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FAMILIES OF FINITE SUBSETS OF $\mathbb{N}$

Abstract. This is an overview of results about families of finite sets of integers. We first examine the fine structure theory for such families and then relate these to the corresponding applications.

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1. Introduction

Results about families of finite subsets of integers show up naturally in many fields of mathematics. From a structural point of view, they have been studied in infinite combinatorics. Perhaps the most well-known result of this type is a result of Ramsey that gives us a pigeonhole for families of the form $[\mathbb{N}]^k := \{s \subseteq \mathbb{N} : \#s = k\}$. In a naive way, one would like to know how any such family $\mathcal{F}$ looks like. The first thing to do is to clarify what we mean by "looks like". As Ramsey result suggests, more than the family $\mathcal{F}$ we will study its version in some infinite subset M of $\mathbb{N}$. There are two natural versions of $\mathcal{F}$ in M. The first one is the *restriction* $\mathcal{F} \upharpoonright M$ of $\mathcal{F}$ to M, and it consists of all elements of $\mathcal{F}$ which are subsets of M. The second is the *trace* $\mathcal{F}[M]$ of $\mathcal{F}$ to M, whose elements are all intersections $s \cap M$ for $s \in \mathcal{F}$. The next to do is to fix the properties to study. There are two basic relations between subsets of $\mathbb{N}$, the inclusion relation $\subseteq$ and the initial part relation $\sqsubseteq$. Related to them, one studies when families are antichains with respect to $\subseteq$ or to $\sqsubseteq$, called *Sperner* and *thin*, respectively, or when they are *hereditary* in both senses (see the beginning of the next section for the definitions). Interestingly, the main tool for this study is the Ramsey property of a family. This was first discovered by Nash-Williams [Na], who made clear the relation between the Sperner and the Ramsey properties, and introduced the central concept of a *barrier*. We expose here this work and what we call the structural theorem for restrictions, together with contribution of Pudlák and Rödl on uniform families.

The next part is devoted to study when families are hereditary. It turns out that small families (such as, for example the precompact families) always have some hereditary trace. This is not the case for an arbitrary family, as it can be easily seen by considering simple families of finite intervals of ℕ. Nevertheless, in our structural theorem on traces, we expose a trichotomy that in particular explains when traces of a family are never hereditary. We present this as a consequence of auxiliary results on families of finite sets of doubletons.

Although restrictions and traces of families give fundamental information of them, they obviously do not fully characterize the families. In Subsection 3.2 we give an example of a compact and hereditary family $\mathcal{F}$ where traces are hereditarily very simple, as they are of the form $[N]^{\leqslant 3}$, but globally $\mathcal{F}$ is equivalent to the *Schreier* family and, moreover, $\mathcal{F}$ has a particular density property which the family $[N]^{\leqslant 3}$ is far to have.

In the last part of the paper, we go one step further and we consider families consisting on vectors of c_0. This choice is not arbitrary, as hereditary properties on this new families correspond to the well-know properties of weakly-null sequences in Banach spaces. Among the results that we present there are Mazur's Lemma, Odell's Schreier unconditionality result, and Rosenthal's ℓ_1-Theorem.

Finally, we want to mention that most of the work presented here comes from several collaborations with Stevo Todorcevic listed in our reference list.

2. Families of finite subsets of ℕ

By a *family* we mean a collection of finite subsets of ℕ. We say that $\mathcal{F}$ is *hereditary* when $s \subseteq t \in \mathcal{F}$ implies that $s \in \mathcal{F}$. The family $\mathcal{F}$ is $\sqsubseteq$-*hereditary* (hereditary for initial subsets) when $s \sqsubseteq t \in \mathcal{F}$ implies that $t \in \mathcal{F}$. On the opposite direction, a family is called *Sperner* when there are no $s \subsetneq t$ both in $\mathcal{F}$. The family $\mathcal{F}$ is *thin* when there are no $s \sqsubset t$ both in $\mathcal{F}$.

Given $J \subseteq I$, let

$$\mathcal{F} \upharpoonright J := \{s \in \mathcal{F} : s \subseteq J\} = \mathcal{F} \cap \mathcal{P}(J) \text{ and } \mathcal{F}[J] := \{s \cap J : s \in \mathcal{F}\}.$$

$\mathcal{F} \upharpoonright J$ and $\mathcal{F}[J]$ are in general different families. It is clear from definition that $\mathcal{F} \upharpoonright J \subseteq \mathcal{F}[J]$; but, in contrast to $\mathcal{F} \upharpoonright J$, the traces $\mathcal{F}[J]$ has tendency of being hereditary. To see this, let $\mathcal{F} := [\mathbb{N}]^n$, let $J := 2\mathbb{N}$. Then $\mathcal{F} \upharpoonright J = [J]^n$ which is Sperner, while $\mathcal{F}[J] = [J]^{\leqslant n}$ is hereditary. Recall that the Classical Ramsey theorem states that if we partition $[\mathbb{N}]^n$ into finitely many colors then one of the colors must contain some $[N]^n$. Generalizing this property, Nash-Williams introduced in [Na] the following. A family $\mathcal{F}$ is *Ramsey* when for every partition $\mathcal{F} = \mathcal{F}_0 \cup \cdots \cup \mathcal{F}_n$ there is M such that there is at most one $0 \leqslant i \leqslant n$ such that $\mathcal{F}_i \upharpoonright M$ is nonempty. It is easy to see that hereditary families does not have, in general the Ramsey property (consider the parity of cardinality as coloring). One of the main achievements in this field is the work of Nash-Williams where the Ramsey and the Sperner properties are proved to be the same. We call this result the First Structural Theorem (see Theorem 2.2). The proof we present here is not the original one from [Na]. Instead, we will combine ideas by Pudlák and Rödl and some basic topological techniques.

Given two sets of integers A and B, we write $A < B$ to denote that $\max A < \min B$, and $A/B := \{n \in A : \max B < n\}$. Recall that a family $\mathcal{F}$ of finite sets of $\mathbb{N}$ has a natural topology by simply identifying a set $s \in \mathcal{F}$ with its characteristic function $\mathbb{1}_s \in 2^{\mathbb{N}}$, and then using the product topology in $2^{\mathbb{N}}$. It is not difficult to see that every sequence $(s_n)_n$ of finite subsets of $\mathbb{N}$ has a subsequence $(s_n)_{n\in M}$ forming a Δ-system with root r, that is $s_m \cap s_n = r$ for every $m < n$ in M and $r < (s_m \setminus r) < (s_n \setminus r)$ for every $m < n$ in M, or a subsequence $(s_n)_{n\in M}$ and some infinite subset $L \subseteq \mathbb{N}$, such that for every $t \sqsubseteq L$ there is n_t such that $t \sqsubseteq s_n$ for every $n \geqslant n_t$. Observe that in the first case, $(s_n)_{n\in M}$ converges to t, while in the second one $(s_n)_{n\in M}$ converges to L. The (topological) closure $\overline{\mathcal{F}}$ of a family $\mathcal{F}$ is simply $\mathcal{F}$ together with the limit sets of sequences in $\mathcal{F}$.

Definition 2.1. A family $\mathcal{F}$ in $\mathbb{N}$ is called *precompact* when $\overline{\mathcal{F}}$ consists only of finite subsets of $\mathbb{N}$.

Equivalently, a family $\mathcal{F}$ is precompact when every sequence in $\mathcal{F}$ has a subsequence forming a Δ-system.

There are various ways to associate an ordinal index to a precompact family $\mathcal{F}$ of finite subsets of $\mathbb{N}$. For example, one may consider the *Cantor–Bendixson index* $r(\mathcal{F})$, the minimal ordinal α for which the iterated *Cantor–Bendixson derivative* $\partial^\alpha(\mathcal{F})$ is equal to $\emptyset$. Recall that $\partial\mathcal{F}$ is the set of all proper accumulation points of $\mathcal{F}$ and that $\partial^\alpha(\mathcal{F}) = \bigcap_{\xi<\alpha} \partial(\partial^\xi(\mathcal{F}))$. Observe that $\partial(\mathcal{F}) = \partial(\overline{\mathcal{F}})$. So, the Cantor–Bendixson index is well defined since $\overline{\mathcal{F}}$ is countable and therefore a scattered compactum and consequently the sequence $(\partial^\xi(\mathcal{F}))_\xi$ of iterated derivatives must vanish for some countable ordinal ξ. Observe that if $\mathcal{F}$ is a nonempty compact, then necessarily $r(\mathcal{F}) = r(\overline{\mathcal{F}})$ is a successor ordinal.

2.1. Uniform families. Mappings on uniform families.

We pass now to present the families introduced by Pudlák and Rödl in [Pu-Ro].

Definition 2.2. Given a countable ordinal α, the family $\mathcal{F}$ is called *α-uniform on M* provided that:
(a) $\alpha = 0$ implies $\mathcal{F} = \{\emptyset\}$,
(b) $\alpha = \beta + 1$ implies that $\mathcal{F}_{\{n\}} := \{s \in \mathrm{FIN} : n < s,\ \{n\} \cup s \in \mathcal{F}\}$ is β-uniform on M/n,
(c) $\alpha > 0$ limit implies that there is an increasing sequence $\{\alpha_n\}_{n\in M}$ of ordinals converging to α such that $\mathcal{F}_{\{n\}}$ is α_n-uniform on M/n for all $n \in M$.

$\mathcal{F}$ is called uniform on M if it is α-uniform on M for some countable ordinal α. This ordinal α is called the uniform rank of $\mathcal{F}$.

Examples of uniform families are $[\mathbb{N}]^n := \{s \in \mathrm{FIN} : \#s = n\}$ which is n-uniform, or $\mathfrak{S} := \{s \in \mathrm{FIN} : \#s = \min s + 1\}$ which is ω-uniform, and it is usually called the Schreier barrier. Its $\subseteq$-closure $\mathcal{S} := \{s \in \mathrm{FIN} : \#s \leqslant \min s + 1\}$ is called the *Schreier family.*

It follows from the definition that if $\mathcal{F}$ is uniform on M, then for every finite set t the family $\mathcal{F}_t := \{u : t < u \text{ and } t \cup u \in \mathcal{F}\}$ is uniform on M/t, and that $\mathcal{F} \restriction N$ is uniform on N for every $N \subseteq M$. The following is proven easily by induction on the uniform rank of a uniform family.

Proposition 2.1. *Every uniform family is Ramsey.* $\square$

A very important property of a uniform family $\mathcal{F}$ on a given set M is that every infinite subset of M has an initial part that belongs to $\mathcal{F}$. This can be easily proved by induction on the uniform rank of the family. Another property of $\mathcal{F}$ is that it is thin, also proved by induction. Families with these two properties are called front.

Definition 2.3. A family $\mathcal{F} \subseteq \mathcal{P}(M)$ is called a front in M when $\mathcal{F}$ is thin and every infinite subset of M has a (unique) initial part in $\mathcal{F}$. If in addition $\mathcal{F}$ is Sperner, then the family $\mathcal{F}$ is called a *barrier* on M.

Proposition 2.2. *Fronts are precompact. In fact, $\overline{\mathcal{B}} = \overline{\mathcal{B}}^{\sqsubseteq}$ for every front $\mathcal{B}$ in M.*

Proof. Suppose that $\mathcal{B}$ is a front on M, and let $t \in \overline{\mathcal{B}}^{\sqsubseteq} \setminus \mathcal{B}$. Let $s_0 \in \mathcal{B}$ be such that $t \sqsubset s_0$. Now let $s_1 \in \mathcal{B}$ be such that $s_1 \sqsubseteq t \cup (M/s_0)$. By the thin property of $\mathcal{B}$, $t \sqsubset s_1$, and so on. In this manner we can find a sequence $(s_n)_n$ in $\mathcal{B}$ forming a Δ-system with root t. Now suppose that $A \in \overline{\mathcal{B}}$. We see first that A is finite; otherwise, using that $\mathcal{B}$ is a front on M, we can find $s \sqsubseteq A$, $s \in \mathcal{B}$. Let $t \in \mathcal{B}$ be such that $t \cap [0,n] = A \cap [0,n]$, where $n = \min M/s$. It follows that $s \sqsubset t$, contradicting the fact that $\mathcal{B}$ is thin. It is clear now that $\overline{\mathcal{B}} \subseteq \overline{\mathcal{B}}^{\sqsubseteq}$. $\square$

Uniform families are comparable in the following sense:

Proposition 2.3. *Suppose that $\mathcal{B}$ and $\mathcal{C}$ are uniform on some M. Then there is some $N \subseteq M$ such that either $\mathcal{B} \upharpoonright N \subseteq \overline{(\mathcal{C} \upharpoonright N)}^{\sqsubseteq}$ or $\mathcal{C} \upharpoonright N \subseteq \overline{(\mathcal{B} \upharpoonright N)}^{\sqsubseteq}$.*

Proof. Color each element s of $\mathcal{B}$ by 1 if there is some $t \in \mathcal{C}$ such that $t \sqsubseteq s$ and by 0 otherwise, and similarly color the elements of $\mathcal{C}$ with respect to $\mathcal{B}$. By the Ramsey property of $\mathcal{B}$, there is some $N \subseteq M$ where both colorings are constant with values $\varepsilon_0, \varepsilon_1 \in \{0,1\}$ respectively. Observe that $\varepsilon_0 = \varepsilon_1 = 0$ is impossible: Let $s \in \mathcal{B}$ and $t \in \mathcal{C}$ be such that $s, t \sqsubseteq N$. Hence either $s \sqsubseteq t$, contradicting the color 0 of t, or $t \sqsubseteq s$, contradicting the color 0 of s. Suppose that $\varepsilon_0 = \varepsilon_1 = 1$. Then, given $s \in \mathcal{B} \upharpoonright N$ there is $t \sqsubseteq s$ and $u \in \mathcal{B} \upharpoonright N$ such that $u \sqsubseteq t$. This means that $s = u = t$, and consequently $\mathcal{B} \upharpoonright N \subseteq \mathcal{C} \upharpoonright N$ and similarly $\mathcal{B} \upharpoonright N = \mathcal{C} \upharpoonright N$. The cases $\varepsilon_0 = 0, \varepsilon_1 = 1$ and $\varepsilon_0 = 1, \varepsilon_1 = 0$ are symmetric one of the other. So, suppose that ε_0 and $\varepsilon_1 = 1$. We claim that $\mathcal{B} \upharpoonright N \subseteq \overline{\mathcal{C} \upharpoonright N}^{\sqsubseteq}$. Fix $s \in \mathcal{B} \upharpoonright N$, and let $t \in \mathcal{C}$ be such that $t \sqsubseteq s \cup (N/s)$. It follows that $t \sqsubseteq s$ or $s \sqsubseteq t$, and the first case is impossible because $\varepsilon_0 = 0$. $\square$

Proposition 2.4. *Suppose that $\mathcal{F}$ is a α-uniform family on M. Then $\partial^{(\alpha)}(\overline{\mathcal{F}}) = \{\emptyset\}$ and consequently, $r(\mathcal{F}) = \alpha + 1$.*

To prove this fact, we use the following.

