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BASIS PROBLEMS FOR BOREL GRAPHS

Abstract. Following [12], we examine dichotomies about graphs defined on standard Borel spaces in which the edge relation is also Borel. More precisely, we study the Borel homomorphisms between such graphs and the corresponding notion of Borel chromatic number. It turns out that this category enjoys structural results not present in the category of all graphs. In particular, the concept of Borel chromatic number frequently does not coincide with the concept of the usual chromatic number. Cases of special interest are provided by graphs defined by the shift operation. We also briefly analyze graphs defined on families of finite sets of natural numbers.

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Contents

1. Introduction

A *graph* $\mathcal{G} = (X, R)$ is a pair where X is a set and R is a binary irreflexive and symmetric binary relation on X. A *subgraph* of $\mathcal{G} = (X, R)$ is a graph of the form $\mathcal{H} = (Y, S)$, where $Y \subseteq X$ and $S \subseteq R$. Such a subgraph $\mathcal{H} = (Y, S)$ is an *induced subgraph* if $S = R \cap Y^2$. Two extremal examples of graphs is the *complete graphs* on the vertex set X, defined as $\mathcal{K}_X = (X, X^2 \setminus \Delta(X))$[1] and the *discrete graph* on the set of vertices X, defined as $\mathcal{D}_X = (X, \emptyset)$. A *homomorphism* from a graph $\mathcal{G} = (X, R)$ into a graph $\mathcal{H} = (Y, R')$ is a mapping $f : X \to Y$ such $(x, y) \in R$ implies $(f(x), f(y)) \in R'$. We shall use the shorthand $\mathcal{G} \leqslant \mathcal{H}$ to denote the fact that there is a homomorphism from $\mathcal{G}$ into $\mathcal{H}$. This leads us to the standard notion of a *chromatic number* of a graph $\mathcal{G}$:

$$\chi(\mathcal{G}) = \min\{|Y| : \mathcal{G} \leqslant \mathcal{K}_Y\}.$$

Put it alternatively, let us say that a *good coloring* of $\mathcal{G}$ is a function $c : X \to Y$ such that $c(x) \neq c(y)$ whenever $x, y \in X$ and $(x, y) \in R$; for a cardinal number k, we say that c is a *good k-coloring* if the cardinality of Y is k. The chromatic number of $\mathcal{G}$, $\chi(\mathcal{G})$, is the least k such that there is a good k-coloring of $\mathcal{G}$. Graph homomorphism and graph coloring problems have been studied extensively by combinatorialists for decades. We will however review here some results on chromatic properties of graphs defined on standard Borel spaces, for which the edge relation is definable in some way. In particular, we will consider edge relations that are a Borel (or

[1]Here, $\Delta(X) = \{(x, x) : x \in X\}$ is the diagonal of X^2.

analytic) subset of the square of the space. Let X be a standard Borel space, i.e., a complete metrizable separable space together with its Borel structure, and let $R \subseteq X^2$ be irreflexive and symmetric. If R is a Borel (or analytic) subset of X^2, we say that the graph (X, R) is a Borel (or analytic) graph. For two such graphs $\mathcal{G} = (X, R)$ and $\mathcal{H} = (Y, S)$, a *Borel homomorphism* from $\mathcal{G} = (X, R)$ into $\mathcal{H} = (Y, S)$ is a Borel mapping $f : X \to Y$ such $(x, y) \in R$ implies $(f(x), f(y)) \in S$. We shall use the shorthand $\mathcal{G} \leqslant_B \mathcal{H}$[2] to denote the fact that there is a Borel homomorphism from $\mathcal{G}$ into $\mathcal{H}$. This leads us to the notion of *Borel chromatic number* of a graph $\mathcal{G}$ first isolated in [12]:

$$\chi_B(\mathcal{G}) = \min\{|Y| : \mathcal{G} \leqslant_B \mathcal{K}_Y\}.$$

Thus, for a cardinal number $k \leqslant \omega$, a *good Borel* k-coloring of (X, R) is a Borel function $c : X \to k$ that is also a good coloring.[3] The Borel chromatic number of (X, R) is the least k for which there is a good Borel k-coloring of (X, R), and we chose to denote it by $\chi_{\mathcal{B}}(X, R)$. A notable fact is the existence of Borel graphs with small chromatic number but large Borel chromatic number, examples of which will be given below.

We will present some results about definable graphs and their chromatic properties. The study of these topics was initiated in [12] from where several research directions have been stemmed (see, for example, [3, 4, 8, 11, 13, 14, 15, 16])

The paper is organized as follows. In Section 2 we give some basic definitions about definable graphs and their chromatic numbers. An interesting dichotomy related to analytic graphs of uncountable Borel chromatic number is discussed in Section 3, together with some comments about the possibility of discerning in a dichotomical way between finite and infinite Borel chromaticity. Section 4 is devoted to the case of graphs defined by a function $F : X \to X$ from a standard Borel space to itself, with $x, y \in X$ forming an edge of the graph if $F(x) = y$ or $F(y) = x$. Particular interest is set on the case in which F is the shift operation S.

In Section 5 we study properties of Borel sets which with the shift give a graph with infinite Borel chromatic number; and in Sections 6, 7 and 8 we study the special case of closed subsets of $\mathbb{N}^{[\infty]}$. In Section 7 we give a combinatorial characterization, in terms of trees and fronts, of the closed sets which have infinite continuous chromatic number; and in Section 8 we prove some facts about the local structure of those sets. The last section presents briefly some results about graphs defined in families of finite sets of natural numbers which might have some bearing on the problem of finite or infinite Borel chromaticity.

2. Borel graphs, their homomorphisms, and their chromatic numbers

In this section we start our detailed presentation by expanding on, and therefore repeating, some of the definitions already given above in the Introduction. We also

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[2]If f is moreover continuous, we use the notation $\mathcal{G} \leqslant_C \mathcal{H}$ and if X supports some measure and f is measurable relative to this measure (which will be clear from the context), we use the notation $\mathcal{G} \leqslant_M \mathcal{H}$.

[3]Here, $k \leqslant \aleph_0$ is given the discrete topology

present an example that shows the difference between the theory of Borel graphs and the classical theory.

A *good coloring* of a graph (X, E) is a function of the form $c : X \to K$ such that if $x, y \in X$ and $\{x, y\} \in E$, then $c(x) \neq c(y)$. Thus, a good coloring is simply a graph-homomorphism from (X, E) into a complete graph $(K, K^2 \setminus \Delta(K))$ in the sense of the definition given in the Introduction which we will reproduce in more details below. We say that c is a *good k-coloring* if the set K has cardinality k. The chromatic number of a graph (X, E), denoted by $\chi(X, E)$, is the least k such that there is a good k-coloring of the graph.

We will consider graphs $\mathcal{G} = (X, E)$ where X is a Hausdorff topological space or a standard Borel space and the edge relation E is definable, for example, Borel, or analytic; in this case, we say that $c : X \to k$, with $k \leqslant \aleph_0$ is a Borel coloring if besides being a coloring of the graph, it is a Borel measurable function (k endowed with the discrete topology). The Borel chromatic number of the graph, denoted by $\chi_{\mathcal{B}}(\mathcal{G})$, is the least $k \leqslant \aleph_0$ for which there is a good Borel k-coloring. If no such Borel coloring exists, it is said that the graph has uncountable Borel chromatic number, and this is expressed by $\chi_{\mathcal{B}}(\mathcal{G}) > \aleph_0$. An ample study of definable graphs on standard Borel spaces and their Borel chromatic numbers was done in [12]. In the rest of the paper we will present some results and problems from [12] or other studies that it has motivated.

Definition 2.1. *Given two graphs $G = (X, R)$ and $G' = (X', R')$, a* graph-homomorphism *from G into G', or simply a* homomorphism *from G into G', is a map $f : X \to X'$ such that $x \, R \, y \Rightarrow f(x) \, R' f(y)$. A* graph-embedding*, or simply an* embedding*, is an injective homomorphism, in other words, an isomorphism of G with a subgraph (not necessarily induced!) of G'.*

If $\mathcal{C}$ is a class of functions, $G \leqslant_{\mathcal{C}} G'$ expresses that there is a homomorphism of G into G' which belongs to $\mathcal{C}$; and $G \sqsubseteq_{\mathcal{C}} G'$ if there is an embedding of G into G' which is in $\mathcal{C}$. If the class $\mathcal{C}$ is the class of Borel functions, then we use $G \leqslant_B G'$ and $G \sqsubseteq_B G'$, and if $\mathcal{C}$ is the class of continuous functions, then we use $G \leqslant_C G'$ and $G \sqsubseteq_C G'$, and if $\mathcal{C}$.

