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PROBABILITY LOGICS

Abstract. The paper summarizes the results of the authors in formalization of uncertain reasoning. A number of probability logics is considered. Their axiomatizations, completeness, compactness and decidability are addressed. Some possible applications of probability logics are analyzed. A historical overview of related works is given.

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Contents

1. Introduction

The problem of reasoning with uncertain knowledge is an ancient problem dating, at least, from Leibnitz and Boole. However, in the last decades there is a growing interest in the field connected with applications to computer science and artificial intelligence. Researchers from those areas have studied uncertain reasoning using different tools. Some of the proposed formalisms for representing, and reasoning with, uncertain knowledge are based on probabilistic logics. That approach extends the classical (propositional or first order) calculus with expressions that speak about probability, while formulas remain true or false. Thus, one is able to make statements of the form (in our notation) $P_{\geqslant s}\alpha$ with the intended meaning "the probability of α is at least s".

The probability operators behave like modal operators and the corresponding semantics consists in special types of Kripke models (possible worlds) with addition of probability measures defined over the worlds. One of the main proof-theoretical problems with that approach is providing an axiom system which would be strongly complete ("every consistent set of formulas has a model", in contrast to the weak completeness "every consistent formula has a model"). This results from the inherent non-compactness of such systems. Namely, in such languages it is possible to define an inconsistent infinite set of formulas, every finite subset of which is consistent (e.g., $\{\neg P_{=0}\alpha\} \cup \{P_{<1/n}\alpha : n \text{is a positive integer}\}$). As it was pointed in [85, 125], there is an unpleasant consequence of finitary axiomatization in that case: there exist unsatisfiable sets of formulas that are consistent with respect to the assumed finite axiomatic system (since all finite subsets are consistent and deductions are finite sequences). Another important theoretical problem is related to the decidability issue.

In this paper we present a number of probabilistic logic. The main differences between the logics are:

- some of the logics are infinitary[1], while the others are finitary,
- the corresponding languages contain different kinds of probabilistic operators, both for unconditional and conditional probability,
- some of the logics are propositional, while the others are based on the first-order logic,
- for most of the logics we start from classical logic, but in some cases the basic logic can be intuitionistic or temporal,
- in some of the logics iterations of probabilistic operators are not allowed,
- for some of the logics restrictions of the following kinds are used: only probability measures with fixed finite range are allowed in models, only one probability measure on sets of possible worlds is allowed in a model, the measures are allowed to be finitely additive.

For all these logics we give the corresponding axiomatizations, prove completeness, and discuss their decidability. More precisely, we consider the following logics (the notation was taken from the corresponding papers):

- LPP_1 (L for logic, the first P for propositional, and the second P for probability), probability logic which starts from classical propositional logic, with iterations of the probability operators and real-valued probability functions [83, 85],

[1] In this paper the terms finitary and infinitary concern meta language only. Object languages are countable, formulas are finite (except where it is explicitly said), while only proofs are allowed to be infinite.

- $LPP_1^{\mathrm{Fr}(n)}$ and LPP_1^S that are similar to LPP_1, but with probability functions restricted to have ranges $\{0, 1/n, \dots, (n-1)/n, 1\}$ and S, respectively [81, 83, 85],
- $LPP_1^{A,\omega_1,\mathrm{Fin}}$, probability logic similar to $LPP_1^{\mathrm{Fr}(n)}$, but with probability functions restricted to have arbitrary finite ranges [26],
- LPP_1^{LTL}, probability logic similar to LPP_1, but the basic logic is discrete linear-time logic LTL [82, 83, 91],
- LPP_2, $LPP_2^{\mathrm{Fr}(n)}$, $LPP_2^{A,\omega_1,\mathrm{Fin}}$ and LPP_2^S, probability logics similar to the above logics, but without iterations of the probability operators [83, 85, 106],
- $LPP_{2,P,Q,O}$, probability logic which extends LPP_2 by having a new kind of probabilistic operators of the form Q_F, with the intended meaning "the probability belongs to the set F" [84],
- $LPP_{2,\preceq}$ and $LPP_{2,\preceq}^{\mathrm{Fr}(n)}$, probability logics similar to LPP_2 and $LPP_2^{\mathrm{Fr}(n)}$, but allowing reasoning about qualitative probabilities [93],
- LPP_2^I, probability logics similar to LPP_2, but the basic logic is propositional intuitionistic logic [74, 75, 76],
- $LFOP_1$, $LFOP_1^{\mathrm{Fr}(n)}$, $LFOP_1^{A,\omega_1,\mathrm{Fin}}$, $LFOP_1^S$ and $LFOP_2$, first-order counterparts of the above logics [85, 110],
- $LPCP_2^{S,\approx}$, propositional Kolmogorov's style-conditional probability logic, without iterations of the probability operators, with probability functions restricted to have the range S and probability operators that can express approximate probabilities [88, 92, 112, 113, 114], and
- $LPCP_2^{\mathrm{Chr}}$, propositional conditional probability logic, which corresponds to de Finetti's view on coherent conditional probabilities [50, 90].

The rest of the paper is organized in the following way. In section 2 we give a short overview of studies relating logic and probability until the mid 1980's, and the work of H. J. Keisler and N. Nilsson [41, 42, 78, 116, 122]. Syntax and semantics, an infinitary axiomatization, the corresponding extended completeness, decidability and complexity of LPP_2 are presented in Section 3.1. As a semantics we introduce a class of models that combine properties of Kripke models and probabilities defined on sets of possible worlds. We consider the class of so called measurable models (which means that all sets of possible worlds definable by classical formulas are measurable) and some of its subclasses: in the first case all subsets of worlds are measurable, then probabilities are required to be σ-additive, while models in the last subclass satisfy that only empty set has the zero probability. The proposed axiomatization is infinitary, i.e., there is an inference rule with countably many premisses and one conclusion. That rule corresponds to the following property of real numbers: if the probability is arbitrary close to s, it is at least s. Thus, proofs with countably many formulas are allowed. The proof of extended completeness follows Henkin procedure: starting from a consistent set we construct its maximal consistent extension and the corresponding canonical model which satisfies the considered set of formulas. Decidability of LPP_2 is proved by reducing the satisfiability problem to linear programming problem. Since the related linear systems can be of exponential sizes, in the same section we describe some heuristical approaches (genetic algorithms and variable neighborhood search) to the probabilistic satisfiability problem [51, 86, 87, 89]. Some variants of LPP_2 ($LPP_2^{\mathrm{Fr}(n)}$, $LPP_2^{A,\omega_1,\mathrm{Fin}}$ and LPP_2^S obtained by putting some restrictions on ranges of probability functions) and the logic LPP_1 are considered in the sections 4 and 5, respectively. In Section 6 we consider some extensions of the basic probability language. The first extension, $LPP_{2,P,Q,O}$, contains probability operators of the form Q_F with the intended meaning "the probability belongs to the set F". It turns out that in a general case neither $P_{\geqslant}$-operators are definable from Q_F-operators, nor are Q_F-operators operators definable from $P_{\geqslant}$-operators. Then, we discuss two logics that allow expressing qualitative probabilities: $LPP_{2,\preceq}$ and $LPP_{2,\preceq}^{\mathrm{Fr}(n)}$. It is proved elsewhere that the set of probability first-order valid formulas is not recursively enumerable and that no recursive complete axiomatization is possible. In Section 7 we extend our approach for the propositional case and give a complete infinitary first order axiomatization. That section also contains a discussion on the (dis)similarities between probability and modal logics. Intuitionistic and temporal probability logics are presented in Section 8. Two logics with conditional probabilities ($LPCP_2^{S,\approx}$ and $LPCP_2^{\mathrm{Chr}}$), and their applications are described in Section 9. One of the infinitary inference rules for $LPCP_2^{S,\approx}$ enables us to syntactically define the range of probability functions. In the case of $LPCP_2^{S,\approx}$, that range is the unit interval of a recursive non-archimedean field which makes it possible to express statements about approximate probabilities: $CP_{\approx s}(\alpha,\beta)$ which means "the conditional probability of α given β is approximately s". Furthermore, formulas of the form $CP_{\approx 1}(\alpha,\beta)$ can be used to model defaults, i.e., expressions of the form "if β, then generally α". It relates $LPCP_2^{S,\approx}$ with the well known system P which forms a core of default reasoning. It is proved that if we restrict attention only to formulas of the form $CP_{\approx 1}(\alpha,\beta)$, the resulting system coincides with P when we work only with finite sets of assumptions. If we allow inferences from infinite sets of such "defaults", our system is somewhat stronger. The main advantage, however, is ability to use $LPCP_2^{S,\approx}$ to combine uncertain knowledge and defaults. Finally, Section 10 discusses some of the more recent related papers.

2. History

Gottfried Wilhelm Leibnitz (1646–1716) investigated universal basis for all sciences and tried to establish logic as a generalized mathematical calculus. He considered probabilistic logic as a tool for the uncertainty estimation, and defined probability as a measure of knowledge. In some of his essays [67, 68, 69] Leibnitz suggested that tools developed for analyzing games of chance should be applied in developing a new kind of logic treating degrees of probability which, in turn, could be used to make rational decisions on conflicting claims. He distinguished two calculi. The first one, forward calculus, was concerned with estimating the probability of an event if the probabilities of its conditions are known. In the second one, called reverse calculus, estimations of probabilities of causes, when the probability of their consequence is known, were considered. Leibnitz's logical works were for the most part published long after his death (by L. Couturat in the early 1900s). However, Leibnitz had some successors, the most important of whom, when the probabilistic logic is in question, were the brothers Jacobus (1654–1705) and Johann (1667–1748) Bernoulli, Thomas Bayes (1702–1761), Johann Heinrich Lambert (1728–1777), Pierre Simon de Laplace (1749–1827), Bernard Bolzano (1781–1848), Augustus De Morgan (1806–1871), George Boole (1815–1864), John Venn (1834–1923), Hugh MacColl (1837–1909), Charles S. Peirce (1839–1887), Platon Sereyevich Poretskiy (1846–1907), etc. We shall briefly mention some of their results.

Jacobus Bernoulli in his unfinished work [7, Part IV, Chapter III], was the first who made advance along the Leibnitz's ideas. Using Huygen's notion of expectation, i.e., the value of a gamble in games of chance, he offered a procedure for determining numerical degrees of certainty of conjectures produced by arguments. The word argument was used to represent statements as well as the implication relation between premises and conclusions. He divided arguments into categories according to whether the premises, and the argumentation from premises to conclusions are contingent or necessary. For example, if an argument exist contingently (i.e., it is true in $b > 0$ cases, while it is not in $c > 0$ cases) and implies a conclusion necessarily, then such an argument establishes $\frac{b}{b+c}$ as the certainty of the conclusion. Bernoulli also discussed the question of computing the degree of certainty when there were more then one argument for the same conclusion.

