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POSETS OF ISOMORPHIC SUBSTRUCTURES OF RELATIONAL STRUCTURES

Abstract. This is a survey of some recent results concerning several classifications of relational structures related to the properties of their self-embedding monoids. For example, if $\preceq^R$ is the right Green's preorder on the monoid $\mathrm{Emb}(\mathbb{X})$ of self-embeddings of a structure $\mathbb{X}$, then the antisymmetric quotient of its inverse is isomorphic to the poset $\mathbb{P}(\mathbb{X})$ of copies of the structure $\mathbb{X}$ contained in $\mathbb{X}$ and, defining two structures to be similar if the Boolean completions of the corresponding posets are isomorphic (or, equivalently, if the inverses of the right Green's preorders are forcing-equivalent) we obtain a classification of structures. Some results concerning the posets of copies of specific structures, the interplay between the properties of structures and the properties of their posets of copies, and the corresponding classification of structures and classification of posets representable in this way will be presented.

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1. Introduction

This paper is a survey of some recent results concerning the partial orders of copies of relational structures. More precisely, assuming that $L = \langle R_i : i \in I \rangle$ is a relational language, where $\mathrm{ar}_L(R_i) = n_i \in \mathbb{N}$, for $i \in I$, that X is a non-empty set, $\langle \rho_i : i \in I \rangle \in \prod_{i \in I} P(X^{n_i}) = \mathrm{Int}_L(X)$ an interpretation of L and $\mathbb{X} = \langle X, \langle \rho_i : i \in I \rangle \rangle \in \mathrm{Mod}_L(X)$ the corresponding L-structure, by $\mathbb{P}(\mathbb{X})$ we denote the set of domains of substructures of $\mathbb{X}$ isomorphic to $\mathbb{X}$, that is

$$\mathbb{P}(\mathbb{X}) = \big\{ A \subset X : \big\langle A, \langle \rho_i \cap A^{n_i} : i \in I \rangle \big\rangle \cong \mathbb{X} \big\}$$

and order it by the inclusion. Clearly we have $\{X\} \subset \mathbb{P}(\mathbb{X}) \subset [X]^{|X|}$ so the partial order $\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ is a suborder of the Boolean lattice $\langle P(X), \subset\rangle$. Since $\mathbb{P}(\mathbb{X}) = \{f[X] : f \in \mathrm{Emb}(\mathbb{X})\}$, where $\mathrm{Emb}(\mathbb{X})$ is the set of self-embeddings of the structure $\mathbb{X}$, and since the poset $\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ is isomorphic to the antisymmetric quotient of the inverse of the right Green's preorder on the monoid $\langle \mathrm{Emb}(\mathbb{X}), \circ, \mathrm{id}_X\rangle$ (see [21]) the investigation of posets of copies can be regarded as a part of the investigation of self-embedding monoids of stuctures, but this aspect will be ignored in this survey.

The correspondence $\mathbb{X} \mapsto \langle \mathbb{P}(\mathbb{X}), \subset\rangle$ can be extended to a functor Π from the category Mod of all relational structures and isomorphisms into its subcategory POSET of all posets: for a structure $\mathbb{X}$, $\Pi(\mathbb{X}) = \langle \mathbb{P}(\mathbb{X}), \subset\rangle$ and, for $f \in \mathrm{Iso}(\mathbb{X}, \mathbb{Y})$, the isomorphism $\Pi(f) \in \mathrm{Iso}(\langle \mathbb{P}(\mathbb{X}), \subset\rangle, \langle \mathbb{P}(\mathbb{Y}), \subset\rangle)$ is given by $\Pi(f)(A) = f[A]$, for all $A \in \mathbb{P}(\mathbb{X})$. In the sequel, in order to simplify notation, instead of $\langle \mathbb{P}(\mathbb{X}), \subset\rangle$, we will write $\mathbb{P}(\mathbb{X})$ whenever the context admits.

Generally speaking we investigate the posets of copies of specific structures, the interplay between the properties of structures and the properties of their posets of copies, and, finally, the corresponding classification of structures and classification of posets representable in this way. In order to obtain a coarse classification of representable posets we transform them into complete Boolean algebras. Namely, to each structure we adjoin a complete Boolean algebra in the following natural way. We remind the reader that a partial order $\mathbb{P} = \langle P, \leqslant\rangle$ is called *separative* iff for each $p, q \in P$ satisfying $p \not\leqslant q$ there is $r \in P$ such that $r \leqslant p$ and $r \perp q$. The *separative modification* of $\mathbb{P}$ is the preorder $\mathrm{sm}\,\mathbb{P} = \langle P, \leqslant^*\rangle$, where $p \leqslant^* q$ iff $\forall r \leqslant p\ \exists s \leqslant r\ s \leqslant q$. The *separative quotient* of $\mathbb{P}$ is the separative partial order $\mathrm{sq}\,\mathbb{P} = \langle P/{=^*}, \trianglelefteq\rangle$, where $p =^* q \Leftrightarrow p \leqslant^* q \wedge q \leqslant^* p$ and $[p] \trianglelefteq [q] \Leftrightarrow p \leqslant^* q$. If $\mathbb{P}$ is a separative partial order, by $\mathrm{ro}\,\mathbb{P}$ we will denote its *Boolean completion.* Thus, if Sep denotes the subcategory of the category POSET containing all separative posets and CBA^+ its subcategory determined by all complete Boolean algebras without zero, then, according to the following diagram,

$$\mathrm{Mod} \xrightarrow{\Pi} \mathrm{POSET} \xrightarrow{\mathrm{sq}} \mathrm{Sep} \xrightarrow{\mathrm{ro}^+} \mathrm{CBA}^+$$

to each relational structure $\mathbb{X}$ we adjoin the posets $\mathbb{P}(\mathbb{X})$, $\mathrm{sq}\,\mathbb{P}(\mathbb{X})$ and $(\mathrm{ro}\,\mathrm{sq}\,\mathbb{P}(\mathbb{X}))^+$. Again, since an isomorphism of posets generates an isomorphism of their separative quotients which generates an isomorphism of their Boolean completions, the mappings sq, ro^+ and $\mathrm{ro}^+ \circ \mathrm{sq} \circ \Pi : \mathrm{Mod} \to \mathrm{CBA}^+$ can be regarded as functors.

Now, if $C \subset \mathrm{Mod}$ is a class of structures, a poset $\mathbb{P}$ will be called *representable (resp. sq-representable, ro-representable) over* C iff $\mathbb{P}$ is isomorphic to the poset $\mathbb{P}(\mathbb{X})$ (resp. $\mathrm{sq}\,\mathbb{P}(\mathbb{X})$, $\mathrm{ro}\,\mathrm{sq}\,\mathbb{P}(\mathbb{X})$) for some $\mathbb{X} \in C$. Clearly a natural question is: Which posets are representable (sq-representable, ro-representable) over Mod or over a given class C?

Although the problem of characterization of representable posets seems to be too general and out of reach, there are some natural restrictions. We remind the reader that a partial order with a largest element $\mathbb{P} = \langle P, \leqslant, 1_{\mathbb{P}}\rangle$ is called *homogeneous* iff it is isomorphic to the principal ideal $p\!\downarrow := \{q \in P : q \leqslant p\}$, for each $p \in P$.

Example 1.1. If $\mathbb{X}$ is the ordinal ω with the natural linear ordering, $\langle \omega, < \rangle$, then, clearly, $\mathbb{P}(\mathbb{X}) = [\omega]^\omega$ and, clearly again, $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is a homogeneous partial order.

The homogeneity of the poset of copies of the structure from Example 1.1 is, in fact, a general rule. Moreover we have the following restrictions.

Theorem 1.1. [14] *For each relational structure* $\mathbb{X}$ *the poset* $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ *is*

(a) *homogeneous,*
(b) *of size* 1 *or infinite,*
(c) *atomic or atomless.*

If, in particular, X *is a countable set, then the set* $\mathbb{P}(\mathbb{X})$ *is*

(d) *analytic, regarded as a subset of the Cantor cube,* 2^X*,*
(e) *of size* 1 *or* ω *or* $\mathfrak{c}$*,*
(f) *equal to* $[X]^\omega$ *or a nowhere dense set in the poset* $\langle [X]^\omega, \subset \rangle$*.*

The following statement gives one restriction for Boolean algebras.

Theorem 1.2. [20] *The algebra* $\operatorname{ro}\operatorname{sq}\mathbb{P}(\mathbb{X})$ *is a homogeneous complete Boolean algebra, for each relational structure* $\mathbb{X}$*.*

Thus all representable Boolean algebras must be homogeneous and

$$(\mathrm{ro}^+ \circ \ \mathrm{sq} \circ \Pi)[\mathrm{Mod}] \subset \mathrm{HCBA}^+,$$

where HCBA^+ denotes the category of homogeneous complete Boolean algebras without zero. Hence the investigation of the algebras representable over the class of relational structures can be regarded as a part of the investigation of the class of homogeneous complete Boolean algebras.

Concerning the problem of representability of Boolean algebras, we note that Theorem 1.2 has the following consequence, which gives a possibility to use the methods of forcing in our investigation. We remind the reader that, if V is a model of set theory, then preorders $\mathbb{P}$ and $\mathbb{Q}$ belonging to V are said to be *forcing equivalent*, in notation $\mathbb{P} \equiv \mathbb{Q}$, iff $\mathbb{P}$ and $\mathbb{Q}$ produce the same generic extensions. It is well known that $\mathbb{P} \equiv \operatorname{sq}\mathbb{P} \equiv \operatorname{ro}\operatorname{sq}\mathbb{P}$, for each poset $\mathbb{P}$.

Theorem 1.3. [20] *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be relational structures. Then*

$$\operatorname{ro}\operatorname{sq}\mathbb{P}(\mathbb{X}) \cong \operatorname{ro}\operatorname{sq}\mathbb{P}(\mathbb{Y}) \quad \Leftrightarrow \quad \mathbb{P}(\mathbb{X}) \equiv \mathbb{P}(\mathbb{Y}).$$

Thus representable Boolean algebras are isomorphic iff the corresponding posets of copies are forcing-equivalent.

We make one more remark concerning the algebras representable over the class of indivisible structures. (A structure $\mathbb{X}$ is called *indivisible* iff for each partition $X = A \cup B$ there is $C \in \mathbb{P}(\mathbb{X})$ such that $C \subset A$ or $C \subset B$).

Theorem 1.4. [14] *A relational structure* $\mathbb{X}$ *is indivisible iff*

$$\mathcal{I}_{\mathbb{X}} := \{I \subset X : I \text{ does not contain a copy of } \mathbb{X}\} \text{ is an ideal in } P(X).$$

Theorem 1.5. [14] *If* $\mathbb{X}$ *is an indivisible relational structure, then*

(a) $\operatorname{ro}\operatorname{sq}\mathbb{P}(\mathbb{X}) \cong \operatorname{ro}(P(X)/\mathcal{I}_{\mathbb{X}})^+$.

If, in addition, $\mathbb{X}$ *is a countable structure, then*

(b) $\mathcal{I}_{\mathbb{X}} = \mathrm{Fin}$ *or* $\mathcal{I}_{\mathbb{X}}$ *is a co-analytic tall ideal;*

(c) $\mathrm{sq}\,\mathbb{P}(\mathbb{X})$ *is an atomless partial order of size* $\mathfrak{c}$*.*

So, the investigation of the algebras representable over the class of countable indivisible structures can be regarded as a part of the investigation of the Boolean algebras of the form $P(\omega)/\mathcal{I}$, where $\mathcal{I} \subset P(\omega)$ is a co-analytic tall ideal.

Four binary structures (that is, the structures of the form $\mathbb{X} = \langle X, \rho\rangle$, where $\rho \subset X^2$) considered in Example 1.2 and the corresponding posets $\mathbb{P}(\mathbb{X})$, $\mathrm{sq}\,\mathbb{P}(\mathbb{X})$ and $(\mathrm{ro}\,\mathrm{sq}\,\mathbb{P}(\mathbb{X}))^+$ are described in Figure 1.

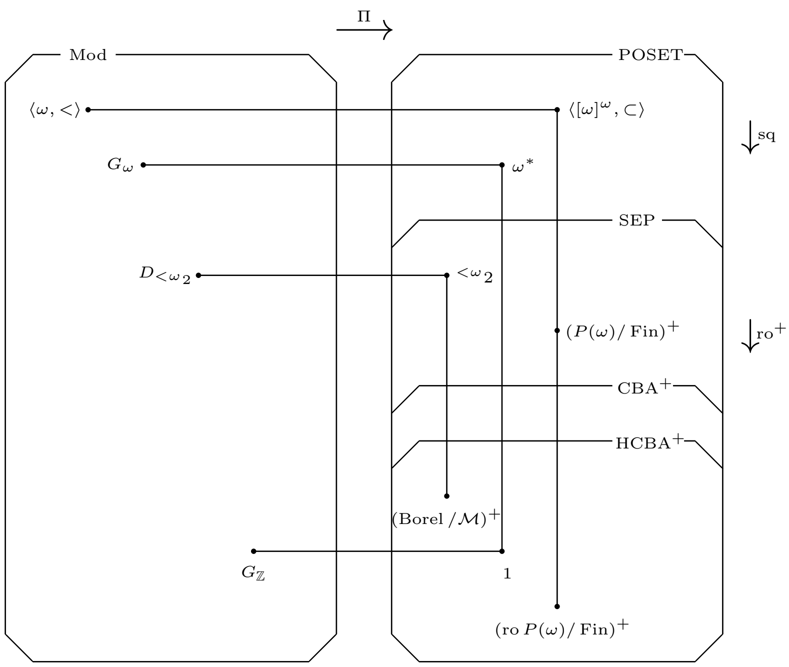

Figure 1. Functors Π, sq and ro^+

Example 1.2. (a) For the linear order $\langle\omega, <\rangle$ we have $\mathbb{P}(\langle\omega, <\rangle) = [\omega]^\omega$ and, hence, $\mathrm{sq}\langle\mathbb{P}(\langle\omega, <\rangle), \subset\rangle \cong (P(\omega)/\mathrm{Fin})^+$.

(b) For the linear graph $G_\omega = \langle\omega, \rho\rangle$, where $m\,\rho\,n$ iff $|m - n| = 1$, we have $\mathbb{P}(G_\omega) = \{[n, \infty)_\omega : n \in \omega\}$, the poset $\langle\mathbb{P}(G_\omega), \subset\rangle$ is isomorphic to the linear order ω^* which is an atomic poset and, hence, $\mathrm{sq}\,\mathbb{P}(G_\omega) \cong (\mathrm{ro}\,\mathrm{sq}\,\mathbb{P}(G_\omega))^+ \cong 1$.