Example 2.1 (see [19])**.** *A Borel graph $\mathcal{G}$ defined on the set of real numbers with the property $\chi(\mathcal{G}) \leqslant 2$ and $\chi_B \mathcal{G} > \aleph_0$. Let two real numbers x and y form an edge if $|x - y| = 3^k$ for some $k \in \mathbb{Z}$. An easy calculation shows this graph has no odd cycles, and thus its chromatic number is 2. To see that its Borel chromatic number cannot be countable, recall that for any measurable set of reals A of positive measure, the difference set $A - A$ contains an interval around 0. If $c : \mathbb{R} \to \mathbb{N}$ is a Borel coloring of the graph, then at least for one $i \in \mathbb{N}$, $c^{-1}\{i\}$ must be of positive measure, and thus for k sufficiently big, we can find $x, y \in c^{-1}\{i\}$ such that $|x - y| = 3^{-k}$.*

In the next section, we shall see that in fact there is minimal Borel graph with these properties, the graph $\mathcal{G}_0$ of [12].

3. The $\mathcal{G}_0$-dichotomy

Section 6 of [12] presents an interesting dichotomy which explains when an analytic graph has uncountable Borel chromatic number. This result shows the existence of a graph $\mathcal{G}_0$ such that $\chi_{\mathcal{B}}(\mathcal{G}_0) > \aleph_0$ and $\mathcal{G}_0$ is the minimal analytic graph with uncountable Borel chromatic number in the following sense.

Theorem 3.1. [12, 6.3] *Let X be a Polish space and $G = (X, R)$ an analytic graph. Then exactly one of the following holds.*

(1) $\chi_{\mathcal{B}}(\mathcal{G}) \leqslant \aleph_0$,
(2) *there is a continuous homomorphism from $\mathcal{G}_0$ into G.*

This dichotomy has been carefully studied in several subsequent papers (see for example [15, 16, 13]) where it is shown that it could be used in explaining several other dichotomies in the context of descriptive set theory, dichotomies that do not necessarily talk about Borel graphs. For example, the $\mathcal{G}_0$-dichotomy could be used to obtain a natural short proof of Silver's dichotomy theorem for co-analytic equivalence relations (see [16]).

The graph $\mathcal{G}_0 = (2^{\mathbb{N}}, R_0)$ is defined as follows. Let $T \subseteq 2^{<\mathbb{N}}$ be a set of finite binary sequences such that T is dense, in the sense that for every $s \in 2^{<\mathbb{N}}$ there is $t \in T$ such that $s \sqsubseteq t$, and T is sparse, in the sense that for every $n \in \mathbb{N}$ there is a unique element of T of length n. Then

$$R_0 = \{(t^\frown i^\frown x, t^\frown (1-i)^\frown x) : i < 2,\ t \in T,\ x \in 2^{\mathbb{N}}\}$$

A Baire category argument using the density of T shows that there is no Baire measurable $\aleph_0$-coloring of $\mathcal{G}_0$; and in particular its Borel chromatic number is uncountable.

The original proof given in [12] of the fact that when T is sparse, and G is an analytic graph defined on a Polish space, then there is a Borel $\aleph_0$-coloring of G or here is a continuous homomorphism from $\mathcal{G}_0$ to G, uses effective descriptive set theory. Subsequently, B. Miller has found a classical proof reminiscent of Cantor's proof of the perfect set theorem for closed sets using his notion derivatives (see [16]).

In section 8 of [12], the problem of the existence of a graph characterizing infinite Borel chromaticity is considered, and in particular it is asked if the following dichotomy is true: If $\mathcal{G} = (X, R)$ is an analytic graph on a Polish space X, then exactly one of the following holds:

(1) $\chi_{\mathcal{B}}(\mathcal{G}) < \aleph_0$,
(2) there is a continuous homomorphism of the shift graph $\mathcal{G}_S$ into $\mathcal{G}$[4].

By taking the direct sum of a sequence of complete finite graphs of unbounded cardinalities it is clear that this dichotomy should be modified a bit by adding some restriction on $\mathcal{G}$ (see the discussion at the end of this section). For example, it is shown in [12] that when the graph is of the form (X, F), defined by a $\leqslant \aleph_0$-to-1 Borel function $F : X \to X$, the problem can be reduced to understanding when $\chi_{\mathcal{B}}(\mathcal{A}, S)$ is infinite, for the graph defined by the shift S on a Borel $\mathcal{A} \subseteq \mathbb{N}^{[\infty]}$.

[4]See definition of $\mathcal{G}_S$ in Section 4

It is thus natural to ask for a characterization of those Borel, or even analytic subsets $\mathcal{A} \subseteq \mathbb{N}^{[\infty]}$ for which the induced graph $(\mathcal{A}, S)$ has infinite Borel chromatic number. Clearly, any Borel set $\mathcal{A} \subseteq \mathbb{N}^{[\infty]}$ which contains a subset of the form $x^{[\infty]}$, where $x \in \mathbb{N}^{[\infty]}$, has infinite Borel chromatic number. In [12] it is asked if the converse is true, that is, if any Borel set with infinite Borel chromatic number contains a set of the form $x^{[\infty]}$. This has been answered by the following result of [6] .

Proposition 3.1. *There is a Borel subset $\mathcal{A}$ of $\mathbb{N}^{[\infty]}$ such that $\chi_{\mathcal{B}}(\mathcal{A}, S)$ is infinite but $\mathcal{A}$ contains no set $x^{[\infty]}$ of all infinite subsets of some infinite set $x \subseteq \mathbb{N}$.*

However, it turns out that the set $\mathcal{A}$ in this proposition is the range of a particular Borel graph-homomorphism $\Phi : (\mathbb{N}^{[\infty]}, S) \to (\mathbb{N}^{[\infty]}, S)$ so one can reformulate the question from [12] by asking if for every induced Borel subgraph $(\mathcal{A}, S)$ of $(\mathbb{N}^{[\infty]}, S)$ such that $\chi_{\mathcal{B}}(\mathcal{A}, S) = \aleph_0$ there is a Borel graph-homomorphism $\Phi : (\mathbb{N}^{[\infty]}, S) \to (\mathcal{A}, S)$.

We have already mentioned that the first question from [12] should also be modified, so let us now see why. Given a sequence of graphs, $\{\mathcal{G}_n = (X_n, E_n)\}_{n=0}^{\infty}$, we define $\mathcal{G} = \bigcup_n \mathcal{G}_n$ the graph obtained juxtaposing the graphs $\mathcal{G}_n$, as follows: $\mathcal{G} = (X, E)$, where X is the disjoint union of the X_n and E is the union of the E_n.

Proposition 3.2. *Let $\{k_n\}_{n=0}^{\infty}$ be an increasing sequence of natural numbers, and let $\{\mathcal{G}_n\}_{n=0}^{\infty}$ be a sequence of graphs with $\mathcal{G}_n = (X_n, E_n)$, and for every $n \in \omega$, $\chi_{\mathcal{B}}(\mathcal{G}_n) = k_n$. Then $\mathcal{G} = \bigcup_n \mathcal{G}_n$ has infinite Borel chromatic number and there is no Borel homomorphism from $\mathcal{G}_S$ into $\mathcal{G}$.*

Proof. Clearly, $\mathcal{G}$ has infinite Borel chromatic number. Since each $\mathcal{G}_n$ has finite Borel chromatic number, there is no homomorphism from $\mathcal{G}_S$ into $\mathcal{G}_n$. Suppose there is a Borel homomorphism $h : \mathbb{N}^{[\infty]} \to \bigcup X_n$ from $\mathcal{G}_S$ into $\mathcal{G}$. Being Borel, there is an infinite set $A \in \mathbb{N}^{[\infty]}$ such that h is continuous on $A^{[\infty]}$. Clearly, each connected component of $\mathcal{G}_S$ has to be mapped within one of the graphs $\mathcal{G}_n$, in particular the connected component of A is mapped within a $\mathcal{G}_n$. Since h is continuous on $A^{[\infty]}$, and the class of A with respect to finite differences is dense in $A^{[\infty]}$, the whole $A^{[\infty]}$ is mapped by h into $\mathcal{G}_n$, which contradicts that this graph has finite Borel chromatic number. $\square$

As a particular example consider the union of a sequence of graphs as those given in the Appendix of [12]. Another example can be obtained considering for each $n \in \omega$ the graph K_n, the full graph defined on an n element set, and the infinite graph $\mathcal{G} = \bigcup_n K_n$.