J. H. Lamber in [65], analyzed syllogistic inference of the form "if three quarters of the A's are B's, and C is A, then with probability $\frac{3}{4}$, C is B". In [5], written by T. Bayes, there was the first occurrence of a result involving conditional probability. In modern notation, he considered the problem of finding the conditional probability $P(A|B)$ where A is the proposition "$P(E) \in [a, b]$", while B is the proposition "an event E happened p and failed q times in $p + q$ independent trials". For B. Bolzano [11] logic was a theory of science, while probability was a part of logic. Using contemporary language it can be said that he understood validity of a proposition $A(x)$ as a measure of the set $\{c : \models A(c)\}$, i.e., as the ratio $\frac{|\{x : x \in U \wedge U \models A(x)\}|}{|\{x : x \in U\}|}$. Relative validity was a relation between propositions and had the same properties as what we call conditional probability. Bolzano derived a number of theorems regarding relative validity. A. De Morgan devoted a chapter of [21], to probability inference offering a defense for the numerical probabilistic approach as a part of logic. Instead of giving a systematic treatment of the field, he rather described some problems and tried to apply logical concepts to them. It is interesting that De Morgan made some mistakes, mainly due to his ignoring of (in)dependence of events.

The calculus inaugurated by G. Boole in [12, 13] initiated rapid development of mathematical logic. Boole sought to make his system the basis of a logical calculus as well as a more general method for the application in the probability theory. He wrote "... Every system of interpretation which does not affect the truth of the relations supposed is equally admissible, and it is thus that the same process may under one scheme of interpretation represent the solution of a question on the properties of numbers, under another that of a geometrical problems, and under the third that of a problem of dynamics or optics..." Since 1854 Boole concentrated on unification of various elements of truth. He hoped to continue the advancement toward probable indications concerning the nature and structure of human thought. The most general problem (originally called "general problem in the theory of probability") Boole claimed that he could solve, concerned an arbitrary set of logical functions $\{f_1(x_1, \ldots, x_m),\ \ldots,\ f_k(x_1, \ldots, x_m),\ F(x_1, \ldots, x_m)\}$ and the corresponding probabilities $p_1 = P(f_1(x_1, \ldots, x_m))$, ..., $p_k = P(f_k(x_1, \ldots, x_m))$, and asked for $P(F(x_1, \ldots, x_m))$ in terms of $p_1, \ldots, p_k$. He explained the relation between the logic of classical connectives and the formal probability properties of compound events using the following assumptions. He restricted disjunctions to the exclusive ones, and believed that any compound proposition can be expressed in terms of, maybe ideal, simple and independent components. Thus, the probability of an or-compound is equal to the sum of the components, while the probability of an and-compound is equal to the product of the components. In such a way, it was possible to convert logical functions of events into a system of algebraic functions of the corresponding probabilities. Boole tried to solve such systems using a procedure equivalent to Fourier–Motzkin elimination. His procedure, although not entirely successful, provided a basis for probabilistic inferences. In [40, 41] a rationale and a correction for the Boole's procedure were given using the linear programming approach. It was noted that analytical expressions of the lower and upper bounds of the probabilities could be obtained.

The successors of Boole tried to improve the form of Boole's ideas. One of them was P. S. Poretskiy [96]. C. S. Pierce in [95] and H. MacColl in [71] clarified the notion of conditional probability, as the chance that a statement is true on the assumption that another statement is true, and introduced the corresponding symbol x_a ($P(x|a)$, in the contemporary formal language).

McColl also developed, contemporaneously with Frege, propositional logic as a branch of logic independent of the class calculus or term logic of the traditional syllogisms. He was the first author who made an attempt, in [72], to augment the two-valued logical formalism with a third truth value. It was a system of propositional logic with certain, impossible, and variable propositions. The propositions of the former two types are either necessary true or necessary false, while the propositions of the last type are sometimes true and sometimes false. MacColl's idea of proceeding along the probabilistic lines in the development of many-valued logic is of particular interest because he applied the calculus of variable propositions to the calculus of probabilities. His truth-values, like probabilities, cannot be combined in a truth-functional way. For example, if p is a variable proposition, so are $p \wedge p$ and $\neg p$, while $p \wedge \neg p$ is impossible rather than variable. Later systems, for example the ones of Lukasiewicz, were deficient in this respect.

In the 1870's J. Venn developed the idea of extending the frequency of occurrence concept of probability to logic. Venn thought that probability logic is the logic of sequence of statements. A single element sequence of this type attributes to the given proposition one of two values 0 or 1, while an infinite sequence attributes any real number which lies in the interval $[0, 1]$. Some of the traditional logicians were dissatisfied with the inclusion of the induction in the definition of the concept of probability, but the others continued to work in that direction.

During the first half of XX century there were at least three directions in the development of theory of probability. The researchers that belonged to the first one, Richard von Miss (1883–1953) and Hans Reichenbach (1891–1953), for example, regarded probability as a relative frequency and derived rules of the theory from that interpretation. The second approach was characterized by the development of formal calculus of probability. Some of the corresponding authors were Georg Bohlmann (1869–1928) [10], Sergei Natanovich Bernstein (1880–1968) [8], andÉmil Borel (1871–1956) [14, 15]. These investigations culminated in A. N. Kolmogorov's (1903–1987) axiomatization of probability [60]. Finally, some of the researcher, like John M. Keynes (1883–1946) [59], Hans Reichenbach [115, 116], and Rudolf Carnap (1891–1970) [18, 19] continued Boole's approach connecting probability and logic.

In work of J. Keynes probability was seen as an undefined primitive concept. He presented an axiomatic analysis of a relation between propositions which behaved like conditional probability. That axiomatic system is not acceptable, at least from the point of the recent logical standards. For example, no specification of syntax was given, there were no inference rules, etc.

R. Carnap's work on logical foundations of probability was an attempt to develop a pure logical concept of probability. Carnap connected the concepts of inductive reasoning, probability and confirmation. He was among the first researchers who clearly acknowledged that there are two distinct concepts of probability. The concept of probability as the relative frequency (in the long run) which is used in statistical investigations is empirical in nature and, therefore, unsuitable for the development of inductive logic. For the development of inductive logic, which in his view is the same as probability logic, he needed the logical concept of probability as a degree of confirmation of some hypothesis on the basis of some evidence, i.e., a logical relation between two propositions, denoted by $c(h, e)$. Carnap fixed an unary first order language to express h and e, and studied properties of c. Even though Carnap's work was not completely successful, it stimulated a line of research on probabilistic first-order logics [33, 34, 120, 123]. In [33] there was a generalization of the notion of a model for a first-order language in which probability values replaced truth-values, and some kind of completeness theorem was proven. Similarly, in [34] a first order language L of arithmetic and a set of its models were considered. To every sentence the set of models in which it is true was associated, and the probability was defined on such definable sets. Then, they studied random sequences and some other notions from the theory of probability defined over L. In [120] the ideas from [33] were extended to infinitary languages. Boolean algebras with attached probability measures were considered as suitable models for reasoning about probability. Let I and m denote an interpretation and a probability defined on a Boolean algebra, respectively. A probability assertion A is a tuple $(a, s_1, \ldots, s_n)$, where a is a formula of the language of real closed fields with n free variables, while s_i's are sentences of an infinitary first order language. A speaks about probabilities such that it holds in a model if $a(m(I(s_1)), \ldots, m(I(s_n)))$ is true in the reals. Then, a probability assertion A is a consequence of a set T of assertions if A holds in every model of T. In [120] a number of results about such a consequence relation were proved.

H. Reichenbach investigated the logical structure of probability statements from the philosophical and technical points of view. He introduced a fundamental probability relation between classes and real numbers using formulas of the form $P(A, B) = p$ which could be read as "for every i, if x_i belongs to the class A, then y_i belongs to the class B with probability p". Reichenbach gave a frequency interpretation for the probability relation, and the corresponding axioms $((A \to B) \to (P(A, B) = 1)$, for example). If $x_i \in A$ for every i, he used $P(B) = p$ instead of $P(A, B) = p$, and constructed truth tables for the classical connectives with a continuous scale of truth (if $P(A) = p$, $P(B) = q$, and $P(A, B) = u$, then $P(A \vee B) = p + q - pu$, for example). However, as can be seen, the value of $P(A \vee B) = p + q - pu$ depends on three values, i.e., on $P(A)$, $P(B)$, and $P(A, B)$, and not on $P(A)$ and $P(B)$ only, as it is the case in the classical two-valued logic.

Aleksandar Kron (1938–2000), Belgrade's logician and philosopher, studied relationship between multi-valued logics and probability theory [64]. He considered a unary operation generating a Boolean algebra of sets of formulas, and a probability function defined on that algebra, and gave some statement connecting notions from probability theory (conditional probability, independence) and logic (implication, proof).

In spite of the mentioned works of Reichenbach, Carnap and their followers, the mainstreams of development of logic and probability theory were almost separated during second half of XX century. Namely, in the last quarter of XIX century, independently of the algebraic approach, there was a development of mathematical logic inspired by the need of giving axiomatic foundations of mathematics. The main representative of that effort was Gottlob Frege (1848–1925). He tried to explain the fundamental logical relationships between the concepts and propositions of mathematics. Truth-values, as special kinds of abstract values, were described by Frege according to whom every proposition is a name for truth or falsity. It is clear that, according to Frege, the truth values had a special status that had nothing to do with probabilities. That approach culminated with Kurt Gödel's (1904–1977) proof of the completeness for the first order logic [37]. Since those works, the first order logic played the central role in the logical community for many years, and only in the late 70's a wider interest in probability logics reappeared.

The most important advancement in probability logic, after work of Leibnitz and Boole, was made by H. Jerome Keisler. The purpose of his famous paper [54] was to give model-theoretic approach to probability theory. Also it is important to emphasize that in this paper he makes use of nonstandard analysis as an useful method.

Keisler introduced several probability quantifiers, as for example $Px > r$. The formula $(Px > r)\phi(x)$ means that the set $\{x : \phi(x)\}$ has probability greater than r. A recursive axiomatization for that kind of logics (the main one denoted by L_{AP}) was given by D. Hoover [46]. He used admissible and countable fragments of infinitary predicate logic (but without ordinary quantifiers $\forall$ and $\exists$). In the following years Keisler and Hoover made very important contributions in the field. They proved Completeness theorem for various kinds of models (probability, graded, analytic, hyperfinite etc.) and many other model-theoretical theorems. The development of probability model theory has engendered the need for the study of logics with greater expressive power than that of the logic L_{AP}. The logic L_{AI}, introduced in [55] as an equivalent of the logic L_{AP}, allows us to express many properties of random variables in an easier way. In this logic the quantifiers $\int \dots dx$ are incorporated instead of the quantifiers $Px > r$. The completeness proof for L_{AI} used the Loeb construction of the Daniell integral (see also [22, 23, 24, 25]).

The logic L_{AI} is not rich enough to express probabilistic notions involving conditional expectations of random variables with respect to σ-algebras, such as martingale, Markov process, Brownian motion, stopping time, optional stochastic process, etc. These properties can be naturally expressed in a language with both integral quantifiers and conditional expectation operators. The logics LAE and Lad introduced by Keisler in [55], are appropriate for the study of random variables and stochastic processes. The model theory of these logics has been developed further by Hoover in [47], Keisler in [57, 58], Rodenhausen in [118] and Fajardo in [29].

