Dušan Zorica*

CREEP AND STRESS RELAXATION IN A VISCOELASTICITY THEORY WITH DERIVATIVES OF FRACTIONAL ORDER

Abstract. We study stress relaxation, creep and forced oscillations of a viscoelastic rod. One end of a rod is fixed, while we prescribe displacement or stress on the other end of a rod. We assume a general form of distributed-order fractional constitutive equation. Then we specify it for the solid and fluid-like viscoelastic body and obtain the displacement and stress. Existence of the solution for displacement and stress is proved via the Laplace transform method. Numerical examples in cases of stress relaxation and creep are presented as well.

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**Mathematical Institute of Serbian Academy of Arts and Sciences, Beograd, Serbia*

Contents

1. Introduction

In this review paper we present the results obtained in [3, 5, 6]. The aim of this paper is generalize classical wave equation for one-dimensional elastic body. Consider the equation of motion

$$\frac{\partial}{\partial x}\sigma(x,t) = \rho\frac{\partial^2}{\partial t^2}u(x,t), \quad x \in [0,L], \ t > 0, \tag{1.1}$$

where ρ, σ and u denote density, stress and displacement of a material at a point positioned at x and at a time t, respectively. It is coupled with a constitutive equation which corresponds to a generalized viscoelastic body:

$$\int_0^1 \phi_\sigma(\eta)\,{}_0\mathrm{D}_t^\eta\sigma(x,t)\,d\eta = E\int_0^1 \phi_\varepsilon(\eta)\,{}_0\mathrm{D}_t^\eta\varepsilon(x,t)\,d\eta, \quad x \in [0,L], \ t > 0, \tag{1.2}$$

where E is a generalized Young modulus (positive constant having dimension of stress), ϕ_σ and ϕ_ε are given functions or distributions and ε is a strain measure, defined by

$$\varepsilon(x,t) = \frac{\partial}{\partial x}u(x,t), \quad x \in [0,L], \ t > 0. \tag{1.3}$$

Recall, ${}_0\mathrm{D}_t^\eta y$ is the left Riemann–Liouville fractional derivative of a function $y \in AC([0,T])$, for every $T>0$, of the order $\eta \in [0,1)$, defined as

$${}_0\mathrm{D}_t^\eta y(t) := \frac{1}{\Gamma(1-\eta)} \frac{d}{dt} \int_0^t \frac{y(\tau)}{(t-\tau)^\eta} d\tau, \quad t>0,$$

where Γ is the Euler gamma function. Recall, $AC([0,T])$ denotes the space of absolutely continuous functions. For a detailed account on fractional calculus we refer to [16]. In the case when ϕ_σ and ϕ_ε are distributions, we assume that ϕ_σ and ϕ_ε are compactly supported by $[0,1]$ ($\phi_\sigma, \phi_\varepsilon \in \mathcal{E}'(\mathbb{R})$, $\operatorname{supp}\phi_\sigma, \operatorname{supp}\phi_\varepsilon \subset [0,1]$). In this case integrals in (1.2) are as

$$\left\langle \int_{\operatorname{supp}\phi} \phi(\eta)\, {}_0\mathrm{D}_t^\eta h(t)\, d\eta, \varphi(t) \right\rangle := \big\langle \phi(\eta), \langle {}_0\mathrm{D}_t^\eta h(t), \varphi(t)\rangle \big\rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}).$$

For details see [4]. Recall, $\mathcal{S}'_+(\mathbb{R})$ denotes the space of tempered distributions supported by $[0,\infty)$ and $\langle h(t), \varphi(t)\rangle$ denotes the action of a distribution $h \in \mathcal{S}'_+(\mathbb{R})$ on a test function $\varphi \in \mathcal{S}(\mathbb{R})$, see [18].

We prescribe initial conditions for system (1.1)–(1.3)

$$u(x,0)=0, \quad \frac{\partial}{\partial t}u(x,0)=0, \quad \sigma(x,0)=0, \quad \varepsilon(x,0)=0, \quad x\in[0,L], \tag{1.4}$$

as well as with two types of boundary conditions

$$u(0,t)=0, \quad u(L,t)=\Upsilon(t), \quad t\in\mathbb{R}, \tag{1.5}$$

$$u(0,t)=0, \quad \sigma(L,t)=\Sigma(t), \quad t\in\mathbb{R}. \tag{1.6}$$

Functions Υ and Σ are locally integrable functions supported in $[0,\infty)$. Note that the $(1.5)_1$, or $(1.6)_1$, means that the one rod's end is fixed at $x=0$, while $(1.5)_2$ ($(1.6)_2$) describes the prescribed displacement (stress) of the other rod's end. If $\Upsilon = \Upsilon_0 H$, then we have the case of stress relaxation, while if $\Sigma = \Sigma_0 H$, then we have the case of creep, where H is the Heaviside function.

In the sequel we shall treat system (1.1)–(1.3), subject to (1.4) and either (1.5), or (1.6) for two types of viscoelastic bodies.

I Solid-like viscoelastic body can be modelled by the distributed-order constitutive equation (1.2) with the choice of constitutive (weight) functions

$$\phi_\sigma(\eta) := a^\eta, \quad \phi_\varepsilon(\eta) := b^\eta, \quad \eta\in(0,1), \ a \leqslant b. \tag{1.7}$$

The restriction $a \leqslant b$ follows from the Second Law of Thermodynamics, see [1, 2]. If $a=b$, then (1.2), with (1.7), reduces to the Hooke Law. The choice of ϕ_σ and ϕ_ε in the form (1.7) is the simplest choice guaranteeing dimensional homogeneity.

II Fluid-like viscoelastic body can be modelled by the constitutive equation (1.2) with the choice of constitutive (weight) distributions

$$\begin{aligned} \phi_\sigma(\eta) &:= \delta(\eta) + \frac{a}{b}\delta(\eta-(\alpha-\beta)), \\ \phi_\varepsilon(\eta) &:= a\delta(\eta-\alpha) + c\delta(\eta-\gamma) + \frac{ac}{b}\delta(\eta-(\alpha+\gamma-\beta)), \end{aligned} \tag{1.8}$$

as proposed in [17], where a, b, c are given positive constants, while $0 < \beta < \alpha < \gamma < \frac{1}{2}$. It should be stressed that in [17] there is no discussion concerning restrictions on the parameters $a, b, c, \alpha, \beta, \gamma$. Instead, only $a, b, c \geqslant 0$, $0 \leqslant \beta < \alpha \leqslant 1$ and $0 \leqslant \gamma \leqslant 1$ were assumed. The constitutive equation (1.2), (1.8) is obtained in [17] via the rheological model that generalizes the classical Zener rheological model by substituting spring and dashpot elements by fractional elements. It is assumed that the stress-strain relation for the fractional element is given by $\sigma(t) = {}_0\mathrm{D}_t^\eta \varepsilon$, $t > 0$, where $\eta \in [0, 1]$. We refer to [17] for more details of the derivation, creep compliance and relaxation modulus.

Remark 1.1. The choice of constitutive functions or distributions ϕ_σ and ϕ_ε is not arbitrary, as implied above. Namely, it has to be checked whether (1.2) satisfies the restrictions following from the Second Law of Thermodynamics. This type of restrictions are obtained in [1, 2]. We refer to [3] for a systematic review of restrictions on parameters if ϕ_σ and ϕ_ε are given in the terms of a sum of the Dirac δ distributions as

$$\phi_\sigma(\gamma) := \sum_{n=0}^{N} a_n \delta(\gamma - \alpha_n), \quad \phi_\varepsilon(\gamma) := \sum_{m=0}^{M} b_m \delta(\gamma - \beta_m), \quad \alpha_n, \beta_m \in [0, 1].$$

Remark 1.2. Physically, the difference between solid and fluid-like materials is observed in the creep test (i.e., when material is subjected to a sudden, but later constant force on its free end). Namely, solid-like materials creep to a finite value of displacement, while the fluid-like materials creep to an infinite value of displacement.

We refer to [11, 12, 15] for the detailed account of applications of fractional calculus in viscoelasticity. Problems similar to (1.1)–(1.3) were also treated in [13, 14] with the constitutive equations related to the distributed-order model (1.2) in the special cases.

2. Formal solution to system (1.1)–(1.3), (1.4) and either (1.5), or (1.6)

Introducing dimensionless quantities

$$\bar{x} = \frac{x}{L}, \quad \bar{t} = \frac{t}{L\sqrt{\rho/E}}, \quad \bar{u} = \frac{u}{L}, \quad \bar{\sigma} = \frac{\sigma}{E}, \quad \bar{\Upsilon} = \frac{\Upsilon}{L}, \quad \bar{\Sigma} = \frac{\Sigma}{E}$$
$$\bar{\phi}_\sigma = \frac{\phi_\sigma}{\left(L\sqrt{\rho/E}\,\right)^\alpha}, \quad \bar{\phi}_\varepsilon = \frac{\phi_\varepsilon}{\left(L\sqrt{\rho/E}\,\right)^\alpha},$$

and using the fact that the fractional derivative transforms as

$${}_0\mathrm{D}_{\bar{t}}^\alpha u(\bar{t}) = \left(L\sqrt{\rho/E}\,\right)^\alpha {}_0\mathrm{D}_t^\alpha u(t),$$

we obtain, after omitting bar over dimensionless quantities, the following system

$$
\begin{aligned}
&\frac{\partial}{\partial x}\sigma(x,t) = \frac{\partial^2}{\partial t^2}u(x,t),\\
&\int_0^1 \phi_\sigma(\alpha)\,{}_0\mathrm{D}_t^\alpha \sigma(x,t)d\alpha = \int_0^1 \phi_\varepsilon(\alpha)\,{}_0\mathrm{D}_t^\alpha \varepsilon(x,t)d\alpha,\\
&\varepsilon(x,t) = \frac{\partial}{\partial x}u(x,t),\ \ x\in[0,1],\ t>0.
\end{aligned}
\tag{2.1}
$$

System (2.1) is subject to initial

$$
u(x,0)=0,\quad \frac{\partial}{\partial t}u(x,0)=0,\quad \sigma(x,0)=0,\quad \varepsilon(x,0)=0,\ \ x\in[0,1], \tag{2.2}
$$

and two types of boundary conditions

$$
u(0,t)=0,\ \ u(1,t)=\Upsilon(t),\ \ t\in\mathbb{R}, \tag{2.3}
$$

$$
u(0,t)=0,\ \ \sigma(1,t)=\Sigma(t),\ \ t\in\mathbb{R}. \tag{2.4}
$$

Formally applying the Laplace transformation to (2.1) and (2.2), we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial x}\tilde{\sigma}(x,s) = s^2\tilde{u}(x,s),\\
&\tilde{\sigma}(x,s)\int_0^1 \phi_\sigma(\alpha)s^\alpha d\alpha = \tilde{\varepsilon}(x,s)\int_0^1 \phi_\varepsilon(\alpha)s^\alpha d\alpha,\\
&\tilde{\varepsilon}(x,s) = \frac{\partial}{\partial x}\tilde{u}(x,s),\ \ x\in[0,1],\ s\in D.
\end{aligned}
\tag{2.5}
$$

Recall, the Laplace transformation of $f\in L^1_{loc}(\mathbb{R})$, $f\equiv 0$ in $(-\infty,0]$ and $|f(t)|\leqslant ce^{at}$, $t>0$, for some $a>0$, is defined by

$$
\tilde{f}(s) = \mathcal{L}[f(t)](s) := \int_0^\infty f(t)e^{-st}dt,\quad \operatorname{Re} s > a
$$

and analytically continued into the appropriate domain D. Domain D for (2.5) is determined after (2.9), bellow.

System (2.5) reduces to

$$
\frac{\partial^2}{\partial x^2}\tilde{u}(x,s) - (sM(s))^2\tilde{u}(x,s) = 0,\ \ x\in[0,1],\ s\in D, \tag{2.6}
$$

where we introduced

$$
M(s) := \sqrt{\frac{\int_0^1 \phi_\sigma(\alpha)s^\alpha d\alpha}{\int_0^1 \phi_\varepsilon(\alpha)s^\alpha d\alpha}},\quad s\in D.
$$

Since our aim is to consider the prescribed displacement (stress) response of two types of viscoelastic bodies, for the solid-like viscoelastic body described by the constitutive functions (1.7) function M becomes

$$
M_s(s) := \sqrt{\frac{\ln(bs)}{\ln(as)}\frac{as-1}{bs-1}},\ \ s\in\mathbb{C}\setminus(-\infty,0], \tag{2.7}
$$

while for the fluid-like viscoelastic body described by the constitutive functions (1.8) the function M takes the form

$$M_f(s) := \sqrt{\frac{1+\frac{a}{b}s^{\alpha-\beta}}{as^{\alpha}+cs^{\gamma}+\frac{ac}{b}s^{\alpha+\gamma-\beta}}} = \frac{1}{\sqrt{as^{\alpha}}}\sqrt{\frac{1+\frac{a}{b}s^{\alpha-\beta}}{1+\frac{c}{a}s^{\gamma-\alpha}+\frac{c}{b}s^{\gamma-\beta}}}. \tag{2.8}$$

Solution of (2.6) is of the form

$$\tilde{u}(x,s) = C_1(s)e^{xsM(s)} + C_2(s)e^{-xsM(s)}, \quad x \in [0,1], \quad s \in D, \tag{2.9}$$

where C_1 and C_2 are functions of s which will be determined from the boundary conditions. Since the power function s^{η} is analytic on the complex plane except the branch cut along the negative axis (including the origin) we take $D := \mathbb{C} \setminus (-\infty, 0]$ to be the domain for variable s in (2.5). Applying either $(2.3)_1$ or $(2.4)_1$, we obtain $C_1 = -C_2 =: C$, and thus

$$\tilde{u}(x,s) = C(s)(e^{xsM(s)} - e^{-xsM(s)}), \quad x \in [0,1], \quad s \in \mathbb{C} \setminus (-\infty, 0]. \tag{2.10}$$

From (2.5) and (2.8) it follows that

$$\tilde{\sigma}(x,s) = \frac{1}{M^2(s)}\frac{\partial}{\partial x}\tilde{u}(x,s), \quad x \in [0,1], \ s \in \mathbb{C} \setminus (-\infty, 0]. \tag{2.11}$$

We shall separately seek solutions in different cases: displacement u and stress σ in the case of prescribed displacement (such as e.g., stress relaxation), and displacement u in the case of prescribed stress (e.g., creep). For the former we supply to the system boundary conditions (2.3), while for the latter we assume (2.4). In the sequel we derive convolution forms of solutions in all these cases.

In the case of prescribed displacement Υ, substituting (2.10) into (2.9) and using (2.3), one obtains

$$\tilde{u}(x,s) = \tilde{\Upsilon}(s)\tilde{P}(x,s), \quad x \in [0,1], \ s \in \mathbb{C} \setminus (-\infty, 0], \tag{2.12}$$

where

$$\tilde{P}(x,s) := \frac{\sinh(xsM(s))}{\sinh(sM(s))}, \quad x \in [0,1], \ s \in \mathbb{C} \setminus (-\infty, 0]. \tag{2.13}$$

Clearly, $P(1,t) = \delta(t)$, $t \in \mathbb{R}$.

Since Υ and P are supported in $[0,\infty)$, displacement u is given by

$$u(x,t) = \Upsilon(t) * P(x,t), \quad x \in [0,1], \ t \in \mathbb{R}, \tag{2.14}$$

and

$$u(x,t) = 0, \quad x \in [0,1], \ t < 0,$$

where $*$ denotes the convolution with respect to t. Recall, if $f, g \in L^1_{loc}(\mathbb{R})$, $\operatorname{supp} f, g \subset [0,\infty)$, then $(f * g)(t) := \int_0^t f(\tau)g(t-\tau)\,d\tau$, $t \in \mathbb{R}$. Explicit calculation of (2.14) will be done by the use of the Laplace inversion formula applied to (2.12).

Further, from (2.11), (2.12) and (2.13) it follows that

$$\tilde{\sigma}(x,s) = s\tilde{\Upsilon}(s)\tilde{T}(x,s), \quad x \in [0,1], \quad s \in \mathbb{C} \setminus (-\infty, 0], \tag{2.15}$$

where

$$\tilde{T}(x,s) = \frac{\cosh(xsM(s))}{M(s)\sinh(sM(s))}, \quad x \in [0,1], \ s \in \mathbb{C} \setminus (-\infty, 0]. \tag{2.16}$$

Applying the Laplace inversion formula to (2.15) we obtain

$$\sigma(x,t) = \frac{d}{dt}(\Upsilon(t) * T(x,t)), \quad x \in [0,1], \ t \in \mathbb{R}, \tag{2.17}$$

where the derivative is understood in the sense of distributions. Again, $\sigma(x,t) = 0$ for $x \in [0,1]$, $t < 0$.

