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APPLICATION OF THE R-FUNCTIONS METHOD FOR VIBRATION AND BUCKLING ANALYSIS OF FUNCTIONALLY GRADED PLATES AND SHALLOW SHELLS WITH COMPLEX PLANFORM. LITERATURE REVIEW FROM 2014 TO 2020

Abstract. A literature review regarding the use of the R-functions theory to solve linear and nonlinear dynamics problems of the functionally graded plates and shallow shells is presented in the work. The paper reviews the main works devoted to analysis of the functionally graded structures with an arbitrary planform and different boundary conditions. In most cases the R-functions theory is applied together with the Ritz method. Therefore questions connected with construction of sets of admissible functions that have to satisfy at least the geometric (essential) boundary conditions are discussed also here. The review focuses on papers using the classical and first order shear deformation theories applied widely in the modeling of laminated and functionally graded plates and shells. In order to demonstrate the effectiveness and universalities of the R-functions method (RFM), some numerical results related to dynamic behavior and stability of the functionally graded plates and shallow shells are presented.

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Contents

1. Introduction

The R-functions theory was created by Vladimir L. Rvachev in the 70-80s of the last century. Dynamic problems of the plates and shells theory were among the first applications of the R-functions method. Later this method received a worldwide recognition and nowadays it is known as a method of RFM (R-functions method). Efficiency and universality of the RFM have been illustrated on a large number of linear vibrations problems and stability of plates and shallow shells with complex forms and various boundary conditions. In particular the laminated shallow shells of an arbitrary shape were investigated by RFM. Results obtained for these classes of problems have been reflected in a lot of papers and monographs [1-14] and others.

In this paper we will review the works published in last decade. In this time the main applications of the RFM were connected with research buckling, linear and nonlinear vibrations of the functionally graded plates and shallow shells [18, 19, 23-27, 31-36, 38-40]. First, we consider the main concept of the R-functions theory.

2. The R-functions theory

2.1. R-functions. The R-functions belong to the class of elementary functions. The distinctive feature of the R-functions from any elementary ones is following: sign of the R-functions is completely determined by the sign of the arguments. It is a feature that connects the R-functions with Boolean functions. Indeed, if the signs of the arguments and the function correspond to the numbers “zero” and “one”, then one can come to Boolean functions [1-4].

Consider the most widely used R-functions of two variables:

$$z_1 = \frac{1}{1+\alpha\left(x_1, x_2\right)}\left(x_1 + x_2 - \sqrt{x_1^2 + x_2^2 - 2\alpha\left(x_1, x_2\right)x_1 x_2}\right),$$

$$z_2 = \frac{1}{1+\alpha(x_1,x_2)}\left(x_1 + x_2 + \sqrt{x_1^2 + x_2^2 - 2\alpha(x_1,x_2)x_1x_2}\right)$$

where variable $\alpha(x_1,x_2)$ belongs to interval $-1<\alpha(x_1,x_2)\le 1$.

It is easy to check that function $z_1(x_1,x_2)$ corresponds to the Boolean function--conjunction $F_1 = X_1 \wedge X_2$, and function $z_2(x_1,x_2)$ corresponds to the disjunction $F_2 = X_1 \vee X_2$. The R-functions corresponding to negation $F_3 = \overline{X_1}$, $\left(F_3 = \overline{X}_2\right)$ are obtained by the following formulas:

$$z_3 = -x_1,\ z_3 = -x_2.$$

Since the R-functions $z_1(x_1,x_2)$ and $z_2(x_1,x_2)$ correspond to Boolean functions of conjunction, disjunction and negation, special notations were introduced for them (R-conjunction, R-disjunction and R-negation), namely:

$$x_1 \wedge_\alpha x_2 \equiv \frac{1}{1+\alpha}\left(x_1 + x_2 - \sqrt{x_1^2 + x_2^2 - 2\alpha\, x_1x_2}\right);$$

$$x_1 \vee_\alpha x_2 \equiv \frac{1}{1+\alpha}\left(x_1 + x_2 + \sqrt{x_1^2 + x_2^2 - 2\alpha\, x_1x_2}\right);$$

$$\overline{x} = -x.$$

In practice, usually the value $\alpha(x_1,x_2)$ is assumed to be zero and the main system R_0 takes more simple form ($\overline{x} \equiv -x$):

$$\overline{x} \equiv -x,$$

$$\begin{aligned} x_1 \wedge_0 x_2 &\equiv x_1 + x_2 - \sqrt{x_1^2 + x_2^2}, \\ x_1 \vee_0 x_2 &\equiv x_1 + x_2 + \sqrt{x_1^2 + x_2^2}. \end{aligned} \tag{1}$$

The described connection between Boolean functions and functions with continuous argument allowed to solve the inverse problem of analytic geometry: for a given geometric object to construct the equation of this object in the form of a single analytic expression. A general rule was obtained by Rvachev [1] for constructing the boundary equation for the given region. With this purposes so-called logic formula (Predicate) $\Omega = F(\Omega_1,\Omega_2,\ldots,\Omega_n)$ is constructed for the given domain, where

$$\Omega_i = \begin{cases} 1,\ if\ \ f_i(x,y) \ge 0, \\ 0,\ if\ \ f_i(x,y) < 0, \end{cases}$$

and inequalities $f_i(x,y) \ge 0$ describe the corresponding subdomain Ω_i. To obtain an equation of the domain boundary $\omega(x,y)=0$ it is sufficient to perform a formal substitution of symbols Ω_i for $f_i(x,y)$ or $\omega_i(x,y)$, symbols $\wedge$ - conjunctions, $\vee$ - disjunctions for the symbols - $\wedge^* - R$ conjunction and $\vee_* - R$ -disjunction, and the negation symbol for R-negation. The function $\omega(x,y)$ will satisfy conditions:

$$\omega(x,y)\big|_{\partial\omega}=0\,, \tag{2}$$
$$\omega(x,y)>0 \qquad \forall(x,y)\in\Omega\,,$$
$$\omega(x,y)<0 \qquad \forall(x,y)\notin\Omega\,.$$

For example, consider the region shown in Fig. 1.

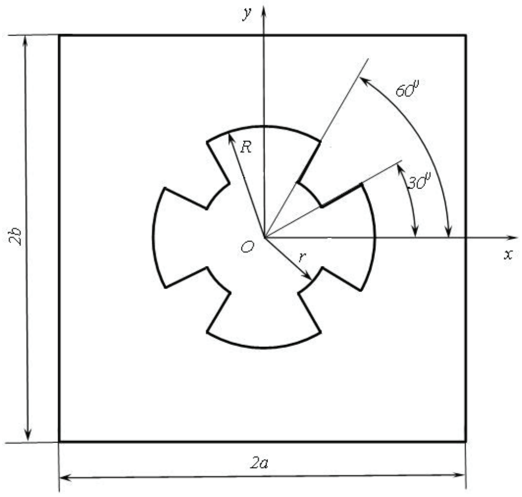


FIGURE 1. Example of the domain with complex form

Using the R-operations [1-4], we build the equation of the border in the following form:

$$\omega=\omega_{inside}\wedge_0\omega_{outside},$$

where

$$\omega_{inside}=\left(-\left(\left(\left(\left(f_1\wedge_0 f_2\right)\vee_0\left(\overline{f_1}\wedge_0\overline{f_2}\right)\vee_0\left(\left(f_3\wedge_0 f_4\right)\vee_0\left(\overline{f_3}\wedge_0\overline{f_4}\right)\right)\right)\vee_0 f_5\right)\wedge_0 f_6\right)\right),$$

$$\omega_{outside}=f_7\wedge_0 f_8\,.$$

The functions $f_i, i=1,\dots,8$ are defined as follows:

$$f_1=\left(y+\frac{1}{\sqrt{3}}x\right)\geq 0,\ f_2=\left(-y+\frac{1}{\sqrt{3}}x\right)\geq 0,\ f_3=\left(y-\sqrt{3}x\right)\geq 0,\ f_4=\left(y+\sqrt{3}x\right)\geq 0,$$

$$f_5=\left(r_1^2-x^2-y^2\right)\geq 0,\ f_6=\left(r_2^2-x^2-y^2\right)\geq 0,\ f_7=\left(a^2-x^2\right)\geq 0,\ f_8=\left(b^2-y^2\right)\geq 0.$$

The ability to construct equations of the boundary of a region in the form of a single analytic expression is a very important step in solving boundary value problems of mathematical physics.