In [97] Rašković introduced new L_{AM} logic which, instead of probability measure, has σ-finite one and give the method how to transfer results from L_{AP} to L_{AM}. In a series of papers [98, 99, 101, 104, 108], he also gave answers to a number of problems proposed by Keisler in [55]. In [98, 99] a new method of using Barwise compactness theorem [4] in proving completeness theorems was presented. It is difficult to mix ordinary and probability quantifiers because of the fact that projection of a measurable set can be nonmeasurable. As a consequence of that it is hard (if not impossible) to expect adequate logic in its full strength. But some effort in that direction has been made in [100, 102, 103, 105]. The notion of a cylindric probability algebra can be considered as a common algebraic abstraction from a geometry associated with basic set-theoretic notions on the one hand and the theory of deductive systems of probability logic on the other. These two sources are connected because models of deductive systems of probability logic give rise in natural way to probability structures within set-theoretical algebras. As is well known, the theory of Boolean algebras is related to the sentential calculus, and theory of cylindric algebras to the first-order predicate logic. The theory of cylindric probability algebras, designed to provide an apparatus for an algebraic study of probability logics, is presented in [49, 109, 111] analogously to Boolean algebras and cylindric algebras. The model theory for probability logic with undetermined finite range is given in [104]. Continuous time probability logic L^t_{AP}, developed in [107], is a logic appropriate for the study of a space with a family of continuous time probability measures. The set of universal conjunctive formulas of L^t_{AP} is the least set containing all quantifier-free formulas and closed under arbitrary $\wedge$, finite $\vee$, and quantifiers $(Px > r)$, $r \in Q \cap [0, 1]$. The completeness theorem and finite compactness theorem (for universal conjunctive formulas) were proven.

Since the middle of 1980's the interest in probabilistic logics started growing because of development of many fields of application of reasoning about uncertain knowledge: in economics, artificial intelligence, computer science, philosophy etc. Researchers attempt to combine probability-based and logic-based approaches to knowledge representation. In the logical framework for modelling uncertainty, probabilities express degrees of belief. For example, one can say that "probability that Homer wrote Iliad is at most a half" expressing one's disbelief that Homer is the real author of Iliad. The first of those papers is [79] (see also: [80]) which resulted from the work on developing an expert system in medicine, where N. Nilsson tried to give a logic with probabilistic operators as a well-founded framework for uncertain reasoning. Sentences of the logic spoke about probabilities. He was able to express a probabilistic generalization of modus ponens as "if α holds with the probability s, and β follows from α with the probability t, then the probability of β is r".

3. Logic LPP_2

In this section we present the logic LPP_2. We describe its syntax and some classes of models, give an infinitary axiomatization and prove that it is sound and complete with respect to the mentioned classes of models.

3.1. Syntax.

Let S be the set of all rational numbers from $[0, 1]$. The language of LPP_2 consists of the denumerable set $\phi = \{p, q, r, \ldots\}$ of primitive propositions, classical propositional connectives $\neg$, and $\wedge$, and a list of probability operators $P_{\geqslant s}$ for every $s \in S$. The set For_C of all classical propositional formulas over the set ϕ is defined as usual. The formulas from the set For_C will be denoted by α, β,... If $\alpha \in \mathrm{For}_C$ and $s \in S$, then $P_{\geqslant s}\alpha$ is a *basic probability formula*. The set For_P of all probability formulas is the smallest set

- containing all basic probability formulas, and
- closed under formation rules: if $A, B \in \mathrm{For}_P$, then $\neg A$, $A \wedge B \in \mathrm{For}_P$.

The formulas from the set For_P will be denoted by A, B,... Let $\mathrm{For}_{LPP_2} = \mathrm{For}_C \cup \mathrm{For}_P$. The formulas from the set For_{LPP_2} will be denoted by Φ, Ψ,...

We use the usual abbreviations for the other classical connectives, and also denote:

- $\neg P_{\geqslant s}\alpha$ by $P_{<s}\alpha$,
- $P_{\geqslant 1-s}\neg\alpha$ by $P_{\leqslant s}\alpha$,
- $\neg P_{\leqslant s}\alpha$ by $P_{>s}\alpha$,
- $P_{\geqslant s}\alpha \wedge P_{\leqslant s}\alpha$ by $P_{=s}\alpha$, and
- both $\alpha \wedge \neg\alpha$ and $A \wedge \neg A$ by $\perp$, letting the context determine the meaning.

As it can be seen, neither mixing of pure propositional formulas and probability formulas, nor nested probability operators are allowed. Thus, $\alpha \wedge P_{\geqslant s}\beta$ and $P_{\geqslant s}P_{\geqslant r}\alpha$ do not belong to the set For_{LPP_2}.

Let $p_1,\ldots, p_n$ be a list of all primitive propositions from $\Phi \in \mathrm{For}_{LPP_2}$. An *atom* a of Φ is a formula of the form $\pm p_1 \wedge \ldots \wedge \pm p_n$, where $\pm p_i$ is either p_i, or $\neg p_i$.

3.2. Semantics.

The semantics for For_{LPP_2} will be based on the possible-world approach.

Definition 1. An LPP_2-*model* is a structure $\mathbf{M} = \langle W, H, \mu, v\rangle$ where:

- W is a nonempty set of objects called worlds,

- H is an algebra of subsets of W, and
- μ is a finitely additive measure, $\mu : H \to [0, 1]$,
- $v : W \times \phi \to \{\text{true}, \text{false}\}$ provides for each world $w \in W$ a two-valued evaluation of the primitive proposition, that is $v(w, p) \in \{\text{true}, \text{false}\}$, for each primitive proposition $p \in \phi$ and each world $w \in W$; a truth-evaluation $v(w, \cdot)$ is extended to classical propositional formulas as usual.

If $\mathbf{M}$ is an LPP_2-model and $\alpha \in \text{For}_C$, the set $\{w : v(w, \alpha) = \text{true}\}$ is denoted by $[\alpha]_{\mathbf{M}}$. We will omit the subscript $\mathbf{M}$ from $[\alpha]_{\mathbf{M}}$ and write $[\alpha]$ if $\mathbf{M}$ is clear from the context. An LPP_2-model $\mathbf{M} = \langle W, H, \mu, v\rangle$ is *measurable* if $[\alpha]_{\mathbf{M}} \in H$ for every formula $\alpha \in \text{For}_C$. In this section we focus on the class of all measurable LPP_2-models (denoted by $LPP_{2,\text{Meas}}$).

Definition 2. *The satisfiability relation* $\models \subseteq LPP_{2,\text{Meas}} \times \text{For}_{LPP_2}$ fulfills the following conditions for every $LPP_{2,\text{Meas}}$-model $\mathbf{M} = \langle W, H, \mu, v\rangle$:

- if $\alpha \in \text{For}_C$, $\mathbf{M} \models \alpha$ iff for every $w \in W$, $v(w, \alpha) = \text{true}$,
- if $\mathbf{M} \models P_{\geqslant s}\alpha$ iff $\mu([\alpha]) \geqslant s$,
- if $A \in \text{For}_P$, $\mathbf{M} \models \neg A$ iff $\mathbf{M}, \not\models A$,
- if $A, B \in \text{For}_P$, $\mathbf{M} \models A \wedge B$ iff $\mathbf{M} \models A$ and $\mathbf{M} \models B$. □

Definition 3. A formula $\Phi \in \text{For}_{LPP_2}$ is *satisfiable* if there is an $LPP_{2,\text{Meas}}$-model $\mathbf{M}$ such that $\mathbf{M} \models \Phi$; Φ is *valid* if for every $LPP_{2,\text{Meas}}$-model $\mathbf{M}$, $\mathbf{M} \models \Phi$; a set of T formulas is satisfiable if there is an $LPP_{2,\text{Meas}}$-model $\mathbf{M}$ such that $\mathbf{M} \models \Phi$ for every $\Phi \in T$.

Example 4. Consider the set $T = \{\neg P_{=0}\alpha\} \cup \{P_{<1/n}\alpha : n \text{ is a positive integer}\}$. Although every finite subset of T is $LPP_{2,\text{Meas}}$-satisfiable, the set T itself is not. So, the compactness theorem "If every finite subset of T is satisfiable, then T is satisfiable" does not hold for LPP_2. □

Example 5. Note that the classical formulas do not behave in the usual way: for some α and $\beta \in \text{For}_C$ and an $LPP_{2,\text{Meas}}$-model $\mathbf{M}$ it can be $\mathbf{M} \models \alpha \vee \beta$, but that neither $\mathbf{M} \models \alpha$, nor $\mathbf{M} \models \beta$. Similarly, it can be simultaneously $\mathbf{M} \not\models \alpha$ and $\mathbf{M} \not\models \neg\alpha$. Nevertheless, the set of all classical formulas that are valid with respect to the above given semantics and the set of all classical valid formulas coincide, because every world from an arbitrary $LPP_{2,\text{Meas}}$-model can be seen as a classical propositional interpretation.

In the sequel we will also consider the following classes of LPP_2-models:

$$LPP_{2,\text{Meas,All}}, \quad LPP_{2,\text{Meas},\sigma} \quad \text{and} \quad LPP_{2,\text{Meas,Neat}}.$$

A model $\mathbf{M} = \langle W, H, \mu, v\rangle$ belongs to the first class if H is the power set of W, i.e., if every subset of W is μ-measurable. A model $\mathbf{M}$ belongs to the second class if it is a σ-additive measurable model, i.e., if μ is a σ-additive probability measure. A model $\mathbf{M}$ belongs to the second class if it is a measurable model such that $\mu(H_1) = 0$ iff $H_1 = \emptyset$, i.e., if only the empty set has the zero probability.

3.3. Complete Axiomatization. The set of all valid formulas can be characterized by the following set of axiom schemata:

(1) all instances of the classical propositional tautologies
(2) $P_{\geqslant 0}\alpha$
(3) $P_{\leqslant r}\alpha \to P_{<s}\alpha$, $s > r$
(4) $P_{<s}\alpha \to P_{\leqslant s}\alpha$
(5) $(P_{\geqslant r}\alpha \wedge P_{\geqslant s}\beta \wedge P_{\geqslant 1}(\neg(\alpha \wedge \beta))) \to P_{\geqslant \min(1,r+s)}(\alpha \vee \beta)$
(6) $(P_{\leqslant r}\alpha \wedge P_{<s}\beta) \to P_{<r+s}(\alpha \vee \beta)$, $r + s \leqslant 1$

and inference rules:

(1) From Φ and $\Phi \to \Psi$ infer Ψ.
(2) From α infer $P_{\geqslant 1}\alpha$.
(3) From $A \to P_{\geqslant s-\frac{1}{k}}\alpha$, for every integer $k \geqslant \frac{1}{s}$, and $s > 0$ infer $A \to P_{\geqslant s}\alpha$.

We denote this axiomatic system by Ax_{LPP_2}.

Definition 6. A formula Φ is *deducible from a set* T of formulas (denoted by $T \vdash \Phi$) if there is an at most denumerable sequence of formulas $\Phi_0, \Phi_1, \ldots, \Phi$, such that every Φ_i is an axiom or a formula from the set T, or it is derived from the preceding formulas by an inference rule. A *proof* for Φ from T is the corresponding sequence of formulas. A formula Φ is a *theorem* (denoted by $\vdash \Phi$) if it is deducible from the empty set. □

Definition 7. A set T of formulas is *consistent* if there are at least a formula from For_C, and at least a formula from For_P that are not deducible from T, otherwise T is *inconsistent*. A consistent set T of formulas is said to be *maximal consistent* if the following holds:

- for every $\alpha \in \mathrm{For}_C$, if $T \vdash \alpha$, then $\alpha \in T$ and $P_{\geqslant 1}\alpha \in T$, and
- for every $A \in \mathrm{For}_P$, either $A \in T$ or $\neg A \in T$.

A set T of formulas is *deductively closed* if for every $\Phi \in \mathrm{For}_{LPP_2}$, if $T \vdash \Phi$, then $\Phi \in T$.