Proposition 2.5. *Suppose that $\mathcal{F}$ is precompact. Then for every n and every $\alpha < \omega_1$ one has that $\partial^{(\alpha)}(\mathcal{F}_{\{n\}}) = (\partial^{(\alpha)}(\mathcal{F}))_{\{n\}}$. Consequently, for every $A \subseteq \mathbb{N}$ and every α, $A \in \partial^{(\alpha)}(\overline{\mathcal{A}})$ if and only if $A \setminus \{\min A\} \in \partial^{(\alpha)}\big(\overline{\mathcal{A}_{\{\min A\}}}\big)$.*

Proof. To simplify the notation, let us assume that $M = \mathbb{N}$. For each n, $\mathcal{F}_{\{n\}}$ is α_n-uniform on $\mathbb{N}/n$, where $\alpha_n+1 = \alpha$ for every n if α is successor, or $(\alpha_n)_n$ is increasing and $\sup_n \alpha_n = \alpha$ when α is limit. By inductive hypothesis, $\partial^{(\alpha_n)}(\mathcal{F}_{\{n\}}) = \{\emptyset\}$ for every n. Let us prove first that $\partial^{(\alpha)}(\mathcal{F}) \subseteq \{\emptyset\}$. Otherwise, let $\emptyset \subsetneq s \in \partial^{(\alpha)}(\mathcal{F})$, and let $n := \min s$. Then, by Proposition 2.5, $s \setminus \{n\} \in \partial^{(\alpha)}(\mathcal{F}_{\{n\}}) = \emptyset$, which is impossible. Let us prove now that $\emptyset \in \partial^{(\alpha)}(\mathcal{F})$. Since $\emptyset \in \partial^{(\alpha_n)}(\mathcal{F}_{\{n\}})$ for each n, it follows from Proposition 2.5 that $\{n\} \in \partial^{(\alpha_n)}(\mathcal{F})$ for every n. Since $\partial^{(\alpha_n)}(\mathcal{F}) \subseteq \partial^{(\alpha_m)}(\mathcal{F})$ for every $n \geqslant m$, it follows then that $\emptyset = \lim_{n\to\infty}\{n\} \in \bigcap_{n\in\mathbb{N}} \partial(\partial^{(\alpha_n)}(\mathcal{F})) = \partial^{(\alpha)}(\mathcal{F})$. □

Definition 2.4. A mapping $\varphi : \mathcal{F} \to \mathrm{FIN}$ defined on a family $\mathcal{F}$ is called *is uniform* when for every $s, u \in \mathcal{F}$ and every $t \sqsubseteq s, u$ we have that

$$\min(s \setminus t) \in \varphi(s) \Leftrightarrow \min(u \setminus t) \in \varphi(u).$$

We say that φ is *Lipschitz* iff for every $s, u \in \mathcal{F}$, if $t \sqsubseteq s, u$ then $\varphi(s)\cap t = \varphi(u)\cap t$.

So, uniform mappings are those $\varphi : \mathcal{F} \to \mathrm{FIN}$ such that given $s \in \mathcal{F}$ and $n \in s$, the value of $\chi_{\varphi(s)}(n) \in \{0, 1\}$ only depends on the initial part $s \cap [0, n)$ of s, while Lipschitz mappings are those that the value of $\varphi(s) \restriction t$ only depends on t for every $t \sqsubseteq s \in \mathcal{F}$. The notion of Lipschitzness has a natural metric interpretation when we consider in FIN the standard distance d defined by $d(s, t) = 1/2^{\min(s\triangle t)}$, where $s\triangle t = (s\setminus t)\cup(t\setminus s)$ is the *symmetric difference* of s and t. This metric defines the topology on FIN we explained in the introduction. With this metric it is easy to see that the Lipschitz notion defined above coincides with the metric 1-Lipschitz condition associated to d. The notion of uniformness has more combinatorial nature. It is easy to see that uniform mappings are always Lipschitz.

The following generalizes the Ramsey property of uniform families.

Proposition 2.6. *Suppose that $\mathcal{B}$ is uniform on M and $\varphi : \mathcal{B} \to \mathrm{FIN}$ is an arbitrary mapping. Then there is $N \subseteq M$ such that $\varphi \restriction (\mathcal{B} \restriction N)$ is uniform.*

Proof. By the Ramsey property of $\mathcal{B}_t$ for every $t \in [M]^{<\infty}$ and using a simple diagonal one can find an infinite subset $N \subseteq M$ such that for every $t \in [N]^{<\infty}$ the mappings $f_t : \mathcal{B}_t \restriction N \to \{0, 1\}$ defined by $f_t(u) = \chi_{\varphi(t\cup u)}(\min u)$ are all constant. This means that N has the property required. □

A consequence of this is that the selection of an initial part of every element of a uniform family, defines essentially a uniform family. More precisely,

Corollary 2.1. *Suppose that $\mathcal{B}$ is uniform on M and suppose that $\varphi : \mathcal{B} \to \mathrm{FIN}$ is such that $\varphi(s) \sqsubseteq s$ for every $s \in \mathcal{B}$. Then there is $N \subseteq M$ such that $\varphi''(\mathcal{B} \restriction N)$ is a uniform on N.*

Proof. The proof is done by induction on the uniform rank of $\mathcal{B}$. Let $N \subseteq M$ be such that φ is uniform on $\mathcal{B} \restriction N$ and such that either $\varphi(s) = \emptyset$ for all $s \in \mathcal{B} \restriction N$ or $\varphi(s) \neq \emptyset$ for all $s \in \mathcal{B} \restriction N$. In the first case $\varphi''(\mathcal{B} \restriction N) = \{\emptyset\}$ is a uniform family on N. Suppose that $\varphi(s) \neq \emptyset$ for all $s \in \mathcal{B} \restriction N$. For each $n \in N$, let $\varphi_n : \mathcal{B}_{\{n\}} \to \mathrm{FIN}$ be defined by $\varphi_n(t) := \varphi(\{n\} \cup t) \setminus \{n\}$ for each $t \in \mathcal{B}_{\{n\}}$.

By inductive hypothesis, and the fact that φ (hence φ_n) is uniform, we can find $P \subseteq N$ such that $\varphi_n''(\mathcal{B}_n \upharpoonright P/n)$ is α_n-uniform on P/n for every $n \in P$. Now choose $Q \subseteq P$ such that either $(\alpha_n)_{n\in Q}$ is constant with value α, or else $(\alpha_n)_{n\in Q}$ is strictly increasing, with limit α. Since by uniformness of φ we have that

$$\varphi_n''(\mathcal{B}_n \upharpoonright P/n) \upharpoonright Q/n = \varphi_n''(\mathcal{B}_n \upharpoonright Q/n),$$

we obtain that $\varphi''(\mathcal{B} \upharpoonright Q)$ is $\alpha + 1$-uniform on Q in the first case, or α-uniform on Q, in the second one. □

Definition 2.5. We say that a mapping $\varphi : \mathcal{F} \to \mathrm{FIN}$ is *inner* when $\varphi(s) \subseteq s$ for every $s \in \mathcal{B}$.

Mapping $\varphi : \mathcal{B} \to \mathrm{FIN}$ defined on a barrier whose range is precompact is "almost" inner, as the next shows.

Lemma 2.1. *Let $\mathcal{B}$ be a barrier on M, and suppose that $\varphi : \mathcal{B} \to \mathrm{FIN}$ is such that its range is a precompact family. Then there is some infinite subset $N \subseteq M$ such that $\varphi(s) \cap N \subseteq s$ for every $s \in \mathcal{B} \upharpoonright N$.*

Proof. By changing φ with $s \mapsto \varphi(s) \setminus s$, we may assume that $\varphi(s) \cap s = \emptyset$, and we have to find $N \subseteq M$ such that $\varphi(s) \cap N = \emptyset$ for every $s \in \mathcal{B} \upharpoonright N$. Now the proof is by induction on the uniform rank of $\mathcal{B}$. Using the inductive hypothesis, we can find $M_0 \subseteq M$ such that for every $s \in \mathcal{B} \upharpoonright M_0$ one has that $\varphi(s) \cap M_0 \subseteq M_0 \cap \min s$. It follows from this, and the Ramsey property of barriers that there is $M_1 \subseteq M_0$ such that $\varphi(s) \cap M_0 = \varphi(t) \cap M_0$ provided that $\min s = \min t$ and $s, t \in \mathcal{B} \upharpoonright M_1$. For each $n \in M_1$, let $t_n := \varphi(s_n) \cap M_0$ for some (any) $s_n \in \mathcal{B} \upharpoonright M_1$ with $\min s_n = n$. Since $\varphi''(\mathcal{B})$ is precompact, there is $M_2 \subseteq M_1$ such that $(t_n)_{n\in M_2}$ forms a Δ-sequence with root r. Now find $N \subseteq M_3 \subseteq M_2$ such that $N \cap \bigcup_{n\in M_3} t_n = \emptyset$. We check that $\varphi(s) \cap N = \emptyset$ for every $s \in \mathcal{B} \upharpoonright N$: $\varphi(s) \cap N \subseteq t_n$ for $n = \min s$, and $N \cap t_n = \emptyset$. Hence $\varphi(s) \cap N = \emptyset$. □

Combining Proposition 2.6 and Lemma 2.1 we obtain the following.

Corollary 2.2. *Let $\mathcal{B}$ be a barrier on M, and suppose that $\varphi : \mathcal{B} \to \mathrm{FIN}$ is such that its range is a precompact family. Then there is some infinite subset $N \subseteq M$ such that the mapping $s \in \mathcal{B} \upharpoonright N \mapsto \varphi(s) \cap N$ is inner and uniform.* □

Perhaps the importance of inner mappings is reflected in the following structural result by Pudlák and Rödl on arbitrary mappings defined on barriers. We refer the reader to the original paper [Pu-Ro] and to [Ar-To].

Theorem 2.1 (Pudlák-Rödl)**.** *Suppose that $f : \mathcal{B} \to X$ is a mapping defined on a barrier $\mathcal{B}$ on M. Then there is $N \subseteq M$, a barrier $\mathcal{C}$ on N, and an inner and uniform mapping $g : \mathcal{B} \upharpoonright N \to \mathcal{C}$ such that for every $s, t \in \mathcal{B} \upharpoonright N$ one has that $f(s) = f(t)$ iff $g(s) = g(t)$.*

2.2. Restrictions of families. The next is the structural result on families modulo restrictions. It is due to Nash-Williams [Na], except the part about uniform families, which was done by Pudlák and Rödl [Pu-Ro].

Theorem 2.2 (Structural Theorem for Restrictions)**.** *The following are equivalent for a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$:*

(a) *There is an infinite $M \subseteq \mathbb{N}$ such that $\mathcal{F} \upharpoonright M$ is either empty or uniform on M.*
(b) *There exists an infinite $M \subseteq \mathbb{N}$ such that $\mathcal{F} \upharpoonright M$ is Ramsey.*
(c) *There is an infinite $M \subseteq \mathbb{N}$ such that $\mathcal{F} \upharpoonright M$ is Sperner.*
(d) *There is an infinite $M \subseteq \mathbb{N}$ such that $\mathcal{F} \upharpoonright M$ is thin.*
(e) *There is an infinite $M \subseteq \mathbb{N}$ such that $\mathcal{F} \upharpoonright M$ is either empty or a front on M.*
(f) *There is an infinite $M \subseteq \mathbb{N}$ such that $\mathcal{F} \upharpoonright M$ is either empty or a barrier on M.*

We note that the fact that Barriers have the Ramsey property is equivalent to the Ramsey property of clopen subsets of $\mathbb{N}^{[\infty]}$.

Proof. We have already proved that uniform families have the Ramsey property.

(b) implies (c): Suppose that $\mathcal{F}$ has the Ramsey property. Let $\mathcal{F}_0$ be the subfamily of $\mathcal{F}$ consisting of the $\subseteq$-minimal subsets of $\mathcal{F}$. By the Ramsey property of $\mathcal{F}$, there is M such that either $\mathcal{F} \upharpoonright M = \mathcal{F}_0 \upharpoonright M$ or $\mathcal{F}_0 \upharpoonright M = \emptyset$. In the first case, $\mathcal{F} \upharpoonright M$ is Sperner, and in the second one $\mathcal{F} \upharpoonright M = \emptyset$ (hence Sperner), because otherwise, if $s \in \mathcal{F} \upharpoonright M$ then there is a $\subseteq$-minimal $t \in \mathcal{F}$ which is subset of s, and then $t \in \mathcal{F}_0 \upharpoonright M$, a contradiction.