2.2. Solution structures. The R-functions method (RFM). The RFM is often referred as a structural variation method because the main concept of this method is a solution structure of the boundary value problem. Solution structure is a family of functions $U = B\left(\Phi, \omega, \omega_{1,} \omega_2, \ldots \omega_j\right)$ satisfying the boundary conditions at any choice of indefinite components $\Phi = \left(\Phi_1, \Phi_2, \ldots \Phi_s\right)$ [1-4].

Function $\omega\left(x, y\right)$ describes equation of a whole domain boundary $\partial\Omega$ and functions $\left(\omega_1, \omega_2, \ldots \omega_j\right)$ describe separate parts $\partial\Omega_r, r = 1, 2, \ldots j$. It is very important that the constructed solution structure must be completed. The simplest example of the complete structure is the solution structure for clamped shells. For example, within the framework of the classical theory boundary conditions for clamped shells are

$u = 0, \quad \mathrm{v} = 0, \quad \mathrm{w} = 0, \quad \frac{\partial w}{\partial n} = 0$, where u, v, w are displacements of the shells in directions *Ox, Oy*, and *Oz* respectively and n is normal to boundary domain. The solution structure, satisfying all boundary conditions is

$$u = \omega\Phi_1, \quad v = \omega\Phi_2, \quad \mathrm{w} = \omega^2\Phi_3. \tag{3}$$

In expressions (3) function $\omega(x, y)$ satisfies conditions (2):

$$\omega\left(x, y\right) > 0, \forall\left(x, y\right) \in \Omega, \qquad \omega\left(x, y\right)\Big|_{\partial\Omega} = 0.$$

Hence $\omega(x, y) = 0$ is an equation of the domain boundary. In formulas (3) the functions $\Phi_i\left(x, y\right), \quad \left(i = \overline{1, n}\right)$ are indefinite components of the structure solutions. These components are expanded in a series on some complete system of functions $\left\{\phi_i^{(k)}\right\}, \quad \left(k = 1, 2, 3\right)$:

$$\Phi_1 = \sum_{i=1}^{N_1} a_i \phi_i^{(1)}, \quad \Phi_2 = \sum_{i=1}^{N_2} b_i \phi_i^{(2)}, \quad \Phi_3 = \sum_{i=1}^{N_3} c_i \phi_i^{(3)}, \tag{4}$$

where a_i, b_i, c_i are indefinite coefficients. Let us substitute expressions (4) into (3). We obtain the following representation of the sought solution:

$$u = \sum_{i=1}^{N_1} a_i u_i, \quad v = \sum_{i=1}^{N_2} b_i v_i, \qquad w = \sum_{i=1}^{N_3} c_i u_i. \tag{5}$$

The functions

$$u_i = \omega\left(x, y\right)\phi_i^{(1)}, \quad v_i = \omega\left(x, y\right)\phi_i^{(2)} u_i = \omega^2\left(x, y\right)\phi_i^{(3)} \tag{6}$$

are admissible functions that satisfy boundary conditions (4) at any choice of the indefinite coefficients. These coefficients are determined by variational methods from condition for corresponding functional to have a minimum.

Solution structures have more complex form for other boundary conditions. Some of them are derived and presented in Ref. [1-4].

3. Mathematical statement of the vibration problems for functionally graded plates and shallow shells

Since 2014 the R-functions method has been applied to static and dynamic problems of plates and shallow shells fabricated of modern composite materials, so-called functionally graded materials (FGM's). This class of composite materials has significant advantages over multilayer composites due to continuous and smooth variation in properties in one or several directions. They restrain a sharp change in the mechanical properties of the layers and, as a consequence, prevent stress concentration, destruction and delamination of layers. In addition they are able to withstand high temperature environments. Due to these reasons, the study of the static and dynamic behavior of FGM structures draws an attention of many researchers. Calculation of FGM structures is among the most important problems of modern mechanics. A huge number of works devoted to this problem and, in particular, to vibration FGM's plates and shells is known. Let us formulate the mathematical statement of the problem about geometrically nonlinear vibrations of the shallow shells, described in Ref. [15-17].

Suppose that shallow shells or plates are fabricated of functionally graded materials (mixture of metal and ceramics). A planform of the shell can be complex and fixed by different ways. It is assumed that the temperature is varied only in the thickness direction. Young's modulus of ceramics and metal depend on temperature according to the law [15, 17]:

$$P_j(T) = P_0\left(P_{-1}T^{-1} + 1 + P_1T + P_2T^2 + P_3T^3\right),$$

where $P_0, P_{-1}, P_1, P_2, P_3$ are coefficients determined for each specific material. A table of values of these coefficients for some materials is presented in Ref. [2, 4, 14].

The effective material properties of the FGMs are calculated by Voigt's model [17], provided that Poisson's ratio ν is a constant. The Young modulus E_f and density ρ of FG structure is defined as:

$$E_f(z,T) = \left(E_c(T) - E_m(T)\right)V_c + E_m(T), \qquad \rho(z) = \left(\rho_c - \rho_m\right)V_c + \rho_m, \tag{7}$$

$$V_c = \left(\frac{2z+h}{2h}\right)^p. \tag{8}$$

In formulas (7, 8) z is the distance between the current point and the shell middle surface, the index p $(0 \le p < \infty)$ denotes the volume fraction exponent of ceramics V_c which is connected with volume fraction of metal V_m by relation $V_c + V_m = 1$. So effective module E_f depends on temperature T and applicate z.

We will consider shells with the same temperatures on the top and bottom of the object. If solution of the problem is carried out within the first order shear deformation theory of shallow shells (FSDT), then according to this theory, the displacements components u_1, u_2, u_3 at a point (x, y, z) are expressed as functions of the middle surface displacements u, v and w in the Ox, Oy and Oz directions and the independent rotations ψ_x, ψ_y of the transverse normal to middle surface about the Oy and Ox axes, respectively [15-17]:

$$u_1 = u + z\psi_x, \quad u_2 = v + z\psi_y, \quad u_3 = w.$$

Strains $\varepsilon = \left\{\varepsilon_{11}; \varepsilon_{22}; \varepsilon_{12}\right\}^T$ at an arbitrary point of the shallow shell are:

$$\varepsilon_{11} = \varepsilon_{11}^0 + z\chi_{11}, \quad \varepsilon_{22} = \varepsilon_{22}^0 + z\chi_{22}, \quad \varepsilon_{12} = \varepsilon_{12}^0 + z\chi_{12}, \tag{9}$$

where

$$\varepsilon_{11}^0 = \frac{\partial u}{\partial x} - k_1 w + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \varepsilon_{22}^0 = \frac{\partial v}{\partial y} - k_2 w + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2,$$

$$\varepsilon_{12}^0 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}, \tag{10}$$

and $k_1 = 1/R_x$, $k_2 = 1/R_y$ are the principal curvatures of the shell along the coordinates *x* and *y*, respectively. Let us present formulas (10) employing the following general notation:

$$\varepsilon_{ij} = \varepsilon_{ij}^L + \varepsilon_{ij}^{ND}, \quad (i, j = 1,2),$$

where

$$\varepsilon_{11}^L = u,_x + w/R_x, \varepsilon_{22}^L = v,_y + w/R_y, \varepsilon_{12}^L = u,_y + v,_x, \varepsilon_{13} = w,_x + \psi_x, \quad \varepsilon_{23} = w,_y + \psi_y$$

$$\varepsilon_{11}^{ND} = \frac{1}{2} w,_x^2, \quad \varepsilon_{22}^{ND} = \frac{1}{2} w,_y^2, \quad \varepsilon_{12}^{ND} = w,_x w,_y,$$

$$\chi_{11} = \psi_x,_x, \quad \chi_{22} = \psi_y,_y \quad \chi_{12} = \psi_x,_y + \psi_y,_x.$$

