Velimir Jurdjević

ELASTIC PROBLEMS AND OPTIMAL CONTROL: INTEGRABLE SYSTEMS

Abstract. This paper, loosely described as the variations on the Euler–Kirchhoff elastic theme, focuses on the class of variational problems on an orthonormal frame bundle of a Riemannian space of constant curvature and introduces optimal control theory as an important ingredient for their solutions. The fundamental spaces are the Euclidean space, the sphere and the hyperboloid, and their orthonormal frame bundles coincide with the isometry groups $\mathrm{SE}_n(R)$, $\mathrm{SO}_{n+1}(R)$ and $\mathrm{SO}(1,n)$.

In each of these cases, the underlying manifold M_ϵ with its curvature $\epsilon = 0, \pm 1$ can be realized as the quotient $M_e p = G_\epsilon / K$, where G_ϵ denotes the appropriate isometry group and where $K = \mathrm{SO}_n(R)$. The pair (G_ϵ, K) induces a Cartan decomposition $\mathfrak{g}_\epsilon = \mathfrak{p}_\epsilon + \mathfrak{k}$ of the Lie algebra $\mathfrak{g}_\epsilon$ of G_ϵ, where $\mathfrak{k}$ is the Lie algebra of K and where $\mathfrak{p}_\epsilon$ is the orthogonal complement of $\mathfrak{k}$ relative to the Cartan–Killing form on $\mathfrak{g}_\epsilon$.

Kirchhoff's formalism used to model the equilibrium configurations of a thin elastic rod subject to bending and twisting torques at its ends admits natural formulation on these groups as an optimal control problem of optimizing the energy integral $\frac{1}{2}\int_0^T \langle u(t), Qu(t)\rangle\, dt$ over the trajectories of the control system $\frac{dg}{dt} = g(t)(A + u(t))$ that satisfy fixed boundary conditions in G_ϵ. Here, A a fixed element in $\mathfrak{p}_\epsilon$, $u(t)$ is an arbitrary curve in $\mathfrak{k}$, Q is a positive definite $n \times n$ matrix and $\langle X, Y\rangle = -\frac{1}{2}\,\mathrm{Tr}(XY)$, $X, Y \in \mathfrak{k}$.

The paper then singles out the integrable cases of the Hamiltonians associated with these optimal problems obtained by the Maximum Principle. The paper also defines a symplectic structure over quasi-periodiic curves on three dimensional spaces of constant curvature and shows that the Heisenberg's magnetic equation corresponds to the Hamiltonian flow associated with $\frac{1}{2}\int_0^T \kappa^2(s)\, ds$ over such curves with κ equal to the curvature of the curve. Finally, the paper gives the exact correspondence between the Heisenberg's magnetic equation and the nonlinear Schroedinger's equation and relates the soliton solutions to the elastic curves.

Mathematics Subject Classification (2010): 49J15, 53D05, 53D15, 93B27, 74B20.

Keywords: optimality, Hamiltonians, elastic curves, symplectic structure, integrability, solitons

Contents

1. Introduction

This exposition fuses optimal control theory with integrable Hamiltonian systems through a class of variational problems loosely inspired by the theory of elastic rods. This selection of problems apart of its own intrinsic importance also serves several larger mathematical objectives. To begin with, it demonstrates the significance of elastic problems for the theory of integrable Hamiltonian systems an area of mathematics traditionally associated with problems of Riemannian geometry and Lagrangian mechanics. Secondly, it illuminates the conceptual novelty of optimal control theory for problems of the calculus of variations. Thirdly, it makes a compelling and original case for Lie groups and Lie algebras in the study of integrable systems.

The class of problems, called elastic for referential convenience, has its origins in a study of Daniel Bernoulli, who in 1742 suggested to L. Euler that the differential equation for the equilibrium shape of a thin elastic inextensible beam subject to bending torques at its ends could be found by making the integral of the square of of the curvature along the beam a minimum. Euler, acting on this suggestion, obtained the differential equation for this problem in 1744 and was able to describe its solutions, known since then as *elastica*, well before the discovery of elliptic functions, [10, 30].

Since a thin inextensible beam could be naturally modeled by a parametrized curve $\gamma(s)$ in $\mathbb{R}^2$ subject to the constraint that $\left\|\frac{d\gamma}{ds}\right\| = 1$, i.e., a curve parametrized by its arc length, in which case the bending moments at its ends are represented by fixed tangential directions at $\gamma(0)$ and $\gamma(L)$ with L equal to the length of the beam. Hence, Euler's problem could be reformulated as a geometric problem of finding the minimum of the integral $\frac{1}{2}\int_0^L \kappa^2(s)\,ds$, where κ is equal to the curvature of γ, over the space of curves parametrized by arc length having fixed tangential directions at their end points. As such this problem has natural extensions to arbitrary Riemannian manifolds.

The passage from Euler's work on elastica to more general theory of elastic plates and rods required new theoretical concepts, and this new subject matter attracted the attention of some of the best mathematical minds of the 19th century (see the Historical introduction to the Mathematical Theory of Elasticity by Love [30]). In this formative period of the theory, the work of A. Cauchy in 1822 on stresses and strains was of central importance for the subsequent generalizations of Euler's elastica to spacial rods, in which the most notable contribution was due to G. Kirchhoff. Kirchhoff in his remarkable paper of 1859 Kirchhoff not only wrote the differential equations for the equilibrium configurations of an elastic rod in $\mathbb{R}^3$ subject to bending and twisting torques at its ends, but he also likened the elastic equations to the motions of the heavy top, a statement known ever since as "Kirchhoff's kinetic analogue" (see [30] for exact reference).

Kirchhoff's elastic rods were modeled by a curve $\gamma(s)$ that corresponds to the central line of the rod and an oriented orthonormal frame $F(s) = (v_1(s), v_2(s), v_3(s))$ along γ that measures the amount of twisting and bending along the central line. The frame deformations were assumed to be confined to the plane perpendicular to the central line of the rod, which meant that the frame was adapted to the curve via the relation $\frac{d\gamma}{ds}(s) = v_1(s)$ for all $s \in (0, L)$, where L stands for the length of the rod. The bending and twisting torques at the ends of the rod were modeled by assigning fixed values to $F(0)$ and $F(L)$. The elastic energy E of the rod was defined by three functions $u_1(s), u_2(s), u_3(s)$, called *strains*, and three constants c_1, c_2, c_3 reflecting the physical and geometric characteristics of the rod, and was assumed to be of the form

$$E = \frac{1}{2}\int_0^L \left(c_1 u_1^2(s) + c_2 u_2^2(s) + c_3 u_3^2(s)\right) ds.$$

The strains are induced via the deformations of the frames according to the following formulas:

$$\frac{dv_1}{ds} = -u_2 v_3 + u_3 v_2, \quad \frac{dv_2}{ds} = -u_3 v_1 + u_1 v_3, \quad \frac{dv_3}{ds} = -u_1 v_2 + u_2 v_1.$$

Similar to Euler's elastica, Kirchhoff's elastic model admits a simple geometric formulations as on the oriented orthonormal frame bundle of $\mathbb{R}^3$, which coincides with the group of motions $SE_3(R)$. More explicitly, the frame curve $F(s)$ can be represented by a curve $R(s)$ with the identification $Re_i = v_i$, $i = 1, 2, 3$, and then the central line $\gamma(s)$ and the frame $R(s)$ can be represented by a curve $g(s) = \left(\begin{smallmatrix} 1 & 0 \\ \gamma(s) & R(s)\end{smallmatrix}\right)$ in the semi-direct product $G = \mathbb{R}^3 \rtimes \mathrm{SO}_3(R)$. The relations

$$\frac{d\gamma}{ds}(s) = v_1(s), \frac{dR}{ds}(s) = R(s)\begin{pmatrix} 0 & -u_3(s) & u_2(s) \\ u_3(s) & 0 & -u_1(s) \\ u_2(s) & u_1(s) & 0 \end{pmatrix}$$

are then expressed by a single equation on G

$$\frac{dg}{ds}(s) = g(s)(E_1 + \sum_{i=1}^3 u_i(s)A_i(s)), \text{ with } E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1}$$

where A_1, A_2, A_3 denotes the standard basis in $\mathrm{so}_3(R)$ imbedded in the Lie algebra of G. The preceding equation can be also written as

$$\frac{dg}{ds}(s) = X_0(g(s)) + \sum_{i=1}^3 u_i(s) X_i(g(s)), \tag{2}$$

where each X_i is a left invariant vector field on G ($X_0(g) = gE_1$, $X_i(g) = gA_i$, $i = 1, 2, 3$). The elastic energy $E = \frac{1}{2}\int_0^L (c_1 u_1^2(s) + c_2 u_2^2(s) + c_3 u_3^2(s))\, ds$ can be also written as $\frac{1}{2}\int_0^L \|u(s)\|^2\, ds$, where $\| \;\|$ is the norm induced by a positive definite quadratic form $\langle u, v\rangle = c_1 u_1 v_1 + c_2 u_2 v_2 + c_3 u_3 v_3$ on $\mathbb{R}^3$. So Kirchhoff's elastic problem can be seen as a left invariant optimal (control) problem of minimizing the integral $\frac{1}{2}\int_0^L \|u(s)\|^2 ds$ over the trajectories $g(s)$ of control system (2) that satisfy fixed boundary conditions $g(0) = g_0$ and $g(L) = g_1$ with the strain functions playing the role of controls. The significance of control theoretic view will be made more explicit later on in the paper; instead, let us single out some special cases relevant for the theory of curves:

1. *Serret–Frenet frames.* The Serret–Frenet equations

$$\frac{d\gamma}{dt} = v_1(t), \; \frac{dv_1}{dt} = -\kappa(t) v_2(t), \; \frac{dv_2}{dt} = \kappa(t) v_1(t) + \tau(t) v_3(t), \; \frac{dv_3}{dt} = -\tau(t) v_2(t), \tag{3}$$

can be considered as a particular Kirchhoff system with $u_1(t) = \tau(t)$, $u_2(t) = 0$ and $u_3(t) = \kappa(t)$. The associated elastic energy E is given by the integral

$$E = \frac{1}{2}\int_0^L (c_1\tau^2(t) + c_3\kappa^2(t))\,dt$$

In this light, $\frac{1}{2}\int_0^L(\kappa^2(t)+\tau^2(t))\,dt$ may be thought of as the "natural elastic energy" of a curve $\gamma(t)$.

2. *Auto-parallel frames.* The frame $v_1(t), v_2(t), v_3(t)$ that is adapted to the curve $\gamma(t)$ by $\frac{d\gamma}{dt}(t) = v(t)$ and deforms according to

$$\text{(4)} \qquad \frac{dv_1}{dt} = u_2(t)v_2(t) + u_3(t)v_3(t), \quad \frac{dv_2}{dt} = -u_1(t)v_1(t), \quad \frac{dv_3}{dt} = -u_3(t)v_1(t)$$

is called *auto-parallel* in the literature on differential geometry. It is a special case of Kirchhoff's system with an additional constraint that $u_1(t) = 0$. Since $\frac{d^2\gamma}{dt^2} = \frac{dv_1}{dt}$, $\kappa^2(t) = \left\|\frac{d^2\gamma}{dt^2}\right\|^2 = u_2^2(t)+u_3^2(t)$. Hence, the elastic energy with $c_2 = c_3 = 1$ coincides with the Euler's functional $E = \frac{1}{2}\int_0^L \kappa^2(t)\,dt$.

It is therefore natural to consider Kirchhoff's problem in this restricted class of curves as an extension of Euler's planar elastic problem to spacial curves with one important modification. For Euler's elastic problem it is natural to fix only the tangential directions at the end points of the curve, and not the entire frame as was done in the problem of Kirchhoff. To reconcile this difference we may replace the boundary conditions $\gamma(0) = x_0$, $\frac{d\gamma}{dt}(0) = v_0$, $\gamma(L) = x_1$, $\frac{d\gamma}{dt}(L) = v_1$ with an initial manifold $S_0 = \left\{\left(\begin{smallmatrix}1 & 0\\ x_0 & R\end{smallmatrix}\right) : Re_1 = v_0\right\}$ and a terminal manifold $S_1 = \left\{\left(\begin{smallmatrix}1 & 0\\ x_1 & R\end{smallmatrix}\right) : Re_1 = v_1\right\}$, and then consider the minimum of $\frac{1}{2}\int_0^L(u_2^2(t)+u_3^2(t))\,dt$ over the solutions $g(t)$ of system (2) (with $u_1(t) = 0$) which satisfy $g(0) \in S_0$ and $g(L) \in S_1$. We will refer to this problem as the *Euler's elastic problem in* $\mathbb{R}^3$. Of course, Euler's elastic problem can be naturally defined in its own right on any Riemannian manifold without any reference to Kirchhoff's problem as follows.

Let M denote any Riemannian manifold. Then the geodesic curvature of any curve $x(t)$, parameterized by arc length can be expressed in terms of the Levi–Civita connection as $\kappa(t) = \left\|\nabla_{x(t)}\frac{dx}{dt}\right\|$. So it is natural to think of the elastic problem as a variational problem on the unit tangent bundle T^1M as follows: let $v_0 \in T^1_{x_0}M$ and $v_1 \in T^1_{x_1}M$ be given and let $\mathcal{A}$ denote the class of curves $x(t)$ in M defined over an interval $[0, L]$, subject to certain smoothness assumptions to be specified later, such that $\left\|\frac{dx}{dt}\right\| = 1$ for all t and $\frac{dx}{dt}(0) = v_0$ and $\frac{dx}{dt}(L) = v_1$. The elastic problem of Euler then can be defined in this more general context as:

Definition 1.1. *The Euler–Griffiths elastic problem* (EG). For a fixed pair of tangential directions v_0 and v_1 and a fixed length L find a curve $x \in \mathcal{A}$ that minimizes $\frac{1}{2}\int_0^L \|\nabla_{x(t)}\frac{dx}{dt}\|^2 dt$ over all other curves in $\mathcal{A}$. The elastic problem is said to be *free* if the requirement of fixed length is dropped, and is said to be relaxed if the terminal condition $\frac{dx}{dt}(L) = v_1$ is omitted.

Remark 1.1. Elastic problem in non-Euclidean spaces was first treated by Griffiths in his book on Exterior Differential Systems and the Calculus of Variations [13] and later more generally in [28]. For the class of problems treated in that study, as well in the related contemporary literature, the Eulere-Lagrange equation for the elastic problem is obtained through the variations in the class of smooth curves(shown later in Section 4). Absolutely continuous curves are natural for problems of optimal control, and are actually needed to describe extrema for problems with bounds on the curvature, as will be made clear below.

Kirchhoff's elastic problem could also be defined in a more general setting over the curves $F(s) = (v_1(s), \ldots, v_n(s))$ in the bundle of positively orthonormal frames of a Riemannian manifold M that are adapted to the projected curve $\gamma(t) \in M$ by $\frac{d\gamma}{dt}(t) = v_1(t)$ but we will not go into this generalizations beyond the spaces of constant curvature.

In what follows we will provide a detailed discussion of the solutions of these problems both in the Euclidean and the non-Euclidean settings. To emphasize further the relevance of optimal control for problems of geometry we will include the problems involving the bounds on controls. The most classical of these, known as the problem of Delaunay, consists of finding the shortest curve that connects two fixed points with fixed tangential directions in the class of curves having constant curvature. Classically, this problem was considered as a particular case of the variational problems known as the problems of Lagrange and, according to Caratheodory, Weierstrass was the first to successfully obtain the Euler–Lagrange equation for this problem (Caratheodory [7, p. 373]). Delauney's problem can be naturally recast as an optimal control problem of minimizing the integral $\int_0^L dt$ over the solutions $g(t)$ of

$$\frac{dg}{dt}(t) = g(t)(E_1 + u_2(t)A_2 + u_3(t)A_3), \text{ with } u_2^2(t) + u_3^2(t) = c$$

satisfying $g(0) \in S_0$ and $g(L) \in S_1$. Here, E_1, A_2, A_3 have the same meaning as in equation (1) and S_0 and S_1 have the same meaning as in the paragraph above.

However, as it stands, Delauney's problem is not well defined, because of the lack of convexity: a geodesic, whose curvature is zero, is the uniform limit of concatenations of curves having constant curvature. To remedy this situation, one needs to convexify the problem, and allow the curves whose curvature is less or equal to the given bound, that is, replace the sphere $u_2^2 + u_3^2 = c$ with the ball $u_2^2 + u_3^2 \leqslant c$.

In $\mathbb{R}^2$ the convexified Delauney problem is known as the problem of Dubins [9]. As demonstrated in Dubins paper, optimal solutions are the concatenations of segments of straight lines and circles, a prototype of a more general situation where the optimal solutions are concatenations of solutions generated by controls that take values on the boundary and those that that are generated by controls in the interior of the control set. Such solutions are not obtainable by the usual Lagrangian methods.

This class of problems introduced above and their extensions to arbitrary Riemannian manifolds together with some solutions forms a core of this paper. As already stated earlier, the exposition is biased in the direction of optimal control as a way of getting to the integrable systems and hence is somewhat novel, considering that optimal control theory has not yet found its proper place in the classical calculus of variations. To help the reader navigate more easily through through the exposition it may be helpful to give an overview of the material presented in the paper.

Section 2 introduces control systems and discusses the basic topological properties of their reachable sets in terms of certain Lie algebraic criteria with the ultimate goal of establishing the existence of solutions that connect any two points in the state space. This theory is then applied to a selection of geometric problems that includes the non-Euclidean extensions of the elastic problems defined above. The exposition then proceeds to optimality, the existence of optimal solutions and the associated Hamiltonians obtained by the Maximum Principle of Pontryagin and his collaborators.

On spaces of constant curvature, the orthonormal frame bundle, on which the elastic problems are defined, is a Lie group. Section 3 is therefore devoted to left invariant Hamiltonians on the cotangent bundle T^*G of a Lie group G. To preserve the left invariant symmetries of the elastic problems, the cotangent bundle T^*G is realized as the product $G \times \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$ of G. Then the Hamiltonian equations of a left-invariant Hamiltonian can be integrated by quadratures once the solution of the projection of the equations on $\mathfrak{g}^*$ is known. This observation makes contact with the theory of integrable systems on the Lie algebra $\mathfrak{g}$ whenever $\mathfrak{g}$ can be identified with $\mathfrak{g}^*$ via an invariant quadratic form (usually a multiple of the Cartan–Killing form).

In this setting the Poisson structure of $\mathfrak{g}^*$ plays an important role and there is a substantial discussion of symmetries, coadjoint orbits and their invariance properties. This formalism is then used to provide a detailed analysis of the Hamiltonians associated with the problem of Kirchhoff, their relation to the equations of the heavy top and the classification of the integrable cases. This material is divided into two sections: Section 4 deals with the geometric case and the elastic curves, while Section 5 deals with general problem on six dimensional Lie groups and the relation to the equations of the heavy top (The kinetic analogue of Kirchhoff).

Section 6 deals with infinite dimensional Hamiltonian systems. It is shown that the space of anchored periodic curves on a three dimensional space of constant curvature can be given a symplectic structure. It is then shown that the Heisenberg's magnetic equation corresponds to the Hamiltonian flow associated with $\frac{1}{2}\int_0^L \kappa^2(s)\,ds$. In order to connect with the Schroedinger's equation, Heisenberg's magnetic equation is then represented in the space of Hermitian 2×2 matrices. Then the adjoint action of SU_2 on the space of Hermitian matrices reveals a correspondence between the Heisenberg's magnetic equation and Schroedinger's nonlinear equation. The section concludes with a discussion of soliton solutions of the nonlinear Schroedinger's equation and their relation to the elastic curves. Section 7 contains concluding remarks and a brief suggestion of open problems.

2. Control systems. Controllability and Optimality

For the purposes of this paper it will be convenient and sufficient to introduce control theory through a particular class of systems, known as *control affine systems.* They are defined as follows:

$$\frac{dx}{dt} = X_0(x) + \sum_{i=1}^{m} u_i(t)X_i(x), \tag{5}$$

where

1. $X_0, \dots, X_m$ are given smooth vector fields on a smooth n dimensional manifold M, which for extra convenience will be assumed complete in the sense that their integral curves are defined globally on $(-\infty, \infty)$. Vector field X_0 is called *the drift* and the remaining vector fields $X_1, \dots, X_m$ are called *controlled* vector fields.
2. The control functions $u(t) = (u_1(t), \dots, u_m(t))$ take values in a prescribed set $C \subseteq \mathbb{R}^m$, which could be a open or closed, possibly governed by inequalities, or even dependent on the time variable t. In what follows C will be assumed symmetric about the origin i.e., that it satisfies $u \in C \Leftrightarrow -u \in C$.
3. The control functions $u(t)$ are assumed to belong to $L^\infty[t_0, t_1]$ on any interval $[t_0, t_1]$.

It then follows from the theory of differential equations that for each control $u(t)$, any initial state x_0 and any initial time t_0 there exists an absolutely continuous curve $x(t)$ in M defined on some interval $[t_0, T]$ that satisfies (1) almost everywhere in $[t_0, T]$ [6]. Such a curve is called a *trajectory* generated by a control $u(t)$.

Since (1) is defined by time invariant vector fields the initial time t_0 can be translated to $t_0 = 0$. Problems of optimality are usually stated as two point boundary value problems, that is, they concern the minima of an integral $\int_0^{t_1} f(u(t), x(t))dt$, where $f : C \times M \to \mathbb{R}$ is a given smooth function, called the *cost*, over the solutions of (1) that initiate at x_0 at time $t_0 = 0$ and terminate at x_1 at time t_1. Both the initial state x_0 and the terminal state x_1 are given a priori while the terminal time t_1 could be either fixed or variable. In some situations the initial and the final states are replaced by submanifolds S_0 and S_1 of M.

Controls that result in optimal trajectories are called *optimal.* In order to ensure that the above optimal problem is well posed it is necessary first to demonstrate that there are trajectories that connect the given boundary conditions before any discussion of optimality could take place. This issue, known as the controllability problem, forms an integral part of optimal control theory and will be addressed below in more detail.

2.1. Controllability and Accessibility.

A control system is an extension of a dynamical system where a single vector field X is replaced by a family of vector fields $\mathcal{F}$. In the case of affine control systems $\mathcal{F} = \{X_0 + \sum_{i=1}^m u_i X_i,\ u \in C\}$. Analogous to the theory of dynamical systems, with its principal object an understanding of the orbits of the one parameter group of diffeomorphisms $\{\Phi_t, t \in \mathbb{R}\}$ induced by X, the main object of control theory is the study of the reachable or accessible sets. There are several types of reachable sets: $\mathcal{A}(x_0,T)$, the set of points in M reachable by the trajectories of (9) at exactly T units of time, $\mathcal{A}(x_0,\leqslant T)$, the set of points reachable in time T or less, and $\mathcal{A}(x_0)$, the set of points reachable in any positive time. If the control functions are restricted to the class of piecewise constant functions with values in U, then the reachable set can be described by the semi-flows $\{\Phi_t : t \geqslant 0\}$ of vector fields in $\mathcal{F}$. For instance, when $\mathcal{F}$ consists of two fields X and Y then the points reachable in time T from x_0 are of the form

$$\left\{\Phi^X_{t_p}\cdot\Phi^Y_{t_{p-1}}\cdots\Phi^Y_{t_1}(x_0),\ t_1+t_2+\cdots+t_p=T,\ t_i\geqslant 0\right\},$$

where $\{\Phi^X_t : t\in R\}$ and $\{\Phi^Y_t : t\in R\}$ denote the one parameter groups corresponding to X and Y. In contrast to the theory of dynamical systems, where the time parameter can be negative or positive, reachable sets of control systems are defined only for positive time, which essentially reduces their study to a study of orbits of semi-groups of diffeomorphisms, rather than the groups of diffeomorphisms as is the case in the theory of dynamical systems. Since it is mathematically easier to deal with groups, rather than semigroups, it is natural to relax the requirement that the time be positive and consider first the group of diffeomorphisms generated by the flows defined by the elements of $\mathcal{F}$ and its action on the points of M.

