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UNIVERSAL OBJECTS AND ASSOCIATIONS BETWEEN CLASSES OF BANACH SPACES AND CLASSES OF COMPACT SPACES

Abstract. In the context of classical associations between classes of Banach spaces and classes of compact Hausdorff spaces, we survey known results and open questions concerning the existence and nonexistence of universal Banach spaces and of universal compact spaces in various classes. This gives quite a complex network of interrelations which quite often depend on additional set-theoretic assumptions.

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CONTENTS

1. Introduction

In this survey note, we would like to look at universal objects in several classes of compact spaces and several classes of Banach spaces. All Banach spaces are over the reals and are assumed to be infinite dimensional. All compact spaces are assumed to be Hausdorff and infinite.

In Section 2 we use the Stone duality between totally disconnected compact Hausdorff spaces with continuous maps and Boolean algebras with homomorphisms as a motivation for seeing the interactions between compact Hausdorff spaces with continuous maps and Banach spaces with linear bounded operators. We interpret what is known in functional analysis as permanence properties in the light of the Stone duality. We define several notions of associated classes of Banach spaces and compact spaces (associated, K-associated, B-associated and strongly associated). We show elementary facts about the relationships among theses notions and note natural ways of defining associated classes.

In Section 3 we list the classes of compact spaces we will be concerned with: uniform Eberlein, Eberlein, Corson, Corson with property M, Radon–Nikodým, Quasi Radon–Nikodým; and classes of Banach spaces: Hilbert generated, weakly

compactly generated, weakly Lindelöf, weakly Lindelöf determined, Asplund generated, subspaces of Asplund generated. We distinguish subclasses of a given density character. We also state known relationships among these classes.

In Section 4 we describe various types of universality (surjective, injective, isomorphic, isometric, weak) and prove elementary results on how the existence of universal objects for some class gives the existence of the universal objects in an associated class depending on the type of the association and the type of universality. We note that unless we are in the case of weakly universal Banach spaces, the existence of various types of universal compact spaces implies the existence of universal Banach spaces but apparently not vice versa. In particular all present in the literature proofs of results on the nonexistence of universal Banach spaces are considerably harder than the results on the nonexistence of universal compact spaces.

Finally in Section 5 we list in a systematic way known results and open problems on the existence and nonexistence of universal compact spaces and Banach spaces in the classes introduced in Section 3. We stress general patterns along the various notions of association and duality from Section 2, in the context of various notions of universality from Section 4.

In Section 6 we mention known results concerning the universality of l_∞/c_0. Throughout the paper we mention many questions which seem to be open. For undefined notions we refer the reader to the standard textbooks like [31, 40, 45, 29]. We are grateful to Antonio Avilés for conversations which allowed the author to improve this paper.

2. Associations between compact Hausdorff spaces and Banach spaces

As the motivation for the associations between compact spaces and Banach spaces, we would like to propose the Stone duality between compact totally disconnected spaces and Boolean algebras. This duality consists of two standard constructions of contravariant functors.

Recall that to each Boolean algebra $\mathcal{A}$ we can associate a totally disconnected compact space

$$K_\mathcal{A} = \{h : h \text{ is a homomorphism from } \mathcal{A} \text{ into } \{0, 1\}\}$$

with the smallest topology making all evaluations of elements of $K_\mathcal{A}$ at a fixed element of $\mathcal{A}$ continuous. This topology is called the Stone topology (see e.g., [40]). Then, to every homomorphism of Boolean algebras $g : \mathcal{B} \to \mathcal{A}$ we associate a continuous map $\phi_g : K_\mathcal{A} \to K_\mathcal{B}$ given for every $h \in K_\mathcal{A}$ by $\phi_g(h) = h \circ g$. This way, by a standard argument, g is onto if and only if ϕ_g is an injection and vice versa. Also to each compact totally disconnected space K we can associate a Boolean algebra

$$\mathcal{A}_K = \{\phi : \phi \text{ is a continuous function from } K \text{ into } \{0, 1\}\}$$

with the operations of taking the minimum, taking the maximum and the difference with the constant function equal to one. Given a continuous function $\psi : L \to K$, where K and L are compact totally disconnected, we can define $h_\psi : \mathcal{A}_K \to \mathcal{A}_L$ by $h_\psi(\phi) = \phi \circ \psi$. We have similar relations among surjections and injections. The

Stone duality gives us that $K_{\mathcal{A}_L}$ is homeomorphic to L for every totally disconnected compact L and $\mathcal{A}_{K_{\mathcal{B}}}$ is isomorphic to $\mathcal{B}$ for every Boolean algebra establishing a complete correspondence between the two categories which is elegant and very fruitful for both the topology and Boolean algebras. Taking continuous images of totally disconnected compact spaces corresponds to taking Boolean subalgebras and taking closed subspaces corresponds to taking quotients of Boolean algebras.

Now let us see what happens if in an analogous way, instead of a Boolean algebra, we associate with a given compact Hausdorff space K a Banach space. We consider

$$C(K) = \{f : f \text{ is a continuous function from } K \text{ into } \mathbb{R}\}$$

with the pointwise linear operations. Given a continuous function $\psi : L \to K$ where K and L are compact we can define

$$T_\psi(f) = f \circ \psi.$$

We have similar relations among surjections and injections as in the case of Boolean algebras. Now, to each Banach space X we associate the dual ball with the weak* topology, that is

$$B_{X^*} = \{x^* : x^* \text{ is linear and bounded map from } X \text{ into } \mathbb{R} \text{ with } \|x^*\| \leqslant 1\}$$

with the smallest topology making all evaluations of elements of B_{X^*} at a fixed element of X continuous. This topology is called the weak* topology (see e.g., [31] for details). Here the analogy to the Stone duality ends, because given a linear bounded operator $T : Y \to X$, if we define $T^* : X^* \to Y^*$ by

$$T^*(x^*) = x^* \circ T,$$

for every $x^* \in X^*$ there is no reason why T^* restricted to B_{X^*} has to end up in B_{Y^*} unless T is an isometry. Now we have two options, either consider only isometries, but this is not what is done in Banach space theory, or consider the duality between X^* and Y^* but then we loose compactness and all of its privileges.

The third way[1] which is behind the theory discussed in this paper is to note that T^* is continuous with respect to the weak* topologies, so the image of B_{X^*} must be compact, and so bounded in Y^*, hence there is $n \in \mathbb{N}$ such that $T^*[B_{X^*}] \subseteq nB_{Y^*}$ and of course nB_{Y^*} is a homeomorphic copy of B_{Y^*}. Then some analogies with the behaviour of a contravariant functor can be saved.

Proposition 2.1. *Suppose K, L are compact spaces, X, Y are Banach spaces.*

(1) *If K is a continuous image of L, then $C(K)$ is isometric to a subspace of $C(L)$.*
(2) *If L is a subspace of K, then $C(L)$ is isometric to a quotient of $C(K)$.*
(3) *If X is a subspace of Y, then B_{X^*} is a continuous image of B_{Y^*}.*
(4) *If X is isomorphic to a quotient of Y, then B_{X^*} is homeomorphic to a subspace of B_{Y^*}.*

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[1]For possible functors between various categories of compact spaces and various categories of Banach spaces see [55].

It should be nevertheless noted, that even isomorphic Banach spaces may have nonhomeomorphic dual balls (with the weak* topologies), this can even happen in the very canonical case of a nonseparable Hilbert space (see [7], here the weak and the weak* topologies agree). On the other hand we have the following analogy of the Stone duality:

Proposition 2.2. *Let K be a compact space and X be a Banach space.*
  (1) *K is homeomorphic to a subspace of $B_{C(K)^*}$.*
  (2) *X is isometric to a subspace of $C(B_{X^*})$.*

Proof. If x is a point of a compact space K then the pointwise measure functionals δ_x are inside the dual ball of the Banach space $C(K)$ and the map that sends $x \in K$ to δ_x is a homeomorphism. For any Banach space the map which sends x to x^{**} in the bidual is an isometry whose image is included in $C(B_{X^*})$. $\qquad\square$

However, unlike in the case of the Stone duality, $B_{C(K)^*}$ is not homeomorphic to K nor $C(B_{X^*})$ is not isomorphic to X in most cases.