The strain resultants $M = (M_{11}, M_{22}, M_{12})^T$, moment resultants $M = (M_{11}, M_{22}, M_{12})^T$, and shear stress resultants $Q = (Q_x, Q_y)^T$ are calculated by integration along the Oz-axis, and they have the following forms:

$$N = [A]\{\varepsilon\} + [B]\{\chi\}, \qquad M = [B]\{\varepsilon\} + [D]\{\chi\},$$

$$[A] = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}, \ [B] = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{33} \end{bmatrix}, \ [D] = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix}.$$

Elements of the matrices are calculated by the following formulas:

$$([A],[B],[D]) = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q(z)\left(1, z, z^2\right)dz, \qquad Q(z) = \frac{E(z)}{1-\nu^2(z)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}.$$

Transverse shear force resultants Q_x, Q_y are defined as

$$Q_x = K_s^2 A_{33}\varepsilon_{13}, \quad Q_y = K_s^2 A_{33}\varepsilon_{23},$$

where K_s^2 denotes the shear correction factor if FSDT is applied. Considering application of the R-functions theory we assumed that Poisson's ratio is independent of the temperature, being the same for ceramic and metal, i.e., $\nu_c = \nu_m$.

The governing differential motion equations for free vibration of shear deformable shallow shell can be expressed as

$$\frac{\partial N_{11}}{\partial x} + \frac{\partial N_{12}}{\partial y} - \frac{Q_x}{R_x} = I_0\frac{\partial^2 u}{\partial t^2} + I_1\frac{\partial^2 \psi_x}{\partial t^2}; \qquad \frac{\partial N_{22}}{\partial y} + \frac{\partial N_{12}}{\partial x} - \frac{Q_y}{R_y} = I_0\frac{\partial^2 v}{\partial t^2} + I_1\frac{\partial^2 \psi_y}{\partial t^2};$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{N_{11}}{R_x} + \frac{N_{22}}{R_y} + N_{11}\frac{\partial^2 w}{\partial x^2} + 2N_{12}\frac{\partial^2 w}{\partial x \partial y} + N_{22}\frac{\partial^2 w}{\partial y^2} = I_0\frac{\partial^2 w}{\partial t^2}; \tag{11}$$

$$\frac{\partial M_{11}}{\partial x} + \frac{\partial M_{12}}{\partial y} - Q_x = I_2\frac{\partial^2 \psi_x}{\partial t^2} + I_1\frac{\partial^2 u}{\partial t^2}; \qquad \frac{\partial M_{22}}{\partial y} + \frac{\partial M_{12}}{\partial x} - Q_y = I_2\frac{\partial^2 \psi_y}{\partial t^2} + I_1\frac{\partial^2 v}{\partial t^2},$$

where

$$\left(I_0, I_1, I_2\right) = \sum_{r=1}^{3}\int_{z_r}^{z_{r+1}} (\rho)_r \left(1, z, z^2\right)dz.$$

4. Solution of linear vibration problem by RFM

The total strain energy U and kinetic energy T of the shells are given by the following formulas:

$$U = \frac{1}{2}\iint_{\Omega}\left(N_{11}^L\varepsilon_{11}^L + N_{22}^L\varepsilon_{22}^L + N_{12}^L\varepsilon_{12}^L + M_{11}^L\chi_{11} + M_{22}^L\chi_{22} + M_{12}^L\chi_{12} + Q_x\varepsilon_{13} + Q_y\varepsilon_{23}\right)dxdx, \tag{12}$$

$$T = \frac{1}{2}\iint_{\Omega} I_0\left(u,_t^2 + v,_t^2 + w,_t^2 + 2I_1\left(u,_t \psi_x,_t + v,_t \psi_y,_t\right) + I_2\left(\psi_x,_t^2 + \psi_y,_t^2\right)\right)dxdy. \tag{13}$$

The total energy functional for FGM shallow shell is defined as follows:

$$J = T - U \tag{14}$$

Assuming that the shell vibrates periodically, the vector of unknown functions can be presented as

$$\vec{U}\left(u(x,y,t), v(x,y,t), w(x,y,t), \psi_x(x,y,t), \psi_y(x,y,t)\right) =$$

$$= \vec{U}\left(u(x,y), v(x,y), w(x,y), \psi_x(x,y), \psi_y(x,y)\right)\sin\lambda t, \tag{15}$$

where λ is a vibration frequency. Using (13) and (14) and Hamilton's principle, we get the variational statement of the problem

$$\delta J = 0, \tag{16}$$

where

$$J = U\left(u, v, w, \psi_x, \psi_y\right) - \lambda^2 T\left(u, v, w, \psi_x, \psi_y\right). \tag{17}$$

Now, expressions for *U* and *T* in formulas (12, 13) depend on (*x,y*) and the kinetic energy takes the following form:

$$T = \frac{1}{2}\iint_{\Omega}\left(I_0\left(u^2 + v^2 + w^2\right) + 2I_1\left(u\psi_x + v\psi_y\right) + I_2\left(\psi_x^2 + \psi_y^2\right)\right)dxdy. \tag{18}$$

Minimization of functional (17) is performed by using the Ritz method, belonging to a powerful tool in the vibration analysis of shells and plates.

Suppose that admissible functions $\{u_i\}, \{v_i\}, \{w_i\}, \{\psi_{xi}\}, \{\psi_{yi}\}$ have been constructed. Then, according to the Ritz method, unknown functions $u(x,y)$, $v(x,y)$, $w(x,y)$, $\psi_x(x,y)$, $\psi_y(x,y)$ are presented as follows:

$$u = \sum_{i=1}^{N_1} a_i u_i, \quad v = \sum_{i=N_1+1}^{N_2} a_i v_i, \quad w = \sum_{i=N_2+1}^{N_3} a_i w_i, \quad \psi_x = \sum_{i=N_3+1}^{N_4} a_i \psi_{xi}, \quad \psi_y = \sum_{i=N_4+1}^{N_5} a_i \psi_{yi}. \tag{19}$$

Coefficients of this expansion $\{a_i\}, i = 1, \dots, N_5$ in (19) are yielded by the Ritz system:

$$\frac{\partial I}{\partial a_i} = 0, \quad i = 1, \dots, N_5.$$

A system of admissible functions satisfying the main (kinematic) boundary conditions is built by the R-functions theory.

5. Solution of the nonlinear problem by RFM

In order to solve the nonlinear vibration problem, the approach proposed in Ref. [5-7, 13] for laminated shells was developed for FGM shallow shells. It is assumed that inertia forces in the middle surface of a shell are ignored while solving the nonlinear problem. In what follows, we introduce unknown functions in the form of expansion with respect to the eigen functions $w_1^{(e)}(x,y)$, $u_1^{(e)}\left(x,y\right)$, $v_1^{(e)}\left(x,y\right)$, $\psi_{x1}^{(e)}(x,y)$, $\psi_{y1}^{(e)}(x,y)$. They correspond to the fundamental vibration form:

$$\begin{cases} u(x,y,t)=y_1(t)u_1^{(e)}(x,y)+y_1^2(t)u_2(x,y), \\ v(x,y,t)=y_1(t)v_1^{(e)}(x,y)+y_1^2(t)v_2(x,y), \\ w(x,y,t)=y_1(t)w_1^{(e)}(x,y), \\ \psi_x(x,y,t)=y_1(t)\psi_{x1}^{(e)}(x,y)+y_1^2(t)\psi_{x2}(x,y), \\ \psi_y(x,y,t)=y_1(t)\psi_{y1}^{(e)}(x,y)+y_1^2(t)\psi_{y2}(x,y). \end{cases} \tag{19}$$