To be more precise, let $\{\Phi^X_t : t\in\mathbb{R}\}$ denote the one-parameter group of diffeomorphisms induced by a vector field X and let $G_{\mathcal{F}}$ denote the subgroup of Diff(M) generated by $\{\Phi^X_t : t\in\mathbb{R},\ X\in\mathcal{F}\}$. A typical element Φ of $G_{\mathcal{F}}$ is of the form

$$\Phi=\Phi^{X_m}_{t_m}\circ\Phi^{X_{m-1}}_{t_{m-1}}\cdots\circ\Phi^{X_1}_{t_1},\ \ X_i\in\mathcal{F},\ \ t_i\in\mathbb{R},\ \ i=1,\dots,m.$$

Then $G_{\mathcal{F}}(x)=\{\Phi(x):\Phi\in G_{\mathcal{F}}\}$ is called the orbit of $G_{\mathcal{F}}$ through a point $x\in M$. Remarkably, each orbit of $G_{\mathcal{F}}(x)$ is an immersed submanifold of M [42].

There is a large class of families of vector fields, called *Lie determined* for which the tangent spaces of orbits are determined by the Lie brackets of elements of $\mathcal{F}$, and as a consequence, the reachable sets of such systems exhibit nice mathematical properties that serve as a basis for geometric control theory. To explain in more detail, let $[X,Y]$ denote the Lie bracket of vector fields X and Y, and let Lie$(\mathcal{F})$ denote the Lie algebra generated by the family $\mathcal{F}$. Elements of Lie$(\mathcal{F})$ are linear combinations of iterated Lie brackets of vector fields in $\mathcal{F}$. In general, Lie$(\mathcal{F})$ is an infinite dimensional sub-Lie algebra of the Lie algebra of all vector fields on M.

Let $\mathrm{Lie}_x(\mathcal{F})$ denote the evaluation of Lie$(\mathcal{F})$ at x, i.e., $\mathrm{Lie}_x(\mathcal{F})=\{X(x): X\in\mathrm{Lie}(\mathcal{F})\}$. Each $\mathrm{Lie}_x(\mathcal{F})$ is a linear subspace of the tangent space T_xM. It is simple to show that Lie$(\mathcal{F})$ is tangent to each orbit of $G_{\mathcal{F}}$, that is, $\mathrm{Lie}_y(\mathcal{F})\subseteq T_y(G_{\mathcal{F}}(x))$ for all points y in an orbit $G_{\mathcal{F}}(x)$. Then Lie determined families are those for which $\mathrm{Lie}_y(\mathcal{F})=T_y(G_{\mathcal{F}}(x))$ for all $y\in G_{\mathcal{F}}(x)$ and for each orbit $G_{\mathcal{F}}(x)$. In the class of Lie determined families each reachable set $\mathcal{A}(x,\leqslant T)$ has a nonempty interior in the topology of the orbit $G_{\mathcal{F}}(x)$ for each $T>0$ (see [20]). In particular, $\mathcal{A}(x,\leqslant T)$ has a nonempty interior in M whenever the dimension of $\mathrm{Lie}_x(\mathcal{F})$ is equal to the dimension of M. Systems for which $\mathrm{Lie}_x(\mathcal{F})=T_xM$ for all $x\in M$ are said to have *the accessibility* property.

Families of analytic vector fields on an analytic manifold M are Lie determined, a fact known as the Hemann–Nagano Theorem [20]. In this paper all control systems will be analytic, so the subsequent discussion will be restricted to Lie determined systems. Then it follows from the above remarks that a necessary and sufficient condition that $\mathcal{A}(x_0, \leqslant T)$ has a nonempty interior in M is that $\mathrm{Lie}_x(\mathcal{F}) = T_x M$ [20, Chapter 3].

Control system is said to be *controllable* if $\mathcal{A}(x_0) = M$ for each x_0 in M, that is, whenever any point x_1 can be reached from x_0 for some time $T > 0$. In general it is very difficult to ascertain if a given control system is controllable. Nevertheless, there are some controllability criteria which can be applied successfully in some situations. One such criterion is based on the notion of the Lie saturate.

The Lie saturate $\mathcal{LS}(\mathcal{F})$ of $\mathcal{F}$ is the largest (in the sense of set inclusion) family of vector fields included in $\mathrm{Lie}(\mathcal{F})$ which leaves the closure of $\mathcal{A}(x)$ invariant. Then it can be shown that $\mathcal{F}$ is controllable if and only if $\mathcal{LS}(\mathcal{F}) = \mathrm{Lie}(\mathcal{F})$ [20, Chapter 3]. The Lie saturate is invariant under the following enlargement procedures:

1. If $Y_1, Y_2, \ldots, Y_p$ are any set of vector fields in $\mathcal{LS}(\mathcal{F})$, then the positive affine hull $\{\alpha_1 Y_1 + \cdots + \alpha_p Y_p : \alpha_i \geqslant 0,\ i = 1, \ldots, p\}$ is also contained in $\mathcal{LS}(\mathcal{F})$.
2. If $\mathcal{V}$ is a vector space of vector fields in $\mathcal{LS}(\mathcal{F})$, then $\mathrm{Lie}(\mathcal{V})$ is in $\mathcal{LS}(\mathcal{F})$.
3. If $\pm Y$ is in $\mathcal{LS}(\mathcal{F})$, then $(\Phi_\lambda^Y)_* X \Phi_{-\lambda}^Y$ is in $\mathcal{LS}(\mathcal{F})$ for any $\lambda \in \mathbb{R}$ and any $X \in \mathcal{LS}(\mathcal{F})$, where $\{\Phi_\lambda^Y : \lambda \in \mathbb{R}\}$ denotes the flow of Y and $\Phi_{\lambda *}$ denotes its tangent map.

In the case of affine control systems with no constraints on the control set U, then it is easy to show that the vector space spanned by $X_1, \ldots, X_m$ is a subspace of the Lie saturate of $\mathcal{F}$. Then by (1) and (3) above, the affine hull spanned by $\{(\Phi_{-\lambda}^{X_i})_* X_0 \Phi_\lambda^{X_i} : \lambda \in \mathbb{R}\}$ is in the Lie saturate of $\mathcal{F}$. A systematic exploitation of these properties could ultimately lead to proving controllability as illustrated by the examples below (There is an extensive discussion of the Lie saturate in [20]).

2.2. Examples. It may be relevant to state the convention adopted in this paper concerning the sign of the Lie bracket: both $[X, Y] = Y \circ X - X \circ Y$ and its negative appear in the literature and each choice carries its own signs in regard to various geometric objects related to the Lie bracket. In this paper, the Lie bracket is taken as $[X, Y] = Y \circ X - X \circ Y$ which means that in local coordinates, the i-th coordinate of $[X, Y]$ is given by the formula $\sum_{j=1}^n \left(\frac{\partial X^i}{\partial x_j} Y^j - \frac{\partial Y^i}{\partial x_j} X^j\right)$. It also means that the Lie bracket of left invariant vector fields $X = gA$ and $Y(g) = gB$ is given by $[X, Y](g) = g(BA - AB)$, while the Lie bracket of right invariant vector fields $X(g) = Ag$ and $Y(g) = Bg$ is given by $[X, Y](g) = (AB - BA)g$. Since only left invariant vector fields appear in this paper, the commutator of matrices is defined by $[A, B] = BA - AB$ for any matrices A and B.

1.Linear control systems. Control system

$$\frac{dx}{dt} = Ax + Bu(t), \tag{6}$$

with A an $n \times n$ matrix and B a matrix with columns $b_1, \ldots, b_m$, central to linear control theory, may be viewed as an affine system with a linear drift $X_0(x) = Ax$ and constant controlled vector fields $X_i = b_i$, $i = 1, \ldots, m$. Then it follows from the preceding remarks that the vector space of constant vector fields spanned by $b_1, \ldots, b_m$

is in $\mathcal{LS}(\mathcal{F})$. Since $\Phi_\lambda^{X_i}(x) = x + \lambda b_i$, $Y_\lambda(x) = (\Phi_{-\lambda}^{X_i})_* X_0 \Phi_\lambda^{X_i}(x) = Ax + \lambda A b_i$ is in $\mathcal{LS}(\mathcal{F})$ for each λ. It then follows that $\lim_{\lambda\to\pm\infty} \frac{1}{|\lambda|}(Ax + \lambda A b_i) = \pm A b_i$. Therefore, the vector space spanned by constant vector fields $b_1, \dots, b_m, Ab_1, \dots, Ab_m$ is in $\mathcal{LS}(\mathcal{F})$.

Successive repetitions of these arguments show that each constant vector field $\pm A^k b_j$, $k \geqslant 0$, $j = 1, \dots, m$ is in $\mathcal{LS}(\mathcal{F})$. We leave it to the reader to show that this linear span is equal to the evaluation of $\mathrm{Lie}(\mathcal{F})$ at the origin. This fact shows that (6) is controllable if and only if the linear span of $\{A^k b_j,\ k \geqslant 0,\ j = 1, \dots, m\}$ is equal to $\mathbb{R}^n$. As a consequence of the Cayley–Hamilton theorem each A^k for $k \geqslant n$ is linearly dependent on $I, A, \dots, A^{n-1}$. Hence, (6) is controllable if and only if the $n \times nm$ matrix $(B\,AB\,A^2B \cdots A^{n-1}B)$ is of rank n. This matrix is known as *the controllability matrix* in linear control theory.

2. Serret–Frenet systems. The differential system $\frac{d\gamma}{dr} = R(t)e_1$, $\frac{dR}{dt} = R(t)\Omega(t)$ with $\Omega(t) = \sum_{i=1}^{n-1} \kappa_i(t) e_i \wedge e_{i+1}$ that is associated with geometric invariants of curves in $\mathbb{R}^n$ can be also regarded as an affine control system in the group of motions $SE_n(R)$ with $\kappa_1, \dots, \kappa_{n-1}$ playing the role of controls. The state variable $g = \left(\begin{smallmatrix} 1 & 0 \\ \gamma & R \end{smallmatrix}\right)$ satisfies the equation $\frac{dg}{dt} = g(t)(E_1 + \tilde{\Omega}(t))$, where $\tilde{\Omega}$ denotes the embedding of Ω in the Lie algebra of $SE_n(R)$, i.e., $\tilde{\Omega} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \Omega \end{smallmatrix}\right)$ and $E_1 = e_1 \otimes e_0$. Therefore, a Serret–Frenet system can be written as a left invariant control system

$$\frac{dg}{dt} = X_0(g) + \sum_{i=1}^{n-1} u_i(t) X_i(g), \tag{7}$$

where $X_0(g) = gE_1$, $X_1(g) = g(e_2 \wedge e_1), \dots,\ X_{n-1}(g) = g(e_n \wedge e_{n-1})$.

Since the Lie brackets of left invariant vector fields correspond to the commutators of matrices, $\mathrm{Lie}_g(\mathcal{F}) = g(\mathrm{Lie}(\Gamma))$, where $\Gamma = \{E_1, A_1 \dots, A_{n-1}\}$ with $A_i = e_{i+1} \wedge e_i$, $i = 1, \dots, n-1$, and $\mathrm{Lie}(\Gamma)$ denotes the Lie algebra generated by the commutators of matrices in Γ. Then $\mathrm{Lie}_g(\mathcal{F}) = T_g G$ if and only if $\mathrm{Lie}(\Gamma) = \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of G.

It is easy to verify that the Lie algebra generated by the matrices $A_1, \dots, A_{n-1}$ is isomorphic to $\mathrm{so}_n(R)$, and that the orbit of E_1 under the adjoint action by $\mathrm{SO}_n(R)$ is equal to the sphere S^n. This implies that $\mathrm{Lie}(\Gamma) = \mathfrak{g}$. It is known that a family of left (or right) invariant vector fields on a semi-direct product $G = V \rtimes K$ is controllable if and only if $\mathrm{Lie}(\Gamma) = \mathfrak{g}$. This fact implies that the Serret–Frenet system (7) is controllable and remains controllable even if the controls are restricted to the sphere $\|u\| \leqslant 1$ [20, p. 179].

3. Non-Euclidean elastic problems. The elastic problems on the sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ and the hyperboloid $\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} : x_0^2 - \sum_{i=1}^n x_i^2 = 1,\ x_0 > 0\}$ can be described simultaneously in terms of a single parameter ϵ through the quadratic forms

$$(x, y)_\epsilon = x_0 y_0 + \epsilon \sum_{i=1}^n x_i y_i, \quad \epsilon = \pm 1.$$

For that reason it will be convenient to write $\mathbb{S}^n_\epsilon$ with $\mathbb{S}^n_1 = \mathbb{S}^n$ and $\mathbb{S}^n_{-1} = \mathbb{H}^n$, and it will be convenient also to refer to $\mathbb{S}^n_\epsilon$ as the unit sphere, Euclidean or hyperbolic. Then $x \otimes_\epsilon y$ denotes rank one matrix defined $(x \otimes_\epsilon y)z = (y, z)_\epsilon x$, $z \in \mathbb{R}^{n+1}$, and $x \wedge_\epsilon y$ denotes the rank two matrix $x \otimes_\epsilon y - y \otimes_\epsilon x$.

Let $G_\epsilon = \mathrm{SO}_\epsilon$ denote the connected component through the group identity of the group O_ϵ that leaves the form $(\,,)_\epsilon$ invariant. It follows that G_ϵ is equal to $\mathrm{SO}_{n+1}(R)$ for $\epsilon = 1$ and to $\mathrm{SO}(1, n)$ for $\epsilon = -1$. Each of these groups acts on the points x in $\mathbb{R}^{n+1}$ via the matrix multiplication $(g, x) \to gx$ for $g \in O_\epsilon$. It follows that each sphere $\{x : \langle x, x\rangle_\epsilon\}$ is invariant under this action. Since G_ϵ acts transitively on the spheres, $\mathbb{S}^n_\epsilon$ can be realized as the orbit of G_ϵ through any point x_0 of $\mathbb{S}^n_\epsilon$.

In this paper we will identify $\mathbb{S}^n_\epsilon$ with the orbit through e_0. Then each $g \in G_\epsilon$ defines a point $x = \pi(g) = ge_0$ in $\mathbb{S}^n_\epsilon$ and the remaining columns of g, $v_1 = ge_1, \dots, v_n = ge_n$ define an orthonormal frame $(v_1, \dots, v_n)$ at x. Conversely, each orthonormal frame at a point x of $\mathbb{S}^n_\epsilon$ can be identified with a unique matrix g in O_ϵ. An orthonormal frame is said to be *positively oriented* if the corresponding g belongs to SO_ϵ. This identification shows that G_ϵ can be regarded as the positively oriented orthonormal frame bundle of $\mathbb{S}^n_\epsilon$.

Let K denote the subgroup of G_ϵ that leaves e_0 invariant, i.e., $Ke_0 = e_0$. It follows that K consists of matrices of the form $\left(\begin{smallmatrix}1 & 0\\ 0 & g\end{smallmatrix}\right)$ with $g \in \mathrm{SO}_n(R)$, hence is independent of ϵ. We will write $K = \{1\} \times \mathrm{SO}_n(R)$ and similarly $\mathfrak{k} = \{0\} \times \mathrm{so}_n(R)$ for the Lie algebra $\mathfrak{k}$ of K.

Each curve $g(t)$ in G_ϵ projects onto the curve $x(t) = g(t)e_0$ in $\mathbb{S}^n_\epsilon$. Then it follows that

$$\frac{dx}{dt} = \frac{dg}{dt}e_0 = g(t)\Lambda(t)e_0 = \sum_{i=1}^n g(t)\Lambda_{i0}(t)e_i = \sum_{i=1}^n \Lambda_{i0}(t)v_i(t),$$

where the curve of matrices $\Lambda(t)$ in the Lie algebra $\mathfrak{g}_\epsilon$ of G_ϵ is defined by $\Lambda(t) = \frac{dg}{dt}(t)g^{-1}(t)$. The framed curve satisfies $\frac{dx}{dt} = v_1$ if and only if $\Lambda_{i0} = e_1$. It follows that the most general framed curve $g(t)$ adapted to the projected curve $x(t) = g(t)e_0$ via $\frac{dx}{dt} = v_1$ is a solution of

$$\frac{dg}{dt} = g(t)(E_\epsilon + U(t)), \tag{8}$$

with $E_\epsilon = e_1 \wedge_\epsilon e_0$ and $U(t) = \sum_{i,j=1}^n u_{ij}(t)(e_j \wedge e_i)$. The most general frame $v_1(t), \dots, v_n(t)$ adapted to the projected curve $x(t)$ via the relation $\frac{dx}{dt}(t) = v_1(t)$ is called *Darboux frame.* The corresponding system of equations (13) will be referred to as a *Darboux system.*

Each Darboux system lends itself to control theoretic interpretations with the deformation matrix $U(t)$ playing the role of a control function as a left invariant affine control system

$$\frac{dg}{dt} = X_0(g) + \sum_{i,j=1}^n u_{ij}(t)X_{ij}(g), \tag{9}$$

where $X_0(g) = gE_\epsilon$ and $X_{ij}(g) = gA_{ij}$, $A_{ij} = e_j \wedge e_i$ and the controls $u(t) \in \mathbb{R}^m$, $m = \frac{1}{2}n(n-1)$.

We will now reveal additional geometric structure in the Lie algebra $\mathfrak{g}_\epsilon$ that is of relevance for framed curves and their controllability properties. In what follows $\langle\,,\rangle$ denote the quadratic form on $\mathfrak{g}_\epsilon$ defined by $\langle A, B\rangle = -\frac{1}{2}\operatorname{Tr}(AB)$, where $\operatorname{Tr}(X)$ stands for the trace of X. This quadratic form is called *the trace form*. The trace form satisfies:

1. $\langle A, B\rangle = \sum_{i>j} a_{ij}b_{ij}$ for any skew-symmetric matrices $A = (a_{ij})$ and $B = (b_{ij})$,
2. $\langle A, B\rangle = -\sum_{i>j} a_{ij}b_{ij}$ for any symmetric matrices $A = (a_{ij})$ and $B = (b_{ij})$,
3. $\langle A, B\rangle = 0$ if A is a symmetric and B is a skew-symmetric matrix.

The trace form identifies $\mathfrak{p}_\epsilon = \sum_{i=1}^n p_i(e_0 \wedge_\epsilon e_i)$, $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$ as the orthogonal complement of the subalgebra $\mathfrak{k}$. It follows that $\mathfrak{g}_\epsilon = \mathfrak{p}_\epsilon \oplus \mathfrak{k}$. We leave it to the reader to show that

$$[\mathfrak{p}_\epsilon, \mathfrak{p}_\epsilon] = \mathfrak{k}, \quad [\mathfrak{p}_\epsilon, \mathfrak{k}] = \mathfrak{p}_\epsilon, \quad [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k} \tag{10}$$

The preceding decomposition of $\mathfrak{g}_\epsilon$, subject to the Lie algebraic relations (15) is known as *the Cartan decomposition*. With (10) at our disposal, consider controllability of (9). Let $\Gamma = \{E_\epsilon + \sum_{i,j=1} u_{ij}A_{ij}\}$ and let $LS(\Gamma)$ denote the family of matrices induced by the Lie saturate $LS(\mathcal{F})$ through $LS(\mathcal{F})(g) = g(LS(\Gamma))$. It follows the earlier remarks that the vector space spanned by the controlled vector fields is in the Lie saturate which implies that $\mathfrak{k} \subseteq LS(\Gamma)$. Then $e^{-\lambda A_{ij}}E_\epsilon e^{\lambda A_{ij}}$ belongs to $LS(\Gamma)$ for each λ (by property 3 of $LS(\mathcal{F})$). For $i = 1$ and $j = 2$,

$$e^{-\lambda A_{ij}}E_\epsilon e^{\lambda A_{ij}} = \cos\lambda E_\epsilon - \sin\lambda(e_0 \wedge_e pe_2).$$

The preceding expression is equal to $-E_\epsilon$ for $\lambda = \pi$. Therefore the vector space spanned by Γ is in $LS(\Gamma)$, which implies that $\operatorname{Lie}(\Gamma) \subseteq LS(\Gamma)$. It is easy to show that $adE_\epsilon(\mathfrak{k}) = \mathfrak{p}_\epsilon$, from which it follows that $LS(\Gamma) = \mathfrak{g}_\epsilon$. Hence, (9) is controllable.

For two dimensional spheres, Darboux curves coincide with the Serret–Frenet curves and are the solutions of

$$\frac{dg}{dt} = g(t)\begin{pmatrix} 0 & -\epsilon & 0 \\ 1 & 0 & -u(t) \\ 0 & u(t) & 0 \end{pmatrix}. \tag{11}$$

It is easy to show that parameter $u(t)$ is equal to the curvature of $x(t)$ relative to the Riemannian metric $\epsilon\langle\,,\rangle_\epsilon$. The argument is simple: first, $\|\frac{dx}{dt}\|_\epsilon^2 = \epsilon\langle\frac{dx}{dt}, \frac{dx}{dt}\rangle_\epsilon = \epsilon\langle g(t)e_1, g(t)e_1\rangle = 1$ and hence $x(t)$ is parametrized by its arc length. Secondly, the covariant derivative $\frac{D_x}{dt}v(t)$ of a tangent vector $v(t)$ along a curve $x(t)$ is given by

$$\frac{D_x}{dt}v(t) = \frac{dv}{dt} + \left\langle v(t), \frac{dx}{dt}\right\rangle_\epsilon x(t),$$

where $\frac{d}{dt}$ denotes the standard derivative in $\mathbb{R}^3$ [22]. Therefore,

$$\frac{D_x}{dt}\frac{dx}{dt}(t) = \frac{d}{dt}g(t)e_1 + \epsilon x(t) = u(t)v_2(t).$$

This shows that $u(t)^2 = \left\|\frac{D_x}{dt}\frac{dx}{dt}(t)\right\|^2$ and hence $u(t)$ is the geodesic curvature of $x(t)$.

In higher dimensions, Serret–Frenet systems are subsystems of Darboux systems in which controls $U(t)$ are restricted to

$$U(t) = \begin{pmatrix} 0 & -u_1(t) & 0 & \cdots & \cdots \\ u_1(t) & 0 & -u_2(t) & 0 & \cdots \\ 0 & u_2(t) & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & u_{n-2}(t) & 0 & -u_{n-1}(t) \\ 0 & \cdots & 0 & u_{n-1}(t) & 0 \end{pmatrix}.$$

Controls $(u_1(t), \ldots, u_{n-1}(t))$ coincide with the curvatures $\kappa_1(t), \ldots, \kappa_{n-1}(t)$ of the projected curve $x(t)$ whenever the covariant derivatives $\frac{D_x^k}{dt^k}(\frac{dx}{dt})$, $k = 1, \ldots, n-1$ are linearly independent. In the case that $\frac{D_x^k}{dt^k}(\frac{dx}{dt})$ is linearly dependent on $\frac{dx}{dt}$, $\frac{D_x}{dt}(\frac{dx}{dt}), \ldots, \frac{D_x^{k-1}}{dt^{k-1}}(\frac{dx}{dt})$ for some $k < n-1$, then $u_1(t) = \kappa_1(t), \ldots, u_{k-1}(t) = \kappa_{k-1}(t)$ and the remaining parameters $u_k(t), \ldots, u_n(t)$ are arbitrary and bear no relation to the projected curve $x(t)$. In particular, Serret–Frenet curves generated by the controls $U(t)$, with $u_1(t) = 0$ project onto the geodesics around which the frames $v_2(t), \ldots, v_n(t)$ spin arbitrarily (a geodesic does not generate a Serret–Frenet frame). Sets of points reachable by controls $U(t)$ with $u_1(t) = 0$, P. Griffiths termed *strange integral manifolds* in his book on the calculus of variations[13].

Due to these defficiencies, Serret–Frenet systems may not offer an ideal settings for studying variational problems involving geometric invariants of curves, as in the Euler–Griffiths or in the Delauney–Dubins problems. For these problems there is another, more suitable, choice of frames, called *auto-parallel*, defined by the controls $U(t)$ of the form

$$U(t) = \begin{pmatrix} 0 & -u_1(t) & -u_2(t) & \cdots & -u_{n-1}(t) \\ u_1(t) & 0 & 0 & \cdots & 0 \\ u_2(t) & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{n-1}(t) & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Every curve $x(t)$ in $\mathbb{S}_\epsilon^n$ can be lifted to a unique curve of auto-parallel frames and, moreover, the geodesic curvature $\kappa(t)$ of $x(t)$ is given by

$$\kappa^2(t) = u_1^2(t) + u_2^2(t) + \cdots + u_{n-1}^2(t) \ \ [22].$$

One can show, using the techniques of the Lie Saturate, that both the Serret–Frenet and the auto-parallel system of frames, adapted to the projected curve $x(t)$ via the relation $\frac{dx}{dt} = v_1(t)$, are controllable on G_ϵ. These systems remain controllable if the controls are further restricted to any ball of radius $r > 0$ in the case $\epsilon = 1$. In the hyperbolic case, the system remains controllable only whenever the radius r is greater than the absolute value of the curvature of the space (the curvature of a hyperbolic spaces is negative).