(c) Let $D_{<\omega 2}$ be the digraph $\langle {}^{<\omega}2, \rho\rangle$, where $\rho = \{\langle\varphi, \varphi^\frown i\rangle : \varphi \in {}^{<\omega}2 \wedge i \in 2\}$. It is easy to check (see [14]) that $\mathbb{P}(D_{<\omega 2}) = \{A_\varphi : \varphi \in {}^{<\omega}2\}$, where the sets A_φ, $\varphi \in {}^{<\omega}2$, are defined by $A_\varphi = \{\psi \in {}^{<\omega}2 : \varphi \subset \psi\}$, and that $\langle\mathbb{P}(D_{<\omega 2}), \subset\rangle$ is a separative poset isomorphic to the reversed binary tree $\langle {}^{<\omega}2, \supset\rangle$ (the forcing which adds one Cohen real). So $\mathrm{sq}\langle \mathbb{P}(D_{<\omega 2}), \subset\rangle \cong \langle {}^{<\omega}2, \supset\rangle$ and $\mathrm{ro}\,\mathrm{sq}\langle \mathbb{P}(D_{<\omega 2}), \subset\rangle \cong$ Borel $/\mathcal{M}$.

(d) Let $G_{\mathbb{Z}} = \langle \mathbb{Z}, \rho\rangle$ be the linear graph, where $m\,\rho\, n$ iff $|m-n| = 1$. Then $|\mathbb{P}(G_{\mathbb{Z}})| = 1$, which implies $\mathrm{sq}\,\mathbb{P}(G_{\mathbb{Z}}) \cong (\mathrm{ro}\,\mathrm{sq}\,\mathbb{P}(G_{\mathbb{Z}}))^+ \cong 1$.

Concerning the ro-representability of Boolean algebras, we note that three simple structures from Figure 1, G_ω, $D_{<\omega 2}$ and $\langle \omega, <\rangle$, determine three Boolean algebras generated by a large subclass of the class of countable binary structures, especially under CH. Namely, in [14], on the basis of Theorems 1.1, 1.4 and some additional statements and examples, the classification of countable binary structures related to the properties of their posets of copies and described in Figure 2 is obtained.

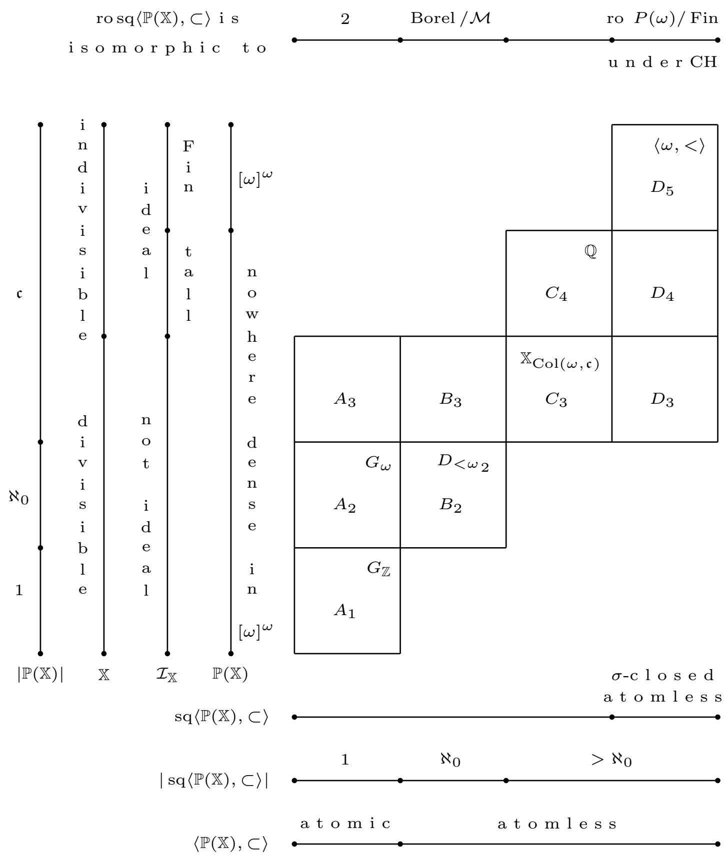

FIGURE 2. A classification of countable binary structures

So, the posets of copies of the structures from column A (resp. B; D) are forcing equivalent to the trivial poset (resp. to the Cohen forcing, $\langle {}^{<\omega}2, \supset\rangle$; to a separative atomless ω_1-closed poset, and under CH to $(P(\omega)/\operatorname{Fin})^+$).

By the results of Section 2 (see Remark 2.1) several statements concerning countable binary structures can be transformed into general ones and, hence, in Sections 4–8 we consider these structures following their classification from Figure 2.

2. Different similarities

The functors Π, $\operatorname{sq}\circ\Pi$ and $\operatorname{ro}\circ\operatorname{sq}\circ\Pi$ induce coarse classifications of relational structures; for example, conditions $\mathbb{P}(\mathbb{X}) = \mathbb{P}(\mathbb{Y})$, $\mathbb{P}(\mathbb{X}) \cong \mathbb{P}(\mathbb{Y})$, $\operatorname{sq}\mathbb{P}(\mathbb{X}) \cong \operatorname{sq}\mathbb{P}(\mathbb{Y})$, $\operatorname{ro}\operatorname{sq}\mathbb{P}(\mathbb{X}) \cong \operatorname{ro}\operatorname{sq}\mathbb{P}(\mathbb{Y})$ and $\mathbb{P}(\mathbb{X}) \equiv \mathbb{P}(\mathbb{Y})$ define equivalence relations ("similarities") on the class Mod of all relational structures and their interplay with the similarities defined by $\mathbb{X} = \mathbb{Y}$, $\mathbb{X} \cong \mathbb{Y}$ and $\mathbb{X} \rightleftarrows \mathbb{Y}$ (equimorphism) was considered in [20]. First, it is natural to ask whether a similarity of structures implies a similarity of their posets of copies, and in particular we have

Theorem 2.1. [20] *If $\mathbb{X}$ and $\mathbb{Y}$ are structures of the same relational language, then*

(a) $\mathbb{X} \cong \mathbb{Y} \;\Rightarrow\; \mathbb{P}(\mathbb{X}) \cong \mathbb{P}(\mathbb{Y})$

(b) $\mathbb{X} \rightleftarrows \mathbb{Y} \;\Rightarrow\; \mathbb{P}(\mathbb{X}) \equiv \mathbb{P}(\mathbb{Y})$.

In general, Figure 3 describes the implications between the mentioned forms of similarity of relational structures; for example, line n denotes the statement that equimorphic structures have forcing-equivalent posets of copies (Theorem 2.1(b)).

Of course we can restrict our consideration to smaller classes of structures. For example, if $L = \langle R_i : i \in I\rangle$ is a relational language and X is a set, then restricting our similarity relations to the set $\operatorname{Mod}_L(X)$ or equivalently, to the corresponding set of interpretations, $\operatorname{Int}_L(X)$, we obtain the following equivalence relations: for $\rho = \langle \rho_i : i \in I\rangle, \sigma = \langle \sigma_i : i \in I\rangle \in \operatorname{Int}_L(X)$ (writing $\mathbb{P}(\rho)$ instead of $\langle \mathbb{P}(\langle X, \rho\rangle), \subset\rangle$, $\rho \cong \sigma$ instead of $\langle X, \rho\rangle \cong \langle X, \sigma\rangle$ and similarly for $\rho \rightleftarrows \sigma$)

$\rho \sim_0 \sigma \Leftrightarrow \rho = \sigma$,
$\rho \sim_1 \sigma \Leftrightarrow \mathbb{P}(\rho) = \mathbb{P}(\sigma) \wedge \rho \cong \sigma$,
$\rho \sim_2 \sigma \Leftrightarrow \mathbb{P}(\rho) = \mathbb{P}(\sigma) \wedge \rho \rightleftarrows \sigma$,
$\rho \sim_3 \sigma \Leftrightarrow \rho \cong \sigma$,
$\rho \sim_4 \sigma \Leftrightarrow \mathbb{P}(\rho) = \mathbb{P}(\sigma)$,
$\rho \sim_5 \sigma \Leftrightarrow \mathbb{P}(\rho) \cong \mathbb{P}(\sigma) \wedge \rho \rightleftarrows \sigma$,
$\rho \sim_6 \sigma \Leftrightarrow \mathbb{P}(\rho) \cong \mathbb{P}(\sigma)$,
$\rho \sim_7 \sigma \Leftrightarrow \operatorname{sq}\mathbb{P}(\rho) \cong \operatorname{sq}\mathbb{P}(\sigma) \wedge \rho \rightleftarrows \sigma$,
$\rho \sim_8 \sigma \Leftrightarrow \operatorname{sq}\mathbb{P}(\rho) \cong \operatorname{sq}\mathbb{P}(\sigma)$,
$\rho \sim_9 \sigma \Leftrightarrow \rho \rightleftarrows \sigma$,
$\rho \sim_{10} \sigma \Leftrightarrow \mathbb{P}(\rho) \equiv \mathbb{P}(\sigma)$,
$\rho \sim_{11} \sigma \Leftrightarrow 0 = 0$.

Then the diagram of implications on the set $\operatorname{Mod}_L(X)$ is displayed in Figure 4 and it is natural to ask are there more implications in it (except the ones which follow from the transitivity)? Are some of the implications a–o, in fact, equivalences?

Concerning the last question the class of all relational structures splits into three parts:

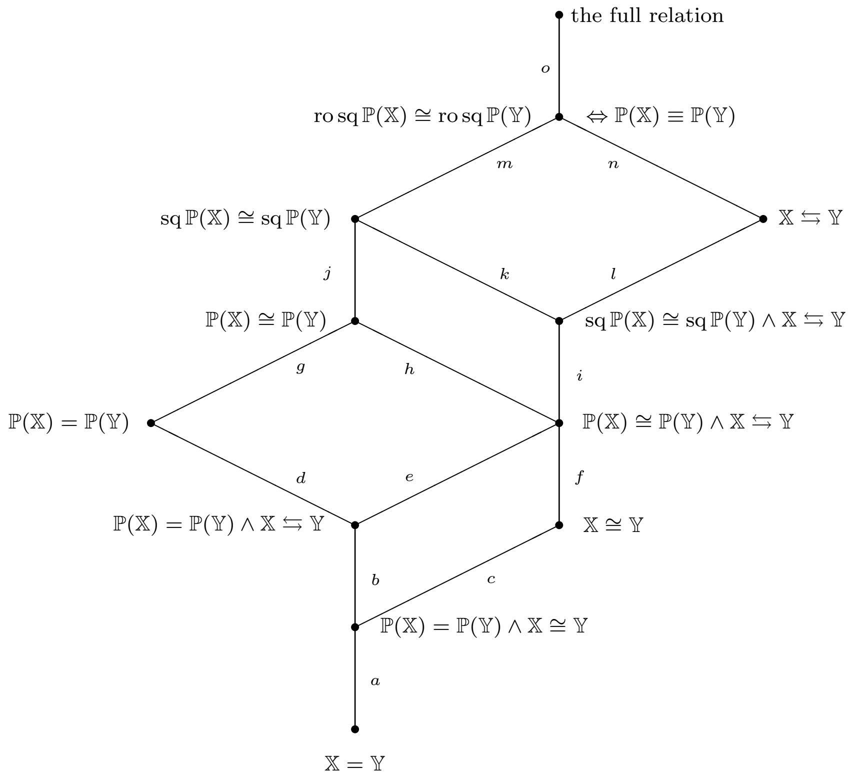

FIGURE 3. The hierarchy of similarities between relational structures

– Finite structures,
– Infinite structures of unary languages,
– Infinite structures of non-unary languages.

(A language $L = \langle R_i : i \in I \rangle$ is called *unary* iff $\mathrm{ar}(R_i) = 1$, for all $i \in I$.)

2.1. Finite structures. For finite structures the posets of copies are trivial and the diagram from Figure 4 collapses significantly. Let us call a class $\mathcal{C}$ of structures a *Cantor–Schröder–Bernstein (CSB) class* iff

$$\forall \mathbb{X}, \mathbb{Y} \in \mathcal{C} \ \ (\mathbb{X} \rightleftarrows \mathbb{Y} \ \Rightarrow \ \mathbb{X} \cong \mathbb{Y}).$$

Theorem 2.2. [20] *For any relational language L and any finite set X we have*

(a) $\mathbb{P}(\mathbb{X}) = \{X\}$, *for each* $\mathbb{X} \in \mathrm{Mod}_L(X)$;

(b) $\mathrm{Mod}_L(X)$ *is a CSB class;*

(c) *Figure* 5 *describes the hierarchy of the similarities* $\sim_k$ *on the set* $\mathrm{Mod}_L(X)$, *for* $|X| > 1$. *In addition,* $\sim_0 = \sim_1$ *iff* $|X| = 1$.

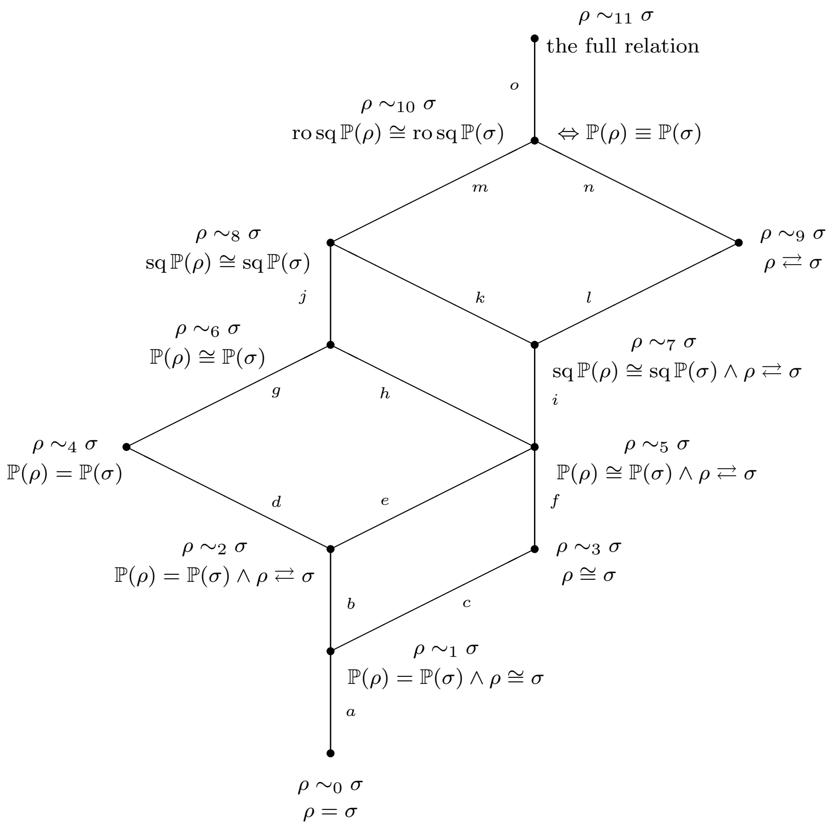

FIGURE 4. The hierarchy of similarities on $\mathrm{Mod}_L(X)$

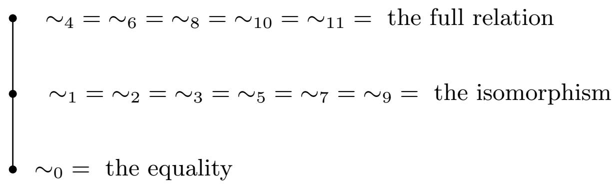

FIGURE 5. The hierarchy of the similarities for finite structures

2.2. Infinite unary structures. The class of posets representable over the class of unary structures is very limited. The following theorem gives a characterization of such partial orders.