In the case of prescribed stress Σ, using (2.11) at $x = 1$ and (2.4), we obtain

$$\frac{\partial}{\partial x}\tilde{u}(1,s) = \tilde{\Sigma}(s)M^2(s), \quad s \in \mathbb{C} \setminus (-\infty, 0].$$

This combined with (2.10) gives

$$\tilde{u}(x,s) = \tilde{\Sigma}(s)\tilde{Q}(x,s), \quad x \in [0,1], \ s \in \mathbb{C} \setminus (-\infty, 0], \tag{2.18}$$

where

$$\tilde{Q}(x,s) = \frac{1}{s}M(s)\frac{\sinh(xsM(s))}{\cosh(sM(s))}, \quad x \in [0,1], \ s \in \mathbb{C} \setminus (-\infty, 0]. \tag{2.19}$$

Again, applying the Laplace inversion formula to (2.18), the displacement reads

$$u(x,t) = \Sigma(t) * Q(x,t), \quad x \in [0,1], \ t > 0, \tag{2.20}$$

and

$$u(x,t) = 0, \quad x \in [0,1], \ t < 0.$$

3. Case of solid-like viscoelastic body

In this section we shall calculate the inverse Laplace transforms of distributions and functions on $\mathbb{R}$ supported by $[0,\infty)$. In the sequel we will write $A(x) \sim B(x)$ if $\lim_{x\to\infty} \frac{A(x)}{B(x)} = 1$. Propositions 3.1, 3.2 and 3.3 are crucial in proving the existence of the solution to system (2.1), (2.2), (2.3). They will be used in proofs of Theorems 3.1 and 3.2.

Proposition 3.1. *We have*

(i) M_s *is an analytic function in* $s \in \mathbb{C} \setminus (-\infty, 0]$*;*

(ii) $\lim\limits_{\substack{s\to 0\\ s\in\mathbb{C}\setminus(-\infty,0]}} M_s(s) = 1$ *and* $\lim\limits_{\substack{|s|\to\infty\\ s\in\mathbb{C}\setminus(-\infty,0]}} M_s(s) = \sqrt{\frac{a}{b}}$.

(iii) *Let* $p \in (0, s_0)$*,* $s_0 > 0$*. Then*

$$M_s(p \pm iR) \sim \sqrt{\frac{a}{b}}\frac{1}{\ln(aR)}\sqrt[4]{(\ln(aR)\ln(bR))^2 + \left(\frac{\pi}{2}\ln\frac{b}{a}\right)^2}\exp\left(\mp i\arctan\frac{\frac{\pi}{2}\ln\frac{b}{a}}{\ln(aR)\ln(bR)}\right)$$

as $R \to \infty$.

Proof. We first prove (i). The only points where M_s could be singular are $s=\frac{1}{a}$ and $s=\frac{1}{b}$. Since,

$$\frac{\ln(bs)}{bs-1}=\frac{\ln(1+(bs-1))}{bs-1}=\sum_{n=0}^{\infty}\frac{(-1)^n(bs-1)^n}{n+1},\quad |bs-1|\in(-1,1],$$

it is obvious that $s=\frac{1}{b}$ is a regular point of M_s. Similar arguments hold for $s=\frac{1}{a}$. Limits in (ii) can easily be calculated.

In order to prove (iii), let us introduce $\mu=\sqrt{p^2+R^2}$ and $\nu=\arctan\frac{R}{p}$. It is obvious that $\mu\sim R$, $\nu\sim\frac{\pi}{2}$, as $R\to\infty$. Then $M_s(p\pm iR)$ becomes

$$\begin{aligned}
&M_s(p\pm iR)\\
&\quad=\sqrt{\frac{\ln(a\mu)\ln(b\mu)+\nu^2\mp i\nu\ln\frac{b}{a}}{\ln^2(a\mu)+\nu^2}}\sqrt{\frac{(ap-1)(bp-1)+abR^2\pm iR(b-a)}{(bp-1)^2+(bR)^2}}\\
&\quad=\Bigg(\frac{\left[abR^2+(ap-1)(bp-1)\right]\left[\ln(a\mu)\ln(b\mu)+\nu^2\right]+R\nu(b-a)\ln\frac{b}{a}}{((bR)^2+(bp-1)^2)\left(\ln^2(a\mu)+\nu^2\right)}\\
&\qquad\quad\pm i\frac{R(b-a)\left[\ln(a\mu)\ln(b\mu)+\nu^2\right]-\nu\ln\frac{b}{a}\left[abR^2+(ap-1)(bp-1)\right]}{((bR)^2+(bp-1)^2)\left(\ln^2(a\mu)+\nu^2\right)}\Bigg)^{\frac{1}{2}}\\
&\quad\sim\sqrt{\frac{a}{b}}\frac{1}{\ln(aR)}\sqrt{\ln(aR)\ln(bR)\mp i\frac{\pi}{2}\ln\frac{b}{a}},\quad\text{as } R\to\infty.
\end{aligned}$$

□

Let us examine the properties of $\tilde{P}$, given by (2.13), with M_s given by (2.7). Clearly, it has complex conjugated poles at ${}_Ps_n^{(\pm)}$, $n\in\mathbb{N}$, which are solutions of

$$\sinh\left(sM_s(s)\right)=0,\quad\text{i.e.,}\quad sM_s(s)=\pm ni\pi. \tag{3.1}$$

The position and the multiplicity of the solutions to (3.1) are as follows.

Proposition 3.2. *There are infinitely many solutions* ${}_Ps_n^{(\pm)}$, $n\in\mathbb{N}$, *of* (3.1), *such that*

$$\operatorname{Re}\left({}_Ps_n^{(\pm)}\right)\approx-\frac{\frac{\pi}{4}\ln\frac{b}{a}\sqrt{b/a}\,n\pi}{\ln\left(\sqrt{ab}\,n\pi\right)\ln\left(b\sqrt{b/a}\,n\pi\right)}, \tag{3.2}$$

$$\operatorname{Im}\left({}_Ps_n^{(\pm)}\right)\approx\pm R\approx\pm\sqrt{b/a}\,n\pi, \tag{3.3}$$

as $n\to\infty$. *Moreover, there exists* $n_0\in\mathbb{N}$, *such that poles* ${}_Ps_n^{(\pm)}$, *for* $n>n_0$, *are simple.*

Proof. Let us square (3.1) and put ${}_Ps_n^{(\pm)}=Re^{i\phi}$, $\phi\in(-\pi,\pi)$. Then, after separation of real and imaginary parts, we obtain

$$R^2\cos(2\phi)\operatorname{Re}\left(M_s^2(Re^{i\phi})\right)-R^2\sin(2\phi)\operatorname{Im}\left(M_s^2(Re^{i\phi})\right)=-n^2\pi^2, \tag{3.4}$$

$$R^2\sin(2\phi)\operatorname{Re}\left(M_s^2(Re^{i\phi})\right)+R^2\cos(2\phi)\operatorname{Im}\left(M_s^2(Re^{i\phi})\right)=0. \tag{3.5}$$

By the use of (2.7), real and imaginary parts of $M_s^2(Re^{i\phi})$ are

$$\operatorname{Re}(M_s^2(Re^{i\phi})) = \frac{(\ln(aR)\ln(bR)+\phi^2)(abR^2-(a+b)R\cos\phi+1)}{(\ln^2(aR)+\phi^2)(b^2R^2-2bR\cos\phi+1)} + \frac{\ln\frac{b}{a}(b-a)R\phi\sin\phi}{(\ln^2(aR)+\phi^2)(b^2R^2-2bR\cos\phi+1)},$$

$$\operatorname{Im}(M_s^2(Re^{i\phi})) = -\frac{\phi\ln\frac{b}{a}(abR^2-(a+b)R\cos\phi+1)}{(\ln^2(aR)+\phi^2)(b^2R^2-2bR\cos\phi+1)} + \frac{+(b-a)R\sin\phi(\ln(aR)\ln(bR)+\phi^2)}{(\ln^2(aR)+\phi^2)(b^2R^2-2bR\cos\phi+1)}.$$

Letting $R\to\infty$, previous expressions are written as

$$\operatorname{Re}(M_s^2(Re^{i\phi})) \approx \frac{abR^2\ln(aR)\ln(bR)}{b^2R^2\ln^2(aR)} = \frac{a}{b}\frac{\ln(bR)}{\ln(aR)}, \tag{3.6}$$

$$\operatorname{Im}(M_s^2(Re^{i\phi})) \approx -\frac{ab\ln\frac{b}{a}R^2\phi}{b^2R^2\ln^2(aR)} = -\frac{a}{b}\ln\frac{b}{a}\phi\frac{1}{\ln^2(aR)}. \tag{3.7}$$

Using (3.5), (3.6) and (3.7), we obtain

$$\tan(2\phi) = -\frac{\operatorname{Im}(M_s^2(Re^{i\phi}))}{\operatorname{Re}(M_s^2(Re^{i\phi}))} \approx \phi\frac{\ln\frac{b}{a}}{\ln(aR)\ln(bR)}. \tag{3.8}$$

Let $\phi\in(0,\pi)$. Then $\frac{\tan(2\phi)}{\phi}>0$ and $\frac{\tan(2\phi)}{\phi}\to 0$ as $R\to\infty$. Hence, $\phi\in\left(0,\frac{\pi}{4}\right)$ or $\phi\in\left(\frac{\pi}{2},\frac{3\pi}{4}\right)$. Since $\phi\neq 0$ and $\tan(2\phi)\to 0$, it follows that $\phi\to\frac{\pi}{2}$ from the interval $\phi\in\left(\frac{\pi}{2},\frac{3\pi}{4}\right)$. Therefore, by (3.8), we have

$$\sin\phi \approx 1-\left(\frac{\frac{\pi}{2}\ln\frac{b}{a}}{2\ln(aR)\ln(bR)}\right)^2 \approx 1, \quad \cos\phi \approx -\frac{\frac{\pi}{2}\ln\frac{b}{a}}{2\ln(aR)\ln(bR)}. \tag{3.9}$$

Inserting (3.6), (3.7) and (3.9) in (3.4), we obtain

$$\frac{1}{\sqrt{\ln^2(aR)\ln^2(bR)+\left(\frac{\pi}{2}\ln\frac{b}{a}\right)^2}}\left(\ln^2(bR)+\frac{\left(\frac{\pi}{2}\ln\frac{b}{a}\right)^2}{\ln^2(aR)}\right) \approx \frac{b}{a}\frac{n^2\pi^2}{R^2} \approx 1. \tag{3.10}$$

Thus, the real and imaginary parts of ${}_Ps_n^{(\pm)}$, as $R\to\infty$, obtained by (3.9) and (3.10), are as stated in the proposition.

In order to prove that solutions to (3.1) are simple for $n>n_0$, we define

$$f(s) := \sinh(sM_s(s)), \quad s\in\mathbb{C}\setminus(-\infty,0].$$

Then

$$\frac{d}{ds}f(s) = M_s(s)\left(1-\frac{\ln\frac{b}{a}}{2\ln(as)\ln(bs)}+\frac{(b-a)s}{2(as-1)(bs-1)}\right)\cosh(sM_s(s)).$$

Solutions to $f(s)=0$ are given by (3.1), and so, as $\left|{}_Ps_n^{(\pm)}\right|\to\infty$,

$$\frac{d}{ds}f(s)\Big|_{s={}_Ps_n^{(\pm)}} \sim (-1)^n\left[M_s(s)\left(1-\frac{\ln\frac{b}{a}}{2\ln(as)\ln(bs)}+\frac{(b-a)s}{2(as-1)(bs-1)}\right)\right]_{s={}_Ps_n^{(\pm)}}.$$

By Proposition 3.1, $M_s \sim \sqrt{a/b}$ and this implies that

$$\frac{d}{ds} f(s)\Big|_{s={}_P s_n^{(\pm)}} \sim (-1)^n \sqrt{a/b} \quad \text{as} \quad \big|{}_P s_n^{(\pm)}\big| \to \infty.$$

Thus, for large $\big|{}_P s_n^{(\pm)}\big|$ we have $\frac{d}{ds} f(s)\big|_{s={}_P s_n^{(\pm)}} \neq 0$ and solutions are simple for $n > n_0$. □

Function $\tilde{Q}$ given by (2.19), with M_s given by (2.7), contains the natural logarithm and consequently has a branch point at $s = 0$. It also have poles, that are solutions of

$$\cosh(sM_s(s)) = 0 \quad \text{i.e.,} \quad sM_s(s) = \pm \frac{2n+1}{2} i\pi, \quad n \in \mathbb{N}_0. \tag{3.11}$$

We state the proposition analogous to the Proposition 3.1.

Proposition 3.3. *There are infinitely many solutions ${}_Q s_n^{(\pm)}$, $n \in \mathbb{N}_0$, of* (3.11), *such that*

$$\operatorname{Re}\big({}_Q s_n^{(\pm)}\big) \approx -\frac{\frac{\pi}{4} \ln \frac{b}{a} \sqrt{b/a}\, \frac{2n+1}{2}\pi}{\ln\big(\sqrt{ab}\, \frac{2n+1}{2}\pi\big) \ln\big(b\sqrt{b/a}\, \frac{2n+1}{2}\pi\big)}, \quad \operatorname{Im}\big({}_Q s_n^{(\pm)}\big) \approx \pm\sqrt{b/a}\, \frac{2n+1}{2}\pi,$$

as $n \to \infty$. Moreover, there exists $n_0 \in \mathbb{N}_0$, such that the poles ${}_Q s_n^{(\pm)}$ are simple for $n > n_0$.

3.1. Determination of the displacement u in a stress relaxation test.

We investigate properties of $\tilde{P}$, given by (2.13) with M_s taken in the form (2.7), and find a solution to (2.1), (2.2), (2.3), with weight functions given by (1.7) in two cases.

I. The case when the boundary condition (2.3) is given by

$$\Upsilon(t) = \Upsilon_0 H(t), \;\; \Upsilon_0 > 0, \; t \in \mathbb{R}, \tag{3.12}$$

is investigated in 3.1.1.

II. The case when the boundary condition (2.3) is given by

$$\Upsilon(t) = \Upsilon_0 H(t) + F(t), \;\; t \in \mathbb{R}, \tag{3.13}$$

where F is a locally integrable function, equal to zero on $(-\infty, 0]$ is investigated in subsubsection 3.1.2.

3.1.1. Case $\Upsilon = \Upsilon_0 H$. This is the case that has physical importance, since we obtain displacement in the case of stress relaxation test. Formally, we write (2.14) with (3.12) as

$$u_H(x,t) = \Upsilon_0 H(t) * P(x,t), \;\; x \in [0,1], \; t \in \mathbb{R}. \tag{3.14}$$

The following theorem is on existence and properties of u_H.

Theorem 3.1. *Let* $\Upsilon = \Upsilon_0 H$ *and let* ϕ_σ *and* ϕ_ε *be given by* (1.7). *Then the solution to* (2.1), (2.2), (2.3) *is given by* (3.14), *where*

$$P(x,t) = \frac{1}{2\pi i}\int_0^\infty \left(\frac{\sinh(xqM_s(qe^{-i\pi}))}{\sinh(qM_s(qe^{-i\pi}))} - \frac{\sinh(xqM_s(qe^{i\pi}))}{\sinh(qM_s(qe^{i\pi}))}\right)e^{-qt}dq$$
$$+\sum_{n=1}^\infty \Big[\operatorname{Res}\big(\tilde{P}(x,s)e^{st}, {}_P s_n^{(+)}\big) + \operatorname{Res}\big(\tilde{P}(x,s)e^{st}, {}_P s_n^{(-)}\big)\Big],\ \ x\in[0,1],\ t>0, \tag{3.15}$$

$$P(x,t) = 0,\ \ x\in[0,1],\ t<0. \tag{3.16}$$

The residues are given by

$$\operatorname{Res}\big(\tilde{P}(x,s)e^{st}, {}_P s_n^{(\pm)}\big) = \left[\frac{\sinh(xsM_s(s))}{\frac{d}{ds}\big[\sinh(sM_s(s))\big]}e^{st}\right]_{s={}_P s_n^{(\pm)}} \tag{3.17}$$

and simple poles ${}_P s_n^{(\pm)}$, *for* $n > n_0$, *are solutions of* (3.1). *The function* P *is real-valued, locally integrable on* $\mathbb{R}$ *and smooth for* $t > 0$.

The explicit form of solution is

$$u_H(x,t) = \frac{\Upsilon_0}{2\pi i}\int_0^\infty \left(\frac{\sinh(xqM_s(qe^{-i\pi}))}{\sinh(qM_s(qe^{-i\pi}))} - \frac{\sinh(xqM_s(qe^{i\pi}))}{\sinh(qM_s(qe^{i\pi}))}\right)\frac{1-e^{-qt}}{q}dq$$
$$+\int_0^t\left(\sum_{n=1}^\infty\Big[\operatorname{Res}\big(\tilde{P}(x,s)e^{s\tau}, {}_P s_n^{(+)}\big)+\operatorname{Res}\big(\tilde{P}(x,s)e^{s\tau}, {}_P s_n^{(-)}\big)\Big]\right)d\tau,\ \ x\in[0,1],\ t>0, \tag{3.18}$$

$$u_H(x,t) = 0,\ \ x\in[0,1],\ t<0. \tag{3.19}$$

Function u_H *is continuous at* $t = 0$.

Proof. We calculate $P(x,t)$, $x \in [0,1]$, $t\in\mathbb{R}$, by the integration over a suitable contour. Let $t > 0$. The Cauchy residues theorem yields

$$\oint_\Gamma \tilde{P}(x,s)e^{st}ds = 2\pi i\sum_{n=1}^\infty\Big[\operatorname{Res}\big(\tilde{P}(x,s)e^{st}, {}_P s_n^{(+)}\big)+\operatorname{Res}\big(\tilde{P}(x,s)e^{st}, {}_P s_n^{(-)}\big)\Big], \tag{3.20}$$

where $\Gamma = \Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_\varepsilon\cup\Gamma_4\cup\Gamma_5\cup\Gamma_6\cup\gamma_0$, so that all poles lie inside the contour Γ (see Figure 1).