4. Graphs defined by a Borel function

An interesting class of graphs is formed by those graphs defined on a space X by a single Borel mapping $F : X \to X$: given $x, y \in X$, the pair $\{x, y\}$ is an edge of the graph (X, F) if $x \neq y$ and $y = F(x)$ or $x = F(y)$.

Theorem 4.1. [12, Theorem 5.1] *Let $F : X \to X$ be a Borel mapping defined on a Borel space X. Then, the Borel chromatic number of the graph (X, F) belongs to the set $\{1, 2, 3, \aleph_0\}$.*

Proof. For a given positive integer $n \geqslant 2$, consider the shift map $s_n : n^{\mathbb{N}} \to n^{\mathbb{N}}$ defined by $s_n(x)(i) = x(i + 1)$. Here, as usual, we identify the number n with the set of its predecessors $\{0, 1, \ldots, n - 1\}$, thus, $n^{\mathbb{N}}$ is the collection of sequences of natural numbers smaller than n.

Lemma 4.1. *For every $n \geqslant 2$, $\chi_{\mathcal{B}}(n^{\mathbb{N}}, s_n) \leqslant 3$.*

Proof. By induction on n.

For $n = 2$, define a Borel partition of $2^{\mathbb{N}}$ in three Borel pieces by

$$A_0 = \{x \in 2^{\mathbb{N}} : x \text{ starts with an odd number of 0's}\}$$
$$A_1 = \{x \in 2^{\mathbb{N}} : x \text{ starts with an odd number of 1's}\}$$
$$A_2 = 2^{\mathbb{N}} \setminus (A_0 \cup A_1).$$

It is easy to verify that this partition is a coloring of the graph $(2^{\mathbb{N}}, s_n)$.

Suppose now that $d : n^{\mathbb{N}} \to 3$ is a Borel coloring of the graph $(n^{\mathbb{N}}, s_n)$. Define $c : (n + 1)^{\mathbb{N}} \to 3$ as follows: given $x \in (n + 1)^{\mathbb{N}}$,

- if there are infinitely many i such that $x(i) \in n$ and infinitely many i such that $x(i) = n$, then put $c(x) = 0$ if x starts with an odd number of n's, $c(x) = 1$ if x starts with an odd number of elements of n, and $c(x) = 2$ otherwise.
- If there are only finitely many i such that $x(i) \in n$, then color x according to the parity of the least k such that for every $i \geqslant k$ $x(i) = n$. This gives a 2-coloring of the connected component formed by these x's
- If there are only finitely many i such that $x(i) = n$, then there is a unique decomposition of x as $x = s^\frown y$ where $s \in (n + 1)^k$ for some $k \geqslant 0$, $s(k - 1) = n$ (if $k > 0$), and $y \in n^{\mathbb{M}}$. Then put $c(x) = (d(y) + k) \pmod 3$.

Then c is a Borel coloring of $((n + 1)^{\mathbb{N}}, s_{n+1})$ with at most 3 colors. □

The rest of the proof of the theorem relies on the following observations.

First notice that $\chi_{\mathcal{B}}(X, F) \leqslant \aleph_0$. Let $\{U_n : n \in \mathbb{N}\}$ be a countable basis of the topology of X, and define $c : X \to \mathbb{N}$ defined by $c(x)$ is the first n such that $x \in U_n$ and $F(x) \notin U_n$. Since F is a Borel map, c is Borel, and it is clearly a coloring of the graph (X, F)[5]

Suppose that $\chi_{\mathcal{B}}(X, F) \leqslant n < \aleph_0$. Let $c : X \to n$ be a Borel coloring of (X, F), and define $p : X \to n^{\mathbb{N}}$ by $p(x) = \langle c(x), c(F(x)), \ldots, c(F^i(x)), \ldots \rangle$.

It should be clear that $p(F(x)) = s_n(p(x))$. Now, if $c' : n^{\mathbb{N}} \to 3$ is a Borel coloring of $(n^{\mathbb{N}}, s_n)$, which exists by Lemma 4.1, the composition $c' \circ p$ is a Borel coloring of (X, F). □

[5]This argument can be applied in a more general context. In particular, for directed graphs (X, P) of finite out degree at each point, where X is a standard Borel space and P is such that for every Borel $A \subseteq X$, the set $P^{-1}(A)$ is Borel. See [12, 4.5].

A particularly interesting subfamily of graphs is obtained from the space $\mathbb{N}^{[\infty]}$ of infinite subsets of $\mathbb{N}$, with the topology inherited from the product topology on $2^{\mathbb{N}}$, and the shift function $S : \mathbb{N}^{[\infty]} \to \mathbb{N}^{[\infty]}$ defined $S(x) = x \smallsetminus \{\min(x)\}$. The chromatic number of the graph $\mathcal{G}_S = (\mathbb{N}^{[\infty]}, S)$ is 2, which is easily seen since this graph is acyclic; and its Borel chromatic number is $\aleph_0$, as we explain now.

As we have already seen, since S is a Borel measurable function, $\chi_{\mathcal{B}}(\mathbb{N}^{[\infty]}, S) \leqslant \aleph_0$ (but more directly, the function $c(x) = \min(x)$ is an $\aleph_0$-coloring). Now, given any finite partition of $\mathbb{N}^{[\infty]}$ into Borel subsets, by a theorem of Galvin–Prikry ([9]), there is $x \in \mathbb{N}^{[\infty]}$ such that all infinite subsets of x lie in the same piece of the partition. In particular x and $x \smallsetminus \{\min(x)\}$ are in the same piece and therefore the partition does not give a coloring of the graph.

By Theorem 4.1, if $F : X \to X$ is a Borel function on a standard Borel space, and the graph (X, F) has finite Borel chromatic number, then there is a graph homomorphism f from (X, F) to $(3^{\mathbb{N}}, s_3)$, and in general, for any Borel F there is a homomorphism from (X, F) to the shift graph $(\mathbb{N}^{[\infty]}, S)$.

7.8 of [12] gives a stronger result for the case that F is $\aleph_0$-to-1 and has at most countably many periodic points. In this case there is a Borel embedding, i.e., an injective homomorphism. Since the periodic part of such a function F, $\{x \in X : \exists n, m > 0\,(F^n(x) = F^{n+m}(x))\}$, has finite Borel chromatic number, it can be assumed that F has no periodic points when just concerned with characterizing when (X, F) has infinite Borel chromatic number. In this case, the graph (X, F) is isomorphic to $(\mathcal{A}, S)$, where $\mathcal{A}$ is a Borel subset of $\mathbb{N}^{[\infty]}$ and the shift function S is restricted to $\mathcal{A}$. For this reason it is interesting to study chromatic properties of graphs of this form, which we do below.

5. Borel subsets of $\mathbb{N}^{[\infty]}$ of Borel chromatic number > 3

It follows from results explained in Section 4 that for every non-empty Borel set $\mathcal{A} \subseteq \mathbb{N}^{[\infty]}$ the possible values of the Borel chromatic number of the graph $(\mathcal{A}, S)$ are $1, 2, 3$, or $\aleph_0$. It is not difficult to see that all these possibilities are attained.

Borel sets for which this chromatic number is 1 or 2 are easy to find. To give an example of a Borel set $\mathcal{A} \subseteq \mathbb{N}^{[\infty]}$ such that $\chi_{\mathcal{B}}(\mathcal{A}, S) = 3$, we first prove the following result about G_δ sets.

Proposition 5.1. *Let $\mathcal{A} \subseteq \mathbb{N}^{[\infty]}$ be a perfect G_δ set closed under the operation S. Then if $(\mathcal{A}, S)$ has finite Borel chromatic number, then this number is equal to* 3.

Proof. Being a G_δ, the set $\mathcal{A}$ is a Polish space with the topology inherited from $\mathbb{N}^{[\infty]}$. We now work within this space.

Suppose that $c : \mathcal{A} \to 2$ is a Borel coloring of $(\mathcal{A}, S)$, and consider the Borel sets $\mathcal{A}_0 = c^{-1}(\{0\})$ and $\mathcal{A}_1 = c^{-1}(\{1\})$. Since both $\mathcal{A}_0$ and $\mathcal{A}_1$ cannot be meager, we can assume that there is a basic open set $[s]$ such that one of $c^{-1}\{0\}$ or $c^{-1}\{1\}$ is comeager in $[s]$. Let s of minimal size with that property, and suppose that $c^{-1}\{0\}$ is comeager in $[s]$. We will see that s must be empty. Otherwise, since for every $x \in [s] \cap c^{-1}\{0\}$, $S(x) \in c^{-1}\{1\}$, we get that $c^{-1}\{1\}$ is comeager in the basic neighborhood $[s \smallsetminus \{\min(s)\}]$, and this contradicts the minimality of s. We have thus that one of $c^{-1}\{0\}$ or $c^{-1}\{1\}$ is comeager in $\mathbb{N}^{[\infty]}$, say the latter is comeager and therefore it is comeager in any basic neighborhood. Let $n \in \omega$; then, $c^{-1}\{1\}$ is comeager on the basic neighborhood $[\{n\}]$. But applying the shift to each element of $[\{n\}] \cap c^{-1}\{1\}$, we obtain a comeager subset of $c^{-1}\{0\}$, a contradiction. □

Below in this section we give examples of perfect G_δ sets, closed under S, for which the shift graph has finite Borel chromatic number, and so by Proposition 5.1, this number is 3.