(c) implies (a): Now suppose that $\mathcal{F}$ is Sperner. If we can find $N \subseteq M$ such that $\mathcal{G} \upharpoonright N = \emptyset$, then we are done. Otherwise, $\mathcal{F}$ must be precompact; if not, let M be a limit point of $\mathcal{F}$, and let $t \in \mathcal{F} \upharpoonright M$. Let $n \in M/t$ and let $s \in \mathcal{F}$ be such that $s \sqsubseteq M$, $n \in s$. It follows that $t \subsetneq s$, contradicting the fact that $\mathcal{F}$ is Sperner. Let $\alpha := r(\mathcal{F})$. Let $\mathcal{H}$ be any α-uniform family on M. Let $\theta_0 : \mathcal{H} \to 2$ be the coloring defined by $\theta_0(s) = 0$ if and only if there is $t \in \mathcal{F}$ such that $s \sqsubseteq t$. Let $N \subseteq M$ be such that θ_0 is constant on $\mathcal{H} \upharpoonright N$ with value $\varepsilon = 0, 1$. Then $\varepsilon = 1$ because otherwise $\mathcal{H} \upharpoonright N \subseteq \overline{\mathcal{F}}^{\sqsubseteq}$, hence

$$\partial^{(\alpha)}(\mathcal{H} \upharpoonright N) \subseteq \partial^{(\alpha)}(\overline{\mathcal{G}}^{\sqsubseteq}) = \partial^{(\alpha)}(\mathcal{G}) \tag{2.1}$$

for every α. Since $\mathcal{H} \upharpoonright N$ is α-uniform on N, $r(\mathcal{H} \upharpoonright N) = \alpha + 1$, and consequently it follows from (2.1) that $r(\mathcal{F}) \geqslant \alpha + 1$, a contradiction. Let $\theta_1 : \mathcal{H} \upharpoonright N \to 2$ be the coloring defined by $\theta_1(s) = 1$ if and only if there is $t \in \mathcal{F}$ such that $t \sqsubseteq s$. Let $P \subseteq N$ and $\varepsilon \in 2$ be such that θ_2 is constant in $\mathcal{H} \upharpoonright P$ with value ε. Then $\varepsilon = 1$: We are assuming that $\mathcal{F} \upharpoonright R \neq \emptyset$ for every $R \subseteq M$. Let $t \in \mathcal{F} \upharpoonright P$. Let $Q \subseteq P$ be such that $t \sqsubseteq Q$. Since $\mathcal{H}$ is a front in M, there is some $s \in \mathcal{H}$ (in fact unique) such that $s \sqsubseteq Q$. Consequently, either $t \sqsubseteq s$ or $s \sqsubseteq t$. The second alternative is impossible as θ_0 is constant in $\mathcal{H} \upharpoonright N$ with value 1. So, $t \sqsubseteq s$, hence $\varepsilon = 1$. Now we can naturally define a mapping $\varphi : \mathcal{H} \upharpoonright P \to \mathrm{FIN}$ such that $\varphi(s) \sqsubseteq s$. By Corollary 2.1, there is some $Q \subseteq P$ such that $\varphi''(\mathcal{H} \upharpoonright Q)$ is uniform on Q. It is easy to see that $\varphi''(\mathcal{H} \upharpoonright Q) = \mathcal{F} \upharpoonright Q$.

(c) implies (d) trivially.

(d) implies (c): Suppose that $\mathcal{F}$ is thin. Let $\mathcal{F}_0$ be the family of all $\subseteq$-minimal elements of $\mathcal{F}$. Let M be such that either $\mathcal{F}_0 \upharpoonright M = \emptyset$ or $\mathcal{F}_0 \upharpoonright M$ is uniform on M. In the first case, it easily follows that $\mathcal{F} \upharpoonright M = \emptyset$. Suppose that $\mathcal{F}_0 \upharpoonright M$ is uniform on M. We claim that $\mathcal{F}_0 \upharpoonright M = \mathcal{F} \upharpoonright M$. Let $t \in \mathcal{F} \upharpoonright M$. Let $s \in \mathcal{F}_0 \upharpoonright M$ be such that $s \sqsubseteq t \cup (M/t)$. Consequently, $s \sqsubseteq t$ or $t \sqsubseteq s$. Since $\mathcal{F}$ is thin, it follows that $t = s \in \mathcal{F}_0 \upharpoonright M$.

Once we have established the equivalence of (a)–(d), equivalence (a)–(f) follows trivially. $\square$

3. Traces of families

As we have already said, the traces of families tends to be hereditary in some sense, in contrast to what happens with restrictions. Although it is not true that every family have some trace which is hereditary (take for example $\mathcal{F} := \{[0,n] : n \in \mathbb{N}\}$, or see Theorem 4.6 for a very strong counterexample), we are going to see that every family has a trace which is $\sqsubseteq$-hereditary, i.e., hereditary for initial subsets.

Theorem 3.1. *Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$. Then there is $M \subseteq \mathbb{N}$ such that $\mathcal{F}[M]$ is $\sqsubseteq$-closed.*

Proof. Suppose otherwise that for every M there is a triple (t_M, u_M, v_M) with the property that

$$\text{(a)}\ t_M \in \mathcal{F}, \quad \text{(b)}\ t_M \cap M = u_M, \quad \text{(c)}\ v_M \sqsubset u_M \text{ and } v_M \notin \mathcal{F}[M].$$

For each M let $s_M := M \cap (\max t_M + 1)$. So, s_M is the minimal initial part of M containing $t_M \cap M$. Let $\mathcal{B}$ be the set of $\sqsubseteq$-minimal elements of $\mathcal{F}$. Observe that $\mathcal{B}$ is a front: It is clear by definition that $\mathcal{B}$ is thin. On the other hand, given M we know that $s_M \sqsubseteq M$, so the $\sqsubseteq$ minimal initial part of s_M in $\mathcal{F}$ is in fact in $\mathcal{F}$ and it is clearly an initial part of M. For each $s \in \mathcal{B}$, fix $M^{(s)}$ be such that $s = s_{M^{(s)}}$ and set $(t_s, u_s, v_s) := (t_{M^{(s)}}, u_{M^{(s)}}, v_{M^{(s)}})$. Observe that $t_s \cap s = t_s \cap M^{(s)} = u_s$. Let N be such that $s \in \mathcal{B} \upharpoonright N \to v_s$ is Lipschitz. Now fix $s \in \mathcal{B} \upharpoonright N$. Let $w_1 := \{m \in s : v_s < m\}$, and $w_0 = s \setminus w_1$. This is well defined because $v_s \sqsubset t_s \cap M^{(s)} \subseteq s$. Let $\bar{s} \in \mathcal{B} \upharpoonright N$ be such that $w_0 \sqsubseteq \bar{s}$ and $s < \bar{s} \setminus w_0$. By the Lipschitzness, we have that $v_{\bar{s}} = v_s$. Since in addition $\bar{s} \sqsubseteq M^{(\bar{s})}$ it follows that

$$t_s \cap M^{(\bar{s})} = t_s \cap \bar{s} = t_s \cap s \cap \bar{s} = u_s \cap \bar{s} = (u_s \cap w_0) \cup (u_s \cap (\bar{s} \setminus w_0)) = v_s = v_{\bar{s}}.$$

Consequently, $v_{M^{(\bar{s})}} = v_{\bar{s}} \in \mathcal{F}[M^{(\bar{s})}]$, which is contradictory with (c) above. $\square$

Examples of families without hereditary traces are collections of intervals.

Definition 3.1. A subset $s \subseteq M$ is called an M-interval when $s = [\min s, \max s] \cap M$.

The following classifies families of intervals.

Proposition 3.1. *Suppose that $\mathcal{F}$ is a family of intervals of some M. Then there is $N \subseteq M$ such that one and only one of the following occurs:*

(a) $\mathcal{F}[N] = \{\emptyset\}$.
(b) $\mathcal{F}[N] = [N]^{\leqslant 1}$.
(c) $\mathcal{F}[N] = \{t \in \mathrm{FIN} : t \sqsubset N\}$.
(d) $\mathcal{F}[N] = \{t \in \mathrm{FIN} : t \sqsubset N\} \cup [N]^{\leqslant 1}$.
(e) $\mathcal{F}[N] = \textsc{Int}(N)$.

Proof. Let c be the color of pairs of M that assign to each pair s color $c(s) = 0$ if there is some $t \in \mathcal{F}$ such that $s \subseteq t$ and $c(s) = 1$ otherwise. Let $N \subseteq M$ be such that $c \restriction [N]^2$ is constant with color $\varepsilon = 0, 1$. Suppose that $\varepsilon = 1$. Then $\mathcal{F}[N] \subseteq [N]^{\leqslant 1}$. It is not difficult to find $P \subseteq N$ such that $\mathcal{F}[P] = \{\emptyset\}$ or $\mathcal{F}[P] = [P]^{\leqslant 1}$. Suppose that $\varepsilon = 0$. It is easy to find $P \subseteq N$ such that $\{t \in \mathrm{FIN} : t \sqsubset P\} \subseteq \mathcal{F}[P]$. Now color triples s of P by $d(s) = 0$ if there is $t \in \mathcal{F}$ such that $_*s \subseteq t$ and $\min s \notin t$, and by $d(s) = 1$ otherwise. Let $Q \subseteq P$ be such that d is constant in $[Q]^3$ with value $\delta = 0, 1$. Now if $\delta = 1$ then $\mathcal{F}[Q] = \{s \in \mathrm{FIN} : s \sqsubset Q\}$ or $\mathcal{F}[Q] = \{s \in \mathrm{FIN} : s \sqsubset Q\} \cup [Q]^{\leqslant 1}$. If otherwise $\delta = 0$, then it is easy to find $R \subseteq Q$ such that $\mathcal{F}[R] = \textsc{Int}(R)$. □

Note that the five possibilities in the previous proposition are preserved by taking further traces.

3.1. Traces of precompact families.

We start with the following structural result for traces of precompact families.

Theorem 3.2. *Suppose that $\mathcal{F} \subseteq \mathrm{FIN}$ is a precompact family. Then for every N there is an infinite set $M \subseteq N$ such that $\mathcal{F}[M]$ is the closure of a uniform barrier on M, and in particular hereditary.*

In order to understand traces of precompact families is essential to study the traces of barriers.

Proposition 3.2. *For every $N \subseteq M$ such that $M \setminus N$ is infinite we have that $\mathcal{B}[N] = \overline{\mathcal{B} \restriction N}$, and in particular $\mathcal{B}[N]$ is downwards closed.*

Proof. Let $s \in \mathcal{B}[N]$, and let $t \sqsubseteq s \cup (N/s)$, $t \in \mathcal{B}$. It follows that $s \sqsubseteq t$, since otherwise, $t \sqsubset s \subseteq u \in \mathcal{B}$, contradicting the fact that $\mathcal{B}$ is Sperner. We have seen in Proposition 2.2 that $\overline{\mathcal{B} \restriction N} = \overline{\mathcal{B} \restriction N}^{\sqsubseteq}$. So, let $t \in \overline{\mathcal{B} \restriction N}^{\sqsubseteq}$. Let $s \in \mathcal{B}$ be such that $s \sqsubseteq t \cup ((M \setminus N)/t)$. It follows that $s \cap M = t$. □

Proposition 3.3. *Suppose that $\mathcal{F}$ is an arbitrary family of finite subsets of $\mathbb{N}$. Then for every M there is $N \subseteq M$ such that $\mathcal{F} \restriction N$ has the property that for every $t < m < n$ with $t \cup \{m, n\} \subseteq N$, one has that $t \cup \{m\} \in \mathcal{F}$ iff $t \cup \{n\} \in \mathcal{F}$.*

Proof. Otherwise, fix M and choose for each $N \subseteq M$ $t_N < m_N \neq n_N$ such that $t_M \cup \{m_N, n_N\} \subseteq N$, $t_M \cup \{m_N\} \in \mathcal{F}$ and $t_M \cup \{n_N\} \in \mathcal{F}$. Let $\mathcal{G}$ be the set of $\subseteq$-minimal elements of $\{t_N \cup \{m_N, n_N\}\}_{N \subseteq M}$. It follows from the first structural theorem that there is $N \subseteq M$ such that $\mathcal{B} := \mathcal{G} \restriction N$ is a barrier on N. Write each $s \in \mathcal{B}$ as $t = t_s \cup \{k_s, l_s\}$ where $t_s < k_s < l_s$. By the Ramsey property of $\mathcal{B}$, and the Corollary 2.1 there is $P \subseteq N$ such that

(a) either for every $s \in \mathcal{B} \restriction P$ one has that $t_s \cup \{k_s\} \in \mathcal{F}$ and $t_s \cup \{l_s\} \notin \mathcal{F}$ or for every $s \in \mathcal{B} \restriction P$ one has that $t_s \cup \{k_s\} \notin \mathcal{F}$ and $t_s \cup \{l_s\} \in \mathcal{F}$.

(b) $\{t_s : s \in \mathcal{B} \restriction P\}$ is a barrier on P.

Now fix $s \in \mathcal{B} \restriction P$. And let $u \in \mathcal{B} \restriction P$ be such that $u \sqsubseteq t_s \cup \{l_s\} \cup (P/s)$. Since t_s and t_u are $\sqsubseteq$-comparable and $\{t_v\}_{v \in \mathcal{B} \restriction P}$ is a barrier on P, it follows that $t_s = t_u$. Hence, $l_s = k_u$, which is a contradiction by (a). □

Proof of Theorem 3.2. Fix $\mathcal{F}$ and M. Suppose first that $\mathcal{F}$ is hereditary. Let $N \subseteq M$ be such that $\mathcal{F} \upharpoonright N$ has the property exposed in Proposition 3.3. Let $\mathcal{B}$ be the set of $\sqsubseteq$-maximal elements in $\mathcal{F} \upharpoonright N$. Because of the property of $\mathcal{F} \upharpoonright N$ in Proposition 3.3, $\mathcal{B}$ is a front in N. Let $P \subseteq N$ be such that $\mathcal{B} \upharpoonright P$ is a uniform barrier on P. By the $\sqsubseteq$-maximality of elements in $\mathcal{B}$ and the fact that $\mathcal{F}$ is hereditary, one has that $\mathcal{F}[P] = \mathcal{F} \upharpoonright P = \overline{\mathcal{B}}^{\subseteq}$, as desired. Now let $\mathcal{F}$ be an arbitrary precompact family. Set $\mathcal{G} := \overline{\mathcal{F}}^{\subseteq}$. We have just proved that there is some $N \subseteq M$ and some uniform barrier $\mathcal{B}$ on N such that $\mathcal{G}[N] = \overline{\mathcal{B}}$. Then

$$\mathcal{B} \subseteq \mathcal{F}[N] \subseteq \overline{\mathcal{B}}: \tag{3.1}$$

The first inclusion is clear because $\mathcal{F} \subseteq \mathcal{G}$. For the second, we fix an arbitrary $s \in \mathcal{B}$. Let $t \in \mathcal{F}$ be such that $s \subseteq t$. Then $s \subseteq t \cap N \in \mathcal{F}[N] \subseteq \overline{\mathcal{B}}$. Hence, $s = t \cap N$, and we are done.