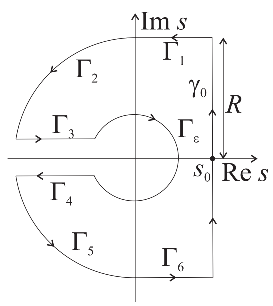

Figure 1. Integration contour Γ

First we show that the series of residues in (3.15) is convergent. By Proposition 3.2, poles ${}_{P}s_n^{(\pm)}$ of $\tilde{P}$, given by (2.13) with M_s in the form (2.7), are simple for $n > n_0$. Then the residues in (3.20) can be calculated using (3.1) and (3.17) as

$$(3.21)\quad \operatorname{Res}\big(\tilde{P}(x,s)e^{st}, {}_{P}s_n^{(\pm)}\big) = (-1)^n \frac{\sin(n\pi x)}{n\pi}\left[\frac{se^{st}}{1-\frac{\ln\frac{b}{a}}{2\ln(as)\ln(bs)}+\frac{s(b-a)}{2(as-1)(bs-1)}}\right]_{s={}_{P}s_n^{(\pm)}}, \quad n > n_0.$$

If ${}_{P}s_n^{(\pm)} = Re^{\pm i\phi}$, then (3.21) transforms into

$$\begin{aligned}\operatorname{Res}\big(\tilde{P}(x,s)e^{st}, {}_{P}s_n^{(\pm)}\big) &= (-1)^n \frac{\sin(n\pi x)}{n\pi}\frac{Re^{Rt\cos\phi}e^{\pm i(\phi+Rt\sin\phi)}}{\big[1-\frac{\ln\frac{b}{a}}{2\ln(as)\ln(bs)}+\frac{s(b-a)}{2(as-1)(bs-1)}\big]_{s=Re^{\pm i\phi}}}\\ &= (-1)^n \frac{\sin(n\pi x)}{n\pi}\frac{Re^{Rt\cos\phi}\left[\cos(\phi+Rt\sin\phi)\pm i\sin(\phi+Rt\sin\phi)\right]}{\big[1-\frac{\ln\frac{b}{a}}{2\ln(as)\ln(bs)}+\frac{s(b-a)}{2(as-1)(bs-1)}\big]_{s=Re^{\pm i\phi}}},\end{aligned}$$

and therefore, for $n > n_0$, we have

$$(3.22)\quad \begin{aligned}&\operatorname{Res}\big(\tilde{P}(x,s)e^{st}, {}_{P}s_n^{(+)}\big)+\operatorname{Res}\big(\tilde{P}(x,s)e^{st}, {}_{P}s_n^{(-)}\big)\\ &= (-1)^n \frac{\sin(n\pi x)}{n\pi}Re^{Rt\cos\phi}\times\left(\frac{\cos(\phi+Rt\sin\phi)+i\sin(\phi+Rt\sin\phi)}{\big[1-\frac{\ln\frac{b}{a}}{2\ln(as)\ln(bs)}+\frac{s(b-a)}{2(as-1)(bs-1)}\big]_{s=Re^{i\phi}}}\right.\\ &\qquad\left.+\frac{\cos(\phi+Rt\sin\phi)-i\sin(\phi+Rt\sin\phi)}{\big[1-\frac{\ln\frac{b}{a}}{2\ln(as)\ln(bs)}+\frac{s(b-a)}{2(as-1)(bs-1)}\big]_{s=Re^{-i\phi}}}\right).\end{aligned}$$

When $n\to\infty$, $\big|{}_{P}s_n^{(\pm)}\big|\to\infty$ (i.e., $R\to\infty$) then

$$\left|\left[1-\frac{\ln\frac{b}{a}}{2\ln(as)\ln(bs)}+\frac{s(b-a)}{2(as-1)(bs-1)}\right]_{s=Re^{\pm i\phi}}\right|\to 1.$$

Also, as $n\to\infty$, (3.22) becomes

$$\big|\operatorname{Res}\big(\tilde{P}(x,s)e^{st}, {}_{P}s_n^{(+)}\big)+\operatorname{Res}\big(\tilde{P}(x,s)e^{st}, {}_{P}s_n^{(-)}\big)\big| \approx 2\Big|\frac{\sin(n\pi x)}{n\pi}Re^{Rt\cos\phi}\cos(\phi+Rt\sin\phi)\Big|.$$

The formula (3.2) implies that

$$\operatorname{Re}\big({}_{P}s_n^{(\pm)}\big)\approx -\frac{\pi}{4}\ln\frac{b}{a}\sqrt{\frac{b}{a}}\,\pi\frac{n}{\ln\big(\sqrt{ab}\,n\pi\big)\ln\big(b\sqrt{b/a}\,n\pi\big)}\leqslant -C\sqrt{n},\quad n>n_0.$$

Also, by (3.3), we have that $R/n\approx\sqrt{b/a}\,\pi$. This implies that summands in (3.20) can be estimated by $Ke^{-Ct\sqrt{n}}$, which implies the convergence of the sum of residues in (3.20).

We calculate now the integral over Γ in (3.20). Consider the integral along contour Γ_1. Then

$$\left|\int_{\Gamma_1} \tilde{P}(x,s)e^{st}ds\right| \leqslant \int_0^{s_0} \left|\tilde{P}(x,p+iR)\right|\left|e^{(p+iR)t}\right| dp.$$

Let $R \to \infty$. In order to investigate the asymptotic behavior of $\left|\tilde{P}(x,p\pm iR)\right|$ as $R \to \infty$, we use Proposition 3.1 (iii) and write

$$M_s(p \pm iR) \sim v \pm iw,$$

$$v = \sqrt{\frac{a}{b}}\frac{1}{\ln(aR)}\frac{\ln(aR)\ln(bR)}{\sqrt[4]{\left(\ln(aR)\ln(bR)\right)^2 + \left(\frac{\pi}{2}\ln\frac{b}{a}\right)^2}},$$

$$w = -\sqrt{\frac{a}{b}}\frac{1}{\ln(aR)}\frac{\frac{\pi}{2}\ln\frac{b}{a}}{\sqrt[4]{\left(\ln(aR)\ln(bR)\right)^2 + \left(\frac{\pi}{2}\ln\frac{b}{a}\right)^2}}.$$

Then, as $R \to \infty$,

$$\left|\tilde{P}(x,p\pm iR)\right| \sim \left|\frac{\sinh\left[x(pv-Rw) \pm ix(pw+Rv)\right]}{\sinh\left[(pv-Rw) \pm i(pw+Rv)\right]}\right| \tag{3.23}$$

$$\leqslant \frac{e^{x(pv-Rw)} + e^{-x(pv-Rw)}}{\left|e^{pv-Rw} - e^{-(pv-Rw)}\right|}$$

$$= e^{-(1-x)(pv-Rw)}\frac{1+e^{-2x(pv-Rw)}}{\left|1-e^{-2(pv-Rw)}\right|} \to 0. \tag{3.24}$$

The previous statement is valid since, as $R \to \infty$,

$$pv - Rw = \sqrt{\frac{a}{b}}\frac{1}{\ln(aR)}\frac{1}{\sqrt[4]{(\ln(aR)\ln(bR))^2 + (\frac{\pi}{2}\ln\frac{b}{a})^2}} \times \left(p\ln(aR)\ln(bR) + R\frac{\pi}{2}\ln\frac{b}{a}\right)$$

$$\sim \sqrt{\frac{a}{b}}\left(p\sqrt{\frac{\ln(bR)}{\ln(aR)}} + \frac{\pi}{2}\ln\frac{b}{a}\frac{R}{\ln(aR)\sqrt{\ln(aR)\ln(bR)}}\right) \to \infty.$$

Therefore, according to (3.23), we have

$$\lim_{R\to\infty}\left|\int_{\Gamma_1} \tilde{P}(x,s)e^{st}ds\right| = 0.$$

By the use of (3.23), we conclude that similar arguments are valid for the integral along the contour Γ_6. Thus,

$$\lim_{R\to\infty}\left|\int_{\Gamma_6} \tilde{P}(x,s)e^{st}ds\right| = 0.$$

Next, we consider the integral along contour Γ_2

$$\left|\int_{\Gamma_2} \tilde{P}(x,s)e^{st}ds\right| \leqslant \int_{\pi/2}^{\pi} R\left|e^{R(1-x)e^{i\phi}M_s(Re^{i\phi})}\right|\left|\frac{e^{2xRe^{i\phi}M_s(Re^{i\phi})}-1}{e^{2Re^{i\phi}M_s(Re^{i\phi})}-1}\right|e^{Rt\cos\phi}d\phi.$$

Since $M_s \sim \sqrt{a/b}$ as $|s| \to \infty$ and $\cos\phi \leqslant 0$ for $\phi \in [\pi/2, \pi]$, by the Lebesgue theorem, we have

$$\lim_{R\to\infty} \left| \int_{\Gamma_2} \tilde{P}(x,s) e^{st} ds \right| \leqslant \lim_{R\to\infty} \int_{\pi/2}^{\pi} R\, e^{R\cos\phi \left(t+(1-x)\sqrt{a/b}\right)} d\phi = 0.$$

Similar arguments are valid for the integral along the contour Γ_5. Thus,

$$\lim_{R\to\infty} \left| \int_{\Gamma_5} \tilde{P}(x,s) e^{st} ds \right| = 0.$$

The integration along contour Γ_ε gives

$$\lim_{\varepsilon\to 0} \left| \int_{\Gamma_\varepsilon} \tilde{P}(x,s) e^{st} ds \right| = \lim_{\varepsilon\to 0} \int_{\pi}^{-\pi} \varepsilon \left| e^{-\varepsilon(1-x)e^{i\phi} M_s(\varepsilon e^{i\phi})} \right| \times \left| \frac{1 - e^{-2x\varepsilon e^{i\phi} M_s(\varepsilon e^{i\phi})}}{1 - e^{-2\varepsilon e^{i\phi} M_s(\varepsilon e^{i\phi})}} \right| e^{\varepsilon t \cos\phi} d\phi = 0.$$

Integrals along Γ_3, Γ_4 and γ_0 give

$$\begin{aligned}
\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_{\Gamma_3} \tilde{P}(x,s) e^{st} ds &= \int_0^\infty \frac{\sinh(xqM_s(qe^{i\pi}))}{\sinh(qM_s(qe^{i\pi}))} e^{-qt} dq, \\
\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_{\Gamma_4} \tilde{P}(x,s) e^{st} ds &= -\int_0^\infty \frac{\sinh(xqM_s(qe^{-i\pi}))}{\sinh(qM_s(qe^{-i\pi}))} e^{-qt} dq, \\
\lim_{R\to\infty} \int_{\gamma_0} \tilde{P}(x,s) e^{st} ds &= 2\pi i P(x,t).
\end{aligned}$$

Now, by the Cauchy residues theorem (3.20), the function P is determined by (3.15).

In order to see that P is a real-valued function, we use the fact that for $q \in [0, \infty)$, $M_s(qe^{-i\pi}) = \overline{M_s(qe^{i\pi})}$, where the bar denotes the complex conjugation. Due to the exponential in the hyperbolic sine, we have $\sinh(xqM_s(qe^{-i\pi})) = \overline{\sinh(xqM_s(qe^{i\pi}))}$ and therefore the integrand in (3.15) is of the form

$$\begin{aligned}
&\frac{\sinh(xqM_s(qe^{-i\pi}))}{\sinh(qM_s(qe^{-i\pi}))} - \frac{\sinh(xqM_s(qe^{i\pi}))}{\sinh(qM_s(qe^{i\pi}))} \\
&\qquad = \overline{\left(\frac{\sinh(xqM_s(qe^{i\pi}))}{\sinh(qM_s(qe^{i\pi}))} \right)} - \frac{\sinh(xqM_s(qe^{i\pi}))}{\sinh(qM_s(qe^{i\pi}))} \\
&\qquad = -2i \operatorname{Im} \left(\frac{\sinh(xqM_s(qe^{i\pi}))}{\sinh(qM_s(qe^{i\pi}))} \right),
\end{aligned}$$

which implies that the first term in (3.15) is real.

Next, we examine $\operatorname{Res}\left(\tilde{P}(x,s)e^{st}, {}_Ps_n^{(\pm)}\right)$ in order to prove that the sum of residues is also real. By (3.21) and the fact that

$$\left[1-\frac{\ln\frac{b}{a}}{2\ln(as)\ln(bs)}+\frac{(b-a)s}{2(as-1)(bs-1)}\right]_{s={}_Ps_n^{(-)}} = \overline{\left(\left[1-\frac{\ln\frac{b}{a}}{2\ln(as)\ln(bs)}+\frac{(b-a)s}{2(as-1)(bs-1)}\right]_{s={}_Ps_n^{(+)}}\right)}$$

we obtain that $\operatorname{Res}\big(\tilde{P}(x,s)e^{st}, {}_Ps_n^{(-)}\big) = \overline{\operatorname{Res}\big(\tilde{P}(x,s)e^{st}, {}_Ps_n^{(+)}\big)}$. It is clear that

$$\operatorname{Res}\big(\tilde{P}(x,s)e^{st}, {}_Ps_n^{(+)}\big)+\operatorname{Res}\big(\tilde{P}(x,s)e^{st}, {}_Ps_n^{(-)}\big) = 2\operatorname{Re}\big(\operatorname{Res}\big(\tilde{P}(x,s)^{st}, {}_Ps_n^{(+)}\big)\big).$$

This implies that the second term in (3.15) is also real for $n\in\mathbb{N}$. Hence, (3.15) is a real-valued function.

Let $t<0$. We prove that the integral over γ_0 does not depend on the choice of s_0 (see Figure 1). Let $\bar{\Gamma}=\gamma_0\cup\gamma_1\cup\gamma_0'\cup\gamma_2$ (see Figure 2), where s_0 and s_0' are chosen so that all poles, i.e., the solutions of (3.1) lie on the left of γ_0. The Cauchy residues theorem yields $\oint_{\bar{\Gamma}}\tilde{P}(x,s)e^{st}ds=0$ for $x\in[0,1]$. This and (3.23) imply

$$\lim_{R\to\infty}\left|\int_{\gamma_1}\tilde{P}(x,s)e^{st}ds\right| \leqslant \lim_{R\to\infty}\int_{s_0}^{s_0'}\big|\tilde{P}(x,v+iR)\big|\big|e^{(v+iR)t}\big|\,dv=0.$$

Similar arguments hold for the integral along γ_2. Therefore, by the Cauchy residues theorem, integrals along γ_0 and γ_0' are equal and the inversion of the Laplace transformation does not depend on the choice of s_0 as well as on the choice of s_0'.

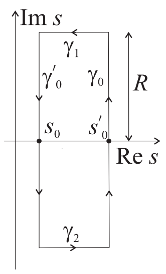

FIGURE 2. Integration contour $\bar{\Gamma}$

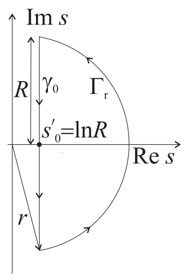

FIGURE 3. Integration contour $\tilde{\Gamma}$

Again, by the Cauchy residues theorem yields $\oint_{\tilde{\Gamma}}\tilde{P}(x,s)e^{st}ds=0$ for $x\in[0,1]$, where $\tilde{\Gamma}=\gamma_0\cup\Gamma_r$ (see Figure 3), with the assumption that all poles, i.e., the solutions of (3.1), lie on the left of γ_0. Let $r(R)=\sqrt{R^2+s_0^2}$. Then

$$\left|\int_{\Gamma_r}\tilde{P}(x,s)e^{st}ds\right| \leqslant \int_{-\phi_0(r(R))}^{\phi_0(r(R))}\sqrt{R^2+s_0^2}\,\left|\tilde{P}\Big(x,\sqrt{R^2+s_0^2}e^{i\phi}\Big)\right|\,e^{t\sqrt{R^2+s_0^2}\cos\phi}d\phi,$$

where $\lim_{R\to\infty}\phi_0(r(R))=\frac{\pi}{2}$. Since $M_s\sim\sqrt{\frac{a}{b}}$ as $|s|\to\infty$, by (2.13) and (2.7), we have, as $|s|\to\infty$,

(3.25)
$$\left|\tilde{P}(x,s)\right|=\left|e^{-(1-x)sM_s(s)}\frac{1-e^{-2xsM_s(s)}}{1-e^{-2sM_s(s)}}\right|\leqslant C,\quad x\in[0,1],\ s\in\mathbb{C}\setminus(-\infty,0].$$

By (3.25), we have

$$\lim_{R\to\infty}\left|\int_{\Gamma_r}\tilde{P}(x,s)e^{st}ds\right|\leqslant C\lim_{R\to\infty}\int_{-\phi_0(r(R))}^{\phi_0(r(R))}\sqrt{R^2+s_0^2}\ e^{t\sqrt{R^2+s_0^2}\cos\phi}d\phi=0,$$

since $t<0$ and $\cos\phi>0$. Therefore, we proved (3.16).

By the use of (3.15) and (3.16) in (3.14) and by calculating the convolution we obtain (3.18) and (3.19).