Definition 5.1. *Let $\mathcal{A} \subseteq \mathbb{N}^{[\infty]}$ be Borel. If the graph $(\mathcal{A}, S)$ has infinite Borel chromatic number, we say $\mathcal{A}$ is infinitely chromatic and express this by $\chi_{\mathcal{B}}(\mathcal{A}, S) = \infty$, otherwise we say $\mathcal{A}$ is finitely chromatic and write $\chi_{\mathcal{B}}(\mathcal{A}, S) < \infty$.*

Fact 5.1. *The collection $\mathcal{B}_F$ of finitely chromatic Borel subsets of $\mathbb{N}^{[\infty]}$ is an ideal of Borel sets.*

Proof. Clearly, if $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathbb{N}^{[\infty]}$ are Borel sets and $\mathcal{B}$ is finitely chromatic, then $\mathcal{A}$ is also finitely chromatic.

Let $\mathcal{A}$ and $\mathcal{B}$ be two finitely chromatic Borel subsets of $\mathbb{N}^{[\infty]}$. Let c_a and c_b be finite Borel colorings of $\mathcal{A}$ and $\mathcal{B}$ respectively. Define a finite Borel coloring c of the union $\mathcal{A} \cup \mathcal{B}$ as follows: Put $c(x) = (a, c_a(x))$ if $x \in \mathcal{A}$, and $c(x) = (b, c_b(x))$ if $x \in \mathcal{B} \setminus \mathcal{A}$. □

It is easily seen that $\mathcal{B}_F$ is not a σ-ideal, for example considering the collection $\{\mathcal{A}_i : i \in \mathbb{N}\}$ where for every i, $\mathcal{A}_i = \{x \in \mathbb{N}^{[\infty]} : \min(x) = i\}$. Each $\mathcal{A}_i$ is 1-chromatic, and the union $\bigcup_{i=0}^{\infty} \mathcal{A}_i$ is $\mathbb{N}^{[\infty]}$.

The following result shows that when working with an infinitely chromatic Borel subset $\mathcal{A}$ of $\mathbb{N}^{[\infty]}$, we can always go to an infinitely chromatic subset $\mathcal{A}^*$ of essentially the same Borel complexity which has the pleasant property of being closed under the shift operation S; that is, $S(x) \in \mathcal{A}^*$ for every $x \in \mathcal{A}^*$.

Proposition 5.2. *For every Borel set $\mathcal{A}$, the subset*

$$\mathcal{A}^* = \{x \in \mathcal{A} : \forall n (S^{(n)}(x) \in \mathcal{A})\}$$

has the following properties.

1. *$\mathcal{A}^*$ is closed under shifts; i.e., $S(x) \in \mathcal{A}^*$ for all $x \in \mathcal{A}^*$.*
2. *$\mathcal{A}^*$ is of the same multiplicative Borel complexity of $\mathcal{A}$, so, in particular, if $\mathcal{A}$ is closed, so is $\mathcal{A}^*$.*
3. *$\mathcal{A} \setminus \mathcal{A}^*$ is Borel 2-chromatic. Thus, if $\mathcal{A}$ is infinitely chromatic, so is $\mathcal{A}^*$.*

Proof. It is clear that $\mathcal{A}^*$ is closed under shifts. Note also that

$$\mathcal{A}^* = \mathcal{A} \cap \left(\bigcap_{n=0}^{\infty} \mathcal{A}_n \right),$$

where $\mathcal{A}_n$ is the preimage of $\mathcal{A}$ under the map $S^{(n)}$, and so $\mathcal{A}$ and $\mathcal{A}^*$ have the same multiplicative Borel complexity.

Define the Borel 2-coloring $c : \mathcal{A} \setminus \mathcal{A}^* \to 2$ putting $c(x)$ equal to the parity of the least i such that $S^{(i)}(x) \notin \mathcal{A}$. Clearly c is a 2-coloring of the graph $(\mathcal{A} \setminus \mathcal{A}^*), S)$. □

For $x, y \in \mathbb{N}^{[\infty]}$, listed in increasing order respectively by

$$x = \{x(0), x(1), \dots\} \text{ and } y = \{y(0), y(1), \dots\},$$

we define

$$x \leqslant^* y \text{ if } x(i) \leqslant y(i) \text{ for all except finitely many } i's.$$

Recall that a set $\mathcal{A} \subseteq \mathbb{N}^{[\infty]}$ is bounded by y if $x \leqslant^* y$ for every $x \in \mathcal{A}$. A set $\mathcal{A}$ is unbounded if it is not bounded by any $y \in \mathbb{N}^{[\infty]}$. A set $\mathcal{A}$ is dominating (or cofinal) if for every $y \in \mathbb{N}^{[\infty]}$, there is $x \in \mathcal{A}$ such that $y \leqslant^* x$.

Definition 5.2. *For a given $x \in \mathbb{N}^{[\infty]}$, listed in increasing order by*

$$x = \{x(0), x(1), \dots\},$$

define $I_0^x = \{n \in \mathbb{N} : n < x(0)\}$, and for $i > 0$ let $I_i^x = [x(i-1), x(i))$. We say that a set $y \in \mathbb{N}^{[\infty]}$ (or $y \in \mathbb{N}^{[<\infty]}$) is a selector for x, if for every $i \in \mathbb{N}$, $|y \cap I_i^x| \leqslant 1$. We say that y is a spread out selector for x if it is a selector for x and for every i, $y \cap I_i^x \neq \emptyset$ implies $y \cap I_{i+1}^x = \emptyset$.

Now we give the promised examples of finitely chromatic G_δ sets without isolated points and closed under S. Fix $y \in \mathbb{N}^{[\infty]}$ such that $y(i+1) - y(i)$ increases to infinity with i, and let

$$\mathcal{A} = \{x \in \mathbb{N}^{[\infty]} : \exists^\infty i \ (|x \cap [y(i), y(i+1))| > 1)\}.$$

To show that $\chi_{\mathcal{B}}(\mathcal{A}, S) < \infty$, a finite coloring of $(\mathcal{A}, S)$ can be defined as we indicate below, then by Proposition 5.1, we obtain that $\chi_{\mathcal{B}}(\mathcal{A}, S) = 3$.

Define a coloring $c : \mathcal{A} \to 4$ as follows. Given $y \in \mathcal{A}$, let j be the least i such that $I_i^x \cap y \neq \emptyset$, then put

$c(y) = (0, \delta)$ if $|I_j^x \cap y| > 1$, and δ is the parity of the number of elements of $I_j^x \cap y$; and

$c(y) = (1, \epsilon)$ if $|I_j^x \cap y| = 1$, and ϵ is the parity of the initial number of intervals I_i^x such that $|I_i^x \cap y| = 1$. It is easy to verify that c is indeed a coloring of the graph $(\mathcal{A}, S)$.

Now we present another example, a perfect closed set with Borel chromatic number 3.

Let $\mathcal{A} = \{x \in \mathbb{N}^{[\infty]} : \forall i (x(i+1) = x(i) + 1 \text{ or } x(i+1) = x(i) + 2\}$.

$\chi_{\mathcal{B}}(\mathcal{A}) \leqslant 3$, since the argument of Theorem 4.1 for $2^{\mathbb{N}}$ works here as well. For this we define a 3 coloring of the shift graph on $\mathcal{A}$ by

$c(x) = 0$ if x begins with an odd number of successive natural numbers, $c(x) = 1$ if x starts with an odd number of numbers congruent mod 2; and $c(x) = 2$ in any other case, except if x is of the form $\{n, n+1, n+2, \dots, n+k, n+k+1, \dots\}$ or $\{n, n+2, n+4, \dots, n+2k, n+(2k+2), \dots\}$, cases which are treated apart. For $x = \{n, n+1, n+2, \dots, n+k, n+k+1, \dots\}$ we color x according to the parity of n, and in case $\{n, n+2, n+4, \dots, n+2k, n+(2k+2), \dots\}$, color x according to the parity of the integer part of $n/2$. It is easy to verify that this is a coloring of the shift graph of $\mathcal{A}$.