Now we use Proposition 3.2 to find $P \subseteq N$ such that $\overline{\mathcal{B} \upharpoonright N} = \mathcal{B}[N]$. By (3.1) we obtain that

$$\overline{\mathcal{B} \upharpoonright P} = \mathcal{B}[P] \subseteq \mathcal{F}[N][P] = \mathcal{F}[P] \subseteq \overline{\mathcal{B}} \upharpoonright P = \overline{\mathcal{B} \upharpoonright P}. \qquad \square$$

The following explains traces of an arbitrary family.

Theorem 3.3 (Structural Theorem for Traces)**.** *Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$. Then for every N there exists $M \subseteq N$ such that $\mathcal{F}[M]$ is $\sqsubseteq$-hereditary and one, and only one, of the following three alternatives holds:*

(a) $\mathcal{F}[M] = [M]^{<\omega}$.
(b) $\mathcal{F}[M]$ *is the closure of a barrier in* M.
(c) $M \in \overline{\mathcal{F}[M]} \setminus \mathrm{FIN} \subseteq \{N \in [M]^{\infty} : M \setminus N$ *is finite*$\}$.

It follows that if $\mathcal{F}$ does not have hereditary traces, then it has a trace $\mathcal{F}[M]$ whose closure is a countable compacta, and consequently homeomorphic to the closure of a barrier.

The proof of this theorem is based on a study of finite families of *doubletons* similar to the one for families of finite subsets of $\mathbb{N}$. We introduce some definitions and notation for it.

Definition 3.2. Let $(\mathbb{N}^{[2]})^{[\leqslant\infty]} \subseteq \mathcal{P}(\mathbb{N}^{[2]})$ be the set of *block sets of doubletons* , i.e., the set of those $A \subseteq P(\mathbb{N}^{[2]})$ such that for every $s, t \in A$, either $s < t$ or $t < s$. Then $(\mathbb{N}^{[2]})^{[\leqslant\infty]}$ is naturally a compact space because it is a closed subspace of $\mathcal{P}(\mathbb{N}^{[2]})$, where $\mathbb{N}^{[2]}$ is endowed with the discrete topology. Let

$$\mathrm{FIN}_2 = \{A \in (\mathbb{N}^{[2]})^{[\leqslant\infty]} : A \text{ is finite}\}.$$

We say that $\mathcal{U} \subseteq \mathrm{FIN}_2$ is precompact iff $\overline{\mathcal{U}} \subseteq \mathrm{FIN}_2$. We say that $\mathcal{U}$ is hereditary iff $A \subseteq B \in \mathcal{U}$ implies that $A \in \mathcal{U}$. Given $\mathcal{U} \subseteq \mathrm{FIN}_2$ and $M \subseteq \mathbb{N}$ infinite, we define

$$\begin{aligned} \mathcal{U}[M] &:= \{A \cap M^{[2]} : A \in \mathcal{U}\} \\ \mathcal{U} \upharpoonright M &:= \mathcal{U} \cap \mathcal{P}(M^{[2]}) \\ \overline{\mathcal{U}}^{\subseteq} &:= \{B \in \mathrm{FIN}_2 : B \subseteq A \in \mathcal{U}\}. \end{aligned}$$

We have the following dichotomy for these families.

Lemma 3.1. [Lo-To] *For every* $\mathcal{U} \subseteq \mathrm{FIN}_2$ *there is* $M \subseteq \mathbb{N}$ *infinite such that either* (a) $\mathcal{U}[M]$ *is precompact or else,* (b) $\mathrm{FIN}_2 \restriction M \subseteq \overline{\mathcal{U}}^{\subseteq}$.

The proof uses the following classical result of Galvin in [Ga] stating that open subsets of $[\mathbb{N}]^\infty$ have the Ramsey property.

Lemma 3.2 (Galvin's Lemma)**.** *For every family* $\mathcal{F} \subseteq \mathrm{FIN}$ *and every infinite set* N *there exists an infinite* $M \subseteq N$ *such that the restriction* $\mathcal{F} \restriction M$ *is either empty or it contains a barrier.* □

Proof of Lemma 3.1. Let $\mathcal{G} = \left\{\bigcup A : A \in \mathrm{FIN}_2 \setminus \overline{\mathcal{U}}^{\subseteq}\right\}$. By Galvin's Lemma, there is M such that either $\mathcal{G} \restriction M$ contains a barrier on M or else $\mathcal{G} \restriction M = \emptyset$. Suppose first that $\mathcal{G} \restriction M$ contain a barrier on M. We claim that in this case $\mathcal{U}[M]$ is precompact. Suppose otherwise that $A \in (\mathrm{FIN}_2 \restriction M) \cap \overline{\mathcal{U}[M]}$, and set $N = \bigcup A \subseteq M$. Since $\mathcal{G} \restriction M$ contain a barrier, there is $s \in \mathcal{G} \restriction M$ such that $s \sqsubseteq N$. Let $B \in \mathrm{FIN}_2 \setminus \overline{\mathcal{U}}^{\subseteq}$ such that $s = \bigcup B$. Since $A \in \overline{\mathcal{U}[M]}$, we can find $B \in \overline{\mathcal{U}[M]}$ such that $A \subseteq B$, and consequently $A \subseteq B \subseteq C$ for some $C \in \mathcal{U}$, i.e., $A \in \overline{\mathcal{U}}^{\subseteq}$, which is impossible. If $\mathcal{G} \restriction M = \emptyset$, then $\mathrm{FIN}_2 \restriction M \subseteq \overline{\mathcal{U}}^{\subseteq}$. □

Corollary 3.1. *Suppose that* $\mathcal{U}_0, \mathcal{U}_1 \subseteq \mathrm{FIN}_2$ *and* $M \subseteq \mathbb{N}$ *are such that*

$$\mathrm{FIN}_2 \restriction M \subseteq \{A_0 \cup A_1 : A_0 \in \mathcal{U}_0,\ A_1 \in \mathcal{U}_1\}. \tag{3.2}$$

Then there is infinite $N \subseteq M$ *and* $i = 0, 1$ *such that* $\mathrm{FIN}_2 \restriction M \subseteq \overline{\mathcal{U}_i}^{\subseteq}$.

Proof. Suppose that (3.2) holds. Observe that $\mathrm{FIN}_2 \times \mathrm{FIN}_2 \to \mathrm{FIN}_2$, $(A, B) \mapsto A \cup B$ is continuous, so the desired result follows from Lemma 3.1. □

We give now a proof of the second structural theorem.

Proof of Theorem 3.3. Let $\mathcal{U}$ be the family of doubletons $A = \{s_1 < \cdots < s_n\}$ such that there is $s \in \mathcal{F}$ with the property that $\{\min s_i\}_{i=1}^n \subseteq s$ and $\{\max s_i\}_{i=1}^n \cap s = \emptyset$. This is clearly a hereditary family. We use Lemma 3.1, and the fact that $\mathcal{U}$ is hereditary, to find M such that either $\mathcal{U}[M] = \mathrm{FIN}_2 \restriction M$ or else $\mathcal{U}[M]$ is compact. Suppose that $\mathcal{U}[M] = \mathrm{FIN}_2 \restriction M$. Then if $N \subseteq M$ is such that $\min M < \min N$ and $]n_0, n_1[\cap M \neq \emptyset$ for every $n_0 < n_1$ in N, then it is easy to conclude that $\mathcal{F}[N] = [N]^{<\infty}$. Suppose that $\mathcal{U}[M]$ is compact. Then, since the mapping $A \in \mathcal{U}[M] \mapsto (\{\min s\}_{s \in A}, \{\max s\}_{s \in A})$ is continuous. It follows that $\mathcal{G}_0 := \{\{\min s\}_{s \in A} : A \in \mathcal{U}[M]\}$ and $\mathcal{G}_1 := \{\{\max s\}_{s \in A} : A \in \mathcal{U}[M]\}$ are both compact families. We note that the fact that $\mathcal{G}_0$ does not necessarily mean that the corresponding trace $\mathcal{F}[M]$ is precompact. Now if $\mathcal{F}[M]$ is precompact, then, by Lemma 3.2, we obtain that $\mathcal{F}[N]$ is the closure of a uniform barrier on some $N \subseteq M$; otherwise, there is some $N \subseteq M$ such that $N \in \overline{\mathcal{F}[M]}$. This implies that indeed $N \in \overline{\mathcal{F}[N]}$. Now the fact that $\mathcal{U}[M]$ is compact implies that $\overline{\mathcal{F}[N]} \setminus \mathrm{FIN} \subseteq \{P \subseteq N : N \setminus P \text{ is finite}\}$. □

3.2. Beyond traces. Of course restrictions/traces does not give full information of families. It is rather easy to find for every integer n a compact and hereditary family $\mathcal{F}$ of rank $n+1$ such that for every infinite set M there is $N \subseteq M$ such that $\mathcal{F} \upharpoonright N = \mathcal{F}[N] = [N]^{\leqslant 1}$: Take a partition $M_0 \cup M_1 \cdots \cup M_{n-1}$ of $\mathbb{N}$ into infinite sets, and let $\mathcal{F}$ be the hereditary closure of the collection of finite sets $\{m_0 < \cdots < m_{n-1}\}$ such that $m_i \in M_i$. From this, an easy diagonal argument gives a compact and hereditary family of rank $\omega+1$ and such that its traces are hereditarily of rank 2, i.e., of the form $[N]^{\leqslant 1}$.

There is a natural way to produce families of rank $\omega+1$: Take a finite-to-one onto mapping $\varphi : \mathbb{N} \to \mathbb{N}$, and now consider the family $\{s \in \mathrm{FIN} : \varphi'' s \in \mathcal{S}\}$. Observe that φ is nothing else but an infinite partition $\{I_n\}_{n\in\mathbb{N}}$ of $\mathbb{N}$ into finite subsets of $\mathbb{N}$, by declaring $I_n := \varphi^{-1}\{n\}$ for each $n \in \mathbb{N}$. On the other hand, it is quite easy to see that those families have (hereditarily) traces with rank $\omega+1$. In order to correct this, and produce a family $\mathcal{F}_0$ of rank $\omega+1$ and traces of rank 2 hereditarily, one can proceed as follows. For each n, let $\mathcal{S}_n := \mathcal{S} \upharpoonright n = \{s \subseteq \{0,1,...,n-1\} : \#s \leqslant \min s + 1\}$. It is nice exercise to prove that $\#\mathcal{S}_n = f_{n+1}$, where f_n is the n^{th} Fibonacci number. Let $(I_n)_{n\in\mathbb{N}}$ be the sequence of consecutive intervals of integers such that $\#I_n = \#\mathcal{S}_n = f_{n+2}$. Let $\theta_n : \mathcal{S}_n \to f_{n+1}$ be a bijection for every n. We are going to define $F : \mathcal{S} \to \mathrm{FIN}$ satisfying

(a) $F(t) \subseteq \bigcup_{n\in t} I_n$ and $\#(F(t) \cap I_n) = 1$ for every $t \in \mathcal{S}$ and every $n \in t$.
(b) For every $t_0, t_1 \in \mathcal{S}$ and every $n \in t_0 \cap t_1 \setminus t_0 \wedge t_1$ one has that $F(t_0) \cap F(t_1) \cap I_n = \emptyset$, where $t_0 \wedge t_1$ is the maximal initial part of both t_0 and t_1.
(c) For every $t_0 \sqsubseteq t_1$ in $\mathcal{S}$ one has that $F(t_0) \sqsubseteq F(t_1)$.

We define $F \upharpoonright \mathcal{S}_n$ recursively on n. Suppose defined $F \upharpoonright \mathcal{S}_n$. Let $t \in \mathcal{S}_{n+1} \setminus \mathcal{S}_n$. This means that $n \in t$. We define $F(t) := F(t \setminus \{n\}) \cup \{\theta_{n+1}(t \setminus \{n\})\}$. It is easy to prove that F has the properties (a) and (c) above. We prove that $F \upharpoonright \mathcal{S}_n$ has the property (b) for every n by induction on n. Suppose then that $F \upharpoonright \mathcal{S}_n$ has the property (b). Let $t_0, t_1 \in \mathcal{S}_{n+1}$, and let $l \in t_0 \cap t_1 \setminus t_0 \wedge t_1$. Set $\bar{t}_i := t_i \setminus \{n\}$, $i = 0, 1$. Suppose first that $l = n$. Then, since $t_0 \neq t_1$, and $n \in t_0 \cap t_1$ it follows that $\bar{t}_0 \neq \bar{t}_1$. By definition, $F(t_0) \cap I_n = \{\theta_{n+1}(\bar{t}_0)\}$, $F(t_1) \cap I_n = \{\theta_{n+1}(\bar{t}_1)\}$ and $\theta_{n+1}(t_0) \neq \theta_{n+1}(t_1)$, since θ_{n+1} is 1-1, and we are done. Suppose that $l < n$. Since $t_0 \wedge t_1 \subseteq n$, it follows that $t_0 \wedge t_1 = \bar{t}_0 \wedge \bar{t}_1$, and consequently, it follows that $l \in \bar{t}_0 \cap \bar{t}_1 \setminus \bar{t}_0 \wedge \bar{t}_1$. By definition and by inductive hypothesis, $F(t_0) \cap I_l = F(\bar{t}_0) \cap I_l \neq F(\bar{t}_1) \cap I_l = F(t_1) \cap I_l$ and since $\#F(t_i) \cap I_l = 1$, $i = 0, 1$, we are done.