The coefficient of this expansion, i.e. the function $y(t)$ depends on time. The functions $u_2, v_2, \psi_{x2}, \psi_{y2}$ have to satisfy the following system of differential equations:

$$\begin{cases} L_{11}u_2(x,y)+L_{12}v_2(x,y)+L_{14}\psi_{x2}(x,y)+L_{15}\psi_{y2}(x,y)=NL_1w_1^{(e)}(x,y) \\ L_{21}u_2(x,y)+L_{22}v_2(x,y)+L_{24}\psi_{x2}(x,y)+L_{25}\psi_{y2}(x,y)=NL_2w_1^{(e)}(x,y) \\ L_{41}u_2(x,y)+L_{42}v_2(x,y)+L_{44}\psi_{x2}(x,y)+L_{45}\psi_{y2}(x,y)=NL_4w_1^{(e)}(x,y) \\ L_{51}u_2(x,y)+L_{52}v_2(x,y)+L_{54}\psi_{x2}(x,y)+L_{55}\psi_{y2}(x,y)=NL_5w_1^{(e)}(x,y) \end{cases} \tag{20}$$

The right part of the system depends on eigen functions $w_1^{(e)}(x,y)$ received after solving linear vibration problems within the framework of the refined theory. Differential operators $L_{ij}, NL_i \;\; i,j=\overline{1,5}$ are described in Ref. [31-34]

System (20) is naturally added by the corresponding inhomogeneous boundary conditions. It is possible to prove that a solution of the system is a point of stationary of the following functional:

$$I(u_2,v_2,\psi_{x2},\psi_{y2})=\iint_{\Omega}\Big(N_{11}^{L_2}\varepsilon_{11}^{L_2}+N_{22}^{L_2}\varepsilon_{22}^{L_2}+N_{12}^{L_2}\varepsilon_{12}^{L_2}+M_{11}^{L_2}\chi_{11}^{L_2}+M_{22}^{L_2}\chi_{22}^{L_2}+M_{12}^{L_2}\chi_{12}^{L_2}+$$

$$+K_s^2A_{33}\left(\psi_{x2}^2+\psi_{y2}^2\right)-2\Big(NL_1\left(w_1^{(e)}\right)u_2+NL_2\left(w_1^{(e)}\right)v_2+NL_4\left(w^{(e)}\right)\psi_{x2}+NL_5\left(w^{(e)}\right)\psi_{y2}\Big)\Big)d\Omega-$$

$$-2\int_{\partial\Omega}\left(F_1\,u_2^{(n)}+F_2\,v_2^{(n)}+F_4\psi_{x2}^{(n)}+F_5\psi_{y2}^{(n)}\right)ds \tag{21}$$

where $N_{ij}^{L_2}$, $\varepsilon_{ij}^{L_2}$, $M_{ij}^{L_2}$ ($i,j=1,2$), $u_2^{(n)}, v_2^{(n)}$ and $\psi_{x2}^{(n)}=\psi_{x2}l+\psi_{y2}m$, $\psi_{y2}^{(n)}=-\psi_{x2}m+\psi_{y2}l$ are defined in Ref. [34];

Values $F_i(i=1,2,4,5)$ are determined by initial boundary conditions. Solution of this problem is carried out by means of the variational Ritz method and the R-functions method (RFM).

The following algorithm holds: calculated functions $u_2(x,y)$, $v_2(x,y)$, $\psi_{x2}(x,y)$, $\psi_{y2}(x,y)$ we substitute into expressions (19) and into equation of motion (11). Then we apply the Bubnov-Galerkin procedure. The following second-order nonlinear differential equation is obtained:

$$\ddot{y}_1(t)+\omega_L^2 y_1(t)+\beta y_1^2(t)+\gamma y_1^3(t)=0\,. \tag{22}$$

It should be mentioned that the values for coefficients of equation (22) have been obtained in an analytical form. The expressions of these coefficients were obtained in work [34]. The solution to equation (22) is defined numerically using the classical 4-th order Runge-Kutta method. Note, that described algorithm takes into account only one mode. If we use a few modes of the linear vibrations then initial problem will be reduced to system of the nonlinear ordinary differential equations. To obtain this system it is necessary to solve consequences of the auxiliary problems as it was shown in Ref. [31]. Obviously, it is better to use larger numbers of eigen functions, but this complicates significantly the algorithm and its numerical implementation. The obtained numerical results for nonlinear vibrations may be viewed as a rather accurate approximation of the desired solution.

6. Some numerical results

The proposed algorithm was validated on many linear and nonlinear problems. The first numerical results obtained by RFM for FGM shallow shells and plates were published in works [18,19]. Dependence of the natural frequency on value of the gradient index for two types of FGMs (Al/Al_2O_3 and Al/ZrO_2) were presented. Obtained results were compared with known ones [20, 21, 22]. As example some comparison for simply supported FGMs (Al/Al_2O_3) shallow shells with square planform is shown in Table 1.

The comparison shows that the results obtained using the refined first order theory (RFM, FSDT) are almost the same as those reported in reference [21]]. A deviation from the results obtained by means of the theory of the higher order (HSDT) [20] does not exceed 4%. Deviation results obtained by using the classical theory (RFM, CST) with the results of [22] does not exceed 2%. In general, it should be noted that the classical theory, in most of cases, overestimates the fundamental frequencies compared with the refined theory. Comparison of the obtained results for nonlinear vibration with results presented in Ref. [21] was carried out in Ref. [19]. Clamped shallow spherical shells with elliptical planform, made of two types of FGM (Al/Al_2O_3 and Al/ZrO_2) were investigated. Testing for shells with square and elliptical planform was performed. It has proved the validation and effectiveness of the developed approach since the maximum deviation does not exceed 1.5%. Illustration of the proposed method for shells with complex planform (Fig. 2) was presented in Ref. [19].

TABLE 1. Comparison of the fundamental frequency parameters $\Omega_1 = \lambda_1 h\sqrt{\rho_c / E_c}$ of square FG shallow shells with simply supported movable edges (Al / Al_2O_3, $a/h = 10$)

$\frac{a}{R_y}$; $\frac{a}{R_x}$	p	RFM (CPT)	RFM (FSDT)	(CPT) [22]	(FSDT) [21]	(HSDT) [20]
(0,0)	0	0.0597	0.0576	0.0597	0.0577	0.0578
	0.5	0.0505	0.0489	0.0506	0.0490	0.0492
	1	0.0455	0.0441	0.0456	0.0442	0.0443
	4	0.0395	0.0382	0.0396	0.0383	0.0381
	10	0.0380	0.0365	0.0380	0.0366	0.0364
(0.5;0.5)	0	0.0770	0.0753	0.0779	0.0762	0.0751
	0.5	0.0665	0.0652	0.0676	0.0664	0.0657
	1	0.0605	0.0593	0.0617	0.0607	0.0601
	4	0.0508	0.0496	0.0519	0.0509	0.0503
	10	0.0472	0.0462	0.0482	0.0471	0.0464
(0;0.5)	0	0.0642	0.0622	0.0648	0.0629	0.0622
	0.5	0.0546	0.0531	0.0553	0.0540	0.0535
	1	0.0494	0.0481	0.0501	0.0490	0.0485
	4	0.0423	0.0411	0.0430	0.0419	0.0413
	10	0.0403	0.0389	0.0408	0.0395	0.0390
(0.5;-0.5)	0	0.0582	0.0562	0.0597	0.0580	0.0563
	0.5	0.0493	0.0477	0.0506	0.0493	0.0479
	1	0.0444	0.0430	0.0456	0.0445	0.0432
	4	0.0385	0.0372	0.0396	0.0385	0.0372
	10	0.0370	0.0356	0.0380	0.0368	0.0355

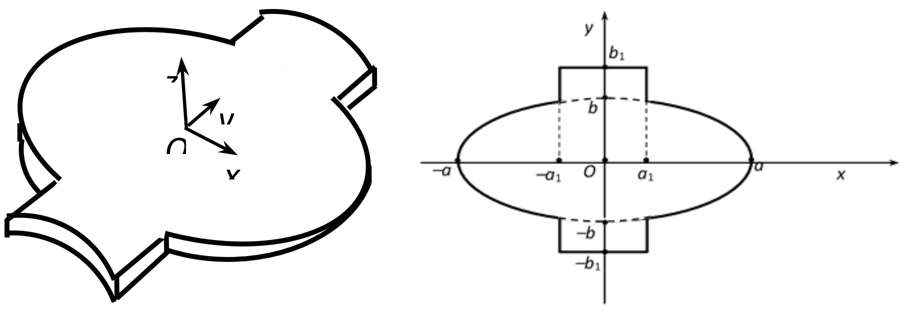


FIGURE 2. FGM shallow shell with complex shape and its planform

Geometrical parameters for shell presented in Fig. 2 are the following: $b/2a = 0.5$; $b_1/2a = 0.35$; $a_1/2a = 0.2$.