Clearly $\mathcal{A}$ is closed and has no isolated points (and therefore it is Polish).

Suppose that $\mathcal{A} = B_0 \cup B_1$ is a Borel coloring of the shift graph of $\mathcal{A}$. Let $[s]$ be a basic neighborhood such that $[s] \cap B_0$ is comeager in $[s] \cap \mathcal{A}$. We can assume that $|s|$ is even. Let $a = \max(s) + 1$, and $b = a + 1$, and consider the neighborhoods $[s \cup \{a, b\}]$ and $[s \cup \{b\}]$. Applying the shift operation $|s|$ times, we obtain that $[\{a, b\}] \cap B_0$ is comeager in $[\{a, b\}] \cap \mathcal{A}$ and also $[\{b\}] \cap B_0$ is comeager in $[\{b\}] \cap \mathcal{A}$. But $S''[\{a, b\}] \cap B_0 \subseteq [\{b\}] \cap B_1$ and since the shift function is a homeomorphism from $[\{a, b\}] \cap \mathcal{A}$ to $[\{b\}] \cap \mathcal{A}$ that sends comeager sets in comeager sets, we get that $[\{b\}] \cap B_1$ is a comeager subset of $[\{b\}] \cap \mathcal{A}$, a contradiction.

The next proposition shows that if a Borel set $\mathcal{A}$ is infinitely chromatic, then given any $x \in \mathbb{N}^{[\infty]}$, almost every element of $\mathcal{A}$, in the sense of chromaticity, is a spread out selector for x.

Proposition 5.3. *Let $\mathcal{A}$ be an infinitely chromatic Borel set, then for every $x \in \mathbb{N}^{[\infty]}$ the set $\{y \in \mathcal{A} : y$ is a spread out selector for $x\}$ is infinitely chromatic.*

Proof. Given $x \in \mathbb{N}^{[\infty]}$, we show that the set of elements of $\mathcal{A}$ which are not spread out selectors for x is finitely Borel colorable. Clearly, the set $\mathcal{B} = \{y \in \mathcal{A} : y$ is not a spread out selector for $x\}$ is a Borel subset of $\mathcal{A}$. For every $y \in \mathcal{B}$, if y is a selector for x, then there is $i \in \mathbb{N}$ such that $y \cap I_i^x \neq \emptyset \neq y \cap I_{i+1}^x$.

In symbols,

$$\forall y \in \mathcal{B} \left[y \text{ is a selector for } x \Rightarrow \exists j (y \cap I_j^x \neq \emptyset \neq y \cap I_{j+1}^x) \right].$$

We define a finite coloring for $\mathcal{B}$ as follows:

(a) y is not a selector for x.

$c(y) = (0, s)$ if y is not a selector for x and for the first i such that $y \cap I_i^x \neq \emptyset$, $|y \cap I_i^x| = 1$ (y starts with a singleton), and in this case s is the parity of the number of initial singletons.

$c(y) = (1, ns)$ if y is not a selector for x and for the first i such that $y \cap I_i^x \neq \emptyset$, $|y \cap I_i^x| > 1$ (y starts with a multiple selection), and in this case ns is the parity of the number of elements of y in that first nonempty interval.

(b) y is a selector for x.

$c(y) = (2, \text{iso})$ if for some i, $y \cap I_i^x = \{y(0)\}$ and $y \cap I_{i+1}^x = \emptyset$, i.e., y starts with an isolated selection, and iso is the parity of the number of initial isolated selections (notice that this number is finite since y is s selector but not a spread out selector).

$c(y) = (3, \text{cs})$ if for some i $y(0) \in I_i^x$ and $y(1) \in I_{i+1}^x$, i.e., y starts selecting from contiguous intervals of x, there is $j > i + 1$ such that $y \cap I_j^x = \emptyset$, and cs is the parity of the number of intervals of x which intersect y in the initial block: the parity of the least n such that $y(0), y(1), \ldots, y(n-1)$ lie in consecutive intervals of x.

$c(y) = (4, e)$ if $y(0) \in I_i^x$ and $\forall j > i (y \cap I_j^x) \neq \emptyset$, i.e., y misses the first i intervals of x and selects from all other intervals of x; and e is the parity of the number of intervals of x missed by y. ∎

We will use the notation

$$\mathrm{Sp}(\mathcal{A}, z) = \{x \in \mathcal{A} : x \text{ is a spread out selector for } z = \{n_i\}_{i=0}^{\infty}\}.$$

Corollary 5.1. *If $\mathcal{A} \subseteq \mathbb{N}^{[\infty]}$ is infinitely chromatic Borel set, then $\mathcal{A}$ is dominating.*

Proof. Let $\mathcal{A} \subseteq \mathbb{N}^{[\infty]}$ is infinitely chromatic Borel set, then for every $x \in \mathbb{N}^{[\infty]}$ the set $\mathrm{Sp}(\mathcal{A}, x) \subseteq \mathcal{A}$ is infinitely chromatic. By Proposition 5.2, we can assume that $\mathcal{A}$ is closed under shifts. Notice that the set $\{y \in \mathcal{A} : y(0) < x(0)\}$ is finitely chromatic (color according to the parity of the number of elements of y below $x(0)$).

If $\mathcal{A}$ is not dominating, there is $x \in \mathbb{N}^{[\infty]}$ such that for every $y \in \mathcal{A}$ the set $\{i : y(i) < x(i)\}$ is infinite. If $y \in \mathrm{Sp}(\mathcal{A}, x)$ and $x(0) < y(0)$, then for every i, $x(i) < y(i)$, a contradiction. □

6. Infinitely chromatic closed subsets of $\mathbb{N}^{[\infty]}$

In this section we prove that infinitely chromatic closed sets are strongly dominating. To this end we need to fix some further notation that will be useful throughout rest of the paper.

We identify sets of positive integers (finite or infinite) with the corresponding strictly increasing sequence. A tree is $T \subseteq \mathbb{N}^{[<\infty]}$ partially ordered by $\sqsubseteq$ (end extension) such that T is downwards closed. Recall that a subset of $\mathbb{N}^{[\infty]}$ is closed if and only of it is the set of branches of some tree. A tree T is perfect if for every $s \in T$ there are incomparable $r, t \in T$ such that $s \sqsubset r$ and $s \sqsubset t$. A tree is superperfect if every $s \in T$ has an extension with infinitely many immediate successors. A tree T is a Laver tree if there is $p \in T$ such that every $q \in T$ is comparable with p and for every $q \in T$, $p \leqslant q$ implies that q has infinitely many immediate successors in T.

A subset $\mathcal{A}$ of $\mathbb{N}^{\mathbb{N}}$ is said to be dominating if for every $x \in \mathbb{N}^{\mathbb{N}}$ there is an element $y \in \mathcal{A}$ and $k \in \mathbb{N}$ such that $x(i) < y(i)$ for all $i \geqslant k$; and $\mathcal{A}$ is said to be strongly dominating if for every $x \in \mathbb{N}^{\mathbb{N}}$ there is $y \in \mathcal{A}$ and $k \in \mathbb{N}$ such that $x(y(i)) < y(i+1)$ for all $i \geqslant k$. It is easy to see that there are dominating sets that are not strongly dominating, for example the set $\{y \in \mathbb{N}^{\mathbb{N}} : y(2k) = y(2k+1)\}$. A set $\mathcal{A} \subseteq \mathbb{N}^{\mathbb{N}}$ is strongly dominating if and only if there is a Laver tree T such that $[T] \subseteq \mathcal{A}$.

Given a tree T, T' is the tree obtained removing the terminal nodes of T. Let $\ker(T)$ be the Cantor–Bendixon kernel of T, i.e., the tree defined by induction removing the terminal nodes in each step. More precisely, define $T^{(0)} = T$, $T^{(\xi+1)} = (T^{(\xi)})'$, and $T^{(\lambda)} = \bigcap_{\xi<\lambda} T^{(\xi)}$, for λ limit. If η is the first ordinal ξ such that $T^{(\xi+1)} = T^{(\xi)}$, then $\ker(T) = T^{(\eta)}$. Given $s, t \in \mathbb{N}^{[<\infty]}$, we write

$$s \lhd t \text{ if and only if } s \setminus \{\min(s)\} \sqsubseteq t.$$

As mentioned above, for $x \in \mathbb{N}^{[\infty]}$ define $S(x) = x \setminus \{\min(x)\}$.