Let $\mathcal{F}_0$ be the $\subseteq$-closure of $\mathrm{Im}\, F$, and let $\varphi := \varphi_{\mathbf{I}} : \mathbb{N} \to \mathbb{N}$ be the mapping canonically defined by the partition $\mathbf{I} = (I_n)_n$; that is, $\varphi(k) = n$ if and only if $k \in I_n$. It follows by the property (a) above that $\mathcal{F}_0 \subseteq \{s \in \mathrm{FIN} : \varphi'' s \in \mathcal{S}\}$. In addition, $\bar{\varphi} : \mathcal{F}_0 \to \mathcal{S}$, $\bar{\varphi}(s) := \varphi''(s)$ is onto, since $\bar{\varphi}(F(t)) = t$ for every $t \in \mathcal{S}$. Clearly, $\bar{\varphi}$ is finite-to-one, hence, $\mathcal{F}_0$ has the same rank $\omega+1$ than $\mathcal{S}$. We finally prove that for every $M \subseteq \mathbb{N}$ there is $N \subseteq M$ such that $\mathcal{F}_0 \upharpoonright N = \mathcal{F}_0[N] = [N]^{\leqslant 1}$. Otherwise, by Proposition 2.3, Proposition 3.2 and the fact that $\mathcal{F}_0$ is hereditary, we can find M such that $[M]^2 \subseteq \mathcal{F}_0 \upharpoonright M$. For each $s \in [M]^2$ let $t_s \in \mathcal{S}$ be such that $s \subseteq F(t_s)$. By the property (c) above, we assume that $\max t_s = \varphi(\max s)$ for every $s \in [M]^2$. A simple argument gives some $N \subseteq M$ such that

(d) for every $s_0, s_1 \in [N]^2$ with the same minimum n one has that $\min t_{s_0} = \min t_{s_1} = k_n$, and
(e) either $(k_n)_n$ is strictly increasing or constant.

Suppose first that $(k_n)_{n\in N}$ is strictly increasing. Let $n_0 < n_1 < n_2$ be in N. Then $\min t_{\{n_0,n_1\}} = k_{n_0} < k_{n_1} = \min t_{\{n_1,n_2\}}$, hence $t_{\{n_0,n_1\}} \wedge t_{\{n_1,n_2\}} = \emptyset$. On the other hand, $k_1 \in F(t_{\{n_0,n_1\}}) \cap F(t_{\{n_1,n_2\}}) \cap I_{\varphi(k_1)}$ and this contradicts that, by the property (b) above, we know that $F(t_{\{n_0,n_1\}}) \cap F(t_{\{n_1,n_2\}}) \cap I_{\varphi(k_1)} = \emptyset$.

Suppose now that $k_n = k$ for every $n \in N$. Let $n_0 < \cdots < n_{k+1}$ be in N. Then there must be $i < j < k+1$ such that $t_{\{n_i,n_{k+1}\}} \neq t_{\{n_j,n_{k+1}\}}$, since otherwise, $\{n_0, \dots, n_{k+1}\} \subseteq t_{\{n_0,n_{k+1}\}}$, that is impossible since $\#t_{\{n_0,n_{k+1}\}} \leqslant k+1$. Observe that $\varphi(n_{k+1}) \notin t_{\{n_i,n_{k+1}\}} \wedge t_{\{n_j,n_{k+1}\}}$, because otherwise, since we are assuming that $\max t_{\{n_i,n_{k+1}\}} = \varphi(n_{k+1}) = \max t_{\{n_j,n_{k+1}\}}$, we would have that $t_{\{n_i,n_{k+1}\}} = t_{\{n_j,n_{k+1}\}}$. Since in addition $\varphi(n_{k+1}) \in t_{\{n_i,n_{k+1}\}} \cap t_{\{n_j,n_{k+1}\}}$, it follows that $F(t_{\{n_i,n_{k+1}\}}) \cap F(t_{\{n_j,n_{k+1}\}}) \cap I_{\varphi(n_{k+1})} = \emptyset$ and this obviously forbids that $n_{k+1} \in F(t_{\{n_i,n_{k+1}\}}) \cap F(t_{\{n_j,n_{k+1}\}}) \cap I_{\varphi(n_{k+1})}$.

A partition $\mathbf{I} = (I_n)_n$ gives the opportunity to define the following density notions.

Definition 3.3. Given $0 < \lambda < 1$ and $s \in \mathrm{FIN}$ let us define

$$s[+] := \{n \in \mathbb{N} : s \cap I_n \neq \emptyset\} = \varphi_{\mathbf{I}}(s)$$
$$s[\lambda] := \{n \in \mathbb{N} : \#(s \cap I_n) \geqslant \lambda \# I_n\}.$$

Given a family $\mathcal{F}$, let

$$\mathcal{G}_+(\mathcal{F}) := \{s[+] : s \in \mathcal{F}\} = \varphi_{\mathbf{I}}(\mathcal{F}) \text{ and } \mathcal{G}_\lambda(\mathcal{F}) := \{s[\lambda] : s \in \mathcal{F}\}.$$

The example $\mathcal{F}_0$ above satisfies that $\mathcal{G}_+(\mathcal{F}_0) = \mathcal{S}$ but $\mathcal{G}_\lambda(\mathcal{F}_0) = \{\emptyset, \{0\}\}$ for every $\lambda > 1$. We present now for every $0 < \lambda < 1$ an example of a family $\mathcal{F}_\lambda$ such that $\mathcal{G}_\lambda(\mathcal{F}_\lambda) = \mathcal{S}$, yet the rank of the traces of $\mathcal{F}_\lambda$ is, hereditarily, 4. We point out that if a family $\mathcal{F}$ is such that $\mathcal{G}_\lambda(\mathcal{F}) = \mathcal{S}$, then the rank of its traces is, at least 3; it seems possible to us that our example is sharp. This family is defined in [Lo-Ru-Tra] where the family $\mathcal{F}_\lambda$ is used to find a weakly compact set of a Banach space with the Banach-Saks property, and contrary to its convex hull.

In order to put this is the right context, we introduce some concepts. In what follows, $\mathbb{N} = \bigcup_{n\in\mathbb{N}} I_n$ is a partition of $\mathbb{N}$ into finite intervals I_n. A *transversal* (relative to $(I_n)_n$) is an infinite subset T of I such that $\#(T \cap I_n) \leqslant 1$ for all n.

The construction of this family is based on the previous example $\mathcal{F}_0$ and on a classical counterexample by Erdős and Hajnal [Er-Ha] to the natural generalization of Gillis' Lemma to double-indexed sequences of large measurable sets.

Lemma 3.3. *For every $\varepsilon > 0$ there is $r := r(\varepsilon) \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ there is probability space (Ω, Σ, μ) and a sequence $(A_{i,j})_{1\leqslant i<j\leqslant n}$ with $\mu(A_{i,j}) \geqslant \varepsilon$ for every $1 \leqslant i < j \leqslant n$ such that for every $s \subseteq \{1, \dots, n\}$ of cardinality strictly bigger than r one has that $\bigcap_{\{i,j\}\in[s]^2} A_{i,j} = \emptyset$.*

Proof. Given $\varepsilon > 0$, let $r \in \mathbb{N}$ be such that $1 - 1/r \geqslant \varepsilon$. Now, given $n \in \mathbb{N}$ let $\Omega := \{1, \dots, r\}^n$, and let μ be the probability counting measure on Ω. Given

$1 \leqslant i < j \leqslant n$ we define the subset of n-tuples

$$A_{i,j}^{(n,r)} := \{(a_l)_{l=1}^n \in \{1, \dots, r\}^n \,:\, a_i \neq a_j\}. \tag{3.3}$$

This is the desired counterexample. □

This example is a discrete version of the original construction by Erdős and Hajnal in the unit interval $[0,1]$ with the Lebesgue measure: Given $\varepsilon > 0$, consider as above $r \in \mathbb{N}$ such that $1 - 1/r \geqslant \varepsilon$, and for two integers $i \neq j$, let $B_{i,j}$ be the set of all real numbers $\theta \in [0,1]$ whose i^{th} and j^{th} terms in its $1/r$-expansion are different (i.e., if $\theta = \sum_{k=1}^{\infty} a_k (1/r)^k$ with $a_k \in \{0, \dots, r-1\}$, then $a_i \neq a_j$). This is a Borel set, and it is easy to see that it has Lebesgue measure bigger than ε. On the other hand, if $s \subseteq \mathbb{N}$ has cardinality bigger than $r+1$, then $\bigcap_{\{i,j\} \in [s]^2} B_{i,j} = \emptyset$.

We come back to the construction of $\mathcal{F}_\lambda$. For practical reasons we will define such family not in $\mathbb{N}$ but in a more appropriate countable set I. Fix $0 < \lambda < 1$. We define first the disjoint sequence $(I_n)_n$. For each $m \in \mathbb{N}$, $m \geqslant 4$, let r_m be such that

$$\big(1 - 1/r_m\big)^{\binom{m-2}{2}} \geqslant \lambda.$$

Let $4 \leqslant m \leqslant n$ be fixed. Let $I_{m,n} := \{1, \dots, r_m\}^{n \times [\{2,\dots,m-1\}]^2}$. Let $I_n = \{n\}$ for $n = 1, 2, 3$. For $n \geqslant 4$ let

$$I_n := \prod_{4 \leqslant m \leqslant n} I_{m,n} = \prod_{4 \leqslant m \leqslant n} \{1, \dots, r_m\}^{n \times [\{2,\dots,m-1\}]^2}.$$

Observe that for $n \neq n'$ one has that $I_n \cap I_{n'} = \emptyset$. Let $I := \bigcup_n I_n$. Now, given $4 \leqslant m_0 \leqslant n$ and $2 \leqslant i_0 < j_0 \leqslant m_0 - 1$, let $\pi_{i_0,j_0}^{(n,m_0)} : I_n \to \{1, 2, \dots, r_{m_0}\}^n$ be the natural projection,

$$\pi_{i_0,j_0}^{(n,m_0)}\big(((b_{i,j}^{(l,m)}))_{4 \leqslant m \leqslant n,\, 1 \leqslant l \leqslant n,\, 2 \leqslant i < j \leqslant m-1}\big) := (b_{i_0,j_0}^{(l,m_0)})_{l=1}^n \in \{1, 2, \dots, r_{m_0}\}^n.$$

We start with the definition of the family $\mathcal{F}$ on I. Recall that $\mathfrak{S} := \{s \subseteq \mathbb{N} : \#s = \min s\}$ is the Schreier barrier. We define $F : \mathfrak{S} \to [I]^{<\infty}$ such that $F(u) \subseteq \bigcup_{n \in u} I_n$ and then we define $\mathcal{F}_\lambda$ as the image of F. Fix $u = \{n_1 < \cdots < n_{n_1}\} \in \mathfrak{S}$:

(i) For $u = \{1\}$, let $F(u) := I_1$.
(ii) For $u := \{2, n\}$, $2 < n$, let $F(u) := I_2 \cup I_n$.
(iii) For $u := \{3, n_1, n_2\}$, $3 < n_1 < n_2$, let $F(u) := I_3 \cup I_{n_1} \cup I_{n_2}$.
(iv) For $u = \{n_1, \dots, n_{n_1}\}$ with $3 < n_1 < n_2 < \cdots < n_{n_1}$, then let

$$F(u) \cap I_{n_k} := I_{n_k} \text{ for } k = 1, 2, 3,$$

and for $3 < k \leqslant n_1$, let

$$F(u) \cap I_{n_k} := \bigcap_{1 < i < j < k} \big(\pi_{i,j}^{(n_k,n_1)}\big)^{-1}\big(A_{n_i,n_j}^{(n_k,r_{n_1})}\big) \tag{3.4}$$

Where the A's are as in (3.3). Explicitly,

$$F(u) \cap I_{n_k} = \big\{((b_{i,j}^{(l,m)}))_{4 \leqslant m \leqslant n_k,\, 1 \leqslant l \leqslant n_k,\, 2 \leqslant i < j \leqslant m-1} \in I_{n_k} : b_{i,j}^{(n_i,n_1)} \neq b_{i,j}^{(n_j,n_1)} \text{ for every } 1 < i < j < k\big\}.$$

Observe that it follows from (3.4) that

$$\pi_{i,j}^{(n_k,n_1)}\big(F(u)\cap I_{n_k}\big) = A_{n_i,n_j}^{(n_k,r_{n_1})} \subset \{1,2,\dots,r_{n_1}\}^{n_k}$$

for every $1 < i < j < k$. It is rather easy to see that $\mathcal{G}_\lambda(\mathcal{F}_\lambda) = \mathcal{S}$. It is less trivial to prove that the traces of $\mathcal{F}_\lambda$ have rank 4, hereditarily. We refer the reader to [Lo-Ru-Tra] for full details.

4. Extensions of finite sets. Compact and bounded subsets of $\mathbb{R}^{\mathbb{N}}$

Let $\mathbb{R}^{\mathbb{N}}$ be endowed with its natural product topology (i.e. determined by the pointwise convergence). We intend to present results extending some of the results we presented for families of finite subsets of $\mathbb{N}$. To do this, first we introduce some well-known notions and notation. Given $f \in R^{\mathbb{N}}$ and $n \in \mathbb{N}$ we write $(f)_n$ to denote $f(n)$. Given $U \subseteq \mathbb{R}$ and $A \subseteq \mathbb{N}$, finite or infinite, we identify U^A with $\{f \in \mathbb{R}^{\mathbb{N}} : \operatorname{supp} f \subseteq A \text{ and } \operatorname{Im} f \subseteq U\}$, where $\operatorname{supp} f := \{n \in \mathbb{N} : (f)_n \neq 0\}$.

Recall c_0 is the subset of $\mathbb{R}^{\mathbb{N}}$ consisting of the sequences $(f_n)_n$ converging to zero. We equip c_0 with the induced topology in $\mathbb{R}^{\mathbb{N}}$. Let c_{00} be the subset of c_0 consisting of the eventually zero sequences. A subset $\mathcal{N}$ of c_0 is compact when every sequence in $\mathcal{N}$ has a pointwise convergent subsequence in $\mathcal{N}$. $\mathcal{N}$ is precompact (or relatively compact) when every sequence in $\mathcal{N}$ has a converging subsequence with limit in c_0.

Given $M \subseteq \mathbb{N}$ we will denote a mapping $f : M \to \mathbb{R}$ sequentially as $(f_n)_{n\in M}$. Given $A \subseteq \mathbb{N}$, finite or infinite, and $f \in \mathbb{R}^{\mathbb{N}}$, let $f \upharpoonright A := f \cdot \mathbb{1}_A$, pointwise multiplication, be the restriction of f to A; that is, $(f \upharpoonright A)_n = f(n)$ when $n \in A$ and $(f \upharpoonright A)_n = 0$ otherwise. Given $\mathcal{N} \subseteq \mathbb{R}^{\mathbb{N}}$ and $A \subseteq \mathbb{N}$, finite or infinite, let $\mathcal{N}[A] := \{f \upharpoonright A : f \in \mathcal{N}\}$.