Alternatively, we can say that T is inconsistent iff $T \vdash \bot$. Also, note that classical and probability formulas are handled in different ways in Definition 7: it is not required that for every classical formula α, either α or $\neg\alpha$ belongs to a maximal consistent set, as it is done for formulas from For_P.

Let us now discuss the above axioms and rules. First note that, by Axiom 1, the classical propositional logic is a sublogic of LPP_2. It is also easy to see that every LPP_2-proof consists of two parts (one of them may be empty). In the first one only classical formulas are involved, while the second one uses formulas from For_P. Two parts are separated by some applications of Rule 2. There is no inverse rule, so we can pass from the classical to the probability level, but we cannot come back. It follows that LPP_2-logic is a conservative extension of the classical propositional logic. The axioms 2– 6 concern the probabilistic aspect of LPP_2. Axiom 2 announces that every formula is satisfied by a set of worlds of the measure at least 0. By substituting $\neg\alpha$ for α in the axiom, the formula $P_{\geqslant 0}\neg\alpha$ is obtained. According to our definition of the operator $P_{\leqslant 1}$, we have the following instance of Axiom 2:

2′. $P_{\leqslant 1}\alpha$ ($= P_{\geqslant 1-s}\neg\alpha$, for $s = 1$).

It forces that every formula is satisfied by a set of time instants of the measure at most 1, and gives the upper bound for probabilities of formulas in $LPP_{2,\mathrm{Meas}}$-models. In a similar way,the axioms 3 and 4 are equivalent to

3′. $P_{\geqslant t}\alpha \to P_{>s}\alpha$, $t > s$

4′. $P_{>s}\alpha \to P_{\geqslant s}\alpha$

respectively. The axioms 5 and 6 correspond to the additivity of measures. For example, in Axiom 5, if sets of worlds that satisfy α and β are disjoint, then the measure of the set of worlds that satisfy $\alpha \vee \beta$ is the sum of the measures of the former two sets. Rule 1 is classical Modus Ponens. Rule 2 can be considered as the rule of necessitation in modal logics, but it can be applied on the classical propositional formulas only. Rule 3 is the only infinitary inference rule in the system, i.e., it has a countable set of assumptions and one conclusion. It corresponds to the Archimedean axiom for real numbers and intuitively says that if the probability is arbitrary close to s, then it is at least s.

3.4. Soundness and completeness.

3.4.1. Soundness. Soundness of our system follows from the soundness of classical propositional logic, as well as from the properties of probabilistic measures, so we give only a sketch of a straightforward but tedious proof.

Theorem 8 (Soundness). *The axiomatic system Ax_{LPP_2} is sound with respect to the class of $LPP_{2,\mathrm{Meas}}$-models.*

Proof. We can show that every instance of an axiom schemata holds in every model, while the inference rules preserve the validity. For example, let us consider Axiom 5. Suppose that $P_{\geqslant r}\alpha$, $P_{\geqslant s}\beta$, and $P_{\geqslant 1}\neg(\alpha \vee \beta)$ hold in a model $\mathbf{M} = \langle W, H, \mu, v\rangle$. It means that $\mu([\alpha]) \geqslant r$, $\mu([\beta]) \geqslant s$, and that $[\alpha]$ and $[\beta]$ are disjoint sets. By the definition of finitely additive measures, the measure of $[\alpha] \cup [\beta]$ (which is $[\alpha \vee \beta]$) is $\mu([\alpha]) + \mu([\beta])$. Hence, $M \models P_{\geqslant \min(1,r+s)}(\alpha \vee \beta)$, and Axiom 5 holds in M. The other axioms can be proved to be valid in a similar way.

Rule 1 is validity-preserving for the same reason as in classical logic. Consider Rule 2 and suppose that a formula $\alpha \in \mathrm{For}_C$ is valid. Then, for every model $\mathbf{M} = \langle W, H, \mu, v\rangle$, $[\alpha] = W$, and $\mu([\alpha]) = 1$. Hence, $P_{\geqslant 1}\alpha$ is valid too. Rule 3 preserves validity because of the properties of the set of real numbers. □

3.4.2. Completeness. In the proof of the completeness theorem the following strategy is applied. We start with a form of Deduction theorem (Theorem 9) and some other auxiliary statements (the lemmas 10, 11, 12). Then, we show how to extend a consistent set T of formulas to a maximal consistent set T^* (Theorem 13). Finally, the canonical model $\mathbf{M}_T$ is constructed using the set T^* (Theorem 14) such that $\mathbf{M}_T \models \varphi$ iff $\varphi \in T^*$ (Theorem 15).

Theorem 9 (Deduction theorem). *If T is a set of formulas and $\varphi, \psi \in \mathrm{For}_C$ or $\varphi, \psi \in \mathrm{For}_P$, then*

$$T \cup \{\varphi\} \vdash \psi \textit{ iff } T \vdash \varphi \to \psi.$$

Proof. The implication from right to left can prove exactly in the same way as in the classical propositional case. For the other direction we use the transfinite induction on the length of the proof of ψ from $T \cup \{\varphi\}$. The cases when either $\vdash \psi$ or $\varphi = \psi$ or ψ is obtained by application of Modus Ponens (Rule 1) are standard.

Thus, let us consider the case where $\psi = P_{\geqslant 1}\alpha$ is obtained from $T \cup \{\varphi\}$ by an application of Rule 2, and $\varphi \in \mathrm{For}_P^S$. In that case:

$T, \varphi \vdash \alpha$
$T, \varphi \vdash P_{\geqslant 1}\alpha$ by Rule 2

However, since $\alpha \in \mathrm{For}_C$, and $\varphi \in \mathrm{For}_P^S$, φ does not affect the proof of α from $T \cup \{\varphi\}$, and we have:

(1) $T \vdash \alpha$
(2) $T \vdash P_{\geqslant 1}\alpha$ by Rule 2
(3) $T \vdash P_{\geqslant 1}\alpha \to (\varphi \to P_{\geqslant 1}\alpha)$
(4) $T \vdash \varphi \to P_{\geqslant 1}\alpha$ by Rule 1.

Next, let us consider the case where $\psi = A \to P_{\geqslant s}\alpha$ is obtained from $T \cup \{\varphi\}$ by an application of Rule 3, and $\varphi \in \mathrm{For}_P$. Then:

(1) $T, \varphi \vdash A \to P_{\geqslant s-\frac{1}{k}}\alpha$, for every integer $k \geqslant \frac{1}{s}$
(2) $T \vdash \varphi \to (A \to P_{\geqslant s-\frac{1}{k}}\alpha)$, for $k \geqslant \frac{1}{s}$, by the induction hypothesis
(3) $T \vdash (\varphi \wedge A) \to P_{\geqslant s-\frac{1}{k}}\alpha$, for $k \geqslant \frac{1}{s}$
(4) $T \vdash (\varphi \wedge A) \to P_{\geqslant s}\alpha$, from (3) by Rule 3
(5) $T \vdash \varphi \to \psi$. □

Lemma 10.

(1) $\vdash P_{\geqslant 1}(\alpha \to \beta) \to (P_{\geqslant s}\alpha \to P_{\geqslant s}\beta)$,
(2) *if* $\vdash \alpha \leftrightarrow \beta$, *then* $\vdash P_{\geqslant s}\alpha \leftrightarrow P_{\geqslant s}\beta$,
(3) $\vdash P_{\geqslant s}\alpha \to P_{\geqslant r}\alpha$, $s \geqslant r$,
(4) $\vdash P_{\leqslant r}\alpha \to P_{\leqslant s}\alpha$, $s \geqslant r$.

Proof. (1) First note that using Rule 2, from $\vdash \neg\alpha \vee \neg\bot$, we obtain

$$\vdash P_{\geqslant 1}(\neg\alpha \vee \neg\bot), \tag{1}$$

and similarly, from $\vdash (\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha$ we have

$$\vdash P_{\geqslant 1}((\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha). \tag{2}$$

By Axiom 5, we have $\vdash (P_{\geqslant s}\alpha \wedge P_{\geqslant 0}\bot \wedge P_{\geqslant 1}(\neg\alpha \vee \neg\bot)) \to P_{\geqslant s}(\alpha \vee \bot)$. Since $\vdash P_{\geqslant 0}\bot$ by Axiom 2, from (1) it follows that

$$\vdash P_{\geqslant s}\alpha \to P_{\geqslant s}(\alpha \vee \bot). \tag{3}$$

The expression $P_{\geqslant s}(\alpha \vee \bot)$ denotes $P_{\geqslant s}\neg(\neg\alpha \wedge \neg\bot)$, $P_{\geqslant 1-(1-s)}\neg(\neg\alpha \wedge \neg\bot)$, and $P_{\leqslant 1-s}(\neg\alpha \wedge \neg\bot)$. Similarly, $\neg P_{\geqslant s}\neg\neg\alpha$ denotes $P_{<s}\neg\neg\alpha$. By Axiom 6, we have

$$\vdash (P_{\leqslant 1-s}(\neg\alpha \wedge \neg\bot) \wedge P_{<s}\neg\neg\alpha) \to P_{<1}((\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha).$$

Since $P_{\geqslant 1}((\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha)$ denotes $\neg P_{<1}((\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha)$, from (2) we obtain

$$\vdash (P_{\leqslant 1-s}(\neg\alpha \wedge \neg\bot) \wedge P_{<s}\neg\neg\alpha) \to \\ (P_{<1}((\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha) \wedge \neg P_{<1}((\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha)).$$

It follows that $\vdash P_{\leqslant 1-s}(\neg\alpha \wedge \neg\bot) \to \neg P_{<s}\neg\neg\alpha$, i.e.,

$$\vdash P_{\geqslant s}(\alpha \vee \bot) \to P_{\geqslant s}\neg\neg\alpha. \tag{4}$$

From (3) and (4) we obtain $\vdash P_{\geqslant s}\alpha \to P_{\geqslant s}\neg\neg\alpha$. The negation of the formula $P_{\geqslant 1}(\alpha \to \beta) \to (P_{\geqslant s}\alpha \to P_{\geqslant s}\beta)$ is equivalent to $P_{\geqslant 1}(\neg\alpha \vee \beta) \wedge P_{\geqslant s}\alpha \wedge P_{<s}\beta$. Since $\vdash P_{\geqslant s}\alpha \to P_{\geqslant s}\neg\neg\alpha$, this formula implies $P_{\geqslant 1}(\neg\alpha \vee \beta) \wedge P_{\geqslant s}\neg\neg\alpha \wedge P_{<s}\beta$ which can be rewritten as $P_{\geqslant 1}(\neg\alpha \vee \beta) \wedge P_{\leqslant 1-s}\neg\alpha \wedge P_{<s}\beta$. From:

- Axiom 6, $P_{\leqslant 1-s}\neg\alpha \wedge P_{<s}\beta \to P_{<1}(\neg\alpha \vee \beta)$, and
- $P_{<1}\alpha = \neg P_{\geqslant 1}\alpha$,

we have

$$\vdash \neg(P_{\geqslant 1}(\alpha \to \beta) \to (P_{\geqslant s}\alpha \to P_{\geqslant s}\beta)) \to P_{\geqslant 1}(\neg\alpha \vee \beta) \wedge \neg P_{\geqslant 1}(\neg\alpha \vee \beta),$$

a contradiction. It follows that

$$\vdash P_{\geqslant 1}(\alpha \to \beta) \to (P_{\geqslant s}\alpha \to P_{\geqslant s}\beta).$$

(2) It is an easy consequence of Lemma 10(1).