In order to prove that u_H is a continuous function at $t=0$, we will use Lebesgue dominated convergence theorem. Let

$$f(q,x):=\frac{\Upsilon_0}{2\pi i}\left(\frac{\sinh(xqM_s(qe^{-i\pi}))}{\sinh(qM_s(qe^{-i\pi}))}-\frac{\sinh(xqM_s(qe^{i\pi}))}{\sinh(qM_s(qe^{i\pi}))}\right),\quad q\in(0,\infty),\ x\in[0,1].$$

Then

$$\left|\int_0^\infty f(q,x)\frac{1-e^{-qt}}{q}dq\right|\to 0\ \text{ as }\ t\to 0. \tag{3.26}$$

By simple calculations we have $\frac{1-e^{-qt}}{q}\leqslant Ct$ if $0<q<1$ and $\frac{1-e^{-qt}}{q}\leqslant 1-e^{-qt}$ if $q\geqslant 1$. Thus $f(q,x)\frac{1-e^{-qt}}{q}\leqslant Cf(q,x)$, $q>0$ and since $\frac{1-e^{-qt}}{q}\to 0$ as $t\to 0$, (3.26) follows.

In proving the continuity of u_H at $t=0$, by (3.14), we estimated $\int_0^t P(x,\tau)d\tau$, $x\in[0,1]$, $t>0$, and actually proved that P is integrable function on any interval $[0,T]$, $T>0$. Thus, P is locally integrable on $\mathbb{R}$. □

3.1.2. Case $\Upsilon=\Upsilon_0 H+F$.

Condition 3.1. *Let F be a locally integrable function, equal to zero for $t\leqslant 0$, such that its Laplace transformation exists in $\mathbb{C}\setminus(-\infty,0]$. Assume:*

(i) *$\tilde{F}$ is analytic and $\tilde{F}\neq 0$ in $\mathbb{C}\setminus(-\infty,0]$;*
(ii) *for some $\alpha>1$, $\tilde{F}(s)\sim\frac{1}{|s|^\alpha}$, $s\in\mathbb{C}\setminus(-\infty,0]$, as $|s|\to\infty$;*
(iii) *$s\tilde{F}(s)\sim o(1)$, $s\in\mathbb{C}\setminus(-\infty,0]$, as $|s|\to 0$.*

If boundary condition (2.3) is given by (3.13), then the solution to (2.1), (2.2), (2.3), given by (2.12) in the Laplace domain, reads formally

$$\tilde{u}(x,s)=\tilde{u}_H(x,s)+\tilde{F}(s)\tilde{P}(x,s),\quad x\in[0,1],\ s\in\mathbb{C}\setminus(-\infty,0],$$

which in the time domain is $u(x,t)=u_H(x,t)+F(t)*P(x,t)$, $x\in[0,1]$, $t\in\mathbb{R}$. The existence of u_H is shown in 3.1.3 therefore it remains to show the existence of

$$u_F(x,t)=F(t)*P(x,t),\quad x\in[0,1],\ t\in\mathbb{R}.$$

For $t > 0$, $x \in [0,1]$, we apply the Cauchy residues theorem and obtain
(3.27)

$$\oint_\Gamma \tilde{u}_F(x,s)e^{st}ds = 2\pi i \sum_{n=1}^{\infty} \Big[\operatorname{Res}\Big(\tilde{u}_F(x,s)e^{st}, {}_Ps_n^{(+)}\Big) + \operatorname{Res}\Big(\tilde{u}_F(x,s)e^{st}, {}_Ps_n^{(-)}\Big)\Big],$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_\varepsilon \cup \Gamma_4 \cup \Gamma_5 \cup \Gamma_6 \cup \gamma_0$ (see Figure 1). Since poles ${}_Ps_n^{(\pm)}$ of $\tilde{u}_F$ are actually the poles of $\tilde{P}$, that are obtained from (3.1) and they are simple for $n > n_0$, the residues in (3.27) can be calculated as

$$\operatorname{Res}\Big(\tilde{u}_F(x,s)e^{st}, {}_Ps_n^{(\pm)}\Big) = \Big[\tilde{F}(s)\frac{\sinh(xsM_s(s))}{\frac{d}{ds}[\sinh(sM_s(s))]}e^{st}\Big]_{s={}_Ps_n^{(\pm)}}. \tag{3.28}$$

The proof that the sum in (3.27) converges is analog to the one presented in 3.1.1.

Consider the integral along contour Γ_1. It reads

$$\left|\int_{\Gamma_1} \tilde{u}_F(x,s)e^{st}ds\right| \leqslant \int_0^{s_0} \big|\tilde{F}(p+iR)\big|\big|\tilde{P}(x,p+iR)\big|\big|e^{(p+iR)t}\big|\,dp.$$

According to (3.25) and Condition 3.1, we have

$$\lim_{R\to\infty}\left|\int_{\Gamma_1} \tilde{u}_F(x,s)e^{st}ds\right| \leqslant C\lim_{R\to\infty}\int_0^{s_0}\frac{e^{pt}}{(p^2+R^2)^{\alpha/2}}\,dp = 0.$$

The integral along contour Γ_2 reads

$$\left|\int_{\Gamma_2} \tilde{u}_F(x,s)e^{st}ds\right| \leqslant \int_{\frac{\pi}{2}}^{\pi} \big|\tilde{F}(Re^{i\phi})\big|\big|e^{R(1-x)e^{i\phi}M_s(Re^{i\phi})}\big| \times \left|\frac{e^{2xRe^{i\phi}M_s(Re^{i\phi})}-1}{e^{2Re^{i\phi}M_s(Re^{i\phi})}-1}\right| e^{Rt\cos\phi}R\,d\phi.$$

In order to apply the Lebesgue theorem, we need $\alpha > 1$ in Condition 3.1 (actually it is enough to have $\alpha \geqslant 1$, but case $\alpha = 1$ is already considered). Since $M_s \sim \sqrt{a/b}$ as $|s| \to \infty$ and $\cos\phi \leqslant 0$ for $\phi \in [\frac{\pi}{2}, \pi]$, we have

$$\lim_{R\to\infty}\left|\int_{\Gamma_2} \tilde{u}_F(x,s)e^{st}ds\right| \leqslant C\lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi} R^{1-\alpha}e^{R\cos\phi\left(t+(1-x)\sqrt{a/b}\right)}d\phi = 0.$$

Similar arguments are valid for the integral along the contour Γ_5. Thus,

$$\lim_{R\to\infty}\left|\int_{\Gamma_5} \tilde{u}_F(x,s)e^{st}ds\right| = 0.$$

The integration along contour Γ_ε gives

$$\left|\int_{\Gamma_\varepsilon} \tilde{u}_F(x,s)e^{st}ds\right| \leqslant \int_{-\pi}^{\pi} \big|\tilde{F}\big(\varepsilon e^{i\phi}\big)\big|\big|e^{-\varepsilon(1-x)e^{i\phi}M_s(\varepsilon e^{i\phi})}\big| \times \left|\frac{1-e^{-2x\varepsilon e^{i\phi}M_s(\varepsilon e^{i\phi})}}{1-e^{-2\varepsilon e^{i\phi}M_s(\varepsilon e^{i\phi})}}\right| e^{\varepsilon t\cos\phi}\varepsilon\,d\phi,$$

and this tends to zero as $\varepsilon \to 0$, according to Condition 3.1. Integrals along of contours Γ_3, Γ_4 and γ_0 give

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_{\Gamma_3} \tilde{u}_F(x,s) e^{st} ds = \int_0^\infty \tilde{F}\big(qe^{i\pi}\big) \frac{\sinh\big(xqM_s\big(qe^{i\pi}\big)\big)}{\sinh\big(qM_s\big(qe^{i\pi}\big)\big)} e^{-qt} dq,$$

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_{\Gamma_4} \tilde{u}_F(x,s) e^{st} ds = -\int_0^\infty \tilde{F}\big(qe^{-i\pi}\big) \frac{\sinh\big(xqM_s\big(qe^{-i\pi}\big)\big)}{\sinh\big(qM_s\big(qe^{-i\pi}\big)\big)} e^{-qt} dq,$$

$$\lim_{R\to\infty} \int_{\gamma_0} \tilde{u}_F(x,s) e^{st} ds = 2\pi i u_F(x,t).$$

Now, by the Cauchy residues theorem (3.27), u_F is determined as

$$\begin{aligned} u_F(x,t) = {} & \frac{1}{2\pi i} \int_0^\infty \bigg(\tilde{F}(qe^{-i\pi}) \frac{\sinh(xqM_s(qe^{-i\pi}))}{\sinh(qM_s(qe^{-i\pi}))} \\ & - \tilde{F}(qe^{i\pi}) \frac{\sinh(xM_s(qe^{i\pi}))}{\sinh(qM_s(qe^{i\pi}))} \bigg) e^{-qt} dq \\ & + \sum_{n=1}^{\infty} \Big[\operatorname{Res}\Big(\tilde{u}_F e^{st}, {}_P s_n^{(+)}\Big) + \operatorname{Res}\Big(\tilde{u}_F e^{st}, {}_P s_n^{(-)}\Big)\Big], \quad x \in [0,1],\ t > 0, \\ u_F(x,t) = {} & 0, \quad x \in [0,1],\ t < 0, \end{aligned} \tag{3.29}$$

where the residues are given by (3.28). Note that u_F is a locally integrable, real-valued function, which can shown similarly as in § **3.1.1**.

Therefore, in the case when the boundary condition takes the form (3.13), the solution to system (2.1), (2.2), (2.3) reads

$$u(x,t) = u_H(x,t) + u_F(x,t), \quad x \in [0,1],\ t > 0,$$

where u_H and u_F are given by (3.18) and (3.29), respectively. Note that u_H and u_F are equal to zero for $t < 0$. Again, we have that u is a smooth function for $x \in [0,1]$, $t > 0$.

3.2. Determination of the stress σ in a stress relaxation test. We see that $\tilde{T}$, given by (2.16) with M_s in the form (2.7), has the branch point at $s = 0$ and poles at the same points as $\tilde{P}$. Therefore, the poles of $\tilde{T}$ are solutions to (3.1). Using the Cauchy residues theorem

$$\oint_\Gamma \tilde{T}(x,s) e^{st} ds = 2\pi i \sum_{n=1}^{\infty} \Big[\operatorname{Res}\Big(\tilde{T}(x,s)e^{st}, {}_P s_n^{(+)}\Big) + \operatorname{Res}\Big(\tilde{T}(x,s)e^{st}, {}_P s_n^{(-)}\Big)\Big], \tag{3.30}$$

where contour Γ is given in Figure 1, we obtain T in the following way. The residues in (3.30) are given by

$$\operatorname{Res}\Big(\tilde{T}(x,s)e^{st}, {}_P s_n^{(\pm)}\Big) = \left[\frac{\cosh(xsM_s(s))}{M_s(s)\frac{d}{ds}[\sinh(sM_s(s))]} e^{st} \right]_{s={}_P s_n^{(\pm)}}, \tag{3.31}$$

where ${}_P s_n^{(\pm)}$, $n \in \mathbb{N}$, are solutions of (3.1).

Evaluating the integral at the left-hand side of (3.30) in the same way as in 3.1.1 we obtain

$$T(x,t) = 1 + \frac{1}{2\pi i}\int_0^\infty \left(\frac{\cosh(xqM_s(qe^{i\pi}))}{M_s(qe^{i\pi})\sinh(qM_s(qe^{i\pi}))} - \frac{\cosh(xqM_s(qe^{-i\pi}))}{M_s(qe^{-i\pi})\sinh(qM_s(qe^{-i\pi}))} \right) e^{-qt} dq$$
$$+ \sum_{n=1}^{\infty} \left[\operatorname{Res}\left(\tilde{T}(x,s)e^{st}, {}_P s_n^{(+)}\right) + \operatorname{Res}\left(\tilde{T}(x,s)e^{st}, {}_P s_n^{(-)}\right) \right], \quad x \in [0,1],\ t > 0,$$
$$T(x,t) = 0, \quad x \in [0,1],\ t < 0,$$

where the residues are given by (3.31). The proof is analogue to the one presented in 3.1.1.

Thus, by (2.17), we have

$$\sigma_H(x,t) = \Upsilon_0 T(x,t), \quad x \in [0,1],\ t > 0, \tag{3.32}$$

if the boundary conditions (2.3) are given by (3.12). Note that σ_H is a locally integrable function with the jump at $t = 0$ and smooth for $t > 0$. Also in the case when the boundary conditions are given by (2.3) and (3.13), we have for $x \in [0,1]$

$$\sigma_F(x,t) = \sigma_H(x,t) + \frac{d}{dt}\big(F(t) * T(x,t)\big), \quad t > 0$$
$$\sigma_F(x,t) = 0, \quad t < 0.$$

This is a smooth function for $t > 0$. Note that σ_H and σ_F are real-valued functions, which can be shown similarly as in 3.1.1.

3.3. Determination of displacement u in a creep test . We are now in a position to state our result concerning the existence of a solution to system (2.1), (2.2), (2.4). Recall, this is the case of the prescribed stress.

Theorem 3.2. *Let ϕ_1 and ϕ_2 be given by* (1.7). *Then the solution to* (2.1), (2.2), (2.4) *is given by* (2.20), *where*

$$Q(x,t) = \frac{1}{2\pi i}\int_0^\infty \left(M_s(qe^{-i\pi})\frac{\sinh(xqM_s(qe^{-i\pi}))}{\cosh(qM_s(qe^{-i\pi}))} - M_s(qe^{i\pi})\frac{\sinh(xqM_s(qe^{i\pi}))}{\cosh(qM_s(qe^{i\pi}))} \right) \frac{e^{-qt}}{q}\, dq$$
$$+ \sum_{n=0}^{\infty} \left[\operatorname{Res}\left(\tilde{Q}(x,s)e^{st}, {}_Q s_n^{(+)}\right) + \operatorname{Res}\left(\tilde{Q}(x,s)e^{st}, {}_Q s_n^{(-)}\right) \right], \quad x \in [0,1],\ t > 0 \tag{3.33}$$

$$Q(x,t) = 0, \quad x \in [0,1],\ t < 0. \tag{3.34}$$

The residues are

$$\operatorname{Res}\left(\tilde{Q}(x,s)e^{st}, {}_Q s_n^{(\pm)}\right) = \left[\frac{1}{s} M_s(s) \frac{\sinh(xsM_s(s))}{\frac{d}{ds}[\cosh(sM_s(s))]} e^{st} \right]_{s = {}_Q s_n^{(\pm)}}, \quad n > n_0,$$

where ${}_{Q}s_n^{(\pm)}$ *are simple poles for* $n > n_0$*, and they are solutions of* (3.11)*. For every fixed* $x \in [0,1]$*, function* Q *is real-valued, locally integrable on* $\mathbb{R}$ *and smooth for* $t > 0$*.*

The proof of Theorem 3.2 is analogous to the proof of Theorem 3.1. For the details we refer to [5].

3.4. Numerical examples. **3.4.1. Stress relaxation experiments.** Here, we give several examples in which the displacement u_H and the stress σ_H, are determined numerically from (3.18) and (3.32), respectively. In Figure 4, we show displacements, determined according to (3.18), for three different positions. Parameters in (3.18) are chosen as follows: $\Upsilon_0 = 1$, $a = 0.045$, $b = 0.5$. From Figure 4, one can see that the further the point is from the free end of the rod, the delay in the start of its oscillation is greater. This indicates that the speed of the disturbance propagation is finite. The displacements of the points close to the free end

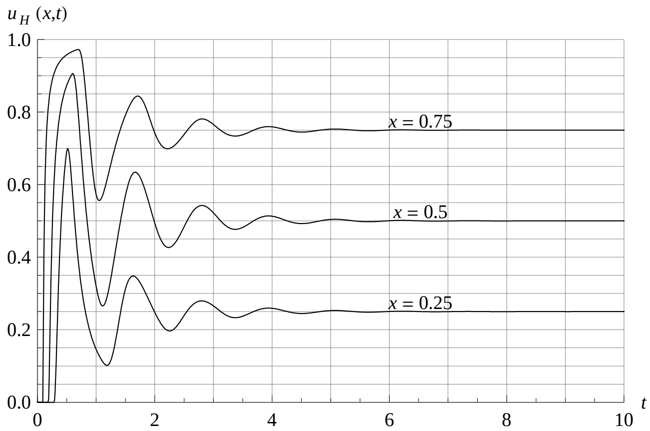

FIGURE 4. Displacements $u_H(x,t)$ in a stress relaxation test, as functions of time at $x \in \{0.25, 0.5, 0.75\}$ for $t \in (0,10)$.

are shown in the Figure 5. The figure shows that the waves approaching the free end deform in shape, so that their amplitudes do not exceed the displacement of the free end, i.e. $u(1,t) = 1$. Also, from Figures 4 and 5, for large times one sees that $\lim_{t\to\infty} u_H(x,t) = x$.