2.3. Optimal problems and the existence of solutions. Kirchhoff's elastic rod problem has a natural formulation on $\mathbb{S}^n_\epsilon$ as the problem of minimizing the integral of $\frac{1}{2}\int_0^T \langle QU(t), U(t)\rangle\, dt$ over the solutions of (9) in the fixed time interval $[0, T]$ subject to the fixed boundary conditions $g(0) = g_0$ and $g(T) = g_1$, where Q is a positive linear operator on $\mathfrak{k}$ relative to the trace form. Any diagonal matrix $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ with positive diagonal entries defines a positive definite operator $Q(U) = DUD$ relative to the trace form. For $n = 3$ with $U = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ u_2 & u_1 & 0 \end{pmatrix}$,

$$\langle DUD, U\rangle = d_3 d_2 u_1^2 + d_1 d_3 u_2^2 + d_1 d_2 u_3^2.$$

It then follows by a simple calculation, that for any positive numbers c_1, c_2, c_3 there is a unique matrix D such that $c_1 = d_3 d_2$, $c_2 = d_1 d_3$, $c_3 = d_1 d_2$.

Analogous formulations over the Serret–Frenet framed curves, or over the auto-parallel framed curves could be considered as minor variations of the original elastic problem of Kirchhoff. In dimension 2 all these problems coalesce to the Euler–Griffiths problem, but not so in larger dimensions. On the spheres $\mathbb{S}^n_\epsilon$ the Euler–Griffiths problem has a natural formulation over the auto-parallel framed curves as follows.

The boundary conditions (x_0, v_0) and (x_1, v_1) for the problem of Euler–Griffiths define submanifolds $S_0 = \{f \in G_\epsilon : fe_0 = x_0,\ fe_1 = v_0\}$ and $S_1 = \{f \in G_\epsilon : fe_0 = x_1,\ fe_1 = v_1\}$ in G_ϵ. Auto-parallel framed curves are the solutions of

$$\frac{dg}{dt} = g(t)(E_\epsilon + U(t)) \text{ with } U(t) = \sum_{i=1}^{n-1} u_i(t)(e_1 \wedge e_i). \tag{12}$$

Matrices $U(t)$ can be written also as $U(t) = \begin{pmatrix} 0 & -u^T(t) \\ u(t) & 0 \end{pmatrix}$, where u^T denotes the vector transpose (row vector) of the column vector u. Consider now

Definition 2.1. Lifted Euler–Griffiths Problem. Let $T > 0$ be given. Find the solution $g(t)$ of (13) in the interval $[0, T]$ that satisfies $g(0) \in S_0$, $G(t) \in S_1$ and minimizes the integral $\frac{1}{2}\int_0^T \langle U(t), U(t)\rangle\, dt$ among all other solutions of (AP) that conform to the same boundary conditions.

Since every curve $x(t)$ in $\mathbb{S}^n_\epsilon$ that is parametrized by its arc length is the projection of a curve $g(t)$ that is a solution of (12) and since $\langle U(t), U(t)\rangle = \kappa^2(t)$, it is easy to show that $x(t)$ is a solution for the Euler–Griffiths problem if and only if $x(t)$ is the projection of the solution for the lifted Euler–Griffiths problem. The Dubins–Delauney problem can be also phrased in terms of the auto-parallel framed curves as the problem of connecting the initial manifold S_0 to the terminal manifold S_1 in the least possible time via the solutions of (12) generated by the controls $U(t)$ subject to the constraint $\langle U(t), U(t)\rangle \leqslant 1$.

All of the above "energy" problems are controllable in the sense that for any boundary conditions h_1 and h_2 there exists a time T and a trajectory $g(t)$ that satisfies $g(0) = h_1$ and $g(T) = h_2$. We will show later that, even for a more general class of problems, there are optimal solutions $(\hat{g}(t), \hat{U}(t))$ on any such interval $[0, T]$ with $\hat{U}(t)$ a square summable function on $[0, T]$. Time optimal solutions for problems of Dubins–Delauney type also exist because the reachable sets $\mathcal{A}(x_0, \leqslant T)$ of the associated control systems are closed [20, p. 119].

2.4. The Maximum Principle. We now come to the necessary conditions of optimality and Pontryagin's Maximum Principle. For simplicity of exposition we will confine the discussion to control affine systems (5) and consider the optimal problem of minimizing a functional of the type $\int_0^T f(x(t), u(t))\,dt$ over its solutions subject to the given boundary conditions. The boundary conditions could consist either of single points, or submanifolds of the state space M. We will be principally interested in the minimal energy problems in which $f(x, u)$ is a given quadratic form $\langle u, Qu\rangle$ and the minimization is done under no constraints on the control functions, and the time optimal problems ($f = 1$) with controls constrained to the unit ball $\|u\| \leqslant 1$.

The Maximum Principle states that each optimal trajectory $x(t)$ is the projection of an *extremal curve* $\xi(t)$ in the cotangent bundle $T^*(M)$ which, moreover, is an integral curve of an appropriate Hamiltonian system on T^*M. To be more precise, let π denote the natural projection $T^*M \to M$ defined by $\pi(\xi) = x$ for any $\xi \in T_x^*(M)$, and let ω denote the canonical symplectic form on $T^*(M)$. For each function h on T^*M let $\vec{h}$ denote the Hamiltonian vector field induced by h, i.e., defined by $dh = i_{\vec{h}}\omega$, where $i_{\vec{h}}$ denotes the contraction along $\vec{h}$.

Let $h_0, \dots, h_m$ denote the Hamiltonians defined by the vector fields $X_0, \dots, X_m$ in (5) by $h_i(\xi) = \xi(X_i(x))$, $x = \pi(\xi)$. Then control system (5) together with the cost functional f can be lifted to the cotangent bundle T^*M via the time varying Hamiltonians

$$h_\lambda(u(t), \xi) = -\lambda f(u(t), \pi(\xi)) + \sum_{i=1}^m u_i(t) h_i(\xi), \quad \xi \in T^*(M), \tag{13}$$

according to the parameter λ which can be either equal to 1 or equal to 0.

Definition 2.2. Control function $u(t), t \in [0, t_1]$ is called *extremal* if there exist a curve $\xi(t)$ in $T^*(M)$ such that

1. $\frac{d\xi}{dt} = \vec{h}_\lambda(u(t), \xi(t))$ for almost all $t \in [0, t_1]$; 2. $\xi(t) \neq 0$ if $\lambda = 0$;
3. $h_\lambda(u(t), \xi(t)) \geqslant h_\lambda(v, \xi(t))$ for all $v \in U$ and almost all $t \in [0, T]$.

The curve $\xi(t)$ is called an *extremal curve.* It is called *normal* if $\lambda = 1$ and *abnormal* if $\lambda = 0$. The pair $(\xi(t), u(t))$ as usually referred to as an *extremal pair.*

Theorem 2.1 (The Maximum Principle)**.** *Each optimal trajectory pair $(x(t), u(t))$ on an interval $[0, t_1]$ is the projection of an extremal curve $\xi(t)$ generated by $u(t)$. If the terminal time is fixed, then the Hamiltonian $h(\xi(t), u(t))$ is constant on the interval $[0, t_1]$, but if the terminal time t_1 is variable, then $h(\xi(t), u(t)) = 0$.*

If the boundary conditions consist of an initial submanifold S_0 and a termial submanifold S_1, then the extremal curve $\xi(t)$ must, in addition, satisfy the transversality conditions*: $\xi(0)(T_{x_0}S_0) = 0$ and $\xi(t_1)(T_{x_1}S_1) = 0$, where $x_0 = \pi(\xi(0))$ and $x_1 = \pi(\xi(t_1))$.*

To understand the mysterious appearance of the multiplier λ one needs to go to the fundamental facts upon which the Maximum Principle is based. The principle is based on the fact that an optimal trajectory pair $(x(t), u(t))$ must be on the boundary of the reachable set from $(0, x_0)$ in the extended space $\tilde{M} = \mathbb{R} \times M$ of the "cost extended" system:

$$\frac{dx_0}{dt} = f(x(t), u(t)), \quad \frac{dx}{dt} = X_0(x(t)) + \sum_{i=1}^{m} u_i(t) X_i(x(t)), \tag{14}$$

The cotangent bundle $T^*\tilde{M}$ is equal to the product $T^*\mathbb{R} \times T^*M$. Since $T^*\mathbb{R}$ is trivial, it can be realized as the product $\mathbb{R}^2$ with coordinates (x_0, λ). Then the Hamiltonian lift $\tilde{h}(u(t), \xi)$ of (14) in $T^*\tilde{M}$ is formally the same as (13). Since $\tilde{h}(u(t), \xi)$ does not explicitly depend on the variable x_0, the variable λ is cyclic, in the sense that it is constant along an integral curve of the Hamiltonian vector field defined by $\tilde{h}(u(t), \xi)$. The Maximum Principle then provides a necessary condition, in terms of $\tilde{h}(u(t), \xi)$, that $(x(t), u(t))$ be on the boundary of the reachable set. Then, it follows from the proof of the Maximum Principle, that the associated coordinate λ must be nonpositive because the cost is minimal. If λ is negative, then it can be normalized to -1. Since $h(u, \xi)$ is the projection of $\tilde{h}(u), \xi$ on T^*M, λ appears as an extra parameter.

Abnormal extremals may occur only when the number of controlled fields is less that the dimension of M because of conditions 2 and 3 that define extremal curves. For when $\lambda = 0$, the maximality condition implies that

$$h_1(\xi(t)) = \cdots = h_m(\xi(t)) = 0 \tag{15}$$

along an abnormal extremal $\xi(t)$. When $m = n$ and $X_1(x(t)), \ldots, X_n(x(t))$ are linearly independent, then (15) implies that $\xi(t) = 0$, which contradicts condition 2. But when the number of controlled fields is less that $\dim(M)$, then it may happen that either an optimal trajectory is the projection of both a normal and an abnormal extremal, or it may also happen that an optimal trajectory is the projection of an abnormal extremal only.

In general, there may be additional constraints on the abnormals encountered by further differentiations of (15). The ultimate resolution of these constraints and the determination of the corresponding extremal control could result in an involved procedure, often reminiscent of Dirac's reduction of Hamiltonian systems under constraints. However, for the problems considered in this paper abnormal extremals do not play a significant role and their presence will be ignored.

For the "energy" problems $f(x, u) = \frac{1}{2}\langle u, Qu\rangle$, $Q > 0$ with no bounds on the controls, normal extremal controls maximize the expression

$$h(u, \xi) = -\frac{1}{2}\langle u, Qu\rangle + h_0(\xi) + \sum_{i=1}^{m} u_i h_i(\xi)$$

over the control set U. Hence, they are necessarily solutions of $\frac{\partial h}{\partial u}(u, \xi) = 0$, i.e., they satisfy

$$-\sum_{j=1}^{m} Q_{ij} u_j(t) + h_i(\xi(t)) = 0, \quad i = 1, \ldots, m, \tag{16}$$

Since $Q > 0$ this linear system of equations yields $u(t) = Q^{-1}h(\xi(t))$, where $h = (h_1, \dots, h_m)^T$. It follows that the normal extremals corresponding to such extremal controls are the integral curves of a single Hamiltonian H given by

$$H(\xi) = \frac{1}{2}\big(Q^{-1}h(\xi), h(\xi)\big) + h_0(\xi). \tag{17}$$

For time optimal problems with controls in the unit ball $\|u\|^2 \leqslant 1$, the Hamiltonian lift is given by

$$h(u, \xi) = \lambda + h_0 + \sum_{i=1}^{m} u_i h_i, \quad \lambda = 0, -1. \tag{18}$$

The maximality condition of the Maximum Principle determines the extremal control only when $(h_1(\xi(t)), \dots, h(\xi(t))) \neq 0$. For when $(h_1(\xi(t)), \dots, h(\xi(t))) \neq 0$ the extremal control is of the form

$$u = \frac{1}{\sqrt{h_1^2 + \cdots + h_m^2}}(h_1(\xi), \dots, h_m(\xi)), \tag{19}$$

i.e, it belongs to the boundary of the unit ball and remains there on an open time interval by the continuity of ξ. The surface

$$\mathcal{S} = \{\xi : h_1(\xi) = h_2(\xi) = \cdots = h_m(\xi) = 0\} \tag{20}$$

is called *the switching surface*. Extremals $\xi(t)$ which reside on the switching surface on open intervals of time, together with the controls that generate them are called *singular*. Singular controls belong to the interior of the unit ball. In addfition to conforming to $h_i(\xi(t)) = 0$, singular extremals also conform to $\{h, h_i\}(\xi(t)) = 0$ which generates another set of constraints

$$\{h_0, h_i\} + \sum_{j=1}^{m} \{h_j, h_i\} u_j(t) = 0, \quad i = 1, \dots, m. \tag{21}$$

In the simplest case, the matrix $\{h_i, h_j\}$ is nonsingular, and the preceding system of equations is solvable for u in terms of $h_0, \dots, h_m$. Otherwise equations (21) imply further constraints and further Poisson brackets are needed before getting a closed form solution for $u(t)$. For instance, for problems with scalar controls ($m = 1$) equation (21) reduces to $\{h_0, h_1\}(\xi(t)) = 0$. Then

$$\{h, \{h_0, h_1\}\}(\xi(t)) = \{h_0, \{h_0, h_1\}(\xi(t))\} + u(t)\{h_1, \{h_0, h_1\}(\xi(t))\}(\xi(t)) = 0. \tag{22}$$

In the case of Dubins–Delauney problem on $\mathbb{S}^2_\epsilon$, equation (22) implies $u(t) = 0$. Therefore, the singular extremals for the problem of Dubins–Delauney project onto the geodesics of $\mathbb{S}^2_\epsilon$.

An extremal curve $\xi(t)$ that belongs to the complement $\mathcal{S}^c$ of $\mathcal{S}$ for some time t belongs to $\mathcal{S}^c$ on an open interval centered around t by continuity of $\xi(t)$. Hence, $\{t : \xi(t) \in \mathcal{S}^c\}$ is an open set. On this set, the control $u(t)$ that generates $\xi(t)$ is a boundary control and is given by (19). This extremal may or may not cross the switching surface, and if it does cross, it may cross it *transversally* or *tangentially.* In the case of transversal crossing $\xi(t)$, the extremal control leaves $\mathcal{S}$ immediately after the crossing and the control maintaiins the form given by (19). In the tangential case, however, the extremal may remain on the switching surface, in which case the extremal control becomes singular. An extremal may cross the switching surface several times.

In the problem of Dubins–Delauney on the sphere $\mathbb{S}^2_\epsilon$, the boundary of the sphere $\|u\| \leqslant 1$ consisits of two points $u = \pm 1$. In that case, the extremals are the concatenations of the integral curves of

$$h_+ = \lambda + h_0 + h_1\,(u=1),\;\; h_- = \lambda + h_0 - h_1\,(u=-1),\; \text{or } h_0\,(u=0).$$

The projections of these extremal curves consist of circles of curvature ± 1 and the geodesics. It is known that optimal curves are the projections of normal extremals with at most two switches (for instance, see [22] and [33]).

Let us now turn our attention to the Hamiltonian equations and their solutions. Since solvability of Hamiltonian equations is intimately linked with symmetries, and since the elastic problems with all their variants admit left invariant representations on Lie groups it is important to choose coordinates that are suitably adapted to these symmetries. The section below deals with Lie groups and their cotangent bundles as a natural setting for variational problems with symmetries.

3. Lie groups and left-invariant Hamiltonians

The elastic problems discussed earlier fit naturally into a larger class of left-invariant optimal problems on any Lie group G that admits an *involutive automorphism* σ. Since much of the subsequent exposition about the Hamiltonian systems associated with elastic problems is equally valid for this larger class, and since this larger class is central for the theory of integrable systems, it seems worthwhile to digress briefly, and introduce the relevant terminology.

3.1. Lie groups with involutive automorphism and the affine problem. An involutive automorphism σ on a Lie group G is an analytic mapping $\sigma : G \to G$ that satisfies 1. $\sigma(g_2g_1) = \sigma(g_2)\sigma(g_1)$ for all g_1, g_2 in G; 2. $\sigma^2 = I, \sigma \neq I$.

Such automorphisms identify a Lie subgroup K_0 consisting of fixed points of σ, i.e., $K_0 = \{g : \sigma(g) = g\}$, and their tangent maps σ_* define a direct sum decomposition $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ of the Lie algebra $\mathfrak{g}$ with

(23) $$\mathfrak{p} = \{M : \sigma_*(M) = -M\} \text{ and } \mathfrak{k} = \{M : \sigma_*(M) = M\}.$$

The latter decomposition follows from the fact that σ_* is an involutive Lie algebra automorphism of $\mathfrak{g}$, in the sense that $\sigma_*([A,B]) = [\sigma_*(A), \sigma_*(B)]$. Since $\sigma_*^2 = I$, $(\sigma_* + I)(\sigma_* - I) = 0$, and hence, ± 1 are the only eigenvalues of σ_* with $\mathfrak{k}$ with $\mathfrak{p}$ the corresponding eigenspaces. The above implies that

(24) $$[\mathfrak{p}, \mathfrak{p}] \subseteq fk,\;\; [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p},\;\; [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k},$$

and hence, the Cartan decomposition on space forms (10) is valid in a more general context. Relations (24) imply that $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$. It can be easily shown that $\mathfrak{k}$ is the Lie algebra of K_0. Since K_0 may not be connected, it will be more convenient to pass to the group K, the connected component of K_0 that contains the group identity I.

Vector space $\mathfrak{p}$ is called the Cartan space. It is invariant under the adjoint action of K in view of the Lie bracket condition $[\mathfrak{p}, \mathfrak{k}] \subseteq \mathfrak{p}$. This fact will be noted by

$$\mathrm{Ad}_K(\mathfrak{p}) \subseteq \mathfrak{p}, \tag{25}$$

Since K acts linearly on $\mathfrak{p}$ by (25), it defines the semidirect product $G_s = \mathfrak{p} \rtimes K$. The semidirect product G_s is the Cartesian product $\mathfrak{p} \times K$ together with the the group operation $(A_1, h_1) \cdot (A_2, h_2) = (A_1 + \mathrm{Ad}_{h_1}(A_2), h_1 h_2)$. It is a Lie group, and its Lie algebra consists of points in $\mathfrak{p} \times \mathfrak{k}$ with the Lie bracket $[\,,]_s$ given by

$$[(A_1, B_1), (A_2, B_2)]_s = \big(ad(B_1)(A_2) - ad(B_2)(A_1), [B_1, B_2]\big)$$

for all $(A_i, B_i) \in \mathfrak{p} \rtimes \mathfrak{k}, i = 1, 2$.

If (A, B) in $\mathfrak{p} \rtimes \mathfrak{k}$ is identified with the sum $A + B$, then $\mathfrak{p} \times \mathfrak{k}$ is identified with $\mathfrak{p} \oplus \mathfrak{k}$. Hence, $[(A_1, B_1), (A_2, B_2)]_s$ is identified with $[B_1, A_2] - [B_2, A_1] + [B_1, B_2]$. This shows that $\mathfrak{g}$ as a vector space carries two Lie algebras: the semisimple Lie algebra and the semidirect Lie algebra. The semisimple Lie bracket $[\,,]$ is related to the semidirect Lie bracket $[\,,]_s$ according to

$$[A_1 + B_1, A_2 + B_2] = [A_1, A_2] + [A_1 + B_1, A_2 + B_2]_s \ \text{ for any } \ A_i \in fp, \ B_i \in \mathfrak{k}.$$

In what follows $\langle\,,\rangle$ will denote the scalar multiple of the Killing form $\langle A, B\rangle = Trace(ad(A) \circ ad(B))$ that is positive definite on $\mathfrak{k}$ and coincides with the trace form on $\mathfrak{g}_\epsilon$ defined in the previous section. It is easy to show that $\langle\,,\rangle$ is invariant relative to any automorphism ϕ on $\mathfrak{g}$ in the sense that $\langle A, B\rangle = \langle \phi(A), \phi(B)\rangle$. This invariance implies that $\langle\,,\rangle$ is Ad_K invariant, and secondly, it implies that $\mathfrak{p}$ and $\mathfrak{k}$ are orthogonal because $\langle A, B\rangle = \big\langle \sigma_*(A), \sigma_*(B)\big\rangle = \langle -A, B\rangle = -\langle A, B\rangle$. Moreover,

$$0 = \frac{d}{dt}(\langle \mathrm{Ad}_{\exp tC}(A), \mathrm{Ad}_{\exp tC}(B)\rangle)|_{t=0} = \langle [C, A], B\rangle + \langle A, [C, B]\rangle$$

for any elements A, B, C in $\mathfrak{g}$. Therefore, $\langle [A, B], C\rangle = \langle A, [B, C]\rangle$.

A Lie group G is said to be semisimple if the Killing form is nondegenerate on $\mathfrak{g}$. For Lie groups that admit involutive automorphisms semisimplicity on $\mathfrak{g}$ implies semisimplicity on $\mathfrak{k}$. Therefore, K is semisimple as well. In addition, K will be assumed compact with finite center. This assumption implies that the Killing form is negative definite on $\mathfrak{k}$ [17] and it also implies that there exists a scalar multiple of the Killing form for which $\langle\,,\rangle$ is positive definite on K.

Semisimplicity of G implies stronger relations in (24), namely, $[\mathfrak{p}, \mathfrak{k}] = \mathfrak{p}$. It is important to note that $\langle\,,\rangle$, as a scalar multiple of the Killing form, is degenerate on the semidirect Lie algebra $\mathfrak{g}_s$.

In the literature of symmetric spaces [17] the pair (G, K) with K a closed subgroup of G obtained as by an involutive automorphism is called a symmetric pair. If in addition this pair admits a positive definite Ad_K invariant quadratic form on the Cartan space $\mathfrak{p}$, then the pair is called Riemannian symmetric pair. We will not go further into the theory of these spaces; the interested reader should see [17] or [11] (also see [25] for examples). An element A in $\mathfrak{p}$ is said to be regular if $\{X \in \mathfrak{p} : [X, A] = 0\}$ is an abelian algebra. Alternatively, A is regular if there is a maximal abelian algebra that contains A.

With these theoretic ingredients at our disposal we come to the fundamental control problems on G, a natural extension of Kirchhoff's elastic problem:

Definition 3.1 (The Affine Problem). Minimize $\frac{1}{2}\int_0^T \langle Qu(t), u(t)\rangle\, dt$ over the trajectories of

$$\frac{dg}{dt} = g(t)(B + U(t)), \quad U(t) \in \mathfrak{k}, \tag{26}$$

that satisfy $g(0) = g_0$ and $g(T) = g_1$, where B is a regular element in $\mathfrak{p}$, Q a positive definite linear mapping on $\mathfrak{k}$ and g_0 and g_1 given points in G.

Remark 3.1. System (26) is formally the same as system (8) when $U(t)$ is written as $U(t) = \sum_{i,j=1}^n u_{ij}(t)A_{ij}$ with $A_{ij} : 1 \leqslant i, j \leqslant n$.

Remark 3.2. The affine control problem has an identical formulation on the semidirect product $G_s = \mathfrak{p} \rtimes K$ with the boundary conditions on G replaced by the boundary conditions on G_s. The affine control problem on G_s is always a "shadow" of the affine problem on G.

Theorem 3.1 (The Existence Theorem). *Given any pair g_0 and g_1 in G, there exist a positive number T and an optimal trajectory $(g(t), U(t))$ on $[0, T]$ of (26) relative to g_0, g_1 and T. The same holds for the affine problem on the semidirect product G_s relative to g_0 and g_1 in G_s.*

Proof. We will merely outline the main points in the proof, since the inclusion of details would take us too far from the central topic of this paper, the relevance to integrable systems.

1. *Controllability.* Regularity of B implies that the positive affine hull generated by $\{\mathrm{Ad}_h(B) : h \in K\}$ is equal to $\mathfrak{p}$. Therefore, both $\mathfrak{p}$ and $\mathfrak{k}$ are in the Lie saturate of the set $\Gamma = \{B + U : U \in \mathfrak{k}\}$. Hence, system (26) is controllable.