Recall that the ℓ_1-norm of a summable sequence $f \in \mathbb{R}^{\mathbb{N}}$ is defined as $\|f\|_{\ell_1} := \sum_n |(f)_n|$. Given $\mathcal{N} \subseteq \mathbb{R}^{\mathbb{N}}$ and $C > 0$, let

$$(\mathcal{N})_C := \{f \in \mathbb{R}^{\mathbb{N}} : \text{there exists } g \in \mathcal{N} \text{ such that } \|f-g\|_{\ell_1} \leqslant C\}$$
$$C \cdot \mathcal{N} := \{f \in \mathbb{R}^{\mathbb{N}} : \text{there exists } \lambda \leqslant C \text{ such that } \lambda f \in \mathcal{N}\}.$$

We say that $\mathcal{N}$ is (pointwise) bounded when there is K such that $\mathcal{N} \subseteq [-K,K]^{\mathbb{N}}$. Compact and bounded subsets of c_0 are exactly the weakly compact subsets of c_0 when considered as Banach space.

Definition 4.1. [Lo-To] Let $\mathcal{F} \subseteq \mathrm{FIN}$ be an arbitrary family, and let $f : \mathcal{F} \to c_0$.
(a) f is inner if for every $s \in \mathcal{F}$ one has that $\operatorname{supp} f(s) \subseteq s$.
(b) f is uniform if for every $t \in \mathrm{FIN}$ one has $|\{f(s)(\min(s/t)) : t \sqsubseteq s,\, s \in \mathcal{F}\}| = 1$
(c) f is called a U-mapping if $\mathcal{F}$ if it is inner and uniform.

The following is a generalization of Corollary 2.2 concerning mappings on barriers with precompact range.

Theorem 4.1. [Lo-To] *Suppose that $\mathcal{B}$ is a barrier on M, $\mathcal{N}$ is a compact and bounded subset of c_0 and suppose that $f : \mathcal{B} \to \mathcal{N}$. Then for every $\varepsilon > 0$ there is $N \subseteq M$ and there is a U-mapping $g : \mathcal{B} \upharpoonright N \to c_{00}$ such that for every $s \in \mathcal{B} \upharpoonright N$ one has that $\|f(s) \upharpoonright N - g(s)\|_{\ell_1} \leqslant \varepsilon$.*

A compact subset $\mathcal{N}$ has associated naturally compact families of finite subsets of $\mathbb{N}$ as follows. Given $\varepsilon > 0$, the family of all $\{n \in \mathbb{N} : |(f)_\varepsilon| \geqslant \varepsilon\}$, $f \in \mathcal{N}$ is compact. Now, Theorem 4.1 follows from the following generalization of Lemma 2.1. We refer the reader to [Lo-To] for more details.

Lemma 4.1. *Suppose that $\{\mathcal{B}_l\}_{l\in\mathbb{N}}$ is a collection of uniform barriers on M, and suppose that for every $k \in \mathbb{N}$ we have $\varphi_l : \mathcal{B}_l \to \mathrm{FIN}$ with precompact range. Then there is some infinite subset N of M such that $(\varphi_l(s) \setminus s) \cap N \subseteq N \cap [0, n]$ for every $n \in N$, $l \leqslant n$, and every $s \in \mathcal{B}_l \upharpoonright N$.* □

We pass now to present some results concerning hereditary properties of subsets of $\mathbb{R}^{\mathbb{N}}$. We introduce more terminology. Given $\mathcal{N} \subseteq \mathbb{R}^{\mathbb{N}}$, let

$$\overline{\mathcal{N}}^{\sqsubseteq} := \{f \upharpoonright I : f \in \mathcal{N} \text{ and } I \text{ is an initial interval of } \mathbb{N}\}$$
$$\overline{\mathcal{N}}^{\subseteq} := \{f \upharpoonright A : f \in \mathcal{N} \text{ and } A \subseteq \mathbb{N}\}.$$

The following is a very useful result in this context.

Lemma 4.2 (Matching Lemma)**.** *Suppose that $\mathcal{B}$ and $\mathcal{C}$ are two barriers on M and $\varphi : \mathcal{B} \to \mathcal{C}$ is an internal mapping. Then there is an infinite subset N of M and a mapping $\sigma : \mathcal{B} \upharpoonright N \to \mathcal{B}$ such that $\sigma(s) \cap N = \varphi(s) = \varphi(\sigma(s))$ for every $s \in \mathcal{B} \upharpoonright N$.*

Proof. First of all, color each $t \in \mathcal{C}$ by 1 when there is $s \in \mathcal{B}$ such that $\varphi(s) = t$, and otherwise. By the Ramsey property of $\mathcal{C}$ there is some $P \subseteq M$ such that $\mathcal{C} \upharpoonright P$ is monochromatic, with color $i = 0, 1$. As for every $s \in \mathcal{B} \upharpoonright P$, $\varphi(s) \in \mathcal{C} \upharpoonright N$ is colored by 1, i must be equal to 1. Define now $\psi : \mathcal{C} \upharpoonright P \to \mathcal{B}$ by $\psi(t) \in \mathcal{B}$ is such that $\varphi(\psi(t)) = t$. Apply Lemma 2.1 to ψ to get some $N \subseteq P$ such that $\psi(t) \cap N \subseteq t$ for every $t \in \mathcal{C} \upharpoonright N$. Observe that this is equivalent to say that $\psi(t) \cap N = t$ ($t \subseteq \psi(t)$ because $t = \varphi(\psi(t)) \subseteq \psi(t)$ by the properties of φ). Finally define $\sigma : \mathcal{B} \upharpoonright N \to \mathcal{B}$ by $\sigma(s) = \psi(\varphi(s))$ for each $s \in \mathcal{B} \upharpoonright N$. Then, for $s \in \mathcal{B} \upharpoonright N$ we have

$$\varphi(\sigma(s)) = \varphi(\psi(\varphi(s))) = \varphi(s),$$
$$\sigma(s) \cap N = \psi(\varphi(s)) \cap N = \varphi(s),$$

as desired. □

4.1. Partial unconditionality.

Theorem 4.2 (Mazur)**.** *Let $\mathcal{N}$ be a compact bounded subset of $c_0 \subseteq \mathbb{R}^{\mathbb{N}}$. Then for every $\varepsilon > 0$ there is M such that $\overline{\mathcal{N}[M]}^{\sqsubseteq} \subseteq (\mathcal{N}[M])_\varepsilon$.*

Proof. Fix all data. Suppose that the desired result is false. For every $M \subseteq \mathbb{N}$, let $t_M \sqsubseteq M$ and $f_M \in \mathcal{N}$ be such that $f_M \upharpoonright t_M \notin (\mathcal{N}[M])_\varepsilon$. Since $\mathcal{N}$ is compact, it follows that for every $M \subseteq \mathbb{N}$ there exists $t_M \sqsubseteq s_M \sqsubseteq M$ such that $f_M \upharpoonright t_M \notin (\mathcal{N}[s_M])_\varepsilon$. Let $\mathcal{F}$ be the set of minimal elements of $\{s_M\}_{M\subseteq\mathbb{N}}$. By the first structural theorem, there exists $M_0 \subseteq \mathbb{N}$ such that $\mathcal{B} := \mathcal{F} \upharpoonright M_0$ is a barrier in M_0. For each $s \in \mathcal{B}$ choose $M_s \subseteq M_0$ such that $s \sqsubseteq M_s$, and set $t_s := t_{M_s}$ and $f_s := f_{M_s}$. It follows that

$$t_s \sqsubseteq s \text{ and } f_s \upharpoonright t_s \notin (\mathcal{N}[s])_\varepsilon \text{ for every } s \in \mathcal{B}. \tag{4.1}$$

We use Theorem 4.1 to find $M \subseteq M_0$ and a U-mapping $g : \mathcal{B} \upharpoonright M \to c_{00}$ such that $\|f_s \upharpoonright M - g(s)\|_{\ell_1} \leqslant \varepsilon/3$ and such that φ and $s \in \mathcal{B} \mapsto t_s$ are uniform on $\mathcal{B} \upharpoonright M$. Now let $s_0 \in \mathcal{B} \upharpoonright M$ and let $s_1 \in \mathcal{B} \upharpoonright M$ be such that $s_1 \sqsubseteq t_{s_0} \cup M/s_0$. Since $t_s \sqsubseteq s$ always, it follows from the uniformness of $s \mapsto t_s$ and g that $t_{s_0} = t_{s_1}$ and $g(s_1) \upharpoonright t_{s_0} = g(s_0) \upharpoonright t_{s_0}$. We claim that $\|f_{s_0} \upharpoonright t_{s_0} - f_{s_1} \upharpoonright s_0\|_{\ell_1} \leqslant \varepsilon$, contradicting (4.1):

$$\begin{aligned}\|f_{s_0} \upharpoonright t_{s_0} - f_{s_1} \upharpoonright s_0\|_{\ell_1} &\leqslant \|f_{s_0} \upharpoonright t_{s_0} - g(s_1) \upharpoonright t_{s_0}\|_{\ell_1} + \|g(s_1) \upharpoonright t_{s_0} - f_{s_1} \upharpoonright s_0\|_{\ell_1} \\ &= \|f_{s_0} \upharpoonright t_{s_0} - g(s_0) \upharpoonright t_{s_0}\|_{\ell_1} + \|g(s_1) \upharpoonright t_{s_0} - f_{s_1} \upharpoonright s_0\|_{\ell_1} \\ &\leqslant \|f_{s_0} \upharpoonright M - g(s_0) \upharpoonright M\|_{\ell_1} + \|g(s_1) \upharpoonright t_{s_0} - f_{s_1} \upharpoonright s_0\|_{\ell_1} \\ &\leqslant \frac{\varepsilon}{3} + \|g(s_1) \upharpoonright t_{s_0} - f_{s_1} \upharpoonright t_{s_0}\|_{\ell_1} \\ &\quad + \|f_{s_1} \upharpoonright M \setminus s_1 - g(s_1) \upharpoonright (M \setminus s_1)\|_{\ell_1} \leqslant \varepsilon\end{aligned}$$

where for the last inequality we have used that g is inner and consequently $g(s_1) \upharpoonright (M \setminus s_1) = 0$. □

Remark 4.1. The previous result corresponds to the classical theorem by Mazur stating that every nontrivial weakly null sequence in a Banach space has a $(1+\varepsilon)$-basic subsequence.

Theorem 4.3 (Odell's unconditionality)**.** *Suppose that $\mathcal{N}$ is a compact and bounded subset of c_0. Then for every $\varepsilon > 0$ there is $M = \{m_0 < m_1 < \cdots < m_k < \cdots\}$ such that*

(4.2) *for every k and every $s \in [\{m_l\}_{l \geqslant k}]^k$ one has that $\mathcal{N}[s] \subseteq (\mathcal{N}[\{m_l\}_{l \geqslant k}])_\varepsilon$.*

Consequently, there is $\{m_0 < m_1 < \cdots < m_k < \cdots\}$ such that for every k and every $s \in [\{m_l\}_{l \geqslant k}]^k$ one has that $\mathcal{N}[s] \subseteq (\mathcal{N}[M] - \mathcal{N}[M])_\varepsilon$.

Proof. We prove first the last part of the statement assuming the first one. By Mazur's Theorem, let M_0 be such that $\overline{\mathcal{N}[M_0]}^{\sqsubseteq} \subseteq (\mathcal{N}[M_0])_\varepsilon$. Let now $M = \{m_0 < m_1 < \cdots < m_k < \cdots\} \subseteq M_0$ be such that (4.2) holds for $\varepsilon/2$. Then given $k \in \mathbb{N}$, $s \in [\{m_l\}_{l \geqslant k}]^k$, and $f \in \mathcal{N}$, we fix $g, h \in \mathcal{N}$ such that $\|f \upharpoonright s - g \upharpoonright \{m_l\}_{l \geqslant k}\|_{\ell_1} \leqslant \varepsilon/2$ and $\|g \upharpoonright \{m_0, \dots, m_{k-1}\} - h \upharpoonright M\|_{\ell_1} \leqslant \|g \upharpoonright \{m_0, \dots, m_{k-1}\} - h \upharpoonright M_0\|_{\ell_1} \leqslant \varepsilon/2$. Then

$$\begin{aligned}\|f \upharpoonright s - (g-h) \upharpoonright M\|_{\ell_1} \leqslant \|f \upharpoonright s - g \upharpoonright \{m_l\}_{l \geqslant k}\|_{\ell_1} \\ + \|g \upharpoonright \{m_0, \dots, m_{k-1}\} - h \upharpoonright M\|_{\ell_1} \leqslant \varepsilon.\end{aligned}$$

We pass now to prove the first part of the statement. Observe that if $s \subseteq N \subseteq M$ and $f \upharpoonright s \in (\mathcal{N}[M])_\varepsilon$, then $f \upharpoonright s \in (\mathcal{N}[N])_\varepsilon$. So, instead of proving the desired result in the statement of the theorem, it suffices to prove that for every n and every M there exists $N \subseteq M$ such that for every $t \in [N]^n$ one has that $\mathcal{N}[t] \subseteq (\mathcal{N}[N])_\varepsilon$. Going towards a contradiction, fix M and $n \in \mathbb{N}$ such that for every $N \subseteq M$ there exists $t \in [N]^n$ such that $\mathcal{N}[t] \not\subseteq (\mathcal{N}[N])_\varepsilon$. For each $N \subseteq M$, choose $t_N \in [N]^n$ and $f_N \in \mathcal{N}$ such that $f_N \upharpoonright t_N \notin (\mathcal{N}[N])_\varepsilon$. Since $\mathcal{N}$ is compact, this last relation means that we can fix also for each $N \subseteq M$ an initial part $t_N \subseteq s_N \sqsubseteq N$ such that $f_N \upharpoonright t_N \notin (\mathcal{N}[s_N])_\varepsilon$. Let $\mathcal{F}$ be the set of $\sqsubseteq$-minimal elements of $\{s_N : N \subseteq M\}$, and let $M_0 \subseteq M$ be such that $\mathcal{B} := \mathcal{F} \upharpoonright M_0$ is a barrier on M_0. For each $s \in \mathcal{B}$ choose $N \subseteq M$ such that $s = s_{N_s}$, and set $t_s := t_{N_s}$, $f_s := f_{N_s}$. We use Theorem 4.1 and the Ramsey property of $\mathcal{B}$ to find $M_1 \subseteq M_0$ and a U-mapping $g : \mathcal{B} \upharpoonright M_1 \to c_{00}$ such that $\|f_s \upharpoonright M_1 - g(s)\|_{\ell_1} \leqslant \varepsilon/2$ for every $s \in \mathcal{B} \upharpoonright M_1$, and such that for every $s_0, s_1 \in \mathcal{B} \upharpoonright M_1$ if $\{k_0^{(j)} < \cdots < k_{n-1}^{(j)}\} = t_{s_j}$, $j = 0, 1$, then $\sum_{i<n} |(f_{s_0})_{k_i^{(0)}} - (f_{s_1})_{k_i^{(1)}}| \leqslant \varepsilon/2$. Now we use Lemma 4.2 to find $s_0, s_1 \in \mathcal{B} \upharpoonright M_1$ such that $t_{s_0} = t_{s_1} = s_0 \cap s_1$. Hence,

$$\begin{aligned}\|f_{s_0} \upharpoonright t_{s_0} - f_{s_1} \upharpoonright s_0\|_{\ell_1} \leqslant \|f_{s_0} \upharpoonright t_{s_0} - f_{s_1} \upharpoonright t_{s_0}\|_{\ell_1} \\ + \|f_{s_1} \upharpoonright (M_1 \setminus s_1) - g(s_1) \upharpoonright (M_1 \setminus s_1)\|_{\ell_1} \leqslant \varepsilon,\end{aligned}$$

so $f_{s_0} \upharpoonright t_{s_0} \in (\mathcal{N}[s_0])_\varepsilon$, a contradiction. □

Remark 4.2. The previous Theorem corresponds to the well-known result by Odell in [Od] stating that every every non-trivial weakly null sequence has a Schreier-unconditional basic subsequence.