(3) This formula expresses monotonicity of probabilities. From Axiom 3' $P_{\geqslant s}\alpha \to P_{>r}\alpha$, $s > r$, and Axiom 4' $P_{>r}\alpha \to P_{\geqslant r}\alpha$, we obtain $\vdash P_{\geqslant s}\alpha \to P_{\geqslant r}\alpha$ for $s > r$. If $s = r$, the formula is trivially a theorem of the form $\vdash \varphi \to \varphi$.

(4) Similarly as (3). □

Lemma 11. *Let T be a consistent set of formulas.*

(1) *For any formula $A \in \mathrm{For}_P$, either $T \cup \{A\}$ is consistent or $T \cup \{\neg A\}$ is consistent.*
(2) *If $\neg(\alpha \to P_{\geqslant s}\beta) \in T$, then there is some $n > \frac{1}{s}$ such that $T \cup \{\alpha \to \neg P_{\geqslant s-\frac{1}{n}}\beta\}$ is consistent.*

Proof. (1) The proof is standard: if $T \cup \{A\} \vdash \bot$, and $T \cup \{\neg A\} \vdash \bot$, by Deduction Theorem we have $T \vdash \bot$.

(2) Suppose that for every $n > \frac{1}{s}$:

$$T, \alpha \to \neg P_{\geqslant s-\frac{1}{n}}\beta \vdash \bot.$$

By Deduction Theorem, and manipulation at the propositional level, we have

$$T \vdash \alpha \to P_{\geqslant s-\frac{1}{n}}\beta,$$

for every $n > \frac{1}{s}$. By application of Rule 3 we obtain $T \vdash \alpha \to P_{\geqslant s}\beta$, a contradiction with the fact that $\neg(\alpha \to P_{\geqslant s}\beta) \in T$. □

Lemma 12. *Let T be a maximal consistent set of formulas. Then,*

(1) *for any formula $A \in \mathrm{For}_P$, exactly one member of $\{A, \neg A\}$ is in T,*
(2) *for all formulas $A, B \in \mathrm{For}_P$, $A \vee B \in T$ iff $A \in T$ or $B \in T$,*
(3) *for all formulas φ, ψ, where either $\varphi, \psi \in \mathrm{For}_C$ or $\varphi, \psi \in \mathrm{For}_P$, $\varphi \wedge \psi \in T$ iff $\{\varphi, \psi\} \subset T$,*
(4) *for every $\varphi \in \mathrm{For}_{LPP_2}$, if $T \vdash \varphi$, then $\varphi \in T$,*
(5) *for all formulas φ, ψ, where either $\varphi, \psi \in \mathrm{For}_C$ or $\varphi, \psi \in \mathrm{For}_P$, if $\{\varphi, \varphi \to \psi\} \subset T$, then $\psi \in T$,*

(6) *for all formulas* φ, ψ*, where either* $\varphi, \psi \in \mathrm{For}_C$ *or* $\varphi, \psi \in \mathrm{For}_P$*, if* $\varphi \in T$ *and* $\vdash \varphi \to \psi$*, then* $\psi \in T$*,*

(7) *for any formula* α*, if* $t = \sup_s\{P_{\geqslant s}\alpha \in T\}$*, and* $t \in S$*, then* $P_{\geqslant t}\alpha \in T$*.*

Proof. Proofs (1)–(6) are standard.

(7) Let $t = \sup_s\{P_{\geqslant s}\alpha \in T\} \in S$. By the monotonicity of the measure (Lemma 10(12)), for every $s \in S$, $s < t$, $T \vdash P_{\geqslant s}\alpha$. Using Rule 3 we have $T \vdash P_{\geqslant t}\alpha$. Since T is a maximal consistent set, it follows from Lemma 12(4) that $P_{\geqslant t}\alpha \in T$. □

Theorem 13. *Every consistent set can be extended to a maximal consistent set.*

Proof. Let T be a consistent set, $Cn_C(T)$ the set of all classical formulas that are consequences of T, and A_0, A_1,... an enumeration of all formulas from For_P. We define a sequence of sets T_i, $i = 0, 1, 2,\dots$ such that:

(1) $T_0 = T \cup Cn_C(T) \cup \{P_{\geqslant 1}\alpha : \alpha \in Cn_C(T)\}$

(2) for every $i \geqslant 0$,

(a) if $T_i \cup \{A_i\}$ is consistent, then $T_{i+1} = T_i \cup \{A_i\}$, otherwise

(b) if A_i is of the form $\beta \to P_{\geqslant s}\gamma$, then $T_{i+1} = T_i \cup \{\neg A_i, \beta \to \neg P_{\geqslant s-\frac{1}{n}}\gamma\}$, for some positive integer n, so that T_{i+1} is consistent, otherwise

(c) $T_{i+1} = T_i \cup \{\neg A_i\}$.

(3) $\mathcal{T} = \bigcup_{i=0}^{\infty} T_i$.

The set T_0 is consistent since it is contains consequences of an consistent set, and similarly for the other members of the family of sets, by Lemma 12 each T_i, $i > 0$, is consistent.

It remains to show that $\mathcal{T}$ is maximal and consistent. The steps 1 and 2 of the above construction fulfill all requirements from Definition 7 which guarantees that $\mathcal{T}$ is maximal. We continue by showing that $\mathcal{T}$ is a deductively closed set which does not contain all formulas, and, as a consequence, that $\mathcal{T}$ is consistent.

First of all, $\mathcal{T}$ does not contain all formulas. If $\alpha \in \mathrm{For}_C$, by the construction of T_0, α and $\neg\alpha$ cannot be simultaneously in T_0. For a formula $A \in \mathrm{For}_P$ the set $\mathcal{T}$ does not contain both $A = A_i$ and $\neg A = A_j$, because $T_{\max(i,j)+1}$ is consistent.

I remains to show that $\mathcal{T}$ is deductively closed. If a formula $\alpha \in \mathrm{For}_C$ and $\mathcal{T} \vdash \alpha$, then by the construction of T_0, $\alpha \in \mathcal{T}$ and $P_{\geqslant 1}\alpha \in \mathcal{T}$. Let $A \in \mathrm{For}_P$. It can be proved by the induction on the length of the inference that if $\mathcal{T} \vdash A$, then $A \in \mathcal{T}$. Note that if $A = A_j$ and $T_i \vdash A$, it must be $A \in \mathcal{T}$ because $T_{\max(i,j)+1}$ is consistent. Suppose that the sequence φ_1, φ_2,..., A forms the proof of A from $\mathcal{T}$. If the sequence is finite, there must be a set T_i such that $T_i \vdash A$, and $A \in \mathcal{T}$. Thus, suppose that the sequence is countably infinite. We can show that for every i, if φ_i is obtained by an application of an inference rule, and all the premises belong to $\mathcal{T}$, then it must be $\varphi_i \in \mathcal{T}$. If the rule is a finitary one, then there must be a set T_j which contains all the premises and $T_j \vdash \varphi_i$. Reasoning as above, we conclude $\varphi_i \in \mathcal{T}$. Next, we consider the only infinitary rule 3. Let $\varphi_i = B \to P_{\geqslant s}\alpha$ be obtained from the set of premises $\{\varphi_i^k = B \to P_{\geqslant s_k}\gamma : s_k \in S\}$. By the induction hypothesis, $\varphi_i^k \in \mathcal{T}$ for every k. If $\varphi_i \notin \mathcal{T}$, by the step 2b of the construction, there are some l and j such that $\neg(B \to P_{\geqslant s}\alpha), B \to \neg P_{\geqslant s-\frac{1}{l}}\gamma \in T_j$. It means that for some $j' \geqslant j$:

- $B \wedge \neg P_{\geqslant s}\alpha \in T_{j'}$,
- $B \in T_{j'}$,
- $\neg P_{\geqslant s-\frac{1}{l}}\gamma, P_{\geqslant s-\frac{1}{l}}\gamma \in T_{j'}$,

which is in contradiction with consistency of $T_{j'}$. □

The set $\mathcal{T}$ is used to define a tuple $\mathbf{M}_T = \langle W, H, \mu, v\rangle$, where:

- $W = \{w \models Cn_C(T)\}$ contains all classical propositional interpretations that satisfy the set $Cn_C(T)$ of all classical consequences of the set T,
- $[\alpha] = \{w \in W : w \models \alpha\}$ and $H = \{[\alpha] : \alpha \in \mathrm{For}_C\}$,
- $\mu : H \to [0, 1]$ such that $\mu([\alpha]) = \sup_s\{P_{\geqslant s}\alpha \in \mathcal{T}\}$, and
- for every world w and every primitive proposition $p \in \phi$, $v(w, p) =$ true iff $w \models p$.

The next theorem states that $\mathbf{M}_T$ is an $LPP_{2,\mathrm{Meas}}$-model.

Theorem 14. *Let $\mathbf{M}_T = \langle W, H, \mu, v\rangle$ be defined as above and α, $\beta \in \mathrm{For}_C$. Then, the following hold:*

(1) *H is an algebra of subsets of W,*
(2) *If $[\alpha] = [\beta]$, then $\mu([\alpha]) = \mu([\beta])$,*
(3) *$\mu([\alpha]) \geqslant 0$.*
(4) *$\mu(W) = 1$ and $\mu(\emptyset) = 0$.*
(5) *$\mu([\alpha]) = 1 - \mu([\neg\alpha])$.*
(6) *$\mu([\alpha] \cup [\beta]) = \mu([\alpha]) + \mu([\beta])$, for all disjoint $[\alpha]$ and $[\beta]$.*

Proof. (1) Let α, α_1, α_2,... α_n be formulas from For_C. It is not hard to see that the following hold:

- $W = [\alpha \vee \neg\alpha]$, and $W \in H$,
- if $[\alpha] \in H$, then its complement $[\neg\alpha]$ belongs to H, and
- if $[\alpha_1]$,..., $[\alpha_n] \in H$, then the union $[\alpha_1] \cup \ldots \cup [\alpha_n] \in H$ because $[\alpha_1] \cup \ldots \cup [\alpha_n] = [\alpha_1 \vee \ldots \vee \alpha_n]$.

Thus, H is an algebra of subsets of W.

(2) It is enough to prove that $[\alpha] \subset [\beta]$ implies $\mu([\alpha]) \leqslant \mu([\beta])$. By the completeness of the propositional logic, $[\alpha] \subset [\beta]$ means that $\alpha \to \beta \in Cn_C(T)$ and $P_{\geqslant 1}(\alpha \to \beta) \in \mathcal{T}$. By Lemma 10(1) we have that for every $s \in S$, $P_{\geqslant s}\alpha \to P_{\geqslant s}\beta \in \mathcal{T}$. Thus, $\mu([\alpha]) \leqslant \mu([\beta])$.

(3) Since $P_{\geqslant 0}\alpha$ is an axiom, $\mu([\alpha]) \geqslant 0$.

(4) Since $p \vee \neg p \in Cn_C(T)$ and $P_{\geqslant 1}(p \vee \neg p) \in \mathcal{T}$ for every $p \in \phi$, we have $W = [p \vee \neg p]$ and $\mu(W) = 1$. On the other hand, obviously, $\mu(\emptyset) \geqslant 0$. Since $P_{\geqslant 1}(p \vee \neg p) = P_{\geqslant 1-0}(p \vee \neg p) = P_{\leqslant 0}\neg(p \vee \neg p) = P_{\leqslant 0}(p \wedge \neg p) = \neg P_{>0}(p \wedge \neg p)$, by Axiom 3', $\sup_s\{P_{\geqslant s}(p \wedge \neg p) \in \mathcal{T}\} = 0$, and $\mu(\emptyset) = 0$.