2. *Optimality.* Let $\mathrm{Tr}(T, g_0, g_1)$ denote the set of all trajectories of (26) that satisfy $g(0) = g_0$, $g(T) = g_1$. It follows from above that there exists $T > 0$ such that $\mathrm{Tr}(T, g_0, g_1)$ is not empty. Let $U_0(t)$ denotes any control whose trajectory is in $\mathrm{Tr}(T, g_0, g_1)$, let c_0 denote its cost $\int_0^T \langle QU_0(t), U_0(t)\rangle\, dt$.

Let (g_n, U_n) denote a sequence in $\mathrm{Tr}(T, g_0, g_1)$ so that $\int_0^T \langle QU_n(t), U_n(t)\rangle\, dt \leqslant c_0$ and

$$\lim \int_0^T \langle QU_n(t), U_n(t)\rangle\, dt = \liminf \left\{ \int_0^T \langle QU(t), U(t)\rangle\, dt, (g, U) \in \mathrm{Tr}(T, g_0, g_1) \right\}.$$

The sequence U_n belongs to the closed ball $\{U(t); t \in [0, T], \int_0^T \langle QU(t), U(t)\rangle\, dt \leqslant c_0\}$ in the Hilbert space H consisting of square summable functions $U(t)$ on $[0, T]$ with values in $\mathfrak{k}$. Since closed balls in a Hilbert space are weakly compact, the sequence $\{U_n\}$ has a weakly convergent subsequence. So there is no loss in generality in assuming that $\{U_n\}$ itself is weakly convergent. It then follows that the corresponding trajectories converge uniformly [20]. If $g(t)$ denotes the uniform limit of $\{g_n\}$, then $g(t)$ is the trajectory generated by the weak limit $U(t)$ of $\{U_n\}$. This argument shows that optimal trajectories exist, but the argument only shows that they are generated by the controls in H. □

Remark 3.3. The above shows that the optimal control is only square summable on the interval $[0, T]$. So additional arguments are needed to show that the Maximum Principle is applicable, since, as it stands, the Maximum Principe is valid only for essentially bounded controls.

3.2. Left-invariant Hamiltonians. For left invariant optimal control problems on a Lie group G it is natural to consider T^*G as the product $G \times \mathfrak{g}^*$, where $\mathfrak{g}^*$ denotes the dual of $\mathfrak{g}$, with each ξ in T^*_gG identified with $(g, l) \in G \times \mathfrak{g}^*$ via the formula $\xi(gA) = l(A)$ for every $gA \in T_gG$. If $\mathfrak{g}$ is semisimple, then the Killing form is nondegenerate, and hence $\langle\, ,\rangle$ is also nondegenerate and can be used to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the formula

$$\ell \in \mathfrak{g}^* \Leftrightarrow L \in \mathfrak{g} \text{ if and only if } \langle L, X\rangle = \ell(X), \quad X \in \mathfrak{g}.$$

The duals $\mathfrak{p}^*$ and $\mathfrak{k}^*$ of $\mathfrak{p}$ and $\mathfrak{k}$ will be identified with the annihilators $\mathfrak{k}^0 = \{l \in \mathfrak{g} : l(A) = 0,\ A \in \mathfrak{k}\}$ and $\mathfrak{p}^0 = \{l \in \mathfrak{g} : l(A) = 0,\ A \in \mathfrak{p}\}$. Because $\mathfrak{p}$ and $\mathfrak{k}$ are orthogonal, $\mathfrak{p}^0$ is identified with $\mathfrak{p}$ and $\mathfrak{k}^0$ is identified with $\mathfrak{k}$. The above implies that if $\ell_\mathfrak{p} \in \mathfrak{p}^*$ corresponds to $L_\mathfrak{p} \in \mathfrak{p}$ and if $\ell_\mathfrak{k} \in \mathfrak{k}^*$ corresponds to $L_\mathfrak{k} \in \mathfrak{k}$, then $\ell = \ell_\mathfrak{p} + \ell_\mathfrak{k}$ corresponds to $L = L_\mathfrak{p} + L_\mathfrak{k}$.

The Poisson structure on $\mathfrak{g}^*$ inherited from the symplectic structure of $T^*G = G \times \mathfrak{g}^*$ is defined in terms of the Poisson bracket $\{f, h\}(\ell) = \ell([df, dh])$, $\ell \in \mathfrak{g}^*$, for functions f and h on $\mathfrak{g}^*$. Functions on $\mathfrak{g}^*$ can be identified with the functions on $G \times \mathfrak{g}^*$ that are invariant under the left translations by the elements of G, and for that reason, will be called *left invariant Hamiltonians*. Any left invariant Hamiltonian h generates a Hamiltonian vector field $\vec{h}$ on $G \times \mathfrak{g}$ whose integral curves $(g(t), l(t))$ are the solutions of

$$\frac{dg}{dt} = g dh(l(t)), \quad \frac{dl}{dt} = -ad^* dh(l(t))(l(t)), \tag{27}$$

where $ad^*(X) : \mathfrak{g}^* \to \mathfrak{g}^*$ is defined by $ad^*(X)(l) = l \circ ad(X)$. On semisimple Lie groups equations (28) can be rephrased as

$$\frac{dg}{dt} = g dh(L(t)), \quad \frac{dL}{dt} = [dh(L(t)), L(t)], \tag{28}$$

due to the invariance of $\langle\, ,\rangle$ (formula (25)). In terms of the decomposition $L = L_\mathfrak{p} + L_\mathfrak{k}$ equations (28) take on the following form

$$\frac{dg}{dt} = g dh(l(t)), \quad \frac{dL_\mathfrak{k}}{dt} = [dh_\mathfrak{k}, L_\mathfrak{k}] + [dh_\mathfrak{p}, L_\mathfrak{p}], \quad \frac{dL_\mathfrak{p}}{dt} = [dh_\mathfrak{k}, L_\mathfrak{p}] + [dh_\mathfrak{p}, L_\mathfrak{k}], \tag{29}$$

where $dh_\mathfrak{p}$ and $dh_\mathfrak{k}$ denote the projections of dh on $\mathfrak{p}$ and $\mathfrak{k}$.

For Lie groups with an involutive automorphism functions on $\mathfrak{g}^*$ may be considered as functions on the dual of the semidirect product $\mathfrak{g}_s$ and vice versa. The Hamiltonian equations of functions on $\mathfrak{g}_s^*$ are different from the Hamiltonian equations on $\mathfrak{g}^*$ because of the difference in the Poisson structures. Recall that $[X, Y]_s = [X_\mathfrak{p}, Y_\mathfrak{k}] + [X_\mathfrak{k}, Y_\mathfrak{p}] + [X_\mathfrak{k}, Y_\mathfrak{k}]$, where $X_\mathfrak{p}$, $X_\mathfrak{k}$ and $Y_\mathfrak{p}$, $Y_\mathfrak{k}$ denote the appropriate projections of X and Y on $\mathfrak{p}$ and $\mathfrak{k}$. Therefore, equation $\frac{dl}{dt} = -ad^*(dh)(l(t))$

relative to the semidirect product implies that

$$\begin{aligned}\Big\langle \frac{dL}{dt}, X\Big\rangle &= \Big\langle \frac{dL_{\mathfrak{p}}}{dt}, X_{\mathfrak{p}}\Big\rangle + \Big\langle \frac{dL_{\mathfrak{k}}}{dt}, X_{\mathfrak{k}}\Big\rangle \\ &= -\big\langle L_{\mathfrak{p}}, [dh_{\mathfrak{p}}, X_{\mathfrak{k}}] + [dh_{\mathfrak{k}}, X_{\mathfrak{p}}]\big\rangle - \big\langle L_{\mathfrak{k}}, [dh_{\mathfrak{k}}, X_{\mathfrak{k}}]\big\rangle \\ &= \big\langle [dh_{\mathfrak{p}}, L_{\mathfrak{p}}] + [dh_{\mathfrak{k}}, L_{\mathfrak{k}}], X_{\mathfrak{k}}\big\rangle + \big\langle [dh_{\mathfrak{k}}, L_{\mathfrak{p}}], X_{\mathfrak{p}}\big\rangle.\end{aligned}$$

Hence,

$$\frac{dL_{\mathfrak{k}}}{dt} = [dh_{\mathfrak{k}}, L_{\mathfrak{k}}] + [dh_{\mathfrak{p}}, L_{\mathfrak{p}}], \quad \frac{dL_{\mathfrak{p}}}{dt} = [dh_{\mathfrak{k}}, L_{\mathfrak{p}}]. \tag{30}$$

Together with $\frac{dg}{dt} = g(t)\, dh(l(t))$ equations (30) constitute the Hamiltonian equations of a function h on the semidirect product $\mathfrak{p} \rtimes \mathfrak{k}$. It will be convenient to combine (29) and (30) into a single equation

(31)

$$\frac{dg}{dt} = g\, dh(l(t)),\ \frac{dL_{\mathfrak{k}}}{dt} = [dh_{\mathfrak{k}}, L_{\mathfrak{k}}] + [dh_{\mathfrak{p}}, L_{\mathfrak{p}}],\ \frac{dL_{\mathfrak{p}}}{dt} = [dh_{\mathfrak{k}}, L_{\mathfrak{p}}] + s[dh_{\mathfrak{p}}, L_{\mathfrak{k}}],\ s = 0, 1$$

3.3. Hamiltonian equations of the Affine problem. It follows from our discussion of the Maximum Principle that the normal extremals for the Affine problem (Aff) are the integral curves of $H = \frac{1}{2}\langle Q^{-1}h(\xi), h(\xi)\rangle + h_0(\xi)$, where $h(\xi)$ is the matrix with entries $h_{ij}(\xi) = \xi(gA_{ij})$ (equation 17). The covector ξ in T_g^*G is identified with $(g, \ell) \in G \times \mathfrak{g}^*$ and hence, $h_{ij}(\xi) = \ell(A_{ij})$. Similarly, $h_0(\xi) = \ell(B)$. Therefore, H is a left-invariant Hamiltonian on $\mathfrak{g}^*$.

Since we are in a semisimple case, ℓ is identified with $L \in \mathfrak{g}$. In the notations used above $L = L_{\mathfrak{k}} + L_{\mathfrak{p}}$ and $\langle L_{\mathfrak{k}}, X\rangle = \ell_{\mathfrak{k}}(X)$ for all $X \in \mathfrak{k}$. Hence, $\langle L_{\mathfrak{k}}, A_{ij}\rangle = \ell_{\mathfrak{k}}(A_{ij}) = h_{ij}(\xi)$. This means that h is identified with $L_{\mathfrak{k}}$ and $h_0(\xi)$ is identified with $\langle L_{\mathfrak{p}}, B\rangle$. It follows that H as a function on $\mathfrak{g}$ is given by

$$H = \frac{1}{2}\langle Q^{-1}L_{\mathfrak{k}}, L_{\mathfrak{k}}\rangle + \langle L_{\mathfrak{p}}, B\rangle \tag{32}$$

(remember, $\mathfrak{p}$ and $\mathfrak{k}$ are orthogonal). Then, $dH_{\mathfrak{k}} = Q^{-1}L_{\mathfrak{k}}$ and $dH_{\mathfrak{p}} = B$ and the Hamiltonian equations (31) take on the following form

$$\begin{aligned}\frac{dg}{dt} &= g(B + Q^{-1}L_{\mathfrak{k}}), \\ \frac{dL_{\mathfrak{k}}}{dt} &= [Q^{-1}L_{\mathfrak{k}}, L_{\mathfrak{k}}] + [B, L_{\mathfrak{p}}], \\ \frac{dL_{\mathfrak{p}}}{dt} &= [Q^{-1}L_{\mathfrak{k}}, L_{\mathfrak{p}}] + s[B, L_{\mathfrak{k}}], \quad s = 0, 1.\end{aligned} \tag{AffHam}$$

4. Kirchhoff's problem on space forms with $Q = I$.

4.1. Elastic curves. Let us begin analyzing the solutions of the Affine Hamiltonian system in the simplest case: $Q = I$ and $G = G_\epsilon$, $\epsilon = \pm 1$. Since these groups are the positively oriented orthonormal frame bundles of the homogeneous space $\mathbb{S}^n_\epsilon = G_\epsilon / K$, with $K = \{1\} \times \mathrm{SO}_n(R)$ the Hamiltonian in (32) stands for the equilibrium energy associated with non-Euclidean Kirchhoff's elastic rod. It will be shown below that when $Q = I$, (AffHam) not only recovers the solutions to the

Euler–Griffiths elastic problem but makes an interesting link with the equations of the heavy top (Lagrange's top).

To set the stage for this analysis, it will be helpful, particularly to the reader not familiar with optimal control, first to digress briefly into the Euler–Lagrange equation associated the Euler–Griffiths problem and acquaint the reader with the known results. The approach will that of Langer and Singer [28].

4.2. The Euler–Lagrange equation. Let us begin with an arbitrary Riemannian manifold M and let ∇ denote the Levi–Civita connection M and let $R(X,Y)$ denote the Riemannian curvature tensor defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

for any vector fields X, Y, Z on M. First, recall the following relations (well known in the literature on Riemannian geometry, for instance see [8])
1. $\nabla_X Y - \nabla_Y X = [X,Y]$, for any vector fields X and Y.
2. $\nabla_{fX} = f\nabla_X$ for any function f and any vector field X.
3. $\nabla_X(fY) = f\nabla_X Y + X(f)Y$ for any function f and any vector fields X and Y.
4. $\langle R(X,Y)Z, W\rangle = \langle R(W,Z)Y, X\rangle$ for any vector fields X, Y, Z, W.
Then

Proposition 4.1. *Let $T(s)$denote the tangent vector of a curve $\gamma(s)$ that is a critical point for the Euler–Griffiths functional $\frac{1}{2}\int_0^L \|\nabla_T T\|^2\,ds$. Then $T(s)$ satisfies*

$$(\nabla_T)^3 T + \nabla_T\Big(\frac{3}{2}\kappa^2 - \lambda\Big)T + R(\nabla_T T, T)T\rangle = 0. \tag{33}$$

with λ a constant.

Equation (33) is known as the Euler–Lagrange equation. A curve $\gamma(s)$ is called elastic if its tangent vector is a solution of the Euler–Lagrange equation.

Proof. Let $\Gamma(w,t)$ denote a family of two parameter smooth regular curves $\gamma_w(t) = \Gamma(w,t)$, with $0 \leqslant t \leqslant 1$, $|w| < \epsilon$, for some $\epsilon > 0$ of fixed length L that satisfy fixed boundary conditions:

$$\gamma_w(0) = x_0,\ \ \gamma(1) = x_1,\ \ \frac{d\gamma_w}{dt}(0) = v_0,\ \ \frac{d\gamma_w}{dt}(1) = v_1.$$

Define $V(w,t) = \frac{\partial\Gamma}{\partial t}(w,t)$ and $W(w,t) = \frac{\partial\Gamma}{\partial w}(w,t)$, and let $v(w,t) = \|V(w,t)\|$. The preceding boundary conditions imply

$$W(w,0) = W(w,1) = 0 \ \text{ and } \ \frac{\partial W}{\partial t}(w,0) = \frac{\partial W}{\partial t}(w,1) = 0.$$

Since each curve $\gamma_w(t)$ is regular, $v(w,t) \neq 0$ and there is a well defined unit tangent vector $T(w,t) = \frac{1}{v(w,t)}V(w,t)$. If s denotes the arc length, then $vdt = ds$. In terms of the Levi–Civita connection, $\nabla_T = \frac{\partial}{\partial s}$ and $\nabla_W = \frac{\partial}{\partial w}$. In particular, $\nabla_T T = \frac{\partial T}{\partial s}$ and therefore, $\kappa^2 = \|\nabla_T T\|^2$.

Since each curve γ_w is of fixed length, Γ satisfies $\int_0^1 (v(w,t)-L)\,dt = 0$. Consider now the critical points of the functional

$$\mathcal{F} = \int_0^L \frac{1}{2}\kappa^2(s)\,ds + \lambda\Big(\int_0^L ds - L\Big) = \int_0^1 \Big(\frac{1}{2}\kappa^2(t) + \lambda\Big)v\,dt - \lambda L$$

over the curves γ_w induced by Γ, where λ is a Lagrange multiplier due to the constraint on the arc length. Then $\gamma(t) = \Gamma(0,t)$ is a critical point of $\mathcal{F}$ over Γ if

$$d\mathcal{F}|_{w=0} = \frac{\partial}{\partial w}\int_0^1 \Big(\frac{1}{2}\kappa^2(t) + \lambda\Big)v\,dt - \lambda L|_{w=0} = 0.$$

Then,

Lemma 4.1. $\frac{\partial}{\partial w}(\frac{1}{2}\kappa^2) = \langle(\nabla_T)^2 W, \nabla_T T\rangle - 2\langle\nabla_T W, T\rangle v\kappa^2 + \langle R(W,T)T, \nabla_T T\rangle$.

Proof. First a few auxiliary facts: $\frac{\partial}{\partial w}\frac{\partial}{\partial t}\Gamma = \frac{\partial}{\partial t}\frac{\partial}{\partial w}\Gamma$ implies that $[V,W] = 0$. Secondly, $v^2 = \langle V,V\rangle$ implies that $v\frac{\partial v}{\partial w} = \langle\frac{\partial V}{\partial w}, V\rangle = \langle\frac{\partial}{\partial t}W, V\rangle = \langle v\nabla_T W, V\rangle$. Hence, $\frac{\partial v}{\partial w} = \langle\nabla_T W, T\rangle v$. Furthermore, $[W,T] = [W, \frac{1}{v}V] = W(\frac{1}{v})V + \frac{1}{v}[W,V] = -\frac{\partial v}{\partial w}T$, and therefore, $\nabla_W T = \nabla_T W + [W,T] = \nabla_T W - \frac{\partial v}{\partial w}T$. So,

$$\begin{aligned}
\frac{\partial}{\partial w}\Big(\frac{1}{2}\kappa^2\Big) &= \langle\nabla_W\nabla_T T, \nabla_T T\rangle \\
&= \langle\nabla_T\nabla_W T, \nabla_T T\rangle + \langle R(W,T)T, \nabla_T T\rangle + \langle\nabla_{[W,T]}T, \nabla_T T\rangle \\
&= \Big\langle\nabla_T\Big(\nabla_T W - \frac{\partial v}{\partial w}T\Big), \nabla_T T\Big\rangle + \big\langle R(W,T)T, \nabla_T T\big\rangle + \Big\langle -\frac{\partial v}{\partial w}\nabla_T T, \nabla_T T\Big\rangle \\
&= \big\langle(\nabla_T)^2 W, \nabla_T T\big\rangle - 2\frac{\partial v}{\partial w}\kappa^2 + \big\langle R(W,T)T, \nabla_T T\big\rangle.
\end{aligned}$$

□

It follows that

$$\begin{aligned}
d\mathcal{F}|_{w=0} &= \frac{\partial}{\partial w}\int_0^1\Big(\frac{1}{2}\kappa^2(t) + \lambda\Big)v\,dt - \lambda L|_{w=0} = \int_0^1\Big(\frac{\partial}{\partial w}\Big(\frac{1}{2}\kappa^2\Big) + \Big(\frac{1}{2}\kappa^2 + \lambda\Big)\frac{\partial v}{\partial w}\Big)\,dt \\
&= \int_0^1 \Big(\langle(\nabla_T)^2 W, \nabla_T T\rangle - \frac{3}{2}\langle\nabla_T W, T\rangle\kappa^2 + \langle R(W,T)T, \nabla_T T\rangle + \lambda\langle\nabla_T W, T\rangle\Big)v\,dt \\
&= \int_0^L \Big(\langle(\nabla_T)^2 W, \nabla_T T\rangle - \frac{3}{2}\langle\nabla_T W, T\rangle\kappa^2 + \langle R(W,T)T, \nabla_T T\rangle + \lambda\langle\nabla_T W, T\rangle\Big)\,ds
\end{aligned}$$

After the appropriate integrations by parts, the boundary terms vanish and the preceding reduces to:

$$d\mathcal{F}|_{w=0} = \int_0^L \Big\langle W, (\nabla_T)^3 T + \nabla_T\Big(\frac{3}{2}\kappa^2 - \lambda\Big)T + R(\nabla_T T, T)T\Big\rangle ds = 0,$$

because $\langle R(W,T)T, \nabla_T T\rangle = \langle R(\nabla_T T, T)T, W\rangle$ by property (4) listed above. But then

$$\Big\langle W, (\nabla_T)^3 T + \nabla_T\Big(\frac{3}{2}\kappa^2 - \lambda\Big)T + R(\nabla_T T, T)T\Big\rangle = 0,$$

provided that the class of variations is sufficiently large. (This part os glossed over in the existing literature). □

Proposition 4.2. *Let M be a space form and suppose that there is a well defined Serret–Frenet frame $v_1, \ldots, v_n$ along an elastic curve γ. Let $\kappa_1 = \kappa$, $\kappa_2 = \tau$, $\kappa_3, \ldots, \kappa_{n-1}$ denote the associated curvatures defined by $\frac{d\gamma}{dt} = v_1(t)$, $\nabla_{v_1} v_1 = \kappa_1 v_2$, $\nabla_{v_1} v_2 = -\kappa_1 v_1 + \kappa_2 v_3$, $\nabla_{v_1} v_k = -\kappa_i v_{i-1} + \kappa_{i+1} v_{i+1}, \ldots,$ $\nabla_{v_1} v_n = -\kappa_{n-1} v_{n-1}$. The Euler–Lagrange equation is given by*

$$\Big(\kappa_{ss} + \frac{1}{2}\kappa^3 - \kappa\tau^2 + (\epsilon - \lambda)\kappa\Big)v_2 + \Big(\kappa_s\tau + \frac{d}{ds}(\kappa\tau)\Big)v_3 + \kappa\tau\kappa_3 v_4 = 0, \tag{34}$$

Proof. It follows that

$$\begin{aligned}
\nabla_T T &= \kappa v_2,\\
(\nabla_T)^2 T &= \kappa_s v_2 + \kappa \nabla_T v_2 = -\kappa_1^2 v_1 + \kappa_s v_2 + \kappa\tau v_3,\\
(\nabla_T)^3 T &= -3\kappa\kappa_s v_1 + (\kappa_{ss} - \kappa\tau^2 - \kappa^3)v_2 + \Big(\kappa_s\tau + \frac{d}{ds}(\kappa\tau)\Big)v_3 + \kappa\tau\kappa_3 v_4,
\end{aligned}$$

On space forms, $R(\nabla_T T, T)T = \epsilon \nabla_T T$, where $\epsilon = \pm 1, 0$ is the curvature of the underlying space. Therefore, equation (33) becomes

$$\Big(\kappa_{ss} + \frac{1}{2}\kappa^3 - \kappa\tau^2 + (\epsilon - \lambda)\kappa\Big)v_2 + \Big(\kappa_s\tau + \frac{d}{ds}(\kappa\tau)\Big)v_3 + \kappa\tau\kappa_3 v_4 = 0. \qquad \square$$

Corollary 4.1. *$\kappa^2(s)\tau(s) = c$ and $\kappa_i = 0$ for $i = 3, \ldots, (n-1)$. It $\xi = \kappa^2$, then*

$$\frac{d\xi}{ds}^2 + \xi^3 + 4(\epsilon - \lambda)\xi^2 - 4c_1\xi + 4c^2 = 0, \tag{35}$$

for some constant c_1

Proof. It follows from above that

$$\kappa_{ss} + \frac{1}{2}\kappa^3 - \kappa\tau^2 + (\epsilon - \lambda)\kappa = 0, \quad \kappa_s\tau + \frac{d}{ds}(\kappa\tau) = 0, \quad \kappa\tau\kappa_3 = 0.$$

Therefore, $0 = \kappa(\kappa_s\tau + \frac{d}{ds}(\kappa\tau)) = \frac{d}{ds}(\kappa^2\tau)$, hence $\kappa^2\tau = c$, c a constant and $\kappa_3 = 0$.