In Figures 6, 7, 8 and 9 we show the stresses determined according to (3.32) for the following values of parameters: $\Upsilon_0 = 1$, $a = 0.045$, $b = 0.5$. In order to emphasize the stress relaxation process, we also show the curve corresponding to quasistatic deformation, usually used in classical stress relaxation tests, see [1, 3]. The quasistatic case is described by $(2.1)_2$ and $(2.1)_3$. In this case we have $u(x,t) = x \cdot u(1,t) = x \cdot u_{QS}(t)$, $t > 0$, $x \in [0,1]$, i.e. $(2.1)_3$ implies $\varepsilon = u_{QS}$. Next, we use the Laplace transformation of the boundary condition (2.3), given in the form (3.12),

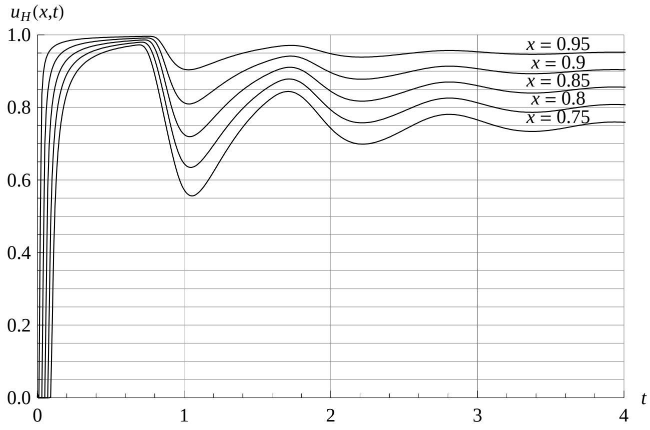

FIGURE 5. Displacements $u_H(x,t)$ in a stress relaxation test, as functions of time at $x \in \{0.75, 0.8, 0.85, 0.9, 0.95\}$ for $t \in (0,4)$.

in $(2.5)_2$ and obtain

$$\tilde{\sigma}(x,s) = \tilde{\varepsilon}(x,s)\frac{\int_0^1 (bs)^\alpha d\alpha}{\int_0^1 (as)^\alpha d\alpha}, \quad s \in \mathbb{C} \setminus (-\infty, 0],$$
$$\tilde{\sigma}_{QS}(s) = \Upsilon_0 \frac{1}{sM_s^2(s)}.$$

Inverting the Laplace transformation $\tilde{\sigma}_{QS}$, as in Theorem 3.1, we obtain

$$\sigma_{QS}(t) = \Upsilon_0 + \frac{\Upsilon_0}{2\pi i}\int_0^\infty \left[\frac{1}{M_s^2(qe^{i\pi})} - \frac{1}{M_s^2(qe^{-i\pi})}\right]\frac{e^{-qt}}{q}\,dq.$$

Figure 6 presents the time evolution of the stress at three points close to the fixed end of the rod. From Figure 6, one sees that there is a delay in the occurrence of the stress and that the delay is greater as the point is closer to the fixed end of the rod. This is a consequence of the finite wave speed. Also, it is evident that the stresses display the damped oscillatory character and that the curves tend to the curve of σ_{QS}, which is monotonically decreasing. Figure 7 shows stresses for the points that are in the middle part of the rod. One sees that the amplitude of the stress increases as the point is closer to the free end of the rod. Finally, in Figures 8 and 9, we present the time evolution of stresses at the points close to the free end. Figure 8 displays stresses at three different points. As could be seen, for the points closer to the free end the time-lag is smaller and the amplitude of stress is higher. For all three points shown, the stresses are positive (extension). Figure 9 displays stresses at the same points as the Figure 8, but at the latter time instant. For these points we have, in the beginning positive stresses, while after some time the stresses become negative, indicating the compressive phase in the axial vibrations of the rod.

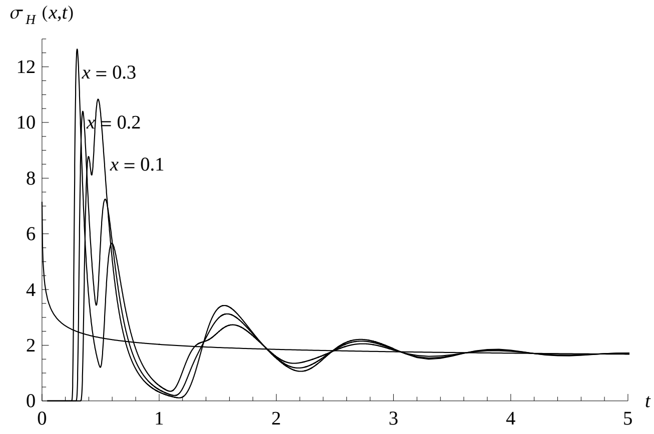

FIGURE 6. Stresses $\sigma_H(x,t)$ in a stress relaxation test, as functions of time at $x \in \{0.1, 0.2, 0.3\}$ for $t \in (0,5)$.

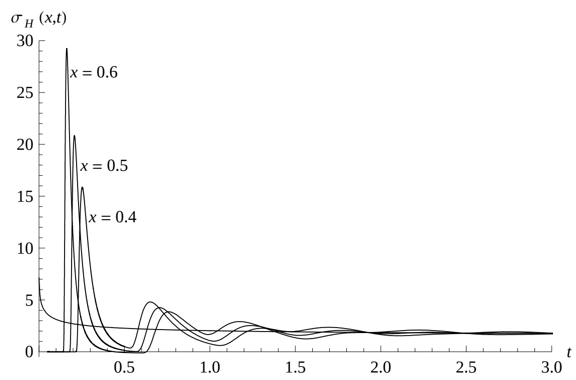

FIGURE 7. Stresses $\sigma_H(x,t)$ in a stress relaxation test, as functions of time at $x \in \{0.4, 0.5, 0.6\}$ for $t \in (0,2.5)$.

All figures show oscillatory character of both displacements and stresses. Oscillations are damped, and for large time displacements show linear dependence on the distance of a particle from a fixed end, while stresses are approaching to the limiting value independently of the position of the particle. In Figure 4 the first minima of $u_H(x,t)$ at $x = 0.75$ is closer to the origin than the first minima corresponding to $x = 0.25$. This shows that the position of the minima is not determined by the waves reflected from the ends of the rod only. The accumulation of material near the ends of the rod (ends are rigid) also influences the position of minima, in the sense that the curves are deformed in shape, and the maxima and minima are not on the positions where one would expect them to be. This is seen form the fact that $u_H(1,t) = H(t)$. Therefore, we have that $\partial u_H(1,t)/\partial t = \delta(t)$ and, in the limit

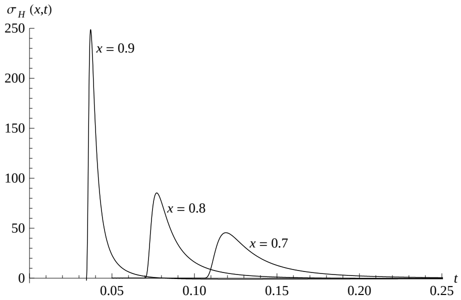

FIGURE 8. Stresses $\sigma_H(x,t)$ in a stress relaxation test, as functions of time at $x \in \{0.7, 0.8, 0.9\}$ for $t \in (0, 0.25)$.

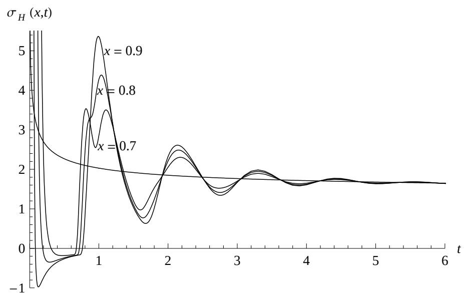

FIGURE 9. Stresses $\sigma_H(x,t)$ in a stress relaxation test, as functions of time at $x \in \{0.7, 0.8, 0.9\}$ for $t \in (0, 6)$.

when $x \to 1$ both minima and maxima of $u_H(x,t)$ are positioned arbitrary close to the origin, since $\delta(t) = 0$ for $t \in (0, \infty)$. For large time, stress relaxation curves show a similar behavior to the curves obtained in the quasistatic case, see [1, 7].

3.4.2. Creep experiments. In this subsubsection, we examine solutions given by (2.20), which correspond to displacement u in the cases of creep and forced oscillations. Choosing $\Sigma(t) = \Sigma_0 H(t)$, $t \in \mathbb{R}$, displacement u, given by (2.20), corresponds to a creep. If $\Sigma(t) = \Sigma_0 \sin(\omega t)$, $t \in \mathbb{R}$ and $\Sigma(t) = 0$, for $t < 0$, then (2.20) describes displacement u in case of forced oscillations of a rod.

In this case, sudden but constant stress is applied at rod's free end. It is mathematically described via boundary condition (2.4) given by

$$\Sigma(t) = \Sigma_0 H(t), \quad t \in \mathbb{R}, \tag{3.35}$$

where Σ_0 is a positive constant. When Σ is given by (3.35), the displacement is determined from (2.20) as

$$u(x,t) = \Sigma_0 H(t) * Q(x,t), \quad x \in [0,1], \ t \in \mathbb{R}, \tag{3.36}$$

where Q is given by (3.33). In Figure 10, we present u given by (3.36) with parameters $\Sigma_0 = 1$, $a = 0.045$, $b = 0.5$; the upper bound in integral is 1000 and the number of residues in the sum is 400. The time interval is $(0, 5)$. Note that there is a delay in displacement of a point $x = 0.5$.

The values of a and b are chosen arbitrary with the only restriction $a \leqslant b$, see (1.7). Note that $a = b$ corresponds to linearly elastic (Hookean) body. Therefore, to stress the viscoelastic property of the body, we selected $a \ll b$. Although, there are experimental data for fractional order Kelvin-Voigt, Maxwell and Zener viscoelastic bodies, see [8, 9, 10, 19], we are not aware of any experimental data corresponding to constitutive equation (1.2) with constitutive functions given by (1.7).

Next, we compare the behavior of displacement of a point at $x = 1$ for large times with the quasistatic case, which is described by $(2.1)_2$ and $(2.1)_3$. In this

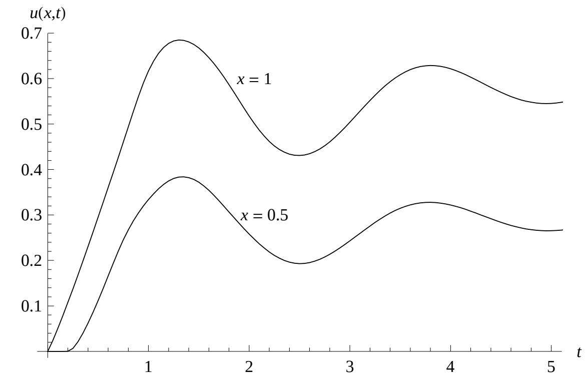

FIGURE 10. Displacements $u(x,t)$ in creep experiment, $\sigma(1,t) = H(t)$, as a function of time at $x = 0.5$, $x = 1$ for $t \in (0,5)$.

case we have $u(x,t) = x \cdot u(1,t) = x \cdot u_{QS}(t)$, $t > 0$, $x \in [0,1]$, i.e., $(2.1)_3$ implies $\varepsilon = u_{QS}$. Using the Laplace transformation of (3.35) in $(2.5)_2$, we obtain

$$\tilde{\varepsilon}(x,s) = \tilde{\sigma}(x,s) M_s^2(s), \quad s \in \mathbb{C} \setminus (-\infty, 0],$$
$$\tilde{u}_{QS}(s) = \Sigma_0 \frac{1}{s} M_s^2(s).$$

This implies, as in Theorem 3.2

$$u_{QS}(t) = \Sigma_0 + \frac{\Sigma_0}{2\pi i} \int_0^\infty \left[M_s^2(qe^{i\pi}) - M_s^2(qe^{-i\pi}) \right] \frac{e^{-qt}}{q} \, dq. \tag{3.37}$$

In Figure 11, we show displacement $u(\cdot,t)$, $t \in (0,30)$, given by (3.36), as well as the quasistatic displacement u_{QS}, given by (3.37), in the same time interval. We see that the curve that represents u displays damped oscillatory character and that the curve tends to the curve of u_{QS}, which is monotonically increasing. It is evident that the displacement is exhibiting creep in the rod, and that the quasistatic approximation (3.37) agrees well with the dynamic solution (3.36), for large times. The stress at the free end is given by (3.35). From Figure 10 we conclude that displacements are zero for a certain time interval at points far from the free end. This is a consequence of the fact that waves in a rod have finite speed of propagation. The displacement depends on the position of a point, it has the character of damped oscillations and tends to a *finite* value for a large time. For large time creep curves tend to curves corresponding to quasistatic analysis, see [1, 7].

Next, we consider the case when the periodic stress is applied at rod's free end. It is mathematically described via boundary condition (2.4) as

$$\Sigma(t) = \begin{cases} \Sigma_0 \sin(\omega t), & t \geqslant 0, \\ 0, & t < 0, \end{cases} \tag{3.38}$$

where Σ_0 is the amplitude and ω is the angular frequency of the applied force.

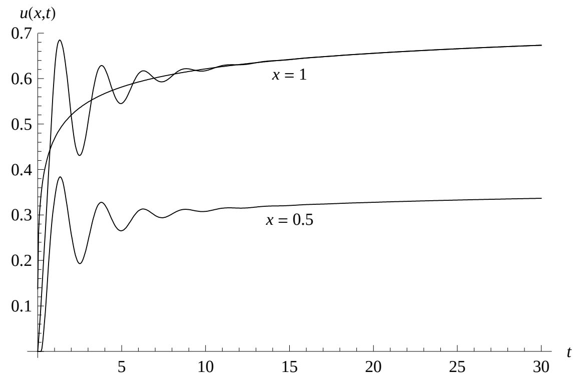

FIGURE 11. Displacements $u(x,t)$ in creep experiment, $\sigma(1,t) = H(t)$, as a function of time at $x = 0.5$, $x = 1$ for $t \in (0,30)$.

When Σ is given by (3.38), the displacement is determined from (2.20) as

$$u(x,t) = \Sigma_0 \sin(\omega t) * Q(x,t), \quad x \in [0,1], \ t \in \mathbb{R}, \tag{3.39}$$

where again, Q is given by (3.33). In Figures 12 and 13, we present u given by (3.39) with parameters $\Sigma_0 = 1$, $a = 0.1$, $b = 0.5$, $\omega = 2$; the upper bound in integral is 1000 and the number of residues in the sum is 400. Figure 12 show the delay

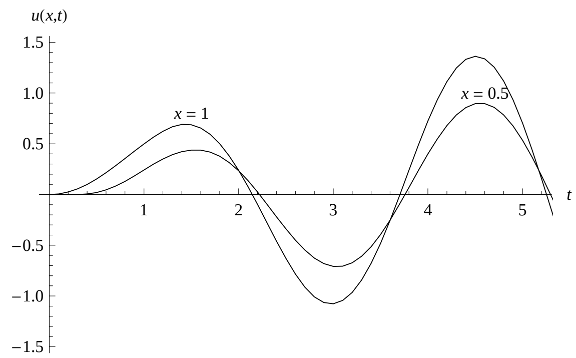

FIGURE 12. Displacements $u(x,t)$ in forced oscillations experiment, $\sigma(1,t) = \sin(2t)$, as a function of time at $x = 0.5$, $x = 1$ for $t \in (0,5)$.

in the displacement function for $x = 0.5$. In Figure 13 we present oscillations for large time. We assume that the stress at free end is given by a periodic function (see (3.38)). The response of the rod, i.e., the displacement is given in Figures 12 and 13. Again, the delay in displacement depends on the position of a point. After the delay the motion is oscillatory, with an increasing amplitude and with angular frequency equal to the angular frequency of the force applied at the free end. For large t the amplitude tends to a constant finite value.

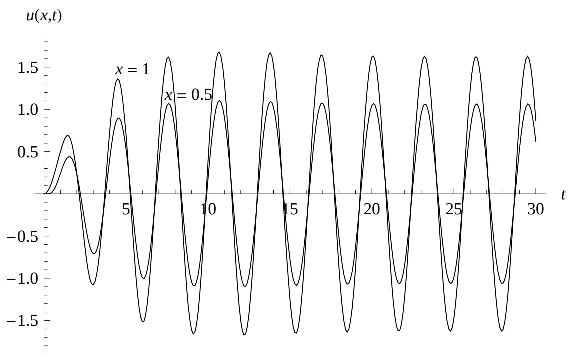

FIGURE 13. Displacements $u(x,t)$ in forced oscillations experiment, $\sigma(1,t) = \sin(2t)$, as a function of time at $x = 0.5$, $x = 1$ for $t \in (0, 30)$.

4. Case of fluid-like viscoelastic body

This section is devoted to the calculation of inverse Laplace transforms of certain distributions and functions introduced above. We begin with examining some basic properties of M_f given in (2.8), which will be important for further investigations. In the sequel we shall again write $A(x) \sim B(x)$ if $\lim_{x\to\infty} \frac{A(x)}{B(x)} = 1$.

Proposition 4.1. *Let M_f be the function defined by* (2.8). *Then*

(i) *M_f is an analytic function on $\mathbb{C} \setminus (-\infty, 0]$ if $0 < \beta < \alpha < \gamma < 1$.*
(ii) *For $s \in \mathbb{C} \setminus (-\infty, 0]$, $\lim_{|s|\to 0} M_f(s) = \infty$, $\lim_{|s|\to 0} sM_f(s) = 0$, $\lim_{|s|\to\infty} M_f(s) = 0$, and $\lim_{|s|\to\infty} sM_f(s) = \infty$.*

Proof. (i) Since $1 + \frac{a}{b}s^{\alpha-\beta} \neq 0$ and $1 + \frac{c}{a}s^{\gamma-\alpha} + \frac{c}{b}s^{\gamma-\beta} \neq 0$ if $\arg s \in (-\pi, \pi)$ and $0 < \beta < \alpha < \gamma < 1$, it follows that the function M_f given in (2.8) is analytic on the complex plane except the branch cut along the negative axis, i.e., on $\mathbb{C} \setminus (-\infty, 0]$.