Note that in 2.1 the isomorphic version of (3) is not true: if $T : X \to Y$ is an isomorphism onto its range, then B_{X^*} may not be a continuous image of B_{Y^*}. This follows from results of Avilés [7, Theorems 2 and 4] even in the case of both of the spaces isomorphic to a Hilbert space. However we have the following.

Proposition 2.3. *Suppose that X and Y are Banach spaces and $T : X \to Y$ is an isomorphism onto its range. Then the family of compact subsets of X^* with the weak* topology is included in the family of continuous images of compact subsets of Y^* with the weak* topology.*

Proof. Let $T^{-1} : T[X] \to X$. $T^* : Y^* \to X^*$ is weak * continuous. Note that

$$\frac{1}{\|T^{-1}\|} B_{X^*} \subseteq T^*[B_{Y^*}] \subseteq \|T\| B_{X^*}.$$

The right hand side inclusion is clear from the fact that $\|T\| = \|T^*\|$. For the left hand side inclusion note that if $\phi \in \frac{1}{\|T^{-1}\|} B_{X^*}$, then $(T^{-1})^*(\phi) \in B_{T[X]^*}$ is at most of norm one, and so it has an extension $\psi \in B_{Y^*}$. Now $T^*(\psi) = \psi \circ T = \phi \circ T^{-1} \circ T = \phi$.

If K is compact in X^*, it is bounded and so $K \subseteq n B_{X^*}$ for some $n \in \mathbb{N}$. It follows that K is a continuous image under T^* of $(T^*)^{-1}[K] \subseteq n\|T^{-1}\| B_{Y^*}$. $\qquad\square$

So, it may be more natural to consider classes of compact spaces associated with a Banach space (all compact weak* subsets of its dual) than just the dual ball. This leads, however, to classes of Banach spaces if we aim at some duality. In the Banach space theory practice one considers quite general classes of Banach spaces and compact spaces which are closed by taking these dual objects. In most cases the classes of compact spaces are closed under taking subspaces and so the family of all weakly* compact subsets of a given dual X^* is included in such a class if and only if the dual ball $n B_{X^*}$ belongs to the class. Here by classes of compact spaces or Banach spaces we mean classes which are homeomorphism or isomorphism invariant respectively.

Definition 2.4. Let $\mathcal{K}$ be a class of compact spaces and $\mathcal{B}$ be a class of Banach spaces. We say that $\mathcal{K}$ and $\mathcal{B}$ are associated if and only if the following two implications hold:

- If $K \in \mathcal{K}$, then $C(K) \in \mathcal{B}$.
- If $X \in \mathcal{B}$, then $B_{X^*} \in \mathcal{K}$.

If instead of the first implication we have equivalence, we say that the classes are K-associated. If instead of the second implication we have equivalence, we say that the classes are B-associated. If instead of both of the implications above we have equivalences, we say that the classes are strongly associated.

Lemma 2.5. *Suppose that $\mathcal{K}$ is a class of compact spaces and $\mathcal{B}$ is a class of Banach spaces such that $\mathcal{K}$ and $\mathcal{B}$ are associated.*

(1) *If $\mathcal{K}$ is closed under taking subspaces, then $\mathcal{K}$ and $\mathcal{B}$ are K-associated.*
(2) *If $\mathcal{B}$ is closed under taking subspaces, then $\mathcal{K}$ and $\mathcal{B}$ are B-associated.*

Proof. If $C(K) \in \mathcal{B}$, by the second implication of 2.4, $B_{C(K)^*} \in \mathcal{K}$. By 2.2 K is a subspace of $B_{C(K)^*}$, so, by the hypothesis on $\mathcal{K}$, we get $K \in \mathcal{K}$. The other part is analogous. $\square$

Lemma 2.6. *Suppose that $\mathcal{K}$ is a class of compact spaces closed under taking continuous images and $\mathcal{B}$ is a class of Banach spaces which is not closed under taking subspaces and suppose that $\mathcal{K}$ and $\mathcal{B}$ are associated. Then $\mathcal{K}$ and $\mathcal{B}$ are not B-associated.*

Proof. Let $X \notin \mathcal{B}$ be a subspace of $Y \in \mathcal{B}$. Then B_{X^*} is a continuous image of $B_{Y^*} \in \mathcal{K}$ by 2.1, hence $B_{X^*} \in \mathcal{K}$ which proves that the second implication of 2.4 cannot be reversed. $\square$

We note here that some natural classes of compact spaces are not associated with classes of Banach spaces. For example B_{X^*} is never totally disconnected, so neither totally disconnected spaces nor scattered spaces are associated with a class of Banach spaces. Also many natural classes of Banach spaces are not associated with classes of compact spaces, for example reflexive spaces are not associated with a class of compact spaces because no infinite dimensional $C(K)$ space is reflexive. As the above classes play important roles, the issues of universality are well investigated for them, e.g., see [59, 62, 35, 15], but we will not be concerned with them in this note. In general we have the following:

Proposition 2.7. *A class of compact spaces $\mathcal{K}$ closed by taking subspaces is associated with a class of Banach spaces if and only if $B_{C(K)^*} \in \mathcal{K}$ whenever $K \in \mathcal{K}$. A class of Banach spaces $\mathcal{B}$ closed under taking isometric copies is associated with a class of compact spaces if and only if $C(B_{X^*}) \in \mathcal{B}$ whenever $X \in \mathcal{B}$.*

Proof. If the above conditions hold define

$$\mathcal{B} = \{X : B_{X^*} \in \mathcal{K}\}, \quad \mathcal{K} = \{K : C(K) \in \mathcal{B}\},$$

respectively. It remains to prove that $\mathcal{B}$ is isomorphism invariant and $\mathcal{K}$ is homeomorphism invariant. In the first case we use the fact that if we have two isomorphic

Banach spaces, then the dual ball of one is a subspace of a homeomorphic copy of the dual ball of the other. In the second case we use the fact that if two compact spaces are homeomorphic, then the spaces of their continuous functions are isometric. $\qquad\square$

There are also elementary ways of obtaining strongly associated classes:

Proposition 2.8. *Suppose that $\mathcal{B}$ and $\mathcal{K}$ are associated classes of Banach spaces and compact spaces respectively. Suppose that $\mathcal{K}$ is closed under taking subspaces of its elements. Let $\mathcal{B}'$ be the class of all subspaces of elements of $\mathcal{B}$ and $\mathcal{K}'$ be the class of continuous images of elements of $\mathcal{K}$. Then $\mathcal{B}'$ and $\mathcal{K}'$ are strongly associated.*

Proof. As $\mathcal{B}'$ and $\mathcal{K}'$ are closed under taking subspaces, by 2.5 it is enough to prove that the classes are associated. Suppose $Y \in \mathcal{B}'$, i.e, $X \subseteq Y$ with $Y \in \mathcal{B}$, then B_{Y^*} is a continuous image od $B_{X^*} \in \mathcal{K}$ by 2.1, and hence $B_{Y^*} \in \mathcal{K}'$. If $L \in \mathcal{K}'$, then L is a continuous image of some $K \in \mathcal{K}$ and so $C(L) \subseteq C(K) \in \mathcal{B}$ by 2.1. $\qquad\square$

It should be noted that given a class of compact spaces or a class of Banach spaces an associated class is not uniquely determined. For example WCG Banach spaces and subspaces of WCG Banach spaces are associated with the class of Eberlein compacta (see the next section). The intersection of two associated classes with a given class is again an associated class, so we can talk about minimal associated classes. However these are pretty trivial and uninteresting, e.g., given associated classes $\mathcal{K}$ and $\mathcal{B}$ consider $\mathcal{B}'$ to be equal to the class of all Banach spaces in $\mathcal{B}$ isomorphic to a space of the form $C(K)$ for $K \in \mathcal{K}$. $\mathcal{B}'$ is the minimal class associated with $\mathcal{K}$. As our paradigmatic link between classes of Banach spaces and compact spaces will be obtaining a universal Banach space for the associated class from a universal compact space in a given class (or in general getting an information about Banach spaces from compact spaces) we are rather interested in a maximal class of Banach spaces associated with minimal classes of compact spaces.