The first equation after a multiplication by $2\kappa_s$ and the substitution $\kappa\tau^2 = \frac{c^2}{\kappa^3}$ becomes

$$\frac{d}{ds}(\kappa_s^2) + 2\kappa^3\kappa_s - 2\frac{c^2}{\kappa^3}\kappa_s + 2(\epsilon - \lambda)\kappa\kappa_s = 0,$$

and can be written in integrated form as

$$\kappa_s^2 + \frac{1}{4}\kappa^4 + \frac{c^2}{\kappa^2} + (\epsilon - \lambda)\kappa^2 = c_1. \tag{36}$$

Equation (36), upon a multiplication by $4\kappa^2$ leads to (35) after the substitution $\xi = \kappa^2$. $\square$

Equation (35) is integrable in terms of elliptic functions. Then the torsion $\tau(s)$ is determined through $\xi\tau = c$. Since all the higher curvatures are zero, each elastic curve resides on a three dimensional manifold defined by the Serret–Frenet triad $T(s), N(s), B(s)$. This three dimensional system is integrable by quadratures, as demonstrated in [29] and [22].

4.3. Elastic curves – Hamiltonian view. Let use now return to the space forms $\mathbb{S}^n_\epsilon = G_\epsilon/K$ and the Hamiltonian H in (32). When $Q = I$ this Hamiltonian reduces to $H = \frac{1}{2}\langle L_{\mathfrak{k}}, L_{\mathfrak{k}}\rangle + \langle L_{\mathfrak{p}}, B\rangle$ and the associated Hamiltonian equations (AffHam) reduce to

$$\frac{dg}{dt} = g(B + L_{\mathfrak{k}}), \quad \frac{dL_{\mathfrak{k}}}{dt} = [B, L_{\mathfrak{p}}], \quad \frac{dL_{\mathfrak{p}}}{dt} = [L_{\mathfrak{k}}, L_{\mathfrak{p}}] + [B, L_{\mathfrak{k}}]. \tag{37}$$

We will now identify two types of functions, called *integrals of motion*, which are constant along the solutions of (37). Equations

$$\frac{dL_{\mathfrak{k}}}{dt} = [B, L_{\mathfrak{p}}], \quad \frac{dL_{\mathfrak{p}}}{dt} = [L_{\mathfrak{k}}, L_{\mathfrak{p}}] + [B, L_{\mathfrak{k}}] \tag{38}$$

can be written as

$$\frac{L(\lambda)}{dt} = [M(\lambda), L(\lambda)], \tag{39}$$

where $M(\lambda) = \frac{1}{\lambda}(L_{\mathfrak{p}} - B)$ and $L(\lambda) = L_{\mathfrak{p}} - \lambda L_{\mathfrak{k}} + (\lambda^2 - 1)B$, and therefore, the spectral invariants of $L(\lambda)$ are integrals of motion for (38) and (39). The fact is easily verified by noticing that $g(t)L(\lambda)g^{-1}(t)$ is constant for $g(t)$ a solution of $\frac{dg}{dt}(t) = gM(\lambda)$. These integrals of motion are called *isospectral.*

To identify the second type of integrals, let $\mathfrak{k}_B = \{X \in \mathfrak{k} : [X, B] = 0\}$ and let $\mathfrak{k}_B^\perp$ denote its orthogonal complement in $\mathfrak{k}$ relative to the trace form. It is easy to see that $\mathfrak{k}_B$ is a Lie subalgebra of $\mathfrak{k}$ and that $[B, \mathfrak{p}_\epsilon] \in \mathfrak{k}_B^\perp$ since $\langle \mathfrak{k}_B, [B, \mathfrak{p}_\epsilon]\rangle = \langle [\mathfrak{k}_B, B], \mathfrak{p}_\epsilon\rangle = 0$. This means that the projection $L_{\mathfrak{k}_B}$ of $L_{\mathfrak{k}}$ on $\mathfrak{k}_B$ is constant along the solutions of (37).

For the rest of this section we will be interested in the case that $L_{\mathfrak{k}_{B=0}}$.

Proposition 4.3. *Suppose that the drift matrix B in* (40) *satisfies $\langle B, B\rangle = \epsilon$. Then the projections of solutions in* (37) *on $\mathbb{S}^n_\epsilon$ that correspond to $L_{\mathfrak{k}_B} = 0$ are elastic. Conversely, every elastic curve in $\mathbb{S}^n_\epsilon$ is the projection of such a solution.*

It will be convenient to adopt the language of principal fiber bundles and regard G_ϵ as the principal bundle over $\mathbb{S}^n_\epsilon$ with K as the structure group. In the terminology of principal bundles, the left invariant distribution $\mathcal{P}_\epsilon(g) = \{g\Lambda : \Lambda \in \mathfrak{p}_\epsilon\}$ is called a connection over $\mathbb{S}^n_\epsilon$ and the integral curves of $\mathcal{P}_\epsilon$ are called horizontal [26]. If $g(t)$ is a horizontal curve so is $\tilde{g}(t) = g(t)h$ for any h in the isotropy group K. Moreover, both $g(t)$ and $\tilde{g}(t)$ project onto the same curve x in the base manifold $\mathbb{S}^n_\epsilon$. Conversely, every parametrized curve $x(t)$ in $\mathbb{S}^n_\epsilon$ is the projection of a horizontal curve $g(t)$ in G_ϵ.

The Riemannian length of the tangent curve $\frac{dx}{dt}$ is given by

$$\left\|\frac{dx}{dt}\right\|^2 = \epsilon\left\langle \frac{dx}{dt}, \frac{dx}{dt}\right\rangle_\epsilon = \sum_{i=1}^n \frac{dx_i}{dt}^2 + \epsilon\frac{dx_0}{dt}^2.$$

If $g(t)$ is a horizontal curve that projects onto $x(t)$, i.e., $x(t) = g(t)e_0$, then $\frac{dx}{dt} = \frac{dg}{dt}e_0 = g(t)\Lambda(t)e_0$, where $\Lambda(t) = a \wedge_\epsilon e_0 = \begin{pmatrix} 0 & -\epsilon a^T \\ a & 0 \end{pmatrix}$. Hence, $\|\frac{dx}{dt}\|^2 = \sum_i^n a_i^2(t) = \epsilon\langle \Lambda(t), \Lambda(t)\rangle = -\epsilon\frac{1}{2}\operatorname{Tr}(\Lambda^2)$. Thus $\frac{dx}{dt}$ is a unit tangent vector if and only if $\langle \Lambda, \Lambda\rangle = \epsilon$.

Suppose now that $v(t)$ is a curve of tangent vectors along a curve $x(t)$ in $\mathbb{S}^n_\epsilon$. Let $g(t)$ be any horizontal curve that projects onto $x(t)$, i.e., $x(t) = g(t)e_0$. Then there exists a curve of matrices $V(t)$ in $\mathfrak{p}$ such that $v(t) = g(t)V(t)e_0$. Then the covariant derivative $\frac{D_x}{dt}v(t)$ of $v(t)$ along $x(t)$ is given by

$$\frac{D_x}{dt}v(t) = g(t)\frac{dV}{dt}e_0. \tag{40}$$

In particular, if $\frac{dx}{dt} = g(t)\Lambda(t)e_0$ for a horizontal curve $g(t)$ with $\Lambda(t) = \frac{dg}{dt}g^{-1}$, then $\frac{D_x}{dt}(\frac{dx}{dt}) = g(t)\frac{d\Lambda}{dt}e_0$. If furthermore, $\langle\Lambda,\Lambda\rangle = \epsilon$, then $\epsilon\langle\frac{d\Lambda}{dt},\frac{d\Lambda}{dt}\rangle$ is the curvature of $x(t)$.

With this terminology at our disposal we come to the proof of the proposition.

Proof. Let $\bar{g}(t)$ be any solution of equations (37) and let $x(t) = \bar{g}(t)e_0$ be the projection on $\mathbb{S}^n_\epsilon$. Let $h(t)$ denote the curve in K that is the solution of $\frac{dh}{dt} = h(t)L_\mathfrak{k}$, $h(0) = I$. Then $g(t) = \bar{g}(t)h(t)$ is a horizontal curve with $h(t)Bh^{-1}(t) = \Lambda(t) = \frac{dg}{dt}g^{-1}(t)$ that projects onto the same curve $x(t)$. Since $\langle\Lambda,\Lambda\rangle = \langle B,B\rangle = \epsilon$, $\|\frac{dx}{dt}\| = 1$ and therefore, the curvature $\kappa(t)$ of $x(t)$ satisfies $\kappa^2 = \epsilon\langle\frac{d\Lambda}{dt},\frac{d\Lambda}{dt}\rangle$.

It follows that $\frac{d\Lambda}{dt} = h[B, L_\mathfrak{k}]h^{-1} = [\Lambda, Q]$ with $Q = hL_\mathfrak{k}h^{-1}$. Further differentiations show that $\frac{dQ}{dt} = [\Lambda, P]$ with $P = hL_\mathfrak{p}h^{-1}$, and $\frac{dP}{dt} = [\Lambda, Q]$. Hence P and Λ differ by a constant matrix in $\mathfrak{p}_\epsilon$. These equations will be referred to as the associated equations of (37).

Recall now the earlier notations $a \wedge_\epsilon b$ for the wedge product relative to the quadratic form $(\,,)_\epsilon$. In this notation, B is equal $b \wedge_\epsilon e_0$ for a vector $b = \sum_{i=1}^n b_i e_i$. The following formulas will be useful for some of the computations below.

$$[a \wedge_\epsilon b, c \wedge_\epsilon d] = \langle b,c\rangle_\epsilon a \wedge_\epsilon d + \langle c,a\rangle_\epsilon d \wedge_\epsilon b + \langle b,d\rangle_\epsilon c \wedge_\epsilon a + \langle a,d\rangle_\epsilon b \wedge_\epsilon c \tag{41a}$$

$$h(a \wedge_\epsilon b)h^{-1} = ha \wedge_\epsilon hb. \tag{41b}$$

We will also make use of the following lemma whose proof will be left to the reader:

Lemma 4.2. *$L_\mathfrak{k} = l \wedge_\epsilon b$ for some vector $l = \sum_{i=1}^n l_i e_i$ orthogonal to b when $L_{\mathfrak{k}_B} = 0$.*

Then $\Lambda = hBh^{-1} = h(b \wedge_\epsilon e_0)h^{-1} = \lambda \wedge_\epsilon e_0$, where $\lambda = hb$, and $Q = h(l \wedge_\epsilon b)h^{-1} = hl \wedge_\epsilon hb = q \wedge_\epsilon \lambda$ with $q = hl$. Since h is orthogonal, q and λ are orthogonal. It follows that $[\Lambda, Q] = [\lambda \wedge_\epsilon e_0, q \wedge_\epsilon \lambda] = (\lambda,\lambda)_\epsilon e_0 \wedge_\epsilon q = \epsilon(e_0 \wedge_\epsilon q)$.

Therefore,

$$\kappa^2 = \epsilon\langle[\Lambda,Q],[\Lambda,Q]\rangle = \epsilon\langle e_0 \wedge_\epsilon q, e_0 \wedge_\epsilon q\rangle = \|q\|^2 = \|Q\|^2.$$

Since $\|Q\|^2 = \|L_\mathfrak{k}\|^2$ the Hamiltonian H in (32) can be also expressed as

$$H = \frac{1}{2}\kappa^2 + \langle B, L_\mathfrak{p}\rangle = \frac{1}{2}\kappa^2 + \langle\Lambda, P\rangle. \tag{42}$$

We will show next that Λ is the lifted version of the Euler–Lagrange equation (33). Easy calculations show that $[[\Lambda,Q],Q] = -\|Q\|^2\Lambda$ and $[\Lambda,[\Lambda,P]] = \langle\Lambda,P\rangle\Lambda - \epsilon P$.

Hence,

$$\frac{d^2\Lambda}{dt^2} = [[\Lambda, Q], Q] + [\Lambda, [\Lambda, P]] = (\langle \Lambda, P\rangle - \|Q\|^2)\Lambda - \epsilon P = \Big(\frac{3}{2}\kappa^2 + H\Big)\Lambda - \epsilon P.$$

But then,

$$\frac{d^3\Lambda}{dt^3} = \frac{d}{dt}\Big(\frac{3}{2}\kappa^2 + H\Big)\Lambda - \epsilon\frac{d\Lambda}{dt}$$

since $\frac{dP}{dt} = \frac{d\Lambda}{dt}$. It follows that the projected tangent vector $T(t) = g(t)\Lambda(t)e_0$ satisfies the Euler–Lagrange equation (33) with $R(\nabla_T T)T = \epsilon\nabla_T T$. □

Let us now recover the contents of the corollary to Proposition 4.2 in a "Hamiltonian" way. Since the spectrum of $L_\mu = L_{\mathfrak{p}} - \mu L_{\mathfrak{k}} + (\mu^2 - 1)B$ and that of $hL_\mu h^{-1}$ are the same, it follows that the spectral invariants of $L_\mu = P - \mu Q + (\mu^2 - 1)\Lambda$ are constants of motion for the associated system.

The characteristic polynomial of the above matrix is given by $s^4 + c_1 s^2 + c_2 = 0$, with $c_1 = \epsilon(\mu^2 - 1)H + \epsilon\langle q, q\rangle_\epsilon + \langle p, p\rangle_\epsilon$, $c_2 = \mu^2(\|q\|^2\|p\|^2) - \langle q, q\rangle_\epsilon\langle \lambda, p\rangle_\epsilon^2 - \langle q, p\rangle_\epsilon^2$.

The above shows that in addition to the Hamiltonian H there are two other integrals of motion $I_1 = \epsilon\langle q, q\rangle_\epsilon + \langle p, p\rangle_\epsilon$ and $I_2 = \|q\|^2\|p\|^2 - (q, q)_\epsilon\langle \lambda, p\rangle_\epsilon^2 - \langle q, p\rangle_\epsilon^2$. These integrals of motion can be expressed more suscinctly as

$$I_1 = \|P\|^2 + \epsilon\|Q\|^2,\ \ I_2 = \|P\|^2\|Q\|^2 - \|[P, Q]\|^2,$$

where all the norms are induced by the quadratic form $\langle\, ,\rangle_\epsilon$ defined by

$$\langle X_1 + Y_1, X_2 + Y_2\rangle_\epsilon = \epsilon\langle X_1, X_2\rangle + \langle Y_1, Y_2\rangle \text{ for any } X_i \in \mathfrak{p}_\epsilon,\ Y_i \in \mathfrak{k},\ i = 1, 2.$$

The first integral I_1, a universal integral also known as a Casimir, is an integral of motion for any left invariant Hamiltonian on $\mathfrak{g}_\epsilon$. However, I_2 is a particular integral of motion which, together with the Hamiltonian H and I_1 is sufficient to recover equation (35). The proof is as follows:

Proof. Since $\kappa^2 = \|Q\|^2$, $\frac{d\kappa^2}{dt} = 2\langle Q, \frac{dQ}{dt}\rangle = 2\langle Q, [\Lambda, P]\rangle = -2\epsilon\langle q, p\rangle_\epsilon$. Hence,

$$\begin{aligned}\frac{d\xi}{dt}^2 &= \Big(\frac{d\kappa^2}{dt}\Big)^2 = 4(p, q)_\epsilon^2 = 4\big(-I_2 + \|P\|^2\|Q\|^2 - \|Q\|^2\langle \Lambda, P\rangle^2\big)\\ &= 4\Big(-I_2 + \|Q\|^2(I_1 - \epsilon\|Q\|^2) - \|Q\|^2\Big(H - \frac{1}{2}\|Q\|^2\Big)^2\Big)\\ &= -4I_2 + 4\|Q\|^2 I_1 - 4\epsilon\|Q\|^4 - 4\|Q\|^2\Big(H^2 - H\|Q\|^2 + \frac{1}{4}\|Q\|^2\Big)\\ &= -\xi^3 + 4(H - \epsilon)\xi^2 + 4(I_1 - H^2)\xi - 4I_2.\end{aligned}$$

□

The equation

$$\frac{d\xi}{dt}^2 + \xi^3 + 4(\epsilon - H)\xi^2 - 4(I_1 - H^2)\xi + 4I_0 = 0, \tag{43}$$

is the same as the corresponding equation (38) with $\lambda = H$, $c_1 = I_1 - H^2$, and $c^2 = I_2$. We leave it to the reader to show that $(\kappa^2\tau)^2 = I_2$.

Remark 4.1. The Euler–Griffiths problem could have been attacked more directly as the problem of minimizing $\frac{1}{2}\int_0^T \|U(t)\|_\epsilon^2\,dt$ over the solutions $g(t)$ of auto-parallel system $\frac{dg}{dt}(t) = g(t)(E_\epsilon + U(t))$ that originate in $S_0 = \{g : ge_0 = x_0,\, ge_1 = v_0\}$ at $t = 0$ and terminate on $S_1 = \{g : ge_0 = x_1\, ge_1 = v_1\}$ as done in [22] and JuPM. The presentation in this paper makes a better link between the elastic problem and the mechanical systems as we will see in the following section.

5. The Kinetic Analogue

Consider now the affine Hamiltonian $H = \frac{1}{2}\langle L_{\mathfrak{k}}, L_{\mathfrak{k}}\rangle + \langle L_{\mathfrak{p}}, B\rangle$ on six dimensional Lie groups G_ϵ, $\mathrm{SO}_4(R)$, $\mathrm{SO}(1,3)$ and the semidirect product $SE_3 = \mathbb{R}^3 \rtimes \mathrm{SO}_3$. These groups are both the isometry groups of and the orthonormal frame bundles (positively oriented) of the three dimensional space forms $S^3_\epsilon, \epsilon = \pm 1$ and $\mathbb{E}^3$, and hence the affine Hamiltonian H could be considered as the elastic energy for the equilibrium configurations of the "generalized" elastic problem of Kirchhoff, in the sense that the center line $x(t)$ of the rod is adapted to the frame deformations of the rod via a general relation

$$\frac{dx}{dt}(t) = b_1 v_1(t) + b_2 v_2(t) + b_3 v_3(t),$$

rather than just $\frac{dx}{dt} = v_1$.

As before, $\langle A_1+B_1, A_2+B_2\rangle_\epsilon = \langle A_1, A_2\rangle + \epsilon\langle B_1, B_2\rangle$ for any matrices $A_i \in \mathfrak{k}$ and $B_i \in \mathfrak{p}_\epsilon$, $i = 1,2$, where $\langle\,,\rangle$ denotes the trace form on $\mathfrak{g}_\epsilon$. Then, $\{B_i = e_i \wedge_\epsilon e_0,\ i = 1,2,3\}$ is an orthonormal basis for $\mathfrak{p}_\epsilon$ and $A_i = e_3 \wedge e_2$, $A_3 = e_1 \wedge e_3$, $A_3 = e_2 \wedge e_1$ is an orthonormal basis for $\mathfrak{k}$ relative to $\langle\,,\rangle_\epsilon$. To make the connection with the equations of the heavy top it will be helpful to express the Hamiltonian equations (AffHam) in terms of the coordinates defined by the above basis. Then each matrix

$$L = \begin{pmatrix} 0 & -\epsilon p_1 & -\epsilon p_2 & -\epsilon p_3 \\ p_1 & 0 & -m_3 & m_2 \\ p_2 & m_3 & 0 & -m_1 \\ p_3 & -m_2 & m_1 & 0 \end{pmatrix}$$

in $\mathfrak{g}_\epsilon$ will be represented by the pair of vectors $(\hat{M}, \hat{P})$, where $\hat{M} = (m_1, m_2, m_3)^T$ and $\hat{P} = (p_1, p_2, p_3)^T$.

The basis elements satisfy the following Lie bracket table:

TABLE 1. $\epsilon = \pm 1$, $s = 0, 1$.

[ ]	A_1	A_2	A_3	B_1	B_2	B_3
A_1	0	$-A_3$	A_2	0	$-B_3$	B_2
A_2	A_3	0	$-A_1$	B_3	0	$-B_1$
A_3	$-A_2$	A_1	0	$-B_2$	B_1	0
B_1	0	$-B_3$	B_2	0	$-s\epsilon A_3$	$s\epsilon A_2$
B_2	B_3	0	$-B_1$	$s\epsilon A_3$	0	$-s\epsilon A_1$
B_3	$-B_2$	B_1	0	$-s\epsilon A_2$	$s\epsilon A_1$	0

It follows that the coordinate vector $(\hat{M},\hat{P})$ of the Lie bracket $[L_1,L_2]$ is given by

$$\hat{M}=\hat{M}_2\times\hat{P}_1+\epsilon(\hat{P}_2\times\hat{P}_1),\quad \hat{P}=\hat{P}_2\times\hat{M}_1+\hat{M}_2\times\hat{P}_1,$$

where $\times$ denotes the vector product in $\mathbb{R}^3$ and where $(\hat{M}_1,\hat{P}_1)$ and $(\hat{M}_2,\hat{P}_2)$ are the coordinate vectors of L_1 and L_2.

The matrix $L_\mathfrak{p}$ which figures in equations (AffHam) is of the form $L_\mathfrak{p}=\begin{pmatrix}0 & -p^T\\ \epsilon p & 0\end{pmatrix}$, hence its coordinate vector is given by $\epsilon\hat{P}$. Recall that $L_\mathfrak{p}$ corresponds to $\ell\in\mathfrak{p}^*$ via the trace form rather than via $\langle\,,\rangle_\epsilon$. In what follows $Q^{-1}L_\mathfrak{k}$ will be assumed of the form $Q_{-1}L_\mathfrak{k} == \sum_{i=1}^3\frac{1}{\lambda_i}m_iA_i$ with $\lambda_1,\lambda_2,\lambda_3$ arbitrary positive numbers. Than $\hat{\Omega}=\left(\frac{1}{\lambda_1}q_1,\frac{1}{\lambda_2}q_2,\frac{1}{\lambda_3}q_3\right)^T$ will denote its coordinate vector. In terms of the coordinates $(\hat{M},\epsilon\hat{P})$,

$$H=\frac{1}{2}\Big(\frac{1}{\lambda_1}m_1^2+\frac{1}{\lambda_2}m_2^2+\frac{1}{\lambda_3}m_3^2\Big)+b_1p_1+b_2p_2+b_3p_3, \tag{44}$$

and (AffHam) associated with H is given by:

$$\frac{d\hat{M}}{dt}=\hat{M}\times\hat{\Omega}+\hat{P}\times\hat{B},\quad \frac{d\hat{P}}{dt}=\hat{P}\times\hat{\Omega}+s\epsilon(\hat{M}\times\hat{B}),\quad \hat{B}=\begin{pmatrix}b_1\\ b_2\\ b_3\end{pmatrix}, \tag{45}$$

together with $\frac{dg}{dt}=g(t)(B+Q^{-1}L_\mathfrak{k})$.

On the semidirect product $\mathfrak{p}_\epsilon\rtimes\mathfrak{k}$ the above equations reduce to

$$\frac{d\hat{M}}{dt}=\hat{M}\times\hat{\Omega}+\hat{P}\times\hat{B},\quad \frac{d\hat{P}}{dt}=\hat{P}\times\hat{\Omega} \tag{46}$$

which formally looks the same as the equation of the heavy top with $\lambda_1,\lambda_2,\lambda_3$ playing the role of the principal moments of inertia of the body, and b_1,b_2,b_3 corresponding to the coordinates of the center of gravity relative to the fixed point of the body. This discovery forms the basis for the "Kinetic Analogue" of Kirchhoff in which he likened the equations of the elastic rod to the motions of the top.

As remarkable as Kirchhoff's observation was, it nevertheless,went in the wrong direction: it is the top that follows the equations of the elastic rod, rather than the other way around, since the equations of the top form an invariant subsystem of the elastic system. Indeed, this observation justifies the long standing ad-hoc practice of treating the equations of the top as the equations in the semidirect product. Let us digress briefly into the equations of the top in order to make this point more explicit.

5.1. The heavy top.

A rigid body in a three dimensional Euclidean space that is free to rotate around a fixed point under a constant gravitational force is known as a heavy top. Its equations of motion are expressed through a rotation matrix $R(t)$ in $\mathrm{SO}_3(R)$ that measures the movements of an orthonormal frame $a_1(t),a_2(t),a_3(t)$ centered at a fixed point O on the body relative to an absolute orthonormal frame e_1,e_2,e_3 in the ambient space $\mathbb{R}^3$. The absolute frame may be rotated so that the gravitational force is of the form $\vec{F}=-Ce_1$, with C the gravitational constant. Then $R(t)$ is the isometry that relates the moving frame to the absolute frame.

More precisely, if $q=(q_1,q_2,q_3)^T$ denotes the coordinate vector of a point $\vec{OP}$ in the absolute space, and if $Q=(Q_1,Q_2,Q_3)^T$ denotes the coordinate vector of the same point relative to the moving frame, then $q=RQ$.