The function S is usually called the "shift", and corresponds to shifting one position upwards in the increasing enumeration of the infinite set.

For $s \in \mathbb{N}^{[<\infty]}$, $[s] = \{x \in \mathbb{N}^{[\infty]} : s \sqsubset x\}$. Similarly, for $t \in 2^{<\infty}$, $[t] = \{x \in 2^{\mathbb{N}} : t \subset x\}$.

If $s \in \mathbb{N}^{[<\infty]}$ and $n \in \mathbb{N}$, $s < n$ is used to express that $i < n$ for every $i \in s$.

Definition 6.1. *Given a tree $T \subseteq \mathbb{N}^{[<\infty]}$, and an element z of $\mathbb{N}^{[\infty]}$ listed increasingly $z = \{n_i : i \in \mathbb{N}\}$, we define $T[z]$ to be the subtree*

$$T[z] = \{t \in T : t(i) < n_i \text{ for all } i \in \mathrm{dom}(t)\}.$$

Analogously, $T[z]$ can be defined for trees $T \subseteq \mathbb{N}^{[<\infty]} \otimes \mathbb{N}^{[<\infty]}$:

$$T[z] = \{(s,t) \in T : \forall i \in \mathrm{dom}(s) = \mathrm{dom}(t)\ s(i), t(i) < n_i\}.$$

We now show that every closed infinitely chromatic set contains the branches of a Laver tree.

We will use the following notion. Given a tree $T \subseteq \mathbb{N}^{[<\infty]}$ and an infinite set $X \subseteq \mathbb{N}$, let T_X be the set $\{s \in T : s \text{ is a spread out selector for } X\}$, it is clear that T_X is a subtree of T since T_X is downwards closed.

Proposition 6.1. *Let $C \subseteq \mathbb{N}^{[\infty]}$ be closed and infinitely chromatic. Then there is a tree $T \subseteq \mathbb{N}^{[<\infty]}$ such that $[T] \subseteq C$, $[T]$ is infinitely chromatic, and every node of T has infinitely many immediate successors in T.*

Proof. Since C is closed, $C = [T]$ for some tree T. Since $C = [T]$ is infinitely chromatic, $\emptyset$ has infinitely many immediate successors.

Take $M_0 \in \mathbb{N}^{[\infty]}$ any infinite set, and let m_0 its minimal element. Notice that $\emptyset$ is infinite branching in T_{M_0} (recall that for X infinite, T_X is the set of nodes of T which do not meet consecutive intervals of X).

Suppose we have defined the sets $M_0, M_1, \ldots, M_k$, and their respective minimal elements $m_0 < m_1 < \cdots < m_k$.

Let M_{k+1} be an infinite subset of $M_k \setminus \{m_k\}$ such that for every $s \subseteq m_k$ which belongs to $\ker(T_{\{m_0,\ldots,m_k\}\cup M_{k+1}})$ either

- (i) s is a finitely branching node of $\ker(T_{\{m_0,\ldots,m_k\}\cup M_k})$ and m_{k+1}, the first element of M_{k+1}, is above the maximum of the immediate successors of s in that tree; or
- (ii) there is no $X \subseteq M_{k+1}$ such that s is finitely branching in the tree $\ker(T_{\{m_0,\ldots,m_k\}\cup X})$.

To find M_{k+1}, list the subsets of m_k which belong to $\ker(T_{\{m_0,\ldots,m_k\}\cup M_k})$ as $\{s_0, \ldots, s_m\}$. If there is $X \subseteq M_k$ such that s_0 is finitely branching in

$$\ker(T_{\{m_0,\ldots,m_k\}\cup M_k}),$$

and $\min(X)$ is above the maximum of the immediate successors of s_0 in

$$\ker(T_{\{m_0,\ldots,m_k\}\cup M_k}),$$

take such a set and call it X_0, otherwise, put $X_0 = M_k$. Having defined X_i, for $i < m$, if there is a set $X \subseteq X_i$ such that s_{i+1} is finitely branching in $\ker(T_{\{m_0,\ldots,m_k\}\cup X_i})$, and $\min(X)$ is above the maximum of the immediate successors of s_{i+1} in $\ker(T_{\{m_0,\ldots,m_k\}\cup X_i})$, take such a set and call it X_{i+1}, otherwise $X_{i+1} = X_i$. Finally, $M_{k+1} = X_m \setminus m_k + 1$.

Take $M_\infty = \{m_0, m_1, \ldots\}$. We claim that T_{M_∞} determines a closed infinitely chromatic set. This is shown, as in Proposition 5.3, defining a finite coloring for the set of branches that contain a node that meets consecutive intervals of M_∞.

We claim also that $\ker(T_{M_\infty})$ is infinitely branching in every node. To see this, let s be a node of $\ker(T_{M_\infty})$. If s is finitely branching in this tree, let m_k be the first element of M_∞ above s. Then the second option of the alternative did not hold for s in step $k+1$, and thus m_{k+1} is above the maximum of the immediate successors of s in $\ker(T_{\{m_0,\dots,m_k\}\cup M_{k+1}})$, and therefore s does not belong to $T_{\{m_0,\dots m_k\}\cup M_{k+1}}$, neither to $T_{\{m_0,\dots,m_k\}\cup M_\infty}$, a contradiction. □

We have shown that every closed infinitely chromatic set contains the set of branches of a Laver tree and therefore it is strongly dominating (see [10]).

7. A combinatorial characterization of continuous infinite chromaticity for closed subsets of $\mathbb{N}^{[\infty]}$

The notion of front on the natural numbers was introduced by Nash-Williams in [17] (see also [20]). Here we will adapt this notion to our purposes.

Definition 7.1. *A collection $\mathcal{F} \subseteq \mathbb{N}^{[<\infty]}$ is a front if*

1. *$s \not\sqsubseteq t$ whenever $s \neq t \in \mathcal{F}$,*
2. *$\bigcup \mathcal{F}$ is infinite and for every infinite $A \subseteq \bigcup \mathcal{F}$ there exists $s \in \mathcal{F}$ such that $s \sqsubseteq A$.*

Given a tree $T \subseteq \mathbb{N}^{[<\infty]}$, we say that $\mathcal{F}$ is a front of T if

1. *$s \not\sqsubseteq t$ whenever $s \neq t \in \mathcal{F}$, and*
2. *every branch of T has an initial segment in $\mathcal{F}$.*

If $\mathcal{F}$ is a front, and $x \in \mathbb{N}^{[\infty]}$, we denote by $\mathcal{F}(x)$ the unique initial segment of x which belongs to $\mathcal{F}$.

Definition 7.2. *A tree $T \subseteq \mathbb{N}^{[<\infty]}$ has property $(*)$ if for every front $\mathcal{F}$ of T and every $c : \mathcal{F} \to F$, where F is a finite set, there are $s, t \in \mathcal{F}$ such that $s \lhd t$, $s \cup t \in T$, and $c(s) = c(t)$.*

We prove several facts regarding trees with this property.

Theorem 7.1. *Let $T \subseteq \mathbb{N}^{[<\infty]}$ be a tree closed under the shift operation, i.e., if $x \in [T]$, then $S(x) \in [T]$. Then, T has property $(*)$ if and only if $\chi_{\mathrm{cont}}([T], S) = \infty$.*

Here, $\chi_{\mathrm{cont}}(\mathcal{A}, S)$ is the continuous chromatic number of the graph $(\mathcal{A}, S)$, in other words, the chromatic number of the graph when only continuous colorings are considered.

Proof. Suppose T has property $(*)$, and let $c : [T] \to F$ be a continuous mapping from $[T]$ to a finite set F. The mapping c determines a front $\mathcal{F}$ of T, and a finite coloring of $\mathcal{F}$. In fact, for every $m \in F$, $c^{-1}(m)$ is an open set and therefore there is a canonical set of maximal pairwise disjoint basic neighborhoods whose union is this open set. The collection of the finite strictly increasing sequences determining these basic open sets corresponding to each $c^{-1}(m)$ for $m \in F$ is a front $\mathcal{F}$ of T. For $s \in \mathcal{F}$, define $c(s) = m$ if $c(x) = m$ for every infinite branch x of T extending s. Let $s, t \in T$ be such that $s \lhd t$, $s \cup t \in T$, and $c(s) = c(t)$; and take $x \in [T]$, a branch of T that extends $s \cup t$. Then, $S(x)$ extends t, and $c(x) = c(S(x))$. Therefore, c cannot be a coloring of $([T], S)$.