Theorem 2.3. [20] *Let L be a unary relational language and $\kappa \geqslant \omega$ a cardinal. Then*

(a) *If* $2^{|L|} < \kappa$, *then a poset* $\mathbb{P}$ *is representable over* $\mathrm{Mod}_L(\kappa)$ *iff*

$$\mathbb{P} \cong \prod_{j\in J} \langle [\kappa_j]^{\kappa_j}, \subset\rangle \tag{2.1}$$

for a family of infinite cardinals $\{\kappa_j : j \in J\}$ *such that* $|J| \leqslant 2^{|L|}$ *and* $\sum_{j\in J} \kappa_j = \kappa$;

(b) *If* $2^{|L|} \geqslant \kappa$, *then a poset* $\mathbb{P}$ *is representable over* $\mathrm{Mod}_L(\kappa)$ *iff* $\mathbb{P} \cong 1$ *or (2.1) holds for some family of infinite cardinals* $\{\kappa_j : j \in J\}$ *such that* $\sum_{j\in J} \kappa_j \leqslant \kappa$.

Concerning the problem of representability of Boolean algebras we note that

$$\mathrm{ro}\,\mathrm{sq} \prod_{j\in J} \langle [\kappa_j]^{\kappa_j}, \subset\rangle = \mathrm{ro} \prod_{j\in J} (P(\kappa_j)/[\kappa_j]^{<\kappa_j})^+.$$

In particular, under GCH, the posets of copies of all L structures of size $\kappa > 2^{|L|}$ are forcing equivalent to one or two collapsing algebras. More precisely, we have

Theorem 2.4. [20] *(GCH) Let* L *be a unary relational language and* $\kappa > 2^{|L|}$ *an infinite cardinal. Then a complete Boolean algebra* $\mathbb{B}$ *is* ro*-representable over* $\mathrm{Mod}_L(\kappa)$ *iff*

$$\mathbb{B} \cong \begin{cases} \mathrm{Coll}(\omega_1, \omega_1) & \text{if } \omega = \kappa, \\ \mathrm{Coll}(\omega, \kappa^+) & \text{if } \omega < \mathrm{cf}\,\kappa = \kappa, \\ \mathrm{Coll}(\omega, \kappa^+) & \text{if } \omega < \mathrm{cf}\,\kappa < \kappa \wedge 2^{|L|} < \mathrm{cf}\,\kappa, \\ \mathrm{Coll}(\omega, \kappa^+) \text{ or } \mathrm{Coll}(\omega_1, \kappa^+) & \text{if } \mathrm{cf}\,\kappa < \kappa \wedge (\omega = \mathrm{cf}\,\kappa \vee 2^{|L|} \geqslant \mathrm{cf}\,\kappa). \end{cases}$$

Regarding the similarities of infinite unary structures we have

Theorem 2.5. [20] *For any unary language* L *and infinite cardinal* κ *we have*

(a) $\mathrm{Mod}_L(\kappa)$ *is a CSB class;*

(b) *Figure* 6 *describes the hierarchy of the similarities* $\sim_k$, *for* $k \neq 8, 10$, *on* $\mathrm{Mod}_L(\kappa)$. *If* κ *is a regular cardinal and* $2^\kappa = \kappa^+$, *then* $\sim_8 \neq \sim_{10}$.

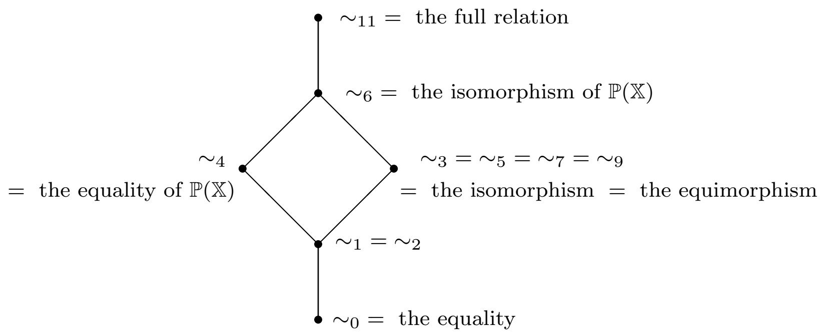

Figure 6. The hierarchy of similarities for unary structures

The following theorem shows that the equivalence of the similarities $\sim_8$ (the isomorphism of $\mathrm{sq}\,\mathbb{P}(\mathbb{X})$) and $\sim_{10}$ (the isomorphism of $\mathrm{ro}\,\mathrm{sq}\,\mathbb{P}(\mathbb{X})$) is independent of ZFC even for the simplest unary language.

Theorem 2.6. [20] *If L is the language containing only one unary relational symbol, then on* $\mathrm{Mod}_L(\omega)$ *we have* $\sim_8 = \sim_6$ *and*

$$\sim_{10} = \begin{cases} \sim_{11} & \text{if the poset } (P(\omega)/\mathrm{Fin})^+ \text{ is forcing equivalent to its square,} \\ \sim_6 & \text{otherwise.} \end{cases}$$

So, under CH, a complete Boolean algebra $\mathbb{B}$ *is representable over* $\mathrm{Mod}_L(\omega)$ *iff* $\mathbb{B} \cong \mathrm{ro}\, P(\omega)/\mathrm{Fin}$.

2.3. Infinite non-unary structures.

For infinite structures of non-unary languages the diagram from Figure 4 does not collapse at all. Namely we have

Theorem 2.7. [20] *If L is a non-unary relational language and κ an infinite cardinal, then in the diagram from Figure 4 describing the similarities $\sim_k$ on the set* $\mathrm{Mod}_L(\kappa)$ *all the implications a–o are proper and there are no new implications (except the ones which follow from transitivity).*

Concerning the proof of the previous theorem we note that first we prove the statement for countable binary structures (constructing, for example, structures $\mathbb{X}$ and $\mathbb{Y}$ such that $\mathrm{sq}\,\mathbb{P}(\mathbb{X}) \cong \mathrm{sq}\,\mathbb{P}(\mathbb{Y})$ but $\mathbb{P}(\mathbb{X}) \not\cong \mathbb{P}(\mathbb{Y})$) and then we apply the following statement, which is of independent interest. Let L_b be the language with exactly one binary relational symbol, λ an infinite cardinal and $\mathrm{Int}^*_{L_b}(\lambda) \subset \mathrm{Int}_{L_b}(\lambda)$ the set of binary relations $\rho \subset \lambda^2$ such that $\rho_{rst} = \lambda^2$ (where ρ_{rst} is the minimal equivalence relation on λ containing ρ, see the next section) and $\rho \cap \Delta_\lambda \neq \emptyset$.

Theorem 2.8. [20] *If $\kappa \geqslant \lambda$ is a cardinal and $L = \langle R_i : i \in I\rangle$ a non-unary relational language, then there is a mapping $\tau : \mathrm{Int}^*_{L_b}(\lambda) \to \mathrm{Int}_L(\kappa)$ such that*

(a) $\mathbb{P}(\kappa, \tau(\rho)) \cong \mathbb{P}(\lambda, \rho)$, *for each* $\rho \in \mathrm{Int}^*_{L_b}(\lambda)$;

(b) *τ preserves all the relations $\sim_k$ from Figure* 4, *that is for each* $k \leqslant 11$

$$\forall \rho, \sigma \in \mathrm{Int}^*_{L_b}(\lambda) \ \ (\rho \sim_k \sigma \Leftrightarrow \tau(\rho) \sim_k \tau(\sigma)).$$

(c) *Each poset representable over the class* $\mathrm{Mod}_{L_b}(\lambda)$ *is representable over the classes* $\mathrm{Mod}^*_{L_b}(\lambda)$ *and* $\mathrm{Mod}_L(\kappa)$.

The previous theorem has the following consequence related to the problem of representability of posets.

Corollary 2.1. [20] *If a poset $\mathbb{P}$ is representable over the class of countable binary structures, then it is representable over the class* $\mathrm{Mod}_L(\kappa)$, *for each non-unary relational language L and each infinite cardinal κ.*

Remark 2.1. By the results of this section, concerning the representability of posets and the hierarchy of similarities over the classes of finite or unary structures we have a sufficiently clear picture. Thus infinite structures of non-unary languages (whose diagram of similarities is given in Figure 4) remain to be explored. By Theorem 2.8(c), the posets representable over the class of binary structures of size λ are representable over the class $\mathrm{Mod}_L(\kappa)$, for any non-unary relational language L and any cardinal $\kappa \geqslant \lambda$, so, it is reasonable to consider binary structures first. So in the rest of the paper we are mainly concerned with binary structures and, in particular, with countable ones, whose rough classification is given in Figure 2.

3. Intermezzo: copies of disconnected binary structures

It turns out that the following natural concept of connectedness of a relational structure (similar to the corresponding concepts in topology and graph theory) gives us an efficient tool for investigation of posets of copies of structures, especially for constructing examples. If $\mathbb{X} = \langle X, \rho \rangle$ is a binary structure, then the transitive closure ρ_{rst} of the relation $\rho_{rs} = \Delta_X \cup \rho \cup \rho^{-1}$ (given by $x \; \rho_{rst} \; y$ iff there are $n \in \mathbb{N}$ and $z_0 = x, z_1, \dots, z_n = y$ such that $z_i \; \rho_{rs} \; z_{i+1}$, for each $i < n$) is the minimal equivalence relation on X containing ρ. For $x \in X$ the corresponding element of the quotient X/ρ_{rst} will be denoted by $[x]$ and called the *component* of $\mathbb{X}$ containing x. The structure $\mathbb{X}$ will be called *connected* iff $|X/\rho_{rst}| = 1$.

If $\mathbb{X}_i = \langle X_i, \rho_i \rangle$, $i \in I$, are binary structures and $X_i \cap X_j = \emptyset$, for different $i, j \in I$, then the structure $\bigcup_{i \in I} \mathbb{X}_i = \left\langle \bigcup_{i \in I} X_i, \bigcup_{i \in I} \rho_i \right\rangle$ will be called the *disjoint union* of the structures $\mathbb{X}_i$, $i \in I$. By [14] we have

Fact 3.1. [14] *If $\mathbb{X} = \langle X, \rho \rangle$ is a binary structure, then*

(a) $\left\langle \bigcup_{x \in X} [x], \bigcup_{x \in X} \rho_{[x]} \right\rangle$ *is the unique representation of $\mathbb{X}$ as a disjoint union of connected structures;*

(b) *at least one of the structures $\mathbb{X}$ and $\mathbb{X}^c$ is connected.*

Fact 3.2. [14] *Let $\langle X, \rho \rangle$ and $\langle Y, \tau \rangle$ be binary structures and $f : X \to Y$ an embedding. Then for each $x \in X$*

(a) $f[[x]] \subset [f(x)]$;

(b) $f \mid [x] : [x] \to f[[x]]$ is an isomorphism;

(c) If, in addition, f is an isomorphism, then $f[[x]] = [f(x)]$.

So, roughly speaking, embeddings of structures must respect their components. In addition, for the embeddings of disconnected structures and their copies, we have the following descriptions from [14].

Theorem 3.1. [14] *Let $\{\mathbb{X}_i : i \in I\}$ and $\{\mathbb{Y}_j : j \in J\}$ be families of disjoint connected binary structures and $\mathbb{X} = \langle X, \rho \rangle$ and $\mathbb{Y} = \langle Y, \sigma \rangle$ their unions. Then*

(a) F is an embedding of $\mathbb{X}$ into $\mathbb{Y}$ if and only if there is a mapping $f : I \to J$ and there are embeddings $g_i : \mathbb{X}_i \hookrightarrow \mathbb{Y}_{f(i)}$, for $i \in I$, such that $F = \bigcup_{i \in I} g_i$ and

$$\forall \{i, i'\} \in [I]^2 \;\; \forall x \in X_i \;\; \forall y \in X_{i'} \;\; \neg \, g_i(x) \, \sigma_{rs} \, g_{i'}(y).$$

(b) $C \in \mathbb{P}(\mathbb{X})$ if and only if there is a mapping $f : I \to I$ and there are embeddings $g_i : \mathbb{X}_i \hookrightarrow \mathbb{X}_{f(i)}$, for $i \in I$, such that $C = \bigcup_{i \in I} g_i[X_i]$ and

$$\forall \{i, i'\} \in [I]^2 \;\; \forall x \in X_i \;\; \forall y \in X_{i'} \;\; \neg \, g_i(x) \, \rho_{rs} \, g_{i'}(y).$$

For some structures (for example, for the disjoint union $\bigcup_{n \in \mathbb{N}} G_n$ of linear graphs (paths) G_n of size n) there are self-embeddings which map more components into one, which produces a chaotic picture of the set of copies. But this is impossible in the three classes of structures described below (and containing some important subclasses) where the investigation of posets of copies and their forcing equivalents becomes easier. Applications of the following three theorems will be given in the rest of the paper.