(5) Let $r = \mu([\alpha]) = \sup_s\{P_{\geqslant s}\alpha \in \mathcal{T}\}$. Suppose that $r = 1$. By Lemma 12(7), $P_{\geqslant 1}\alpha) \in \mathcal{T}$. Thus, $\neg P_{>0}\neg\alpha (= P_{\leqslant 0}\neg\alpha = P_{\geqslant 1}\alpha)$ belongs to $\mathcal{T}$. If for some $s > 0$, $P_{\geqslant s}\neg\alpha \in \mathcal{T}$, by Axiom 3' it must be $P_{>0}\neg\alpha \in w$, a contradiction. It follows that $\mu([\neg\alpha]) = 1$. Next, suppose that $r < 1$. Then, for every rational number $r' \in (r, 1]$, $\neg P_{\geqslant r'}\alpha = P_{<r'}\alpha$, and $P_{<r'}\alpha \in \mathcal{T}$. By Axiom 4, $P_{\leqslant r'}\alpha$ and $P_{\geqslant 1-r'}\neg\alpha$

belong to $\mathcal{T}$. On the other hand, if there is a rational number $r'' \in [0, r)$ such that $P_{\geqslant 1-r''}\neg\alpha \in \mathcal{T}$, then $\neg P_{>r''}\alpha \in \mathcal{T}$, a contradiction. Hence, $\sup_s\{P_{\geqslant s}(\neg\alpha) \in \mathcal{T}\} = 1 - \sup_s\{P_{\geqslant s}\alpha \in \mathcal{T}\}$, i.e., $\mu([\alpha]) = 1 - \mu([\neg\alpha])$.

(6) Let $[\alpha] \cap [\beta] = \emptyset$, $\mu([\alpha]) = r$ and $\mu([\beta]) = s$. Since $[\beta] \subset [\neg\alpha]$, by the above steps (2) and (5), we have $r + s \leqslant r + (1 - r) = 1$. Suppose that $r > 0$, and $s > 0$. By the well known properties of the supremum, for every rational number $r' \in [0, r)$, and every rational number $s' \in [0, s)$, we have $P_{\geqslant r'}\alpha$, $P_{\geqslant s'}\beta \in \mathcal{T}$. It follows by the axiom 5 that $P_{\geqslant r'+s'}(\alpha \vee \beta) \in \mathcal{T}$. Hence, $r + s \leqslant t_0 = \sup_t\{P_{\geqslant t}(\alpha \vee \beta) \in \mathcal{T}\}$. If $r + s = 1$, then the statement trivially holds. Suppose $r + s < 1$. If $r + s < t_0$, then for every rational number $t' \in (r + s, t_0)$ we have $P_{\geqslant t'}(\alpha \vee \beta) \in \mathcal{T}$. We can choose rational numbers $r'' > r$ and $s'' > s$ such that:

$$\neg P_{\geqslant r''}\alpha,\ \ P_{<r''}\alpha \in \mathcal{T}, \quad \neg P_{\geqslant s''}\beta,\ \ P_{<s''}(\beta) \in \mathcal{T} \quad \text{and} \quad r'' + s'' = t' \leqslant 1.$$

By Axiom 4, $P_{\leqslant r''}\alpha \in \mathcal{T}$. Using Axiom 6 we have

$$P_{<r''+s''}(\alpha \vee \beta) \in \mathcal{T}, \quad \neg P_{\geqslant r''+s''}(\alpha \vee \beta) \in \mathcal{T} \quad \text{and} \quad \neg P_{\geqslant t'}(\alpha \vee \beta) \in \mathcal{T},$$

a contradiction. Hence, $r+s = t_0$ and $\mu([\alpha] \cup [\beta]) = \mu([\alpha]) + \mu([\beta])$. Finally suppose that $r = 0$ or $s = 0$. Then we can reason as above, with the only exception that $r' = 0$ or $s' = 0$. □

Theorem 15 (Extended completeness theorem for $LPP_{2,\mathrm{Meas}}$)**.** *A set T of formulas is Ax_{LPP_2}-consistent iff it is $LPP_{2,\mathrm{Meas}}$-satisfiable.*

Proof. The ($\Leftarrow$)-direction follows from the soundness of the above axiomatic system. In order to prove the ($\Rightarrow$)-direction we can construct the $LPP_{2,\mathrm{Meas}}$-model $\mathbf{M}_T$, and show that for every $\varphi \in \mathrm{For}_{LPP_2}$, $\mathbf{M}_T \models \varphi$ iff $\varphi \in \mathcal{T}$.

To begin the induction, let $\varphi = \alpha \in \mathrm{For}_C$. If $\alpha \in Cn_C(T)$, then by the definition of $\mathbf{M}_T$, $\mathbf{M}_T \models \alpha$. Conversely, if $\mathbf{M}_T \models \alpha$, by the completeness of classical propositional logic, $\alpha \in Cn_C(T)$.

Next, let $\varphi = P_{\geqslant s}\alpha$. If $P_{\geqslant s}\alpha \in \mathcal{T}$, then $\sup_r\{P_{\geqslant r}(\alpha) \in \mathcal{T}\} = \mu([\alpha]) \geqslant s$, and $\mathbf{M}_T \models P_{\geqslant s}\alpha$. For the other direction, suppose that $\mathbf{M}_T \models P_{\geqslant s}\alpha$, i.e., that $\sup_r\{P_{\geqslant r}(\alpha) \in \mathcal{T}\} \geqslant s$. If $\mu([\alpha]) > s$, then, by the well known property of supremum and monotonicity of μ, $P_{\geqslant s}\alpha \in \mathcal{T}$. If $\mu([\alpha]) = s$, then by Lemma 12(7), $P_{\geqslant s}\alpha \in \mathcal{T}$.

Let $\varphi = \neg A \in \mathrm{For}_P$. Then $\mathbf{M}_T \models \neg A$ iff $\mathbf{M}_T \not\models A$ iff $A \notin \mathcal{T}$ iff (by Lema 12(1)) $\neg A \in \mathcal{T}$.

Finally, let $\varphi = A \wedge B \in \mathrm{For}_P$. $\mathbf{M}_T \models A \wedge B$ iff $\mathbf{M}_T \models A$ and $\mathbf{M}_T \models B$ iff A, $B \in \mathcal{T}$ iff (by Lema 12(3)) $A \wedge B \in \mathcal{T}$. □

In the last part of this section the canonical model $\mathbf{M}_T$ from Theorem 15 will be used as a weak model, i.e., as a tool in proving completeness with respect to the classes: $LPP_{2,\mathrm{Meas,All}}$, $LPP_{2,\mathrm{Meas},\sigma}$ and $LPP_{2,\mathrm{Meas,Neat}}$.

Theorem 16 (Extended completeness theorem for $LPP_{2,\mathrm{Meas,All}}$)**.** *A set T of formulas is Ax_{LPP_2}-consistent iff it is $LPP_{2,\mathrm{Meas,All}}$-satisfiable.*

Proof. The proof can be obtained by applying the extension theorem for additive measure[2]. on the measure μ from the weak canonical model $\mathbf{M}_T$. Thus, there is a finitely additive measure $\overline{\mu}$ defined on the power set of W that is an extension of the measure μ. □

Theorem 17 (Extended completeness theorem for $LPP_{2,\mathrm{Meas},\sigma}$). *A set T of formulas is Ax_{LPP_2}-consistent iff it is $LPP_{2,\mathrm{Meas},\sigma}$-satisfiable.*

Proof. By the Loeb process and a bounded elementary embedding [46] we can transform the weak canonical model $\mathbf{M}_T$ into a σ-additive probability model ${}^*\mathbf{M}_T$ such that for every formula Φ, $\mathbf{M}_T \models \Phi$ iff ${}^*\mathbf{M}_T \models \Phi$. □

Theorem 18 (Extended completeness theorem for $LPP_{2,\mathrm{Meas},\mathrm{Neat}}$). *A set T of formulas is Ax_{LPP_2}-consistent iff it is $LPP_{2,\mathrm{Meas},\mathrm{Neat}}$-satisfiable.*

Proof. In this proof we use a slightly changed construction of the set $\mathcal{T}$ from Theorem 13. Using the same notation as above, the sequence of sets T_i, $i = 0, 1, 2, \ldots$ is now defined in the following way:

(1) $T_0 = T \cup Cn_C(T) \cup \{P_{\geqslant 1}\alpha : \alpha \in Cn_C(T)\}$
(2) for every $i \geqslant 0$,
  (a) if $T_i \cup \{A_i\}$ is consistent, then $T_{i+1} = T_i \cup \{A_i\}$, otherwise
  (b) if A_i is of the form $\beta \to P_{\geqslant s}\gamma$, then $T_{i+1} = T_i \cup \{\neg A_i, \beta \to \neg P_{\geqslant s-\frac{1}{n}}\gamma\}$, for some positive integer n, so that T_{i+1} is consistent, otherwise
  (c) $T_{i+1} = T_i \cup \{\neg A_i\}$.
  (d) if T_i is enlarged by a formula of the form $P_{=0}\alpha$, add $\neg\alpha$ to T_{i+1} as well.
(3) $\mathcal{T} = \bigcup_{i=0}^{\infty} T_i$.

As it can be seen, the only new step is 2d. We can show that it produces consistent sets, too. So, suppose that for some $\alpha \in \mathrm{For}_C$, $(T_i \cup \{P_{=0}\alpha\}) \cup \{\neg\alpha\} \vdash \bot$. By Deduction theorem, we have that $T_i \cup \{P_{=0}\alpha\} \vdash \alpha$. Since $\alpha \in \mathrm{For}_C$, α belongs to $Cn_C(T)$, and by the construction, we have that $P_{\geqslant 1}\alpha \in T_0$ which leads to inconsistency of $T_i \cup \{P_{=0}\alpha\}$ since:

(1) $T_i, P_{=0}\alpha \vdash P_{\geqslant 1}\alpha$, since $P_{\geqslant 1}\alpha \in T_i$,
(2) $T_i, P_{=0}\alpha \vdash P_{\leqslant 0}\alpha$, by the definition of $P_{=0}$,
(3) $T_i, P_{=0}\alpha \vdash P_{<1}\alpha$, by Axiom 3

and $P_{<1}\alpha = \neg P_{\geqslant 1}\alpha$. The rest of the completeness proof is the same as in Theorem 17. □

The situation that the axiomatic system Ax_{LPP_2} is sound and complete with respect to three different classes of models is similar to the one from the modal framework where, for example, the system K is characterized by the class of all

[2]**Theorem 3.2.10 from** [9]. Let C be an algebra of subsets of a set Ω and $\mu(w)$ a positive bounded charge-a finitely additive measure-on C. Let F be an algebra on Ω containing C. Then there exists a positive bounded charge $\overline{\mu(w)}$ on F such that $\overline{\mu(w)}$ is an extension of $\mu(w)$ from C to F and that the range of $\overline{\mu(w)}$ is a subset of the closure of the range of $\mu(w)$ on C

models, but also by the class of all irreflexive models. In other words, LPP_2-formulas cannot express the differences between the mentioned classes of probability models.