Theorem 4.4. *Suppose that $\mathcal{N} \subseteq c_0$ is precompact and bounded such that*

$$\inf\{(f)_n : f \in \mathcal{N}, n \in \operatorname{supp} f\} = \delta > 0.$$

Then there is a subset M of $\mathbb{N}$ such that $\overline{\mathcal{N}[M]}^{\subseteq} \subseteq K/\delta \operatorname{conv}(\mathcal{N}[M])$, where $K = \sup_{f \in \mathcal{N}} \|f\|_\infty$.

Proof. Consider the family $\mathcal{F}$ whose elements are $\{n \in \mathbb{N} : (f)_n \geqslant \delta\}$, $f \in \mathcal{N}$. It follows that $\mathcal{F}$ is a precompact family. Let M be such that $\mathcal{F}[M]$ is hereditary. In particular, given $s \in \mathcal{F}[M]$ and $t \subseteq s$ there exists $f \in \mathcal{N}$ such that $\operatorname{supp} f \upharpoonright M = \{n \in M : (f)_n \neq 0\} = \{n \in M : (f)_n \geqslant \delta\} = t$. This means that, given $s \in \mathcal{F}[M]$ there is for each $t \in s$, $f_t \in \mathcal{N}$ such that $\operatorname{supp} f_t = t$ and $(f_t)_n \geqslant \delta$ for every $n \in t$; it follows that $[0, \delta]^s \subseteq \operatorname{conv}(\{f_t \upharpoonright M\}_{t \subseteq s}) \subseteq \operatorname{conv} \mathcal{N}[M]$. Hence, given $f \in \mathcal{N}$ and $N \subseteq M$, we know that $f \upharpoonright N \subseteq [0, K]^s$, where $s = \operatorname{supp} f \upharpoonright M$. This means that $f \upharpoonright N \subseteq K/\delta \operatorname{conv} \mathcal{N}[M]$. □

Remark 4.3. This results is equivalent to the fact that if $(x_n)_n$ is a weakly-null sequence in ℓ_∞ such that $\inf\{|(x_n)_k| : n \in \mathbb{N}$ and $(x_n)_k \neq 0\} > 0$ then it has an unconditional subsequence.

Theorem 4.5 (Elton's unconditionality)**.** *Let $\mathcal{N}$ be a compact and bounded subset of c_0, $\varepsilon > 0$. Then there is some M such that for every $f \in \mathcal{N}$ and every $t \subseteq s \subseteq M$ such that f has constant sign on t there exists $g \in \mathcal{N}$ and $u \subseteq t$ such that*

(a) *g has constant sign on u,*
(b) $|\sum_{n \in t} (f)_n| \leqslant (1 + \varepsilon) |\sum_{n \in u} (g)_n|$
(c) $\sum_{n \in s \setminus u, n \geqslant \min u} |(g)_n| \leqslant \varepsilon$.

It readily follows the following.

Corollary 4.1. *Let $\mathcal{N}$ be a compact and bounded subset of c_0, $\varepsilon > 0$. Then there is some M such that for every $f \in \mathcal{N}$ and every $t \subseteq s \subseteq M$ there exists $g \in \mathcal{N}$ such that*

(a) $\|f \upharpoonright t\|_{\ell_1} \leqslant (1 + \varepsilon) \|g \upharpoonright t\|_{\ell_1}$
(b) $\|g \upharpoonright \{n \in s \setminus t : n \geqslant \min t\}\|_{\ell_1} \leqslant \varepsilon$.

Proof. Apply the previous theorem to the compact and bounded set $\{(|(f)_n|)_n : f \in \mathcal{N}\}$. $\square$

We use the following fact extracted from [Di-Od-Sch-Zsa].

Proposition 4.1. *Let $\mathfrak{X}$ be a bounded subset of c_0. Then for every $\varepsilon > 0$ there is $\theta : \mathfrak{X} \to \mathrm{FIN}$ and $\varphi : \mathbb{N} \to \mathbb{N}$ such that*

(a) *$\theta(x)$ is a final subset of $\operatorname{supp} x$ for every $x \in \mathfrak{X}$.*
(b) *$\|x\|_{\ell_1} \leqslant (1+\varepsilon)\|x \upharpoonright \theta(x)\|_{\ell_1}$ and $\|x \upharpoonright \theta(x)\|_{\ell_1} \leqslant \varphi(\min \theta(x))$ for every $x \in \mathfrak{X}$.*

Proof. Let $D > 0$ be such that $\mathfrak{X} \subseteq [-D, D]^{\mathbb{N}}$. Let n_0 be such that

$$n^2 + Dn \leqslant (1+\varepsilon)(n-1)^2 \text{ for every } n \geqslant n_0 \text{ and } n_0^2 - \frac{D}{\varepsilon} n_0 \geqslant \frac{D}{\varepsilon}.$$

Let $\varphi_0(n) := n^2$ if $n \geqslant n_0$, and $\varphi_0(n) := n_0^2$ otherwise. Observe that

$$D(n+1) + \varphi_0(n) \leqslant (1+\varepsilon)\varphi_0(n-1) \text{ for every } n > 0. \tag{4.3}$$

Set $\varphi(n) := \varphi_0(n) + D$ for every n. Given $x \in \mathfrak{X}$, let $k_x \in \mathbb{N}$ be such that

$$\sum_{n > k_x} (x)_n \leqslant \varphi(k_x).$$

observe that k_x exists because φ is unbounded. Define then $\theta(x) := \operatorname{supp} x \cap [k_x, \infty[$. Let us prove that θ and φ have the desired properties: Fix $x \in \mathfrak{X}$. Suppose first that $k_x = 0$. Then $\theta(x) = \operatorname{supp} x$ and

$$\sum_n (x)_n = \sum_{n \in \theta(x)} (x)_n \leqslant (x)_0 + \varphi_0(0) \leqslant \varphi(0) \leqslant \varphi(\min \theta(x)).$$

Suppose now that $k_x > 0$. It follows from (4.3) that

$$\begin{aligned} \sum_n (x)_n = \sum_{n \leqslant k_x} (x)_n + \sum_{n > k_x} (x)_n &\leqslant D(k_x+1) + \varphi(k_x) \\ &\leqslant (1+\varepsilon)\varphi(k_x - 1) \leqslant (1+\varepsilon) \sum_{n \geqslant k_x} (x)_n, \end{aligned}$$

$$\sum_{n \geqslant k_x} (x)_n \leqslant (x)_{k_x} + \varphi_0(k_x) \leqslant \varphi(k_x) \leqslant \varphi(\min \theta(x)).$$ $\square$

Proof of Theorem 4.5. Suppose otherwise, and for each M fix $t_M \subseteq s_M \subseteq M$ and $f_M \in \mathcal{N}$ such that f_M has constant sign on t_M and such that (a), (b) and (c) in the statement of Theorem does not work hold for any $g \in \mathcal{N}$. Use now Proposition 4.1 to find $\varphi : \mathbb{N} \to \mathbb{N}$ and for each M a final subset $u_M \subseteq t_M$ such that

$$\frac{1}{1+\varepsilon}\Big|\sum_{n \in t_M} (f_M)_n\Big| \leqslant \Big|\sum_{n \in u_M} (f_M)_n\Big| \leqslant \varphi(\min u_M). \tag{4.4}$$

For each M, let $r_M := s_M \cap [\min u_M, \infty[$. Let now $\mathcal{B}$ be the set of $\sqsubseteq$-minimal subsets of $\{r_M\}_M$, and for each $r \in \mathcal{B}$, fix M_r such that $r = r_{M_r}$. It follows from

(4.4) and the Ramsey property of $\mathcal{B}$ that there is some M such that

$$\max\left\{\frac{\|f_{M_r}\restriction u_{M_r}\|_{\ell_1}}{\|f_{M_p}\restriction u_{M_p}\|_{\ell_1}},\frac{\|f_{M_p}\restriction u_{M_p}\|_{\ell_1}}{\|f_{M_r}\restriction u_{M_r}\|_{\ell_1}}\right\}\leqslant 1+\varepsilon \text{ for every } r,p\in\mathcal{B}\restriction M \text{ with } \min r=\min p \tag{4.5}$$

Let $\mathcal{C}$ be the set of minimal elements of $\{u_{M_r} : r \in \mathcal{B}\restriction M\}$. For every $u \in \mathcal{C}$, choose $r_u \in \mathcal{B}\restriction M$ such that $u = u_{M_{r_u}}$. It follows from Theorem 4.1 that there is $N \subseteq M$ such that $\|f_{M_{r_u}} \restriction (N \setminus u)\|_{\ell_1} \leqslant \varepsilon$ for every $u \in \mathcal{C}\restriction N$. Now let $r \in \mathcal{B}\restriction N$. Let $u \in \mathcal{C}\restriction N$ be such that $u \sqsubseteq u_{M_r}$. Since $\min r_u = \min u = \min u_{M_r} = \min r$, it follows from (4.5) that

$$\|f_{M_r}\restriction u_{M_r}\|_{\ell_1}\leqslant(1+\varepsilon)\|f_{M_{r_u}}\restriction u\|_{\ell_1}\leqslant(1+\varepsilon)\|f_{M_{r_u}}\restriction t_{M_{r_u}}\|_{\ell_1}.$$

Since in addition

$$\sum_{n\in s_{M_r}\setminus u,\,n\geqslant\min t_{M_r}}|(f_{M_{r_u}})_n|\leqslant\|f_{M_{r_u}}\restriction(N\setminus u)\|_{\ell_1}\leqslant\varepsilon,$$

the vector $g := f_{M_{r_u}}$ and u contradict the fact that s_{M_r}, t_{M_r} and f_{M_r} are a counterexample to (a), (b) and (c). □

Remark 4.4. The previous theorem is the combinatorial core of the result of Elton [El] stating that for every $\delta > 0$ every normalized weakly null sequence $(x_n)_n$ has a subsequence $(y_n)_n$ such that

$$\Big\|\sum_{n\in t}a_ny_n\Big\|\leqslant 17\big(\log_2(1/\delta)+1\big)\Big\|\sum_n a_ny_n\Big\|$$

for every sequence of scalars $(a_n)_n$ such that $\sup_n |a_n| \leqslant 1$ and every $t \subseteq \{n \in \mathbb{N} : |a_n| \geqslant \delta\}$. The proof goes as follows: Fix $\delta < 1$, and fix a normalized weakly null sequence $(x_n)_n$ in some Banach space X. Let $\varepsilon > 0$ be small enough such that

$$\varepsilon\big(\log_2(1/\delta)+1\big)\leqslant\frac{1}{16(1+\varepsilon)^2}-\frac{1}{17}.$$

Apply Mazur's Theorem 4.2 and Theorem 4.5 to $\mathcal{N} := \{(x^*(x_n))_n : x^* \in B_{X^*}\}$ and $\varepsilon\delta/(2(1+\varepsilon))$ to find an infinite set M with the properties (a), (b) and (c) in Theorem 4.5 and such that $(x_n)_{n\in M}$ is a $(1+\varepsilon)$-basic sequence. We claim that $(x_n)_{n\in M}$ is the desired subsequence: Fix scalars $(a_n)_{n\in s}$, $s \subseteq M$, with $\max_{n\in s}|a_n| \leqslant 1$ and $t \subseteq \{n \in s : |a_n| \geqslant \delta\}$. Let $f \in B_{X^*}$ be such that

$$f\Big(\sum_{n\in t}a_nx_n\Big)=\Big\|\sum_{n\in t}a_nx_n\Big\|.$$

Let $n_0 := \mathrm{L}_2(1/\delta)$. We discretize the log function by considering for $x \geqslant 1$, $\mathrm{L}_2(x)$ as the minimal integer n such that $x \leqslant 2^n$. Observe that $\mathrm{L}_2(x) \leqslant \log_2(x) + 1$. Let $u \subseteq t$ be such that

1. $f(x_n)\cdot f(x_m) \geqslant 0$ for every $m, n \in u$.
2. $a_n \cdot a_m \geqslant 0$ for every $m, n \in u$.
3. $\max_{m,n\in u} |a_n|/|a_m| \leqslant 2$.
4. $|\sum_{n\in u} a_n f(x_n)| \geqslant 1/(4\mathrm{L}_2(1/\delta)) f(\sum_{n\in t} a_n x_n) = 1/(4\mathrm{L}_2(1/\delta))\|\sum_n a_n x_n\|$.

