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FROM CLONES TO HYPERCLONES

Abstract. This paper is a survey of different approaches to the study of the lattice of clones of hyperoperations, summarising contributions of the authors in the field. We present three embeddings that are suitable for analysis of the lattice of hyperclones and give more details on embedding of the lattice of hyperclones on a finite set into the lattice of clones on its power set. On a two-element set, we give description of its atoms, coatoms, and the interval generated by unary hyperoperations. On a set, we describe four classes of coatoms, determined by four classes of Rosenberg's relations. Finally, we analyse several Galois connections between particular sets of hyperoperations and relations.

Mathematics Subject Classification (2010): Primary: 08-02, 08A40; Secondary 03C13.

Keywords: clone, clone lattice, Galois connection, partial clones, hyperoperation, hyperclone lattice

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1. Introduction

Nondeterminism is present in every concept where we can obtain different outputs for a same input value. In software systems, repeated executions of a program may produce different results, or concurrent processes may have different reductions. This behaviour can be formally modelled as a function that assigns a set of values to a given argument. These functions are called *hyperoperations*. A widely studied example where such a function appears is the transition function in the definition of non-deterministic finite automata.

Another well-known example is the Kleene's strong three-valued logic [10], that is a multiple-valued logic often considered as a mathematical representation of uncertainty. Besides the logic values `true` and `false`, it includes the third value `unknown`. In this logic, we have

$$\texttt{unknown AND false} = \texttt{false}$$

which is expected result since AND takes the value false if at least one variable is false. If the third value is considered as any of the two possible values, instead of unknown we can write {true, false}. Kleene's logic operations are presented in Figure 1, where $\{0\}, \{1\}$ and $\{0, 1\}$ denote false, true and unknown, respectively.

	NOT$^{\#}$
{0}	{1}
{1}	{0}
{0,1}	{0,1}

AND$^{\#}$	{0}	{1}	{0,1}
{0}	{0}	{0}	{0}
{1}	{0}	{1}	{0,1}
{0,1}	{0}	{0,1}	{0,1}

OR$^{\#}$	{0}	{1}	{0,1}
{0}	{0}	{1}	{0,1}
{1}	{1}	{1}	{1}
{0,1}	{0,1}	{1}	{0,1}

FIGURE 1. Kleenean logic functions NOT$^{\#}$, AND$^{\#}$ and OR$^{\#}$.

We can consider a hyperoperation as a generalisation of three-valued Kleene's logic to 2^k-valued logic, where we have k specified values (singletons) and $2^k - k$ unspecified values that are non-singletons. We assume that input is always specified, and thus we define a hyperoperation as a restriction of such a function to the k-valued input.

This survey is mainly concerned with the composition of hyperoperations and the lattice of clones of hyperoperations generated under such a composition. We recall main notions from clone and hyperclone theory in Section 2. Different approaches to the study of hyperclone lattice properties are presented in Sections 3-6. Section 3 presents three lattice embeddings:

(i) from the lattice of clones on A to the lattice of hyperclones on A;
(ii) from the lattice of hyperclones on A' to the lattice of hyperclones on A, where $\emptyset \neq A' \subseteq A$; and
(iii) from the lattice of hyperclones on A to the lattice of clones on P_A^*, where P_A^* is the set of all nonvoid subsets of A.

Cardinality of the hyperclone lattice on a two-element set, its minimal and maximal elements, and interval generated by unary hyperoperations are contents of Section 4. Section 5 is devoted to three Galois connections between sets of hyperoperations and relations on A. Description of four classes of maximal hyperclones, determined by four classes of maximal total clones, is presented in Section 6.

2. Preliminaries

This section is divided into three parts, the first one devoted to the clones of operations, the second one to relational clones and the third one contains some introductory results on hyperclones. We recall only some main notions and results that will be used throughout the survey. An interested reader can find more details on clone theory in [11], [18] and [26].

2.1. Clones of operations. Let $\mathbb{N}$ be the set of positive integers, A be a finite set with $|A| \geqslant 2$, and $m, n, \ell \in \mathbb{N}$. Let $O_A^{(n)} = A^{A^n}$ be the set of all n-ary operations on A and $O_A = \bigcup_{n \geqslant 1} O_A^{(n)}$ be the set of all finitary operations on A. For $F \subseteq O_A$ let $F^{(n)} = F \cap O_A^{(n)}$. We specify some operations that are significant for this survey:

(i) For $i \in \{1, \dots, n\}$, *i-th n-ary projection* e_i^n is defined by

$$e_i^n(x_1, \dots, x_i, \dots, x_n) = x_i.$$

We denote by J_A the set of all projections on A;

(ii) A ternary *majority operation*, $\mathtt{ma} \in O_A^{(3)}$, is defined by

$$\mathtt{ma}(x, x, y) = \mathtt{ma}(x, y, x) = \mathtt{ma}(y, x, x) = x, \text{ for all } x, y \in A;$$

(iii) A ternary *minority operation*, $\mathtt{mi} \in O_A^{(3)}$, is given by

$$\mathtt{mi}(x, x, y) = \mathtt{mi}(x, y, x) = \mathtt{mi}(y, x, x) = y, \text{ for all } x, y \in A;$$

(iv) For $n > 2$ and $i \in \{1, \dots, n\}$, an n-ary operation s on A is called an *i-th n-ary semiprojection* if it is not a projection, but it holds $s(x_1, \dots, x_n) = x_i$ whenever $|\{x_1, \dots, x_n\}| < n$.

Composition of operations $f \in O_A^{(n)}$ and $g_1, \dots, g_n \in O_A^{(m)}$ is the m-ary operation $f(g_1, \dots, g_n) \in O_A^{(m)}$, defined as follows

$$f(g_1, \dots, g_n)(x_1, \dots, x_m) = f(g_1(x_1, \dots, x_m), \dots, g_n(x_1, \dots, x_m)).$$

Definition 2.1. Set $C \subseteq O_A$ is called *clone of operations* on A if the following two conditions are satisfied:

(i) C contains all projections and
(ii) C is closed with respect to composition.

For a set $F \subseteq O_A$ the least clone containing F will be denoted by $\langle F \rangle_A$ and it holds that

$$\langle F \rangle_A = \bigcap \{C : C \text{ is a clone and } C \supseteq F\}.$$

Whenever F is a finite set, that is if $F = \{f_1, \dots, f_k\}$, we will write $\langle f_1, \dots, f_k \rangle_A$ instead of $\langle \{f_1, \dots, f_k\} \rangle_A$.

Theorem 2.1. *Clones of operations on a finite set A form an algebraic lattice $\mathcal{L}_A$ with respect to the set inclusion. The least element of the lattice is J_A, and the greatest element is O_A. Lattice operations are defined as follows*

$$C_1 \wedge C_2 = C_1 \cap C_2 \quad \text{and} \quad C_1 \vee C_2 = \langle C_1 \cup C_2 \rangle_A.$$

Atoms of this lattice are called *minimal clones*, while the coatoms are called *maximal clones*.

Minimal clones are generated by singletons and usually represented as clones generated by operations of minimal possible arities that generate them. These operations are called *minimal operations* and each of them belongs to one of five types described in the following theorem.

Theorem 2.2. [22] *For every minimal clone $C \subseteq O_A$ we have $C = \langle f \rangle_A$, where operation f is a nontrivial operation (not a projection) of one of the following types:*

(i) *a unary operation, and $f^2(x) = f(x)$ or $f^p(x) = x$ for some prime p;*
(ii) *a binary idempotent operation, i.e., $f(x, x) = x$;*
(iii) *a ternary majority operation;*
(iv) *a ternary minority operation;*
(v) *an n-ary semyprojection, $n > 2$.*

2.2. Relational clones. Another approach to a clone of operations is considering common properties of its elements. We specify particular properties of operations by relations on A. An m-ary relation ρ on A is a subset of A^m. Let $R_A^{(m)}$ be the set of all m-ary relations on A, and $R_A = \bigcup_{m\geqslant 1} R_A^{(m)}$ the set of all relations on A.

Definition 2.2. Let $\delta_{1,2}^{3,A} = \{(x,x,y) : x,y \in A\}$ and let us define on R_A unary operations ζ, τ, pr and binary operations $\cap, \times$ as follows:
(i) for $\rho \in R_A^{(1)}$ or $\rho = \emptyset$ put $\zeta\rho = \tau\rho = \rho$ and $\mathrm{pr}\,\rho = \emptyset$;
(ii) for $\rho \in R_A^{(m)}$, $m \geqslant 2$, define $\zeta\rho, \tau\rho \in R_A^{(m)}$ and $\mathrm{pr}\,\rho \in R_A^{(m-1)}$ as

$$\zeta\rho = \{(a_1,\dots,a_{m-1},a_m) : (a_m,a_1,\dots,a_{m-1}) \in \rho\};$$
$$\tau\rho = \{(a_1,a_2,a_3,\dots,a_m) : (a_2,a_1,a_3,\dots,a_m) \in \rho\};$$
$$\mathrm{pr}\,\rho = \{(a_2,\dots,a_m) : (\exists a_1 \in A)(a_1,a_2,\dots,a_m) \in \rho\};$$

(iii) for $\rho_1,\rho_2 \in R_A^{(m)}$ define $\rho_1 \cap \rho_2 \in R_A^{(m)}$ as

$$\rho_1 \cap \rho_2 = \{(a_1,\dots,a_m) : (a_1,\dots,a_m) \in \rho_1 \text{ and } (a_1,\dots,a_m) \in \rho_2\};$$

(iv) for $\rho_1 \in R_A^{(m)}$ and $\rho_2 \in R_A^{(n)}$ define $\rho_1 \times \rho_2 \in R_A^{(m+n)}$ as

$$\rho_1 \times \rho_2 = \{(a_1,\dots,a_m,b_1,\dots,b_n) : (a_1,\dots,a_m) \in \rho_1 \text{ and } (b_1,\dots,b_n) \in \rho_2\}.$$

A subuniverse of the algebra $\mathcal{R}_A = (R_A; \times, \cap, \zeta, \tau, \mathrm{pr}, \delta_{1,2}^{3,A})$ is called a *relational clone* (or *co-clone*) on A.

For an m-ary relation ρ we shall denote by ρ_ℓ^* the set of $m \times \ell$ matrices over A whose columns are elements of ρ, and by $\rho^* = \bigcup_{\ell\geqslant 1} \rho_\ell^*$ the set of all such matrices over A. Moreover, let us introduce the following useful denotation: if $f \in O_A^{(\ell)}$ and $M = (a_{ij})_{m\times\ell}$, $a_{ij} \in A, 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant \ell$, then

$$f(M) = \begin{pmatrix} f(a_{11},a_{12},\dots,a_{1\ell}) \\ f(a_{21},a_{22},\dots,a_{2\ell}) \\ \dots \\ f(a_{m1},a_{m2},\dots,a_{m\ell}) \end{pmatrix}.$$

We say that an operation $f \in O_A$ *preserves* a relation $\rho \in R_A$, or that ρ is an *invariant* relation of f, if the following implication holds:

$$M \in \rho^* \Rightarrow f(M) \in \rho.$$

The set of all operations preserving ρ is denoted by $\mathrm{Pol}\,\rho$, and dually, the set of all relations that are preserved by f is denoted by $\mathrm{Inv}\,f$. The pair $(\mathrm{Pol}, \mathrm{Inv})$ is a Galois connection between relations and operations on A. Furthermore, every clone is of the form $\mathrm{Pol}\,Q$, for some $Q \subseteq R_A$, and every relational clone is of the form $\mathrm{Inv}\,F$, for some $F \subseteq O_A$ (see [1, 9]).

We list some interesting properties that a relation can possess:

(i) We say that $\rho \subseteq A^m$, $m \geqslant 1$, is *totaly reflexive* if $i \neq j$ and $a_i = a_j$ implies $(a_1,\dots,a_i,\dots,a_j,\dots,a_m) \in \rho$ for all $i,j \in \{1,\dots,m\}$.
(ii) An m-ary relation ρ on A, $m \geqslant 1$, is *totaly symmetric* if $(a_1,\dots,a_m) \in \rho$ implies $(a_{\pi(1)},\dots,a_{\pi(m)}) \in \rho$ for every permutation π on $\{1,\dots,m\}$.

(iii) For a relation $\rho \subseteq A^m$, an element $c \in A$ is *central* if $(c, x_2, \ldots, x_m) \in \rho$ for all $x_2, \ldots, x_m \in A$. The set C_ρ of all central elements is called *center* of ρ. If a relation $\rho \subsetneq A^m$ is totaly reflexive, totaly symmetric and the center $C_\rho \neq \emptyset$, we say that ρ is *central relation.*

(iv) An m-ary relation ρ on A, $m \geqslant 3$, is *m-regular* if there is a nonempty family $\Theta = \{\theta_1, \ldots, \theta_h\}$ of equivalence relations on A, having the following properties:
  (a) each θ_i, $1 \leqslant i \leqslant h$, has exactly m equivalence classes,
  (b) if A_i is an equivalence class of θ_i, $i \in \{1, \ldots, h\}$, then $\bigcap_{i=1}^h A_i \neq \emptyset$,
  (c) $(a_1, \ldots, a_m) \in \rho$ if and only if for every $1 \leqslant i \leqslant h$ there exist $1 \leqslant k < \ell \leqslant m$ such that $(a_k, a_\ell) \in \theta_i$.

Description of all maximal clones, giving an effective characterisation, is due to Rosenberg and is given by the following theorem.

Theorem 2.3. [21] *A clone $C \subseteq O_A$ is maximal if and only if there is a relation ρ such that $C = \operatorname{Pol} \rho$ and ρ belongs to one of the following classes:*

(R_1) *bounded partial orders;*
(R_2) *relations of the form $\rho = \{(x, \pi(x)) : x \in A\}$, where π is a p-regular permutation of A and p is prime;*
(R_3) *relations of the form $\rho = \{(a_1, a_2, a_3, a_4) : a_1 + a_2 = a_3 + a_4\}$, for an elementary Abelian p-group $(A, +, -, 0)$ on A;*
(R_4) *nontrivial equivalence relations;*
(R_5) *central relations;*
(R_6) *m-regular relations ($m \geqslant 3$).*

2.3. Clones of hyperoperations. We can finally introduce the notion of the hyperoperation and consider clones of hyperoperations.

Definition 2.3. Let $\mathcal{P}(A)$ be the power set of A. An n-ary *hyperoperation* f on A is a mapping

$$f : A^n \to \mathcal{P}(A) \setminus \{\emptyset\}.$$

We will write P_A^* for $\mathcal{P}(A) \setminus \{\emptyset\}$. Let $H_A^{(n)} = (P_A^*)^{A^n}$ be the set of all n-ary hyperoperations on A, $n \geqslant 1$, and $H_A = \bigcup_{n \geqslant 1} H_A^{(n)}$ be the set of all finitary hyperoperations on A. For $F \subseteq H_A$ let $F^{(n)} = F \cap H_A^{(n)}$.