(ii) Limits in (ii) can easily be calculated. □

To begin with, we examine properties of $\tilde{P}$ given by (2.13). $\tilde{P}$ has isolated singularities at ${}_P s_n^{(\pm)}$, $n \in \mathbb{N}$, where ${}_P s_n^{(\pm)}$ denotes solutions of the equation

$$\sinh(sM_f(s)) = 0, \quad \text{i.e.,} \quad sM_f(s) = \pm ni\pi. \tag{4.1}$$

Let us examine their position and multiplicity.

Proposition 4.2. *There are infinitely many complex conjugated solutions ${}_P s_n^{(\pm)}$, $n \in \mathbb{N}$, of* (4.1), *which all lie in the left complex half plane. Moreover, each ${}_P s_n^{(\pm)}$, $n \in \mathbb{N}$, is a simple pole.*

Proof. Let us square (4.1) and define

$$\Phi(s, n) := (sM_f(s))^2 + (n\pi)^2 = s^2 \frac{1 + \frac{a}{b}s^{\alpha-\beta}}{as^{\alpha} + cs^{\gamma} + \frac{ac}{b}s^{\alpha+\gamma-\beta}} + (n\pi)^2.$$

Writing $s = Re^{i\varphi}$ it follows that

$$\Phi(Re^{i\varphi}, n) = R^2 e^{2i\varphi} \frac{1 + \frac{a}{b} R^{\alpha-\beta} e^{i(\alpha-\beta)\varphi}}{aR^\alpha e^{i\alpha\varphi} + cR^\gamma e^{i\gamma\varphi} + \frac{ac}{b} R^{\alpha+\gamma-\beta} e^{i(\alpha+\gamma-\beta)\varphi}} + (n\pi)^2$$
$$= R^2(\cos(2\varphi) + i\sin(2\varphi)) \frac{A + iB}{C + iD} + (n\pi)^2,$$

where

$$A := 1 + \frac{a}{b} R^{\alpha-\beta} \cos((\alpha - \beta)\varphi),$$
$$B := \frac{a}{b} R^{\alpha-\beta} \sin((\alpha - \beta)\varphi),$$
$$C := aR^\alpha \cos(\alpha\varphi) + cR^\gamma \cos(\gamma\varphi) + \frac{ac}{b} R^{\alpha+\gamma-\beta} \cos((\alpha + \gamma - \beta)\varphi),$$
$$D := aR^\alpha \sin(\alpha\varphi) + cR^\gamma \sin(\gamma\varphi) + \frac{ac}{b} R^{\alpha+\gamma-\beta} \sin((\alpha + \gamma - \beta)\varphi).$$

Real and imaginary parts of Φ are then given by

$$\text{(4.2)} \quad \operatorname{Re}\Phi(Re^{i\varphi}, n) = \frac{R^2}{C^2 + D^2}[(AC + BD)\cos(2\varphi) + (AD - BC)\sin(2\varphi)] + (n\pi)^2,$$

$$\text{(4.3)} \quad \operatorname{Im}\Phi(Re^{i\varphi}, n) = \frac{R^2}{C^2 + D^2}[(AC + BD)\sin(2\varphi) - (AD - BC)\cos(2\varphi)],$$

and

$$AC + BD = aR^\alpha \cos(\alpha\varphi) + cR^\gamma \cos(\gamma\varphi) + \frac{a^2}{b} R^{2\alpha-\beta} \cos(\beta\varphi)$$
$$+ \frac{a^2 c}{b^2} R^{2(\alpha-\beta)+\gamma} \cos(\gamma\varphi) + 2\frac{ac}{b} R^{\alpha+\gamma-\beta} \cos((\alpha - \beta)\varphi) \cos(\gamma\varphi),$$
$$AD - BC = aR^\alpha \sin(\alpha\varphi) + cR^\gamma \sin(\gamma\varphi) + \frac{a^2}{b} R^{2\alpha-\beta} \sin(\beta\varphi)$$
$$+ \frac{a^2 c}{b^2} R^{2(\alpha-\beta)+\gamma} \sin(\gamma\varphi) + 2\frac{ac}{b} R^{\alpha+\gamma-\beta} \cos((\alpha - \beta)\varphi) \sin(\gamma\varphi).$$

Let (R, φ) be a solution to $\Phi(s, n) = 0$ (or equivalently, $\operatorname{Re}\Phi = \operatorname{Im}\Phi = 0$). Then changing $\varphi \to -\varphi$, we again obtain that $\operatorname{Re}\Phi = \operatorname{Im}\Phi = 0$, which implies that solutions of (4.1) are complex conjugated.

Further, we shall prove that the function Φ has no zeros in the half plane $\arg s \in [0, \frac{\pi}{2}]$. For that purpose we shall use the argument principle. Recall, if Φ is an analytic function inside and on a regular closed curve c and nonzero on c, then the number of zeros of Φ is given by $N_Z = \frac{1}{2\pi}\Delta \arg \Phi(s)$. Let $\gamma = \gamma_a \cup \gamma_b \cup \gamma_c$ be parametrized as

$$\gamma_a : s = x, \quad x \in [0, R],$$
$$\gamma_b : s = Re^{i\varphi}, \quad \varphi \in [0, \tfrac{\pi}{2}],$$
$$\Gamma_c : s = xe^{i\frac{\pi}{2}}, \quad x \in [0, R],$$

and let $R \to \infty$. Along γ_a function Φ becomes a real-valued function, hence $\Delta \arg \Phi(s, n) = 0$. Along γ_b we have that $AC + BD, AD - BC \geqslant 0$ for $\varphi \in$

$[0,\pi]$, since $\sin(\eta\varphi), \cos(\eta\varphi) > 0$, for $\eta \in \{\alpha, \beta, \gamma, \alpha-\beta\} \subset (0, \frac{1}{2})$, and $\varphi \in [0,\pi]$. Therefore, (4.2) and (4.3) imply that

$$\operatorname{Re}\Phi(Re^{i\varphi}, n) > 0, \quad \varphi \in [0, \pi/4] \tag{4.4}$$

$$\operatorname{Im}\Phi(Re^{i\varphi}, n) > 0, \quad \varphi \in [\pi/4, \pi/2] \tag{4.5}$$

(inequalities at boundary points are easily checked by inserting them into (4.2) and (4.3)). Along γ_c, we obtain

$$\operatorname{Re}\Phi(xe^{i\frac{\pi}{2}}, n) = -\frac{R^2}{C^2+D^2}\Big|_{R=x,\varphi=\frac{\pi}{2}} (AC+BD)|_{R=x,\varphi=\frac{\pi}{2}} + (n\pi)^2, \tag{4.6}$$

$$\operatorname{Im}\Phi(xe^{i\frac{\pi}{2}}, n) = \frac{R^2}{C^2+D^2}\Big|_{R=x,\varphi=\frac{\pi}{2}} (AD-BC)|_{R=x,\varphi=\frac{\pi}{2}} > 0. \tag{4.7}$$

From (4.4), (4.5) and (4.7), we may now conclude that along γ_b and γ_c,

$$\Delta \arg\Phi(s,n) = 0.$$

Indeed, this follows from the following: Along γ_b, in the case$\varphi \in [0, \frac{\pi}{4}]$, $\operatorname{Im}\Phi(Re^{i\varphi}, n)$ can change its sign, but $\operatorname{Re}\Phi(Re^{i\varphi}, n) > 0$, while for $\varphi \in [\frac{\pi}{4}, \frac{\pi}{2}]$, $\operatorname{Im}\Phi(Re^{i\varphi}, n) > 0$. Along γ_c, $\operatorname{Im}\Phi(xe^{i\frac{\pi}{2}}, n) > 0$. Therefore, there is no change in the argument of Φ along the whole γ, which implies that Φ has no zeros for $\varphi \in [0, \frac{\pi}{2}]$. This further implies that (4.1) has no solutions in the right complex half plane, since its solutions are complex conjugated.

In order to prove that for fixed $n \in \mathbb{N}$ equation (4.1) has one solution (and its complex conjugate), we again use function Φ and the argument principle. Consider now the contour $\Gamma = \Gamma_A \cup \Gamma_B \cup \Gamma_C$ parametrized by

$$\Gamma_A : s = xe^{i\frac{\pi}{2}}, \quad x \in [0, R],$$
$$\Gamma_B : s = Re^{i\varphi}, \quad \varphi \in [\pi/2, \pi],$$
$$\Gamma_C : s = xe^{i\pi}, \quad x \in [0, R],$$

and let $R \to \infty$. Along Γ_A real and imaginary parts of Φ are given by (4.6) and (4.7). Along Γ_B, using (4.2) and (4.3), we conclude that

$$\operatorname{Re}\Phi(Re^{i\varphi}, n) < 0, \quad \varphi \in [\pi/2, 3\pi/4], \quad \text{for } R \to \infty, \text{ and } n \text{ fixed.}$$
$$\operatorname{Im}\Phi(Re^{i\varphi}, n) < 0, \quad \varphi \in [3\pi/4, \pi].$$

Along Γ_C we have

$$\operatorname{Re}\Phi(Re^{i\varphi}, n) = \frac{R^2}{C^2+D^2}\Big|_{R=x,\varphi=\pi} (AC+BD)|_{R=x,\varphi=\pi} + (n\pi)^2 > 0,$$
$$\operatorname{Im}\Phi(Re^{i\varphi}, n) = -\frac{R^2}{C^2+D^2}\Big|_{R=x,\varphi=\pi} (AD-BC)|_{R=x,\varphi=\pi} < 0.$$

Along Γ_A we have that $\operatorname{Im}\Phi > 0$, while $\operatorname{Re}\Phi$ changes its sign (since $\operatorname{Re}\Phi(0,n) = (n\pi)^2$ and $\lim_{x\to\infty}\operatorname{Re}\Phi(xe^{i\frac{\pi}{2}}, n) = -\infty$, for fixed $n \in \mathbb{N}$). Along the part of Γ_B where $\varphi \in [\frac{3\pi}{4}, \pi]$, and along Γ_C, $\operatorname{Im}\Phi < 0$. Also, $\lim_{R\to\infty}\operatorname{Re}\Phi(Re^{i\frac{3\pi}{4}}, n) = -\infty$

and $\operatorname{Im}\Phi(Re^{i\frac{3\pi}{4}},n)<0$. This implies that the argument of Φ changes from 0 to 2π.

As a conclusion one obtains that along Γ

$$\Delta\arg\Phi(s,n)=2\pi,$$

which implies, by the argument principle, that function Φ has exactly one zero in the upper left complex plane, for each fixed $n\in\mathbb{N}$. Since the zeros of Φ are complex conjugated, it follows that Φ also has one zero in the lower left complex plane, for each fixed $n\in\mathbb{N}$. □

In the next proposition we examine behavior of simple poles ${}_P s_n^{(\pm)}$, $n\in\mathbb{N}$.

Proposition 4.3. *The solutions* ${}_P s_n^{(\pm)}$, $n\in\mathbb{N}$, *of* (4.1), *are such that*

$$\operatorname{Re}\left({}_P s_n^{(\pm)}\right)=R\cos\varphi\sim\sqrt[2-\gamma]{c(n\pi)^2}\cos\left(\frac{\pi}{2-\gamma}\right)<0,$$
$$\operatorname{Im}\left({}_P s_n^{(\pm)}\right)=R\sin\varphi\sim\pm\sqrt[2-\gamma]{c(n\pi)^2}\sin\left(\frac{\pi}{2-\gamma}\right),$$

as $n\to\infty$.

Proof. Let us square (4.1) and insert ${}_P s_n^{(\pm)}=Re^{i\varphi}$, $\varphi\in(-\pi,\pi)$. Then, after separation of real and imaginary parts, we obtain

$$R^2\cos(2\varphi)\operatorname{Re}(M_f^2(Re^{i\varphi}))-R^2\sin(2\varphi)\operatorname{Im}(M_f^2(Re^{i\varphi}))=-(n\pi)^2, \tag{4.8}$$
$$R^2\sin(2\varphi)\operatorname{Re}(M_f^2(Re^{i\varphi}))+R^2\cos(2\varphi)\operatorname{Im}(M_f^2(Re^{i\varphi}))=0. \tag{4.9}$$

Using notation from the proof of Proposition 4.2 we can write

$$\operatorname{Re}(M_f^2(Re^{i\varphi}))=\frac{AC+BD}{C^2+D^2},\quad \operatorname{Im}(M_f^2(Re^{i\varphi}))=\frac{BC-AD}{C^2+D^2}.$$

Letting $R\to\infty$, one has

$$\operatorname{Re}(M_f^2(Re^{i\varphi}))\sim\frac{\frac{a^2c}{b^2}R^{2(\alpha-\beta)+\gamma}\cos(\gamma\varphi)}{\frac{a^2c^2}{b^2}R^{2(\alpha-\beta)+2\gamma}}=\frac{1}{cR^\gamma}\cos(\gamma\varphi), \tag{4.10}$$

$$\operatorname{Im}(M_f^2(Re^{i\varphi}))\sim-\frac{\frac{a^2c}{b^2}R^{2(\alpha-\beta)+\gamma}\sin(\gamma\varphi)}{\frac{a^2c^2}{b^2}R^{2(\alpha-\beta)+2\gamma}}=-\frac{1}{cR^\gamma}\sin(\gamma\varphi). \tag{4.11}$$

It now follows from (4.9), (4.10) and (4.11) that

$$\tan(2\varphi)=-\frac{\operatorname{Im}(M_f^2(Re^{i\varphi}))}{\operatorname{Re}(M_f^2(Re^{i\varphi}))}\sim\tan(\gamma\varphi)\;\Rightarrow\;\frac{\sin((2-\gamma)\varphi)}{\cos(2\varphi)\cos(\gamma\varphi)}\sim 0\;\Rightarrow\;\varphi\sim\pm\frac{\pi}{2-\gamma} \tag{4.12}$$

Inserting (4.12) into (4.10) and (4.11), and subsequently into (4.8), we obtain

$$\frac{R^{2-\gamma}}{c}\cos\left(\frac{2\pi}{2-\gamma}\right)\cos\left(\frac{\gamma\pi}{2-\gamma}\right)+\frac{R^{2-\gamma}}{c}\sin\left(\frac{2\pi}{2-\gamma}\right)\sin\left(\frac{\gamma\pi}{2-\gamma}\right)\sim-(n\pi)^2$$

$$R\sim\sqrt[2-\gamma]{c(n\pi)^2}. \tag{4.13}$$

Thus, real and imaginary parts of ${}_P s_n^{(\pm)}$, as $R\to\infty$, are as claimed. □

Proposition 4.4. *Let* $p \in (0, s_0)$, $s_0 > 0$. *Then* $M_f(p \pm iR) \sim \frac{1}{\sqrt{cR^\gamma}} e^{\mp i \frac{\gamma\pi}{4}}$ *as* $R \to \infty$.

Proof. Set $\mu := \sqrt{p^2 + R^2}$ and $\nu := \arctan \frac{\pm R}{p}$. Then $\mu \sim R$ and $\nu \sim \pm\frac{\pi}{2}$, as $R \to \infty$. By (4.10) and (4.11), we have

$$M_f(\mu e^{i\nu}) \sim \frac{1}{\sqrt{c\mu^\gamma}} e^{\mp i \frac{\gamma\nu}{4}}, \quad \mu \to \infty,$$

as claimed. □

The function $\tilde{Q}$ given by (2.19) is analytic on the complex plane except the branch cut $(-\infty, 0]$, and has isolated singularities at solutions ${}_Q s_n^{(\pm)}$ of the equation

$$\cosh(sM_f(s)) = 0 \text{ i.e., } sM_f(s) = \pm\frac{2n+1}{2} i\pi, \quad n \in \mathbb{N}_0. \tag{4.14}$$

We state a proposition that is analogous to Propositions 4.2 and 4.3. The proof is omitted since it follows the same lines as those of Propositions 4.2 and 4.3.

Proposition 4.5. (i) *There are infinitely many complex conjugated solutions* ${}_Q s_n^{(\pm)}$, $n \in \mathbb{N}_0$, *of* (4.14), *which all lie in the left complex half plane. Moreover, each* ${}_Q s_n^{(\pm)}$, $n \in \mathbb{N}_0$, *is a simple pole.*

(ii) *The solutions* ${}_Q s_n^{(\pm)}$, $n \in \mathbb{N}_0$, *of* (4.14), *are such that*

$$\operatorname{Re}({}_Q s_n^{(\pm)}) = R\cos\varphi \sim \sqrt[2-\gamma]{c\left(\tfrac{1}{2}(2n+1)\pi\right)^2} \cos\left(\frac{\pi}{2-\gamma}\right) < 0, \tag{4.15}$$

$$\operatorname{Im}({}_Q s_n^{(\pm)}) = R\sin\varphi \sim \pm \sqrt[2-\gamma]{c\left(\tfrac{1}{2}(2n+1)\pi\right)^2} \sin\left(\frac{\pi}{2-\gamma}\right), \tag{4.16}$$

as $n \to \infty$.

4.1. Determination of the displacement u in a stress relaxation test .

In order to obtain the explicit form of solution u to initial-boundary value problem (2.1), (2.2), (2.3), it remains to calculate function P.