With each movement of the body a point $q(t)$ undergoes velocity $\frac{dq}{dt}=\frac{dR}{dt}Q$. Since $R(t)$ is a curve in $\mathrm{SO}_3(R)$, $\frac{dR}{dt}=V(t)R(t)$ for some skew-symmetric matrix $V(t)$. The kinetic energy then is given by $\frac{1}{2}m\left\|\frac{dq}{dt}\right\|^2=\frac{1}{2}m\|V(t)R(t)Q\|^2$. The total kinetic energy of the body is the aggregate of the contributions due to point masses and can be expressed as a 3-dimensional integral in terms of the mass density ρ as follows:

$$T=\frac{1}{2}\int_{\text{Body}}\|V(t)R(t)Q\|^2\rho(Q)\,dQ=\frac{1}{2}\int_{\text{Body}}\|R^{-1}V(t)R(t)Q\|^2\rho(Q)\,dQ.$$

To simplify the preceding integral let $U(t)=R^{-1}(t)V(t)R(t)=R^{-1}\frac{dR}{dt}$. Matrix $U(t)$ is called the angular velocity of the body relative to the moving frame (in contrast to V which is the angular velocity relative to the absolute frame). This transformation of angular velocities transforms right-invariant paths $\frac{dr}{dt}=V(t)R(t)$ into left-invariant paths $\frac{dR}{dt}=R(t)U(t)$ with the associated kinetic energy given by $\frac{1}{2}\int_{\text{Body}}\|U(t)Q\|^2\rho(Q)\,dQ$.

The kinetic energy may be related to the positive definite left-invariant quadratic form $\langle\langle U_1,U_2\rangle\rangle$ in the Lie algebra $\mathrm{so}_3(\mathbb{R})$

$$\langle\langle U_1,U_2\rangle\rangle=\int_{\text{Body}}(U_1Q,U_2Q)\rho(Q)\,dQ,$$

defined by the shape of the body, where $(\,,)$ denotes the Euclidean product in $\mathbb{E}^3$. Then, $T(U)=\dfrac{1}{2}\langle\langle U,U\rangle\rangle=\frac{1}{2}\langle QU,U\rangle$, where Q is a positive definite linear operator on $\mathrm{so}_3(R)$ and $\langle\,,\rangle$ is the trace form on $\mathrm{so}_3(R)$. In the literature on mechanics, operator Q is called the inertia tensor and its eigenvalues $\lambda_1,\lambda_2,\lambda_3$ are called the principal moments of inertia [3].

To find the expression for the potential energy it is convenient to introduce the center of mass q defined by $q\int_{\text{Body}}\rho(Q)\,dQ=\int_{\text{Body}}Q\rho(Q)\,dQ$. Then the potential energy is V is given by

$$V=C\int_0^t\int_{\text{Body}}\Big(e_1,\frac{dR}{d\tau}Q\Big)\rho(Q)\,dQ\,d\tau=Cm(e_1,R(t)q),$$

where $m=\int_{\text{Body}}\rho(Q)\,dQ$ stands for the total mass of the body.

The Principle of Least Action states that actual motions minimize the action integral $\int_0^T(T-V)\,dt$. We will paraphrase the Principle of east Action as the optimal control problem of minimizing the action integral

$$\int_0^T(T-V)\,d\tau=\int_0^T\Big(\frac{1}{2}\langle QU(t),U(t)\rangle-Cm\left(R^{-1}(t)e_1,q\right)\Big)\,dt \tag{47}$$

over all trajectories $(R(t),U(t))$ of the left-invariant system $\frac{dR(t)}{dt}=R(t)\,U(t)$.

By choosing a basis A_1,A_2,A_3 in $\mathrm{so}_3(R)$ that is orthonormal relative to the trace form such that $\langle QA_i,A_j\rangle=\lambda_j\delta_{ij}$ each angular velocity $U(t)$ can be written as $U(t) = u_1(t)A_1 + u_2(t)A_2 + u_3(t)A_3$ and the preceding optimal problem can be rephrased as the problem of minimizing the integral

$$\int_0^T \Big(\frac{1}{2}\sum_{i=1}^3 \lambda_i u_i^2(t) + Cm(R^{-1}e_1, q)\Big)\, dt$$

over the trajectories of the left-invariant control system $\frac{dR}{dt}(t) = \sum_{i=1}^3 u_i(t)R(t)A_i$. Then

$$h_{u(t)}(\xi) = -\frac{1}{2}\sum_{i=1}^3 \lambda_i u_i^2(t) + Cm(R^{-1}e_1, q) + \sum_{i=1}^3 u_i \xi(RA_i), \xi \in T_R^* \operatorname{SO}_3(R)$$

is the corresponding Hamiltonian lift of (47). The Maximum Principle then identifies the maximal Hamiltonian

$$H(\xi) = \frac{1}{2}\sum_{i=1}^3 \frac{1}{\lambda_i} m_i^2(\xi) + Cm(R^{-1}e_1, q) \tag{48}$$

in the cotangent bundle $T^*(\operatorname{SO}_3(R))$ as the energy Hamiltonian that determines the motions of the top, in the sense that each motion of the top is the projection of an integral curve of the corresponding Hamiltonian vector field $\vec{H}$. In this notation each momentum m_i is equal to $m_i(\xi) = \xi(RA_i)$, $\xi \in T_R^*(\operatorname{SO}_3(R))$ and the extremal angular velocities are related to the momenta m_i via the relation $u_i = \frac{1}{\lambda_i} m_i$, $i = 1, 2, 3$.

In order to be able to make a connection with the elastic problem and the Kinetic Analogue of Kirchhoff trivialize the cotangent bundle $T^* \operatorname{SO}_3(R)$ from the left and consider it as the product $\operatorname{SO}_3(R) \times so_3^*(R)$. To facilitate this connection further, let K denote $\operatorname{SO}_3(R)$ and let $\mathfrak{k}$ to denote its Lie algebra $\operatorname{so}_3(R)$. In this realization of $T^* \operatorname{SO}_3(R)$ each momentum m_i becomes a linear function on $\mathfrak{k}$ and m_1, m_2, m_3 can be regarded as the coordinates of any $\ell \in \mathfrak{k}^*$ relative to the dual basis A_1^*, A_2^*, A_3^*. The extremal control $U = \sum_{i=1}^3 \frac{1}{\lambda_i} m_i A_i$ is recognized as a linear mapping from $\mathfrak{k}^*$ onto $\mathfrak{k}$.

Since H is not left-invariant the Hamiltonian equations are not given by (28), rather, they are given by:

$$\frac{dR}{dt}(t) = R(t)U(\ell(t)),\ \frac{d\ell}{dt}(X) = -Cm(R^{-1}e_1, Xq) - ad^*\Omega(\ell(t))(\ell(t)),\ X \in \mathfrak{k}, \tag{49}$$

(See [20] or [22] for additional details).

The linear function $X \to -Cm(R^{-1}e_1, Xq)$ is the torque-momentum due to the potential energy. This torque-momentum is zero precisely when the center of gravity coincides with the fixed point. In that case, the heavy top is no longer "heavy" and turns into "the top of Euler". Since $\operatorname{SO}_3(R)$ is simple, the Hamiltonian equations on the dual of the Lie algebra can be expressed on the Lie algebra via the trace form. Then each point $\ell \in \mathfrak{k}^*$ corresponds to the matrix $L_{\mathfrak{k}} = \begin{pmatrix} 0 & m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix}$, $U(\ell))$ corresponds to $Q^{-1}L_{\mathfrak{k}}$ and the linear function $X \to -Cm(R^{-1}e_1, Xq)$ corresponds to the matrix $(CmR^{-1}e_1) \wedge q$. Hence,

$$\frac{dR}{dt}(t) = R(t)Q^{-1}L_{\mathfrak{k}}, \quad \frac{dL_{\mathfrak{k}}}{dt}(t) = [Q^{-1}L_{\mathfrak{k}}, L_{\mathfrak{k}}(t)] + (CmR^{-1}e_1) \wedge q \tag{50}$$

are the corresponding Hamiltonian equations for the top. In the literature on mechanics, these equations are usually written in terms of the vector product as

$$\frac{d\hat{M}}{dt} = \hat{M}(t) \times \hat{\Omega}(t) + \hat{P}(t) \times q, \quad \frac{d\hat{P}}{dt}(t) = \hat{P}(t) \times \hat{\Omega}(t), \quad \hat{\Omega}(t) = \begin{pmatrix} \frac{1}{\lambda_1}m_1 \\ \frac{1}{\lambda_2}m_2 \\ \frac{1}{\lambda_3}m_3 \end{pmatrix} \tag{51}$$

and $\hat{P} = CmR^{-1}e_1$.

Hamiltonian equations (50) can be extended to the semidirect product $\mathbb{E}^3 \rtimes \mathrm{SO}_3(R)$ by identifying the center of mass q with $\hat{B}$. Then the curve $x(t) = \int_0^t R(\tau)Q_0 d\tau$ can be be interpreted as the central line of the rod, which together with the rotation matrix R can then be embedded in a curve $g(t) = \begin{pmatrix} 1 & 0 \\ x(r) & R(t) \end{pmatrix}$ in $\mathbb{E}^3 \rtimes \mathrm{SO}_3(R)$. It is easy to check that $g(t)$ satisfies $\frac{dg}{dt} = g(t)(B + Q^{-1}L_{\mathfrak{k}})$ which, in turn, implies that equations (51) can be interpreted as the Hamiltonian system of Kirchhoff's elastic rod.

The passage from the top to the elastic problem of KIrchhoff, although formally correct, seems in many ways mysterious. To begin with, the Hamiltonian for the top is neither left nor right invariant on $T^*S_3(R)$, but its extension to the the semidirect product $\mathbb{R}^3 \rtimes \mathrm{SO}_3(R)$ is left-invariant. So why should the left-invariant symmetries of the elastic problem be relevant for the equations of the top? Further, why is $\hat{P}(t)$ significant for the purposes of integration of the top since it is not a dependent variable in its configuration space?

Nevertheless, the connection with the elastic problem of Kirchhoff illuminates much of the theory of the top. The passage to the semidirect product gives credence to the long standing tradition in the literature of the top to regard equations (51) as a system of equations in six variables $m_1, m_2, m_3, p_1, p_2, p_3$ with six parameters $q_1, q_2, q_3, \lambda_1, \lambda_2, \lambda_3$. It will be shown below that the Lie algebraic symmetries of the elastic problem clarify much of the integrability theory of the top.

5.2. Symmetry, Coadjoint orbits and Integrals of motion. A Hamiltonian function H on a symplectic manifold M of dimension $2n$ is said to be *integrable* (or completely integrable) if there exist functions $\varphi_2, \ldots, \varphi_n$ on M that together with $\varphi_1 = H$ satisfy the following two properties:

(i) $\varphi_1, \ldots, \varphi_n$ are functionally independent. Functional independence is understood in the local sense, that is, that the differentials $d\varphi_1, \ldots, d\varphi_n$ are linearly independent for an open (often dense) subset of M.

(ii)The functions $\varphi_1, \ldots, \varphi_n$ are in involution, that is, they Poisson commute among each other.

Recall that the Poisson bracket $\{\varphi, \psi\}$ is a function defined by the symplectic form ω on M through the following equivalent conditions:

$$\{\varphi, \psi\}(x) = \omega_x\big(\vec{\varphi}(x), \vec{\psi}(x)\big) = \frac{d}{dt}\big(\varphi \cdot \exp t\vec{\psi}\big)(x)\big|_{t=0}$$

with $\vec{\varphi}$ and $\vec{\psi}$ denoting the Hamiltonian vector fields induced by the functions φ and ψ.

In the literature on mechanics, a function φ is called an integral of motion for the Hamiltonian H if $\{\varphi, H\} = 0$, that is, if φ is constant along the flow of $\vec{H}$. In this terminology then functions $\varphi_1, \dots, \varphi_n$, defined by (ii) above are integrals of motion for each other. The maximal number of functions $\varphi_1, \dots, \varphi_m$ that are functionally independent and in involution is equal to $\frac{1}{2}\dim M$.

If H is an integrable system, then each level set $\{x : \varphi_1(x) = c_1, \dots, \varphi_n(x) = c_n\}$ is an n-dimensional submanifold of M. Such submanifolds are called Lagrangian. They are submanifolds of M of maximal dimension on which the symplectic form vanishes. The connected component through each point x_0 of each level set $\{x : \varphi_i(x) = c_i\ i \leqslant n\}$ is equal to the orbit through x_0 of the commutative family of Hamiltonian vector fields $\{\vec{\varphi}_1, \dots, \vec{\varphi}_n\}$.

In the case that ω is the canonical symplectic form on the cotangent bundle of a manifold M then a vector field X on M is said to be a symmetry for a Hamiltonian H if the Hamiltonian lift h_X of X satisfies $\{H, h_X\} = 0$. The Hamiltonian lifts h_X and h_Y of any vector fields X and Y on M conform to the following formula: $\{h_X, h_Y\} = h_{[X,Y]}$. Hence, any commuting vector fields on M are symmetries for their Hamiltonian lifts.

On Lie groups the Lie bracket of a right and a left invariant vector field is zero since their flows commute. Therefore, their Poisson brackets in the cotangent bundle are in involution. It is a consequence of this symmetry that in the left tirivialization of the cotangent bundle of G the left-invariant Hamiltonian systems evolve on coadjoint orbits of G, while in the right trivialization, right-invariant Hamiltonian systems evolve on coadjoint orbits of G. Below are the relevant details.

In the left-trivialization Hamiltonian lifts of left-invariant vector fields are linear functions on $\mathfrak{g}^*$ while the Hamiltonian lifts h_X of the right-invariant vector fields $X(g) = Ag$ are given by $h_X(\ell, g)) = \ell(g^{-1}Ag)$. If $A_1, \dots, A_n$ is any basis in $\mathfrak{g}$, then the Hamiltonian lifts $h_1, \dots, h_n$ of the corresponding left-invariant vector fields are given by $h_i(\ell) = \ell(A_i)$, $i = 1, \dots, n$ and can be regarded as the coordinates in $\mathfrak{g}^*$ relative to the dual basis $A_1^*, \dots, A_n^*$. If h_X is any Hamiltonian lift of a right invariant vector field, then $\{h_i, h_X\} = 0$ by the remarks above. It follows that any left invariant Hamiltonian H can be regarded as a function of $h_1, \dots, h_n$, in which case

$$\{H, h_X\} = \sum_{i=1}^{n} \frac{\partial H}{\partial h_i} \{h_i, h_X\} = 0,$$

and hence $h_X(g(t), \ell(t)) = \ell(t)(g(t)Ag^{-1}(t))$ is constant along the motions $(g(t), \ell(t))$ of $\vec{H}$. Since, $\ell(g^{-1}Ag) = \ell \circ \mathrm{Ad}_g(A) = (\mathrm{Ad}_g^*(\ell))(A)$, it follows that $Ad^*g(t)(\ell(t)) = Ad^*g(0)(\ell(0))$. This implies that $\ell(t)$ evolves on the coadjoint orbit $\{Ad^*g(\ell(0)) : g \in G\}$.

We finally recall one more fact that each coadjoint orbit is symplectic with its symplectic form induced by the Poisson structure of $\mathfrak{g}$ [3].

Return now to the left-invariant Hamiltonian systems (AffHam). On six dimensional Lie algebras $\mathfrak{g}_\epsilon$ equations (46) can be represented on $\mathrm{so}_3(R) \times \mathrm{so}_3(R)$

as

$$\frac{dM}{dt} = [\Omega, M] + [B, P], \quad \frac{dP}{dt} = [\Omega, P] + s\epsilon[B, M], \tag{52}$$

under the isomorphism

$$\hat{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Longleftrightarrow X = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \text{ and } [Y,\hat{} X] = \hat{X} \times \hat{Y}.$$

Moreover, the Euclidean inner product $(\hat{X}, \hat{Y})$ is the same as the trace product $\langle X, Y \rangle = -\frac{1}{2}\operatorname{Tr}(XY)$.

Let us now address the integrability properties of (52):

1.Casimirs: $I_1 = \|P\|^2 + s\epsilon\|M\|^2$ and $I_2 = \langle M, P\rangle = \|M\|^2\|P\|^2 - \|[M,P]\|^2$ are integrals of motion as can be easily checked. We have already remarked in the previous section, that they are integrals of motion for any left invariant Hamiltonian on $\mathfrak{g}_\epsilon$.

2. Right invariant Hamiltonians. A Cartan algebra $\mathfrak{h}$ is a maximal commutative sub algebra of a Lie algebra $\mathfrak{g}$. It is known that all Cartan sub-algebras of a semi-simple Lie algebra are conjugate, and hence all have the same dimension. The dimension of any Cartan algebra is called the rank of $\mathfrak{g}$. The rank of $\mathfrak{g}_\epsilon$ is two as can be easily verified from Table 1. Hence there two right invariant Hamiltonians h_1 and h_2 that Poisson commute with each other and also Poisson commute with any left invariant Hamiltonians. In particular, they Poisson commute with I_1 and I_2. On the semidirect product the situation is slightly different since $\mathfrak{p}$ is a commutative algebra, hence the rank of the semidirect product $\mathbb{R}^3 \rtimes \mathrm{so}_3(R)$ is three. However, their Hamiltonians are functionally dependent and generate only two functionally independent Hamiltonians. So in all cases there are two functionally independent integrals of motion in involution with each other.

3. The Hamiltonian H itself is also a constant of motion. So there are always five functionally independent integrals of motion all in involution with each other. The maximal number of such integrals of motion is six, since the cotangent bundle of SO_ϵ is 12 dimensional. Thus H is completely integrable whenever there is another left-invariant integral of motion (it is automatically in involution with all others).

4. On these three Lie algebras, generic coadjoint orbits are 4 dimensional. Hence, complete integrability on on T^*G is equivalent on complete integrability on coadjoint orbits (the Casimirs are constant on coadjoint orbits).

Definition 5.1. System (51) is said to be algebraically integrable if the solutions of its complexified system are meromorphic functions of complex time.

This definition has its origins in Kowalewska's famous paper of 1889 in which she discovers an extra integral of motion for the top by classifying the solutions of the complexified equations of the top which are meromorphic functions of complex time. Remarkably, this procedure yields an extra integral of motion for the problem of Kirchhoff, much in the same way as it did in the original paper of Kowalewska.

Proposition 5.1. *The following are the only algebraically integrable cases:*

1. $B = 0$. Euler's case. *Both* $\|M\|^2$ *and* $\|P\|^2$ *are constant. The solutions are the intersections of the energy ellipsoid* $H = \frac{1}{2}(\frac{1}{\lambda_1}m_1 + \frac{1}{\lambda_2}m_2 + \frac{1}{\lambda_3}m_3)$ *and the momentum sphere* $\|M\|^2 = m_1^2 + m_2^2 + m_3^2$. *These solutions coincide with the solutions for the top of Euler on the surface* $P = 0$.
2. Spherical pendulum. $B \neq 0$, $\lambda_1 = \lambda_2 = \lambda_3$. *Then* $\frac{dM}{dt} = [B, P]$ *as in the Euler–Griffiths case. Then* $\mathfrak{k}_B = \{M : [B, M] = 0\}$ *is one dimensional and the projection of* M *on* $\mathfrak{k}_B$ *is constant. If this constant is zero we are in the Euler–Griffiths case. Otherwise, the equations coincide with the equations of the spherical pendulum.*
3. Lagrange's case. $\lambda_2 = \lambda_3$, $b_2 = b_3 = 0$. *Then* $I_3 = m_1$ *is an integral of motion. The corresponding top is known as the top of Lagrange.*
4. Kowalewski's case. $\lambda_1 = \lambda_2 = 2\lambda_3$, $b_3 = 0$. *Then*

$$I_3 = (z^2 - b(w - s\epsilon b))(\bar{z}^2 - \bar{b}(\bar{w} - s\epsilon\bar{b}))$$

is an integral of motion where $z = \frac{1}{2}(m_1 + im_2)$, $w = p_1 + ip_2$, $b = b_1 + ib_2$.

6. Infinite dimensional Hamiltonian systems: Elastic Problem and the nonlinear Schroedinger's equation

This remarkable extension of Hamiltonian theory to infinite dimensional systems is based on the general theory of Fréchet manifolds developed by Hamilton [14]. Here are the essential theoretic ingredients required for the main results of the section.

Recall first that a topological Hausdorff vector space V is called a Fréchet space if its topology is induced by a countable family of semi-norms $\{p_n\}$ and if it is complete relative to the semi-norms in $\{p_n\}$. A Fréchet manifold is a topological Hausdorff space equipped with an atlas whose charts take values in open subsets of a Fréchet space V such that any change of coordinate charts is smooth.

The paper of Hamilton [14] singles out an important class of Fréchet manifolds, called tame, in which the implicit function theorem is true. One of the main theorems is that the set of smooth mappings from a compact interval into a finite-dimensional Riemannian manifold M is a tame Fréchet manifold. Hamilton's theorem remains true if Riemannian manifold is replaced by a sub-Riemannian manifold (Hamilton's arguments carry over to the sub-Riemannian case with minor alterations).

It then follows from the implicit function theorem that closed subsets of tame Fréchet manifolds $\mathcal{M}$, defined by the zero sets of finitely many smooth functions on $\mathcal{M}$ are tame sub-manifolds of $\mathcal{M}$.

Return now to the horizontal curves in G_ϵ over the spheres S_ϵ^3 discussed in section 4. Each horizontal curve can be given its sub-Riemannian length $\int_0^T \|\Lambda(t)\|_\epsilon \, dt$, where $\Lambda(t) = \frac{dg}{dt}g^{-1}(t)$. This length coincides with the Riemannian length of the projected curve $x(t) = g(t)e_0$ in S_ϵ^3. In what follows $\mathcal{H}_\epsilon(L)$ will denote the space of horizontal curves $g(t)$ in G_ϵ on a fixed interval $[0, L]$ that satisfy:

$$g(0) = I, \ \left\| g^{-1}(t)\frac{dg}{dt}(t) \right\|_\epsilon = 1, \ \ t \in [0, L], \text{ and } g^{-1}(0)\frac{dg}{dt}(0) = \Lambda_0, \tag{53}$$

where Λ_0 is a fixed element in $\mathfrak{p}_\epsilon$. Since the isotropy group K acts transitively (by conjugation) on the sphere $\{\Lambda \in \mathfrak{p}_\epsilon : \|\Lambda\| = 1\}$ there is no loss in generality in assuming that $\Lambda_0 = B_1 = e_1 \wedge_\epsilon e_0$. These curves will be called *anchored horizontal curves of length L*. They are the solutions of

$$\frac{dg}{dt} = g(t)\Lambda(t), \;\; g(0) = I, \;\; \|\Lambda(t)\| = 1, \;\; \Lambda(0) = B_1,$$

on a fixed interval $[0, L]$. Curves $g(t)$ in $\mathcal{H}_\epsilon(L)$ generated by periodic curves $\Lambda(t)$ of period L are called *quasi-periodic* and will be denoted by $\mathcal{PH}_\epsilon(L)$. Then we have the following proposition

Proposition 6.1. *The space of anchored horizontal curves of length L is an infinite dimensional tame Frećhet manifold. The space of quasi-periodic curves, when restricted to the interval* $[0, L]$ *is a tame submanifold of* $\mathcal{H}_\epsilon(L)$.

This proposition is a simple consequence of the general remarks about Hamilton's theorem.

In general, tangent vectors and tangent bundles of Fréchet manifolds are defined in the same manner as for finite dimensional manifolds. In particular tangent vectors at a point x in a Fréchet manifold $\mathcal{M}$ are the equivalence classes of curves $\sigma(t)$ in $\mathcal{M}$ all emanating from x (i.e., $\sigma(0) = x$), and all having the same tangent vector $\frac{d\sigma}{dt}(0)$ in each equivalence class. The set of all tangent vectors at x denoted by $T_x\mathcal{M}$ constitutes the tangent space at x.