An i-th n-ary *(hyper)projection* on A, $1 \leqslant i \leqslant n$, is the n-ary hyperoperation $e_i^n \in H_A^{(n)}$ defined by

$$e_i^n(x_1, \ldots, x_i, \ldots, x_n) = \{x_i\}.$$

Let $f \in H_A^{(n)}$ and $g_1, \ldots, g_n \in H_A^{(m)}$, for positive integers m and n. The composition of hyperoperations f and $g_1, \ldots, g_n$ is the m-ary hyperoperation $f[g_1, \ldots, g_n]$ defined by

$$f[g_1, \ldots, g_n](x_1, \ldots, x_m) = \bigcup_{\substack{y_i \in g_i(x_1, \ldots, x_m) \\ 1 \leqslant i \leqslant n}} f(y_1, \ldots, y_n). \tag{2.1}$$

Definition 2.4. Set $C \subseteq H_A$ is called a *clone of hyperoperations* (or *hyperclone*) on A if the following two conditions are satisfied:

(i) C contains all (hyper)projections and
(ii) C is closed with respect to composition.

For a set F of hyperoperations, the least hyperclone containing F is denoted by $\langle F \rangle_h$.

Analogous to the case of clones, the set of all hyperclones, ordered by set inclusion, and denoted by $\mathcal{L}_A^h$, is an algebraic lattice. Notice that here letter h in superscript emphasises the fact that we are considering a hyperclone lattice, and should not be mistaken for the h-th power of a set.

3. Embeddings

The following notions will be used throughout this section.

Definition 3.1. Let $\varphi : L \to L'$ be an injective mapping from the latice $(L, \leqslant)$ into the lattice $(L', \leqslant)$. Then:

- φ is an *order embedding* if for all $a, b \in L$ it holds
$$a \leqslant b \Rightarrow \varphi(a) \leqslant \varphi(b);$$
- φ is a *full order embedding* if for all $a, b \in L$ and $c' \in L'$ satisfying $\varphi(a) \leqslant c' \leqslant \varphi(b)$, there exists $c \in L$ such that $\varphi(c) = c'$.

3.1. Embedding of $\mathcal{L}_A$ into $\mathcal{L}_A^h$.

Let us define a mapping $\lambda : L_A \to L_A^h$ by
$$\lambda(C) = \bigcup_{n \geqslant 1} \big\{ f \in H_A^{(n)} : (\exists f' \in C)(\forall \mathbf{x} \in A^n)\, f(\mathbf{x}) = \{f'(\mathbf{x})\} \big\}, \quad C \in L_A.$$
It is easy to show that $\lambda(C)$ is a hyperclone, for every clone C.

Proposition 3.1. *The mapping λ is a full order embedding.*

Proof. Trivially, λ is an order embedding. Let H be a hyperclone with the property $\lambda(J_A) \subseteq H \subseteq \lambda(O_A)$. Since $f \in H^{(n)}$ implies $|f(x_1, \dots, x_n)| = 1$, for all $(x_1, \dots, x_n) \in A^n$, it is obvious that
$$C = \bigcup_{n \geqslant 1} \big\{ f' \in O_A^{(n)} : (\exists f \in H)(\forall \mathbf{x} \in A^n)\, f(\mathbf{x}) = \{f'(\mathbf{x})\} \big\}$$
is a clone for which it holds $\lambda(C) = H$. □

We usually identify the hyperclone $\lambda(C)$ with the clone C. Especially, we will denote by J_A both the set of projections and the set of hyperprojections on A.

Lemma 3.1. *Let A be a finite set with $|A| \geqslant 2$. Every hyperclone generated by a unary constant hyperoperation on A is minimal.*

Proof. If B is a nonempty subset of A, then $c_B : A \to P_A^*$ given by $c_B(x) = B$, $x \in A$, is a unary constant hyperoperation. Evidently,
$$\langle c_B \rangle_h = \bigcup_{n \geqslant 1} \big\{ f \in H_A^{(n)} : (\forall (x_1, \dots, x_n) \in A^n)\, f(x_1, \dots, x_n) = B \big\} \cup J_A.$$

Now we will show that $\langle c_B\rangle_h$ is a minimal hyperclone. Suppose that C is a hyperclone on A such that $J_A \subsetneq C \subseteq \langle c_B\rangle_h$. From this we conclude that there exists $f \in C \setminus J_A \subseteq \langle c_B\rangle_h \setminus J_A$. Therefore, $f(x_1,\dots,x_n)=B$, for all $x_1,\dots,x_n \in A$, which yields $c_B = f[e_1^1,\dots,e_1^1] \in C$, and consequently $\langle c_B\rangle_h \subseteq C$. □

Since there are $2^{|A|}-1$ nonempty subsets of A and $\langle c_{B_1}\rangle_h \neq \langle c_{B_2}\rangle_h$ for $B_1 \neq B_2$, we immediately obtain the following

Corollary 3.1. *Let A be a finite set with $|A| \geqslant 2$. There are at least $2^{|A|}-1$ minimal clones in the lattice L_A^h.*

Let us consider the following interesting theorems which are extensions of the known results from the clone theory.

Theorem 3.1. *On any finite set A, with $|A| \geqslant 2$, there are three minimal hyperclones such that their join contains all hyperoperations.*

Proof. It is proved in [5] that there are two minimal clones such that their join is the clone of all total operations O_A. Nevertheless, $\lambda(O_A)$ (which may be identified with O_A) is a maximal hyperclone, as proved by Romov in [19]. From the previous lemma we can conclude that the set of minimal hyperclones that are not minimal in the lattice L_A is not empty. Hence, it is sufficient to choose the third minimal hyperclone from that set. □

Theorem 3.2. *There are finite maximal chains in the hyperclone lattice.*

Proof. It is known that there are finite maximal chains in the interval $[J_A, \langle O_A^{(1)}\rangle_h]$. The interval $[\langle O_A^{(1)}\rangle_h, O_A]$ is also a maximal chain. With the hyperclone H_A (since O_A is maximal in the hyperclone lattice), we get the finite maximal chain in the hyperclone lattice. □

3.2. Embedding of $\mathcal{L}_{A'}^h$ into $\mathcal{L}_A^h$.

Let A' be a nonempty subset of A and let us define a mapping $\mu : L_{A'}^h \to L_A^h$ by

$$\mu(C) = \{f \in H_A : f|_{A'} \in C\}, \quad C \in L_{A'}^h.$$

For every hyperclone C on A', $\mu(C)$ is a hyperclone on A. This holds since for every $n \in \mathbb{N}$ and every $1 \leqslant i \leqslant n$ we have $e_i^{n,A'} = e_i^{n,A}|_{A'} \in C$, i.e., $J_A \subseteq \mu(C)$, and also if we take hyperoperations $f \in \mu(C)^{(n)}$ and $g_1,\dots,g_n \in \mu(C)^{(m)}$, then $f|_{A'}[g_1|_{A'},\dots,g_n|_{A'}] = f[g_1,\dots,g_n]|_{A'}$.

Proposition 3.2. *The mapping μ is a full order embedding.*

Proof. It can easily be proven that μ is an order embedding. If H is a hyperclone on A that satisfies $\mu(J_{A'}) \subseteq H \subseteq \mu(H_{A'})$, we will prove that $C = H|_{A'}$ is a hyperclone on A' for which it holds $\mu(C) = H$. Inclusion $H \subseteq \mu(C)$ trivially holds. Now let us suppose $g \in \mu(C)^{(n)}$. Since $g|_{A'} \in C = H|_{A'}$, there exists hyperoperation $f \in H^{(n)}$ such that $g|_{A'} = f|_{A'}$. We define a hyperoperation $h \in H_A^{(n+1)}$ as follows

$$h(x,x_1,\dots,x_n) = \begin{cases} \{x\}, & x_1,\dots,x_n \in A' \\ g(x_1,\dots,x_n), & \text{otherwise.} \end{cases}$$

Obviously, $h|_{A'} = e_1^{n+1,A}|_{A'}$, which implies $h \in \mu(J_{A'}) \subseteq H$. It also holds

$$h(f(x_1,\dots,x_n), x_1,\dots,x_n) = g(x_1,\dots,x_n),$$

for all $x_1,\dots,x_n \in A$, and thus $g = h[f, e_1^n,\dots,e_n^n]$, which means that g can be represented as a composition of the hyperoperations from H, implying $g \in H$. Therefore, $\mu(C) \subseteq H$. □

3.3. Embedding of $\mathcal{L}_A^h$ into $\mathcal{L}_{P_A^*}$. This is the most studied embedding and it independently attracted attention of several authors [17, 23, 24, 27]. To every hyperoperation f on A we can assign an operation $f^\#$ on P_A^* defined by

$$f^\#(X_1,\dots,X_n) = \bigcup \big\{f(x_1,\dots,x_n) \,:\, x_i \in X_i, 1 \leqslant i \leqslant n\big\},\ X_1,\dots,X_n \in P_A^*.$$

For a set F of hyperoperations, let $F^\# = \{f^\# : f \in F\}$.

Notice that for $f \in H_A^{(n)}$ and $g_1,\dots,g_n \in H_A^{(m)}$ we can write

$$f[g_1,\dots,g_n](x_1,\dots,x_m) = f^\#\big(g_1(x_1,\dots,x_m),\dots,g_n(x_1,\dots,x_m)\big)$$

instead of (2.1), and we will use this notation throughout the paper whenever it is convenient.

Although it seems natural that we should map C into $C^\#$, it is not the desired embedding from the lattice of hyperclones on A into the lattice of clones on P_A^*, since C being a hyperclone on A does not imply that $C^\#$ is a clone on P_A^*. More precisely, it is easy to see that $(e_i^{n,A})^\# = e_i^{n,P_A^*}$, but composition of hyperoperations is not compatible with the operator #. Namely, we have

$$(f[g_1,\dots,g_n])^\#(X_1,\dots,X_n) \subseteq f^\#\big(g_1^\#(X_1,\dots,X_n),\dots,g_n^\#(X_1,\dots,X_n)\big),$$

and generally equality does not hold, which we illustrate by the following example.

Example 3.1. Let us define binary hyperoperations f, g_1 and g_2 on $A = \{0,1\}$ by

x_1	x_2	f	g_1	g_2
0	0	{1}	{1}	{1}
0	1	{1}	{0,1}	{0}
1	0	{1}	{0,1}	{0}
1	1	{0}	{0}	{0,1}

Then we calculate

$$(f[g_1,g_2])^\#(\{1\},\{0,1\}) = \{1\} \text{ and } (f^\#(g_1^\#,g_2^\#))(\{1\},\{0,1\}) = \{0,1\}.$$

Consequently, we define the mapping

$$\eta : L_A^h \to L_{P_A^*},\ \ C \mapsto \langle C^\# \rangle_{P_A^*}.$$

This mapping is an order embedding, though not a full one, i.e., there are $C_1, C_2 \in L_A^h$ such that $[\eta(C_1), \eta(C_2)] \smallsetminus im\,\eta \neq \emptyset$. (See [17, 23, 24]). We prove that the cardinality of the set of such clones is the continuum.

Let A be $\{0,1,2,\ldots\}$, $|A| \geqslant 3$, $m \geqslant 2$ and $g_m \in H_A^{(m)}$ the hyperoperation defined by

$$g_m(x_1,\ldots,x_m) = \begin{cases} \{2\}, & (x_1,\ldots,x_m) \in J_m \\ \{0\}, & \text{otherwise,} \end{cases}$$

where J_m is the set of all m-tuples with one coordinate equal 2 and all other coordinates equal 1. Let us define the hyperoperation $f_{m+1} \in H_A^{(m+1)}$ by

$$f_{m+1}(x_1, x_2 \ldots, x_{m+1}) = \begin{cases} A, & x_1 \neq x_2 \\ g_m(x_2,\ldots,x_{m+1}), & x_1 = x_2. \end{cases}$$

Let $\mathcal{F} = \{f_i : i \geqslant 3\}$ and $F_m = \mathcal{F} \smallsetminus \{f_m\}$. Thus, extended operation from f_{m+1} is the operation $f^{\#}_{m+1} \in (O_{P^*_A})^{(m+1)}$ defined by

$$\begin{aligned} f^{\#}_{m+1}(X_1, X_2 \ldots, X_{m+1}) &= \bigcup_{1 \leqslant i \leqslant m+1} \{f_{m+1}(x_1, x_2 \ldots, x_{m+1}) : x_i \in X_i\} \\ &= \begin{cases} g^{\#}_m(X_2,\ldots,X_{m+1}), & X_1 = X_2,\ |X_1| = 1 \\ A, & \text{otherwise.} \end{cases} \end{aligned}$$

Lemma 3.2. [16] *Let $C \in L^h_A$ and $\emptyset \neq Q \subseteq \mathcal{F}$. Then $\eta(C) \neq \langle Q^{\#}\rangle_{P^*_A}$.*

Proof. Let Q be a nonvoid subset of $\mathcal{F}$. Then, there is $m \geqslant 2$ such that $f_{m+1} \in Q$. Suppose to the contrary, that there is a hyperclone C such that its η image $\eta(C)$ is the clone generated by $Q^{\#}$, i.e., $\eta(C) = \langle Q^{\#}\rangle_{P^*_A}$. Since C is a hyperclone and $f_{m+1} \in C$, the hyperoperation $g_m \in H_A^{(m)}$, defined by $g_m(x_1,\ldots,x_m) = f_{m+1}(x_1,x_1,\ldots,x_m)$, also belongs to C. Therefore, $g^{\#}_m \in C^{\#} \subseteq \eta(C)$. However, we shall prove that $g^{\#}_m \notin \langle Q^{\#}\rangle_{P^*_A}$. For every $i \geqslant 2$ we have $f^{\#}_i(A,A,\ldots,A) = A$, and $g^{\#}_m \notin \mathrm{Pol}_{P^*_A}(A)$, because $g^{\#}_m(A,A,\ldots,A) = \{0,2\} \neq A$. Since $\langle Q^{\#}\rangle_{P^*_A} \subset \mathrm{Pol}_{P^*_A}(A)$, it holds $g^{\#}_m \notin \langle Q^{\#}\rangle_{P^*_A}$. □

The proof of the following lemma is a modification of Rónyi's proof of Yanov's and Mučnik's statement that there is a countable infinite set of operations on a finite set A with $|A| \geqslant 3$.

Lemma 3.3. [16] *If $i \geqslant 3$, then $f^{\#}_i \notin \langle F^{\#}_i\rangle_{P^*_A}$.*

Proof. Let us define for every $m \geqslant 2$ a relation $\rho^{(m)} \subseteq (P^*_A)^m$ by $\rho^{(m)} = A_m \cup B_m$, where A_m is the set of all m-tuples with exactly one coordinate equal $\{2\}$ and all others equal to $\{1\}$ and $B_m = \{\{0\},\{2\},\{0,2\},A\}^m \smallsetminus \{(\{2\},\{2\},\ldots,\{2\})\}$. We are going to show that $f^{\#}_{m+1} \notin \mathrm{Pol}_{P^*_A}\,\rho^{(m)}$ and $f^{\#}_{i+1} \in \mathrm{Pol}_{P^*_A}\,\rho^{(m)}$, $i \neq m$.

For the $m \times (m+1)$ matrix

$$M = \begin{pmatrix} \{2\} & \{2\} & \{1\} & \cdots & \{1\} \\ \{1\} & \{1\} & \{2\} & \cdots & \{1\} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \{1\} & \{1\} & \{1\} & \cdots & \{2\} \end{pmatrix} \in (\rho^{(m)})^*$$

(columns of this matrix are all in A_m), it holds $f^{\#}_{m+1}(M) = (\{2\}, \{2\}, \dots, \{2\}) \notin \rho^{(m)}$. Therefore, $f^{\#}_{m+1} \notin \mathrm{Pol}_{P^*_A}\, \rho^{(m)}$.

