanović) Hedrih.
http://www.mi.sanu.ac.rs/projects/results144002.htm

[P.9] Project 174001: *Dynamics of hybrid systems with complex structures. Mechanics of materials*, (2011-2019). Project Leader: Katica (Stevanović) Hedrih., 13 reseatchers defended Ph,D. and 4 Doctoral Disertations. after Magistar of Sciences. http://www.mi.sanu.ac.rs/novi_sajt/research/projects/174001a.php.