For the other implication, suppose that $\chi_{\mathrm{cont}}([T], S) = \infty$. Let $\mathcal{F}$ be a front of T and $c : \mathcal{F} \to F$ be a mapping from $\mathcal{F}$ to a finite set F. Define $\bar{c} : [T] \to F$ by $\bar{c}(x) = c(\mathcal{F}(x))$. Since $\bar{c}$ is not a coloring of $([T], S)$, there is $x \in [T]$ such that $\bar{c}(x) = c(S(x))$, thus $c(\mathcal{F}(x)) = c(\mathcal{F}(S(x))$. Therefore $s = \mathcal{F}(x)$ and $t = \mathcal{F}(S(x))$ witness that t has property $(*)$. □

Fact 7.1. *If T has property $(*)$ and $[T]$ is closed under shifts, then for every front $\mathcal{F}$ of T and every mapping $c : \mathcal{F} \to F$ into a finite set F, there is a sequence $s_0, s_1, \dots$ of elements of $\mathcal{F}$ such that $s_0 \lhd s_1 \lhd s_2 \lhd \dots$ and c is constant on $\{s_i : i \in \mathbb{N}\}$*

Proof. Suppose there is a front $\mathcal{F}$ of T and a mapping $c : \mathcal{F} \to F$ into a finite set F such that c does not take constant value on any sequence $s_0 \lhd s_1 \lhd s_2 \lhd \dots$ of elements of $\mathcal{F}$.

Define a finite coloring d of $([T], S)$ as follows: Let $x \in [T]$, x determines a sequence $s_0 \lhd s_1 \lhd s_2 \lhd \dots$ of elements of $\mathcal{F}$. In fact, given $x \in [T]$, let $\mathcal{F}(x)$ be the unique element of $\mathcal{F}$ extended by x. The sequence of elements of $\mathcal{F}$ determined by x is then $\langle \mathcal{F}(x), \mathcal{F}(S(x)), \dots, \mathcal{F}(S^{(i)}(x), \dots \rangle$. Let $d(x) = (c(\mathcal{F}(x)), p)$ where p is the parity of the least i such that $c(s_i) \neq c(s_{i+1})$. This is a finite continuous coloring of $[T]$, by Proposition 7.1 this contradicts that T has property $(*)$. □

Observe that for any tree T such that $[T]$ is closed under shifts, if $\mathcal{F}$ is a front of T and $c : \mathcal{F} \to \mathrm{Ord}$, there are $s, t \in \mathcal{F}$ such that $s \lhd t$ and $f(s) \leqslant f(t)$. This is so because for any branch $x[\in T]$, the sequence $\langle t_0, t_1, \dots \rangle$ given by $t_i = \mathcal{F}(S^{(i)}(x)$, is such that for every i, $s_i \lhd s_{i+1}$ and the sequence $\langle c(t_i) : i \in \mathbb{N} \rangle$ cannot be strictly decreasing.

Fact 7.2. *If T has property $(*)$ and $[T]$ is closed under shifts, then for every front $\mathcal{F}$ of T and every $f : \mathcal{F} \to \mathrm{Ord}$, there is a sequence $s_0 \lhd s_1 \lhd s_2 \lhd \dots$ of elements of $\mathcal{F}$ such that f is constant on $\{s_i : i \in \mathbb{N}\}$ or f is strictly increasing on $\{s_i : i \in \mathbb{N}\}$, that is, for every i, $f(s_i) < f(s_{i+1})$.*

Proof. We use Proposition 7.1 to prove this. Suppose T is as in the statement and that for a front $\mathcal{F}$ of T, the mapping $f : \mathcal{F} \to \mathrm{Ord}$ is not constant nor strictly increasing on any sequence $s_0 \lhd s_1 \lhd s_2 \lhd \cdots$ of elements of $\mathcal{F}$. Define a continuous coloring of $([T], S)$ as follows: for $x \in [T]$, if $s_0 \lhd s_1 \lhd s_2 \lhd \cdots$ is the sequence of elements of $\mathcal{F}$ determined by x as above, define $c(x) = (0, p)$ if $f(s_0) < f(s_1)$ and p is the parity of the least i such that $f(s_i) \geqslant f(s_{i+1})$; $c(x) = (1, q)$ if $f(s_0) \geqslant f(s_1)$ and q is the parity of the least i such that $f(s_i) < f(s_{i+1})$. By our hypothesis, c is a coloring of $[T]$, and it is continuous, contradicting that $\chi_{\mathrm{cont}}([T], S)$ is infinite. □

8. The local structure of infinitely chromatic closed subsets of $\mathbb{N}^{[\infty]}$

In this section we examine "local" chromaticity properties of infinitely chromatic trees $T \subseteq \mathbb{N}^{[<\infty]}$, that is, trees for which $[T]$ is an infinitely chromatic closed set.

Consider the equivalence relation on $\mathbb{N}^{[\infty]}$ given by $x \sim y$ if and only if the symmetric difference $x \,\Delta\, y$ is finite. We study density properties of the classes of branches of T.

Proposition 8.1. *Suppose $f : \mathbb{N}^{[\infty]} \to \mathbb{N}^{[\infty]}$ is a continuous mapping which commutes with the shift S. Suppose its range $f"\mathbb{N}^{[\infty]}$ is a closed subset of $\mathbb{N}^{[\infty]}$ of the form $[T]$ for some tree $T \subseteq \mathbb{N}^{[<\infty]}$. Then the equivalence class of each branch of T is dense. That is, for every $y \in [T]$, for every $t \in T$, there is $z \in [T]$ such that $t \sqsubset z$ and $y\Delta z$ is finite.*

Proof. The function f is determined by a front $\mathcal{F}$ and a function $\varphi : \mathcal{F} \to \mathbb{N}$.

Given $y \in [T]$, there is $x \in \mathbb{N}^{[\infty]}$ such that $f(x) = y$. And given $t \in T$, $t = \{t(0), t(1), \dots, t(k)\}$, there is a sequence $s_0, s_1, \dots, s_k$ of elements of $\mathcal{F}$ such that $s_0 \triangleleft s_1 \triangleleft \cdots \triangleleft s_k$ and for every $i = 0, \dots, k$, $t(i) = \varphi(s_i)$.

Let n be (the least) such that $\max s_k < \min S^{(n)}(x)$, and consider $v = \bigcup_{i=0}^{k} s_i \cup S^{(n)}(x)$. Let j be the least such that for some l, $S^{(j)}(v) = S^{(l)}(x)$; then for some $s \in \mathbb{N}^{[<\infty]}$, $f(v) = t^\frown s^\frown S^{(l)}(y)$. □

Proposition 8.2. *Let T be a tree such that the closed set $[T]$ is infinitely chromatic. Then, for every $k \in \mathbb{N}$, the set*

$$C_k = \{x \in [T] : |\{n \in \mathbb{N} : n < \min(x) \text{ and } \{n\} \cup x \in [T]\}| \geqslant k\}$$

is infinitely chromatic.

Proof. The complement of C_k is the set of all the $x \in [T]$ such that

$$\big|\{n \in \mathbb{N} : n < \min(x) \text{ and } \{n\} \cup x \in [T]\}\big| < k.$$

The degree of the graph defined by the shift on this set is k and therefore it is finitely chromatic, and thus 3 chromatic. □

For every $x \in [T]$, let $P(x)$ be the sequence

$$P(x) = \langle |\{k : k < \min(S^{(n)}(x)) \text{ and } \{k\} \cup S^{(n)}(x) \in [T]\}| : n \in \mathbb{N}\rangle.$$

Proposition 8.3. *The set $C = \{x \in [T] : P(x)$ is constant$\}$ is finitely chromatic.*

Proof. For the graph (C, S) construct a sequence $B_0, B_1, \dots$ of Borel kernels using repeatedly Proposition 4.2 of [12] so that $C = \bigcup_n B_n$.

For each $x \in C$, let $[x]$ be the connected component of x in the graph (C, S), i.e., $[x] = \{y \in C : x \,\Delta\, y$ is finite$\}$. If $x, y \in C$ and $[x] = [y]$, then $P(x) = P(y)$, since the two constant sequences must have identical tails. Notice then that for every $x \in C$, there exists a number $n_x \in \mathbb{N}$ such that the component $[x]$ is contained in $\bigcup_{i=0}^{n_x} B_i$. Otherwise there would be an element in the class $[x]$ connected to more than the constant value taken by the sequence $P(x)$.

Consider the map $\varphi : C \to \mathbb{N}^{\mathbb{N}}$ defined putting $\varphi(x) = \langle n_0, n_1, \dots\rangle$ if for every $i \in \mathbb{N}$, $S^{(i)}(x) \in B_{n_i}$.