Suppose that there is $i \neq m$ such that $f^{\#}_{i+1} \notin \mathrm{Pol}_{P^*_A}\, \rho^{(m)}$. Then, there is a matrix $M_i = (\mathbf{X}_1\ \mathbf{X}_2\ \cdots\ \mathbf{X}_{i+1}) \in (\rho^{(m)})^*$, such that $f^{\#}_{i+1}(M_i) = (Y_1, Y_2, \dots, Y_m) \notin \rho^{(m)}$. Since $im f^{\#}_{i+1} = \{\{0\}, \{2\}, \{0,2\}, A\}$, it follows $(Y_1, Y_2, \dots, Y_m) = (\{2\}, \{2\}, \dots, \{2\})$ and it is possible only for $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{i+1} \in A_m, \mathbf{X}_1 = \mathbf{X}_2$ and $i = m$, which is obviously a contradiction. □

Theorem 3.3. [16] *The interval* $[\eta(J_A), \eta(H_A)] \smallsetminus im\, \eta$ *has the cardinality of the continuum.*

Proof. Since η is an order embedding, η is injective and for $F, G \in L^h_A$, $F \subseteq G$ implies $\eta(F) \subseteq \eta(G)$. So, for $\langle Q \rangle_h \subseteq H_A$ it follows

$$\langle Q^{\#} \rangle_{P^*_A} \subseteq \eta(\langle Q \rangle_h) \subseteq \eta(H_A),\ Q \subseteq \mathcal{F}.$$

On the other hand, $\eta(J_A) \subseteq \langle Q^{\#} \rangle_{P^*_A}$, because $\eta(J_A) = J_{P^*_A}$. By Lemma 3.2, we have that $\langle Q^{\#} \rangle_{P^*_A} \notin im\, \eta$, for any $Q \subseteq \mathcal{F}$. Finally, Lemma 3.3 implies that for all $Q_1, Q_2 \subseteq \mathcal{F}$ if $Q^{\#}_1 \neq Q^{\#}_2$ then $\langle Q^{\#}_1 \rangle_{P^*_A} \neq \langle Q^{\#}_2 \rangle_{P^*_A}$. Since $\mathcal{F}$ is countable, we obtain the proof. □

4. Lattice $\mathcal{L}^h_2$ of hyperclones on a two-element set

A significant difference between the lattice of clones and that of hyperclones is evident even on a two-element set. Namely, it is a well known result of Post, who gave the complete description of the lattice of clones on E_2, that the lattice $\mathcal{L}_2$ is countable. However, Machida in [12] proved that the lattice of hyperclones on E_2 is of continuum cardinality. In the first part of this section we will present this result in more detail. In the remainder of the section we describe atoms and coatoms of the lattice $\mathcal{L}_2$, as well as one interval in this lattice generated by unary hyperoperations.

4.1. Cardinality of the lattice of hyperclones on E_2. Since the cardinality of the set of hyperoperations on a two-element set is countable, the lattice of hyperclones has at most the cardinality of continuum. Therefore, to show that the lattice $\mathcal{L}^h_2$ is exactly of continuum cardinality, it suffices to find at least continuum many distinct hyperclones on E_2. In order to do that we define the following sequence of hyperoperations.

Definition 4.1. [12] For every $n \geqslant 1$, let g_n be an n-ary hyperoperation on E_2

$$g_n(x_1, \dots, x_n) = \begin{cases} \{1\}, & \text{if } x_1 + \cdots + x_n \leqslant 1 \\ \{0,1\}, & \text{otherwise.} \end{cases}$$

Let us denote $\mathcal{G} = \{g_n : n \geqslant 1\}$ and $G_n = \mathcal{G} \smallsetminus \{g_n\}$, $n \geqslant 1$.

Lemma 4.1. [12] *For every* $n \geqslant 1$, *it holds* $g_n \notin \langle G_n \rangle_h$.

Proof. Let J_2 be the set of all projections on E_2. Note that for every m-ary hyperoperation $f \in \langle G_n\rangle_h \setminus J_2$, it holds $1 \in f(a_1,\ldots,a_m)$ for all $a_1,\ldots,a_m \in E_2$. Suppose that $g_n \in \langle G_n\rangle_h$. Then for some $m \neq n$ there exist $g_m, h_1,\ldots,h_m \in \langle G_n\rangle_h$ such that $g_n = g_m[h_1,\ldots,h_m]$. We can distinguish the following cases:

1) Suppose there exist distinct $i, j \in \{1,\ldots,m\}$ such that $h_i, h_j \in \langle G_n\rangle_h \setminus J_2$. Since g_m is totally symmetric, without loss of generality we can assume $h_1, h_2 \in \langle G_n\rangle_h \setminus J_2$. Then for $\mathbf{0} = (0,\ldots,0)$ we have

$$\begin{aligned}\{1\} = g_n(\mathbf{0}) &= g_m[h_1,h_2,h_3,\ldots,h_m](\mathbf{0})\\ &= g_m^{\#}(h_1(\mathbf{0}),h_2(\mathbf{0}),h_3(\mathbf{0}),\ldots,h_m(\mathbf{0}))\\ &\supseteq g_m^{\#}(\{1\},\{1\},h_3(\mathbf{0}),\ldots,h_m(\mathbf{0}))\\ &= \{0,1\},\end{aligned}$$

which is a contradiction.

2) Suppose that exactly one $h_i \in \langle G_n\rangle_h \setminus J_2$. As in the previous case we may assume $h_i = h_1$. Then $h_2 \in J_2$, i.e., $h_2 = e_j^n$, for some $j \in \{1,\ldots,n\}$. Let $\mathbf{a} = (a_1,\ldots,a_n)$, such that $a_j = 1$ and $a_\ell = 0$, $\ell \in \{1,\ldots,n\} \setminus \{j\}$.

$$\begin{aligned}\{1\} = g_n(\mathbf{a}) &= g_m[h_1,e_j^n,e_{j_3}^n,\ldots,e_{j_m}^n](\mathbf{a})\\ &= g_m^{\#}(h_1(\mathbf{a}),e_j^n(\mathbf{a}),e_{j_3}^n(\mathbf{a}),\ldots,e_{j_m}^n(\mathbf{a}))\\ &\supseteq g_m^{\#}(\{1\},\{1\},a_{j_3},\ldots,a_{j_m})\\ &= \{0,1\},\end{aligned}$$

which is also a contradiction.

3) Suppose that $\{h_1,\ldots,h_m\} \subseteq J_2$. Hence there exist $i_1,\ldots,i_m \in \{1,\ldots,n\}$ such that $g_n(x_1,\ldots,x_n) = g_m(x_{i_1},\ldots,x_{i_m})$, for every $(x_1,\ldots,x_n) \in E_2^n$. If for distinct $j,k \in \{1,\ldots,m\}$ we have $i_j = i_k$, than for the n-tuple $\mathbf{a} = (a_1,\ldots,a_n)$ such that $a_{i_j} = 1$ and $a_\ell = 0$ for $\ell \in \{1,\ldots,n\} \setminus \{i_j\}$ we easily obtain

$$\{1\} = g_n(\mathbf{a}) = g_m(a_{i_1},\ldots,1,\ldots,1,\ldots,a_{i_m}) = \{0,1\}.$$

which is obviously impossible.

The only other possibility is that all of $i_1,\ldots,i_m$ are mutually distinct. Thus, $g_n \neq g_m$ implies $m < n$. Therefore we can pick the least $j \in \{1,\ldots,n\} \setminus \{i_1,\ldots,i_m\}$, and then choose n-tuple $\mathbf{a} = (a_1,\ldots,a_n)$, such that $a_j = a_{i_1} = 1$ and $a_\ell = 0$, for $\ell \in \{1,\ldots,n\} \setminus \{j,i_1\}$. Now we have

$$\{0,1\} = g_n(\mathbf{a}) = g_m(a_{i_1},\ldots,a_{i_m}) = \{1\},$$

and once again it is a contradiction.

Since for all the cases the assumption $g_n \in \langle G_n\rangle_h$ yields a contradiction, we conclude that $g_n \notin \langle G_n\rangle_h$. □

Now we can easily prove the following theorem.

Theorem 4.1. [12] *The lattice $\mathcal{L}_2^h$ of all hyperclones on E_2 has the cardinality of continuum.*

Proof. Let F_1 and F_2 be distinct nonempty subsets of $\mathcal{G}$. It follows immediately from the previous lemma that they generate distinct hyperclones, i.e., $\langle F_1\rangle_h \neq \langle F_2\rangle_h$. Since $\mathcal{G}$ is countable, we have $|\mathcal{P}(\mathcal{G}) \setminus \{\emptyset\}| = \mathfrak{c}$, which means that $\mathcal{L}_2^h$ contains a set of hyperclones with the cardinality of continuum, and consequently $\left|\mathcal{L}_2^h\right| = \mathfrak{c}$. □

4.2. Maximal hyperclones on E_2. In 1974 Tarasov [27] introduced a new composition of partial functions on E_2. For $n \geqslant 1$ and $D_f \subseteq \{0,1\}^n$, a mapping $f : D_f \to \{0,1\}$ is a partial operation on $\{0,1\}$ with domain D_f. Here f can be extended to the n-ary operation f^p on $\{0,1,2\}$ by setting for all $\mathbf{x} \in \{0,1,2\}^n$

$$f^p(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \mathbf{x} \in D_f \\ 2, & \text{otherwise.} \end{cases}$$

The standard composition of partial Boolean functions is the usual composition of their images under $f \mapsto f^p$ followed by restriction to $\{0,1\}$. With respect to this composition one can define clones of partial functions. Freivald [8] completely described all maximal partial clones on E_2, showing that there are exactly 8 such clones. Tarasov introduced a different composition of partial functions on a two-element set. With this new definition, a partial function essentially coincide with a hyperoperation on E_2.

For a set F of hyperoperations on E_2 and for every mapping $\alpha : \{1,\dots,n\} \to \{1,\dots,m\}$, the *place transformation* $\delta_\alpha : F^{(n)} \to F^{(m)}$, is defined by

$$\delta_\alpha(f)(x_1,\dots,x_m) = f(x_{\alpha(1)},\dots,x_{\alpha(n)}).$$

The δ-closure of the set F is the set

$$\delta(F) = \bigcup_{n\in\mathbb{N}} \left\{\delta_\alpha(f) \mid f \in F^{(n)}, \alpha : \{1,\dots,n\} \to \{1,\dots,m\}, m \leqslant n\right\}.$$

In [14], the authors adjust this description to the language of clones, by defining the set of all operations on P_A^* that *r-preserves* $\rho \subseteq E_3^m$ as follows

$$r\,\mathrm{Pol}\;\rho = \left\{f \in O_{P_A^*} : (\delta(\{f\}))^\# \subseteq \mathrm{Pol}\;\rho\right\}.$$

If there is a relation $\rho_1 \subseteq E_2^m$ such that $\rho = \rho_1 \cup (E_3^m \setminus E_2^m)$, we will write $rPOL\,\rho_1$.

Theorem 4.2. [27] *There are nine maximal hyperclones on E_2. They are of the form $M_i' = M_i|_{\{0,1\}}$, $i = 1,\dots,9$, where*
$M_1 = r\,\mathrm{Pol}(0\;\;1)$,
$M_2 = rPOL\{(x_1,x_2) \in E_2^2 : (x_1,x_2) \neq (1,0)\}$,
$M_3 = rPOL\{(x_1,x_2) \in E_2^2 : (x_1,x_2) \notin \{(0,0),(1,1)\}$,
$M_4 = rPOL\{(x_1x_2,x_3,x_4) \in E_2^4 : x_1+x_2+x_3+x_4 = 0\}$,
$M_5 = rPOL\{(x_1,x_2,x_3) \in E_2^3 : (x_1,x_2,x_3) \notin \{(0,1,1),(1,0,0)\}\}$,
$M_6 = rPOL(0)$,
$M_7 = rPOL(1)$,
$M_8 = rPOL\{(x_1,x_2) \in E_2^2 : (x_1,x_2) \notin \{(0,0),(1,0),(1,1)\}\}$,
$M_9 = rPOL\{(x_1,x_2,x_3,x_4) \in E_2^4 : x_1 = x_2 = x_3 = x_4 \vee (x_2 \neq x_3 \wedge x_1 \neq x_4)\}$.

The proof of the previous theorem consists of two parts:

(i) the proof that any set $C \subseteq H_2^{\#}$ such that $C \not\subseteq M_i$ for each $i \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is complete in $H_2^{\#}$ and

(ii) the proof that sets M_i, $1 \leqslant i \leqslant 9$, are pairwise disjoint. This was done by creating suitable representatives.

There is a minor mistake in the part (ii) of the proof of Tarasov. The representative given for the set $M_5 \setminus M_9$ is the operation $g_4 \in E_3^4$ defined as

$$g_4(x_1,x_2,x_3,x_4) = \begin{cases} 0, & (x_1,x_2,x_3,x_4) \in \{(0,1,0,1),(1,0,0,1),(0,1,1,0)\} \\ 1, & (x_1,x_2,x_3,x_4) = (1,0,1,0) \\ 2, & \text{otherwise.} \end{cases}$$

This operation does not belong to M_5 since $(1, 0, 1)$, $(0, 1, 0)$, $(1, 1, 0)$, $(0,0,1) \in \rho_5$ and $(g_4(1,0,1,0),\ g_4(0,1,1,0),\ g_4(1,0,0,1)) = (1,0,0) \notin \rho_5$. Nevertheless, this can be improved by taking the following function:

$$g_4'(x_1,x_2,x_3) = \begin{cases} 0, & (x_1,x_2,x_3) \in \{(0,0,0),(0,0,1),(0,1,0),(1,0,0), \\ & \qquad (0,0,2),(0,2,0),(2,0,0)\} \\ 1, & (x_1,x_2,x_3) = (1,1,1) \\ 2, & \text{otherwise.} \end{cases}$$

It does not belong to M_9 since $(0,0,1,1)$, $(0,1,0,1)$, $(2,0,0,1) \in \rho_9$ and $(g_4'(0,0,2),\ g_4'(0,1,0),\ g_4'(1,0,0),\ g_4'(1,1,1)) = (0,0,0,1) \notin \rho_9$. We shall prove that it preserves ρ_5. Without loss of generality, we can suppose to the contrary that $(g_4'(1,1,1),\ g_4'(a,b,c),\ g_4'(d,e,f)) = (1,0,0)$ for some $a,b,c,d,e,f \in E_3$. From the previous definition of g_4' we can conclude that (a,b,c) and (d,e,f) both have at least two components equal to 0, and therefore $(1,0,0) \in \{(1,a,d),\ (1,b,e),\ (1,c,f)\}$.