3.1. Structures with embedding-incomparable components. Relational structures $\mathbb{X}$ and $\mathbb{Y}$ will be called *embedding-incomparable*, in notation $\mathbb{X} \parallel \mathbb{Y}$, iff $\mathbb{X} \not\hookrightarrow \mathbb{Y}$ and $\mathbb{Y} \not\hookrightarrow \mathbb{X}$. Since each self-embedding of a structure with embedding-incomparable components must map each component into itself we have

Theorem 3.2. [14] *Let $\{\mathbb{X}_i : i \in I\}$ be a family of disjoint connected and pairwise embedding-incomparable binary structures and $\mathbb{X}$ its union. Then the elements of $\mathbb{P}(\mathbb{X})$ are of the form $\bigcup_{i\in I} C_i$, where $C_i \in \mathbb{P}(\mathbb{X}_i)$, for $i \in I$, and, hence*

$$\mathbb{P}(\mathbb{X}) \cong \prod_{i\in I} \mathbb{P}(\mathbb{X}_i) \text{ and } \operatorname{sq}\mathbb{P}(\mathbb{X}) \cong \prod_{i\in I} \operatorname{sq}\mathbb{P}(\mathbb{X}_i)$$

3.2. Structures with maximally embeddable components. We will say that a structure $\mathbb{X}$ is *maximally embeddable* into a structure $\mathbb{Y}$ if $\mathbb{P}(\mathbb{X}, \mathbb{Y}) = [Y]^{|X|}$. Clearly, equivalence relations are structures with maximally embeddable components and the following theorem is related to the posets of copies of such structures. We recall that $\mathrm{Fin} \times \mathrm{Fin}$ denotes the Fubini product of the ideal $\mathrm{Fin} = [\omega]^{<\omega}$, that $\Delta := \{\langle m, n\rangle \in \mathbb{N} \times \mathbb{N} : n \leqslant m\}$ and that the ideal $\mathcal{ED}_{\mathrm{fin}} \subset P(\Delta)$ is defined by

$$\mathcal{ED}_{\mathrm{fin}} = \{S \subset \Delta : \exists r \in \mathbb{N}\ \forall m \in \mathbb{N}\ |S \cap (\{m\} \times \{1, 2, \dots, m\})| \leqslant r\}.$$

Let $\mathfrak{h}(\mathbb{P})$ denote the distributivity number of a separative poset $\mathbb{P}$ and let $\mathfrak{h}_n = \mathfrak{h}(((P(\omega)/\mathrm{Fin})^+)^n)$; thus $\mathfrak{h} = \mathfrak{h}_1$. The following results will be used in the sequel.

Theorem 3.3. (a) $(P(\omega \times \omega)/(\mathrm{Fin} \times \mathrm{Fin}))^+$ *is an ω_1-closed, but not ω_2-closed poset* (Szymański and Zhou [36]).

(b) $\mathrm{Con}[\mathfrak{h}((P(\omega \times \omega)/(\mathrm{Fin} \times \mathrm{Fin}))^+) < \mathfrak{h}]$ (Hernández-Hernández [9]).

(c) $\mathrm{Con}[\mathfrak{h}_{n+1} < \mathfrak{h}_n]$, *for each $n \in \mathbb{N}$* (Shelah and Spinas [34, 35]).

(d) $\mathrm{Con}[\mathfrak{h}((P(\Delta)/\mathcal{ED}_{\mathrm{fin}})^+) < \mathfrak{h}]$ (Brendle [2]).

Theorem 3.4. [15] *Let the components $\mathbb{X}_i$, $i \in I$, of a countable binary structure $\mathbb{X} = \langle X, \rho\rangle$ be pairwise maximally embeddable and such that for each $i, j \in I$ and each $A, B \in [\mathbb{X}_j]^{|\mathbb{X}_i|}$ there are $a \in A$ and $b \in B$ such that $a\ \rho_{rs}\ b$. If we define $N = \{|X_i| : i \in I\}$, $N_{\mathrm{fin}} = N \setminus \{\omega\}$, $I_\kappa = \{i \in I : |X_i| = \kappa\}$, for $\kappa \in N$, $|I_\omega| = \mu$ and $Y = \bigcup_{i\in I\setminus I_\omega} X_i$, then $\operatorname{sq}\mathbb{P}(\mathbb{X})$ is an ω_1-closed atomless poset of size $\mathfrak{c}$ and*

$$\operatorname{sq}\mathbb{P}(\mathbb{X}) \cong \begin{cases} (P(\omega)/\mathrm{Fin})^+)^\mu & \text{if } 1 \leqslant \mu < \omega,\ |N_{\mathrm{fin}}| < \omega, |Y| < \omega,\\ ((P(\omega)/\mathrm{Fin})^+)^{\mu+1} & \text{if } 0 \leqslant \mu < \omega,\ |N_{\mathrm{fin}}| < \omega, |Y| = \omega,\\ \mathbb{P} \times ((P(\omega)/\mathrm{Fin})^+)^\mu & \text{if } 0 \leqslant \mu < \omega,\ |N_{\mathrm{fin}}| = \omega,\\ (P(\omega \times \omega)/(\mathrm{Fin} \times \mathrm{Fin}))^+ & \text{if } \mu = \omega, \end{cases}$$

where $\mathbb{P}$ is an ω_1-closed atomless poset, forcing equivalent to $(P(\Delta)/\mathcal{ED}_{\mathrm{fin}})^+$. The structure $\mathbb{X}$ is indivisible iff $N \in [\mathbb{N}]^\omega$ or $N = \{1\}$ or $|I| = 1$ or $|I_\omega| = \omega$. Also

If $\mathbb{X}$ satisfies	$\mathbb{P}(\mathbb{X})$ *is forcing equivalent to*	$\operatorname{sq}\mathbb{P}(\mathbb{X})$ *is*	*ZFC* $\vdash \operatorname{sq}\mathbb{P}(\mathbb{X})$ *is $\mathfrak{h}$-distributive*
$\mu < \omega \wedge \lvert N_{\mathrm{fin}}\rvert < \omega$	$((P(\omega)/\mathrm{Fin})^+)^n$, *for some* $n \in \mathbb{N}$	$\mathfrak{t}$-*closed*	*yes iff* $n = 1$
$\mu < \omega \wedge \lvert N_{\mathrm{fin}}\rvert = \omega$	$(P(\Delta)/\mathcal{ED}_{\mathrm{fin}})^+ \times ((P(\omega)/\mathrm{Fin})^+)^\mu$	ω_1-*closed*	*no*
$\mu = \omega$	$(P(\omega \times \omega)/(\mathrm{Fin} \times \mathrm{Fin}))^+$	ω_1 *but not* ω_2-*closed*	*no*

where $n = 1$ iff $N \in [\mathbb{N}]^{<\omega} \vee (|Y| < \omega \wedge \mu = 1)$.

3.3. Structures with strongly connected components. A connected binary structure $\mathbb{X} = \langle X, \rho \rangle$ will be called *strongly connected* iff for each $A, B \in \mathbb{P}(\mathbb{X})$ there are $a \in A$ and $b \in B$ such that $a \, \rho_{rs} \, b$. It is easy to check that linear orders, full relations, complete graphs and connected copy-atomic structures (see the next section) are strongly connected. The binary tree $\langle {}^{<\omega}2, \supset \rangle$ is a connected, but not strongly connected partial order.

Theorem 3.5. [18] *Let κ be a cardinal and $\mathbb{X} = \bigcup_{\alpha<\kappa} \mathbb{X}_\alpha$ the union of pairwise disjoint, isomorphic and strongly connected binary structures. Then*
(a) $\mathbb{P}(\mathbb{X}) \cong \mathbb{P}(\mathbb{X}_0)^\kappa$ *and* $\operatorname{sq} \mathbb{P}(\mathbb{X}) \cong (\operatorname{sq} \mathbb{P}(\mathbb{X}_0))^\kappa$, *if* $\kappa < \omega$;
(b) $\operatorname{sq} \mathbb{P}(\mathbb{X})$ *is an atomless poset, if* $\kappa \geqslant \omega$;
(c) $\operatorname{sq} \mathbb{P}(\mathbb{X})$ *is an* ω_1*-closed poset, if* $\kappa = \omega$;
(d) $\operatorname{sq} \mathbb{P}(\mathbb{X}) \equiv (P(\omega)/\mathrm{Fin})^+$, *if* $\kappa = \omega$ *and* $|\mathbb{P}(\mathbb{X}_0)| \leqslant 2^\omega = \omega_1$.

The following examples show that for infinite cardinals κ the statements of Theorem 3.5 are the best possible.

Example 3.1. $\kappa = \omega$ and $|\mathbb{P}(\mathbb{X}_0)| \leqslant 2^\omega$, but the poset $\operatorname{sq} \mathbb{P}(\mathbb{X})$ is not ω_2-closed and it is consistent that $\mathbb{P}(\mathbb{X}) \not\equiv (P(\omega)/\mathrm{Fin})^+$.

If $\mathbb{X} = \bigcup_{i<\omega} \mathbb{X}_i$, where $\mathbb{X}_i = \langle X_i, <_i \rangle$, $i < \omega$, are disjoint copies of the linear order $\langle \omega, < \rangle$, then the components of $\mathbb{X}$ are isomorphic, strongly connected and maximally embeddable, so, by Theorem 3.4 we have $\operatorname{sq} \mathbb{P}(\mathbb{X}) \cong (P(\omega \times \omega)/(\mathrm{Fin} \times \mathrm{Fin}))^+$, this poset is not ω_2-closed and, consistently, not $\mathfrak{h}$-distributive and, hence, not forcing equivalent to $(P(\omega)/\mathrm{Fin})^+$.

Example 3.2. Uncountable sums. Let $\kappa > \omega$ and $\mathbb{X} = \bigcup_{\alpha<\kappa} \mathbb{X}_\alpha$, where $\mathbb{X}_\alpha = \langle \{\alpha\}, \emptyset \rangle$, for $\alpha < \kappa$. Then, clearly, $\mathbb{P}(\mathbb{X}) = [\kappa]^\kappa$ and $\operatorname{sq} \mathbb{P}(\mathbb{X}) = (P(\kappa)/[\kappa]^{<\kappa})^+$. It is known [1, p. 377] that, under the GCH, the Boolean completion of the algebra $P(\kappa)/[\kappa]^{<\kappa}$ is isomorphic to the algebra
– $\mathrm{Coll}(\omega, 2^\kappa)$, if $\mathrm{cf}(\kappa) > \omega$; then the poset $\operatorname{sq} \mathbb{P}(\mathbb{X})$ is not ω_1-closed;
– $\mathrm{Coll}(\omega_1, 2^\kappa)$, if $\mathrm{cf}(\kappa) = \omega$; then the poset $\operatorname{sq} \mathbb{P}(\mathbb{X})$ is not ω_2-closed.

In the sequel we consider countable binary structures following their classification from Figure 2.

4. Column A

The column A is the class of countable binary structures $\mathbb{X}$ such that the poset $\mathbb{P}(\mathbb{X})$ is atomic and such structures will be called *copy-atomic*. First we consider its subclass A_1.

4.1. A$_1$: copy-minimal structures. A_1 is the class of countable binary structures $\mathbb{X}$ having only the trivial copy (i.e., satisfying $\mathbb{P}(\mathbb{X}) = \{X\}$) and such structures will be called *copy-minimal*. Concerning the hierarchy of similarities on the class A_1 (a subclass of $\mathrm{Mod}_{L_b}(\omega)$, see Figure 4), the relation $\sim_4$ is the full relation and, hence, the same holds for $\sim_k$, for $k \in \{6, 8, 10, 11\}$. In addition, if $\mathbb{X}$ and $\mathbb{Y}$ are equimorphic structures from A_1 and $f : \mathbb{X} \hookrightarrow \mathbb{Y}$ and $g : \mathbb{Y} \hookrightarrow \mathbb{X}$, then $g \circ f : \mathbb{X} \hookrightarrow \mathbb{X}$ is a surjection which implies that g is a surjection and, hence, an isomorphism; thus $\mathbb{X} \cong \mathbb{Y}$ and A_1 is a CSB class. Consequently, the relations $\sim_k$, for $k \in \{1, 2, 3, 5, 7, 9\}$, are equal and the hierarchy of the similarities $\sim_k$ on the class A_1 is the same as the hierarchy for the class of finite structures, see Figure 5. So, restricted to the class A_1, all the similarities from Figure 4 except the isomorphism are trivial.

We comment on some properties of structures related to the copy-minimality. A relational structure $\mathbb{X}$ is called:

– *rigid* iff $\mathrm{Aut}(\mathbb{X}) = \{\mathrm{id}_X\}$;
– *copy-minimal* iff $\mathbb{P}(\mathbb{X}) = \{X\}$ (iff $\mathrm{Emb}(\mathbb{X}) = \mathrm{Aut}(\mathbb{X})$);
– *embedding-rigid* iff $\mathrm{Emb}(\mathbb{X}) = \{\mathrm{id}_X\}$ (iff it is rigid and copy-minimal);
– *endomorphism-rigid* iff $\mathrm{End}(\mathbb{X}) = \{\mathrm{id}_X\}$;
– a *Jónsson structure* iff $\{A \subset X : A \prec \mathbb{X} \wedge |A| = |X|\} = \{X\}$.

First we note that there are copy-minimal structures of arbitrary size, which follows from the following stronger statement.

Theorem 4.1 (Vopěnka, Pultr, Hedrlín [37])**.** *On any cardinal κ there is a binary relation ρ such that $\langle \kappa, \rho\rangle$ is an endomorphism-rigid structure.*

Concerning the relationship between the rigidity and copy-minimality, we note that the linear order $\langle \mathbb{Z}, <\rangle$ is neither rigid nor copy-minimal; the graph $G_{\mathbb{Z}} = \langle \mathbb{Z}, \rho\rangle$ (see Example 1.2) is copy-minimal, but not rigid; the linear order $\langle \omega, <\rangle$ is a rigid Jónsson structure but not copy-minimal. A simple embedding-rigid (and, hence rigid and copy-minimal) structure is constructed in Example 4.1(c), using the following characterization of rigid, copy-minimal and embedding-rigid disconnected structures.

Theorem 4.2. [19] *Let $\mathbb{X} = \langle X, \rho\rangle$ be a disconnected binary structure. Then*

(a) $\mathbb{X}$ *is rigid iff its components are rigid and pairwise non-isomorphic;*

(b) $\mathbb{X}$ *is copy-minimal iff its components are copy-minimal and there is no sequence $\langle \mathbb{X}_n : n \in \omega\rangle$ of different components of $\mathbb{X}$ such that $\mathbb{X}_n \hookrightarrow \mathbb{X}_{n+1}$, for each $n \in \omega$.*

(c) $\mathbb{X}$ *is embedding-rigid iff its components are embedding-rigid, pairwise nonisomorphic and there is no sequence $\langle \mathbb{X}_n : n \in \omega\rangle$ of different components of $\mathbb{X}$ such that $\mathbb{X}_n \hookrightarrow \mathbb{X}_{n+1}$, for each $n \in \omega$.*

Example 4.1. Applications of Theorem 4.2.

(a) The disjoint union $\mathbb{X} = \bigcup_{n\in\mathbb{N}} L_n$, where L_n is a chain of size n, is a rigid, but not a copy-minimal poset, since by Theorem 3.4, the poset $\mathbb{P}(\mathbb{X})$ is forcing equivalent to the poset $(P(\Delta)/\mathcal{ED}_{\mathrm{fin}})^+$ and, under CH, to the poset $(P(\omega)/\mathrm{Fin})^+$.

(b) The union of disjoint copies of cycle graphs C_n, $n \geqslant 3$, is a copy-minimal, but not a rigid structure.