By the remarks made above, for every $x \in C$, $\varphi(y)$ for every $y \in [x]$ is a sequence in $n_x^{\mathbb{N}}$,

$$\varphi \restriction [x] : [x] \to n_x^{\mathbb{N}}.$$

Lemma 5.2 of [12] establishes that for every $n \in \mathbb{N}$, there is a Borel coloring $\psi_n : n^{\mathbb{N}} \to 3$, and if we define $c : C \to 3$ by $c(x) = \psi_{n_x}(\varphi(x))$, we obtain a 3-coloring of the graph (C, S). □

Corollary 8.1. *The set* $\{x \in [T] : P(x)$ *is strictly increasing*$\}$ *is infinitely chromatic.*

9. Graphs on families of finite sets of natural numbers

In this section we will describe some results about graphs defined on collections of finite subsets of $\mathbb{N}$. We will consider precompact families $\mathcal{F} \subseteq [\omega]^{<\omega}$ as the vertex set of the graph $(\mathcal{F}, \triangleleft)$ where $\triangleleft$ is the shift relation

$$s \triangleleft t \text{ iff } s \setminus \min(s) \sqsubseteq t,$$

and $\sqsubseteq$ is the initial segment relation between finite subsets of ω, i.e., $u \sqsubseteq v$ if and only if u is an initial segment of v relative to the usual ordering of ω. We say that a subset $\mathcal{F}'$ of $\mathcal{F}$ is independent if there are no $s, t \in \mathcal{F}'$ such that $s \triangleleft t$. The chromatic number of a graph of the form $(\mathcal{F}, \triangleleft)$, denoted by $\chi(\mathcal{F}, \triangleleft)$, is the least $k \leqslant \aleph_0$ such that there is a partition of $\mathcal{F}$ into k-many independent sets.

Recall that a family $\mathcal{F}$ of finite subsets of ω is precompact if its topological closure (seen as a subset of the Cantor set $2^{\mathbb{N}}$) contains only finite sets. It is *thin* if no element of $\mathcal{F}$ is a proper initial segment of another element of $\mathcal{F}$. Recall also that $\mathcal{F}$ is a *front* on an infinite set $X \subseteq \omega$ if $\mathcal{F}$ is thin and every infinite $Y \subseteq X$ has an initial segment in $\mathcal{F}$.

A family $\mathcal{F}$ is a barrier on an infinite set $X \subseteq \omega$ if no element of $\mathcal{F}$ is a proper subset of another element of $\mathcal{F}$ and every infinite $Y \subseteq X$ has an initial segment in $\mathcal{F}$.

A family $\mathcal{F}$ of finite subsets of ω is Ramsey if for every finite partition $\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_k$ and for every infinite $A \subseteq \omega$ there is an infinite $B \subseteq A$ such that at most one of the restrictions $\mathcal{F}_1 \upharpoonright B, \ldots, \mathcal{F}_k \upharpoonright B$ is nonempty. Here, $\mathcal{F} \upharpoonright Y := \mathcal{F} \cap \mathcal{P}(Y)$.

Theorem 9.1. [17] *Every thin family of finite subsets of* ω *is Ramsey.*

If $\mathcal{F}$ is a family of finite sets of natural numbers, $n \in \omega$, and $A \subseteq \omega$, then

$$\mathcal{F}_{\{n\}} = \{t : n < \min(t) \text{ and } \{n\} \cup t \in \mathcal{F}\}$$
$$\mathcal{F} \upharpoonright A = \{s \in \mathcal{F} : s \subseteq A\}.$$

Graphs defined by the shift relation $\triangleleft$ on families of finite sets of natural numbers are interesting objects that appear in different contexts (see for example, [1, 2]). It is shown in [7] that the study of graphs of this kind that have infinite chromatic number can be restricted to graphs defined on precompact families which are thin.

It is not difficult to find families of finite subsets of $\mathbb{N}$ for which the shift graph obtained has infinite chromatic number. For example, we have the following propositions of [7].

Proposition 9.1. *If* $\mathcal{F}$ *is a front on an infinite set* X *of natural numbers, then the chromatic number of the shift graph* $(\mathcal{F}, \triangleleft)$ *is* $\aleph_0$.

Proposition 9.2. *For a family $\mathcal{F} \subseteq [\omega]^2$,*

$$\chi(\omega, \mathcal{F}) = \infty \text{ if and only if } \chi(\mathcal{F}, \lhd) = \infty.$$

We end the section with some questions that relate chromatic properties of graphs defined on precompact families of finite subsets of $\mathbb{N}$ and the problem of characterizing when the shift graph defined on a Borel subset of $\mathbb{N}^{[\infty]}$ has infinite Borel chromatic number (see [7]). First some definitions.

Definition 9.1. *Given a family $\mathcal{F} \subseteq [\omega]^{<\omega}$, define*

(1) $\mathcal{F}^{[1]} = \mathcal{F}$, *and*
(2) $\mathcal{F}^{[k+1]} = \{s \cup t : s \in \mathcal{F}, t \in \mathcal{F}^{[k]}, s \lhd t\}$

Lemma 9.1. *For every thin family $\mathcal{F} \subseteq \mathbb{N}^{[<\infty]}$, $\chi(\mathcal{F}, \lhd) = \infty$ implies $\chi(\mathcal{F}^{[k]}, \lhd) = \infty$ for every positive integer k.*

$\mathcal{F}^{[<\infty]}$ is the family of all finite sets of the form $s_0 \cup s_1 \cup \cdots \cup s_n$ where $n \in \mathbb{N}$ and $s_0 \lhd s_1 \lhd s_1 \lhd \cdots \lhd s_n$ is a finite shift-increasing sequence of elements of $\mathcal{F}$. We consider $\mathcal{F}^{[<\infty]}$ as a tree ordered by the relation $\sqsubseteq$ of end-extension. By $[\mathcal{F}^{[<\infty]}]$, we denote the collection of all infinite branches of this tree which we identify with the collection of all infinite subsets X of ω all of whose finite initial segments belong to the tree $\mathcal{F}^{[<\infty]}$. Clearly, $[\mathcal{F}^{[<\infty]}]$ is a closed subset of $[\omega]^\omega$. A *front* of $\mathcal{F}^{[<\infty]}$ is an antichain $\mathcal{H}$ of the tree $\big(\mathcal{F}^{[<\infty]}, \sqsubseteq\big)$ (and, therefore, a thin family of finite subsets of ω) with the property that every branch $X \in [\mathcal{F}^{[<\infty]}]$ has an initial segment in $\mathcal{H}$.

We finish with four natural questions that we think are relevant to any further study in this subject.

Question 9.1. *Suppose $\mathcal{F}$ is a thin precompact family of finite subsets of ω such that $\chi(\mathcal{H}, \lhd) = \infty$ for every front $\mathcal{H}$ of its tree $\big(\mathcal{F}^{[<\infty]}, \sqsubseteq\big)$. Is $\chi_{\mathcal{B}}\big([\mathcal{F}^{[<\infty]}], S\big) = \infty$?*

Question 9.2. *Suppose $\mathcal{F}$ is a thin precompact family of finite subsets of ω such that $\chi(\mathcal{H}, \lhd) = \infty$ for every front $\mathcal{H}$ of its tree $\big(\mathcal{F}^{[<\infty]}, \sqsubseteq\big)$. Does there exist a barrier $\mathcal{B}$ on ω and a mapping $f : \mathcal{B} \to \mathcal{F}$ such that $s \lhd t$ implies $f(s) \lhd f(t)$?*

Question 9.3. *Suppose that for some closed set $X \subseteq \mathbb{N}^{[\infty]}$ we have $\chi_{\mathcal{B}}(X, S) = \infty$. Does there exist a precompact thin family $\mathcal{F} \subseteq \mathbb{N}^{[<\infty]}$ such that*

$$\chi_{\mathcal{B}}\big([\mathcal{F}^{[<\infty]}], S\big) = \infty \text{ and } [\mathcal{F}^{[<\infty]}] \subseteq X?$$

Question 9.4. *Suppose $\mathcal{O} \subseteq \mathbb{N}^{[\infty]}$ is an open set with the property $\chi_{\mathcal{B}}(\mathcal{O}, S) = \infty$ and let $\mathcal{F} \subseteq \mathbb{N}^{[<\infty]}$ be a thin family such that $\mathcal{O} = \bigcup_{s \in \mathcal{F}}[s]$. By Proposition 2.6 of [7] there is a precompact $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $\chi(\mathcal{F}_0, \lhd) = \infty$. Can one choose such $\mathcal{F}_0$ to have the stronger property the continuous chromatic number of $\big([\mathcal{F}_0^{[<\infty]}], S\big)$ is infinite?*

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