4.3. Minimal hyperclones on E_2. *Minimal hyperclones* are atoms in the lattice of hyperclones $\mathcal{L}_A^h$, i.e., a hyperclone $C \neq J_A$ is minimal if

$$(\forall f \in C \setminus J_A)\langle f\rangle_h = C.$$

Lemma 4.2. [15] *Let f be a hyperoperation on E_2. $\langle f\rangle_h$ is a minimal hyperclone on E_2 iff $\langle f^{\#}\rangle_{P_2^*}$ is a minimal clone on P_2^*.*

Proof. Suppose on the contrary that $\langle f\rangle_h$ is a minimal hyperclone on E_2 and $\langle f^{\#}\rangle_{P_2^*}$ is not a minimal clone on P_2^*. Then, there is $g \in \langle f^{\#}\rangle_{P_2^*}$ such that $\langle g\rangle_{P_2^*}$ is minimal on P_2^* and there is no $h \in H_2$ with $h^{\#} = g$. It is proved in [23] that there exists $q \geqslant n$ and $h \in \langle f\rangle_h \cap H_2^{(q)}$ such that for all $X_1,\dots,X_n \in P_2^*$, $g(X_1,\dots,X_n) = h^{\#}(X_1,\dots,X_1,\dots,,X_n,\dots,X_n)$, i.e., g is obtained from $h^{\#}$ by identification of variables. Without loss of generality, we can assume that h is minimal. If h is minimal binary hyperoperation, every hyperoperation obtained from h by identification of variables is projection and the same also holds for g if X is a one-element subset. Hence, $g(\{0,1\}) = \{0,1\}$ and g is not minimal. If h is a ternary minimal hyperoperation, we obtain a projection by identification of variables, and the same holds for $h^{\#}$ on one-element subsets. Suppose that $h(x,x,y) = \{x\}$ (the proof is similar for other cases). Then, it holds $h^{\#}(X,X,Y) = g(X,Y) = X$, for $|X| = |Y| = 1$, $h^{\#}(X,X,Y) = g(X,Y) = X$, for $|X| = 1$ and $h^{\#}(\{0,1\},\{0,1\},Y) = g(\{0,1\},Y) = \{0,1\}$. This again gives a contradiction with the assumption that g is minimal. □

Using the fact that every minimal clone on A is a minimal hyperclone on A, previous lemma and the description of all minimal clones on $E_3 = \{0,1,2\}$ of Csákány (in [4]), we can conclude that there are 13 minimal hyperclones on E_2.

Theorem 4.3. [15] *Each minimal hyperclone Min_i, $i \in \{1,\ldots,13\}$, on E_2 is of the form $\langle f\rangle_h$, where f belongs to one of the following sets:*

$$\mathrm{Min}^{(1)} = \{c_0, c_1, c_2, \bar{x}, f_0, f_1\};$$
$$\mathrm{Min}^{(2)} = \{\mathtt{max}^h, \mathtt{min}^h, g_1, g_2, g_3\};$$
$$\mathrm{Min}^{(3)} = \{\mathtt{ma}^h, \mathtt{mi}^h\},$$

where

$c_0(x) = \{0\}$, $c_1(x) = \{1\}$, $c_2(x) = \{0,1\}$, $\bar{x}(x) = \{\bar{x}\}$, $f_0(x) = \{0,x\}$, $f_1(x) = \{x,1\}$,

$$\mathtt{max}^h(x,y) = \{\mathtt{max}(x,y)\},\ \mathtt{min}^h(x,y) = \{\mathtt{min}(x,y)\},$$

x	y	g_1	g_2	g_3
0	0	$\{0\}$	$\{0\}$	$\{0\}$
0	1	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	0	$\{1\}$	$\{0,1\}$	$\{0,1\}$
1	1	$\{1\}$	$\{1\}$	$\{1\}$

$$\mathtt{ma}^h(x,y,z) = \{\mathtt{ma}(x,y,z)\} \text{ and } \mathtt{mi}^h(x,y,z) = \{\mathtt{mi}(x,y,z)\}.$$

4.4. Hyperclones on E_2 generated by unary hyperoperations. In Figure 2, using results from 4.3, we describe the interval

$$[J_2, \langle H_2^{(1)}\rangle_h] = \{C \in L_2^h : J_2 \subseteq C \subseteq \langle H_2^{(1)}\rangle_h\}.$$

Theorem 4.4. [14] *Elements of the interval $[J_2, \langle H_2^{(1)}\rangle_h]$ are the following hyperclones:*

$I_1 = \langle c_0, c_1, f_0, f_1\rangle_h$ $\qquad$ $C = \langle c_0, c_1, c_2\rangle_h$

$I_2 = \langle c_0, c_2, \bar{x}\rangle_h = \langle c_1, c_2, \bar{x}\rangle_h = \langle c_0, c_1, c_2, \bar{x}\rangle_h$

$I_3 = \langle \bar{x}, f_0\rangle_h = \langle \bar{x}, f_1\rangle_h = \langle \bar{x}, f_0, f_1\rangle_h = \langle \bar{x}, c_2, f_0, f_1\rangle_h$

$C_1 = \langle c_0, c_2\rangle_h$ $\qquad$ $C_2 = \langle c_0, c_1\rangle_h$

$G_1 = \langle c_2, f_1\rangle_h$ $\qquad$ $C_3 = \langle c_1, c_2\rangle_h$

$G_2 = \langle c_1, f_1\rangle_h$ $\qquad$ $F_1 = \langle c_0, f_0\rangle_h$

$G_3 = \langle c_1, c_2, f_1\rangle_h$ $\qquad$ $F_2 = \langle c_2, f_0\rangle_h$

$G_4 = \langle c_0, c_2, f_1\rangle_h$ $\qquad$ $F_3 = \langle c_0, c_2, f_0\rangle_h$

$G_5 = \langle c_0, c_1, f_1\rangle_h = \langle c_0, c_1, c_2, f_1\rangle_h$

$F_4 = \langle c_1, f_0\rangle_h = \langle c_1, c_2, f_0\rangle_h$

$G_6 = \langle c_1, f_0, f_1\rangle_h = \langle c_1, c_2, f_0, f_1\rangle_h$

$$F_5 = \langle c_0, c_1, f_0\rangle_h = \langle c_0, c_1, c_2, f_0\rangle_h$$
$$H = \langle f_0, f_1\rangle_h = \langle c_2, f_0, f_1\rangle_h$$
$$F_6 = \langle c_0, f_0, f_1\rangle_h = \langle c_0, c_2, f_0, f_1\rangle_h$$
$$\langle O_2^{(1)}\rangle_h = \langle c_0, \bar{x}\rangle_h = \langle c_1, \bar{x}\rangle_h = \langle c_0, c_1, \bar{x}\rangle_h$$

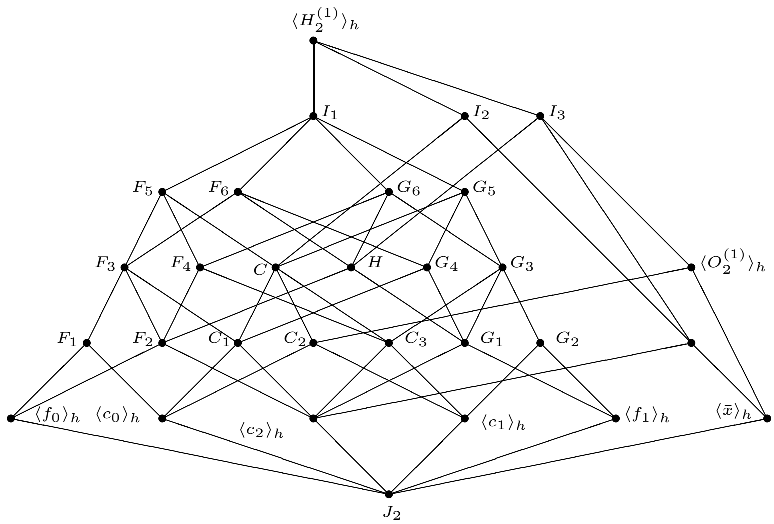

FIGURE 2. Interval $[J_2, \langle H_2^{(1)}\rangle_h]$.

Consequently, a set F of unary hyperoperations generates the hyperclone $\langle H_2^{(1)}\rangle_h$ generated by all unary hyperoperations if and only if F is contained in no submaximal hyperclone of $\langle H_2^{(1)}\rangle_h$.

Corollary 4.1. *Let* $F \subseteq H_2^{(1)}$. $\langle F\rangle_h = \langle H_2^{(1)}\rangle_h$ *if and only if for every* $i \in \{1,2,3\}$ $\langle F\rangle_h \setminus I_i \neq \emptyset$ *holds.*

Corollary 4.2. *There are* 4 *bases of unary hyperoperations in* $\langle H_2^{(1)}\rangle_h$. *These are* $\{c_0,\, \bar{x},\, f_0\}$, $\{c_0,\, \bar{x},\, f_1\}$, $\{c_1,\, \bar{x},\, f_0\}$ *and* $\{c_1,\, \bar{x},\, f_1\}$.

5. Galois connections for hyperclones

In this section we will present several Galois connections between hyperoperations and relations on a finite set. With this aim, we introduce the following relation on the set of all hyperoperations.

Definition 5.1. If $f, g \subseteq H_A^{(n)}$ satisfy

$$g(x_1, \ldots, x_n) \subseteq f(x_1, \ldots, x_n) \text{ for all } (x_1, \ldots, x_n) \in A^n,$$

then g is said to be a *sub-hyperoperation* of f (or f is said to be a *super hyperoperation* of g). We write $g \subseteq f$.

It is known that the composition of hyperoperations is monotone with respect to inclusion, i.e., if $g'_1 \subseteq g_1, \ldots, g'_n \subseteq g_n$ and $f' \subseteq f$ then $f'[g'_1, \ldots, g'_n] \subseteq f[g_1, \ldots, g_n]$.

5.1. Down closed hyperclones. The first Galois connection that will be presented here was independently studied by Börner (in [2]) and Romov (in [19, 20]). Corresponding Galois closed sets are hyperclones that contain all sub-hyperoperations of their elements, and they are called down (or restriction) closed hyperclones.

Let $F \subseteq H_A$, and let us denote by $\lfloor F \rfloor$ the set of all sub-hyperoperations of all the elements of F, i.e.,

$$\lfloor F \rfloor = \{g \in H_A : (\exists f \in F)\ g \subseteq f\}.$$

Lemma 5.1. *If $C \subseteq H_A$ is a hyperclone, then $\lfloor C \rfloor$ is also a hyperclone.*

Proof. Since obviously $C \subseteq \lfloor C \rfloor$, projections are all contained in $\lfloor C \rfloor$. For hyperoperations $f, g_1, \ldots, g_n \in \lfloor C \rfloor$, we have $f', g'_1, \ldots, g'_n \in C$ such that $f \subseteq f'$, $g_1 \subseteq g'_1$, …, $g_n \subseteq g'_n$. C being a hyperclone implies $f'[g'_1, \ldots, g'_n] \in C$, and then from $f[g_1, \ldots, g_n] \subseteq f'[g'_1, \ldots, g'_n]$ it follows that $f[g_1, \ldots, g_n] \in \lfloor C \rfloor$. □

Let us define a mapping $d : \mathcal{P}(H_A) \to \mathcal{P}(H_A)$ by $d(F) = \lfloor \langle F \rangle_h \rfloor$.

Lemma 5.2. *The mapping d is an algebraic closure operator.*

Proof.
- (i) Trivially, $F \subseteq \langle F \rangle_h \subseteq \lfloor \langle F \rangle_h \rfloor$.
- (ii) If $F \subseteq G$, then obviously $\langle F \rangle_h \subseteq \langle G \rangle_h$. For $f \in \lfloor \langle F \rangle_h \rfloor$ there is $g \in \langle F \rangle_h$ such that $f \subseteq g$. However, g is also in $\langle G \rangle_h$, which implies $f \in \lfloor \langle G \rangle_h \rfloor$, i.e., $d(F) \subseteq d(G)$.
- (iii) Let us prove that $d(d(F)) = d(F)$. For any $f \in d(d(F))$, there exist $g \in \langle d(F) \rangle_h (= d(F))$ such that $f \subseteq g$. Thus, there is $h \in \langle F \rangle_h$ with $g \subseteq h$. Therefore, since $h \in \langle F \rangle_h$ and $f \subseteq h$, we obtain $f \in d(F)$. The opposite inclusion holds by (i).
- (iv) We are going to show that $d(F) = \bigcup \{d(G) : G \subseteq F$ and G is finite$\}$.
  - ($\subseteq$) If $f \in d(F)$, there is $g \in \langle F \rangle_h$ such that $f \subseteq g$. If we choose $G = \{g\}$, then $f \in d(G)$.
  - ($\supseteq$) Assume that there is a finite subset G of F such that $f \in d(G)$. It means that there is $g \in \langle G \rangle_h$ such that $f \subseteq g$. However, since $\langle G \rangle_h \subseteq \langle F \rangle_h$, it follows that $g \in \langle F \rangle_h$ and $f \in d(F)$.

□

Definition 5.2. A set $F \subseteq H_A$ is called *down closed hyperclone* if $d(F) = F$.

Every clone is trivially a down closed hyperclone, and obviously H_A is the largest down closed hyperclone. It is also straightforward from Lemma 5.1 and Lemma 5.2 (iii) that if C is a hyperclone then $\lfloor C \rfloor$ is down closed hyperclone.

Let $C \subseteq H_A$ be a down closed hyperclone. Then $L_A^{\Downarrow}(C)$ is the set of all down closed sub-hyperclones of C. If $C = H_A$, we will denote by $L_A^{\Downarrow}$ set of all down closed hyperclones on A.

The following theorem is also an immediate corollary of the previous lemmas.

Theorem 5.1. *Down closed hyperclones form an algebraic lattice $\mathcal{L}_A^{\Downarrow}$ with respect to the set inclusion. The lattice operations on $\mathcal{L}_A^{\Downarrow}$ are defined as follows*

$$C_1 \wedge_{\Downarrow} C_2 = C_1 \cap C_2 \quad \textit{and} \quad C_1 \vee_{\Downarrow} C_2 = d(C_1 \cup C_2).$$

From the definition of the join operation it is easily deduced that $\mathcal{L}_A^{\Downarrow}$ is not the sublattice of the lattice od all hyperclones on A, which we are going to illustrate by the next example.

Example 5.1. Let $A = \{0,1,2\}$, $C_1 = d(\{f_1\})$ i $C_2 = d(\{f_2\})$, and hyperoperations $f_1, f_2 \in H_A^{(1)}$ are defined by

	0	1	2
f_1	{0}	{0}	{0,2}
f_2	{1,2}	{1}	{2}

We know that $\langle C_1 \cup C_2 \rangle_h$ contains hyperoperation $f_2[f_1]$:

	0	1	2
$f_2[f_1]$	{1,2}	{1,2}	{1,2}

Obviously, $d(C_1 \cup C_2)$ contains sub-hyperoperation h of $f_2[f_1]$, given by $h(x) = \{2\}$, $x \in A$, but h is not generated by $C_1 \cup C_2$, since no composition of elements from $C_1 \cup C_2$ yields $h(1) = \{2\}$. Therefore, $C_1 \vee_{\Downarrow} C_2 \neq C_1 \vee_h C_2$.

Next we will introduce the Galois connection for which down closed hyperclones are Galois closed sets.

Let $\ell \geqslant 1$ and $\rho \in R_A^{(\ell)}$. The *strong extension* of ρ is the relation ρ_d defined by

$$\rho_d = \{(A_1, \ldots, A_\ell) \in (P_A^*)^\ell : A_1 \times \cdots \times A_\ell \subseteq \rho\},$$

which means that for $(A_1, \ldots, A_\ell)$ to be in ρ_d it is necessary that all ℓ-tuples $(a_1, \ldots, a_\ell)$, such that $a_i \in A_i$ $(i = 1, \ldots, \ell)$, are contained in ρ.