Theorem 4.1. *The solution* u *to initial-boundary value problem* (2.1), (2.2), (2.3) *is given by* (2.14), *i.e.,* $u(x,t) = \Upsilon(t) * P(x,t)$, *where* P *takes the form*

$$\begin{aligned} P(x,t) &= \frac{1}{2\pi i}\int_0^\infty \left(\frac{\sinh(xqM_f(qe^{-i\pi}))}{\sinh(qM_f(qe^{-i\pi}))} - \frac{\sinh(xqM_f(qe^{i\pi}))}{\sinh(qM_f(qe^{i\pi}))}\right) e^{-qt} dq \\ &+ \sum_{n=1}^{\infty} \left[\operatorname{Res}\left(\tilde{P}(x,s)e^{st}, {}_P s_n^{(+)}\right) + \operatorname{Res}\left(\tilde{P}(x,s)e^{st}, {}_P s_n^{(-)}\right)\right], \quad x \in [0,1], \ t > 0. \end{aligned} \tag{4.17}$$

The residues at simple poles ${}_P s_n^{(\pm)}$, $n \in \mathbb{N}$, *are given by*

$$\operatorname{Res}\left(\tilde{P}(x,s)e^{st}, {}_P s_n^{(\pm)}\right) = \left[\frac{\sinh(xsM_f(s))}{\frac{d}{ds}[\sinh(sM_f(s))]} e^{st}\right]_{s = {}_P s_n^{(\pm)}}. \tag{4.18}$$

Proof. Function $P(x,t)$, $x \in [0,1]$, $t > 0$, will be calculated by integration over a suitable contour. The Cauchy residues theorem yields
(4.19)
$$\oint_\Gamma \tilde{P}(x,s)e^{st}ds = 2\pi i \sum_{n=1}^{\infty} \left[\operatorname{Res}\left(\tilde{P}(x,s)e^{st}, {}_P s_n^{(+)}\right) + \operatorname{Res}\left(\tilde{P}(x,s)e^{st}, {}_P s_n^{(-)}\right)\right],$$
where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_\varepsilon \cup \Gamma_4 \cup \Gamma_5 \cup \Gamma_6 \cup \gamma_0$ is such a contour that all poles lie inside the contour Γ (see Figure 1).

First we show that the series of residues in (4.17) is convergent. By Proposition 4.3, ${}_P s_n^{(\pm)}$ are simple poles of $\tilde{P}$, and therefore also simple poles of $e^{st}\tilde{P}$. The residues in (4.19) can be calculated as it is given in (4.18), so
$$\operatorname{Res}\left(\tilde{P}(x,s)e^{st}, {}_P s_n^{(\pm)}\right) = \left[\frac{1}{\frac{d}{ds}(sM_f(s))}\frac{\sinh(xsM_f(s))}{\cosh(sM_f(s))}e^{st}\right]_{s={}_P s_n^{(\pm)}}.$$
Using (4.1) we have that $\sinh(xsM_f(s)) = \pm i\sin(xn\pi)$, while $\cosh(sM_f(s)) = (-1)^n$. Also, a calculation gives that $\frac{d}{ds}(sM_f(s)) = M_f(s) + M_f(s)\cdot A(s)$, where
$$A(s) := 1 + \frac{\frac{a}{b}(\alpha-\beta)s^{\alpha-\beta}}{2(1+\frac{a}{b}s^{\alpha-\beta})} - \frac{a\alpha s^\alpha + c\gamma s^\gamma + \frac{ac}{b}(\alpha+\gamma-\beta)s^{\alpha+\gamma-\beta}}{2(as^\alpha + cs^\gamma + \frac{ac}{b}s^{\alpha+\gamma-\beta})}.$$
Take now that ${}_P s_n^{(\pm)} = Re^{\pm i\varphi}$. Then
$$\operatorname{Res}\left(\tilde{P}(x,s)e^{st}, {}_P s_n^{(\pm)}\right) = (-1)^n \frac{\sin(n\pi x)}{n\pi}\frac{Re^{Rt\cos\varphi}e^{\pm i(\varphi + Rt\sin\varphi)}}{A(Re^{\pm i\varphi})}$$
Let $n \to \infty$. Then also $|{}_P s_n^{(\pm)}| \to \infty$, i.e., $R \to \infty$, and $|A(Re^{\pm i\varphi})| \to 1 - \frac{\gamma}{2}$. Further, it follows from Proposition 4.3 that
$$\operatorname{Re}({}_P s_n^{(\pm)}) = R\cos\varphi \sim \sqrt[2-\gamma]{c(n\pi)^2}\cos\Big(\frac{\pi}{2-\gamma}\Big) \leqslant -Cn, \text{ for some } C > 0,$$
and by (4.13), $\frac{R}{n} \sim \sqrt[2-\gamma]{c\pi^2}\cdot n^{\frac{\gamma}{2-\gamma}}$. Therefore, as $n \to \infty$,
$$\begin{aligned}\Big|\operatorname{Res}\left(\tilde{P}(x,s)e^{st}, {}_P s_n^{(+)}\right) &+ \operatorname{Res}\left(\tilde{P}(x,s)e^{st}, {}_P s_n^{(-)}\right)\Big| \\ &\leqslant \Big|\frac{\sin(n\pi x)}{n\pi}\frac{Re^{Rt\cos\varphi}}{A(Re^{+i\varphi})}\Big| + \Big|\frac{\sin(n\pi x)}{n\pi}\frac{Re^{Rt\cos\varphi}}{A(Re^{-i\varphi})}\Big| \\ &\leqslant \frac{1}{\pi}\frac{R}{n}e^{-Cnt}\Big(\frac{1}{|A(Re^{+i\varphi})|} + \frac{1}{|A(Re^{-i\varphi})|}\Big) \\ &\sim \frac{4}{\pi(2-\gamma)}\sqrt[2-\gamma]{c\pi^2}\cdot n^{\frac{\gamma}{2-\gamma}}e^{-Cnt},\end{aligned}$$
which implies the convergence of the sum of residues in (4.19).

It remains to calculate the integral over Γ in (4.19). Consider the integral along contour $\Gamma_1 : s = p + iR$, $s_0 > p > 0$. Then
$$\left|\int_{\Gamma_1}\tilde{P}(x,s)e^{st}ds\right| \leqslant \int_0^{s_0}\left|\tilde{P}(x,p+iR)\right|\left|e^{(p+iR)t}\right|dp.$$

Let $R \to \infty$. In order to estimate $|\tilde{P}(x, p \pm iR)|$, using Proposition 4.4, we write

$$M_f(p \pm iR) \sim v \pm iw, \quad v = \frac{1}{\sqrt{cR^\gamma}} \cos\left(\frac{\gamma\pi}{4}\right), \quad w = -\frac{1}{\sqrt{cR^\gamma}} \sin\left(\frac{\gamma\pi}{4}\right).$$

Then

$$\begin{aligned} \left|\tilde{P}(x, p \pm iR)\right| &\sim \left|\frac{\sinh[x(pv - Rw) \pm ix(pw + Rv)]}{\sinh[(pv - Rw) \pm i(pw + Rv)]}\right| \\ &\leqslant \frac{e^{x(pv-Rw)} + e^{-x(pv-Rw)}}{|e^{pv-Rw} - e^{-(pv-Rw)}|} \\ &= e^{-(1-x)(pv-Rw)} \frac{1 + e^{-2x(pv-Rw)}}{|1 - e^{-2(pv-Rw)}|} \to 0, \quad \text{as} \quad R \to \infty. \end{aligned} \tag{4.20}$$

The above convergence is valid since

$$pv - Rw = p\frac{1}{\sqrt{cR^\gamma}} \cos\left(\frac{\gamma\pi}{4}\right) + R\frac{1}{\sqrt{cR^\gamma}} \sin\left(\frac{\gamma\pi}{4}\right) \to \infty, \quad \text{as} \quad R \to \infty.$$

Therefore, according to (4.20), we have

$$\lim_{R\to\infty} \left|\int_{\Gamma_1} \tilde{P}(x, s) e^{st} ds\right| = 0.$$

The similar argument is valid for the integral along Γ_6, thus

$$\lim_{R\to\infty} \left|\int_{\Gamma_6} \tilde{P}(x, s) e^{st} ds\right| = 0.$$

Next, we consider the integral along contour $\Gamma_2 : s = Re^{i\varphi}$, $\frac{\pi}{2} < \varphi < \pi$:

$$\left|\int_{\Gamma_2} \tilde{P}(x, s) e^{st} ds\right| \leqslant \int_{\frac{\pi}{2}}^{\pi} R|e^{-R(1-x)e^{i\varphi} M_f(Re^{i\varphi})}| \left|\frac{1 - e^{-2xRe^{i\varphi} M_f(Re^{i\varphi})}}{1 - e^{-2Re^{i\varphi} M_f(Re^{i\varphi})}}\right| e^{Rt\cos\varphi} d\varphi.$$

Since $sM_f(s) \to \infty$ as $|s| \to \infty$ (see Proposition 4.1 (ii)) and $\cos\varphi \leqslant 0$ for $\varphi \in [\frac{\pi}{2}, \pi]$, we have

$$\lim_{R\to\infty} \left|\int_{\Gamma_2} \tilde{P}(x, s) e^{st} ds\right| \leqslant \lim_{R\to\infty} \int_{\frac{\pi}{2}}^{\pi} R|e^{-R(1-x)e^{i\varphi} M_f(Re^{i\varphi})}| e^{Rt\cos\varphi} d\varphi = 0.$$

The similar argument is valid for the integral along Γ_5, thus

$$\lim_{R\to\infty} \left|\int_{\Gamma_5} \tilde{P}(x, s) e^{st} ds\right| = 0.$$

Since $sM_f(s) \to 0$ as $|s| \to 0$ (see Proposition 4.1 (ii)), the integration along contour $\Gamma_\varepsilon : s = \varepsilon e^{i\varphi}$, $\pi > \varphi > -\pi$, gives

$$\lim_{\varepsilon\to 0} \left|\int_{\Gamma_\varepsilon} \tilde{P}(x, s) e^{st} ds\right| \leqslant \lim_{\varepsilon\to 0} \int_{\pi}^{-\pi} \varepsilon \left|\frac{\sinh(x\varepsilon e^{i\varphi} M_f(\varepsilon e^{i\varphi}))}{\sinh(\varepsilon e^{i\varphi} M_f(\varepsilon e^{i\varphi}))}\right| e^{\varepsilon t\cos\varphi} d\varphi = 0.$$

Integrals along parts of contour $\Gamma_3 : s = qe^{i\pi}$, $R > q > \varepsilon$, $\Gamma_4 : s = qe^{-i\pi}$, $\varepsilon < q < R$, and $\gamma_0 : s = s_0 + ir$, $-R < r < R$, give

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_{\Gamma_3} \tilde{P}(x,s)e^{st}ds = \int_0^\infty \frac{\sinh(xqM_f(qe^{i\pi}))}{\sinh(qM_f(qe^{i\pi}))}e^{-qt}dq,$$

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_{\Gamma_4} \tilde{P}(x,s)e^{st}ds = -\int_0^\infty \frac{\sinh(xqM_f(qe^{-i\pi}))}{\sinh(qM_f(qe^{-i\pi}))}e^{-qt}dq,$$

$$\lim_{R\to\infty} \int_{\gamma_0} \tilde{P}(x,s)e^{st}ds = 2\pi i P(x,t).$$

(4.17) now follows from (4.19). □

Corollary 4.1. *In the case of stress relaxation, i.e., when* $\Upsilon(t) = \Upsilon_0 H(t)$, $\Upsilon_0 > 0$, $t \in \mathbb{R}$, *the solution takes the form*

$$u_H(x,t) = \Upsilon_0 H(t) * P(x,t), \quad x \in [0,1],\ t \in \mathbb{R}. \tag{4.21}$$

We shall numerically examine it in the sequel.

4.2. Determination of the stress σ in a stress relaxation test . We determined stress σ (cf. (2.17)) that is a solution to (2.1), (2.2), (2.3). In order to obtain an explicit form of σ we need to calculate function T. As in previous subsection 4.1 it will be done by inversion of the Laplace transform of $\tilde{T}$.

Function $\tilde{T}$, which is given by (2.16), is analytic on the complex plane except the branch cut $(-\infty, 0]$, and has simple poles at the same points as $\tilde{P}$, i.e., ${}_Ps_n^{(\pm)}$, $n \in \mathbb{N}$.

Using the similar arguments as in the proof of Theorem 4.1, one can prove the following theorem:

Theorem 4.2. *The solution σ to initial-boundary value problem* (2.1), (2.2), (2.3) *is given by* (2.17), *i.e.,* $\sigma(x,t) = \frac{d}{dt}(\Upsilon(t) * T(x,t))$, *where T takes the form*

$$T(x,t) = \frac{1}{2\pi i}\int_0^\infty\left(\frac{\cosh(xqM_f(qe^{i\pi}))}{M_f(qe^{i\pi})\sinh(qM_f(qe^{i\pi}))} - \frac{\cosh(xqM_f(qe^{-i\pi}))}{M_f(qe^{-i\pi})\sinh(qM_f(qe^{-i\pi}))}\right)e^{-qt}dq$$
$$+\sum_{n=1}^{\infty}\left[\operatorname{Res}\left(\tilde{T}(x,s)e^{st}, {}_Ps_n^{(+)}\right) + \operatorname{Res}\left(\tilde{T}(x,s)e^{st}, {}_Ps_n^{(-)}\right)\right], \quad x \in [0,1],\ t > 0.$$

The residues at simple poles ${}_Ps_n^{(\pm)}$, $n \in \mathbb{N}$, *are given by*

$$\operatorname{Res}\left(\tilde{T}(x,s)e^{st}, {}_Ps_n^{(\pm)}\right) = \left[\frac{\cosh(xsM_f(s))}{M_f(s)\frac{d}{ds}[\sinh(sM_f(s))]}e^{st}\right]_{s={}_Ps_n^{(\pm)}}.$$

Corollary 4.2. *Similarly, in the case of stress relaxation* $\Upsilon = \Upsilon_0 H$ *we obtain the solution*

$$\sigma_H(x,t) = \Upsilon_0 T(x,t), \quad x \in [0,1],\ t > 0, \tag{4.22}$$

In order to check our results for large times, we compare them with the quasistatic case. In the quasistatic case one uses only the constitutive equation $(2.1)_3$ with the constitutive distributions taken in as in (1.8), i.e., the dynamics of the process is neglected. Taking the Laplace transform of the constitutive equation we obtain $(2.5)_2$, and define the relaxation modulus G via its Laplace transform, as follows:

$$G(t) := \mathcal{L}^{-1}[\tilde{G}(s)](t), \quad t > 0,$$
$$\tilde{G}(s) := \frac{\tilde{\sigma}^{(QS)}(s)}{\tilde{\varepsilon}^{(QS)}(s)} := \frac{1}{M_f^2(s)}, \quad s \in \mathbb{C} \setminus (-\infty, 0],$$

where M_f is given by (2.8). Then the stress in the quasistatic case is

$$\sigma^{(QS)} = G * \varepsilon^{(QS)}. \tag{4.23}$$

Following the proof of Theorem 4.1, we obtain that

$$G(t) = \frac{1}{2\pi i} \int_0^\infty \left[\tilde{G}(qe^{-i\pi}) - \tilde{G}(qe^{i\pi})\right] e^{-qt} dt.$$

In the quasistatic case it holds that $u(x,t) = x \cdot u(1,t), \quad x \in [0,1]\ t > 0$, and consequently by (2.1),

$$\varepsilon(x,t) = u(1,t) =: \varepsilon^{(QS)}(t), \quad x \in [0,1]\ t > 0.$$

Since according to boundary condition (2.3), $u(1,t) = \Upsilon(t)$, it follows from (4.23) that $\sigma^{(QS)} = \Upsilon * G$, which, in the case of stress relaxation, i.e., when $\Upsilon = \Upsilon_0 H$, $\Upsilon_0 > 0$, becomes

$$\sigma_H^{(QS)} = \Upsilon_0 H * G. \tag{4.24}$$

4.3. Determination of the displacement u in a creep test . We determined displacement u (cf. (2.20)) that is a solution to (2.1), (2.2), (2.4). As above, we now want to find u explicitly, by calculating the inverse Laplace transform of $\tilde{Q}$.

In the following theorem we calculate explicitly displacement u.