(c) Let $\mathbb{X}$ be the union of disjoint copies of the digraphs $\mathbb{X}_n = \langle n, \rho_n\rangle$, $n \geqslant 3$, where $\rho_n = \big\{\langle k, k+1\rangle : 0 \leqslant k \leqslant n-2\big\} \cup \big\{\langle 0, n-1\rangle\big\}$. Since the digraphs $\mathbb{X}_n$ are embedding-rigid, connected and pairwise embedding-incomparable, $\mathbb{X}$ is an embedding-rigid countable structure.

By Theorem 4.2(b), the class A_1 is closed under finite disjoint unions but, by Example 4.1(a), it is not closed under infinite disjoint unions. The class A_1 is not closed under substructures since, for example, the countable empty graph embeds in $G_{\mathbb{Z}}$. The following theorem shows that the class A_1 is not closed under infinite products.

Theorem 4.3. [19] *If κ is an infinite cardinal and $\mathbb{X}$ a binary structure of size >1, then the power $\mathbb{X}^\kappa$ is not a copy-minimal structure.*

Finally, the following theorem shows that the subclass of the class A_1 consisting of (embedding-)rigid structures contains $\mathfrak{c}$ many nonisomorphic elements and, hence, the quotient $A_1/\cong$ is of cardinality $\mathfrak{c}$. A binary structure $\mathbb{X} = \langle X, \rho\rangle$ will be called *very connected* iff for each two different elements x and y of X we have $x\rho y$ and $y\rho x$, or there is $z \in X$ such that $x\,\rho\,z$, $z\,\rho\,x$, $y\,\rho\,z$, and $z\,\rho\,y$.

Theorem 4.4. [19] *For each infinite cardinal κ less than the first strongly inaccessible cardinal there is a family of 2^κ-many irreflexive, embedding-rigid, very connected, and pairwise embedding-incomparable binary structures of size κ.*[1]

Although the only nontrivial similarity from Figure 4 over the class A_1 is the isomorphism, there are natural, coarser classifications. Namely, if $\mathbb{X} \sim_b \mathbb{Y}$ denotes that the structures $\mathbb{X}$ and $\mathbb{Y}$ are quantifier free bi-interpretable, then, by [21],

$$\mathbb{X} \cong \mathbb{Y} \Rightarrow \mathbb{X} \sim_b \mathbb{Y} \Rightarrow \mathrm{Emb}(\mathbb{X}) \cong \mathrm{Emb}(\mathbb{Y}) \Rightarrow \mathbb{P}(\mathbb{X}) \cong \mathbb{P}(\mathbb{Y}),$$

which in A_1 means $\mathbb{X} \cong \mathbb{Y} \Rightarrow \mathbb{X} \sim_b \mathbb{Y} \Rightarrow \mathrm{Aut}(\mathbb{X}) \cong \mathrm{Aut}(\mathbb{Y}) \Rightarrow 0 = 0$. So we obtain new similarities between the isomorphism and the full relation. The following, more general theorem describes the automorphism group of the disconnected structures from the class A_1.

Theorem 4.5. [19] *Let $\mathbb{X} = \bigcup_{\alpha<\kappa} \mathbb{X}_\alpha$ be a copy-minimal structure, $\sim\, \subset \kappa \times \kappa$, where $\alpha_1 \sim \alpha_2$ iff $\mathbb{X}_{\alpha_1} \cong \mathbb{X}_{\alpha_2}$ and $\kappa/\sim = \{[\alpha_\beta] : \beta < \lambda\}$. Then $n_\beta := |[\alpha_\beta]| < \omega$, for all $\beta < \lambda$, and $\mathrm{Aut}(\mathbb{X}) \cong \prod_{\beta<\lambda}(\mathrm{Sym}(n_\beta) \times \mathrm{Aut}(\mathbb{X}_{\alpha_\beta})^{n_\beta})$.*

For example, if $\mathbb{X}$ is the disjoint union of one oriented triangle, two oriented quadrangles, etc., then $\mathrm{Aut}(\mathbb{X}) \cong \prod_{n\geqslant 3}(\mathrm{Sym}(n-2) \times (\mathbb{Z}/n)^{n-2})$.

4.2. Other copy-atomic structures.

By Theorem 4.3(b) of [14] the poset $\mathbb{P}(\mathbb{X})$ is atomic iff $\mathrm{sq}\,\mathbb{P}(\mathbb{X}) \cong 1$. So, restricting the similarities from Figure 4 to the class A, we have $\sim_8=\sim_{10}=\sim_{11}=$ the full relation and $\sim_7=\sim_9$ is the equimorphism. From the following simple example from [20] it follows that the class A is not a CSB class.

Example 4.2. The implications b and f in Figure 4 can not be reversed.

$$\mathbb{X} = \langle \omega, \{\langle n, n+1\rangle : n \in \omega\} \cup \{\langle 2n, 2n\rangle : n \in \omega\}\rangle,$$
$$\mathbb{Y} = \langle \omega, \{\langle n, n+1\rangle : n \in \omega\} \cup \{\langle 2n+1, 2n+1\rangle : n \in \omega\}\rangle.$$

Then $\mathbb{P}(\mathbb{X}) = \mathbb{P}(\mathbb{Y}) = \{[2n, \infty) : n \in \omega\}$ and $\mathbb{X} \rightleftarrows \mathbb{Y}$ but $\mathbb{X} \not\cong \mathbb{Y}$.

With some extra work (see [19]), it can be shown that Figure 7 describes the hierarchy of the similarities $\sim_k$ on the class A (that is, all the implications are proper).

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[1]by Theorem 4.4, there are copy-minimal structures of size $\mathfrak{c}$ and, since it is relatively consistent that $\mathfrak{c}$ is a Jónsson cardinal (i.e., that Jónsson structures of size $\mathfrak{c}$ do not exist; see [7]), it is relatively consistent that these structures are not Jónsson.

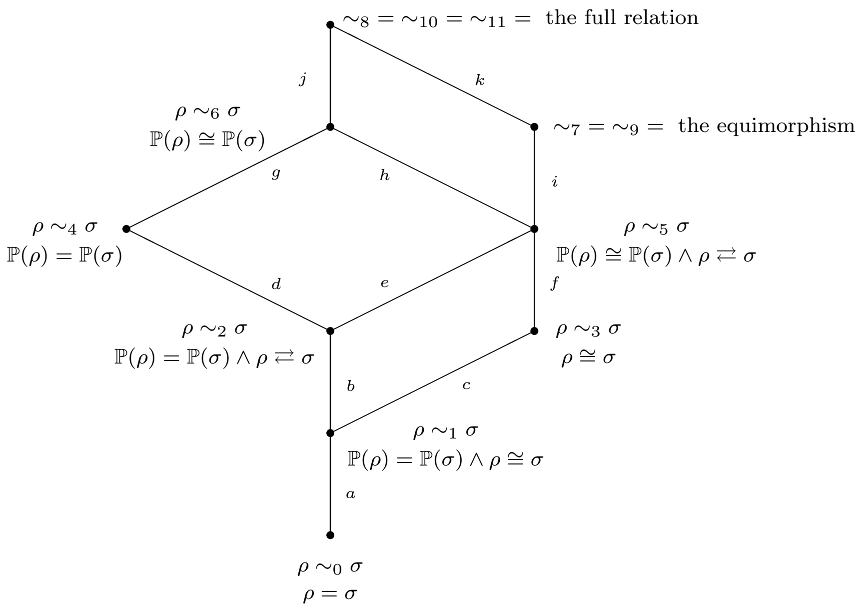

FIGURE 7. The hierarchy of similarities on the class A

The following theorem provides a new tool for the analysis of copy-minimal structures. We recall that, for a set $X \neq \emptyset$, a nonempty family $\mathcal{B} \subset [X]^{|X|}$ is called a *uniform filter base* on X iff for each $A, B \in \mathcal{B}$ there is $C \in \mathcal{B}$ such that $C \subset A \cap B$. Then $\mathcal{F}_{\mathcal{B}} = \{F \subset X : \exists B \in \mathcal{B}\ B \subset F\}$ is a uniform filter on X.

Theorem 4.6. [14] *A relational structure* $\mathbb{X}$ *is copy-atomic iff the set* $\mathbb{P}(\mathbb{X})$ *is a uniform filter base on* X*. Then we have* $\bigcap \mathbb{P}(\mathbb{X}) \in \mathbb{P}(\mathbb{X})$ *iff* $\mathbb{X}$ *is copy-minimal.*

Thus, refining our analysis, we can say that copy-atomic structures $\mathbb{X}$ and $\mathbb{Y}$ are similar if the corresponding filters $\mathcal{F}_{\mathbb{P}(\mathbb{X})}$ and $\mathcal{F}_{\mathbb{P}(\mathbb{Y})}$ are similar (in a way). We remind the reader that the *character of a filter* $\mathcal{F} \subset P(X)$ is the cardinal $\chi(\mathcal{F}) = \min\{|\mathcal{B}| : \mathcal{B} \subset \mathcal{F} \wedge \mathcal{B} \text{ is a base for } \mathcal{F}\}$. For a filter $\mathcal{F}$ on a countable set which does not contain a pseudointersection of $\mathcal{F}$ let us define the cardinal $\mathfrak{p}(\mathcal{F}) = \min\{|\mathcal{P}| : \mathcal{P} \subset \mathcal{F} \wedge \neg\exists F \in \mathcal{F}\ \forall P \in \mathcal{P}\ |F \setminus P| < \omega\}$.

Example 4.3. (a) If X is a copy-minimal structure, then $\mathcal{F}_{\mathbb{P}(\mathbb{X})} = \{X\}$.

(b) The linear graph G_ω from Example 1.2(b) is copy-atomic and the set $\mathbb{P}(G_\omega) = \{[n, \infty)_\omega : n \in \omega\}$ is a base for the Frechét filter on ω.

(c) The graph $\mathbb{X} = C_3 \cup G_\omega$ is copy-atomic since $\mathbb{P}(\mathbb{X}) = \{C_3 \cup [n, \infty)_\omega : n \in \omega\}$ and $\mathbb{P}(\mathbb{X}) \cong \omega^*$. Here we have $\mathcal{F}_{\mathbb{P}(\mathbb{X})} = \{C_3 \cup (\omega \setminus K) : K \in [\omega]^{<\omega}\}$.

(d) Let $\mathbb{X}$ be the union of n disjoint copies of the graph G_ω. Since the graph G_ω is strongly connected, by Theorem 3.5(a) we have $\mathbb{P}(\mathbb{X}) \cong (\omega^*)^n$. Thus $\mathbb{P}(\mathbb{X})$ is an atomic poset (a lattice) and $\mathbb{X}$ is a copy-atomic structure. Here $\mathcal{F}_{\mathbb{P}(\mathbb{X})}$ is the Frechét filter on X.

(e) Let $\mathbb{X} = \langle X, \rho\rangle$, where $X = (\omega \times 3) \smallsetminus \{\langle 0,2\rangle\}$ and

$$\rho = \{\langle\langle m,0\rangle, \langle m+1,0\rangle\rangle : m \in \omega\} \cup \{\langle\langle 0,0\rangle, \langle 0,1\rangle\rangle\} \cup \bigcup_{m\geqslant 1} \{\langle\langle m,n\rangle, \langle m,k\rangle\rangle : 0 \leqslant n < k \leqslant 2\}.$$

It is easy to see that the poset $\mathbb{P}(\mathbb{X})$ is isomorphic to the poset $\langle P, \trianglelefteq\rangle$ where $P = (\omega \times 2) \smallsetminus \{\langle 0,1\rangle\}$ and $\langle p,q\rangle \trianglelefteq \langle p',q'\rangle$ iff $q' \leqslant q$. Thus $\mathbb{P}(\mathbb{X})$ is not a lattice and $\mathcal{F}_{\mathbb{P}(\mathbb{X})}$ is the Frechét filter on X again and, clearly, $\chi(\mathcal{F}_{\mathbb{P}(\mathbb{X})}) = \omega$.

(f) For $n \in \omega$ let $\mathbb{X}_n$ be the digraph obtained by gluing an oriented $(n+3)$-angle to each node of G_ω and let $\mathbb{X}$ be the disjoint union of these digraphs. By Theorem 3.2 the poset $\mathbb{P}(\mathbb{X})$ is isomorphic to the lattice $(\omega^*)^\omega$ thus $\mathbb{X}$ is a copy-atomic structure and $\mathbb{P}(\mathbb{X}) = \mathfrak{c}$. It is easy to see that the copies of $\mathbb{X}$ are coded by functions $f : \omega \to \omega$ and that $\chi(\mathcal{F}_{\mathbb{P}(\mathbb{X})}) = \mathfrak{d}$ and $\mathfrak{p}(\mathcal{F}_{\mathbb{P}(\mathbb{X})}) = \mathfrak{b}$ where $\mathfrak{d}$ (resp. $\mathfrak{b}$) is the dominating (resp. unbounding) number, the minimal size of a dominating (resp. unbounded) family in the preorder $\langle \omega^\omega, \leqslant^*\rangle$.

Concerning the representability of filters by countable binary structures from column A, we note that, by Theorem 3.1(b) of [14], for a countable relational structure $\mathbb{X}$ the set $\mathbb{P}(\mathbb{X})\!\uparrow = \{S \subset X : \exists B \in \mathbb{P}(\mathbb{X})\ B \subset S\}$ is analytic. Thus $\mathcal{F}_{\mathbb{P}(\mathbb{X})}$ is never an ultrafilter.

It is easy to prove that the class of copy-atomic structures is closed under finite disjoint unions but it is not closed under countable unions. Generally, we have

Theorem 4.7. [19] *If κ is an infinite cardinal and $\mathbb{X} = \bigcup_{\alpha<\kappa} \mathbb{X}_\alpha$ a disjoint union of isomorphic, connected and copy-atomic structures, then the poset $\mathbb{P}(\mathbb{X})$ is densely embeddable in the poset $\langle [\kappa]^\kappa, \subset\rangle$ and, hence, $\mathbb{P}(\mathbb{X}) \equiv (P(\kappa)/[\kappa]^{<\kappa})^+$.*

Example 4.4. Let $\mathbb{X}$ be the union of ω disjoint copies of the graph G_ω. By Theorem 4.7, $\mathbb{P}(\mathbb{X}) \equiv (P(\omega)/\operatorname{Fin})^+$ and, hence, $\mathbb{X}$ is not a copy-atomic structure.

5. Column C

Column C contains the countable binary structures $\mathbb{X}$ such that the poset $\operatorname{sq}\mathbb{P}(\mathbb{X})$ is uncountable atomless separative but not ω_1-closed. It can be shown that restricting the similarities $\sim_k$, $k \leqslant 11$, to the class C, we obtain the same picture as in Figure 4 (all the implications are proper) and we can use all of them for classification.