Example 5.2. If ρ is a binary relation on $A = \{0,1\}$ given by $\rho = \left(\begin{smallmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, then the strong extension of ρ is

$$\rho_d = \begin{pmatrix} \{0\} & \{1\} & \{1\} & \{1\} & \{0,1\} \\ \{0\} & \{0\} & \{1\} & \{0,1\} & \{0\} \end{pmatrix}$$

Definition 5.3. We say that hyperoperation $f \in H_A^{(n)}$ *d-preserves* relation $\rho \in R_A^{(\ell)}$, or ρ is *d-invariant* for f, if for every $\ell \times n$ matrix M in ρ^* it holds $f(M) \in \rho_d$.

Let $d\,\mathrm{Pol}\,\rho$ denote the set of all hyperoperations on A which d-preserve relation ρ, a $d\,\mathrm{Inv}\,f$ be the set of all relations on A which are d-invariant for hyperoperation f. We can now define the mappings

$$d\,\mathrm{Pol} : \mathcal{P}(R_A) \to \mathcal{P}(H_A) \quad \text{and} \quad d\,\mathrm{Inv} : \mathcal{P}(H_A) \to \mathcal{P}(R_A)$$

by

$$d\,\mathrm{Pol}\,Q = \bigcap_{\rho \in Q} d\,\mathrm{Pol}\,\rho = \{f \in H_A : f \ d\text{-preserves every } \rho \in Q\}, \quad Q \subseteq R_A,$$

$$d\,\mathrm{Inv}\,F = \bigcap_{f\in F} d\,\mathrm{Inv}\,f = \{\rho \in R_A : \text{every } f \in F \ d\text{-preserves } \rho\}, \quad F \subseteq H_A.$$

Clearly, the pair $(d\,\mathrm{Pol}, d\,\mathrm{Inv})$ is a Galois connection between relations and hyperoperations.

Theorem 5.2. [2]

(i) *For any* $Q \subseteq R_A$, $d\,\mathrm{Pol}\,Q$ *is a down closed hyperclone.*

(ii) *If* $C \subseteq H_A$ *is a down closed hyperclone, then* $C = d\,\mathrm{Pol}(d\,\mathrm{Inv}\,C)$.

5.2. Hyperclones determined by relation ρ_m. The following relation on P_A^* and the corresponding Galois connection between hyperoperations and relations were studied by Pöschel and Drescher in [17].

Let ρ be an ℓ-ary relation on A. We shall define relation $\rho_m \subseteq (P_A^*)^\ell$ by

$$\rho_m = \{(A_1, \ldots, A_\ell) \in (P_A^*)^\ell : (\forall\, i \in \{1,\ldots,\ell\})(\forall a \in A_i)(\exists \mathbf{a} \in (A_1 \times \cdots \times A_\ell) \cap \rho) e_i^{\ell,A}(\mathbf{a}) = a\}.$$

The condition "$\exists\, \mathbf{a} \in (A_1 \times \cdots \times A_\ell) \cap \rho$ such that $e_i^{\ell,A}(\mathbf{a}) = a$" is equivalent to "$\forall j \in \{1,\ldots,\ell\} \setminus \{i\}\ \exists a_j \in A_j$ such that $(a_1,\ldots,a_{i-1},a_j,a_{i+1},\ldots,a_\ell) \in (A_1 \times \cdots \times A_\ell) \cap \rho$".

Example 5.3. Let $A = \{0,1\}$ and $\rho \in R_A^{(2)}$ is given by $\rho = \left(\begin{smallmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)$.
Then we have

$$\rho_m = \begin{pmatrix} \{0\} & \{1\} & \{1\} & \{1\} & \{0,1\} & \{0,1\} \\ \{0\} & \{0\} & \{1\} & \{0,1\} & \{0\} & \{0,1\} \end{pmatrix}.$$

Definition 5.4. Hyperoperation $f \in H_A^{(n)}$ *m-preserves* relation $\rho \in R_A^{(\ell)}$, or ρ is *m-invariant* of f, if for every $\ell \times n$ matrix M in ρ^* it holds $f(M) \in \rho_m$.

The set of all hyperoperations that m-preserve relation ρ is denoted by $m\,\mathrm{Pol}\,\rho$, and dually, the set of all relations which are m-preserved by hyperoperation f is denoted by $m\,\mathrm{Inv}\,f$. Let us define the mappings

$$m\,\mathrm{Pol} : \mathcal{P}(R_A) \to \mathcal{P}(H_A) \quad \text{and} \quad m\,\mathrm{Inv} : \mathcal{P}(H_A) \to \mathcal{P}(R_A)$$

by

$$m\,\mathrm{Pol}\,Q = \bigcap_{\rho\in Q} m\,\mathrm{Pol}\,\rho = \{f \in H_A : f\ m\text{-preserves every } \rho \in Q\},\ Q \subseteq R_A,$$

$$m\,\mathrm{Inv}\,F = \bigcap_{f\in F} m\,\mathrm{Inv}\,f = \{\rho \in R_A : \text{every } f \in F\ m\text{-preserves } \rho\},\ F \subseteq H_A.$$

Obviously, the pair $(m\,\mathrm{Pol}, m\,\mathrm{Inv})$ is a Galois connection between the sets R_A and H_A.

Proposition 5.1. [17] *Set of hyperoperations* $m\,\mathrm{Pol}\,Q$ *is a hyperclone, for every* $Q \subseteq R_A$.

5.3. Upward saturated hyperclones. Dually to the case of down closed hyperclones, described in 5.1, we may concider hyperclones that contain all super hyperoperations of their elements. Here we present part of the results from [7].

For $G \subseteq H_A$, let $\lceil G \rceil$ denote the set of all super hyperoperations of hyperoperations from G, i.e.,

$$\lceil G \rceil := \{f \in H_A : (\exists g \in G) g \subseteq f\}.$$

For the following two lemmas proofs are dual to those of Lemma 5.1 and Lemma 5.2.

Lemma 5.3. [7] *If $C \subseteq H_A$ is a hyperclone, then $\lceil C \rceil$ is a hyperclone.*

Let us define a mapping $u : \mathcal{P}(H_A) \to \mathcal{P}(H_A)$ by $u(F) = \lceil \langle F \rangle_h \rceil$.

Lemma 5.4. *The mapping u is an algebraic closure operator.*

Definition 5.5. A set $F \subseteq H_A$ is called *upward saturated hyperclone* if $u(F) = F$.

Analogous to the case of down closed hyperclones, if C is a hyperclone, then $\lceil C \rceil$ is an upward saturated hyperclone.

We will write $L_A^{\Uparrow}$ for the set of all upward saturated hyperclones on A and $\mathcal{L}_A^{\Uparrow}$ for the poset $(L_A^{\Uparrow}, \subseteq)$.

From the previous lemmas, we can prove directly the following theorem.

Theorem 5.3. [7] *$\mathcal{L}_A^{\Uparrow}$ is an algebraic lattice with the lattice operations defined by*

$$C_1 \wedge_{\Uparrow} C_2 = C_1 \cap C_2 \quad \textit{and} \quad C_1 \vee_{\Uparrow} C_2 = u(C_1 \cup C_2).$$

We will describe one subset of upward saturated hyperclones, by introducing a particular Galois connection. Let $\ell \geqslant 1$ and $\rho \subseteq A^\ell$. The *weak extension* of ρ is the relation ρ_h defined by

$$\rho_h = \{(A_1, \ldots, A_\ell) \in (P_A^*)^\ell : (A_1 \times \ldots \times A_\ell) \cap \rho \neq \emptyset\},$$

i.e., ρ_h consists of ℓ-tuples $(A_1, \ldots, A_\ell)$ of subsets of A such that $(a_1, \ldots, a_\ell) \in \rho$ for some $a_i \in A_i$ $(i = 1, \ldots, \ell)$. (See [25])

Example 5.4. Let $A = \{0, 1\}$ and $\rho \in R_A^{(2)}$ is given by $\rho = \left(\begin{smallmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)$.
Then we have

$$\rho_h = \begin{pmatrix} \{0\} & \{0\} & \{1\} & \{1\} & \{1\} & \{0,1\} & \{0,1\} & \{0,1\} \\ \{0\} & \{0,1\} & \{0\} & \{1\} & \{0,1\} & \{0\} & \{1\} & \{0,1\} \end{pmatrix}.$$

Definition 5.6. Hyperoperation $f \in H_A^{(n)}$ *h-preserves* relation $\rho \in R_A^{(\ell)}$, or ρ is *h-invariant* of f, if for every $\ell \times n$ matrix M in ρ^* it holds $f(M) \in \rho_h$.

Set $h \operatorname{Pol} \rho$ consists of all the hyperoperations that h-preserve relation ρ, and by $h \operatorname{Inv} f$ we denote the set of all relations that hyperoperation f h-preserves. We may now define the mappings

$$h \operatorname{Pol} : \mathcal{P}(R_A) \to \mathcal{P}(H_A) \quad \text{and} \quad h \operatorname{Inv} : \mathcal{P}(H_A) \to \mathcal{P}(R_A),$$

as follows:

$$h \operatorname{Pol} Q = \bigcap_{\rho \in Q} h \operatorname{Pol} \rho = \{f \in H_A : f \ h\text{-preserves each } \rho \in Q\}, \ Q \subseteq R_A,$$

$$h\,\mathrm{Inv}\,F = \bigcap_{f\in F} h\,\mathrm{Inv}\,f = \{\rho \in R_A : \text{each } f \in F\ h\text{-preserves } \rho\},\ \ F \subseteq H_A.$$

It is clear that the pair $(h\,\mathrm{Pol}, h\,\mathrm{Inv})$ is a Galois connection between relations and hyperoperations on A.

Proposition 5.2. [7] *Let $\ell \geqslant 1$ and $\rho \subseteq A^\ell$. Then $h\,\mathrm{Pol}\,\rho$ is an upward saturated hyperclone.*

Proof. Evidently $\mathrm{Pol}\,\rho \subseteq h\,\mathrm{Pol}\,\rho$ and so $h\,\mathrm{Pol}\,\rho$ contains all projections. Let $h = f[g_1,\dots,g_n]$, where $f, g_1,\dots,g_n \in h\,\mathrm{Pol}\,\rho$. Suppose that M is an $\ell \times m$ matrix over A in ρ^* such that $(h(M_{1*}),\dots,h(M_{\ell *})) \notin \rho_h$, where $M_{1*},\dots,M_{\ell *}$ are the rows of M. It follows that $(h(M_{1*}) \times \ldots \times h(M_{\ell *})) \cap \rho = \emptyset$. From $f \in h\,\mathrm{Pol}\,\rho$ there is $i \in \{1,\dots,n\}$ such that $(g_i(M_{1*}),\dots,g_i(M_{\ell *})) \notin \rho_h$, i.e., $(g_i(M_{1*}) \times \ldots \times g_i(M_{\ell *})) \cap \rho = \emptyset$. Since $g_i \in h\,\mathrm{Pol}\,\rho$, we deduce that for some $j \in \{1,\dots,m\}$ the column $M_{*j} \notin \rho$, which is not possible by the choice of matrix M. Therefore, $h \in h\,\mathrm{Pol}\,\rho$, and consequently $h\,\mathrm{Pol}\,\rho$ is a hyperclone.

We shall now prove that it is upward saturated. Let us suppose that $f \in (h\,\mathrm{Pol}\,\rho)^{(n)}$ and $f \subseteq g$. If we assume that $g \notin h\,\mathrm{Pol}\,\rho$, than there is an $\ell \times n$ matrix M in ρ^* such that $(g(M_{1*}),\dots,g(M_{\ell *})) \notin \rho_h$. Therefore, we have $(f(M_{1*}),\dots,f(M_{\ell *})) \subseteq (g(M_{1*}),\dots,g(M_{\ell *}))$ and $(g(M_{1*}) \times \ldots \times g(M_{\ell *})) \cap \rho = \emptyset$. It implies $(f(M_{1*}),\dots, f(M_{\ell *})) \cap \rho = \emptyset$, which gives a contradiction. □

5.4. Relations between the sets $d\,\mathrm{Pol}, m\,\mathrm{Pol}$ and $h\,\mathrm{Pol}$. A natural question to ask is how the sets $d\,\mathrm{Pol}\ \rho$, $m\,\mathrm{Pol}\ \rho$ and $h\,\mathrm{Pol}\ \rho$ are related.
The following inclusion is straightforward from the definitions of the relations ρ_d, ρ_m and ρ_h.

Lemma 5.5. *For $\rho \in R_A^{(\ell)}$ we have $\rho_d \subseteq \rho_m \subseteq \rho_h$.*

We illustrate by the following example that for some relations equalities are possible, but it is in general not true (see examples 5.2–5.4).

Example 5.5. Let $A = \{0,1\}$.

(a) If $\rho = A^2$, then $\rho_d = \rho_m = \rho_h$.

(b) If $\rho = \{(0,0),(0,1)\}$ then

$$\rho_d = \rho_m = \begin{pmatrix} \{0\} & \{0\} & \{0\} \\ \{0\} & \{1\} & \{0,1\} \end{pmatrix},$$

$$\rho_h = \begin{pmatrix} \{0\} & \{0\} & \{0\} & \{0,1\} & \{0,1\} & \{0,1\} \\ \{0\} & \{1\} & \{0,1\} & \{0\} & \{1\} & \{0,1\} \end{pmatrix}.$$

Theorem 5.4. [6]

(a) *Let $\rho \in R_A^{(\ell)}$. Then $d\,\mathrm{Pol}\,\rho \subseteq m\,\mathrm{Pol}\,\rho \subseteq h\,\mathrm{Pol}\,\rho$.*

(b) *There exist $\rho \in R_A^{(\ell)}$ such that $d\,\mathrm{Pol}\,\rho \neq m\,\mathrm{Pol}\ \rho$ and $m\,\mathrm{Pol}\ \rho \neq h\,\mathrm{Pol}\,\rho$.*

Proof. (a) Due to Lemma 5.5 we have $\rho_d \subseteq \rho_m \subseteq \rho_h$. Clearly, for each matrix $M \in \rho^*$ we have that $f(M) \in \rho_d$ implies $f(M) \in \rho_m$, and $f(M) \in \rho_m$ implies $f(M) \in \rho_h$.

(b) Let $A = \{0,1\}$, $\rho = \{(0,0,0),(1,1,0)\}$ and $f, g \in H_A^{(2)}$ are defined by

x_1	x_2	f	g
0	0	{0}	{0,1}
0	1	{0,1}	{0}
1	0	{0}	{0,1}
1	1	{0}	{0,1}

Now it is easy to show that $f \in m\,\mathrm{Pol}\,\rho \setminus d\,\mathrm{Pol}\,\rho$ and $g \in h\,\mathrm{Pol}\,\rho \setminus m\,\mathrm{Pol}\,\rho$. □

5.5. Sets of relations of the form $x\,\mathrm{Inv}\,F$. We will show that in general neither of the three sets of the form $x\,\mathrm{Inv}\,F$, $x \in \{d, m, h\}$, is a relational clone. The first case has already been solved by Börner.

Theorem 5.5. [3]

(a) *If $F \subseteq H_A$, then $d\,\mathrm{Inv}\,F$ is closed with respect to $\cap, \times, \zeta, \tau$ and* pr.