3. Classes of compact and Banach spaces

Definition 3.1. Let κ be a cardinal. We will use the following notation:

- $\mathbb{UE}_\kappa$ – uniform Eberlein compact spaces (i.e., compact subspaces of Hilbert spaces with the weak topology) of weight $\leqslant \kappa$,
- $\mathbb{E}_\kappa$ – Eberlein compact spaces (i.e., compact subspaces of Banach spaces with the weak topology) of weight $\leqslant \kappa$,
- $\mathbb{C}_\kappa$ – Corson compact spaces (i.e., compact subspaces of Σ-products of $\mathbb{R}$) of weight $\leqslant \kappa$,
- $\mathbb{CM}_\kappa$ – Corson compact spaces with property M (i.e., Corson compacta where every Radon measure on K has a separable support) of weight $\leqslant \kappa$,
- $\mathbb{RN}_\kappa$ – Radon–Nikodým compact spaces (i.e., fragmented by a lower semicontinuous metric) of weight $\leqslant \kappa$,
- $\mathbb{QRN}_\kappa$ – Quasi Radon–Nikodým compact spaces (i.e., fragmented by a lower-semi continuous quasi-metric) of weight $\leqslant \kappa$,
- $\mathbb{K}_\kappa$ – all compact spaces of weight $\leqslant \kappa$.

We have $\mathbb{UE}_\kappa \subseteq \mathbb{E}_\kappa \subseteq \mathbb{CM}_\kappa \subseteq \mathbb{C}_\kappa \subseteq \mathbb{K}_\kappa$ and $\mathbb{E}_\kappa \subseteq \mathbb{RN}_\kappa \subseteq \mathbb{QRN}_\kappa \subseteq \mathbb{K}_\kappa$ for each cardinal κ and $\mathbb{E}_\kappa = \mathbb{C}_\kappa \cap \mathbb{RN}_\kappa$ by [46]. Note that metrizability in the class of compact spaces is equivalent to countable weight, so actually the class $\mathbb{K}_\omega$ is the class of metrizable compact spaces. More details concerning these classes of spaces can be found in [45] or in [33]. In particular, all the classes above are closed under taking subspaces and with the exception of the Radon Nikodým compacta, are closed under taking continuous images (see [10]). The first of these facts is trivial and the second relays on the results of Benyamini, Rudin and Wage from [16] and Argyros, Mercourakis, Negrepontis from [4]. By of [4, 3.2.(3)] we have that $\mathbb{C}_\kappa = \mathbb{CM}_\kappa$ for for every cardinal κ if Martin's axiom and the negation of the continuum hypothesis CH holds. If CH holds then $\mathbb{C}_\kappa \neq \mathbb{CM}_\kappa$ for any uncountable κ [4, 3.12] By a result of Avilés [6] $\mathbb{QRN}_\kappa = \mathbb{RN}_\kappa$ if $\kappa < \mathfrak{b}$, however $\mathbb{QRN}_{2^\omega} \neq \mathbb{RN}_{2^\omega}$ by the main result of [10].

Definition 3.2. Let κ be a cardinal. We will use the following notation:
- $\overline{\mathcal{H}}_\kappa$ – Hilbert generated Banach spaces of density character $\leqslant \kappa$,
- $\mathcal{WCG}_\kappa$ – Weakly compactly generated Banach spaces of density character $\leqslant \kappa$,
- $\mathcal{WLD}_\kappa$ – Weakly Lindelöf determined Banach spaces of density character $\leqslant \kappa$,
- $\mathcal{L}_\kappa$ – Banach spaces of density character $\leqslant \kappa$ which are Lindelöf in the weak topology,
- $\overline{\mathcal{A}}_\kappa$ – Asplund generated Banach spaces of density character $\leqslant \kappa$,
- $\mathcal{S}\overline{\mathcal{A}}_\kappa$ – subspaces of Asplund generated Banach spaces of density character $\leqslant \kappa$,
- $\mathcal{B}_\kappa$ – all Banach spaces of density character $\leqslant \kappa$.

Good sources of information about the above classes of Banach spaces are: [31] for $\overline{\mathcal{H}}_\kappa$, $\mathcal{L}_\kappa$ and for $\mathcal{WCG}_\kappa$; [37] for $\mathcal{WLD}_\kappa$; [33] for $\overline{\mathcal{A}}_\kappa$ and $\mathcal{S}\overline{\mathcal{A}}_\kappa$. We have $\overline{\mathcal{H}}_\kappa \subseteq \mathcal{WCG}_\kappa \subseteq \mathcal{WLD}_\kappa \subseteq \mathcal{B}_\kappa$ and $\mathcal{WCG}_\kappa \subseteq \overline{\mathcal{A}}_\kappa \subseteq \mathcal{S}\overline{\mathcal{A}}_\kappa \subseteq \mathcal{B}_\kappa$. All the classes are closed under taking quotients. An example of Rosenthal from [52] is a space from $\overline{\mathcal{H}}_{2^\omega}$ whose subspace is not in $\mathcal{WCG}_{2^\omega}$. We have the following

Theorem 3.3. *The pairs of classes*
1. $\mathbb{UE}_\kappa$ *and* $\overline{\mathcal{H}}_\kappa$ *are K-associated and are not strongly associated.*
2. $\mathbb{E}_\kappa$ *and* $\mathcal{WCG}_\kappa$, *are K-associated and are not strongly associated.*
3. $\mathcal{L}_{\omega_1}$ *is not associated with* $\mathbb{C}_{\omega_1}$.
4. $\mathbb{RN}_\kappa$ *and* $\overline{\mathcal{A}}_\kappa$ *are K-associated, strongly associated if* $\kappa < \mathfrak{b}$ *and not strongly associated if* $\kappa \geqslant \mathfrak{b}$.
5. $\mathbb{QRN}_\kappa$ *and* $\mathcal{S}\overline{\mathcal{A}}_\kappa$ *are strongly associated.*
6. $\mathbb{K}_\kappa$ *and* $\mathcal{B}_\kappa$ *are strongly associated.*

Proof. The equivalence of $K \in \mathbb{UE}_\kappa$ and $C(K) \in \overline{\mathcal{H}}_\kappa$ is the result of Benyamini and Starbird from [17]. The implication from $X \in \overline{\mathcal{H}}_\kappa$ to $B_{X^*} \in UE_\kappa$ is proved in [36, Theorem 6.30]. Since continuous images of uniformly Eberlein compact space are uniformly Eberlein [16], 2.6 implies that $\mathbb{UE}_\kappa$ and $\overline{\mathcal{H}}_\kappa$ are not strongly associated since the example of [52] shows that $\overline{\mathcal{H}}_\kappa$ is not closed under taking subspaces. Facts on $\mathbb{E}_\kappa$ and $\mathcal{WCG}_\kappa$ can be found e.g., in [31], and are due to Amir and Lindenstrauss [1]. The fact that the classes are not strongly associated follows as for the previous class. This gives (1) and (2).

For (3) consider the example of Pol [51] of a scattered compact K such that $C(K)$ is in $\mathcal{L}_{\omega_1} \smallsetminus \mathcal{WCG}_{\omega_1}$. By (2) K cannot be an Eberlein compact, so by a result of Alster ([2]) K is not Corson compact, and so $B_{C(K)^*}$ is not either, however $C(K) \in \mathcal{L}_{\omega_1}$ as proved in [51].