5.1. C_4: forcing with $P(\omega)/\mathcal{I}$.

The structures from the class C_4 are indivisible and the corresponding posets of copies are forcing equivalent to the posets of the form $(P(\omega)/\mathcal{I})^+$, where $\mathcal{I}$ is a co-analytic tall ideal on ω. The class C_4 contains, for example, the rational line $\langle \mathbb{Q}, <\rangle$, and, moreover, all countable non-scattered linear orders. The following two theorems are obtained in collaboration with Stevo Todorčević in [26, 27, 28] and the first of them is related to the similarity class $[\mathbb{Q}]_{\sim_{10}}$ determined by the relation $\sim_{10}$ (isomorphism of Boolean completions adjoined to a structure) and containing all countable non-scattered linear orders. Thus, if $\mathbb{S}$ denotes the Sacks forcing and $\operatorname{sh}(\mathbb{S})$ the size of the continuum in the Sacks extension, then, as a consequence of the main result of [26] we have the following statement.

Theorem 5.1. [26] *For each countable nonscattered linear order L, we have*

$$\mathbb{P}(L) \equiv \mathbb{S} * \pi,$$

where $1_{\mathbb{S}} \Vdash$ *"π is a separative atomless and ω_1-closed poset". If the equality* $\mathrm{sh}(\mathbb{S}) = \aleph_1$ *or PFA holds in the ground model, then*

$$\mathbb{P}(L) \equiv \mathbb{S} * \pi_1,$$

where $1_{\mathbb{S}} \Vdash \pi_1 = (P(\check{\omega})/\,\mathrm{Fin})^+$.

Thus restricted to the class of countable non-scattered linear orders, the similarities $\sim_9$ and $\sim_{10}$ are equal to the full relation. Moreover, it can be shown (see [20]) that, for example, the linear orders $\mathbb{X} = \langle (0,1)_{\mathbb{Q}}, < \rangle$ and $\mathbb{Y} = \langle (0,1]_{\mathbb{Q}}, < \rangle$ are similar in a stronger way: $\mathrm{sq}\,\mathbb{P}(\mathbb{X}) \cong \mathrm{sq}\,\mathbb{P}(\mathbb{Y})$ that is $\mathbb{X} \sim_8 \mathbb{Y}$. But $\sim_6$ sees the difference: $\mathbb{P}(\mathbb{X}) \not\cong \mathbb{P}(\mathbb{Y})$, since the poset $\mathbb{P}(\mathbb{Y})$ is not chain complete.

While the previous theorem is related to the rational line, characterized as the unique countable ultrahomogeneous universal linear order or as the Fraïssé limit of the amalgamation class of all finite linear orders, the following statement shows that the poset of copies of the corresponding object in the class of graphs, the countable random graph (the Rado graph, the Erdős Rényi graph) introduced by Erdős and Rényi [8] (see also [3]) has similar properties. More generally we have

Theorem 5.2. [27, 28] *Let $\mathbb{G}$ be a countable graph containing a copy of the countable random graph. Then*

$$\mathbb{P}(\mathbb{G}) \equiv \mathbb{P} * \pi,$$

where $1_{\mathbb{P}} \Vdash$ *"π is an ω-distributive forcing" and the forcing $\mathbb{P}$ adds a generic real, has the $\aleph_0$-covering property (thus preserves ω_1), has the 2-localization (and, hence, the Sacks) property and does not produce splitting reals.*

5.2. C_3: divisible structures. The class C_3 contains divisible structures from column C. Since an indivisible binary structure is reflexive or irreflexive, it is easy to construct structures from C_3, for example we can reflexify one point of the rational line. The class C_3 also contains structures having posets of copies with extreme forcing theoretic properties, for example a structure $\mathbb{X}$ such that the algebra $\mathrm{ro}\,\mathrm{sq}\,\mathbb{P}(\mathbb{X})$ is isomorphic to the collapsing algebra $\mathrm{Coll}(\omega, 2^\omega)$ (see [14]). We note that the class C contains a large diversity of structures, even when the ultrahomogeneous structures are in question; for example the class C_3 contains the posets $\mathbb{B}_n$, $n \in \mathbb{N}$, from the Schmerl list (see Section 7).

6. Column D

The separative quotients of the posets of copies of the structures from column D are atomless and ω_1-closed and, under CH, all of these posets are forcing equivalent (for example, to the poset $(P(\omega)/\,\mathrm{Fin})^+$) and have Boolean completions isomorphic. Thus, restricting the similarities from Figure 4 to the class D, under CH, we have $\sim_{10} = \sim_{11} =$ the full relation. But in the analysis of the structures from column D we can use the remaining similarities. For example, the structures $\mathbb{X} = \langle \omega, < \rangle$ and $\mathbb{Y} = \langle \omega, \emptyset \rangle$ are not equimorphic but $\mathbb{P}(\mathbb{X}) = \mathbb{P}(\mathbb{Y})$ and, hence, the implications

d, h, k, and n in the diagram for the class D are proper. The same holds for the implications b and f, which implies that the class D is not a CSB class. Namely, if $\mathbb{X}$ and $\mathbb{Y}$ are the structures from Example 4.2, K_ω is the complete graph on ω, and $\mathbb{X}_1$ and $\mathbb{Y}_1$ are the disjoint unions $\mathbb{X} \cup K_\omega$ and $\mathbb{Y} \cup K_\omega$ respectively, then $\mathbb{P}(\mathbb{X}_1) = \mathbb{P}(\mathbb{Y}_1)$ and, by Theorem 3.2, $\operatorname{sq}\mathbb{P}(\mathbb{X}_1) \cong 1 \times (P(\omega)/\operatorname{Fin})^+$ (and both structures are in the class D_3); in addition, $\mathbb{X}_1 \rightleftarrows \mathbb{Y}_1$ but $\mathbb{X}_1 \not\cong \mathbb{Y}_1$. If G'_ω and G''_ω are disjoint copies of G_ω, then for the structures $\mathbb{X} = G'_\omega \cup K_\omega$ and $\mathbb{Y} = G'_\omega \cup G''_\omega \cup K_\omega$ we have $\operatorname{sq}\mathbb{P}(\mathbb{X}) \cong \operatorname{sq}\mathbb{P}(\mathbb{Y})$ but $\mathbb{P}(\mathbb{X}) \not\cong \mathbb{P}(\mathbb{Y})$ and the implications i and j are proper and so on.

In the sequel we show that column D contains several basic relational structures and that it is consistent that the posets of copies of some of these structures are not forcing equivalent.

6.1. Countable scattered linear orders. First, concerning linear orders, we note that in the simplest case, if L is the ordinal ω, then $\langle \mathbb{P}(L), \subset \rangle = \langle [\omega]^\omega, \subset \rangle$ is a homogeneous atomless partial order of size $\mathfrak{c}$ and its separative quotient, the poset $(P(\omega)/\operatorname{Fin})^+$, is ω_1-closed. In [16] using a well known theorem of Laver [31] (stating that a countable scattered linear order is a finite sum of hereditarily indecomposable linear orders) it is shown that the same holds for each countable scattered linear order.

Theorem 6.1. [16] *For each countable scattered linear order L the poset $\langle \mathbb{P}(L), \subset \rangle$ is homogeneous, atomless, of size $\mathfrak{c}$ and its separative quotient is ω_1-closed.*

By Theorems 5.1 and 6.1, concerning the forcing equivalence of posets of copies of countable linear orders under CH we have a clear picture: For each countably infinite linear order L we have

$$\mathbb{P}(L) \equiv \begin{cases} (P(\omega)/\operatorname{Fin})^+ & \text{if } \mathbb{X} \text{ is scattered,} \\ \mathbb{S} * \pi & \text{if } \mathbb{X} \text{ is nonscattered,} \end{cases}$$

where $\mathbb{S}$ is the Sacks forcing and $1_{\mathbb{S}} \Vdash$ "$\pi = (P(\check{\omega})/\operatorname{Fin})^+$".

6.2. Countable ordinals. By [17], if α is a countable ordinal, then the poset $\operatorname{sq}\langle \mathbb{P}(\alpha), \subset \rangle$ is isomorphic to a forcing product of iterated reduced products of Boolean algebras of the form $P(\omega^\gamma)/\mathcal{I}_{\omega^\gamma}$, where γ is a countable limit ordinal or 1. In order to state this we introduce the following notation. For a Boolean lattice $\mathbb{B} = \langle B, \leqslant \rangle$, by $\operatorname{rp}(\mathbb{B})$, we will denote the *reduced power* $\langle B^\omega / \equiv, \leqslant_\equiv \rangle$, where for $\langle b_i \rangle, \langle c_i \rangle \in B^\omega$, $\langle b_i \rangle \equiv \langle c_i \rangle$ (resp. $[\langle b_i \rangle]_\equiv \leqslant_\equiv [\langle c_i \rangle]_\equiv$) iff $b_i = c_i$ (resp. $b_i \leqslant c_i$), for all but finitely many $i \in \omega$. For $n \in \omega$ we define $\operatorname{rp}^n(\mathbb{B})$ by: $\operatorname{rp}^0(\mathbb{B}) = \mathbb{B}$ and $\operatorname{rp}^{n+1}(\mathbb{B}) = \operatorname{rp}(\operatorname{rp}^n(\mathbb{B}))$.

Theorem 6.2. [17] *If $\alpha = \omega^{\gamma_n + r_n} s_n + \cdots + \omega^{\gamma_0 + r_0} s_0 + k$ is a countable ordinal presented in the Cantor normal form, where $k \in \omega$, $r_i \in \omega$, $s_i \in \mathbb{N}$, $\gamma_i \in \operatorname{Lim} \cup \{1\}$ and $\gamma_n + r_n > \cdots > \gamma_0 + r_0$, then*

$$\operatorname{sq}\langle \mathbb{P}(\alpha), \subset \rangle \cong \prod_{i=0}^{n} \Big(\big(\operatorname{rp}^{r_i}(P(\omega^{\gamma_i})/\mathcal{I}_{\omega^{\gamma_i}}) \big)^+ \Big)^{s_i}.$$

By Theorem 6.2, the poset $\mathrm{sq}\langle \mathbb{P}(\alpha), \subset \rangle$ is generated by Boolean algebras of the form $P(\omega^\gamma)/\mathcal{I}_{\omega^\gamma}$. Considering such algebras as forcing notions we assume that $\gamma \geqslant \omega$ is a countable limit ordinal, $\langle \delta_n : n \in \omega \rangle$ a fixed increasing cofinal sequence in $\gamma \setminus \{0\}$ and $\mathcal{L} = \langle L, < \rangle = \sum_{n\in\omega} \langle L_n, <_n \rangle$, where $\langle L_n, <_n \rangle \cong \langle \omega^{\delta_n}, \in \rangle$, for $n \in \omega$, and $L_m \cap L_n = \emptyset$, for $m \neq n$. For $A \subset L$ and $m \in \omega$ let

$$S_A^m = \{n \in \omega : \mathrm{type}(A \cap L_n) \geqslant \omega^{\delta_m}\} \text{ and } \operatorname{supp} A = \{n \in \omega : A \cap L_n \neq \emptyset\}.$$

The ideal $\mathcal{I}_\mathcal{L} = \{A \subset L : \mathcal{L} \not\hookrightarrow A\}$ will be denoted by $\mathcal{I}$ and, if $G \subset P(\omega)$ is an ultrafilter, let $\mathcal{I}_G := \{A \subset L : \exists I \in \mathcal{I} \ \operatorname{supp}(A \setminus I) \notin G\}$. Γ will be the canonical name for a $(P(\omega)/\operatorname{Fin})^+$-generic filter over the ground model V and $q : P(\omega) \to P(\omega)/\operatorname{Fin}$ the quotient mapping. Then concerning forcing with posets of copies of countable ordinals we have

Theorem 6.3. [17] *For each countable limit ordinal γ we have*

$$\langle \mathbb{P}(\omega^\gamma), \subset \rangle \equiv (P(\omega^\gamma)/\mathcal{I}_{\omega^\gamma})^+ \equiv (P(\omega)/\operatorname{Fin})^+ * \big(P(\check{L})/\check{\mathcal{I}}_{\check{q}^{-1}[\Gamma]}\big)^+$$

and $[\omega] \Vdash$ *"$\big(P(\check{L})/\check{\mathcal{I}}_{\check{q}^{-1}[\Gamma]}\big)^+$ is a separative, atomless and ω_1-closed poset".*

Theorem 6.4. [17] *For each countable ordinal $\alpha \geqslant \omega + \omega$ we have*

$$\langle \mathbb{P}(\alpha), \subset \rangle \equiv (P(\omega)/\operatorname{Fin})^+ * \pi,$$

where $[\omega] \Vdash$ *"π is a separative, atomless and ω_1-closed forcing". If, in particular, $\mathfrak{h} = \omega_1$, then $\langle \mathbb{P}(\alpha), \subset \rangle \equiv (P(\omega)/\operatorname{Fin})^+$, for each countable ordinal $\alpha \geqslant \omega$.*

Example 6.1. If $\mathfrak{h}_n = \mathfrak{h}(((P(\omega)/\operatorname{Fin})^+)^n)$, then, clearly, $\mathfrak{h} \geqslant \mathfrak{h}_2 \geqslant \mathfrak{h}_3 \geqslant \ldots \geqslant \omega_1$ and, by Theorem 6.2, $\mathfrak{h}(\mathrm{sq}\langle \mathbb{P}(\omega n), \subset \rangle) = \mathfrak{h}_n$. By Theorem 3.3(c), it is consistent that $\mathfrak{h}_{n+1} < \mathfrak{h}_n$ and, hence, $\langle \mathbb{P}(\omega n), \subset \rangle \not\equiv \langle \mathbb{P}(\omega(n+1)), \subset \rangle$ is consistent as well.