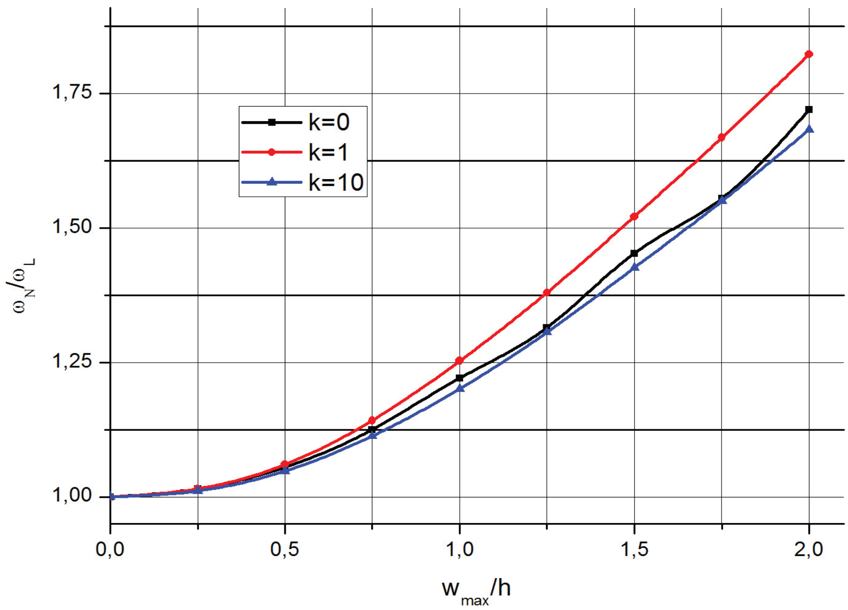


FIGURE 3. Backbone curves for clamped shperical shells made of FGMs Al/Al_2O_3

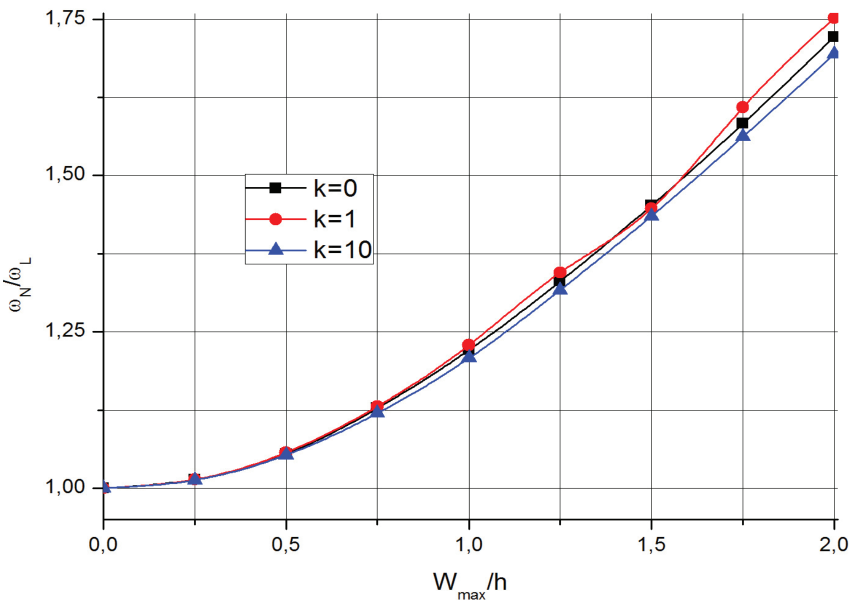


FIGURE 4. Backbone curves for clamped shperical shells made of FGMs Al/ZrO_2

Backbone curves for clamped spherical shells made of FGMs Al/Al_2O_3 and Al/ZrO_2 for different gradient indexes $p = 0; 1; 10$ are drawn in Fig. 3 and Fig. 4 relatively.

In Ref. [23] the vibration problems FGMs (Al/Al_2O_3) shallow shells with complex planform (Fig. 5) were considered. The spline approximation of the undetermined components in solution structures was used.

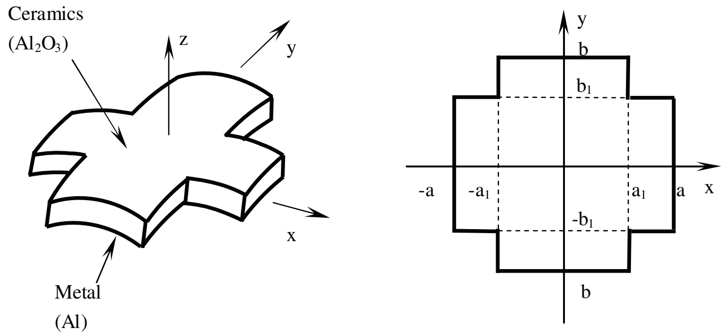
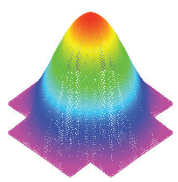
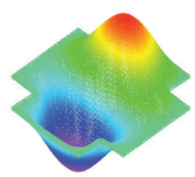
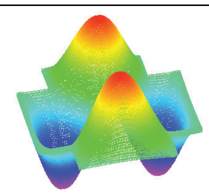
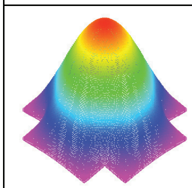
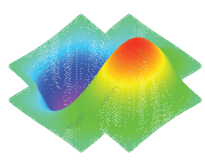
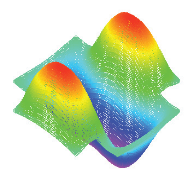


FIGURE 5. FGM shallow shell with cutout and its planform

Comparison of the obtained results with polynomial and spline approximation of the indefinite components was fulfilled for shells with the following geometric parameters: $h/(2a) = 0.2;\quad b/a = 1;\quad b_1/a_1 = 1;\quad b_1/(2a) = 0.3$. The deviation between them does not exceed 12%. Modes of free vibration of the spherical shallow shell $(a/R_y = a/R_x = 0.5)$ with different boundary conditions are presented in Table 2.

TABLE 2. The first three modes and frequencies $\Omega_i = \omega_i h\sqrt{\rho_c/E_c}$ free vibration of the spherical shallow shells from FGM (Fig.5, Al/Al_2O_3, $p = 4$)

Clamped		
		
$\Omega_1 = 0.2615$	$\Omega_2 = 0.4157$	$\Omega_3 = 0.5774$
Simply supported		
		
$\Omega_1 = 0.2143$	$\Omega_2 = 0.3683$	$\Omega_3 = 0.4863$

In Ref [24, 25, 26, 27] authors considered vibration problems for FGMs shallow shells with different boundary conditions. Mathematical statement of the problem has been done with the framework of the first order shear deformation shallow shells. Numerical values of the natural frequencies for clamped and simply supported functionally graded cylindrical and spherical square shell panels were compared with the published results in works [28, 29]. For example, in Table 3 comparison of the fundamental frequency for square clamped FG cylindrical shell panels with side-to-thickness ratios $h/a=10$ is presented for various side-to-radius ratio R/a and power law exponents *p*. Together with testing problems here were solved the problems for shells of the planform shown in Fig. 6, Fig. 7. The effect of boundary conditions, the shape of the plan, curvatures on the fundamental frequencies has been examined. In particular fundamental frequencies for clamped shallow shells (Fig. 7) for different values of gradient index and different curvatures are presented in Table 4.