For associations of $\mathbb{RN}_\kappa$ with $\overline{\mathcal{A}}_\kappa$ and $\mathbb{QRN}_\kappa$ with $\mathcal{S}\overline{\mathcal{A}}_\kappa$ see [33]. As $\mathcal{S}\overline{\mathcal{A}}_\kappa$, $\mathbb{QRN}_\kappa$ and $\mathbb{RN}_\kappa$ are closed under subspaces, 2.5 implies that the association in (5) is strong and in (4) it is a K-association. By a result of Avilés [6] $\mathbb{QRN}_\kappa = \mathbb{RN}_\kappa$ if $\kappa < \mathfrak{b}$. The strong association in (4) for Banach spaces of density $< \mathfrak{b}$ is Theorem 5 of [6]. It is also shown in [6] (bottom of page 79) that there is a WCG Banach space Y with a subspace X of density character $\mathfrak{b}$ which is not Asplund generated. It follows that $B_{X^*} \in \mathbb{RN}_\mathfrak{b}$ because it is an Eberlein compact as a continuous image of an Eberlein compact B_{Y^*}, but X is not Asplund generated, so $\mathbb{RN}_\mathfrak{b}$ and $\overline{\mathcal{A}}_\mathfrak{b}$ are not B-associated. $\hfill\square$

The situation for Corson compacta and weakly Lindelöf determined spaces is more complicated because fundamental relations depend on extra set-theoretic axioms:

Theorem 3.4.
   (1) $K \in \mathbb{CM}_\kappa$ if and only if $C(K) \in \mathcal{WLD}_\kappa$.
   (2) $X \in \mathcal{WLD}_\kappa$ if and only if $B_{X^*} \in \mathbb{C}_\kappa$.
   (3) *Assuming* MA+¬CH *the classes* $\mathbb{C}_\kappa = \mathbb{CM}_\kappa$ *and* $\mathcal{WLD}_\kappa$ *are strongly associated for any uncountable cardinal* κ.
   (4) *Assuming* CH
      (a) $\mathbb{C}_{2^\omega}$ *is not associated with any class of Banach spaces;*
      (b) $\mathcal{L}_{2^\omega}$ *is not associated with any class of compact spaces;*
      (c) $\mathcal{WLD}_{2^\omega}$ *is not associated with any class of compact spaces;*
   (5) $\mathbb{CM}_\kappa$ *is associated with a class of Banach spaces for any uncountable cardinal* κ.

Proof. (1) is 5.4 and (2) is 4.17 of [37] they are originally from [5]. (3) follows from the above and the fact that $\mathbb{C}_\kappa = \mathbb{CM}_\kappa$ under MA+¬CH 3.2.(3) [4, 3.2.(3)]

The results (4) under CH are proved based on two examples, of Argyros, Mercourakis and Negrepontis of a Corson compact $K_0 \in \mathbb{C}_{2^\omega}$ which does not have property M [4, 3.12] and its strengthening [50, 4.4] due to Kalenda based on examples of Plebanek from [50] of a Banach space X_0 whose dual ball is a Corson compact without property M.

Using the fact that for a Corson compact $K \in \mathbb{C}_\kappa$ having property M is equivalent to $C(K) \in \mathcal{L}_\kappa$ [4, 3.5], we conclude that $C(K_0) \notin \mathcal{L}_{2^\omega}$, $C(B_{X_0}^*) \notin \mathcal{L}_{2^\omega}$. And so by the above equivalences we have

$$K_0 \in \mathbb{C}_{2^\omega}, \quad B_{C(K_0)^*} \notin \mathbb{C}_{2^\omega}, \quad X_0 \in \mathcal{WLD}_{2^\omega}, \quad C(B_{X_0}) \notin \mathcal{L}_{2^\omega}.$$

So using the fact that $\mathcal{WLD}_{2^\omega} \subseteq \mathcal{L}_{2^\omega}$ and 2.7 we conclude (4).

For (5) by 2.7 it is enough to have that $K \in \mathbb{CM}_\kappa$ implies $B_{C(K)^*} \in \mathbb{CM}_\kappa$, and this is exactly 4.3 of [50]. $\hfill\square$

However we do not know the answer to the following:

Question 3.5. *Is it consistent[2] that the class $\mathcal{L}_\kappa$ of Lindelöf Banach spaces in the weak topology of density $\leqslant \kappa$ is associated with a class of compact spaces for an uncountable κ?*

By 2.7 one needs to know if $C(B_{X^*})$ is weakly Lindelöf if X is. A class of compact spaces as above would need to contain spaces which are not Corson compact, as for example, the ladder system space K of [51]. Another question of a dual sort is the following:

Question 3.6. *What is the smallest class $\mathcal{K}$ of compact spaces which is associated with a class $\mathcal{B}$ of Banach spaces such that $\mathcal{K}$ contains the Stone spaces of minimally generated Boolean algebras of cardinality 2^ω?*

Minimally generated Boolean algebras were introduced in [39]. It is shown there that their Stone spaces contain all dispersed compacta. But such Stone spaces may not be sequentially compact, so $\mathcal{K}$ must be bigger than the class of all Radon–Nikodým compacta of appropriate weight. See also [18, 44]. A similar question for scattered spaces seems also open:

Question 3.7. *What is the smallest class $\mathcal{K}$ of compact spaces which is associated with a class $\mathcal{B}$ of Banach spaces such that $\mathcal{K}$ contains all scattered compact spaces of uncountable weight κ?*

This is related to Problem 24 of [8] where among others it is asked if every Radon–Nikodým compact space appears as a subspace of the dual ball $B_{C(K)^*}$ where K is a scattered compact. Our question seems to be a much weaker version of this question. 3.4 (5) suggest also the following:

Question 3.8. *Characterize internally in ZFC the class of Banach spaces whose dual balls are Corson compacta with property M.*

4. Types of universality

Definition 4.1. Let $\mathcal{B}$ be a class of Banach spaces. A Banach space $X \in \mathcal{B}$ is said to be injectively isomorphically (isometrically) universal for $\mathcal{B}$ if and only if for every $Y \in \mathcal{B}$ there is an isomorphic (isometric) embedding of Y into X.

A Banach space $X \in \mathcal{B}$ is said to be surjectively isomorphically (isometrically) universal for $\mathcal{B}$ if and only if for every $Y \in \mathcal{B}$ there is a closed subspace $Z \subseteq X$ such that X/Z is isomorphic (isometric) to Y.

Note that X/Z is isomorphic to Y for some closed subspace $Z \subseteq X$ if and only if there is a linear surjection $T : X \to Y$.

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[2]Many of the constructions mentioned in this paper are not absolute, that is the usual axioms of mathematics (ZFC) are not sufficient to carry them out. Many follow from additional axioms which were shown to be equiconsistent with ZFC (they do not lead to contradiction if ZFC does not lead itself) but the consistency of another group was established only using the method of forcing. The readers less familiar with these matters should consult when needed, for example, the textbook [43]. This lack of absoluteness is known to be unavoidable as it is shown in most cases that some other axioms or forcing arguments imply the nonexistence of the constructions.

Definition 4.2. Let $\mathcal{K}$ be a class of compact Hausdorff spaces. A compact space $K \in \mathcal{K}$ is said to be surjectively (injectively) universal for $\mathcal{K}$ if and only if for every $L \in \mathcal{K}$ there is a continuous surjection of K onto L (L homeomorphically maps into K).

We will omit some of the adjectives: for a Banach space universal means injectively isomorphically universal, isometrically universal means injectively isometrically universal, surjectively universal means isomorphically surjectively universal; for compact space universal means surjectively universal.