The ideals $\mathcal{I}_{\omega^\delta} = \{I \subset \omega^\delta : \omega^\delta \not\hookrightarrow I\}$, where $0 < \delta < \omega_1$, are called *ordinal* or *indecomposable ideals*. Let $\mathfrak{h}_{\omega^\delta} = \mathfrak{h}((P(\omega^\delta)/\mathcal{I}_{\omega^\delta})^+)$ and $\mathfrak{t}_{\omega^\delta} = \mathfrak{t}((P(\omega^\delta)/\mathcal{I}_{\omega^\delta})^+)$. Then we have

Theorem 6.5. [17] *For each $\gamma \in \mathrm{Lim} \cup \{1\}$ we have*

(a) $\mathfrak{h} \geqslant \mathfrak{h}_{\omega^\gamma} \geqslant \mathfrak{h}_{\omega^{\gamma+1}} \geqslant \ldots \geqslant \mathfrak{h}_{\omega^{\gamma+r}} \geqslant \ldots \geqslant \omega_1$ *and, hence, there is $r_0 \in \omega$ such that $\mathfrak{h}_{\omega^{\gamma+r}} = \mathfrak{h}_{\omega^{\gamma+r_0}}$, for each $r \geqslant r_0$;*

(b) $\mathfrak{t} \geqslant \mathfrak{t}_{\omega^\gamma} \geqslant \mathfrak{t}_{\omega^{\gamma+1}} \geqslant \ldots \geqslant \mathfrak{t}_{\omega^{\gamma+r}} \geqslant \ldots \geqslant \omega_1$ *and, hence, there is $r_0 \in \omega$ such that $\mathfrak{t}_{\omega^{\gamma+r}} = \mathfrak{t}_{\omega^{\gamma+r_0}}$, for each $r \geqslant r_0$.*

Example 6.2. It is easy to show that $\mathcal{I}_{\omega^2} \cong \operatorname{Fin} \times \operatorname{Fin}$ and, consequently, $\mathfrak{h}_{\omega^2} = \mathfrak{h}((P(\omega \times \omega)/(\operatorname{Fin} \times \operatorname{Fin}))^+)$. In [9] Hernández-Hernández proved that in the Mathias model $\mathfrak{h}((P(\omega \times \omega)/(\operatorname{Fin} \times \operatorname{Fin}))^+) = \omega_1$, while $\mathfrak{h} = \mathfrak{c} = \omega_2$. So, by Theorem 6.5, in this model we have $\omega_2 = \mathfrak{c} = \mathfrak{h} = \mathfrak{h}_{\omega^1} > \mathfrak{h}_{\omega^2} = \mathfrak{h}_{\omega^3} = \cdots = \omega_1$.

By Theorem 3.3(a) the poset $(P(\omega \times \omega)/(\operatorname{Fin} \times \operatorname{Fin}))^+$ is not ω_2-closed. Thus, by Theorem 6.5(b), $\mathfrak{t}_{\omega^2} = \mathfrak{t}_{\omega^3} = \cdots = \omega_1$ holds in ZFC.

The position of countable linear orders in the A_1–D_5 classification (Figure 2) is described in Figure 8.

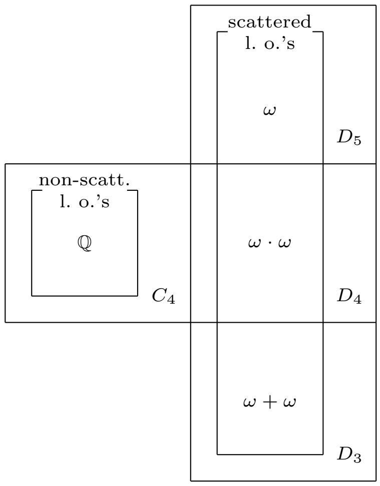

FIGURE 8. Countable linear orders

6.3. Equivalence relations and similar structures. The structures satisfying the assumptions of Theorem 3.4 belong to column D and here we list some typical classes of such structures (see [15]).

Example 6.3. Equivalence relations on countable sets. If $\mathbb{X} = \langle X, \rho \rangle$, where ρ is an equivalence relation on a countable set X, then, clearly, the components X_i, $i \in I$, of $\mathbb{X}$ are the equivalence classes determined by ρ and for each $i \in I$ the restriction ρ_{X_i} is the full relation on X_i, which implies that the assumptions of Theorem 3.4 are satisfied. Thus the poset $\operatorname{sq}\mathbb{P}(\mathbb{X})$ is ω_1-closed and atomless and, hence, $\mathbb{X}$ belongs to the column D. Some examples of such structures are given in Figure 9, where $\bigcup_m F_n$ denotes the union of m disjoint copies of the full relation on a set of size n.

We note that $\mathbb{X}$ is a ultrahomogeneous structure iff all equivalence classes are of the same size, so the following countable equivalence relations are ultrahomogeneous and by Theorem 3.4 have the given properties.

$\bigcup_\omega F_n$. It is indivisible iff $n = 1$ (the diagonal) and the poset $\operatorname{sq}\mathbb{P}(\mathbb{X})$ is isomorphic to the poset $(P(\omega)/\operatorname{Fin})^+$, which is $\mathfrak{t}$-closed and $\mathfrak{h}$-distributive.

$\bigcup_n F_\omega$. It is indivisible iff $n = 1$ (the full relation) and the poset $\operatorname{sq}\mathbb{P}(\mathbb{X})$ is isomorphic to the poset $((P(\omega)/\operatorname{Fin})^+)^n$, which is $\mathfrak{t}$-closed, but, by Theorem 3.3(b), for $n > 1$ not $\mathfrak{h}$-distributive in, for example, the Mathias model.

$\bigcup_\omega F_\omega$ (the ω-homogeneous-universal equivalence relation). It is indivisible and the poset $\operatorname{sq}\mathbb{P}(\mathbb{X})$ is isomorphic to the poset $(P(\omega \times \omega)/(\operatorname{Fin} \times \operatorname{Fin}))^+$, which is ω_1-closed, but not ω_2-closed and consistently neither $\mathfrak{t}$-closed nor $\mathfrak{h}$-distributive.

Example 6.4. Disjoint unions of complete graphs. The same picture as in Example 6.3 is obtained for the countable graphs $\mathbb{X} = \bigcup_{i \in I} \mathbb{X}_i$, where $\mathbb{X}_i = \langle X_i, \rho_i \rangle$, $i \in I$,

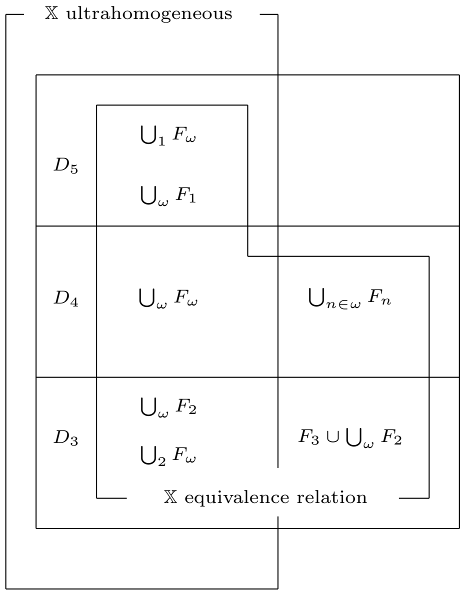

FIGURE 9. Equivalence relations on countable sets

are disjoint complete graphs (that is $\rho_i = (X_i \times X_i) \setminus \Delta_{X_i}$) since, clearly, the assumptions of Theorem 3.4 are satisfied. Also, by a well known characterization of Lachlan and Woodrow [30] all disconnected countable ultrahomogeneous graphs are of the form $\bigcup_m K_n$ (the union of m-many complete graphs of size n), where $mn = \omega$ and $m > 1$. So in Figure 9 we can replace F_n with K_n.

Example 6.5. Disjoint unions of ordinals $\leqslant \omega$. A similar picture is obtained for the countable partial orders $\mathbb{X} = \bigcup_{i\in I} \mathbb{X}_i$, where $\mathbb{X}_i$'s are disjoint copies of ordinals $\alpha_i \leqslant \omega$. (Clearly, linear orders are strongly connected and $\mathbb{P}(\alpha, \beta) = [\beta]^{|\alpha|}$, for each two ordinals $\alpha, \beta \leqslant \omega$.) So in Figure 9 we can replace F_n with L_n, where $L_n \cong n \leqslant \omega$, but these partial orderings are not ultrahomogeneous.

6.4. D_5: copy-maximal structures.

The class D_5 consists of countable binary structures $\mathbb{X}$ having the maximal possible set of copies, $[X]^\omega$. Generally, a relational structure $\mathbb{X} = \langle X, \dots\rangle$ will be called *copy-maximal* iff $\mathbb{P}(\mathbb{X}) = [X]^{|X|}$. The following statement is a generalization of Theorem 6.1 of [14] (characterizing copy-maximal countable binary structures).

Theorem 6.6. *If κ is an infinite cardinal, then for a binary relational structure $\mathbb{X} = \langle \kappa, \rho\rangle$ the following conditions are equivalent:*

(a) $\mathbb{P}(\mathbb{X}) = [\kappa]^\kappa$;
(b) $\mathbb{P}(\mathbb{X})$ *is a dense set in* $\langle [\kappa]^\kappa, \subset\rangle$;
(c) $\mathbb{X} = \langle \kappa, \rho\rangle$ *is isomorphic to one of the following relational structures:*

1 *The empty relation,* $\langle \kappa, \emptyset \rangle$,
2 *The complete graph,* $\langle \kappa, \kappa^2 \setminus \Delta_\kappa \rangle$,
3 *The natural strict linear order on* κ, $\langle \kappa, < \rangle$,
4 *The inverse of the natural strict linear order on* κ, $\langle \kappa, <^{-1} \rangle$,
5 *The diagonal relation,* $\langle \kappa, \Delta_\kappa \rangle$,
6 *The full relation,* $\langle \kappa, \kappa^2 \rangle$,
7 *The natural linear order on* κ, $\langle \kappa, \leqslant \rangle$,
8 *The inverse of the natural linear order on* κ, $\langle \kappa, \leqslant^{-1} \rangle$;

(d) $\mathbb{P}(\mathbb{X})$ *is a somewhere dense set in* $\langle [\kappa]^\kappa, \subset \rangle$;
(e) $\mathcal{I}_\mathbb{X} = [\kappa]^{<\kappa}$.

7. Nonbiconnected ultrahomogeneous structures

Roughly speaking, the main result of [18] is that a classification of posets of copies of biconnected ultrahomogeneous digraphs would provide such classification inside a much wider class of structures. We recall that a structure $\mathbb{X} = \langle X, \rho \rangle$ is a *directed graph (digraph)* iff ρ is an irreflexive and asymmetric binary relation on X. If, in addition, $x\rho y \vee y\rho x$, for each different $x, y \in X$, then $\mathbb{X}$ is a *tournament*. The countable ultrahomogeneous digraphs have been classified by Cherlin [4, 5], see also [32]. Cherlin's list is infinite and includes Schmerl's list of countable ultrahomogeneous strict partial orders [33]:

– $\mathbb{A}_\omega$, a countable antichain (that is, the empty relation on ω),
– $\mathbb{B}_n = n \times \mathbb{Q}$, for $n \in [1, \omega]$, where $\langle i_1, q_1 \rangle < \langle i_2, q_2 \rangle \Leftrightarrow i_1 = i_2 \ \wedge \ q_1 <_\mathbb{Q} q_2$,
– $\mathbb{C}_n = n \times \mathbb{Q}$, for $n \in [1, \omega]$, where $\langle i_1, q_1 \rangle < \langle i_2, q_2 \rangle \Leftrightarrow q_1 <_\mathbb{Q} q_2$,
– $\mathbb{D}$, the unique countable homogeneous universal poset (the random poset),

and Lachlan's list of ultrahomogeneous tournaments [29]:

– $\mathbb{Q}$, the rational line,
– $\mathbb{T}^\infty$, the countable universal ultrahomogeneous tournament,
– $S(2)$, the circular tournament (the local order).

Also we recall the classification of countable ultrahomogeneous graphs given by Lachlan and Woodrow [30]:

– $\mathbb{G}_{\mu,\nu}$, the union of μ disjoint copies of $\mathbb{K}_\nu$, where $\mu\nu = \omega$,
– $\mathbb{G}_{\mathrm{Rado}}$, the unique countable homogeneous universal graph, the Rado graph,
– $\mathbb{H}_n$, the unique countable homogeneous universal $\mathbb{K}_n$-free graph, for $n \geqslant 3$,
– the complements of these graphs.

For convenience we introduce the following notation. If $\mathbb{X} = \langle X, \rho \rangle$ is a binary structure, then its *complement*, $\langle X, \rho^c \rangle$, where $\rho^c = X^2 \setminus \rho$, will be denoted by $\mathbb{X}^c$, its *inverse*, $\langle X, \rho^{-1} \rangle$, by $\mathbb{X}^{-1}$, its *reflexification*, $\langle X, \rho \cup \Delta_X \rangle$, by $\mathbb{X}_{\mathrm{re}}$ and its *irreflexification*, $\langle X, \rho \setminus \Delta_X \rangle$, by $\mathbb{X}_{\mathrm{ir}}$. The binary relation ρ_e on X defined by

$$x \,\rho_e\, y \ \Leftrightarrow \ x \,\rho\, y \vee (x \neq y \wedge \neg\, x \,\rho\, y \wedge \neg\, y \,\rho\, x)$$

will be called the *enlargement* of ρ and the corresponding structure, $\langle X, \rho_e \rangle$, will be denoted by $\mathbb{X}_e$. A structure $\mathbb{X}$ will be called *biconnected* iff both $\mathbb{X}$ and $\mathbb{X}^c$ are connected structures. Using Ramsey's theorem and Theorem 3.5 the following statements are proved in [18].

Theorem 7.1. [18] *For each countable ultrahomogeneous reflexive or irreflexive binary structure* $\mathbb{X}$ *we have*

- *either* $\mathbb{X}$ *is biconnected*
- *or there is an ultrahomogeneous digraph* $\mathbb{Y}$ *and a cardinal* $2 \leqslant \kappa \leqslant \omega$ *such that* $\mathbb{X}$ *is isomorphic to one of the following structures:* $\bigcup_\kappa \mathbb{Y}_e$, $(\bigcup_\kappa \mathbb{Y}_e)^c$, $(\bigcup_\kappa \mathbb{Y}_e)_{\mathrm{re}}$, $((\bigcup_\kappa \mathbb{Y}_e)_{\mathrm{re}})^c$.

In the second case we have

- *either* $\mathbb{P}(\mathbb{X}) \cong \mathbb{P}(\mathbb{Z})^n$, *for some biconnected* $\mathbb{Z}$ *from Cherlin's list and* $n \geqslant 2$,
- *or* $\mathrm{sq}\,\mathbb{P}(\mathbb{X})$ *is an atomless* ω_1*-closed poset (and* $\mathbb{X}$ *belongs to column D).*

Theorem 7.2. [18] *An irreflexive disconnected binary structure is ultrahomogeneous iff its components are isomorphic to the enlargement of an ultrahomogeneous digraph.*

Example 7.1. The posets $\mathbb{B}_n$, $n \in [2, \omega]$, from the Schmerl list are disconnected ultrahomogeneous digraphs (they are disjoint unions of copies of $\mathbb{Q}$) and, by Theorem 7.2, the structures of the form $\bigcup_\kappa (\mathbb{B}_n)_e$ (or its other three variations given in Theorem 7.2) are ultrahomogeneous. For example, by Theorems 7.1 and 5.1 we have $\mathbb{P}(\bigcup_3 (\mathbb{B}_2)_e) \cong \mathbb{P}(\mathbb{Q})^6 \equiv (\mathbb{S} * \pi)^6$. Also under CH we have $\mathbb{P}((\bigcup_\omega (\mathbb{B}_2)_e)^c) \equiv \mathbb{P}(((\bigcup_2 (\mathbb{B}_\omega)_e)_{\mathrm{re}})^c) \equiv (P(\omega)/\mathrm{Fin})^+$.