(b) *There exists $F \subseteq H_A$ such that $\delta_{1,2}^{3,A} \notin d\,\mathrm{Inv}\,F$.*

Example 5.6. Let f be a unary hyperoperation on $A = \{0,1\}$ given by $f(0) = \{0\}$ and $f(1) = \{0,1\}$. Then $(1,1,0) \in \delta_{1,2}^{3,A}$ and $(f(1), f(1), f(0)) = (\{0,1\},\{0,1\},\{0\})$. Obviously, $\{0,1\} \times \{0,1\} \times \{0\} \not\subseteq \delta_{1,2}^{3,A}$, and thus $\delta_{1,2}^{3,A} \notin d\,\mathrm{Inv}\,f$.

There are two more cases to consider.

Theorem 5.6. [6] *Let $x \in \{m, h\}$.*

(a) *If $F \subseteq H_A$, then $x\,\mathrm{Inv}\,F$ is closed with respect to $\times, \zeta, \tau,$ pr and $x\,\mathrm{Inv}\,F \ni \delta_{1,2}^{3,A}$.*

(b) *There exists $F \subseteq H_A$ such that $x\,\mathrm{Inv}\,F$ is not closed under $\cap$.*

Proof. (a) A simple verification shows that $\zeta\rho_x = (\zeta\rho)_x$, $\tau\rho_x = (\tau\rho)_x$, $\mathrm{pr}\,\rho_x = (\mathrm{pr}\,\rho)_x$, and $(\rho_1)_x \times (\rho_2)_x = (\rho_1 \times \rho_2)_x$. Furthermore, $F \subseteq x\,\mathrm{Pol}\,\delta_{1,2}^{3,A}$ because $(B, B, C) \in (\delta_{1,2}^{3,A})_x$ for all $B, C \subseteq A$.

(b) **Case** $x = m$: Let ρ_1 and ρ_2 be binary relations on $A = \{0,1\}$ given by

$$\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

and let $f \in H_A^{(1)}$ be defined by $f(x) = \{0,1\}$, $x \in \{0,1\}$. It holds that $\rho_1, \rho_2 \in m\,\mathrm{Inv}\,f$, while $\rho_1 \cap \rho_2 = \{(0,1)\} \notin m\,\mathrm{Inv}\,f$, since $(f(0), f(1)) = (\{0,1\},\{0,1\}) \notin (\rho_1 \cap \rho_2)_m = \{(\{0\},\{1\})\}$.

Case $x = h$: Let ρ_1 and ρ_2 be binary relations on $A = \{0,1\}$ given by

$$\rho_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \rho_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and let $f \in H_A^{(1)}$ be defined by $f(0) = \{0,1\}$, $f(1) = \{0\}$. It holds that $\rho_1, \rho_2 \in h\,\mathrm{Inv}\,f$, while $\rho_1 \cap \rho_2 \notin h\,\mathrm{Inv}\,f$, since $(1,0) \in \rho_1 \cap \rho_2$, $(f(1), f(0)) = (\{0\},\{0,1\})$, and

$$(\{0\} \times \{0,1\}) \cap (\rho_1 \cap \rho_2) = \{(0,0),(0,1)\} \cap \{(1,0)\} = \emptyset.$$

Therefore, $h\,\mathrm{Inv}\,f$ is in general not closed with respect to $\cap$. □

6. Four classes of maximal hyperclones

The following theorem gives a sufficient condition for a hyperclone $h\operatorname{Pol}\rho$ to be maximal in the hyperclone lattice on A.

Theorem 6.1. [13] *Let* $\operatorname{Pol}\rho$ *be a maximal clone on* A *such that*

$$(\forall f \in H_A \setminus h\operatorname{Pol}\rho)\,(\exists f' \in O_A \setminus \operatorname{Pol}\rho)\ f' \in \langle \operatorname{Pol}\rho \cup \{f\}\rangle_h. \tag{6.1}$$

Then $h\operatorname{Pol}\rho$ *is a maximal hyperclone.*

Proof. Let ρ be an m-ary relation on A such that $\operatorname{Pol}\rho$ is a maximal clone on A and let $f \in H_A \setminus h\operatorname{Pol}\rho$. Then by condition (6.1) there exists $f' \in O_A \setminus \operatorname{Pol}\rho$ such that $f' \in \langle \operatorname{Pol}\rho \cup \{f\}\rangle_h$. Since $\operatorname{Pol}\rho$ is a maximal clone and $f' \notin \operatorname{Pol}\rho$, we have $\langle \operatorname{Pol}\rho \cup \{f'\}\rangle_A = O_A$. Let c_A be an n-ary constant hyperoperation, defined by $c_A(x_1,\dots,x_n) = A$, for all $(x_1,\dots,x_n) \in A^n$. Clearly, $c_A \in h\operatorname{Pol}\rho$ because $\rho \neq \emptyset$, thus $(A,\dots,A) \in \rho_h$. Considering that O_A is a maximal hyperclone, we obtain $\langle O_A \cup \{c_A\}\rangle_h = H_A$. Therefore,

$$\begin{aligned} H_A = \langle O_A \cup \{c_A\}\rangle_h &= \langle \operatorname{Pol}\rho \cup \{f'\} \cup \{c_A\}\rangle_h \subseteq \\ &\subseteq \langle \operatorname{Pol}\rho \cup \{f\} \cup \{c_A\}\rangle_h \subseteq \langle h\operatorname{Pol}\rho \cup \{f\}\rangle_h \subseteq H_A, \end{aligned}$$

and hence $\langle h\operatorname{Pol}\rho \cup \{f\}\rangle_h = H_A$, i.e., $h\operatorname{Pol}\rho$ is a maximal hyperclone. □

In the remainder of this section we will prove that $h\operatorname{Pol}\rho$ is a maximal hyperclone whenever ρ is a bounded partial order, nontrivial equivalence relation, central relation or regular relation.

6.1. Hyperclones determined by bounded partial orders.

In this subsection we will show that for every bounded partial order ρ hyperclone $h\operatorname{Pol}\rho$ is maximal because it satisfies the conditions of Theorem 6.1, that is for every hyperoperation f which is not in $h\operatorname{Pol}\rho$ we are going to determine an operation f' which does not preserve ρ, but can be represented as a composition of some operations from $\operatorname{Pol}\rho$ and the hyperoperation f.

Let $\rho \subseteq A^2$ be a bounded partial order, with the least element $\mathbf{0}$, and the greatest element $\mathbf{1}$. Next we choose $B, C \in P_A^*$ such that $(B \times C) \cap \rho = \emptyset$. Relation ρ being reflexive implies $B \cap C = \emptyset$. Let

$$\begin{aligned} B' &= \{x \in A : (b',x) \in \rho \text{ and } (x,b'') \in \rho \text{ for some } b',b'' \in B\},\\ C' &= \{x \in A : (c',x) \in \rho \text{ and } (x,c'') \in \rho \text{ for some } c',c'' \in C\},\\ C'' &= \{x \in A : x \notin B' \text{ and } (c',x) \in \rho \text{ and } (x,b') \in \rho \text{ for some } b' \in B \text{ and } c' \in C\},\\ G &= \{x \in A : x \notin B' \cup C' \cup C'' \text{ and } ((b',x) \in \rho \text{ for some } b' \in B\\ &\qquad \text{or } (c',x) \in \rho \text{ for some } c' \in C' \cup C'')\},\\ L &= \{x \in A : x \notin B' \cup C' \cup C'' \text{ and } ((x,b') \in \rho \text{ for some } b' \in B\\ &\qquad \text{or } (x,c') \in \rho \text{ for some } c' \in C' \cup C'')\}. \end{aligned}$$

Lemma 6.1. [7] *For all* $b \in B'$ *and* $c \in C' \cup C''$ *it holds* $(b,c) \notin \rho$.

Proof. Suppose that there exist $b \in B'$ and $c \in C' \cup C''$ such that $(b, c) \in \rho$. There are two possibilities.

- If $c \in C'$, then for some $b' \in B$ and $c' \in C$ we have $(b', b), (b, c), (c, c') \in \rho$. Hence, by the transitivity of relation ρ, $(b', c') \in \rho$, which contradicts the choice of the sets B and C.
- If, on the other hand, $c \in C''$, then there exist $b', b'' \in B$ such that $(b', b), (b, c), (c, b'') \in \rho$, thus $(b', c), (c, b'') \in \rho$. However, this would imply $c \in B'$, which is impossible since $B' \cap C'' = \emptyset$. □

Choose arbitrary $b \in B$ and $c \in C$ if $(C \times B) \cap \rho = \emptyset$, or choose $(b, c) \in B \times C$ such that $(c, b) \in \rho$, otherwise. Let us define the unary operation $g^{\rho}_{b,c}$ as follows

$$g^{\rho}_{b,c}(x) = \begin{cases} b, & x \in B' \\ c, & x \in C' \cup C'' \\ \mathbf{1}, & x \in G \\ \mathbf{0}, & x \in L \\ x, & \text{otherwise.} \end{cases} \tag{6.2}$$

Proposition 6.1. [7] *Operation $g^{\rho}_{b,c} \in O^{(1)}_A$ is well defined.*

Proof. We shall prove that the sets $B', C' \cup C'', G$ and L are pairwise disjoint. As G and L by definition satisfy

$$(G \cup L) \cap (B' \cup C' \cup C'') = \emptyset,$$

we only have to show that $B' \cap (C' \cup C'') = \emptyset$ and $G \cap L = \emptyset$. B' and C'' are disjoint by definition of C''. Suppose that there is $x \in B' \cap C'$. Then we have $(c', x), (x, b') \in \rho$, for some $b' \in B$ and $c' \in C$, and therefore $x \in C''$, which is not possible. If $x \in G \cap L$, there are $b', b'' \in B$ and $c', c'' \in C' \cup C''$ such that

$$((b', x) \in \rho \vee (c', x) \in \rho) \wedge ((x, b'') \in \rho \vee (x, c'') \in \rho).$$

We discuss the following cases:

i) $(b', x), (x, b'') \in \rho$, implying $x \in B'$, which is impossible since $(G \cup L)$ andB' are disjoint sets;

ii) $(b', x), (x, c'') \in \rho$, implying $(b', c'') \in \rho$, for $b' \in B \subseteq B'$ and $c'' \in C' \cup C''$, which is not possible by Lemma 6.1;

iii) $(c', x), (x, b'') \in \rho$. For $c' \in C' \cup C''$ there is $c_1 \in C$ such that $(c_1, c') \in \rho$, i.e., $(c_1, x), (x, b'') \in \rho$, for some $b'' \in B$ and $c_1 \in C$. Therefore, $x \in C''$, giving a contradictions with the definition of G and L;

iv) $(c', x), (x, c'') \in \rho$. For $c' \in C' \cup C''$ there is $c_1 \in C$ such that $(c_1, c') \in \rho$. If $c'' \in C'$, we will take $c_2 \in C$ such that $(c'', c_2) \in \rho$, and if $c'' \in C''$, we will take $b_2 \in B$ such that $(c'', b_2) \in \rho$. In both cases $x \in C' \cup C''$, which gives a contradiction with the definition of G and L. □

Operation $g^{\rho}_{b,c}$ is one of the operations that we are going to use for the construction of the operation $f' \in O_A \setminus \operatorname{Pol}\rho$, and therefore it is essential that $g^{\rho}_{b,c}$ preserves ρ, which we will prove next.

Lemma 6.2. [7] *Let* $g^\rho_{b,c}$ *be defined by (6.2). Then* $g^\rho_{b,c} \in \operatorname{Pol}\rho$.

Proof. Let $x, y \in A$. We distinguish the following 25 cases:

$x \setminus y$	B'	$C' \cup C''$	G	L	other
B'	1	2	3	4	5
$C' \cup C''$	6	7	8	9	10
G	11	12	13	14	15
L	16	17	18	19	20
other	21	22	23	24	25

We are going to show that in each of the cases $(x, y) \notin \rho$, or $(x, y) \in \rho$ implies $(g^\rho_{b,c}(x), g^\rho_{b,c}(y)) \in \rho$.

In the following table we denote by ✓ all the cases with $(x, y) \in \rho$ implying $(g^\rho_{b,c}(x), g^\rho_{b,c}(y)) \in \rho$, and by $\times$ all those cases in which assumption that $(x, y) \in \rho$ leads to contradiction.

$x \setminus y$	B'	$C' \cup C''$	G	L	other
B'	✓	$\times$	✓	$\times$	$\times$
$C' \cup C''$	✓	✓	✓	$\times$	$\times$
G	$\times$	$\times$	✓	$\times$	$\times$
L	✓	✓	✓	✓	✓
other	$\times$	$\times$	✓	$\times$	✓

1) Let $x, y \in B'$ and suppose that $(x, y) \in \rho$. Then $(g^\rho_{b,c}(x), g^\rho_{b,c}(y)) = (b, b) \in \rho$, since ρ is reflexive.
2) If $x \in B'$ and $y \in C' \cup C''$, then $(x, y) \notin \rho$ by Lemma 6.1.
3) For $x \in B'$ and $y \in G$, if $(x, y) \in \rho$, then $(g^\rho_{b,c}(x), g^\rho_{b,c}(y)) = (b, \mathbf{1}) \in \rho$, since $\mathbf{1}$ is the greatest element.
4) Let $x \in B'$ and $y \in L$, and assume that $(x, y) \in \rho$. We have the following possibilities:
   (a) There is $c' \in C' \cup C''$ such that $(y, c') \in \rho$. Then $(x, c') \in \rho$ for $x \in B'$ and $c' \in C' \cup C''$, which is not possible by Lemma 6.1.
   (b) There are $b', b'' \in B$ such that $(b', x), (x, y), (y, b'') \in \rho$, which means that $y \in B'$, contradicting the fact that $B' \cap L = \emptyset$.
5) In case $x \in B'$ and $y \notin B' \cup C' \cup C'' \cup G \cup L$ we would have that $(x, y) \in \rho$ implies $y \in G$.
6) If $x \in C' \cup C''$ and $y \in B'$, then $(c', x), (x, y), (y, b') \in \rho$, and therefore $(c', b') \in \rho$, for some $c' \in C$ and $b' \in B$. Hence, $(C \times B) \cap \rho \neq \emptyset$ implies $(g^\rho_{b,c}(x), g^\rho_{b,c}(y)) = (c, b) \in \rho$.
7) From $(x, y) \in \rho$ for $x, y \in C' \cup C''$ it holds that $(g^\rho_{b,c}(x), g^\rho_{b,c}(y)) = (c,\ c) \in \rho$, since ρ is reflexive.
8) For $x \in C' \cup C''$ and $y \in G$, as in the case (3), $(x, y) \in \rho$ implies $(g^\rho_{b,c}(x), g^\rho_{b,c}(y)) = (c, \mathbf{1}) \in \rho$.
9) If $x \in C' \cup C'', y \in L$ and $(x, y) \in \rho$, we consider the following cases:
   (a) $(c', x), (x, y), (y, b') \in \rho$, for some $b' \in B$ and $c' \in C$, meaning that $(c', y), (y, b') \in \rho$, for $b' \in B$ and $c' \in C$, i.e., $y \in C''$.

(b) $(c', x), (x, y), (y, c'') \in \rho$, for some $c' \in C$ and $c'' \in C' \cup C''$. Then $(c', y), (y, c'') \in \rho$ for $c' \in C$ and $c'' \in C' \cup C''$ and therefore $y \in C' \cup C''$.