TABLE 3. Comparison of the fundamental frequencies $\left(\overline{\omega}_L = \omega_L a^2 \sqrt{\frac{\rho_m h}{D}}\right)$ of clamped square cylindrical shell panels (Al/Al_2O_3), *h/a=10* for various *R/a* and *p*

p	Source	R/a=1	R/a=5	R/a=10	R/a=50	Plate
	RFM	96.6235	73.6575	72.8029	72.5271	72.5156
0	Ref.[28]	96.0131	73.6436	72.8141	72.5465	72.5353
	Ref.[29]	94.4973	71.8861	71.0394	70.766	70.7546
	RFM	81.3031	60.8468	60.0817	59.8397	59.8315
0.5	Ref.[28]	80.3049	60.6568	59.9353	59.7178	59.7142
	Ref.[29]	79.5689	63.1896	62.4687	62.238	62.2291
	RFM	72.8309	54.0093	53.3031	53.0821	53.0755
1	Ref.[28]	71.9167	53.9340	53.2759	53.0841	53.0835
	Ref.[29]	71.2453	56.5546	55.8911	55.6799	55.6722
	RFM	53.1347	41.5894	41.1738	41.0456	41.0424
10	Ref.[28]	52.278	41.0985	40.7046	40.5923	40.5229
	Ref.[29]	51.3803	33.6611	33.1474	32.9812	32.9743

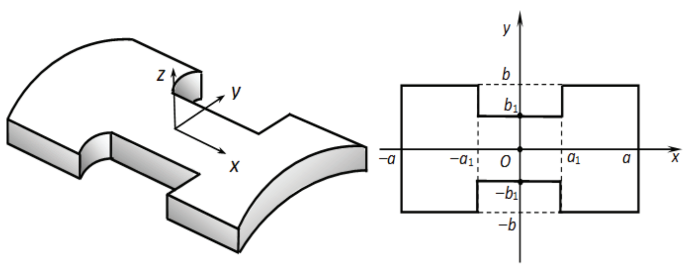


FIGURE 6. FGM shell panel with rectangular cuts

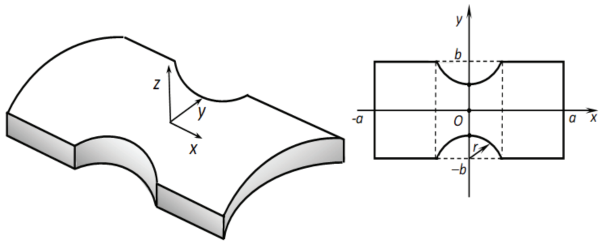


FIGURE 7. FGM shell panel with circular cuts

TABLE 4. Fundamental frequencies of clamped shell panels, *h/2a*=10 for various *p* and types of shell with circular cuts *r*=0.2 (Al/Al_2O_3)

p	Cylindrical shell $\frac{R_x}{2a}=0$, $\frac{R_y}{2a}=10$	Spherical shell $\frac{R_x}{2a}=\frac{R_y}{2a}=10$	Parabolical shell $\frac{R_x}{2a}=10$, $\frac{R_y}{2a}=-10$
0	116.76	117.03	116.87
0.5	97.01	97.25	97.13
1	86.33	86.53	86.44
10	65.40	65.52	65.48
∞	52.76	52.88	52.81

From conducted numerical experiment for clamped shallow shells with different curvatures (cylindrical, spherical, parabolical ones) it follows that for different shapes of shell the values of fundamental frequencies are differed starting with the third sign.

It is absolutely agreed with the physical statement of problem: it is obvious that curvature does not essentially influence on frequency for a case of clamped shell.

In Ref. [31-35] RFM was applied to three layered FGM shallow shells. In particular the shells for the scheme lamination of layers shown in Fig. 8 were studied. Shallow shells of Type 1-2 correspond to sandwich shallow shells with FGM face sheets and isotropic (metal) core. The shells of the Type 2-2 correspond to sandwich shallow shells with FGM core and ceramics on the top face sheet and metal on the bottom face sheet.

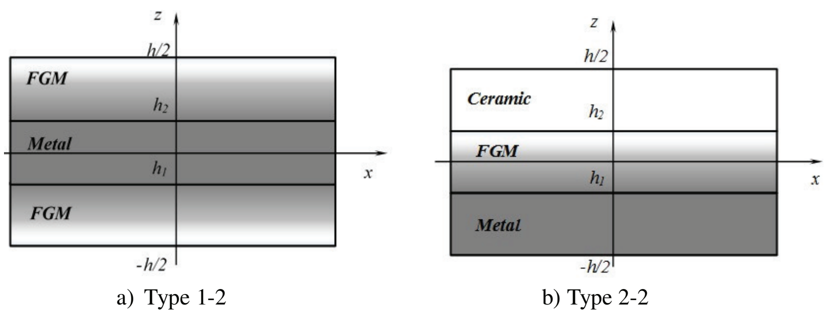


FIGURE 8. Schemes of lamination of the layers

Developed software was tested on a lot of testing problems. Some results of comparison with known ones are shown in Table 5. Note that results presented in Table 5 correspond to shells of Type 1-2 fabricated of FGM Al/Al_2O_3. The shell is either clamped (CCCC) or simply supported (SSSS) or have the mixed boundary conditions (SFSF and SCSC). It can be observed that presented results are in an excellent agreement with those reported in Ref. [30]. Comprehensive comparison of the obtained results with available ones shows the accuracy and reliability of the proposed approach and developed software.

TABLE 5. Comparison of fundamental frequency parameter $\Omega_L^{(1)} = \lambda_1 a^2 / h \sqrt{\rho_0 / E_0}$ $(\rho_0 = 1 \quad kg/m^2, E_0 = 1 \quad GPa)$ of cylindrical and spherical shallow shells with square planform and various boundary conditions

Scheme	p	Method	Cylindrical shell k_1=0.2, k_2=0			Spherical shell k_1=k_2=0.2		
			SSSS	CCCC	SCSC	SSSS	CCCC	SCSC
1-0-1	0.6	[30]	1.6862	3.0433	2.4855	1.8643	3.1027	2.5465
		RFM	1.6919	3.0691	2.5032	1.8689	3.1278	2.5636
	5	[30]	1.2742	2.4252	1.9894	1.4982	2.4657	2.0308
		RFM	1.2781	2.4493	2.0061	1.5028	2.4894	2.0471
	20	[30]	1.0605	1.947	1.5919	1.1948	1.9840	1.6296
		RFM	1.0631	1.9644	1.6036	1.1979	2.0007	1.6409
1-1-1	0.6	[30]	1.6862	2.9305	2.4023	1.8071	2.9807	2.4535
		RFM	1.6919	2.9590	2.4220	1.8131	3.0085	2.4726
	5	[30]	1.2742	2.2461	1.8414	1.3857	2.2845	1.8806
		RFM	1.2781	1.2680	1.8565	1.3899	2.3059	1.8953
	20	[30]	1.0605	1.8506	1.5109	1.1331	1.8871	1.5485
		RFM	1.0631	1.8660	1.5215	1.1358	1.9022	1.5587
1-2-1	0.6	[30]	1.6862	1.6862	2.8005	1.7330	2.8462	2.3455
		RFM	1.6919	1.6919	2.8291	1.7385	2.8742	2.3649
	5	[30]	1.2742	1.2742	2.1318	1.3129	2.1693	1.7845
		RFM	1.2781	1.2781	2.1519	1.3167	2.3189	1.7980
	20	[30]	1.0605	1.0605	1.7969	1.0989	1.8335	1.5036
		RFM	1.0631	1.0631	1.8115	1.1014	1.8476	1.5132

As practice shows, a special attention should be paid to the study of plates and shells with holes and cutouts. Cutouts are often required in the shell elements due to practical necessity, for instance, in order to facilitate a structure, provide an access and compound with other parts, for venting, and other reasons. Cutouts can be free and fixed on their border. Their form can also be arbitrary (not only circle). In Ref. [33] vibration of the shallow shells with complex clamped cutout (see Fig. 1) was studied. It is assumed that the shell is clamped at the internal border of the region.