Proposition 4.3. *Let $\mathcal{B}$ and $\mathcal{K}$ be associated classes of Banach spaces and compact spaces respectively. Suppose K is universal for $\mathcal{K}$; then $C(K)$ is isometrically universal for $\mathcal{B}$.*

Proof. Let $X \in \mathcal{B}$, and so by the association 2.4 we have $B_{X^*} \in \mathcal{K}$ and then there is a continuous surjection from K onto B_{X^*}, and so $C(B_{X^*})$ is isomorphic to a subspace of $C(K)$ by 2.1(1). As X is isomorphic to a subspace of $C(B_{X^*})$ by 2.2, we get that $C(K)$ is isometrically universal for $\mathcal{B}$. $\square$

Under the above assumptions on $\mathcal{B}$ and $\mathcal{K}$ it is not true that if K is injectively universal for $\mathcal{K}$, then $C(K)$ is surjectively universal for $\mathcal{B}$. This perhaps explains less interest in these properties, at least in the context of the duality. Indeed, there cannot be a linear bounded surjection $T : C(K) \to l_1$ for some compact K because it would be a complemented subspace of such a space [23, Theorem VII.5] and it would contradict classical results of Pełczyński from [49]. It follows that there is no surjectively universal Banach space of the form $C(K)$ for the class of separable Banach spaces, and of course $[0,1]^{\mathbb{N}}$ is an injectively universal compact space for metrizable compact spaces. The following is also a classical result:

Theorem 4.4. *$l_1(\kappa)$ is surjectively universal for $\mathcal{B}_{\kappa}$.*

We may consider some weaker versions of universality relevant in the context of compact spaces.

Definition 4.5. Let $\mathcal{B}$ be a class of Banach spaces. A Banach space $X \in \mathcal{B}$ is said to be weakly universal for $\mathcal{B}$ if and only if every $Y \in \mathcal{B}$ is isomorphic to a quotient of a subspace of X.

Let $\mathcal{K}$ be a class of compact Hausdorff spaces. A compact space $K \in \mathcal{K}$ is said to be weakly universal for $\mathcal{K}$ if and only if every $L \in \mathcal{K}$ is a continuous image of a closed subspace of K.

Proposition 4.6. *Let $\mathcal{B}$ be a class of Banach spaces. A Banach space $X \in \mathcal{B}$ is weakly universal for $\mathcal{B}$ if and only if every $Y \in \mathcal{B}$ is isomorphic to a subspace of a quotient of X.*

Let $\mathcal{K}$ be a class of compact Hausdorff spaces. A compact space $K \in \mathcal{K}$ is weakly universal for $\mathcal{K}$ if and only if every $L \in \mathcal{K}$ is a closed subspace of a continuous image of K.

Proof. If $f : A \to B \supseteq C$ and f is a surjection, then $f \upharpoonright f^{-1}[C] \to C$ is a surjection and $A \supseteq f^{-1}[C]$ so we have one way implications.

To obtain the other implication in the case of Banach spaces, note that if X is isomorphic to a quotient Z/W where Z is a subspace of Y, then X is isomorphic to a subspace of the quotient Y/W. In the case of compact spaces, first embed a compact K into a product of intervals and note that if K is a continuous image of a closed subspace L of L', then the continuous map can be extended to L' into the product of intervals using the Tietze extension theorem coordinatewise, so K becomes a subspace of a continuous image of L'.   □

Proposition 4.7. *Let $\mathcal{B}$ and $\mathcal{K}$ be associated classes of Banach spaces and compact spaces respectively.*

    (1) *If $C(K)$ is isometrically universal for $\mathcal{B}$, then K is weakly universal for $\mathcal{K}$.*
    (2) *If K is weakly universal for $\mathcal{K}$, then $C(K)$ is weakly universal for $\mathcal{B}$.*

Proof. Let $L \in \mathcal{K}$. We can use the Holsztyński theorem (see e.g., [54]) which says that an isometry $T : C(L) \to C(K)$ yields a continuous map from a closed subspace of K onto L as required.

Now, let $X \in \mathcal{B}$. We have that $B_{X^*} \in \mathcal{K}$ and so there is a closed subspace K' of K which maps onto B_{X^*}. By 2.1 and 2.2 we have X in $C(B_{X^*})$ in $C(K')$ which is a quotient of $C(K)$ as required.   □

The proposition below is used essentially in a couple of important papers concerning the nonexistence of universal Banach spaces [60, 3]: Having proved that there is no weakly universal compactum, we conclude that there is no universal Banach space. It seems that it is the only topological method used in the literature for proving the nonexistence of universal Banach spaces.

Proposition 4.8. *Suppose that $\mathcal{B}$ and $\mathcal{K}$ are associated classes of Banach spaces and compact spaces respectively. If X is weakly universal for $\mathcal{B}$, then B_{X^*} is weakly universal for $\mathcal{K}$.*

Proof. Let $K \in \mathcal{K}$. Since X is weakly universal for $\mathcal{B} \ni C(K)$, there is a subspace Y of X and a surjective operator $T : Y \to C(K)$. Restrictions of functionals from B_{X^*} defined on X to Y give a continuous surjection from B_{X^*} onto B_{Y^*} by 2.1. But T^* continuously embeds $B_{C(K)^*}$ into $\|T\| B_{Y^*}$. Using the fact that K is homeomorphic to a subspace of $B_{C(K)^*}$ by 2.2, we get that K is homeomorphic to a subspace of B_{Y^*} which is a a continuous image of B_{X^*} as required.   □

Weakly universal above cannot be replaced by universal: $C([0,1])$ is universal for separable Banach spaces but neither $[0,1]$ nor $B_{C([0,1])^*}$ are surjectively universal for metrizable compact spaces because they do not map on disconnected spaces. Finally let us make a few remarks on how little is known on the relation between isomorphically universal and isometrically universal Banach spaces in the nonseparable case. First note the following:

Proposition 4.9. *There is a separable Banach space which is isomorphically universal for separable Banach spaces and is not isometrically universal for separable Banach spaces.*

Proof. Consider a strictly convex renorming ($\|x + y\| = 2$ for $\|x\| = \|y\| = 1$ implies $x = y$) of $C([0,1])$ (which exists by [22, Theorem 9]) and notice that this space continues to be an isomorphically universal space but cannot isometrically include spaces whose norm is not strictly convex. □

However we do not know any corresponding result for nonseparable Banach spaces:

Question 4.10. *Is every universal Banach space of density $\leqslant 2^\omega$ isomorphic to an isometrically universal Banach space of density $\leqslant 2^\omega$? Is it isometrically universal itself?*

This question is also open for any of the classes of nonseparable Banach spaces we consider in this paper. Actually it is known that for example ℓ_∞/c_0 does not have a strictly convex renorming [19]. In particular we do not know the following general:

Question 4.11. *Is there a property P of norms such that:*
- *If $(X, \|\|\|)$ has P and Y is a closed subspace of X, then $(Y, \|\|\| \upharpoonright Y)$ has P as well;*
- *Not all norms on Banach spaces of density $\leqslant 2^\omega$ have P;*
- *$C(K)$ spaces have equivalent norm with property P?*

The above may not be a difficult question but perhaps P as above is not natural in the context of the renorming theory. This is because we usually seek as good renormings as possible for a restricted class of Banach spaces and above we want a bad renorming for all spaces. A renorming of an isometrically universal Banach space with property P would give a universal space which is not an isometrically universal Banach space. However, there are some limitations here, for example any Banach space isomorphic to ℓ_∞/c_0 contains a subspace isometric to ℓ_∞ [48]. As the existence of universal nonseparable Banach spaces is undecidable problem for many classes (see the next section), the above issue of isometrically universal versus isomorphically universal can be expressed in a much stronger way. One example is the following:

Question 4.12. *Are any of the statements below equivalent in ZFC:*
- *there is a universal Banach space of density $\leqslant 2^\omega$,*
- *there is an isometrically universal Banach space of density $\leqslant 2^\omega$,*
- *there is a universal compact space of weight $\leqslant 2^\omega$?*

This question is also open for any of the classes of nonseparable Banach spaces we consider in this paper. Krupski and Marciszewski constructed in [41] the first consistent example of a Banach space which isomorphically embeds in l_∞/c_0 but does not embed isometrically. This does not answer Question 4.10 because in the model considered in [41] the space l_∞/c_0 is not universal.