The tangent bundle of a Fréchet manifold $\mathcal{M}$ is a Fréchet manifold. A vector field X on $\mathcal{M}$ is a smooth mapping from $\mathcal{M}$ into the tangent bundle $T\mathcal{M}$ such that $X(x) \in T_x\mathcal{M}$ for each $x \in \mathcal{M}$. On tame Fréchet manifolds vector fields can be defined as derivations in the space of smooth functions on $\mathcal{M}$. With these concepts at our disposal then

Proposition 6.2. *Let* $g(s)$ *be a curve in* $\mathcal{H}_\epsilon(L)$ *defined by* $\Lambda(s) = g^{-1}(s)\frac{dg}{ds}(s)$. *The tangent space* $T_g(\mathcal{H}_\epsilon(L)$ *at* $g(s)$ *consists of curves* $v(s) = g(s)V(s)$ *such that* $V(s) \in \mathfrak{p}_\epsilon$, $V(0) = \frac{dV}{ds}(0) = 0$, *and* $\langle \Lambda(s), \frac{dV}{ds}(s)\rangle_\epsilon = 0$. *For quasi-periodic curves the curve* $\frac{dV}{ds}(s)$ *must be smoothly periodic having the period equal to L* ($\frac{dV}{ds}(L) = \frac{dV}{ds}(0) = 0$).

Proof. Let $h(s,t) = h_t(s)$ denote a family of anchored horizontal curves such that $h(s, 0) = g(s)$. Then $v(s) = \frac{\partial h}{\partial t}(s,t)_{t=0}$ is a tangent vector at $g(s)$. It follows that $v(0) = 0$ since $h(0, t) = I$.

Let $x(s, t) = h(s, t)e_0$ and let $Z(s, t)$ and $W(s, t)$ denote the matrices in $\mathfrak{g}_\epsilon$ defined by

$$Z(s,t) = h(s,t)^{-1}\frac{\partial h}{\partial s}(s,t), \quad W(s,t) = h(s,t)^{-1}\frac{\partial h}{\partial t}(s,t).$$

It follows that $\frac{\partial x}{\partial s} = h(s,t)Z(s,t)e_0$, and $\frac{\partial x}{\partial t} = h(s,t)W(s,t)e_0$, and $\Lambda(s) = Z(s, 0)$ and $v(s) = g(s)V(s)$ with $V(s) = W(s, 0)$.

On any Riemannian manifold the mixed derivatives $\frac{D_x}{\partial s}\frac{\partial x}{\partial t}$ and $\frac{D_x}{\partial t}\frac{\partial x}{\partial s}$ are equal to each other. Therefore, $\frac{\partial Z}{\partial t}(s,t) = \frac{\partial W}{\partial s}(s,t)$. For $t = 0$ the above equation reduces to $\frac{dV}{ds}(s) = \frac{\partial W}{\partial s}(s,0) = \frac{\partial Z}{\partial t}(s,0)$. Then $\langle Z(s,t), Z(s,t)\rangle_\epsilon = 1$, and $Z(0,t) = B_1$ imply that $\langle Z(s,t), \frac{\partial Z}{\partial t}(s,t)\rangle_\epsilon = 0$ and $\frac{\partial Z}{\partial t}(0,t) = 0$. At $t = 0$, $Z = \Lambda$ and $\frac{\partial Z}{\partial t} = \frac{dV}{ds}$, hence $\langle \Lambda(s), \frac{dV}{ds}\rangle_\epsilon = 0$ and $\frac{dV}{ds}(0) = 0$. For quasi-periodic curves there is an extra condition $\frac{dV}{ds}(L) = 0$.

It remains to show that any curve $V(s)$ that satisfies the conditions above can be realized by the perturbations $h(s,t)$. Define $h(s,t)$ the solution of $\frac{\partial h}{\partial s}(s,t) = h(s,t)Z(s,t)$, $h(0,t) = I$, where

$$Z(s,t) = \frac{1}{1+\phi^2(t)\left\|\frac{dV}{ds}\right\|^2}\Big(\Lambda(s) + \phi(t)\frac{dV}{ds}\Big)$$

for some function ϕ that satisfies $\phi(0) = 0$ and $\frac{d\phi}{dt}(0) = 1$. It is easy to verify that $Z(s,t)$ is a unit vector in $\mathfrak{p}_\epsilon$ for each s and t and that $\frac{\partial Z}{\partial t}(s,0) = \frac{dV}{ds}$. □

We will next show that both $\mathcal{H}_\epsilon(L)$ and $\mathcal{PH}_\epsilon(L)$ may be considered as symplectic Frećhet manifolds. The basic notions of symplectic geometry of infinite-dimensional Fréchet manifolds are defined through differential forms in the same manner as for the finite-dimensional situations. In particular, differential forms ω of degree n are the mappings $\omega : \mathcal{X}(M) \times \cdots \times \mathcal{X}(M)_n \to C^\infty(\mathcal{M})$ that are $C^\infty(\mathcal{M})$ multilinear and skew-symmetric where $\mathcal{X}(M)$ denotes the space of all smooth vector fields on $\mathcal{M}$. Then the exterior derivative $d\omega$ of a form of degree n is a differential form of degree $n+1$ defined by

$$\begin{aligned} d\omega(X_1,\ldots,X_{n+1}) = &\sum_{i=1}^{n+1}(-1)^{i+1}X_i(\omega(X_1,\ldots,\hat{X}_1,\ldots,X_n)) \\ &-\sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],\ldots\hat{X}_i,\ldots\hat{X}_j,X_{n+1}), \end{aligned}$$

where the roof sign above an entry indicates its absence from the expression (i.e., $w(\hat{X}_1, X_2) = w(X_2)$ and $w(X_1, \hat{X}_2) = w(X_1)$).

A differential form ω is said to be closed if its exterior derivative $d\omega$ is equal to zero. A differential form ω of degree 2 is said to be symplectic whenever it is closed and nondegenerate, in the sense that the induced form $(i_X\omega)(Y) = \omega(X,Y)$ is nonzero for each nonzero vector field X.

The differential df of a smooth function f is a form of degree one defined by $df(v) = \frac{d}{dt}f\circ\sigma(t)|_{t=0}$ for any smooth curve in $\mathcal{M}$ such that $\sigma(0) = x$, and $\frac{d\sigma}{dt}(0) = v$.

A vector field X is said to be Hamiltonian if there exists a smooth function f such that $df(Y) = \omega(X,Y)$ for all vector fields Y on $\mathcal{M}$. The dependence of X on f shall be noted explicitly by X_f.

Our principal objective is to demonstrate that $\mathcal{H}_\epsilon(L)$ and $\mathcal{PH}_\epsilon(L)$ are infinite dimensional symplectic manifolds. We will first need some notations. We will continue with the notations used in the previous section and denote each matrix $L = \sum_{i=1}^3 p_iB_i + m_iA_i$ by the pair of vectors $(\hat{M}, \hat{P})$, where $\hat{M} = (m_1, m_2, m_3)^T$ and $\hat{P} = (p_1, p_2, p_3)^T$.

We will make use of this fact that $\mathfrak{p}_\epsilon$ and $\mathfrak{k}$ are of the same dimension and introduce the mapping $\phi : \mathfrak{k} \to \mathfrak{p}_\epsilon$ defined by $\phi(M) = m_1B_1 + m_2B_2 + m_3B_3$, for any $M = m_1A_1+m_2A_2+m_3A_3$. Then for any $P \in \mathfrak{p}_\epsilon$ and any $M \in \mathfrak{k}$, $\langle P, \phi(M)\rangle_\epsilon$ is equal to the Euclidean inner product $\hat{P}\cdot\hat{M}$ in $\mathbb{R}^3$. Moreover, the following equality holds

$$\langle P_1, \phi([P_2, P_3])\rangle_\epsilon = \phi([P_1, P_2]), P_3\rangle_\epsilon, \tag{54}$$

for any P_1, P_2, P_3 in $\mathfrak{p}_\epsilon$.

Proposition 6.3. *6 Let $g(s)$ denote a horizontal curve in either $\mathcal{H}_\epsilon(L)$ or in $\mathcal{PH}_\epsilon(L)$ and let $g(s)V_1(s)$ and $g(s)V_2(s)$ denote a pair of tangent vectors at $g(s)$. Suppose that $\Lambda(s) = g^{-1}(s)\frac{dg}{ds}(s)$. Then*

$$\omega_\Lambda(V_1, V_2) = \int_0^L \Big\langle \Lambda(s), \phi\Big(\Big[\frac{dV_1}{ds}, \frac{dV_2}{ds}\Big]\Big)\Big\rangle_\epsilon ds = \int_0^L \hat{\Lambda}(s)\cdot\Big(\frac{d\hat{V}_1}{ds}\times\frac{d\hat{V}_2}{ds}\Big)\, ds, \tag{55}$$

is a symplectic form on $\mathcal{H}_\epsilon(L)$ (respectively on $\mathcal{PH}_\epsilon(L)$).

The proof of this proposition can be found in [21] with the minor difference that S^3_ϵ in that paper is represented by SU_2 and $SL_2(\mathbb{C})/SU_2$, rather than G_ϵ/K as in this paper.

6.1. The Hamiltonian flow of $\frac{1}{2}\int_0^L \kappa^2(s)\,ds$ and Heisenberg's magnetic equation.

Consider now the function $f(\Lambda) = \frac{1}{2}\int_0^L \left\|\frac{d\Lambda}{ds}\right\|^2 ds$ on $\mathcal{PH}_\epsilon(L)$ and its Hamiltonian vector field X_f induced by the form ω in (55).

To calculate the directional derivative df_Λ, let $g(s)V(s)$ denote a tangent vector at $g(s)$ in $\mathcal{PH}_\epsilon(L)$. Let $h_t(s) = h(s,t))$ denote a family of curves in $\mathcal{PH}_\epsilon(L)$ that are the solutions of $\frac{\partial h}{\partial s} = h(s,t)\Gamma(s,t)$ such that $h(s,0) = g(s)$, $\Gamma(s,0) = \Lambda(s)$, $\frac{\partial\Gamma}{\partial t}(s,0) = \frac{dV}{ds}(s)$. The directional derivative $df_\Lambda(V)$ is given by

$$\begin{aligned}
df_\Lambda(V) &= \frac{1}{2}\frac{\partial}{\partial t}\int_0^L\Big\langle\frac{\partial\Gamma}{\partial s}(s,t), \frac{\partial\Gamma}{\partial s}(s,t)\Big\rangle ds\Big|_{t=0} = \frac{1}{2}\frac{\partial}{\partial t}\int_0^L\Big\langle\frac{\partial\Gamma}{\partial s}(s,t), \frac{\partial\Gamma}{\partial s}(s,t)\Big\rangle ds\Big|_{t=0}\\
&= \int_0^L\Big\langle\frac{\partial\Gamma}{\partial s}(s,t), \frac{\partial}{\partial s}\frac{\partial\Gamma}{\partial t}(s,t)\Big\rangle ds\Big|_{t=0} = \int_0^L\Big\langle\frac{d\Lambda}{ds}, \frac{d}{ds}\Big(\frac{dV}{ds}\Big)\Big\rangle ds\\
&= -\int_0^L\Big\langle\frac{d^2\Lambda}{ds^2}, \frac{dV}{ds}\Big\rangle ds + \Big\langle\frac{d\Lambda}{ds}, \frac{dV}{ds}\Big\rangle\Big|_{s=0}^{s=L} = -\int_0^L\Big\langle\frac{d^2\Lambda}{ds^2}, \frac{dV}{ds}\Big\rangle ds.
\end{aligned}$$

The Hamiltonian vector field X_f is of the form $X_f(g) = gF$ for some matrices $\{F(s),\ s\in[0,L]\}$ in $\mathfrak{p}_\epsilon$ that satisfy $F(0) = 0$ and $\big\langle\Lambda(s), \frac{dF}{ds}(s)\big\rangle_\epsilon = 0$. Then it follows from (55) that

$$df_\Lambda(V) = \int_0^L\Big\langle\Lambda(s), \phi\Big(\Big[\frac{dF}{ds}, \frac{dV}{ds}\Big]\Big)\Big\rangle_\epsilon ds = \int_0^L\Big\langle\phi\Big(\Big[\Lambda(s), \frac{dF}{ds}\Big]\Big), \frac{dV}{ds}\Big\rangle_\epsilon ds$$

which implies that

$$\int_0^L\Big\langle\frac{d^2\Lambda}{ds^2} + \phi\Big(\Big[\Lambda(s), \frac{dF}{ds}\Big]\Big), \frac{dV}{ds}\Big\rangle ds,$$

and since $V(s)$ is an arbitrary tangent vector

$$\frac{d^2\Lambda}{ds^2} + \phi\Big(\Big[\Lambda(s), \frac{dF}{ds}\Big]\Big) = 0.$$

Then $\big[\Lambda, \phi^{-1}\frac{d^2\Lambda}{ds^2}\big] + \big[\Lambda, \big[\Lambda, \frac{dF}{ds}\big]\big] = 0$. On spaces of curvature ϵ, $\big[\Lambda, \big[\Lambda, \frac{dF}{ds}\big]\big] = -\epsilon\frac{dF}{ds}$. Also, note that $[Q, \phi^{-1}P] = \epsilon\phi([Q,P])$ for any P, Q in $\mathfrak{p}_\epsilon$. The above imply that

$$F = \epsilon\Big[\Lambda, \phi^{-1}\frac{d^2\Lambda}{ds^2}\Big] = \phi\Big[\Lambda, \frac{d^2\Lambda}{ds^2}\Big]. \tag{56}$$

Hence $F(s) = \int_0^s \phi\big(\big[\Lambda(x), \frac{d^2\Lambda}{dx^2}\big]\big)\,dx$ since $F(0) = 0$.

The integral curves $X_f(g)(s) = g(s)\phi\big(\big[\Lambda(s), \frac{d^2\Lambda}{ds^2}(s)\big]\big)$ of the Hamiltonian vector field are the solutions of the following system of equations:

$$\frac{\partial g}{\partial t}(s,t) = g(s,t)\int_0^s \phi\Big(\Big[\Lambda(x,t), \frac{d^2\Lambda}{dx^2}(x,t)\Big]\Big)dx, \quad \frac{\partial g}{\partial s}(s,t) = g(s,t)\Lambda(s,t).$$

The equality of mixed partial derivatives $\frac{D_g}{ds}\big(\frac{\partial g}{\partial t}\big) = \frac{D_g}{dt}\big(\frac{\partial g}{\partial s}\big)$ implies that the matrices $\Lambda(s,t)$ evolve according to

$$\frac{\partial\Lambda}{\partial t}(s,t) = \phi\Big(\Big[\Lambda(s,t), \frac{\partial^2\Lambda}{\partial s^2}\Big]\Big). \tag{57}$$

Equation (57) can be also expressed in terms of the coordinate vector $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ relative to the basis B_1, B_2, B_3 as

$$\frac{\partial\lambda}{\partial t}(s,t) = \epsilon\Big(\frac{\partial^2\lambda}{\partial s^2}(s,t) \times \lambda(s,t)\Big).$$

In the hyperbolic case $\epsilon = -1$, and

$$\frac{\partial\lambda}{\partial t}(s,t) = \lambda(s,t) \times \frac{\partial^2\lambda}{ds^2}(s,t) \tag{58}$$

Equation (58) can be also considered as an equation in the space of Hermitian matrices of the form

$$\frac{\partial\Lambda}{\partial t}(s,t) = i\Big[\frac{\partial^2\Lambda}{\partial s^2}(s,t), \Lambda(s,t)\Big], \tag{59}$$

because, as we have already remarked before, the hyperboloid $\mathbb{H}^3$ can be realized as the quotient $SL_2(\mathbb{C})/SU_2$, in which case the horizontal distribution $\mathfrak{p}$ is identified with the Hermitian matrices in $sl_2(\mathbb{C})$. Then the basis elements B_1, B_2, B_3 then correspond to the Pauli matrices

$$B_1 = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad B_2 = \frac{1}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad B_3 = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

matrix Λ takes on the form $\frac{1}{2}\begin{pmatrix}\lambda_1 & \lambda_2 - i\lambda_3\\ \lambda_2 + i\lambda_3 & -\lambda_1\end{pmatrix}$, and the bijection ϕ corresponds to the matrix multiplication by i.

Equation (59) is known as *continuous isotropic Heisenberg's ferromagnetic equation* [12]. This equation is also related to the *vortex filament equation*

$$\frac{\partial\gamma}{\partial t}(s,t) = \kappa(s,t)B(s,t), \tag{60}$$

where $\gamma_t(s) = \gamma(s,t)$ is a continuum of curves γ_t in $\mathbb{R}^3$ parametrized by arc length, $\kappa(s,t)$ is the geodesic curvature and $B(s,t)$ is the binormal vector in the Serret–Frenet triad associated with γ. This equation can be also rephrased as

$$\frac{\partial\gamma}{\partial t}(s,t) = \frac{\partial\gamma}{\partial s} \times \frac{\partial^2\gamma}{\partial s^2} \tag{61}$$

since $\frac{\partial\gamma}{\partial s} = T$, $\frac{\partial^2\gamma}{\partial s^2} = \frac{\partial T}{\partial s} = \kappa N$ and $B = T \times N$. Then the tangent vector $T(s,t) = \frac{\partial\gamma}{\partial s}(s,t)$ satisfies equation (58).

6.2. The nonlinear Schroedinger equation. We come back once more to the adjoint action of the isotropy group K on the unit sphere $S^2 = \{\Lambda : \|\Lambda\|_\epsilon = 1\}$ in $\mathfrak{p}_\epsilon$. Our aim is to show the relation between the nonlinear Schroedinger equation and the Heisenberg's magnetic equation. It will be convenient to represent the Heisenberg's magnetic equation in the space of Hermitian matrices as explained in the previous section. Then it is natural to replace K by SU_2. Since K is isomorphic to $\mathrm{SO}_3(R)$, SU_2 is a double cover of K, but for our purposes this fact will be irrelevant. The isotropy subgroup K_0 of SU_2 that leaves the matrix $B_1 = \frac{1}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ fixed is isomorphic to $\mathrm{SO}_2(R)$. It consists of matrices of the form $\left(\begin{smallmatrix} z & 0 \\ 0 & \bar{z} \end{smallmatrix}\right)$ with $|z| = 1$. Then S^2 will be considered as the quotient $SU_2/\mathrm{SO}_2(R)$. This realization of S^2 identifies a left invariant connection with values in the two dimensional vector space $\mathfrak{k}_1 = \left\{U = \frac{1}{2}\left(\begin{smallmatrix} 0 & \bar{u} \\ u & 0 \end{smallmatrix}\right) : u \in \mathbb{C}\right\}$.

Then for each curve $\Lambda(s)$ in S^2 such that $\Lambda(0) = B_1$ there exists a curve $R(s)$ in SU_2 such that $\Lambda(s) = R(s)B_1R^*(s)$ for all $s \in [0, L]$. Any two such curves $R_1(s)$ and $R_2(s)$ differ by a curve $R_0(s)$ in K_0, i.e., $R_2(s) = R_1(s)R_0(s)$. However, if $R(s)$ is restricted to curves which are solutions of $\frac{dR}{ds} = R(s)U(s)$ such that $U(s)$ takes values in $\mathfrak{k}_1$ subject to fixed initial condition $R(0) = I$, then $R(s)B_1R^*(s) = \Lambda(s)$ has a unique solution $R(s)$. In particular, each solution $\Lambda(s,t)$ of Heisenberg's magnetic equation generate a family of matrices $R(s,t)$, $R(0,t) = I$, through the relations $\Lambda(s,t) = R(s,t)B_1R^*(s,t)$. Curves $R(s,t)$ then evolve according to

$$\frac{\partial R}{\partial s}(s,t) = R(s,t)U(s,t) \quad \text{and} \quad \frac{\partial R}{\partial t} = R(s,t)V(s,t)$$

for some matrices $U(s,t)$, $V(s,t)$ in $\mathfrak{g} = su_2$, which further conform to $V(0,t) = 0$ for all t because of the boundary condition $R(0,t) = I$. Matrices U and V satisfy

$$\frac{\partial U}{\partial t}(s,t) - \frac{\partial V}{\partial s}(s,t) + [U(s,t), V(s,t)] = 0, \tag{62}$$

known as *the zero curvature equation* [12]. The zero curvature equation is a consequence of two facts: the first fact is that the covariant derivative of a curve of tangent vectors $v(s) = R(s)V(s)$ along a curve $R(s)$ in SU_2 is given by

$$\frac{D_R}{ds}(V)(s) = R(s)\Big(\frac{dU}{ds} + \frac{1}{2}[V(s), \Lambda(s)]\Big), \tag{63}$$

where $\Lambda(s) = R^*(s)\frac{dR}{ds}(s)$, and the second fact, that on any Riemannian manifold

$$\frac{\partial D_{x(s,t)}}{\partial s}\frac{\partial x}{\partial t} = \frac{\partial D_{x(s,t)}}{\partial t}\frac{\partial x}{\partial s}$$

for any field of curves $x(s,t)$.

Proposition 6.4. *Let $\Lambda(s,t)$ denote a solution of Heisenberg's magnetic equation and let $R(s,t)$ denote a field of matrices in SU_2 with $T(0,t) = I$ such that $R(s,t)B_1R^*(s,t) = \Lambda(s,t)$. If $U(s,t) = \frac{1}{2}\begin{pmatrix} u_1 & \bar{u} \\ u & -u_1 \end{pmatrix}$ is the matrix defined by $U(s,t) = R^*(s,t)\frac{\partial R}{\partial s}(s,t)$, then $\psi(s,t) = u(s,t)\exp\left(i\int_0^s u_1(x,t)\,dx\right)$ is a solution of the nonlinear Schroedinger's equation*

$$\frac{\partial}{\partial t}\psi(s,t) = i\frac{\partial^2\psi}{\partial s^2}(s,t) + i\Big(\frac{1}{2}|\psi(s,t)|^2 + c\Big)\psi(s,t) \quad \text{with} \quad c(t) = -\frac{1}{2}|u(0,t)|^2.$$

Proof. As before B_1, B_2, B_3 denote the Hermitian Pauli matrices. Then $A_1 = iB_1$, $A_2 = iB_2$, $A_3 = -B_3$ are the skew-Hermitian Pauli matrices. The reader can readily check that the Lie brackets of Pauli matrices are given by Table 1 with $\epsilon = -1$ and $s = 1$. This observation simplifies the calculations of Lie brackets below

$$\begin{aligned}
\frac{\partial\Lambda}{\partial t} &= \frac{\partial}{\partial t}(R(s,t)B_1R^*(s,t)) = R[B_1, V]R^*,\\
\frac{\partial\Lambda}{\partial s} &= \frac{\partial}{\partial s}(R(s,t)B_1R^*(s,t)) = R[B_1, U]R^*,\\
\frac{\partial^2\Lambda}{\partial s^2} &= R\Big([[B_1,U],U] + \Big[B_1, \frac{\partial U}{\partial s}\Big]\Big)R^*.
\end{aligned}$$

Then $\frac{\partial\Lambda}{\partial t}(s,t) = i\big[\frac{\partial^2\Lambda}{\partial s^2}(s,t), \Lambda(s,t)\big]$ implies that

$$[B_1, V] = i\Big(\big[[[B_1,U],U],B_1\big] + \Big[\Big[B_1, \frac{\partial U}{\partial s}\Big], B_1\Big]\Big). \tag{64}$$

An easy computation shows that

$$\big[[B_1,U],U\big] = \langle U, B_1\rangle iU - \langle U,U\rangle B_1 = -(u_2^2 + u_3^2)B_1 + u_1u_2B_2 + B_3u_1u_3,$$

and $\big[[[B_1,U],U],B_1\big] = u_1u_3A_2 - u_1u_2A_3$. Similarly,

$$\Big[B_1, \frac{\partial U}{\partial s}\Big] = \frac{\partial u_3}{\partial s}B_2 - \frac{\partial u_2}{\partial s}B_3 \quad \text{and} \quad \Big[\Big[B_1, \frac{\partial U}{\partial s}\Big], B_1\Big] = -\frac{\partial u_3}{\partial s}A_3 - \frac{\partial u_2}{\partial s}A_2.$$

Equation (64) then reduces to

$$\begin{aligned}
[B_1, V] &= i\Big(u_1(u_3A_2 - u_2A_3) - \frac{\partial u_3}{\partial s}A_3 - \frac{\partial u_2}{\partial s}A_2\Big)\\
&= -u_1(u_3B_2 - u_2B_3) + \frac{\partial u_3}{\partial s}B_3 + \frac{\partial u_2}{\partial s}B_2.
\end{aligned}$$

If $V = v_1A_1 + v_2A_2 + v_3A_3$, then $[B_1, V] = v_3B_2 - v_2B_3$, which, when combined with the above, yields $v_2 = -u_1u_2 - \frac{\partial u_3}{\partial s}$ and $v_3 = -u_1u_3 + \frac{\partial u_2}{\partial s}$. These relations can be rephrased as $v(s,t) = -u_1(s,t)u(s,t) + i\frac{\partial u}{\partial s}(s,t)$, where $u = u_2 + iu_3$ and $v = v_2 + iv_3$.