On the outer boundary of the region, the shell can be either clamped or simply supported or have the mixed boundary conditions (CCCC, SSSS, SFSF and SCSC).

The following geometric parameters are fixed:

$$b/a=1,\ k_1=R_x/2a=0.2,\ k_2=R_y/2a=(0,0.2,-0.2),$$
$$r/2a=0.125,\ R/2a=0.25,\ h/2a=0.1.$$

The admissible functions were constructed by the R-functions theory. Effects yielded by the gradient index p on the fundamental frequency parameter $\Omega_L^{(1)}=\lambda_1 a^2\sqrt{\rho_c/E_c}/h$ for cylindrical, spherical and hyperbolic paraboloidal shells of Type 1-2 and Type 2-2 with different boundary conditions were analysed. As example the obtained results for the spherical shells with thickness scheme (2-1-2) are presented in Fig. 9.

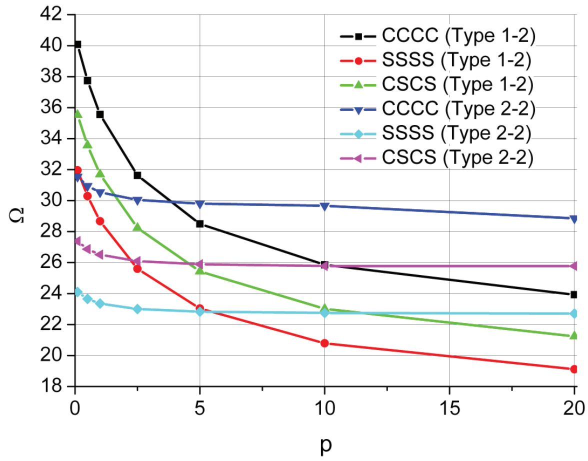


FIGURE 9. Variation of the fundamental frequency parameter $\Omega_L^{(1)}=\lambda_1 a^2\sqrt{\rho_c/E_c}/h$ of the spherical shells with increasing gradient index p

It was shown that the value of fundamental frequency parameters essentially depends on the material type, thickness schemes, and boundary conditions, and the fundamental frequencies parameters for all considered cases decrease with increasing power-law exponent. For the shells of Type 1-2 the decrease is more essential than for the shells of Type 2-2.

Geometrically nonlinear vibrations of sandwich FGM shallow shells were studied in Ref. [34]. The effect of gradient index and thickness schemes is studied for linear and geometrically nonlinear vibrations of the shells with a complex shape, and engineering oriented results are presented and discussed. As complex form of the shells it was accepted the shell drawn in Fig.10. The geometrical parameters are taken as follows:

$$k_1=R_x/2a=0.2,\ \ k_2=R_y/2a=(0;0.2;-0.2),\quad b/a=1,$$
$$a_1/2a=0.25,\ \ b_1/2a=0.35,\ \ h/2a=0.1.$$

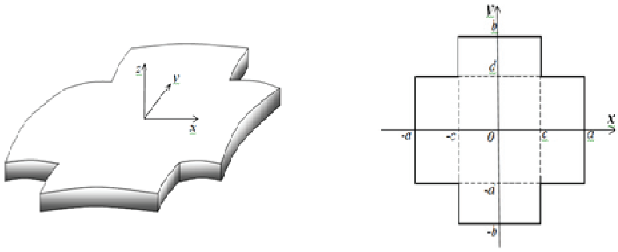


FIGURE 10. Shape of the plan of the laminated FGM shallow shell

Fundamental frequency parameters for clamped and simply supported plates, cylindrical, spherical and hyperbolic paraboloidal shells for different Types 1-1; 1-2; 2-2 and different thickness schemes were calculated. The dependence between the maximum deflection and ratio of the nonlinear frequency to the linear one for spherical and cylindrical shells of the different types and fixed thickness scheme 1-8-1 has been obtained. For example, Fig.11 shows the variation of the nonlinear frequency amplitude relationships (backbone curves) for clamped cylindrical shallow shells of Types 1-1; 1-2; 2-2. Hardening type of nonlinearity is observed for both shells and their different types.

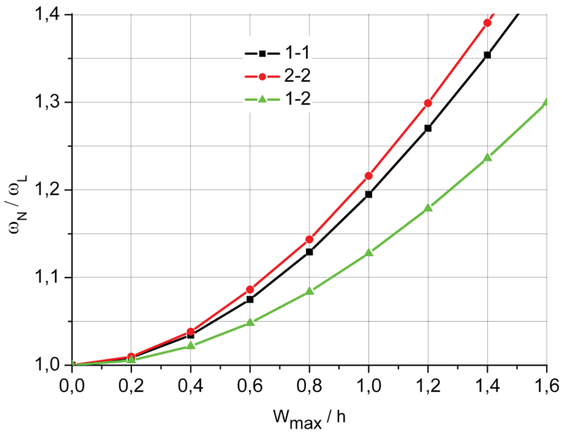


FIGURE 11. Backbone curves for clamped cylindrical shells (1-8-1, p=*0.5*, Al/Al_2O_3)

Variation of the nonlinear frequencies of the maximum amplitude of vibration for simply supported shells is shown in Fig.12.

As follows from Fig. 12 curvature of the shells slightly influences the nonlinear frequencies, while the type of the shell yields significant changes. Shells of the Type 1-1 have the largest values of nonlinear frequencies, and shells of the Type 1-2 have the lowest values of frequencies from another cases considered before.

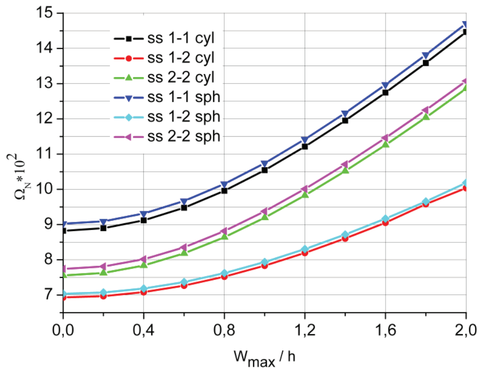


FIGURE 12. Backbone curves for simply-supported (SS) spherical (sph) and cylindrical (cyl) shells with lamination scheme (1-8-1, p=0.5, Al/Al_2O_3)

Papers [35, 36] are devoted to application of the R-functions theory to the study of free vibration of FGMs of shallow shells with nonlinear temperature variation of the properties of the constituent materials. In these papers vibrations of panels with square planform (Fig. 13) and central square cutout, made of FGM ($Si_3N_4/SUS304$) is considered. Similar problem for clamped FG cylindrical panel with free cutout was solved by Malekzaden et al [37].