5. The existence and the nonexistence of universal spaces

As seen in the previous section we have general results (4.3) allowing us to obtain universal Banach spaces from universal compact spaces. All positive universality

results for Banach spaces which we include here come from the universality of compact space via the arguments described above. In some cases they could be also obtained by general model-theoretic arguments, but as far as we know, in all these cases we can also build the corresponding universal compact space.

However the negative universality results require usually some other ideas, unless we can prove that there is no weakly universal compact space of our class like in [3, 60] and then apply 4.8. The other ideas exploited in the relevant literature come from graphs [14], ordinal indices in the spirit of Cantor–Bendixon height [59, 62, 35], or forcing genericity [20, 21].

5.1. Uniformly Eberlein compacta and Hilbert generated Banach spaces.
First we note that the definition of a uniform Eberlein compactum is so restrictive that it is not surprising that there are weakly universal objects in this class.

Proposition 5.1. $B_{l_2(\kappa)}$ *is injectively universal for* $\mathbb{UE}_\kappa$. *In particular* $C(B_{l_2(\kappa)})$ *is weakly universal for* $\overline{\mathcal{H}}_\kappa$.

Proof. Note that by taking appropriate quotients, every uniform Eberlein compact space of weight κ can be continuously embedded in the Hilbert space $l_2(\kappa)$ with the weak topology and the image of this embedding must be a bounded set. As the unit ball in a Hilbert space is weakly compact, it is injectively universal, and so weakly universal uniform Eberlein compact space. By 4.8, this gives that $C(B_{l_2(\kappa)})$ is weakly universal for $\overline{\mathcal{H}}_\kappa$. □

A result of Benyamini, Rudin, Wage provides another weakly universal compact space.

Proposition 5.2. [16] $A(\kappa)^{\mathbb{N}}$ *is weakly universal in* $\mathbb{UE}_\kappa$. *In particular* $C(A(\kappa)^{\mathbb{N}})$ *is weakly universal for* $\overline{\mathcal{H}}_\kappa$.

Proof. Apply 4.7 (2). □

However, we have the following result of M. Bell:

Theorem 5.3. [13] *There is a compact* $K \in \mathbb{UE}_{\omega_1}$ *which is not a continuous image of any space* $A(\kappa)^{\mathbb{N}}$. *In particular the space* $A(\kappa)^{\mathbb{N}}$ *is not universal in* $\mathbb{UE}_\kappa$ *for an uncountable* κ.

Avilés proved in [7] that $B_{l_2(\kappa)}$ is a continuous image of $A(\kappa)^{\mathbb{N}}$. Moving from weakly universal objects to universal objects we encounter undecidability of their existence.

Theorem 5.4. [14] *It is consistent that there is no universal uniform Eberlein compact space of weight* ω_1.

The argument used above employs the consistency of the nonexistence of a universal graph of cardinality ω_1 proved by Shelah [56, 57]. It turns out one can associate graphs with uniform Eberlein compacta (see [14]). See Question 5.22.

Under some cardinal arithmetics hypothesis, using a general procedure applicable in other categories, Bell also proved a positive result for compact spaces which gives, by 4.3, a positive result for associated class of Banach spaces:

Theorem 5.5. [14] *Assume $\kappa = 2^{<\kappa}$. Then, there is a universal uniform Eberlein compact space UE_κ for $\mathbb{UE}_\kappa$ and so $C(UE_\kappa)$ is isometrically universal Banach space for $\overline{\mathcal{H}}_\kappa$. In particular, the above results hold for $\kappa = 2^\omega = \omega_1$ if we assume the continuum hypothesis.*

The negative result 5.4 of Bell cannot be directly applied via 4.8 for the class of Banach spaces $\overline{\mathcal{H}}_\kappa$ because weakly universal compact spaces exist here (5.2). C. Brech and the author obtained the Banach space version of this negative result employing a forcing genericity argument. Actually the result is much stronger and has other universal consequences which we will mention later:

Theorem 5.6. [21] *It is consistent that there is no Banach space of density 2^ω or ω_1 which contains isomorphic copies of all Banach spaces from $\mathbb{UE}_{2^\omega}$ or from $\mathbb{UE}_{\omega_1}$ respectively. In particular, it is consistent that there is no universal Banach space either for $\overline{\mathcal{H}}_{\omega_1}$ or for $\overline{\mathcal{H}}_{2^\omega}$.*

The question of the possible nonexistence of universal spaces for $\overline{\mathcal{H}}_\kappa$ for κ different than ω_1 or 2^ω was not addressed in the literature yet. There are other negative results on universal Eberlein compact spaces, e.g., in [27], which were not dualized to Banach spaces yet.

5.2. Eberlein compacta and weakly compactly generated Banach spaces. It is well known that all Eberlein compacta are subspaces of dual balls of reflexive Banach spaces with the weak* topology. However this class of compact spaces is much reacher than the dual balls of Hilbert spaces and it is not possible to obtain weakly universal objects as in the case of uniform Eberlein compacta.

Theorem 5.7. [3] *If $\kappa^\omega = \kappa$ or $\kappa = \omega_1$, then there is no weakly universal Eberlein compact of weight κ nor a universal WCG Banach space of density κ. If κ is a strong limit cardinal of countable cofinality, then there is a universal Eberlein compact of weight κ, and so, there is a universal WCG Banach space of density κ.*

Here Argyros and Benyamini could conclude the negative result for the associated class of Banach spaces via 4.8. Thus under G.C.H. the situation is completely settled. However many consistent cardinal arithmetics are not covered by the above result. It seems that the following questions were not settled in the literature:

Question 5.8. *Is it consistent that there is no universal space in $\mathbb{E}_{\omega_\omega}$, in $\mathcal{WCG}_{\omega_\omega}$?*

Question 5.9. *Is it consistent that there is a universal space in $\mathbb{E}_{\omega_2}$, in $\mathcal{WCG}_{\omega_2}$?*

There are also results on universal uniform Eberlein compacta from more restrictive classes, e.g., in [15], which did not find their Banach space theory versions as yet.

5.3. Corson compacta and weakly Lindelöf determined and weakly Lindelöf Banach spaces. Todorcevic proved the following:

Theorem 5.10. [60] *For every countably tight compact space K of weight $\leqslant 2^\omega$ there is a Corson compact K' of weight $\leqslant 2^\omega$ where all Radon measures have*

separable supports and which is not a continuous image of any closed set of K. In particular there is no universal weakly Lindelöf determined Banach spaces for $\mathcal{WLD}_{2^\omega}$.

Knowing that every Corson compact space has countable tightness, the result implies that there is no Corson compact of weight continuum whose closed subspaces map onto all Corson compacta with property M of weight continuum. Using (1) and (2) of 3.4 as in 4.8, if there were a universal Banach space X for $\mathcal{WLD}_{2^\omega}$, then B_{X^*} were a Corson compact whose subspaces mapped onto all Corson compacta with property M: Given $K \in \mathbb{CM}_\kappa$, $C(K) \in \mathcal{WLD}_{2^\omega}$, so $C(K)$ isomorphically embeds in X and so B_{X^*} maps onto $B_{C(K)^*}$ by 2.1 which includes K by 2.2.

In Section 4 of [60] we have the following:

Theorem 5.11. *There is no universal weakly Lindelöf Banach space of density 2^ω.*

As we do not have (1), (2) of 3.4 for the class of weakly Lindelöf Banach spaces an approach presented in Section 4 of [60] is necessarily different than that leading to 5.10.

Question 5.12. (1) *Are there (consistently) universal Banach spaces in WLD_κ for $\kappa \neq 2^\omega$, for example consistently for $\kappa = \omega_1$?*

   (2) *Are there (consistently) universal Corson compact spaces (where all Radon measures have separable supports) of weight $\leqslant \kappa$ for $\kappa \neq 2^\omega$, for example consistently for $\kappa = \omega_1$?*

   (3) *Are there (consistently) universal Banach spaces in L_κ for $\kappa \neq 2^\omega$, for example consistently for $\kappa = \omega_1$?*

5.4. (Quasi) Radon–Nikodým compacta and (subspaces of) Asplund generated Banach spaces.