3.5. Decidability and Complexity. In this subsection we will consider the problem of satisfiability of For_{LPP_2} formulas. Since there is a procedure for deciding satisfiability and validity for classical propositional formulas, we will consider For_P-formulas only.

So, let $A \in \mathrm{For}_P$. Recall that an atom a of A is a formula of the form $\pm p_1 \wedge \ldots \wedge \pm p_n$, where $\pm p_i$ is either p_i, or $\neg p_i$, and $p_1, \ldots, p_n$ are all primitive propositions appearing in A. Note that for different atoms a_i and a_j we have $\vdash a_i \to \neg a_j$. Thus, in every $LPP_{2,\mathrm{Meas}}$-model $\mu(a_i \vee a_j) = \mu(a_i) + \mu(a_j)$. It is easy, using propositional reasoning and Lemma 10(2), to show that A is equivalent to a formula

$$DNF(A) = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{k_i} X^{i,j}(p_1, \ldots, p_n)$$

called a disjunctive normal form of$A$, where:

- $X^{i,j}$ is a probability operator from the set $\{P_{\geqslant s_{i,j}}, P_{<s_{i,j}}\}$, and
- $X^{i,j}(p_1, \ldots, p_n)$ denotes that the propositional formula which is in the scope of the probability operator $X^{i,j}$ is in the complete disjunctive normal form, i.e., the propositional formula is a disjunction of the atoms of A.

Theorem 19 (Decidability theorem)**.** *The logic LPP_2 is decidable.*

Proof. As it is noted above, a For_P-formula A is equivalent to

$$DNF(A) = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{k_i} X^{i,j}(p_1, \ldots, p_n).$$

A is satisfiable iff at least one disjunct from $DNF(A)$ is satisfiable. Let the measure of the atom a_i be denoted by y_i. We use an expression of the form $a_t \in X(p_1, \ldots, p_n)$ to denote that the atom a_t appears in the propositional part of $X(p_1, \ldots, p_n)$. A disjunct $D = \bigwedge_{j=1}^{k} X^j(p_1, \ldots, p_n)$ from $DNF(A)$ is satisfiable iff the following system of linear equalities and inequalities is satisfiable:

$$\sum_{i=1}^{2^n} y_i = 1$$
$$y_i \geqslant 0 \text{ for } i = 1, \ldots, 2^n$$
$$\sum_{a_t \in X^1(p_1,\ldots,p_n) \in D} y_t \begin{cases} \geqslant s_1 & \text{if } X^1 = P_{\geqslant s_1} \\ < s_1 & \text{if } X^1 = P_{<s_1} \end{cases} \tag{5}$$
$$\ldots$$
$$\sum_{a_t \in X^k(p_1,\ldots,p_n) \in D} y_t \begin{cases} \geqslant s_k & \text{if } X^k = P_{\geqslant s_k} \\ < s_k & \text{if } X^k = P_{<s_k} \end{cases}$$

Since the problem of $LPP_{2,\mathrm{Meas}}$-satisfiability of A is reduced to the linear systems solving problem, the satisfiability problem for LPP_2-logic is decidable. Finally, since A is $LPP_{2,\mathrm{Meas}}$-valid iff $\neg A$ is not $LPP_{2,\mathrm{Meas}}$-satisfiable, the validity problem is also decidable. □

We can show that the $LPP_{2,\mathrm{Meas}}$-satisfiability problem is NP-complete.

Theorem 20. *The $LPP_{2,\mathrm{Meas}}$-satisfiability problem is* NP*-complete.*

Proof. The lower bound follows from the complexity of the same problem for classical propositional logic. The upper bound is a consequence of the NP-complexity of the satisfiability problem for weight formulas from [27, Theorem 2.9][3]. □

3.6. A heuristical approach to the $LPP_{2,\mathrm{Meas}}$-satisfiability problem. Since the $LPP_{2,\mathrm{Meas}}$-satisfiability problem is NP-complete, it is natural to try to solve its instances using heuristics. In this section we describe such an approach which is based on genetic algorithms.

Genetic algorithms (GA) use populations of individuals. Each individual (also called chromosome) is seen as a possible solution in the search space for the particular problem. Thus, a GA can be seen as a searching procedure for the global optima of the corresponding problem. Individuals are represented by genetic code over a finite alphabet. An evaluation function assigning fitness values to individuals has to be defined. Fitness values indicate quality of the corresponding individuals, while average fitness of entire populations may be good measures of obtained quality of the procedures. GA's consist of applications of the genetic operators to populations that must ensure that average fitness values are continually improved from each generation to subsequent. Basic genetic operators are selection, crossover and mutation, but some additional operators such as inversion, local search, etc., may be used.

Selection mechanism favourizes highly fitted individuals (as well as parts of genetic code of individuals, i.e., genes) to have better chances for reproduction into

---

[3] Statements about complexity of the satisfiability problem for weight formulas from [27]. $|A|$ and $\|A\|$ denote the length of A (the number of symbols required to write A), and the length of the longest coefficient appearing in A, when written in binary, respectively. The size of a rational number a/b, where a and b are relatively prime, is defined to be the sum of lengths of a and b, when written in binary.

Theorem 2.6 Suppose A is a weight formula that is satisfied in some measurable probability structure. Then A is satisfied in a structure (S, H, μ, v) with at most $|A|$ states where every set of states is measurable, and where the probability assigned to each state is a rational number with size $O(|A|\|A\| + |A|\log(|A|))$.

Lemma 2.7 If a system of r linear equalities and/or inequalities with integer coefficients each of length at most l has a nonnegative solution, then it has a nonnegative solution with at most r entries positive, and where the size of each member of the solution is $O(rl + r\log(r))$.

Lemma 2.8 Let A be a weight formula. Let $M = (S, H, \mu, v)$ and $M_0 = (S, H, \mu, v')$ be probability structures with the same underlying probability space (S, H, μ). Assume that $v(w, p) = v'(w, p)$ for every state w and every primitive proposition p that appears in A. Then $M \vDash A$ iff $M_0 \vDash A$.

Theorem 2.9 The problem of deciding whether a weight formula is satisfiable in a measurable probability structure is NPcomplete.

```
InputData();
PopulationInit();
while(not FinishedGA()){
    for (i = 0 ; i < N_pop ; i++) p_i = ObjectiveFunction();
    HeuristicImprovement();
    ComputeFitnesses();
    Selection();
    Crossover();
    Mutation();
}
OutputResults();
```

FIGURE 1. A general description of GA's

next generations. On the other hand, chances for reproduction for less fitted members are reduced, and they are gradually wiped out from populations. Crossover operator partitions a population into a set of pairs of individuals named parents. For each pair a recombination of their genetic material is performed with some probability. In that way nondeterministic exchange of genetic material in populations is obtained. Multiple usage of selection and crossover operators may produce that the variety of genetic materials is lost. It means that some areas of search spaces become not reachable. This usually causes the convergence in local optimums far from the global optimal values. Mutation operator can help to avoid this shortcoming. Parts of individuals (genes) can be changed with some small probability to increase diversibility of genetic material. An initial population is usually generated by random, although sometimes it may be fully or partially produced by an initial heuristic. A general description of GA's is given in Figure 1, where N_{pop} and p_i denote the number of individuals and their objective values, respectively. The objective value of an individual corresponds to the value which the individual owns in the case of the considered problem. The for-loop is repeated until a finishing criterion (the global optima is found, the maximal number of iterations is reached,...) is satisfied. Since the procedure is not complete, if the maximal number of iterations is reached, we do not know whether the considered problem is solvable. HeuristicImprovement() can be optionally included to improve efficiency of GA and/or to help the procedure to escape from local optima.

In this section, we slightly change syntax of probabilistic formulas. Namely, as we will mention below in Section 10, sometimes is suitable to consider boolean combinations of basic weight formulas of the form: $a_1w(\alpha_1) + \cdots + a_nw(\alpha_n) \geqslant c$, where a_i's and c are rational numbers, and α_i's are classical propositional formulas containing primitive propositions from ϕ. The intended meaning of $w(\alpha)$ is "the probability of α". Note that $w(\alpha) \geqslant s$ can be written as $P_{\geqslant s}\alpha$ in our notation. A weight literal is an expression of the form $\sum_i a_iw(\alpha_i) \geqslant c$ or $\sum_i a_iw(\alpha_i) < c$. The logic that allows such kind of formulas is still NP-complete-which can be proved as above, i.e., by reducing the $LPP_{2,\mathrm{Meas}}$-satisfiability problem to linear programming problem – so by using this logic we just add some expressiveness to our language.

Since For_P-formulas can be equivalently translated into their disjunctive normal forms, and a disjunction is satisfiable if at least one disjunct is satisfiable, in the sequel we will only consider formulas of the following form:

$$\bigwedge_{j=1}^{k} a_1^j w(\mathrm{CDNF}(\alpha_1^j)) + \cdots + a_{n_j}^j w(\mathrm{CDNF}(\alpha_{n_j}^j)) rho_j \ c^j,$$

where $\rho_j \in \{\geqslant, <\}$, a_i^j's and c^j are rational numbers, and $\mathrm{CDNF}(\alpha)$ denotes the complete disjunctive normal form of α. We say that such a formula is in the weight conjunctive form (wfc-form). Also, we will use $at \in \mathrm{CDNF}(\alpha)$ to denote that the atom at appears in $\mathrm{CDNF}(\alpha)$.

Example 21. The formula $w(p \to q) + w(p) \geqslant 1.7 \wedge w(q) \geqslant 0.6$ is satisfiable since the same holds for the linear system

$$\begin{array}{l}
\mu(p \wedge q) + \mu(p \wedge \neg q) + \mu(\neg p \wedge q) + \mu(\neg p \wedge \neg q) = 1 \\
\mu(p \wedge q) \geqslant 0 \\
\mu(p \wedge \neg q) \geqslant 0 \\
\mu(\neg p \wedge q) \geqslant 0 \\
\mu(\neg p \wedge \neg q) \geqslant 0 \\
\mu(p \wedge \neg q) + \mu(\neg p \wedge q) + \mu(\neg p \wedge \neg q) + 2\mu(p \wedge q) \geqslant 1.7 \\
\mu(p \wedge q) + \mu(\neg p \wedge q) \geqslant 0.6.
\end{array}$$

□

The input for the $LPP_{2,\mathrm{Meas}}$-satisfiability checker based on genetic algorithms is a weight formula f in the wfc-form with L weight literals. Without loss of generality, we demand that classical formulas appearing in weight terms are in disjunctive normal form. Let $\phi(f) = \{p_1, \ldots, p_N\}$ denote the set of all primitive propositions from f, and $|\phi(f)| = N$.

An individual M consists of L pairs of the form (atom, probability) that describe a probabilistic model. The first coordinate is given as a bit string of length N, where 1 at the position i denotes $\neg p_i$, while 0 denotes p_i. Probabilities are represented by floating point numbers.

For an individual $M = \big((at_1, \mu(at_1)), \ldots, (at_N, \mu(at_N))\big)$, the linear system is equivalent to: $\bigvee_{i=1}^{L} \big(\sum_{j=1}^{L} a_{ij}\mu(at_j))\rho_i \, c_i\big)$. Note that it is possible that some $a_{ij} = 0$, though $[a_{ij}]$ matrix is usually not sparse.