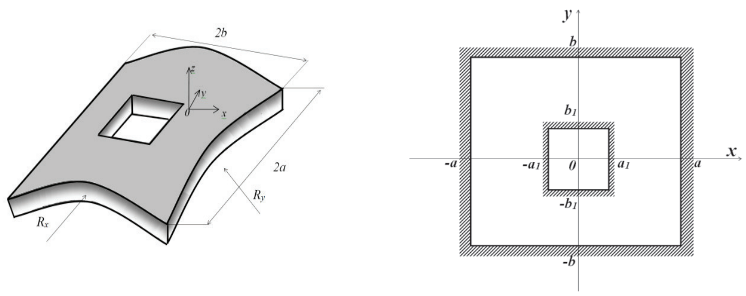


FIGURE 13. FGM shallow shell with inner cutout and its planform

Comparison of the obtained results with corresponding results obtained in Ref. [37] is in a good agreement. The new results for shallow shells shown in Fig. 13, that are clamped over all the border including cutout were obtained in this paper. Geometrical parameters are taken as:

$$a=b,\quad a_1=b_1,\quad a_1/2b=0.3,\quad h/2b=0.1,\ k_1=0;0.1;\ k_2=0;0.1;-0.1.$$

Figure 14 shows influence of volume fraction index and temperature on natural frequencies of the fully clamped FG spherical shell. It should be noted that frequencies for fully clamped FG plate, cylindrical, spherical and hyperbolic paraboloid panels are closed enough to each other. But the magnitude of the volume fraction index effects significantly on the behavior of frequencies. As follows from

Fig. 14 the natural frequencies decrease significantly provided volume fraction of ceramics decreases.

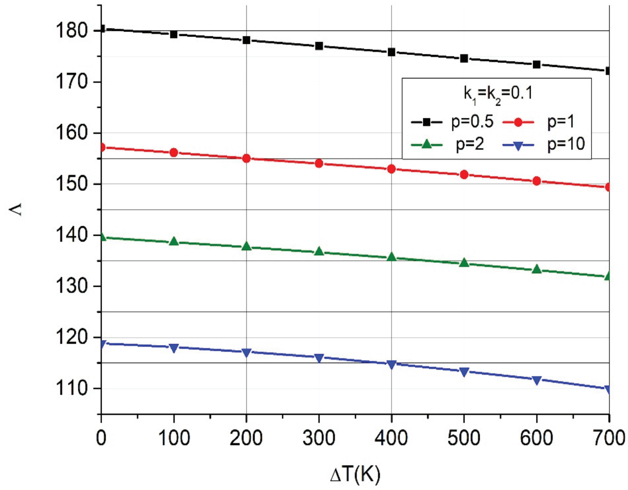


FIGURE 14. Effect of volume fraction index p on the natural frequencies parameters of the clamped FG spherical panels with clamped square cutout subjected to nonlinear temperature rise

Application of the R-functions method to the buckling and stability problems of FGM plates and shallow shells were considered in Ref. [38, 39, 40].

It is assumed that shell consists of three layers and is loaded in the middle plane. The load may be both uniformly or non-uniformly. The power-law distribution in terms of volume fractions is applied to get effective material properties for layers. These properties for different arrangements and thicknesses of layers were calculated in the work due to obtained analytical expressions.

Mathematical formulation is carried out with the framework of the first order shear deformation theory. The proposed approach consists of two steps. The first step is a definition of the pre-buckling state as result of the elasticity problem solution. The critical buckling load and frequencies of FGM shallow shells are determined on the second step.

In order to show the potential of the proposed method, numerical results for frequencies and buckling load of the plate and shallow shells with complex form, and different curvatures are presented. Effects of the power law index, boundary conditions, thickness of core and face sheet layers on fundamental frequencies and critical loads are studied in Ref. [38, 39, 40]. Comparison of the obtained results with known ones for laminated FGM square shallow shells with different boundary conditions confirmed the validation of the developed approach and created software. In particular the buckling and vibration problems for shell shown in Fig. 15 were solved.

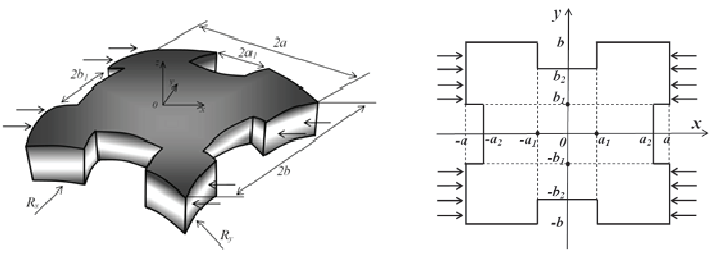


Figure 15. FGM shallow shell with complex planform and its planform

Here geometric parameters are:

$$a_2/2a = 0.4;\ b_2/2a = 0.4;\ k_1 = R_x/2a = 0.2,\ k_2 = R_y/2a = (0;0.2;-0.2),\ b/a = 1,$$
$$b/a = 1;\quad a_1/2a = 0.2;\quad b_1/2a = 0.125;\quad h/2a = 0.1\ .$$

The FGM shell consists of the different mixtures:
M1 (Al/Al_2O_3), M2 ($SiN_4/SUS304$), M3 (Al/ZrO_3).

Properties for these materials are presented in Ref. [40]. Suppose that shell is compressed by uniform uniaxial load in the middle plane as shown in Fig 15. As it was shown in Ref. [40] values of the natural frequencies for cylindrical, spherical and hyperbolic paraboloidal shells are closed each to other for considered curvature.

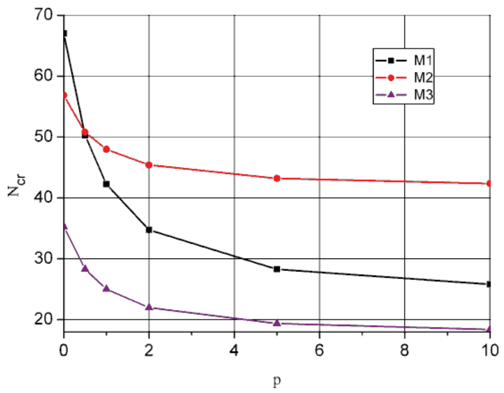


Figure 16. Effect of gradient index on critical load of the clamped (CL) shells for different types of material

Effect of gradient index on critical load $N_{cr} = \dfrac{N_{x0}(2a)^2}{100h^3E_0}$ is shown in Fig. 16 for clamped (CL) and Fig. 17 for simply supported (SS) FGM cylindrical shallow shells. Here $E_0 = 1GPa$. Distribution of layers thickness is taken as 1-2-1. It is observed that critical load is also decreases with increasing gradient index. Behavior of the critical load is the same for both types of the boundary conditions: clamped and simply supported cases. Note that starting from p=0.5 the value of the critical load for material M2 ($Si_3N_4/SUS\,304$) is greater than for FGM M1 and M3. The values of the critical parameter for clamped FGM shell exceeds the corresponding values for simply supported shell in 1,3-1,5 times.

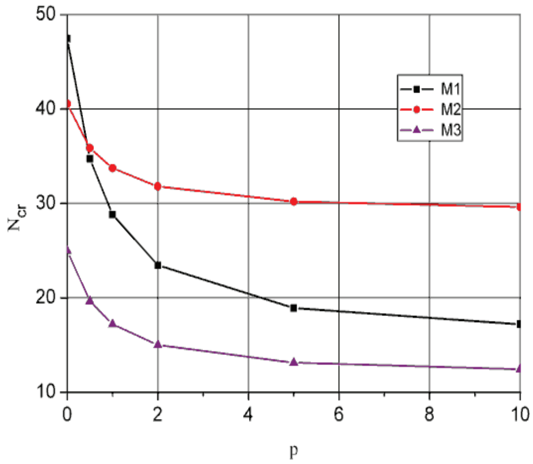


FIGURE 17. Effect of gradient index on critical load of the simply supported (SS) shells for different types of material

7. Conclusion

Review of articles devoted to solving the problems of vibrations and stability of functionally gradient plates and shallow shell by the R-functions method is presented in the given paper. Significant advantages of the developed method are universality in the case of a complex geometric shape of the region; a unified approach to the construction of admissible functions for energy functional; accounting for the subcritical state of the plate or shell; presentation of the solution in an analytical form. The last statement is very important for solution of the nonlinear problems.

The obtained numerical results have a good agreement with available in literature. New numerical results for laminated FGM plates, cylindrical, spherical and hyperbolic paraboloidal shallow shells with complex form, were obtained. Effects of different mechanical and geometrical parameters on frequencies and buckling load were studied.

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