The zero curvature equation implies that

$$\frac{\partial u_1}{\partial t} = \frac{\partial v_1}{\partial s} + \frac{1}{2}\frac{\partial}{\partial s}(u_2^2 + u_3^2), \tag{65}$$

$$\frac{\partial u}{\partial t} = i\frac{\partial^2 u}{\partial s^2} - 2u_1\frac{\partial u}{\partial s} - \frac{\partial u_1}{\partial s}u - i(v_1 + u_1^2)u. \tag{66}$$

Equation (65) implies that

$$\frac{\partial}{\partial t}\int_0^s u_1(x,t)\,dx = v_1(s,t) + \frac{1}{2}\big(u_2^2(s,t) + u_3^2(s,t)\big) + c(t),$$

where $c(t) = -v_1(0,t) - \frac{1}{2}(u_2^2(0,t) + u_3^2(0,t)) = -\frac{1}{2}(u_2^2(0,t) + u_3^2(0,t))$, since $V(0,t) = 0$. The substitution of $v_1(s,t) = \frac{\partial}{\partial t}\int_0^s u_1(x,t)\,dx - \frac{1}{2}|u(s,t)|^2 - c$ into (57) leads to

$$\frac{\partial u}{\partial t} + iu\frac{\partial}{\partial t}\int^s u_1(t,x)\,dx = i\frac{\partial^2 u}{\partial s^2} - 2u_1\frac{\partial u}{\partial s} - u\frac{\partial u_1}{\partial s} - i\Big(-\frac{1}{2}|u|^2 - c + u_1^2\Big)u. \tag{67}$$

After the multiplication by $\exp\big(i\int_0^s u_1(x,t)\,dx\big)$ (67) can be expressed as

$$\frac{\partial}{\partial t}\psi(s,t) = \Big(i\frac{\partial^2 u}{\partial s^2} - 2u_1\frac{\partial u}{\partial s} - u\frac{\partial u_1}{\partial s} - i\Big(u_1^2 - \frac{1}{2}|u|^2 - c\Big)u\Big)\exp\Big(i\int_0^s u_1(x,t)\,dx\Big)e^{-ict},$$

where $\psi(s,t) = u(s,t)\exp(i\int_0^s u_1(x,t)\,dx)$. In addition

$$i\frac{\partial^2\psi}{\partial s^2} = \Big(i\frac{\partial^2 u}{\partial s^2} - 2u_1\frac{\partial u}{\partial s} - u\frac{\partial u_1}{\partial s} - iu_1^2 u\Big)\exp\Big(i\int_0^s u_1(x,t)\,dx\Big).$$

as can be verified by differentiating $\frac{\partial\psi}{\partial s} = \big(\frac{\partial u}{\partial s} + iuu_1\big)\exp\big(i\int_0^s u_1(x,t)\,dx\big)$. Therefore,

$$\frac{\partial}{\partial t}\psi(t,s) = i\frac{\partial^2\psi}{\partial s^2} + i\Big(\frac{1}{2}|\psi|^2 + c(t)\Big)\psi. \tag{68}$$ □

The steps taken in the passage from Heisenberg's equation to the Schroedinger's equation are reversible. Any solution $\psi(s,t)$ of (68) generates matrices

$$U = \frac{1}{2}\begin{pmatrix} 0 & \psi \\ -\overline{\psi} & 0\end{pmatrix} \text{ and } V = \frac{1}{2}\begin{pmatrix} -\frac{1}{2}i(|\psi|^2 + c(t)) & i\frac{\partial\psi}{\partial s} \\ i\frac{\partial\bar\psi}{\partial s} & \frac{1}{2}i(|\psi|^2 + c(t))\end{pmatrix}$$

that satisfy the zero-curvature equation. Therefore, there exist unique curves $R(s,t)$ in SU_2 with boundary conditions $R(0,t) = I$ that evolve according to the differential equations:

$$\frac{\partial R}{\partial s}(s,t) = R(s,t)U(s,t), \quad \frac{\partial R}{\partial t}(s,t) = R(s,t)V(s,t).$$

Such curves define $\Lambda(s,t)$ through familiar formulas $\Lambda(s,t) = R(s,t)B_1R^*(s,t)$. It then follows that Λ is a solution of the Heisenberg's magnetic equation because $\psi = u$ and $v = i\frac{\partial u}{\partial s}$.

The transformation $\psi(s,t) = u(s,t)\exp\big(i\int_0^s u_1(x,t)\,dx\big)$ is a slight generalization of the Hasimoto function $\kappa(s,t)\exp\big(i\int_0^s \tau(x,t)\,dx\big)$, for when $R(s,t)$ corresponds to the Serret–Frenet frame, u_1 is equal to the torsion of the projected curve ([15] and [21]). The reduced curves are defined by $u_1 = 0$, and they set up a bijective correspondence between Heisenberg's equation and the nonlinear Schroedinger equation via the map $\Lambda = RB_1R^*$.

6.3. Soliton solutions and the elastic curves. For mechanical systems the Hamiltonian function represents the total energy of the system and its critical points correspond to the equilibrium configurations. In an infinite-dimensional setting the behavior of a Hamiltonian system at a critical point of a Hamiltonian system seem does not lend itself to such simple characterizations. For the Hamiltonian function $f = \frac{1}{2}\int_0^L k^2 ds$ it is natural to expect that the critical points correspond to the Hamiltonian associated with the Euler–Griffiths problem, with one minor exception: curves in $\mathcal{PH}_\epsilon(L)$ satisfy fixed boundary conditions in G_ϵ, while the curves that project onto elastic curves are only partially fixed at the terminal points. To reconcile these differences it will be necessary to consider the Hamiltonian equations associated with a minor variant of the EulerGriffiths problem, called Euler–Griffiths problem 2 in [22], in which the integral $\frac{1}{2}\int_0^L (u_2^2(t) + u_3^2(t))\,dt$ is minimized over the solutions $g(s)$ in G_ϵ of $\frac{dg}{ds} = g(s)(B_1 + u_2(s)A_2 + u_3(s)A_3)$ that satisfy the boundary conditions $g(0) = I$, $g(L) = g_1$ (rather than $g(0) \in S_0$ and $g(L) \in S_1$ as explained in the earlier sections). It turns out that the extremal curves of this modified Euler–Griffiths problem form traveling waves, known as *solitons*, for the nonlinear Heisenberg's equation.

It is easy to see that the Hamiltonian associated with the preceding problem is given by $H = \frac{1}{2}(m_2^2 + m_3^2) + p_1$, where the variables m_2, m_3 and p_1 have the same meaning as in (46).

Remarkably, this Hamiltonian coincides with the Hamiltonian for the spherical pendulum [22]. The Hamiltonian equations take on the same form as equations (45) with $\hat{\Omega} = (0, m_2, m_3)^T$ and $B = B_1$. More explicitly, for $\epsilon = -1$ these equations are

$$\begin{aligned} &\frac{dm_1}{ds} = 0, && \frac{dm_2}{ds} = -m_3 m_1 + p_3, && \frac{dm_3}{ds} = m_2 m_1 - p_2, \\ &\frac{dp_1}{ds} = m_3 p_2 - m_2 p_3, && \frac{dp_2}{ds} = -m_3 p_1 - m_3, && \frac{dp_3}{ds} = m_2 p_1 + m_2. \end{aligned} \tag{69}$$

It follows that m_1 is a constant of motion for (69). The remaining solutions define complex functions $u(s) = m_2(s) + im_3(s)$ and $w(s) = p_2(s) + ip_3(s)$. Then,

Proposition 6.5. *Let* $u(s) = m_2(s) + im_3(s)$. *Then* $\psi(s,t) = u(s+\xi t)$ *is a solution of the nonlinear Schroedinger's equation with* $c = 0$ *precisely when* $H = -1$ *and* $\xi = -m_1$.

Proof. It follows from equations (69) that $\frac{du}{ds}(s) = im_1 u(s) - iw(s)$ and $\frac{dw}{ds} = i(p_1 + 1)u(s)$. Therefore,

$$\frac{\partial \psi}{\partial t} = i\xi(m_1\psi - w) \quad \text{and} \quad \frac{\partial^2 \psi}{\partial s^2} = -m_1^2\psi + m_1 w + (p_1 + 1)\psi.$$

Since $H = \frac{1}{2}|\psi|^2 + p_1$, $\frac{1}{2}|\psi|^2\psi = (H - p_1)\psi$ and

$$\begin{aligned} -i\frac{\partial \psi}{\partial t} - \Big(\frac{\partial^2 \psi}{\partial s^2} + \frac{1}{2}|\psi|^2\psi\Big) &= \xi(m_1\psi - w) - (-m_1^2\psi + m_1 w + (p_1+1)\psi + \psi(H - p_1)) \\ &= -(\xi + m_1)w + (\xi m_1 + m_1^2 + H + 1)\psi. \end{aligned}$$

The above is equal to zero when $\xi = -m_1$ and $H = -1$. □

Thus the extremals which reside on energy level $H = -1$ generate soliton solutions of the nonlinear Schroedinger's equation traveling with speed equal to the constant of motion $m_1 = -\xi$. These soliton solutions degenerate to the stationary solution when $m_1 = 0$, i.e., when the projected curve is elastic.

6.3.1. Complete Integrability. It has been known now for some time that the nonlinear Heisenberg's equation is integrable relative to the solutions $u(s,t)$ that vanish at infinity [41] and [12]. The paper Shabat and Zacharov exhibited an infinite family $\{C_1, C_2, C_3, \dots\}$ of integrals of motion all in involution with each other with

$$C_1 = \int_{-\infty}^{\infty} |u(s,t)|^2 ds, \quad C_2 = \int_{-\infty}^{\infty} \big(u(s,t)\,\dot{\bar{u}}(s,t) - \bar{u}(s,t)\,\dot{u}(s,t)\big)\,ds,$$

$$C_3 = \int_{-\infty}^{\infty} \Big(\Big|\frac{\partial u}{\partial s}(s,t)\Big|^2 - \frac{1}{4}|u(s,t)|^4\Big) ds,$$

corresponding to the total number of particles, their momentum and the energy. Subsequently, Magri gave a recursive scheme for generating these integrals of motion and he showed that these integrals of motion must necessarily Poisson commute with each other [31]. Quite remarkably, these physical integrals of motion are in a correspondence with the functionals reflecting the geometric invariants of curves in $\mathbb{H}_3$, as first noticed by Langer and Perline [27] for curves that vanish rapidly at infinity. It turns out that these findings are unaltered when the boundary conditions at infinity are replaced by the quasi-periodic conditions of this paper, as will be shown in the paragraph below.

For that purpose let $f_1(\Lambda), f_2(\Lambda), f_3(\Lambda), \dots$ denote the functionals in $\mathcal{PH}_\epsilon(L)$ that are related to $C_1(u), C_2(u), C_3(u), \dots$ via the relations

$$\Lambda(s) = R(s)B_1R^*(s), \quad \frac{dR}{ds} = R(s)U(s), \quad U(s) = u_2(s)A_2 + u_3(s)A_3.$$

Then,

$$f_1(\Lambda) = \frac{1}{2}\int_0^L \|\dot{\Lambda}(s)\|^2 ds = \frac{1}{2}\int_0^L \|[B_1, U(s)]\|^2 ds = \frac{1}{2}\int_0^L |u(s)|^2 ds = \frac{1}{2}C_1.$$

To show that C_2 is a scalar multiple of $f_2(\Lambda) = \int_0^L \kappa^2(s)\tau(s)\,ds$ note that

$$\kappa^2\tau = -i\Big\langle \Big[\Lambda, \frac{d\Lambda}{ds}\Big], \frac{d^2\Lambda}{ds^2}\Big\rangle.$$

Therefore,

$$\int_0^L \kappa^2(s)\tau(s)\,ds = i\int_0^L \Big\langle \Big[\Lambda, \frac{d\Lambda}{ds}\Big], \frac{d^2\Lambda}{ds^2}\Big\rangle = i\int_0^L \big\langle \big[[B_1, U], [B_1, \dot{U}]\big], B_1\big\rangle\,ds$$

$$= \int_0^L Im(\bar{u}\dot{u})\,ds = \frac{1}{2i}\int_0^L (u(s)\dot{\bar{u}}(s) - \bar{u}(s)\dot{u}(s))\,ds = \frac{1}{2i}C_2.$$

We leave it to the reader to show that

$$f_3(\Lambda) = \int_0^L \Big(\|\ddot{\Lambda}(s)\|^2 - \frac{5}{4}\|\dot{\Lambda}(s)\|^4\Big) ds = \int_0^L \Big(\frac{\partial\kappa}{\partial s}(s)^2 + \kappa^2(s)\tau^2(s) - \frac{1}{4}\kappa^4(s)\Big) ds$$

corresponds to C_3.

My earlier paper [21] shows several intriguing facts related to integrability properties of the Heisenberg's equation. To begin with, it shows that f_1, f_2, f_3 Poisson commute relative to the symplectic form in $\mathcal{PH}_\epsilon(L)$. Furthermore, it demonstrates that the Hamiltonian flow of f_3 is given by

$$\frac{\partial\Lambda}{\partial t} = 2\Big(\frac{\partial^3\Lambda}{\partial t^3} - \Big\langle\frac{\partial^3\Lambda}{\partial t^3}, \Lambda\Big\rangle\Lambda\Big) - 3\Big\langle\Lambda, \frac{\partial^2\Lambda}{\partial t^2}\Big\rangle\frac{\partial\Lambda}{\partial t},$$

which is in correspondence with $\frac{\partial u}{\partial t} - 3|u|^2\frac{\partial u}{\partial s} - 2\frac{\partial^3 u}{\partial s^3} = 0$, an equation that bears striking resemblance to the modified Korteweg–de Vries equation (Abraham and Marsden [1]).

Finally, it shows that the functional $f(\tau) = -i\int_0^L \frac{1}{\kappa^2}\big\langle\big[\Lambda, \frac{d\Lambda}{ds}\big], \frac{d^2\Lambda}{ds^2}\big\rangle$ is in the hierarchy $\{f_1, f_2, f_3, \dots\}$ and generates the curve shortening equation

$$\frac{\partial\Lambda}{\partial t}(s,t) = \frac{\partial\Lambda}{\partial s}(s,t) = \kappa(s,t)N(s,t).$$

These findings are in accordance with the results of Langer and Perline, and suggest that the recursive scheme of Magri could be translated into the realm of the Heisenberg's equation, with the ultimate goal of demonstrating its bi-Hamiltonian character. In such a case, complete integrability of Heisenberg's equation would automatically follow. But, more generally, it seems that the symplectic formalism of this paper could be exploited for other equations of mathematical physics, such as, for instance, the Korteweg–de Vries equation.

7. Concluding Remarks and Open Problems

The exposition of this paper, loosely described as "the variations on the Euler-Kirchhoff elastic theme", focuses on the class of variational problems on an orthonormal frame bundle of a Riemannian space which is a Lie group (notably spaces of constant curvature), as prototypes of differential systems with symmetries, and introduces optimal control theory as an important ingredient required for its solutions. With the exception of the last section on infinite dimensional Hamiltonian systems, which stands somewhat apart from the rest of the material, it is shown that all these variational problems could be seen as the variants of the generalized problem of Kirchhoff and could be formulated in terms of a single left invariant optimal control problem on Lie group G_ϵ which is either the group of motions of the Euclidean space, or the rotation group or $\mathrm{SO}(1,n)$ in the case of hyperboloid. In the special case when the elastic energy is defined by the Cartan-Killing form and when certain constants of motion are equal to zero ($L_{\mathfrak{k}_B} = 0$), this problem reduces to the Euler–Griffiths problem of minimizing the integral $\frac{1}{2}\int_0^T \kappa^2(t)\,dt$. The corresponding Hamiltonian system is integrable and the solutions are found by quadrature in terms of elliptic functions. The projections of these solutions on the underlying space forms are known as the elastic curves in the existing literature.

The "elastic bias" reveals the significance of six dimensional Lie groups $SE_3(R)$, $\mathrm{SO}_4(R)$ and $\mathrm{SO}(1,3)$ for equations of mathematical physics. To begin with, these groups provide the natural settings for understanding the solvability of the equations of the heavy top, as explained in Section 5 on the "kinetic analogue" of Kirchhoff, and secondly, they provide the appropriate symplectic structure required for understanding the Hamiltonian character of the Heisenberg's magnetic equation and its relation to the Schroedinger's nonlinear equation. In this context, it should be noted that the existence of the symplectic structure is crucially dependent on the fact that the dimension of the Cartan space $\mathfrak{p}_\epsilon$ is equal to the dimension of the isotropy algebra $\mathfrak{k}$.

Regrettably, the problem of Dubins–Dealauney did not get as much attention in this paper as it deserves since it is the only problem in the paper that can be tackled only by control theoretic methods: optimal solutions which are the concatenations of singular and boundary controls are outside the scope of the classical calculus of variations. In contrast to the two dimensional situation, where the optimal solutions are known to consist of at most three arcs, no such results are available for the three dimensional system and its optimal synthesis still remains as an open problem. The interested reader may consult [22] for a (very) partial analysis of the solutions.

It may be fitting to remark that the Affine Problem remains very relevant for the theory of integrable systems beyond the spaces of constant curvature, even though the contact with the elastic problem of Kirchhoff is lost. The spectral matrix $L_\lambda = -L_\mathfrak{p} - \lambda L_\mathfrak{k} + (\lambda^2 - 1)B$ is of central importance in the loop algebras and the work of Reyman and Semenov Tian Shansky [39], although it is still unknown exactly how many functionally independent spectral invariants it produces. Remarkably, on coadjoint orbits of low dimensions in $sl_n(R)$ the affine Hamiltonian is completely integrable and reduces to the Hamiltonian associated to the mechanical problem of Newmann [36], see also [38] and [25]. In general, however, the solutions of the affine Hamiltonian system remain largely unknown.

References

1. R. Abraham and J. Marsden, *Foundations of Mechanics*, Benjamin-Cummings, Reading, Mass, 1978
2. A. Agrachev and Y. Sachkov, *Control Theory from the Geometric Point of View*, Encyclopedia of Mathematical Sciences 87, Springer-Verlag, New York, 2004
3. V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Graduate Texts in Mathematics 60, Springer-Verlag, New York, 1978
4. A.V. Bolsinov, *A completeness criterion for a family of functions in involution obtained by the shift method*, Soviet Math. Dokl. 38 (1989), 161–165
5. R. Bryant and P. Griffiths, *Reduction of order for constrained variational problems and* $\frac{1}{2}\int_\gamma \kappa^2 ds$, Amer. Jour. Math. 108 (1986), 525–570
6. E. Coddington and N. Levinson, *Theory of Ordinary Differential equations*, McGraw Hill, New York, 1955
7. C. Carathéodory, *Calculus of Variations*, Teubner, 1935; Reprinted by Chelsea in 1982
8. M. P. DoCarmo, *Riemannian Geometry*, Birkhäuser, Boston, 1992
9. L. E. Dubins,, *On curves of minimal ength with a constraint on the average curvature and with prescribed initial positions and tangents* Amer. J. Math. 79 (1957), 497–616
10. L. Euler, *Methodus inveniendi lineas curvas maximi minmive proprieatate gaudentes, sive solutio problematis isoperimtrici lattisimo sensu accepti*, Opera Omnia Ser. I, Lausannae 24, 1744

11. P. Eberlein, *Geometry of Nonpositively Curved Manifolds*, University of Chicago Press, Chicago, 1966
12. L. Faddeev and L. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Springer-Verlag, Berlin, 1980.
13. P. Griffiths, *Exterior Differential Systems and the Calculus of Variations*, Birkhäuser, Boston, 1983
14. R. S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. 7 (1972), 65–221
15. H. Hasimoto, *Motion of a vortex filament and its relation to elastica*, J. Phys. Soc. Japan 31 (1971), 293–294
16. H. Hasimoto, *A soliton on a vortex filament,*, J. Fluid Mech. 51 (1972), 477–485
17. S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York, 1978
18. C. G. J. Jacobi, *Vorlesungen uber Dynamik*, Druck und Verlag von G. Reimer, Berlin, 1884
19. V. Jurdjevic, *Non-Euclidean Elasticae*, Amer. J. Math. 117 (1995), 93–125
20. V. Jurdjevic, *Geometric Control Theory*, Cambridge Studues in Advanced Mathematics 52, Cambridge University Press, 1997, New York
21. V. Jurdjevic, *Integrable Hamiltonian Systems on Lie groups: Kowalewski type*, Annals Math. 150 (1999), 605–644
22. V. Jurdjevic, *Hamiltonian Systems on Complex Lie groups and their Homogeneous Spaces*, Memoirs AMS (836) 178 (2005)
23. V. Jurjdevic, *The symplectic structure of curves in three dimensional spaces of constant curvature and the equations of mathematical physics*, Ann. I. H. Poincaré (2009), 1843–1515
24. V. Jurdjevic and F. Perez-Monroy, *Variational problems on Lie groups and their homogeneous spaces: elastic curves, tops and constrained geodesic problems*; in: *Contemporary trends in Non-linear control theory and its Applications* (editors B. Bonnard et al.) World Scientific Press, 2002, 3–51
25. V. Jurdjevic, *Optimal Control on Lie groups and Integrable Hamiltonian Systems*, Regular and Chaotic Dyn. (2011), 514–535
26. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol I*, Interscienc Publishers, John Wiley and Sons, New York, 1963
27. J. Langer and R. Perline, *Poisson Geometry of the Filament Equation*, J. Nonlinear Sci. 1 (1978), 71–93
28. J. Langer and D. Singer, *The total squared curvature of closed curves*, J. Diff. Geometry 20 (1984), 1–22.
29. J. Langer and D. Singer, *Knotted elastic curves in* IR^3, J. London Math. Soc. 2, 30 (1984), 512–534
30. A. E. Love, *A Treatise on the Mathematical Theory of Elasticity*, 4th edition, Dover, New York, 1927
31. F. A. Magri, *A simple model for the integrable Hamiltonian equation*, J. Math. Phys. 19 (1978), 1156–1162
32. J. Millson and B. A. Zombro, *A Kähler structure on the moduli spaces of isometric maps of a circle into Euclidean spaces*, Invent. Math. 123(1) (1996), 35–59
33. Mittenhuber, *Dubins' problem in hyperbolic spaces*, Geometric Control and non-Holonomic Mechanics (Ed. by V. Jurdjevic et al.), Vol 25, CMS Conference Proceedings, Canadian Math. Soc., 1998, 110–115.
34. J. Moser, *Geometry of quadrics and spectral theory*, Proceeding of the International symposium on Differential Geometry held in honor of S. S. Chern, 1978, 147–188
35. J. Moser, *Integrable Hamiltonian systems and Spectral Theory*, Lezioni Fermiane, Academia Nazionale dei Lincei, Scuola Normale Superiore, Pisa, 1981
36. C. Newmann, *De probleme quodam mechanico, quod ad primam integralium ultra-ellipticoram classem revocatum*, J. Reine Angew. Math. (1856), 345–378
37. T. Popiel and L. Noakes, *Elastica in* SO(3), J. Australian Math. Soc. 83(1) (2007), 105–125

38. T. Ratiu, *The C. Newmann problem as a completely integrable system on a coadjoint orbit*, Trans. Amer. Mat. Soc. 264(2) (1981), 321–329
39. A. G. Reyman and Semenov Tian-Shansky, *Group theoretic methods in the heory of finite dimensional integrable systems*; in: *Dynamical Systems VII, Chapter 2, encyclopedia of Mathematical Sciences, Vol 16*, Springer-Verlag, 1994, 116–259
40. A. G. Reyman, *Integrable Hamiltonian systems connected with graded Lie algebras*, J. Soviet. Math. 19 (1980), 1507–1545
41. C. Shabat and V. Zakharov, *Exact theory of two dimensional self-focusing and one dimensional self-modulation of waves in non-linear media*, Sov. Phys. JETP 34 (1972), 62–69
42. H. J. Sussmann, *Orbits of families of vector fields and integrability of distributions*, Trans. Amer. Math. Soc. 180 (1973), 171–188