Perhaps one should start this subsection with looking at the class of scattered compact spaces whose spirit is present in the classes considered here. One has the Cantor–Bendixon height $\mathrm{ht}(K)$ for all compact scattered spaces K so that $\mathrm{ht}(K)$ is an ordinal such that $|\mathrm{ht}(K)|$ is not bigger than the weight of K, if L is a closed subset of K or if L is a continuous image of K, then $\mathrm{ht}(L) \leqslant \mathrm{ht}(K)$, and moreover for every cardinal κ and every $\alpha < \kappa^+$ there is a compact scattered space K of weight κ such that $\mathrm{ht}(K) > \alpha$. This directly implies that there is no weakly universal scattered compact space of any fixed weight.

This idea was first transferred in the realm of Banach spaces by Szlenk in [59] who defined a similar index for the dual balls of separable reflexive Banach spaces obtaining the nonexistence of a universal reflexive separable Banach space. This was generalized to all Asplund spaces:

Theorem 5.13. [35, 62] *There is no universal Banach space for the class of Asplund spaces of density character κ for any infinite cardinal κ*

The dual balls of Asplund spaces allow to define a version of Cantor–Bendixon height. However the class of Asplund spaces is not associated with a class of compact spaces: if we were to have $C(K)$ Asplund, K would be scattered but dual balls are always connected, so never scattered. Radon–Nikodým compacta

are exactly the weakly* compact subspaces of dual balls of Asplund spaces and are associated with the class of Asplund generated Banach spaces. Thus we can also define Cantor–Bendixon or Szlenk index on them, however it depends on a particular representation as a weakly* compact subspace of the dual ball of an Asplund space. It creates the difficulty which probably explains why the question of universal object in these classes was not addressed in the literature. As Eberlein and uniform Eberlein compacta are Radon–Nikodým, it follows from results of [21] i.e., 5.6 that it is consistent that there are no universal objects in $\mathbb{RN}_{2^\omega}$, $\mathbb{RN}_{\omega_1}$, $\mathbb{QRN}_{2^\omega}$, $\mathbb{QRN}_{2^\omega}$, $\overline{\mathcal{A}}_{2^\omega}$, $\overline{\mathcal{A}}_{\omega_1}$, $\mathcal{S}\overline{\mathcal{A}}_{2^\omega}$, $\mathcal{S}\overline{\mathcal{A}}_{\omega_1}$.

The fragmentability of (quasi) Radon–Nikodým compacta mentioned above suggest, however the possibility of introducing an ordinal index which could be used for proving the nonexistence of universal spaces like Cantor–Bendixon height of a scattered space can be used to prove the nonexistence of a universal metrizable scattered compact space. Remarks concerning this topic are implicitly present in [3, p. 308] or in [35, p. 2034]. Here we make this problem explicit taking into account the result of [10]:

Question 5.14. *Is it possible to associate to each quasi Radon–Nikodým compact K an ordinal index $i(K)$ having the following properties:*

- *$|i(K)|$ is not bigger than the weight of K,*
- *If L is a closed subset of K or if L is a continuous image of K, then $i(L) \leqslant i(K)$,*
- *For every $\alpha < \kappa^+$ there is a quasi Radon–Nikodým compactum K of weight κ such that $i(K) > \alpha$.*

In [3] an index having the properties above was introduced for Eberlein compacta. To avoid the dependence of the representation of a given Eberlein compact space, the authors exploited the possibility of embedding them as weakly compact subspaces in $c_0(\kappa)$. However Radon–Nikodým compacta which are not Eberlein compact do not admit such embeddings. It is clear that the positive answer to the above question would give the negative answer to:

Question 5.15. *Is it consistent that there are universal spaces in one of the classes $\mathbb{RN}_{2^\omega}$, $\mathbb{RN}_{\omega_1}$, $\mathbb{QRN}_{2^\omega}$, $\mathbb{QRN}_{2^\omega}$?*

Question 5.16. *Is it consistent that there are universal spaces in one of the classes $\overline{\mathcal{A}}_{2^\omega}$, $\overline{\mathcal{A}}_{\omega_1}$, $\mathcal{S}\overline{\mathcal{A}}_{2^\omega}$, $\mathcal{S}\overline{\mathcal{A}}_{\omega_1}$?*

5.5. All compact spaces and Banach spaces of a given size.
Under many additional set-theoretic assumptions there exist universal compact spaces of a given weight and so applying 4.3 one can obtain universal Banach spaces of a given density.

Theorem 5.17. [30] *Assume GCH. Then there is a universal compact space for $\mathbb{K}_\kappa$ for each cardinal number κ and so there is an isometrically universal Banach space for each $\mathcal{B}_\kappa$.*

Theorem 5.18. [47] *Assume CH. Then $\mathbb{N}^*$ is universal for $\mathbb{K}_{2^\omega}$ and so ℓ_∞/c_0 is isometrically universal for $\mathcal{B}_{2^\omega}$.*

However, the above results require some additional set theoretic assumptions as seen from the following theorem:

Theorem 5.19. [26] *It is consistent that there is no universal compact space for* $\mathbb{K}_{2^\omega}$.

The consistency of the nonexistence of a universal space in $\mathbb{K}_{\omega_1}$ follows from the next result and 4.3 but it was probably known before.

Theorem 5.20. [20] *The existence of an isomorphically universal Banach space in* $\mathcal{B}_{2^\omega}$ *or in* $\mathcal{B}_{\omega_1}$ *is undecidable.*

Surjectively universal Banach spaces exist without any special set-theoretic assumptions for any density character κ. It is a classical result that $\ell_1(\kappa)$ is such a space. We can also obtain weak universal Banach spaces of the form $C(K)$ without any special set-theoretic assumptions for any density character. This is a consequence of the existence of injectively universal compact spaces:

Proposition 5.21. *Let* κ *be a cardinal.* $[0,1]^\kappa$ *is injectively universal for* $\mathbb{K}_{2^\omega}$. *In particular, any space that maps onto* $[0,1]^{2^\omega}$, *and so,* N^*, *is weakly universal for* $\mathbb{K}_{2^\omega}$. *In particular* ℓ_∞/c_0 *is weakly universal for* $\mathcal{B}_{2^\omega}$.

Proof. Clearly any compact space of weight κ embeds into $[0,1]^\kappa$. $\mathbb{N}^*$ maps onto $[0,1]^{2^\omega}$ because $\wp(\mathbb{N})$ has an independent family of infinite sets which gives a continuous surjection of $\mathbb{N}^*$ onto 2^{2^ω}. Using the standard continuous mapping of the Cantor set onto $[0,1]$, we obtain the desired mapping of $\mathbb{N}^*$ onto $[0,1]^{2^\omega}$. To go to the Banach spaces use 4.8. $\square$

However, there are no surjectively universal Banach spaces of the form $C(K)$ as noted after 4.3. Let us finish this section with a different kind of a problem. There are many results about the existence and nonexistence of universal structures obtained by methods from a field of mathematical logic which is called model theory which deals with structures, languages and theories expressible in these languages which are true in theses structures (models of theories). There is a big limitation of applications of model theory in the theory of Banach spaces (or even topology of compact spaces). The Banach space theory is not a first order theory which is best understood and investigated type of a language in model theory. It may happen however that the existence of universal Banach spaces can be equivalent to the existence of simpler structures describable in terms of a first order language (which would code Banach spaces). Links between universality of Boolean algebras (a first order structure) and Banach spaces are well know (see e.g., [20, 1.1]) but as far as now equivalences were not obtained. Also the existence of universal graphs, which are first order structures is implied by the existence of universal uniform Eberlein compact spaces as shown at the end of [14]. Of course any equivalence would give a possibility of transferring model-theoretic results to questions concerning universal Banach spaces. One concrete question could be:

Question 5.22. *Is it consistent that there is a universal graph of size* ω_1 *(*2^ω*) but there is no universal Banach space for* $\mathcal{B}_{\omega_1}$ *(*$\mathcal{B}_{2^\omega}$*)?*

It should be noted that there are some interesting results where first order model theory is used to conclude the universality results concerning Banach spaces like for example in the work of Shelah and Usvyatsov [58] where it was first proved that it is consistent that there is no isometrically universal Banach space for $\mathcal{B}_{2^\omega}$. However these results concern only the isometric theory as far as now.