10) Let $x \in C' \cup C''$ and $y \notin B' \cup C' \cup C'' \cup G \cup L$. Then $(x, y) \in \rho$ would imply $y \in G$.
11) Let us suppose that $(x, y) \in \rho$ for $x \in G$ and $y \in B'$. We have:
   (a) There are $b', b'' \in B$ such that $(b', x), (x, y), (y, b'') \in \rho$, and so $x \in B'$.
   (b) There are $c' \in C' \cup C''$ and $b'' \in B$ such that $(c', x), (x, y), (y, b'') \in \rho$, i.e., $(c'', c'), (c', x), (x, b'') \in \rho$, for some $c'' \in C, c' \in C' \cup C''$ and $b'' \in B$. Thus $x \in C''$.
12) As $x \in G \wedge y \in C' \cup C'' \Leftrightarrow (x \in G \wedge y \in C') \vee (x \in G \wedge y \in C'')$, we distinguish the following four cases:
   (a) $(b', x), (x, y), (y, c'') \in \rho$, for some $b' \in B$ and $c'' \in C$, i.e., $(b', c'') \in \rho$, for $b' \in B$ and $c'' \in C$, which is not possible by the choice of B and C.
   (b) There are $c' \in C' \cup C''$ and $c'' \in C$ such that $(c', x), (x, y), (y, c'') \in \rho$. Hence, there is $c''' \in C$ such that $(c''', c') \in \rho$, i.e., $(c''', x), (x, c'') \in \rho$, for some $c'', c''' \in C$, which implies $x \in C'$.
   (c) It holds $(b', x), (x, y), (y, b'') \in \rho$, for some $b', b'' \in B$, and therefore $x \in B'$.
   (d) There are $c' \in C' \cup C''$ and $b'' \in B$ such that $(c', x), (x, y), (y, b'') \in \rho$. Then $x \in C''$.
13) If $x, y \in G$ and $(x, y) \in \rho$, then $(g^{\rho}_{b,c}(x), g^{\rho}_{b,c}(y)) = (\mathbf{1}, \mathbf{1}) \in \rho$.
14) Let $x \in G$ and $y \in L$. Then, by the definition of G and L, the following cases are possible:
   (a) There are $b', b'' \in B$, such that $(b', x), (x, y), (y, b'') \in \rho$, and therefore $x \in B'$.
   (b) It holds $(b', x), (x, y), (y, c'') \in \rho$ for some $b' \in B$ and $c'' \in C' \cup C''$, implying that $(b', c'') \in \rho$, for $b' \in B'$ and $c'' \in C' \cup C''$ which is not possible by Lemma 6.1.
   (c) $(c', x), (x, y), (y, b'') \in \rho$, for some $c' \in C' \cup C''$ and $b'' \in B$, implies $x \in C''$.
   (d) There are $c', c'' \in C' \cup C''$ such that $(c', x), (x, y), (y, c'') \in \rho$. From this condition, we have $x \in C' \cup C''$.
15) For $x \in G$ and $y \notin B' \cup C' \cup C'' \cup G \cup L$, from $(x, y) \in \rho$ it follows that $y \in G$.
16) If $x \in L$ and $y \in B'$, then $(x, y) \in \rho$ and $(g^{\rho}_{b,c}(x), g^{\rho}_{b,c}(y)) = (\mathbf{0}, b) \in \rho$, since $\mathbf{0}$ is the least element.
17) Similarly as in the previous case, $x \in L$ and $y \in C' \cup C''$ imply $(x, y) \in \rho \Rightarrow (g^{\rho}_{b,c}(x), g^{\rho}_{b,c}(y)) = (\mathbf{0}, c) \in \rho$.
18) If $x \in L, y \in G$ and $(x, y) \in \rho$, then $(g^{\rho}_{b,c}(x), g^{\rho}_{b,c}(y)) = (\mathbf{0}, \mathbf{1})$, which is obviously in ρ.
19) In case $(x, y) \in \rho$ for $x, y \in L$, we have $(g^{\rho}_{b,c}(x), g^{\rho}_{b,c}(y)) = (\mathbf{0}, \mathbf{0}) \in \rho$.
20) Let $x \in L$ and $y \notin B' \cup C' \cup C'' \cup G \cup L$. Then, as in cases (16) and (17), from $(x, y) \in \rho$ we obtain $(g^{\rho}_{b,c}(x), g^{\rho}_{b,c}(y)) = (\mathbf{0}, y) \in \rho$.
21) For $x \notin B' \cup C' \cup C'' \cup G \cup L$ and $y \in B'$ the assumption $(x, y) \in \rho$ implies $x \in L$.
22) As in the previous case, from $(x, y) \in \rho$ for $x \notin B' \cup C' \cup C'' \cup G \cup L$ and $y \in C' \cup C''$, we have $x \in L$.

23) If $x \notin B' \cup C' \cup C'' \cup G \cup L$ and $y \in G$, as in the cases (3) and (8), from $(x, y) \in \rho$ we get $(g^{\rho}_{b,c}(x), g^{\rho}_{b,c}(y)) = (x, \mathbf{1}) \in \rho$.
24) For $x \notin B' \cup C' \cup C'' \cup G \cup L$ and $y \in L$, $(x, y) \in \rho$ implies $x \in L$.
25) If $(x, y) \in \rho$ for $x, y \notin B' \cup C' \cup C'' \cup G \cup L$, then $(g^{\rho}_{b,c}(x), g^{\rho}_{b,c}(y)) = (x, y) \in \rho$. □

Now we can prove that if ρ is a bounded partial order, then $h\,\mathrm{Pol}\,\rho$ satisfies the conditions of Theorem 6.1.

Theorem 6.2. [7] *Let $\rho \subset A^2$ be a nontrivial bounded partial order relation on A. Then $h\,\mathrm{Pol}\,\rho$ is a maximal hyperclone on A.*

Proof. Let $f \in H_A \setminus h\,\mathrm{Pol}\,\rho$ be an n-ary hyperoperation. Then there exists a matrix

$$M = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \in \rho^*$$

such that $f(M) = (B, C)$, where $(b, c) \notin \rho$, for all $b \in B$ and $c \in C$.

For every $i \in \{1, \dots, n\}$ define $f_i \in O_A^{(1)}$ as follows

$$f_i(x) = \begin{cases} a_i, & x = \mathbf{0} \\ b_i, & \text{otherwise.} \end{cases}$$

We shall prove that $f_1, \dots, f_n \in \mathrm{Pol}\,\rho$. For $(x, y) \in \rho$ we get

$$(f_i(x), f_i(y)) = \begin{cases} (a_i, a_i), & \text{for } x = y = \mathbf{0} \\ (a_i, b_i), & \text{for } x = \mathbf{0} \text{ and } y \neq \mathbf{0} \\ (b_i, b_i), & \text{for } x \neq \mathbf{0} \text{ and } y \neq \mathbf{0}, \end{cases}$$

hence $(f_i(x), f_i(y)) \in \rho$ for all $i \in \{1, \dots, n\}$.

Let $g^{\rho}_{b,c}$ be the unary operation defined by (6.2). We proved (Lemma 6.2) that $g^{\rho}_{b,c} \in \mathrm{Pol}\,\rho$, and therefore we can define operation $f' \in \langle \mathrm{Pol}\,\rho \cup \{f\} \rangle_h$ by

$$f' = g^{\rho}_{b,c}[f[f_1, \dots, f_n]\,].$$

Then for all $x \in A$ we have

$$f'(x) = g^{\rho}_{b,c}[f[f_1, \dots, f_n]\,](x) = (g^{\rho}_{b,c})^{\#}(f^{\#}(f_1(x), \dots, f_n(x))).$$

From this definition and since $B \subseteq B'$ and $C \subseteq C'$ trivially holds, we obtain

$$\begin{aligned} f'(\mathbf{0}) &= (g^{\rho}_{b,c})^{\#}(f^{\#}(f_1(\mathbf{0}), \dots, f_n(\mathbf{0}))) = (g^{\rho}_{b,c})^{\#}(f^{\#}(\{a_1\}, \dots, \{a_n\})) \\ &= (g^{\rho}_{b,c})^{\#}(f(a_1, \dots, a_n)) = (g^{\rho}_{b,c})^{\#}(B) = \{b\}, \\ f'(x) &= (g^{\rho}_{b,c})^{\#}(f^{\#}(f_1(x), \dots, f_n(x))) = (g^{\rho}_{b,c})^{\#}(f^{\#}(\{b_1\}, \dots, \{b_n\})) \\ &= (g^{\rho}_{b,c})^{\#}(f(b_1, \dots, b_n)) = (g^{\rho}_{b,c})^{\#}(C) = \{c\}, \quad x \neq \mathbf{0}. \end{aligned}$$

Since $\mathbf{0}$ is the least element, it follows that $(\mathbf{0}, x) \in \rho$ for every $x \in A$. For $x \neq \mathbf{0}$ we have that $(f'(\mathbf{0}), f'(x)) = (b, c) \notin \rho$. Hence, $f' \in O_A$ and $f' \notin \mathrm{Pol}\,\rho$.

Therefore, the conditions of Theorem 6.1 are satisfied and we may conclude that $h\,\mathrm{Pol}\,\rho$ is a maximal hyperclone for every bounded partial order ρ. □

Example 6.1. Let $A = \{0, 1, 2, 3, 4\}$ and let $\rho \in R_A$ be a nonlinear order given by

$$\rho = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 2 & 4 & 4 & 4 \end{pmatrix}$$

If we take $M = \left(\begin{smallmatrix} 0 & 2 \\ 1 & 4 \end{smallmatrix}\right) \in \rho^*$ and $f \in H_A^{(2)}$ such that $f(M) = (\{1, 2\}, \{3\}) = (B, C) \notin \rho_h$, $b = 1$, $c = 3$, then we can define operations $f_1, f_2, g_{1,3}^{\rho} \in O_A^{(1)}$ by

	0	1	2	3	4
f_1	0	1	1	1	1
f_2	2	4	4	4	4
$g_{1,3}^{\rho}$	0	1	1	3	4

Clearly, $f_1, f_2, g_{1,3}^{\rho} \in \operatorname{Pol}\rho$ and for the operation $f' = g_{1,3}^{\rho}[f[f_1, f_2]\,]$ we have

$$\begin{aligned} f'(0) &= g_{1,3}^{\rho}[f[f_1, f_2]\,](0) = (g_{1,3}^{\rho})^{\#}(f^{\#}(f_1(0), f_2(0))) \\ &= (g_{1,3}^{\rho})^{\#}(f^{\#}(\{0\}, \{2\})) = (g_{1,3}^{\rho})^{\#}(\{1, 2\}) = \{1\}, \\ f'(1) &= g_{1,3}^{\rho}[f[f_1, f_2]\,](1) = (g_{1,3}^{\rho})^{\#}(f^{\#}(f_1(1), f_2(1))) \\ &= (g_{1,3}^{\rho})^{\#}(f^{\#}(\{1\}, \{4\})) = (g_{1,3}^{\rho})^{\#}(\{3\}) = \{3\}. \end{aligned}$$

Therefore, $(0, 1) \in \rho$ and $(f'(0), f'(1)) = (1, 3) \notin \rho$ imply $f' \in O_A \setminus \operatorname{Pol}\rho$. Finally, applying Theorem 6.1 we get $\langle h \operatorname{Pol}\rho \cup \{f\}\rangle_h = H_A$.

6.2. Hyperclones determined by equivivalence relations. In this subsection, as in the previous one, we will prove that $h \operatorname{Pol}\rho$ is a maximal hyperclone for ρ being a nontrivial equivalence relation, by showing that $h \operatorname{Pol}\rho$ satisfies the requirements of Theorem 6.1. However, construction of operations from $\operatorname{Pol}\rho$ that are necessary for defining the operation $f' \notin \operatorname{Pol}\rho$ is much simpler in this case, so we can immediately proceed to the following theorem.

Theorem 6.3. [13] *Let $\rho \subseteq A^2$ be a nontrivial equivalence relation. Then $h \operatorname{Pol}\rho$ is a maximal hyperclone on A.*

Proof. Suppose that $f \in H_A \setminus h \operatorname{Pol}\rho$. Then there exists a matrix

$$M = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \in \rho^*$$

such that $f(M) = (A', B') \notin \rho_h$, i.e., $(A' \times B') \cap \rho = \emptyset$.

Reflexivity of the relation ρ implies that sets A' and B' are disjoint. Thus we can choose $a_i \neq b_i$, for some $i \in \{1, \dots, n\}$. Denote $a = a_i$ and $b = b_i$. For $j = 1, \dots, n$ define operations $f_j \in O_A^{(1)}$ by

$$f_j(x) = \begin{cases} a_j, & x = a \\ b_j, & x = b \\ a_j, & \text{otherwise.} \end{cases}$$

Next we need to prove that $f_j \in \operatorname{Pol}\rho$, for $j = 1, \dots, n$. Let $(x, y) \in \rho$. Discuss the following possibilities:

1) $x, y \in A \setminus \{b\} \Rightarrow (f_j(x), f_j(y)) = (a_j, a_j) \in \rho$, since ρ is reflexive;

2) $x = y = b \Rightarrow (f_j(x), f_j(y)) = (b_j, b_j) \in \rho$, again since ρ is reflexive;
3) $x \neq b$, $y = b \Rightarrow (f_j(x), f_j(y)) = (a_j, b_j) \in \rho$;
4) $x = b$, $y \neq b \Rightarrow (f_j(x), f_j(y)) = (b_j, a_j) \in \rho$, since ρ is symmetric.

Denote by C_q the equivalence class of the relation ρ containing element q. For arbitrary $a' \in A'$ and $b' \in B'$ we define an operation $g \in O_A^{(1)}$ by

$$g(x) = \begin{cases} a', & x \in C_q, \text{ for some } q \in A' \\ b', & x \in C_q, \text{ for some } q \in B' \\ a', & \text{otherwise.} \end{cases}$$

As $(A', B') \notin \rho_h$, operation g is well defined since there are no $x \in A'$ and $y \in B'$ which belong to the same class. Obviously, $(a', b') \notin \rho$. Let us assume that $g \notin \operatorname{Pol}\rho$, which means that $(x, y) \in \rho$ and $(g(x), g(y)) \notin \rho$. However, ρ is reflexive, implying $(g(x), g(y)) \in \{(a', b'), (b', a')\}$. Hence, x and y are not in the same equivalence class, which contradicts $(x, y) \in \rho$. Therefore, $g \in \operatorname{Pol}\rho$.

Now we can define $f' \in \langle \operatorname{Pol}\rho \cup \{f\}\rangle_h$ by $f' = g[f[f_1, \ldots, f_n]]$. Then $f' \in O_A^{(1)}$ and we have

$$\begin{aligned} f'(a) &= g[f[f_1, \ldots, f_n]](a) = g^{\#}(f^{\#}(f_1(a), \ldots, f_n(a))) \\ &= g^{\#}(f^{\#}(\{a_1\}, \ldots, \{a_n\})) = g^{\#}(A') = \{a'\}, \\ f'(b) &= g[f[f_1, \ldots, f_n]](b) = g^{\#}(f^{\#}(f_1(b), \ldots, f_n(b))) \\ &= g^{\#}(f^{\#}(\{b_1\}, \ldots, \{b_n\})) = g^{\#}(B') = \{b'\}. \end{aligned}$$

Since $(a, b) \in \rho$ and $(a', b') \notin \rho$, it follows that $f' \notin \operatorname{Pol}\rho$.