Theorem 4.3. *The solution u to initial-boundary value problem* (2.1), (2.2), (2.4) *is given by* (2.20), *i.e.,* $u(x,t) = \Sigma(t) * Q(x,t)$, *where Q takes the form*

$$\begin{aligned} Q(x,t) = \frac{1}{2\pi i} \int_0^\infty & \left(M_f(qe^{-i\pi}) \frac{\sinh(xqM_f(qe^{-i\pi}))}{\cosh(qM_f(qe^{-i\pi}))} \right. \\ & \left. - M_f(qe^{i\pi}) \frac{\sinh(xqM_f(qe^{i\pi}))}{\cosh(qM_f(qe^{i\pi}))} \right) \frac{e^{-qt}}{q} dq \\ & + \sum_{n=0}^{\infty} \left[\operatorname{Res}\left(\tilde{Q}(x,s)e^{st}, {}_Q s_n^{(+)}\right) + \operatorname{Res}\left(\tilde{Q}(x,s)e^{st}, {}_Q s_n^{(-)}\right) \right], \quad t > 0. \end{aligned}$$

The residues at simple poles ${}_Q s_n^{(\pm)}$, $n \in \mathbb{N}_0$, are given by

$$\operatorname{Res}\left(\tilde{Q}(x,s)e^{st}, {}_Q s_n^{(\pm)}\right) = \left[\frac{1}{s} M_f(s) \frac{\sinh(xsM_f(s))}{\frac{d}{ds}[\cosh(sM_f(s))]} e^{st} \right]_{s = {}_Q s_n^{(\pm)}}.$$

Corollary 4.3. *The case of creep is described by the boundary condition* $\Sigma(t) = \Sigma_0 H(t)$, $\Sigma_0 > 0$, $t \in \mathbb{R}$, *in which displacement u, given by* (2.20), *reads*

$$u(t) = \Sigma_0 H(t) * Q(x,t), \quad x \in [0,1], \ t \in \mathbb{R}. \tag{4.25}$$

Similarly as in subsection 4.2, we examine the quasistatic case that corresponds to displacement u. Again, using only the constitutive equation $(2.1)_3$ and its Laplace transform $(2.5)_2$, we define the creep compliance J via its Laplace transform as

$$J(t) := \mathcal{L}^{-1}[\tilde{J}(s)](t), \quad t > 0,$$
$$\tilde{J}(s) := \frac{\tilde{\varepsilon}^{(QS)}(s)}{\tilde{\sigma}^{(QS)}(s)} := M_f^2(s), \quad s \in \mathbb{C} \setminus (-\infty, 0],$$

where M_f is given by (2.8). Strain measure in the quasistatic case now equals

$$\varepsilon^{(QS)} = J * \sigma^{(QS)}. \tag{4.26}$$

Following the proof of Theorem 4.1 one obtains

$$J(t) = \frac{1}{2\pi i} \int_0^\infty \left[\tilde{J}(qe^{-i\pi}) - \tilde{J}(qe^{i\pi}) \right] e^{-qt} dt.$$

In the quasistatic case we have $u(x,t) = x \cdot u(1,t) = x \cdot u^{(QS)}(t)$, $x \in [0,1]$, $t > 0$, and consequently by (2.1),

$$\varepsilon^{(QS)} = u^{(QS)}. \tag{4.27}$$

Also, $\sigma^{(QS)}(t) := \sigma(1,t) = \Sigma(t)$, $t > 0$, which is the boundary condition (2.4), hence by (4.26) and (4.27), $u^{(QS)} = \Sigma * J$. In the case of creep (cf. Corollary 4.3) we have

$$u_H^{(QS)} = \Sigma_0 H * J. \tag{4.28}$$

4.4. Numerical examples. In this subsection we give several numerical examples of displacement u_H and stress σ_H, given by (4.21) and (4.22) respectively, which correspond to the case of stress relaxation, and examine solutions (4.25), which correspond to displacement u in the case of creep. In addition, we investigate solutions u_H, σ_H and u for different orders of fractional derivatives.

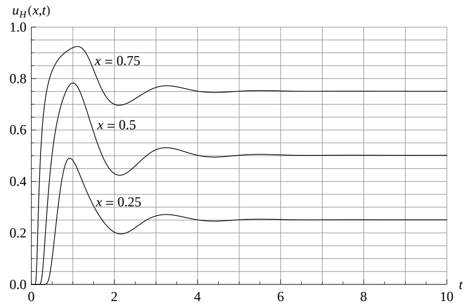

FIGURE 14. Displacements $u_H(x,t)$ in a stress relaxation experiment as functions of time t at $x \in \{0.25, 0.5, 0.75\}$ for $t \in (0, 10)$.

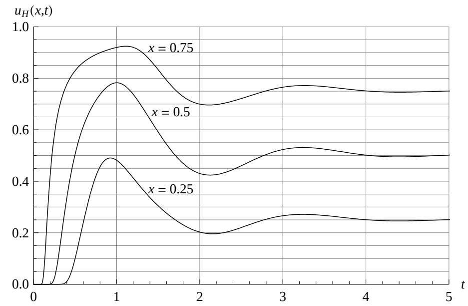

FIGURE 15. Displacements $u_H(x,t)$ in a stress relaxation experiment as functions of time t at $x \in \{0.25, 0.5, 0.75\}$ for $t \in (0, 5)$.

Figure 14 presents displacements in a stress relaxation experiment, determined according to (4.21), for three different positions. Parameters in (4.21) are chosen as follows: $\Upsilon_0 = 1$, $a = 0.2$, $b = 0.6$, $c = 0.45$, $\alpha = 0.3$, $\beta = 0.1$, $\gamma = 0.4$. From Figure 14, one sees that the displacements in the case of stress relaxation show damped oscillatory character and that they tend to a constant value for large times, namely $\lim_{t\to\infty} u_H(x,t) = x$, $x \in [0,1]$. Figure 15 presents the same displacements as Figure 14, but close to initial time instant. It is evident that there is a delay in displacement that increases as the point is further from the end where the prescribed displacement is applied. This is a consequence of the finite wave propagation speed.

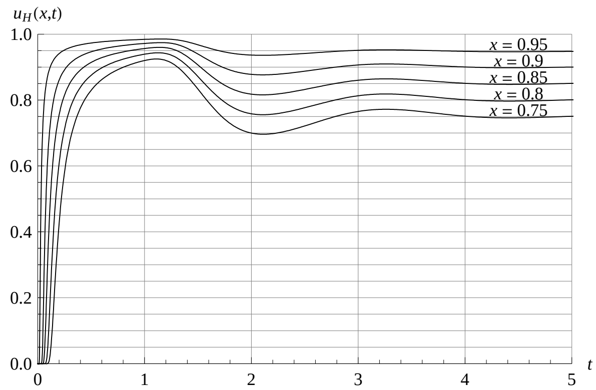

FIGURE 16. Displacements $u_H(x,t)$ in a stress relaxation experiment as functions of time t at $x \in \{0.75, 0.8, 0.85, 0.9, 0.95\}$ for $t \in (0,5)$.

Figure 16 presents the displacements u_H of the points close to the end of the rod where the sudden but afterwards constant displacement is applied (i.e., $u(1,t) = H(t)$, $t > 0$). One sees that the amplitudes of these points deform in shape so that they do not exceed the prescribed value of the displacement of the rod's free end.

In order to examine the influence of the orders of fractional derivatives in the constitutive equation $(2.1)_2$, with (1.8), on the displacement u_H and stress σ_H in a stress relaxation experiment, we plot the displacement u_H and stress σ_H obtained by (4.21) and (4.22) for the following sets of parameters

$$(\alpha, \beta, \gamma) \in \{(0.1, 0.05, 0.15), (0.3, 0.1, 0.4), (0.45, 0.4, 0.49)\}$$

while we fix $x = 0.5$ and leave other parameters as before. In the case of the first set, the constitutive equation $(2.1)_2$, with (1.8), describes a body in which the elastic properties are dominant, since the orders of the fractional derivatives of stress and strain are close to zero, i.e., the fractional derivatives of stress and strain almost coincide with the stress and strain. This is also evident from Figure 17, since the oscillations of the point $x = 0.5$ for the first set of parameters vanish quite slowly comparing to the second set and in particular comparing with the third set of parameters. Note that the third set of parameters describes a body in which the fluid properties of the fractional type dominate, since in the constitutive equation $(2.1)_2$, with (1.8), we have the low-order derivative of stress (almost the stress itself), while almost all the derivatives of strain are of order 0.5. Figure 17 also shows that the dissipative properties of a material grow as the orders of the fractional derivatives increase. Figure 18 shows that the delay in displacement depends on the order of the fractional derivative, so that a material which has dominant elastic properties (the first set) has the longest delay, compared to a material with the dominant fluid properties (the third set).

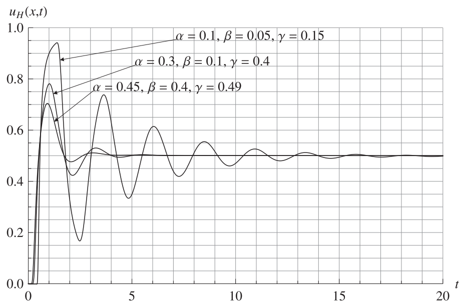

FIGURE 17. Displacements $u_H(x,t)$ in a stress relaxation experiment as functions of time t at $x = 0.5$ for $t \in (0, 20)$.

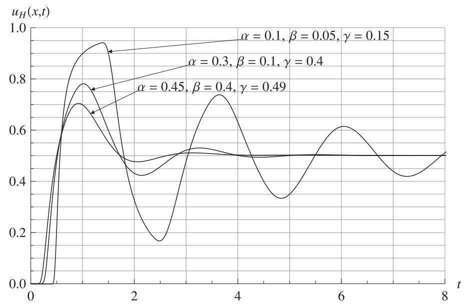

FIGURE 18. Displacements $u_H(x,t)$ in a stress relaxation experiment as functions of time t at $x = 0.5$ for $t \in (0, 8)$.

Figure 19 presents stresses σ_H in the case of stress relaxation, determined according to (4.22), for different points of the rod. Parameters are the same as in the previous case. Also, Figure 19 presents the quasistatic curve $\sigma_H^{(QS)}$ obtained by (4.24).

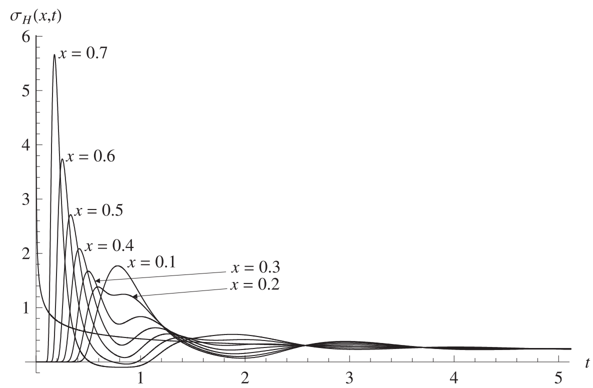

FIGURE 19. Stresses $\sigma_H(x,t)$ and $\sigma_H^{(QS)}(t)$ in a stress relaxation experiment as functions of time t at $x \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7\}$ for $t \in (0, 5)$.

Stresses, as it can be seen from Figure 19, show damped oscillatory character and for large times, in each point $x \in [0, 1]$, tend to the quasistatic curve, i.e., to the same value. Eventually, the stresses in all points of the rod tend to zero, namely $\lim_{t\to\infty} \sigma_H(x,t) = 0$, $x \in [0,1]$. From Figure 19 it is evident that as further the point is from the rod free end the greater is the delay. This is again the consequence of the finite wave speed.

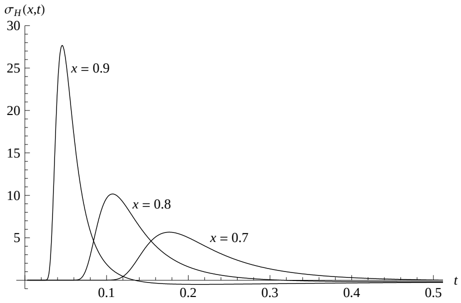

FIGURE 20. Stresses $\sigma_H(x,t)$ in a stress relaxation experiment as functions of time t at $x \in \{0.7, 0.8, 0.9\}$ for $t \in (0, 0.5)$.

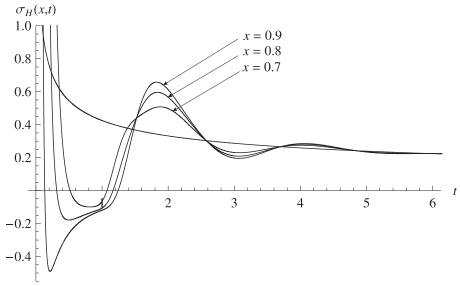

FIGURE 21. Stresses $\sigma_H(x,t)$ and $\sigma_H^{(QS)}(t)$ in a stress relaxation experiment as functions of time t at $x \in \{0.7, 0.8, 0.9\}$ for $t \in (0, 6)$.

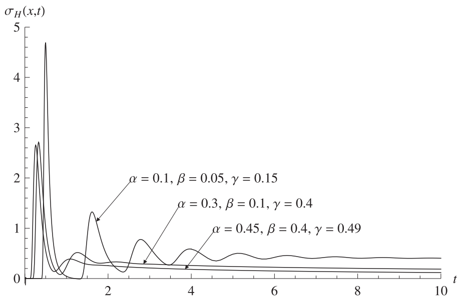

FIGURE 22. Stresses $\sigma_H(x,t)$ in a stress relaxation experiment as functions of time t at $x = 0.5$ for $t \in (0, 10)$.

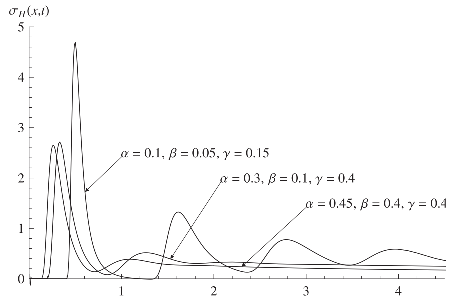

FIGURE 23. Stresses $\sigma_H(x,t)$ in a stress relaxation experiment as functions of time t at $x = 0.5$ for $t \in (0, 4.5)$.

Figures 20 and 21 present the stresses of the points close to the free end. One notices from Figures 19 and 20 that as the point is closer to the end whose displacement is prescribed, the peak of stress is higher and the peak's width is smaller. Figure 21 presents the compressive phase in the stress relaxation process and the quasistatic curve as well.

Again, we fix the midpoint of the rod and investigate the influence of the change of orders of the fractional derivatives (for the same three sets as before) on the stress σ_H in a stress relaxation experiment. From Figure 22 one notices that there is a stress relaxation in a material regardless of the order of fractional derivatives.

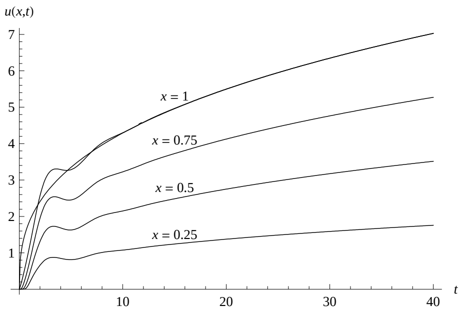

FIGURE 24. Displacements $u(x,t)$ and $u_H^{(QS)}(t)$ in a creep experiment as functions of time t at $x \in \{0.25, 0.5, 0.75, 1\}$ for $t \in (0, 40)$.

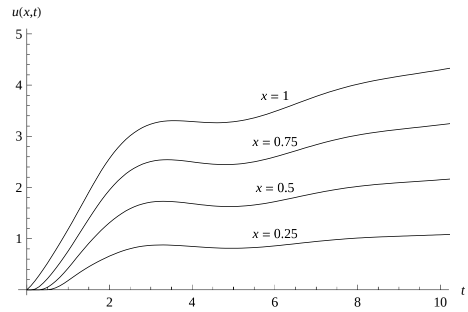

FIGURE 25. Displacements $u(x,t)$ in creep experiment as functions of time t at $x \in \{0.25, 0.5, 0.75, 1\}$ for $t \in (0, 10)$.

However, the relaxed stress depends on the order of the derivatives, as expected from the analysis of the stress $\sigma_H^{(QS)}$ in the quasistatic case, given by (4.24). The material with the dominant elastic properties (the first set) relaxes to the highest stress, and as the fluid properties of the material become more and more dominant, the relaxed stress decreases. Again, the oscillations of the value of the stress for the first set (material with dominant elastic properties) are the least damped comparing to the second and third set. Figure 23 shows that the conclusion about the dependence of delay on the orders of fractional derivatives drown earlier holds.

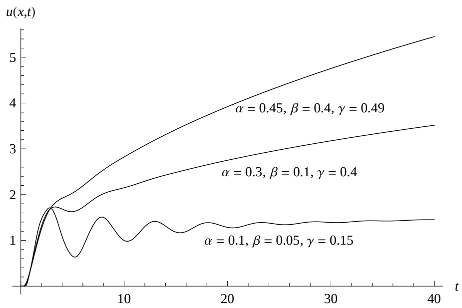

FIGURE 26. Displacements $u(x,t)$ in creep experiment as functions of time t at $x = 0.5$ for $t \in (0, 40)$.

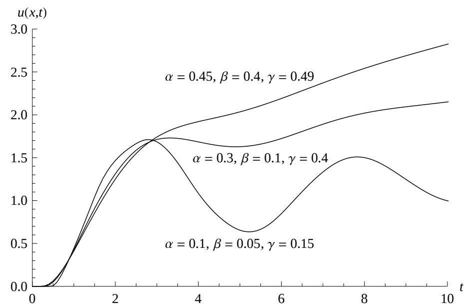

FIGURE 27. Displacements $u(x,t)$ in creep experiment as functions of time t at $x = 0.5$ for $t \in (0, 10)$.

Figure 24 presents displacements u in the creep experiment, determined according to (4.25), for four different points. Parameters are the same as in the previous cases (except that in this case instead of $\Upsilon_0 = 1$ we have $\Sigma_0 = 1$). For large times, as it can be seen from Figure 24, the displacement curves are monotonically increasing. This indicates that we deal with the viscoelastic fluid. Figure 24 also shows good agreement between the displacements obtained by the dynamic model (displacement is given by (4.25)) and the quasistatic model (displacement is given by (4.28)). Figure 25 presents the same displacements as Figure 24, but close to initial time instant and it is evident that, again, there is a delay, due to the finite speed of wave propagation.

In order to examine the dependence of the displacement u in the case of creep experiment, given by (4.25), on the orders of fractional derivatives in the constitutive equation (1.8), we again fix the point $x = 0.5$ of the rod and plot the displacement u for the same set of values of α, β and γ as before. Figure 26 clearly shows that all of the materials exhibits creep, but the displacements do not tend to a constant value. However, the material which has the dominant elastic properties (the first set) creeps slower comparing to the material with the dominant fluid properties (the third set). Figure 27 show the same plots as Figure 27, but for smaller time.

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