6. The nonuniversality of ℓ_∞/c_0 and $\mathbb{N}^*$

The spaces ℓ_∞/c_0 and $\mathbb{N}^*$ play a special role in the classes $\mathcal{B}_{2^\omega}$ and $\mathbb{K}_{2^\omega}$. One of the reasons is the universality result 5.18. Of course ℓ_∞/c_0 is isometric to $C(\mathbb{N}^*)$. After Kunen proved that $[0, 2^\omega]$ may not be a continuous image of $\mathbb{N}^*$ the question under which assumptions which compact spaces must be continuous images of $\mathbb{N}^*$ was considered by many authors. Recently the dual question, which Banach spaces must embed isomorphically or isometrically in ℓ_∞/c_0 was considered be several authors as well. In this section we survey these results. It is not difficult to see the following well known facts:

Proposition 6.1. *All separable compact spaces are continuous images of $\mathbb{N}^*$ and all Banach spaces whose dual balls are weakly* separable isometrically embed in ℓ_∞/c_0.*

Proof. Consider a separable compact K. Let L be a totally disconnected separable compact space which maps onto L (for example take a totally disconnected L which maps irreducibly onto K, like a Gleason space [29, 6.3.19]). By sending the dense countable subset of L onto $\mathbb{N}$, we can embed the Boolean algebra of clopen subsets of L into $\wp(\mathbb{N})$, so by the Stone duality L is a continuous image of $\beta\mathbb{N}$ which is a continuous image of $\mathbb{N}^*$.

If X has separable dual ball, then $\mathbb{N}^*$ maps onto B_{X^*}, so $\ell_\infty/c_0 \equiv C(\mathbb{N}^*)$ isometrically embeds $C(B_{X^*})$ which isometrically embeds X by 2.2. $\qquad\square$

Note that the class of Banach spaces with weakly* separable duals is not closed under isomorphisms [31, Ex. 12.40] and includes all HI spaces. In [53] Przymusiński proved that all perfectly normal compact spaces are continuous images of $\mathbb{N}^*$. The topological part of the following theorem of van Douwen and Przymusiński already can be dualized to Banach spaces:

Theorem 6.2. [24] *Assume MA+¬CH. Let $\kappa < 2^\omega$. Then each compact $K \in \mathbb{K}_\kappa$ is a continuous image of $\mathbb{N}^*$. In particular each Banach space of density character $\kappa < 2^\omega$ isometrically embeds into ℓ_∞/c_0.*

Actually, the Parovichenko property of $\wp(\mathbb{N})/\mathrm{Fin}$ gives that every Boolean algebra of cardinality not bigger than ω_1 embeds into $\wp(\mathbb{N})/\mathrm{Fin}$ which by the Stone duality and 2.1 gives:

Proposition 6.3. *Every compact space in $\mathbb{K}_{\omega_1}$ is a continuous image of $\mathbb{N}^*$, every Banach space in $\mathcal{B}_{\omega_1}$ embeds isometrically in ℓ_∞/c_0.*

Polyadic spaces are by the definition continuous images of products of the form $A(\kappa)^\tau$. They form the smallest class of topological spaces containing metric compact spaces and closed under products and continuous images, they also contain all dyadic spaces i.e., continuous images of 2^κ (see [34]). We have the following.

Theorem 6.4. [12] *Every polyadic space K of weight $\leqslant 2^\omega$ is a continuous image of $\mathbb{N}^*$. In particular $C(K)$ for such a K isometrically embeds in ℓ_∞/c_0.*

Note however that $C(K)$s may not embed into ℓ_∞/c_0 for some continuous images K of subspaces of polyadic spaces by 5.2 and 5.6. This completes our list of absolute results. It follows from our results with Brech from [20] (see 5.20) that it is consistent that some Banach spaces of density 2^ω do not isomorphically embed in ℓ_∞/c_0. In [20] we also proved the following:

Theorem 6.5. [20] *It is consistent that there exist universal Banach spaces for $\mathcal{B}_{2^\omega}$ but ℓ_∞/c_0 is not among them.*

The proof of the above results consist of showing that $C([0, 2^\omega])$ does not embed isomorphically into ℓ_∞/c_0 in a suitable model of set theory dualizing the result of Kunen.

At the end of [20] we describe how to construct a zero-dimensional compact K such that $[0, 2^\omega]$ is not a continuous image of K but $C([0, 2^\omega])$ isomorphically embeds in $C(K)$ but we do not know the answer to the following.

Question 6.6. *Is it consistent that $C([0, 2^\omega])$ isomorphically embeds in ℓ_∞/c_0 but $[0, 2^\omega]$ is not a continuous image of $\mathbb{N}^*$?*

There is another example concerning similar issues. In [25] it is proved that the Stone space K of the measure algebra (Boolean algebra of Borel sets of $[0, 1]$ divided by Lebesgue measure sets zero) consistently may not be a continuous image of $\mathbb{N}^*$. However it is well known of the dual ball $B^*_{L_\infty([0,1])}$ is separable in the weak* topology. This gives that $C(K)$ isomerically embeds into ℓ_∞/c_0 by 6.1.

Let us mention that Krupski and Marciszewski prove in [41] that consistently there could be a uniform Eberlein compact K such that $C(K)$ embeds isomorphically in ℓ_∞/c_0 but does not embed isometrically.

In [61] Todorčević presented an axiomatic approach to the problem of isomorphic (and in fact isometric) embeddings into l_∞/c_0. To express these results we need a few definitions: $\wp^n(\kappa)$ denotes the σ-field of sets generated by sets of the form $A_1 \times \cdots \times A_n$ for $A_1, \ldots, A_n \subseteq \kappa$. If $E \subseteq \kappa^2$ is a binary relation, then $E^{[n]} = \{(x_1, \ldots, x_n) \in \kappa^n : \forall i < j \ (x_i, x_j) \in E\}$. We will say that κ is an n-Ulam cardinal if and only if for every binary relation $E \subseteq \kappa^2$, the sets $E^{[n]}$ and $(\kappa^2 \smallsetminus E)^{[n]}$ can be separated by an element of $\wp^n(\kappa)$. It can be proved that 2-Ulam cardinal is an Ulam cardinal in the sense that $\wp(\kappa \times \kappa) = \wp^2(\kappa)$. More information on Ulam cardinals can be found in [9] where they are called Kunen cardinals. In particular it is known (and implicit in [42]) that it s consistent that 2^ω is not an N-Ulam cardinal for any $n \in \mathbb{N}$.

Theorem 6.7. [61] *If 2^ω is not an n-Ulam cardinal for infinitely many $n \in \mathbb{N}$, then there is a first countable compact space K and a Corson compact space L such that neither $C(K)$ nor $C(L)$ isomorphically embed into l_∞/c_0.*

Using the same idea the Corson part of the above result was improved in [41] in the following way:

Theorem 6.8. [41] *If 2^ω is not an n-Ulam cardinal for infinitely many $n \in \mathbb{N}$, then there is a uniform Eberlein compact space K such that $C(K)$ does not embed isomorphically into ℓ_∞/c_0.*

This is strictly related to 6.4 and 5.2, 5.3, i.e., the gap between what always can be embedded in ℓ_∞/c_0 and what consistently cannot be embedded is quite tight. It is also related to results of [21] (see 5.6). We note that [28] also contains consistency results improving the results of [20] to Corson compacta and use a more technical approach than in [61]. We close this section by the following:

Question 6.9. *Is it consistent that $\mathbb{N}^*$ is not universal for $\mathbb{K}_{2^\omega}$ but ℓ_∞/c_0 is universal for $\mathcal{B}_{2^\omega}$?*

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