Example 7.2. For a cardinal ν, the empty structure of size ν, $\mathbb{A}_\nu = \langle \nu, \emptyset \rangle$, can be regarded as an (empty) digraph with ν components. Then $(\mathbb{A}_\nu)_e \cong \mathbb{K}_\nu$ and for the graphs $\mathbb{G}_{\mu,\nu}$ from the Lachlan and Woodrow list we have $\mathbb{G}_{\mu,\nu} = \bigcup_\mu (\mathbb{A}_\nu)_e$. The posets of copies of these graphs were considered in Example 6.4.

Let $\mathcal{U}$ denote the class of all countable reflexive or irreflexive ultrahomogeneous binary structures, $\mathcal{B} = \{\mathbb{X} \in \mathcal{U} : \mathbb{X} \text{ is biconnected}\}$, $\mathcal{D} = \{\mathbb{X} \in \mathcal{U} : \mathbb{X} \text{ is a digraph}\}$, $\mathcal{D}_e = \{\mathbb{X}_e : \mathbb{X} \in \mathcal{D}\}$, $\mathcal{G} = \{\mathbb{X} \in \mathcal{U} : \mathbb{X} \text{ is a graph}\}$, and let $\mathcal{T} = \{\mathbb{X} \in \mathcal{U} : \mathbb{X} \text{ is a tournament}\}$. By Theorem 7.3, the relations between these classes are displayed in Figure 10.

Theorem 7.3. [18] *Let* $\mathbb{Y} \in \mathcal{D}$. *Then*
(a) $\mathbb{Y} \in \mathcal{B}$ *iff* $\mathbb{Y}$ *is connected iff* $\mathbb{Y}_e \in \mathcal{B}$;
(b) $\mathbb{Y} \in \mathcal{D}_e$ *iff* $\mathbb{Y}$ *is a tournament;*
(c) $\mathbb{Y} \in \mathcal{G}$ *iff* $\mathbb{Y} = \mathbb{A}_\omega$ *iff* $\mathbb{Y}_e = \mathbb{K}_\omega$ *iff* $\mathbb{Y}_e \in \mathcal{G}$.

By Theorem 7.1, the class $\mathcal{D}$ of digraphs generates all structures from $\mathcal{U} \setminus \mathcal{B}$ in a very simple way and a classification of the posets $\mathbb{P}(\mathbb{X})$ for the structures $\mathbb{X} \in \mathcal{D} \cap \mathcal{B}$ would provide a classification for the structures $\mathbb{X}$ belonging to a much wider class: $\mathcal{D} \cup \mathcal{D}_{\mathrm{re}} \cup \mathcal{D}_e \cup (\mathcal{D}_e)_{\mathrm{re}} \cup \mathcal{U} \setminus \mathcal{B}$ (where $\mathcal{X}_{\mathrm{re}} := \{\mathbb{X}_{\mathrm{re}} : \mathbb{X} \in \mathcal{X}\}$). So, if, in addition, we obtain a corresponding classification for $\mathbb{X} \in \mathcal{G} \cap \mathcal{B}$ and hence, for $\mathcal{G} \cup \mathcal{G}_{\mathrm{re}}$, it remains to investigate the posets $\mathbb{P}(\mathbb{X})$ for biconnected irreflexive structures $\mathbb{X}$ which are not: graphs (and, hence, $\mathbb{T}_2 \hookrightarrow \mathbb{X}$), digraphs (and, hence, $\mathbb{K}_2 \hookrightarrow \mathbb{X}$), enlarged digraphs (and, hence, $\mathbb{A}_2 \hookrightarrow \mathbb{X}$), thus they do not have forbidden substructures of size 2.

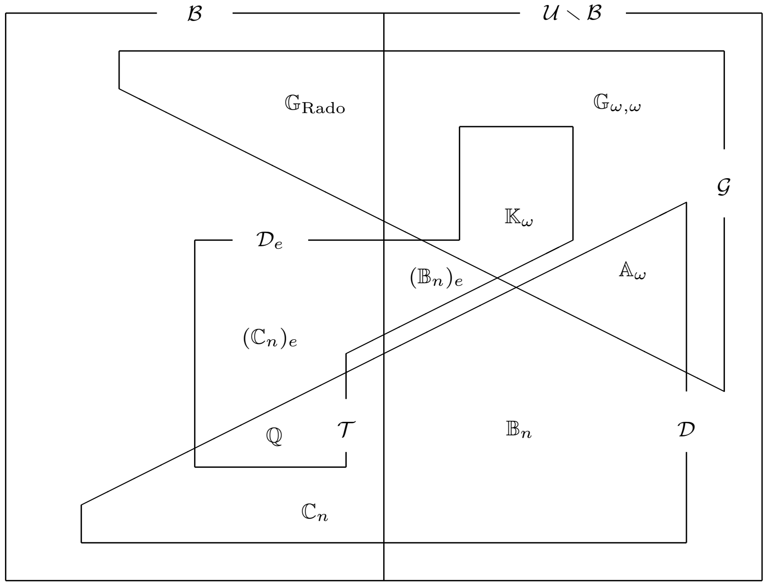

FIGURE 10. Countable reflexive or irreflexive ultrahomogeneous binary structures

8. Maximal chains and antichains of copies

The general concept – to explore the class of posets of copies of relational structures and to obtain the corresponding classifications – can be developed in several ways. Regarding the order-theoretic aspect, one of the extensively investigated order invariants of a poset $\mathbb{P}$ is the class of order types of its maximal chains, $\mathcal{M}(\mathbb{P})$. We mention three related results. Sabine Koppelberg in [10] characterized the class $\mathcal{M}(\text{Intalg}[0,1)_\mathbb{R})$ as the class of types of dense σ-compact subsets of $[0,1]_\mathbb{R}$ containing 0 and 1. By a theorem of Kuratowski [11], the class $\mathcal{M}(P(\kappa))$ is the class of types of the orders $\langle \text{Init}(L), \subset \rangle$, for linear orders L of size κ, where $\text{Init}(L)$ denotes the set of all initial segments of L. Day in [6] proved that a linear order is isomorphic to a maximal chain in a $< \kappa$-complete atomic Boolean algebra iff it is $< \kappa$-complete, has 0 and 1 and has dense jumps.

Regarding the partial orders of the form $\mathbb{P}(\mathbb{X})$, where $\mathbb{X}$ is a relational structure, the equality $\mathcal{M}(\mathbb{P}(\mathbb{X})) = \mathcal{M}(\mathbb{P}(\mathbb{Y}))$ defines a certain similarity of structures and induces their classification. Concerning the class of countable binary structures, the class $\mathcal{M}(\mathbb{P}(\mathbb{X}))$ was characterized for (in particular) copy-maximal structures [12], for the rational line [13], for the Rado graph [22], and, finally, for all ultrahomogeneous posets (Schmerl's list) [23] and all ultrahomogeneous graphs (the list of Lachlan and Woodrow) [24]. The last three are joint results with Boriša Kuzeljević. In order to state these results we introduce the following notation. Let $\mathbb{R}$ be the real line, $\mathcal{C}_{\mathbb{R}}$ the class of order types of sets of the form $K \setminus \{\min K\}$, where $K \subset \mathbb{R}$ is a compact set, such that $\min K$ is an accumulation point of K and let $\mathcal{B}_{\mathbb{R}}$ be the subclass of $\mathcal{C}_{\mathbb{R}}$ determined by nowhere dense sets. Now, if $\mathbb{X}$ is a copy-maximal countable structure, then $\mathcal{M}(\mathbb{P}(\mathbb{X})) = \mathcal{B}_{\mathbb{R}}$ and for the ultrahomogeneous graphs and posets we have

Theorem 8.1. [24] *For a countable ultrahomogeneous graph* $\mathbb{X}$ *we have*

$$\mathcal{M}(\mathbb{P}(\mathbb{X})) = \begin{cases} \mathcal{C}_{\mathbb{R}} & \text{if } \mathbb{X} = \mathbb{G}_{\text{Rado}} \text{ or } \mathbb{X} = \mathbb{H}_n, \text{ for some } n \geqslant 3, \\ \mathcal{B}_{\mathbb{R}} & \text{if } \mathbb{X} = \mathbb{G}_{\mu,\nu}, \text{ where } \mu\nu = \omega. \end{cases}$$

Theorem 8.2. [23] *For a countable ultrahomogeneous partial order* $\mathbb{X}$ *we have*

$$\mathcal{M}(\mathbb{P}(\mathbb{X})) = \begin{cases} \mathcal{C}_{\mathbb{R}} & \text{if } \mathbb{X} = \mathbb{B}_n \text{ or } \mathbb{X} = \mathbb{C}_n, \text{ or } \mathbb{X} = \mathbb{D}, \\ \mathcal{B}_{\mathbb{R}} & \text{if } \mathbb{X} = \mathbb{A}_\omega. \end{cases}$$

One more invariant of a poset $\mathbb{P}$ is the set of cardinalities of maximal antichains in $\mathbb{P}$. This cardinal invariant was investigadted in a collaboration with Petar Marković in [25] and, in particular, it is shown that the poset of copies of the Rado graph contains maximal antichains of size $\mathfrak{c}$, ω and n, for each positive integer n. It is easy to see that the same holds for the rational line, while, clearly, in the posets of copies of the countable copy-maximal binary structures, maximal antichains of size ω do not exist.

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References

1. B. Balcar, P. Simon, *Disjoint refinement*, in: J. D. Monk and R. Bonnet (eds.), *Handbook of Boolean algebras*, Vol. 2, Elsevier, Amsterdam, 1989, 333–388.
2. J. Brendle, *Cardinal invariants of analytic quotients*, a lecture at the Bonn University, 16.6.2009.
3. P. J. Cameron, *The random graph*, in: *The Mathematics of Paul Erd˝s II*, Algorithms Combin. **14**, Springer, Berlin, 1997, 333–351.
4. G. Cherlin, *Homogeneous directed graphs. The imprimitive case*, in: *Logic colloquium '85* (Orsay, 1985), Stud. Logic Found. Math. **122**, North-Holland, Amsterdam, 1987, 67–88.
5. ———, *The classification of countable homogeneous directed graphs and countable homogeneous n-tournaments*, vol. 131, Mem. Am. Math. Soc. **621**, Am. Math. Soc., 1998.
6. G. W. Day, *Maximal chains in atomic Boolean algebras*, Fund. Math. **67** (1970), 293–296.
7. K. J. Devlin, *Some weak versions of large cardinal axioms*, Ann. Math. Logic **5** (1972–1973), 291–325.
8. P. Erdős, A. Rényi, *Asymmetric graphs*, Acta Math. Acad. Sci. Hungar. **14** (1963), 295–315.
9. F. Hernández-Hernández, *Distributivity of quotients of countable products of Boolean algebras*, Rend. Istit. Mat. Univ. Trieste **41** (2009), 27–33.
10. S. Koppelberg, *Maximal chains in interval algebras*, Algebra Univers. **27**(1) (1990), 32–43.
11. K. Kuratowski, *Sur la notion de l'ordre dans la théorie des ensembles*, Fund. Math. **2** (1921), 161–171.
12. M. S. Kurilić, *Maximal chains in positive subfamilies of* $P(\omega)$, Order **29**(1) (2012), 119–129.
13. ———, *Maximal chains of copies of the rational line*, Order **30**(3) (2013), 737–748.
14. ———, *From* A_1 *to* D_5*: Towards a forcing-related classification of relational structures*, J. Symbolic Logic **79**(1) (2014), 279–295.

15. ______, *Maximally embeddable components*, Arch. Math. Logic **52**(7) (2013), 793–808.
16. ______, *Posets of copies of countable scattered linear orders*, Ann. Pure Appl. Logic **165** (2014), 895–912.
17. ______, *Forcing with copies of countable ordinals*, Proc. Am. Math. Soc. **143**(4) (2015), 1771–1784.
18. ______, *Isomorphic and strongly connected components*, Arch. Math. Logic **54**(1–2) (2015), 35–48.
19. ______, *Embedding-minimal and embedding-rigid structures*, (to appear).
20. ______, *Different Similarities*, (to appear).
21. ______, *Monoids of self embeddings*, (to appear).
22. M. S. Kurilić, B. Kuzeljević, *Maximal chains of isomorphic subgraphs of the Rado graph*, Acta Math. Hungar. **141**(1) (2013), 1–10.
23. ______, *Maximal chains of isomorphic suborders of countable ultrahomogeneous partial orders*, Order **32**(1) (2015), 83–99.
24. ______, *Maximal chains of isomorphic subgraphs of countable ultrahomogeneous graphs*, Adv. Math. **264** (2014), 762–775.
25. M. S. Kurilić, P. Marković, *Maximal antichains of isomorphic subgraphs of the Rado graph*, Filomat (in print).
26. M. S. Kurilić, S. Todorčević, *Forcing by non-scattered sets*, Ann. Pure Appl. Logic **163** (2012), 1299–1308.
27. ______, *Copies of the Rado graph*, (to appear).
28. ______, *Forcing by copies of the Rado graph*, (to appear).
29. A. H. Lachlan, *Countable homogeneous tournaments*, Trans. Am. Math. Soc. **284** (1984), 431–461.
30. A. H. Lachlan, R. E. Woodrow, *Countable ultrahomogeneous undirected graphs*, Trans. Am. Math. Soc. **262**(1) (1980), 51–94.
31. R. Laver, *On Fraïssé's order type conjecture*, Ann. of Math. **93**(2) (1971), 89–111.
32. D. Macpherson, *A survey of homogeneous structures*, Discrete Math. **311**(15) (2011), 1599–1634.
33. J. Schmerl, *Countable homogeneous partially ordered sets*, Alg. Univ. **9** (1970), 317–321.
34. S. Shelah, O. Spinas, *The distributivity numbers of $P(\omega)/fin$ and its square*, Trans. Am. Math. Soc. **352**(5) (2000), 2023–2047.
35. ______, *The distributivity numbers of finite products of $P(\omega)/fin$*, Fund. Math. **158**(1) (1998), 81–93.
36. A. Szymański, Zhou Hao Xua, *The behaviour of ω^{2^*} under some consequences of Martin's axiom*, in: *General topology and its relations to modern analysis and algebra V* (Prague, 1981), Sigma Ser. Pure Math. **3**, Heldermann, Berlin, 1983, 577–584.
37. P. Vopěnka, A. Pultr, Z. Hedrlín, *A rigid relation exists on any set*, Comment. Math. Univ. Carolinae **6** (1965), 149–155.