For a hyperoperation $f \notin h\operatorname{Pol}\rho$ we determined operation $f' \notin \operatorname{Pol}\rho$, which is a composition of f and some elements of $\operatorname{Pol}\rho$, and now using Theorem 6.1 we may conclude that $h\operatorname{Pol}\rho$ is a maximal hyperclone. □

Example 6.2. Let ρ be an equivalence relation on $A = \{0, 1, 2\}$ with classes $\{0, 1\}$ and $\{2\}$, i.e.,

$$\rho = \begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \end{pmatrix}$$

If we choose $M = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 2 \end{smallmatrix}\right) \in \rho^*$ and a binary hyperoperation f such that $f(M) = (\{2\}, \{0, 1\}) \notin \rho_h$, then $f \notin h\operatorname{Pol}\rho$. Let us define $f_1, f_2, g \in O_A^{(1)}$ by

	0	1	2
f_1	0	1	1
f_2	2	2	2
g	1	1	2

For the operation $f' = g[f[f_1, f_2]]$ it holds $(1, 0) \in \rho$ and $(f'(1), f'(0)) = (2, 1) \notin \rho$, hence $f' \notin \operatorname{Pol}\rho$.

6.3. Hyperclones determined by central relations.

Using the same method as in the previous two cases, we are going to prove that $h\operatorname{Pol}\rho$ is a maximal hyperclone whenever ρ is a central relation.

Theorem 6.4. [13] *Let $\rho \subseteq A^m, m \geqslant 1$, be a central relation. Then $h\operatorname{Pol}\rho$ is a maximal hyperclone on A.*

Proof. If we take $f \notin h\operatorname{Pol}\rho$, then we have a matrix

$$M = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \in \rho^*$$

such that $f(M) = (A_1, A_2, \dots, A_m) \notin \rho_h$.

Notice that the sets $A_1, \dots, A_m$ are pairwise disjoint, because if we have $a \in A_i \cap A_j$, for $i, j \in \{1, \dots, m\}$, $i \neq j$, total reflexivity of ρ would imply

$$(x_1, \dots, x_{i-1}, a, \dots, x_{j-1}, a, \dots, x_m) \in \rho, \quad x_k \in A_k,$$

thus $(A_1 \times \dots \times A_m) \cap \rho \neq \emptyset$, which is not possible.

Choose distinct elements $c \in C(\rho)$ and $b_2, \dots, b_m \in A$. Hence, $(c, b_2, \dots, b_m) \in \rho$. Define operations $f_i \in O_A^{(1)}$, $i = 1, \dots, n$, as follows

$$f_i(x) = \begin{cases} a_{1i}, & x = c \\ a_{ki}, & x = b_k,\ 2 \leqslant k \leqslant m \\ c, & \text{otherwise.} \end{cases}$$

We will show that $f_1, \dots, f_n \in \operatorname{Pol}\rho$. Let $(x_1, \dots, x_m) \in \rho$. We can distinguish the following cases.

1) If there exists $j \in \{1, \dots, m\}$ such that $x_j \notin \{c, b_2, \dots, b_m\}$, then $f_i(x_j) = c$, $i = 1, \dots, n$, and $(f_i(x_1), \dots, f_i(x_m)) \in \rho$, since c is a central element and ρ is totally symmetric relation.
2) If $x_1, \dots, x_m \in \{c, b_2, \dots, b_m\}$, but there are some $j \neq k$ such that $x_j = x_k$, then $f_i(x_j) = f_i(x_k), i = 1, \dots, n$, and by total reflexivity $(f_i(x_1), \dots, f_i(x_m)) \in \rho$.
3) The remaining possibility is when all x_j, $j = 1, \dots, m$, belong to $\{c, b_2, \dots, b_m\}$, and they are all distinct. Then we have $\{f_i(x_1), \dots, f_i(x_m)\} = \{a_{1i}, \dots, a_{mi}\}$, and using total symmetry of the relation ρ, we get $(f_i(x_1), \dots, f_i(x_m)) \in \rho$.

For arbitrary $d_1 \in A_1, d_2 \in A_2 \dots, d_m \in A_m$ define operation $g \in O_A^{(1)}$ by

$$g(x) = \begin{cases} d_1, & x \in A_1 \\ d_2, & x \in A_2 \\ & \dots \\ d_m, & x \in A_m \\ c, & \text{otherwise.} \end{cases}$$

We need to prove that $g \in \operatorname{Pol}\rho$. If we assume the opposite, i.e., that there exists $(x_1, \dots, x_m) \in \rho$ such that $(g(x_1), \dots, g(x_m)) \notin \rho$, and since ρ is central relation, we may assume that $x_i \in A_i$, for $i = 1, \dots, m$. However, $(A_1, \dots, A_m) \notin \rho_h$ implies $(x_1, \dots, x_m) \notin \rho$, which is a contradiction.

Define $f' \in \langle \operatorname{Pol}\rho \cup \{f\}\rangle_h$ as usual by $f' = g[f[f_1, \dots, f_n]\,]$. Obviously $f' \in O_A^{(1)}$ and we get

$$\begin{aligned} f'(c) &= g^{\#}(f^{\#}(f_1(c), \dots, f_n(c))) \\ &= g^{\#}(f^{\#}(\{a_{11}\}, \dots, \{a_{1n}\})) = g^{\#}(A_1) = \{d_1\}, \\ f'(b_2) &= g^{\#}(f^{\#}(f_1(b_2), \dots, f_n(b_2))) \\ &= g^{\#}(f^{\#}(\{a_{21}\}, \dots, \{a_{2n}\})) = g^{\#}(A_2) = \{d_2\}, \\ &\dots \\ f'(b_m) &= g^{\#}(f^{\#}(f_1(b_m), \dots, f_n(b_m))) \\ &= g^{\#}(f^{\#}(\{a_{m1}\}, \dots, \{a_{mn}\})) = g^{\#}(A_m) = \{d_m\}. \end{aligned}$$

Clearly, $(c, b_2, \dots, b_m) \in \rho$, but $(d_1, \dots, d_m) \notin \rho$ since $(A_1, \dots, A_m) \notin \rho_h$, which means that $f' \notin \operatorname{Pol}\rho$.

Thus, we have shown that for a central relation ρ hyperclone $h\operatorname{Pol}\rho$ satisfies the conditions of Theorem 6.1, and therefore is a maximal hyperclone. □

Example 6.3. Let ρ be a central relation on $A = \{0, 1, 2\}$ with the center $C(\rho) = \{0\}$, i.e.,

$$\rho = \begin{pmatrix} 0 & 1 & 2 & 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 1 & 0 & 2 & 0 \end{pmatrix}$$

Let us choose $M = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right) \in \rho^*$ and $f \in H_A^{(2)}$ such that $f(M) = (\{1\}, \{2\})$. Then $f \notin h\operatorname{Pol}\rho$. If we define operations $f_1, f_2, g \in O_A^{(1)}$ by

	0	1	2
f_1	1	0	0
f_2	0	2	0
g	0	2	1

and then define operation $f' = g[f[f_1, f_2]\,]$, we have $(0, 1) \in \rho$, but $(f'(0), f'(1)) = (2, 1) \notin \rho$, which implies $f' \notin \operatorname{Pol}\rho$.

6.4. Hyperclones determined by regular relations. In this subsection we are going to show that $h\operatorname{Pol}\rho$ is a maximal hyperclone if ρ is a regular relation.

Notice that every m-regular relation is totally reflexive and totally symmetric. Namely, every m-tuple $(a_1, \dots, a_m)$ such that $a_i = a_j$ for some $1 \leqslant i < j \leqslant m$, belongs to ρ, since by reflexivity (a_i, a_j) is in every relation from Θ. On the other hand, if $(a_1, \dots, a_m) \in \rho$, then also $(a_{\pi(1)}, \dots, a_{\pi(m)}) \in \rho$, where π is a permutation of the set $\{1, \dots, m\}$, because for every $i \in \{1, \dots, h\}$ we have $k \neq \ell$ such that $(a_k, a_\ell) \in \theta_i$, $a_k, a_\ell \in \{a_{\pi(1)}, \dots, a_{\pi(m)}\}$ and θ_i is symmetric.

Theorem 6.5. [13] *Let $\rho \subset A^m, m \geqslant 3$, be m-regular relation. Then $h\operatorname{Pol}\rho$ is a maximal hyperclone on A.*

Proof. Let f be an n-ary hyperoperation on A which is not in $h\,\mathrm{Pol}\,\rho$. Then there exists a matrix

$$M = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \in \rho^*$$

such that $f(M) = (A_1, A_2, \dots, A_m) \notin \rho_h$.

As in the case of central relations, since ρ is totally reflexive, sets $A_1, \dots, A_m$ are pairwise disjoint.

Choose distinct elements $d_1, \dots, d_m \in A$, such that $(d_1, \dots, d_m) \in \rho$. Define operations $f_j \in O_A^{(1)}$, $j = 1, \dots, n$ as follows:

$$f_j(x) = \begin{cases} a_{ij}, & x = d_i \\ a_{1j}, & \text{otherwise.} \end{cases}$$

We need to prove that $f_1, \dots, f_n \in \mathrm{Pol}\,\rho$. Let $(x_1, \dots, x_m) \in \rho$. If $x_k = x_\ell$, for some $1 \leqslant k < \ell \leqslant m$, then $f_j(x_k) = f_j(x_\ell)$, and by total reflexivity of ρ we get $(f_j(x_1), \dots, f_j(x_m)) \in \rho$. If, on the other hand, all $x_1, \dots, x_m$ are distinct, we can distinguish the following cases:

1) $\{x_1, \dots, x_m\} = \{d_1, \dots, d_m\}$, and since ρ is totally symmetric, we may assume that $(x_1, \dots, x_m) = (d_1, \dots, d_m)$, which for all $j = 1, \dots, n$ yields

$$(f_j(x_1), \dots, f_j(x_m)) = (f_j(d_1), \dots, f_j(d_m)) = (a_{1j}, \dots, a_{mj}) \in \rho;$$

2) There is exactly one $k \in \{1, \dots, m\}$ such that $x_k \notin \{d_1, \dots, d_m\}$, which means that $f_j(x_k) = a_{1j}$. Depending on whether d_1 belongs to $\{x_1, \dots, x_m\}$ or not, it is possible for $(f_j(x_1), \dots, f_j(x_m))$ to have same two coordinates, which puts it in ρ using total reflexivity, or

$$\{f_j(x_1), \dots, f_j(x_m)\} = \{a_{1j}, \dots, a_{mj}\},$$

which implies $(f_j(x_1), \dots, f_j(x_m)) \in \rho$ by total symmetry of ρ.

3) There are $k \neq \ell$ such that $x_k, x_\ell \notin \{d_1, \dots, d_m\}$. Then $f_j(x_k) = f_j(x_\ell) = a_{1j}$, and thus $(f_j(x_1), \dots, f_j(x_m)) \in \rho$, since ρ is totally reflexive.

Hence, $f_j \in \mathrm{Pol}\,\rho$, $j = 1, \dots, n$.

If we choose arbitrary $b_i \in A_i$, $i = 1, \dots, m$, then $(b_1, \dots, b_m) \notin \rho$. Since ρ is m-regular relation there exists an equivalence relation θ^* such that $(b_i, b_j) \notin \theta^*$ for all $i \neq j$. Denote by C_q^* equivalence class of the relation θ^* which includes $q \in A$. According to the previous discussion, there are no $x \in A_i$ and $y \in A_j$, for $i \neq j$, such that $(x, y) \in \theta^*$, i.e., their equivalence classes C_x^* and C_y^* are disjoint. Define operation $g \in O_A^{(1)}$ by

$$g(x) = \begin{cases} b_1, & x \in C^*_{q_1}, \text{ for some } q_1 \in A_1 \\ b_2, & x \in C^*_{q_2}, \text{ for some } q_2 \in A_2 \\ & \dots \\ b_m, & x \in C^*_{q_m}, \text{ for some } q_m \in A_m \\ b_1, & \text{otherwise.} \end{cases}$$

Operation g is well defined because all the equivalence classes that are used in this definition are, as we already said, disjoint. Suppose that $g \notin \operatorname{Pol}\rho$, i.e., there exists $(x_1, \dots, x_m) \in \rho$ such that $(g(x_1), \dots, g(x_m)) \notin \rho$. If for some x_i and x_j we have $g(x_i) = g(x_j)$, then, by total reflexivity of ρ, $(g(x_1), \dots, g(x_m)) \in \rho$. Thus, $g(x_1), \dots, g(x_m)$ are all distinct, and the only possibility is that $\{g(x_1), \dots, g(x_m)\} = \{b_1, \dots, b_m\}$, which implies that each x_i, $i = 1, \dots, m$, belongs to distinct class of θ^*, contradicting the assumption that $(x_1, \dots, x_m) \in \rho$. Therefore, $g \in \operatorname{Pol}\rho$.

Define $f' \in \langle \operatorname{Pol}\rho \cup \{f\}\rangle_h$ by $f' = g[f[f_1, \dots, f_n]]$. Clearly, $f' \in O_A^{(1)}$ and since $A_i \subseteq \bigcup_{q \in A_i} C^*_q$, $i = 1, \dots, m$, it holds

$$\begin{aligned} f'(d_i) &= g^{\#}(f^{\#}(f_1(d_i), \dots, f_n(d_i))) \\ &= g^{\#}(f^{\#}(\{a_{i1}\}, \dots, \{a_{in}\})) = g^{\#}(A_i) = \{b_i\}. \end{aligned}$$

Now we have $(d_1, \dots, d_m) \in \rho$ and $(b_1, \dots, b_m) \notin \rho$, which implies $f' \notin \operatorname{Pol}\rho$.

We have proved that for every $f \in H_A \setminus h\operatorname{Pol}\rho$ there exists $f' \in O_A \setminus \operatorname{Pol}\rho$ such that $f' \in \langle \operatorname{Pol}\rho \cup \{f\}\rangle_h$, and thus, by Theorem 6.1, $h\operatorname{Pol}\rho$ is a maximal hyperclone. □

Example 6.4. Let ρ be a regular relation on $A = \{0, 1, 2, 3\}$, corresponding to the equivalence relation with classes $\{0, 1\}, \{2\}$ and $\{3\}$, i.e.,

$$\rho = A^3 \setminus \begin{pmatrix} 0 & 0 & 2 & 2 & 3 & 3 & 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 3 & 0 & 3 & 0 & 2 & 2 & 3 & 1 & 3 & 1 & 2 \\ 3 & 2 & 3 & 0 & 2 & 0 & 3 & 2 & 3 & 1 & 2 & 1 \end{pmatrix}.$$

If $M = \left(\begin{smallmatrix} 0 & 1 \\ 2 & 1 \\ 1 & 3 \end{smallmatrix}\right) \in \rho^*$ and f is a binary hyperoperation such that $f(M) = (\{3\}, \{0, 1\}, \{2\})$, then $f \notin h\operatorname{Pol}\rho$. Let $f_1, f_2, g \in O_A^{(1)}$ be defined by

	0	1	2	3
f_1	2	1	0	0
f_2	1	3	1	1
g	1	1	2	3

Define operation $f' = g[f[f_1, f_2]]$. Then $(3, 0, 1) \in \rho$, and $(f'(3), f'(0), f'(1)) = (3, 1, 2) \notin \rho$, hence $f' \notin \operatorname{Pol}\rho$.

Acknowledgement. This work was partly supported by the Serbian Ministry of Education, Science, and Technological Development (projects OI174026 and III44006).

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