Now use the properties of M to find $g \in B_{X^*}$ and $v \subseteq u$ such that $g(x_n) \cdot g(x_m) \geqslant 0$ for every $m, n \in v$, $|\sum_{n \in u} f(x_n)| \leqslant (1+\varepsilon)|\sum_{n \in v} g(x_n)_n|$, and in addition such that $\sum_{n \in s \setminus v, n \geqslant \min v} f(x_n)| \leqslant \varepsilon/\delta(2(1+\varepsilon))$. Observe that it follows

$$\sum_{n \in s \setminus v, n \geqslant \min v} |g(x_n)| \leqslant \frac{\varepsilon\delta}{2(1+\varepsilon)} \leqslant \varepsilon \Big\| \sum_{n \in t} a_n x_n \Big\|,$$

since $(x_n)_{n \in M}$ is $(1+\varepsilon)$-basic, and $\min_{n \in t} |a_n| \geqslant \delta$. Use 3. to find $0 \leqslant k < \mathrm{L}_2(1/\delta)$ such that $1/2^{k+1} \leqslant |a_n| \leqslant 1/2^k$ for every $k \in u$. Hence,

$$\begin{aligned}
\Big\| \sum_{n \in s} a_n x_n \Big\| &\geqslant \frac{1}{2(1+\varepsilon)} \Big\| \sum_{n \geqslant \min v} a_n x_n \Big\| \\
&\geqslant \frac{1}{2(1+\varepsilon)} \Big| g\Big(\sum_{n \geqslant \min v} a_n x_n \Big) \Big| \\
&\geqslant \frac{1}{2(1+\varepsilon)} \Big| \sum_{n \in v} a_n g(x_n) \Big| - \varepsilon \Big\| \sum_{n \in t} a_n x_n \Big\| \\
&\geqslant \frac{1}{2(1+\varepsilon)2^{k+1}} \Big| \sum_{n \in v} g(x_n) \Big| - \varepsilon \Big\| \sum_{n \in t} a_n x_n \Big\| \\
&\geqslant \frac{1}{2(1+\varepsilon)^2 2^{k+1}} \Big| \sum_{n \in u} f(x_n) \Big| - \varepsilon \Big\| \sum_{n \in t} a_n x_n \Big\| \\
&\geqslant \frac{1}{4(1+\varepsilon)^2} \Big| \sum_{n \in u} a_n f(x_n) \Big| - \varepsilon \Big\| \sum_{n \in t} a_n x_n \Big\| \\
&\geqslant \frac{1}{16(1+\varepsilon)^2 \mathrm{L}_2(1/\delta)} \Big\| \sum_{n \in t} a_n x_n \Big\| - \varepsilon \Big\| \sum_{n \in t} a_n x_n \Big\| \\
&\geqslant \frac{1}{17 \mathrm{L}_2(1/\delta)} \Big\| \sum_{n \in t} a_n x_n \Big\| \\
&\geqslant \frac{1}{17(\log_2(1/\delta)+1)} \Big\| \sum_{n \in t} a_n x_n \Big\|.
\end{aligned}$$

Problem 1. Does there exist a constant C such that for every $\delta > 0$ every normalized weakly null sequence $(x_n)_n$ has a subsequence $(y_n)_n$ such that

$$\Big\| \sum_{n \in t} a_n y_n \Big\| \leqslant C \Big\| \sum_{n} a_n y_n \Big\|$$

for every sequence of scalars $(a_n)_n$ such that $\sup_n |a_n| \leqslant 1$ and every $t \subseteq \{n \in \mathbb{N} : |a_n| \geqslant \delta\}$?

4.2. Maurey–Rosenthal example. We give now an example of a compact subset without hereditary traces in a very strong sense.

Theorem 4.6. *There exists a compact and bounded subset $\mathcal{N} \subseteq c_0$ such that for every $M \subseteq \mathbb{N}$ one has that $\overline{\mathcal{N}[M]}^{\subseteq} \not\subseteq C \cdot \overline{\mathrm{conv}(\pm\mathcal{N}[M])}$ for every $C > 0$.*

Proof. Choose a fast increasing sequence (m_i) such that

$$\sum_{i=0}^{\infty}\sum_{j\neq i}\min\left(\sqrt{m_i/m_j},\sqrt{m_j/m_i}\right)\leqslant 1.$$

Let $\mathrm{FIN}^{[<\infty]}$ be the collection of all finite block sequences $s_0 < s_1 < \cdots < s_k$ of nonempty finite subsets of $\mathbb{N}$. Now choose a 1–1 function $\sigma : \mathrm{FIN}^{[<\infty]} \to \{m_i\}$ such that $\sigma((s_i)_{i=0}^n) > s_n$ for all $(s_i) \in \mathrm{FIN}^{[<\infty]}$. Now let $\mathcal{B}_{\mathrm{MR}}$ be the family of unions $s_0 \cup s_1 \cup \cdots \cup s_n$ of finite sets such that

(a) $n = \min s_0 = \# s_0$,
(b) (s_i) is block and
(c) $\#(s_i) = \sigma(s_0, \dots, s_{i-1})$ $(1 \leqslant i \leqslant n)$.

It is not difficult to see that $\mathcal{B}_{\mathrm{MR}}$ is a ω^2-uniform barrier on $\mathbb{N}$. Observe that by definition, every $s \in \mathcal{B}_{\mathrm{MR}}$ has a unique decomposition $s = s_0 \cup \cdots \cup s_n$ satisfying (a), (b) and (c) above. Now define the mapping $\varphi : \mathcal{B}_{\mathrm{MR}} \to c_{00}$,

$$\varphi(s) = \sum_{i=0}^{n} \frac{1}{(\# s_i)^{1/2}} \mathbb{1}_{s_i}.$$

Then φ is a U-mapping. It follows easily that φ extends uniquely to a U-mapping $\bar{\varphi} : \overline{\mathcal{B}_{\mathrm{MR}}}^{\sqsubseteq} \to c_{00}$. Let $\mathcal{N} := \bar{\varphi}''(\overline{\mathcal{B}_{\mathrm{MR}}}^{\sqsubseteq})$. Fix $M \subseteq \mathbb{N}$ and $C > 0$. We work to prove that $\overline{\mathcal{N}[M]}^{\subseteq} \not\subseteq C \cdot \overline{\mathrm{conv}(\mathcal{N}[M])}$. Now let $s = s_0 \cup \cdots \cup s_n \in \mathcal{B}_{\mathrm{MR}} \restriction M$ be such that $n \geqslant 6C + 2$. Suppose that $g := \sum_{i \leqslant n,\, i \text{ even}} (\# s_i)^{-1/2} \mathbb{1}_{s_i} \in C \cdot \overline{\mathrm{conv}(\mathcal{N}[M])}$. Let $f_1, \dots f_k \in \mathcal{N}$, $a_1, \dots, a_k$ be such that $|a_1| + \cdots + |a_k| \leqslant C$ and be such that

$$\left\| g - \left(\sum_{i=1}^{k} a_i f_i \right) \restriction s \right\|_{\ell_1} \leqslant 1 \tag{4.6}$$

Notice that

$$\left\langle g, \sum_{i=0}^{n} (-1)^i \frac{1}{(\# s_i)^{1/2}} \mathbb{1}_{s_i} \right\rangle = \lceil n/2 \rceil$$

Set $f := (\sum_{i=1}^{k} a_i f_i) \restriction s$ and $x := \sum_{i=0}^{n} (-1)^i (\# s_i)^{-1/2} \mathbb{1}_{s_i}$. It follows from (4.6) that $|\langle f, x\rangle| \geqslant \lceil n/2 \rceil - \varepsilon$. So, there must be $i \leqslant k$ such that $|\langle f_i, x\rangle| \geqslant (2/C)(\lceil n/2\rceil - 1)$, because it can be proved that for every $h \in \mathcal{N}$ one has that $|\langle h, x\rangle| \leqslant 3$ We refer the reader to [Lo-To] for more details. □

Remark 4.5. The previous example corresponds to the first in literature weakly-null sequence in a Banach space without unconditional subsequences by Maurey and Rosenthal [Ma-Ro].

4.3. Rosenthal's ℓ_1-dichotomy. The following is an structural result for compact and bounded families. It corresponds to the classical result by Rosenthal [Ro] stating that every bounded sequence in a Banach space has a subsequence equivalent to the unit basis of ℓ_1 or a weakly-Cauchy subsequence. The proof we present here is extracted from [Lo-To], and it uses some of the results on FIN_2 we exposed in Subsection 3.1.

Theorem 4.7 (Rosenthal's ℓ_1-dichotomy)**.** *Suppose that $\mathcal{N} \subseteq \mathbb{R}^{\mathbb{N}}$ is compact and bounded. Then there is an infinite subset $M \subseteq \mathbb{N}$ such that*

(a) *either every sequence in $\mathcal{N}$ is convergent in M (i.e., $((f)_n)_{n\in M}$ is convergent for every $f \in \mathcal{N}$), or*

(b) *there is a closed non-trivial interval $I \subseteq \mathbb{R}$ such that $I^M \subseteq \overline{(\mathrm{conv}(\mathcal{N}[M]))}$.*

Proof. Let $\mathcal{U}$ be the collection of all $A \in \mathrm{FIN}_2$ with the property that there is some $(x_n)_n \in \mathcal{N}$ such that $|x_{\min s} - x_{\max s}| \geqslant 1/2^{\min\min A}$. This is clear an hereditary family, so we use Lemma 3.1 to find $M \subseteq \mathbb{N}$ such that either $\mathcal{U} \upharpoonright M$ is compact or $\mathcal{U} \upharpoonright M = \mathrm{FIN}_2 \upharpoonright M$. Suppose first that $\mathcal{U} \upharpoonright N$ is compact. We claim that $(x_n)_{n\in M}$ is convergent for every $(x_n)_n \in \mathcal{N}$. Otherwise, we could find $\varepsilon > 0$ and $\{k_n\}_{n\in\mathbb{N}} \subseteq M$ such that $|x_{k_{2n}} - x_{k_{2n+1}}| \geqslant \varepsilon$. Going to a subsequence if needed, we assume that $\varepsilon \geqslant (2^{-k_0})$. This means that $\{\{k_{2n}, k_{2n+1}\}\}_{n\leqslant m} \in \mathcal{U} \upharpoonright M$ for every m, and consequently, $\mathcal{U} \upharpoonright M$ is not compact.

Suppose now that $\mathcal{U} \upharpoonright M = \mathrm{FIN}_2 \upharpoonright M$. Let $m_0 < m_1$ be the first two elements of M, and let $M_0 = M \setminus \{m_0, m_1\}$. Let $\mathcal{U}_\varepsilon$ be the collection of all $A \in \mathrm{FIN}_2$ such that there is some $(x_n)_n \in \mathcal{N}$ with the property that $|x_{\min s} - x_{\max s}| \geqslant \varepsilon$ for every $s \in A$. It follows from our assumptions that $\mathcal{U}_\varepsilon \upharpoonright M_0 = \mathrm{FIN}_2 \upharpoonright M_0$. Let D be a finite $\varepsilon/3$-net of the interval $[-K, K]$, where $K > 0$ is such that $\mathcal{N} \subseteq [-K, K]^{\mathbb{N}}$. We define, for $(d_0, d_1) \in D^{[2]}$, the sets

$$\mathcal{U}^{<}_{(d_0,d_1)} = \{A \in \mathrm{FIN}_2 \upharpoonright M_0 : \text{there is } (x_n)_n \in \mathcal{N} \text{ with } x_{\min s} \leqslant d_0 \text{ and } x_{\max s} \geqslant d_1 \ \forall s \in A\}.$$

$$\mathcal{U}^{>}_{(d_0,d_1)} = \{A \in \mathrm{FIN}_2 \upharpoonright M_0 : \text{there is } (x_n)_n \in \mathcal{N} \text{ with } x_{\min s} \geqslant d_1 \text{ and } x_{\max s} \leqslant d_0 \ \forall s \in A\}.$$

Observe that every $A \in \mathrm{FIN}_2 \upharpoonright M_0$ is the union of elements of $\mathcal{U}_{(d_0,d_1)}$'s, and that each $\mathcal{U}_{(d_0,d_1)}$ is hereditary. By Corollary 3.1 there is $N \subseteq M_0$ and $(d_0, d_1) \in D^{[2]}$ and $* \in \{<, >\}$ such that $\mathrm{FIN}_2 \upharpoonright P = \mathcal{U}^{*}_{(d_0,d_1)} \upharpoonright P$. We assume that $*$ is $<$, because the other case when $*$ is $>$ is treated in a similar manner. Now set $P = \{n_{2k+1}\}_k$, where $\{n_k\}_k$ is the increasing enumeration of N. We claim that for every disjoint s, t subsets of N there is $(x_n)_n \in \mathcal{N}$ such that

$$x_n \leqslant d_0 \text{ and } x_m \geqslant d_1 \text{ for every } n \in s \text{ and } m \in t:$$

This follows from the fact that there is $A \in \mathrm{FIN}_2 \upharpoonright N$ such that $s = P \cap \{\min u : u \in A\}$ and $t = P \cap \{\max u : u \in A\}$. This implies that $[d_0, d_1]^P \subseteq \overline{(\mathrm{conv}(\mathcal{N}[P]))}$, because it follows that for every sequence $(\varepsilon_n)_{n\in P}$ of $0, 1$ there exists $(x_n)_n \in \overline{\mathcal{N}}$ such that $x_n \geqslant d_{\varepsilon_n}$ if $\varepsilon_n = 1$ and $x_n \leqslant d_{\varepsilon_n}$ if $\varepsilon_n = 0$ for every $n \in P$. Consequently, for every $s \sqsubseteq P$, every $(t_n)_{n\in s} \in [d_0, d_1]^s$, and every sequence of $0, 1$ $(\varepsilon_n)_{n\in P/s}$ there is $(x_n)_n \in \mathrm{conv}(\overline{\mathcal{N}})$ such that $x_n = t_n$ if $n \in t$, and $x_n \leqslant d_0$ if $\varepsilon_n = 0$ and $x_n \geqslant d_1$ if $\varepsilon_n = 1$ for every $n \in P/t$. This easily implies that $[d_0, d_1]^P \subseteq \overline{\mathrm{conv}(\overline{\mathcal{N}[P]})} = \overline{\mathrm{conv}(\mathcal{N}[P])}$. □

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