The individuals are evaluated using function $d(M)$, which measures a degree of unsatisfiability of an individual M. Function $d(M)$ is defined as the distance between left and right hand side values of the weight literals not satisfied in the model described by M:

$$d(M) = \sqrt{\sum_{M \not\models t_i \ \rho_i \ c_i} \left[a_1^i \sum_{at \in \mathrm{CDNF}(\alpha_1^i)} \mu(at) + \cdots + a_{n_i}^i \sum_{at \in \mathrm{CDNF}(\alpha_{n_i}^i)} \mu(at) - c_i \right]^2}.$$

If $d(M) = 0$, all the inequalities in the linear system are satisfied, hence the individual M is a solution.

Some features of GA have been set for all tests:

- the population consists of 10 individuals,
- one set of tests has been performed with a population of 20 individuals,
- selection is performed using the rank-based roulette operator (with the rank from 2.5 for the best individual to 1.6 for the worst individual-the step is 0.1),
- The crossover operator is one-point, with the probability 0.85
- the elitist strategy with one elite individual is used in the generation replacement scheme,
- multiple occurrences of an individual are removed from the population.

Two problem-specific *two-parts* mutation operator were used. The first operator (*TP1*) features two different probabilities of mutation for the two parts (*atoms, probabilities*) of an individual; after mutation, the real numbers in *probabilities* part of an individual have to be scaled since their sum must equal 1. The second operator (*TP2*) is a combination of ordinary mutation on *atoms* part, and a special mutation on *probabilities* part of an individual. Instead of performing mutation on two bits in the representation of *probabilities* part, two members p_{i1}, p_{i2} of *probabilities* part are chosen randomly and then replaced with random p'_{i_1}, p'_{i_2}, such that $p_{i_1} + p_{i_2} = p'_{i_1} + p'_{i_2}$ and $0 \leqslant p'_{i_1}, p'_{i_2} \leqslant 1$. The sum of probabilities does not change and no scaling is needed.

We have experimented with the following choices in the local search procedure:

LS1 (LS denotes "local search"): For an individual M all the weight literals are divided into two sets: the first set (B) contains all satisfied literals, while the second one (W) contains all the remaining literals. The literal $t_B\ \rho_B\ c_B \in B$ (called the best one) with the biggest difference $|\mu(t_B) - c_B|$ between the left and the right side, and the literal $t_W\ \rho_W\ c_W \in W$(the worst one) with the biggest difference $|\mu(t_W) - c_W|$ are found. Two sets of atoms are determined: the first set $B_{\mathrm{At}(f)}$ contains all the atoms from M satisfying at least one classical formula α_i^B from $t_B = a_1^B w(\alpha_1^B) + \cdots + a_{k_B}^B w(\alpha_{k_B}^B)$, while the second one $W_{\mathrm{At}(f)}$ contains all the atoms from M satisfying at least one classical formula α_i^W from $t_W = a_1^W w(\alpha_1^W) + \cdots + a_{k_W}^W w(\alpha_{k_W}^W)$. The probabilities of a randomly selected atom from $B_{\mathrm{At}(f)} \smallsetminus W_{\mathrm{At}(f)}$ and a randomly selected atom from $W_{\mathrm{At}(f)} \smallsetminus B_{\mathrm{At}(f)}$ are changed so that $t_B\ \rho_B\ c_B$ remains satisfied, while the distance $|\mu(t_W) - c_W|$ is decreased or $t_W\ \rho_W\ c_W$ is satisfied.

LS2: For na individual M, the *worst* weight literal $t_W\ \rho_W\ c_W$ from W (the set of unsatisfied literals) with the biggest difference $|\mu(t_W) - c_W|$ is found. The literal can be represented as $\sum_{j=1}^{L} a_{Wj}\mu(at_j)\rho_W c_W$. We try to change the vector of probabilities $[\mu(at_j)]$, so that the linear equation $\sum_{j=1}^{L} a_{Wj}\mu(at_j) = c_W$ is satisfied. The equation $\sum_{j=1}^{L} a_{Wj}\mu(at_j) = c_W$ represents a hyper-plane in R^n while $[a_{Wj}]$ denotes a vector normal to the hyper-plane. The projection of $[\mu(at_j)]$ to the hyper-plane, which satisfies the equation, is $[\mu'(at_j)] = [\mu(at_j)] + k_W[a_{Wj}]$. The calculation of k and the projection vector is simple and straightforward, and gives

$$k = \frac{c_w - a_W \circ [\mu(at_j)]}{|a_W|^2} = \frac{c_w - \sum_{j=1}^{L} \mu(at_j) a_{Wj}}{\sum_{j=1}^{L} {a_{Wj}}^2}.$$

We set the new vector of probabilities to be

$$[\mu''(at_j)] = \frac{[\max\{\mu'(at_j), 0\}]}{\sum_{k=1}^{L} \max\{\mu'(at_k), 0\}}$$

(negative coordinates are replaced with 0, and the vector is scaled so that the sum of its coordinates $\sum_{j=1}^{L} \mu''(at_j)$ equals 1).

LS3 is similar to LS2, with the difference being made when choosing the weight literal $t_W\ \rho_W\ c_W$ from W (the set of unsatisfied literals). The chosen literal is the one with the smallest difference $|\mu(t_W) - c_W|$; it is the *best bad literal.*

LS4 is similar to LS2 and LS3. Instead of calculating the projection $[\mu'(at_j)] = [\mu(at_j)] + k_W[a_{Wj}]$ for one chosen weight literal $t_W\ \rho_W\ c_W$ from W, we calculate $k_{W_i}[a_{W_ij}]$ for each literal $t_{W_i}\ \rho_{W_i}\ c_{W_i}$ from W (the set of unsatisfied literals) and calculate the *intermediate* vector $[\mu'(at_j)]$, by adding the linear combination to the original vector: $[\mu'(at_j)] = [\mu(at_j)] + \sum_{W_i} k_{W_i}[a_{W_ij}]$. The new vector of probabilities $[\mu''(at_j)]$ is then calculated in same fashion as in LS2.

In our methodology, introduced in [86], the performance of the system is evaluated on a set of PSAT-instances, i.e., on a set of randomly generated formulas in the wfc-form (with classical formulas in disjunctive normal form). The advantage of this approach is that a formula can be randomly generated according to the following parameters: N-the number of propositional letters, L-the number of weight literals, S-the maximal number of summands in weight terms, and D-the maximal number of disjuncts in DNF's of classical formulas. The considered set of test problems contains 27 satisfiable formulas. Three PSAT-instances were generated for each of 9 pairs of (N, L), where $N \in \{50, 100, 200\}$, and $L \in \{N, 2N, 5N\}$. For every instance $S = D = 5$. Having the above parameters, L atoms and their probabilities (with the constraint that the sum of probabilities must be equal to 1) are chosen. Next, a formula f containing L basic weight formulas is generated. It contains primitive propositions from the set $\{p_1, \ldots, p_N\}$ only. Every weight literal contains at most S summands in its weight term. Every classical formula is in disjunctive normal form with at most D disjuncts, while every disjunct is a conjunction of at most N literals. For every weight term t coefficients are chosen, and the value of t is computed. Next, the sum $sp(t)$ of positive coefficients and the sum $sn(t)$ of negative coefficients are computed. Finally, the right side value of the weight literals between $sp(t)$ and $sn(t)$, and the relation sign are chosen such that f is satisfiable.

We prefer to test more problem instances of different sizes (even very large scale instances) rather than making more trials on a smaller set of instances (of smaller or average size). Since the tests are of large sizes, the necessity to perform them in a reasonable time imposed to set the maximal number of generations to be: 10000 for $N = 50$, 7000 for $N = 100$ and 5000 for $N = 200$.

L, N, inst. no.	Table 1 TP2(12,4) 10 ind. No LS	Table 1 TP2(12,4) 20 ind. No LS	Table 2 TP2(12,4) 10 individuals LS's applied in each generation LS1	LS2	LS3	LS4	Table 3 TP2(12,4) 10 individuals LS's applied in each third generation LS1	LS2	LS3	LS4	Table 5 10 individuals Combination of LS's applied in each generation TP2(12,4)	TP2(24,8)	TP2(48,16)
50, 50, 1	0	1	0	0	0	0	0	0	0	0	0	0	0
50, 50, 2	0	1	0	0	0	0	0	0	0	0	0	0	0
50, 50, 3	0	1	0	1	1	0	0	0	0	0	0	0	0
50, 100, 1	1	1	1	0	0	2	1	0	0	1	1	0	1
50, 100, 2	1	2	1	1	2	2	1	1	1	2	2	2	3
50, 100, 3	3	3	1	2	7	10	1	2	3	4	1	3	3
50, 250, 1	16	20	28	16	16	39	22	14	11	21	40	35	42
50, 250, 2	51	56	24	38	34	97	26	35	30	50	68	70	132
50, 250, 3	18	20	18	9	17	25	10	8	13	14	15	16	19
100, 100, 1	0	1	0	0	0	0	0	0	0	0	0	0	0
100, 100, 2	0	1	0	0	0	0	0	0	0	0	0	0	0
100, 100, 3	0	1	0	0	0	1	1	0	0	0	1	1	1
100, 200, 1	8	12	10	3	8	9	6	3	8	7	5	5	7
100, 200, 2	2	3	1	3	2	4	1	3	1	4	4	1	2
100, 200, 3	1	3	4	1	2	26	2	1	1	2	2	2	2
100, 500, 1	187	236	170	130	149	384	94	145	244	228	269	294	271
100, 500, 2	295	309	242	241	298	333	169	306	151	228	236	260	480
100, 500, 3	484	575	326	509	416	775	296	390	355	461	1019	777	671
200, 200, 1	58	91	71	108	56	134	34	78	66	3471	146	270	202
200, 200, 2	5	6	11	7	7	14	11	7	10	9	13	11	9
200, 200, 3	2	3	4	1	2	2	4	1	1	4	4	2	3
200, 400, 1	12	11	4	6	5	25	6	7	5	14	8	11	7
200, 400, 2	238	286	N/A	195	163	484	N/A	171	161	296	479	686	1128
200, 400, 3	205	230	N/A	174	205	247	N/A	153	201	208	419	334	374
200, 1000, 1	1593	2173	3064	888	1347	2972	2307	811	1271	1865	2363	2087	2032
200, 1000, 2	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	19582	19977
200, 1000, 3	1489	1861	3298	1364	792	3548	2456	1135	1080	2023	2818	2770	2778

TABLE 1. Average time (rounded to seconds) used by the test computer to execute *successful* tests for some selected parameters. (*Note: Value* 0 *means that the average time was less than half second.*)

As an illustration of the corresponding results we give Table 1 which contains the average running time of successful tests as measured on our test computer (a Pentium P4 2.4GHz, 512MB-based Linux station). The table shows running times only for selected tests. Columns 2 and 3 show times for tests without LS's, with different population size (10 individuals vs 20 individuals). Increased population size does result in smaller number of iterations needed to find the solution, but the computational cost for each iteration is increased and the overall computational cost is greater than with smaller population size. In columns 4-7 and 8-11 we can compare the efficiency of various LS's. It is clear that LS2 and LS3 are more efficient than LS1 and LS4 when used for large problem instances, however it is not clear which of them is the most efficient. The running times in columns 8–11 (LS's applied in each third generation) are on average smaller than times in columns 4-7 (LS's applied in each generation). However, this does not mean that the principle of reducing application of LS's to each third generation is always more efficient. Finally, columns 12-14 show execution times for tests using combination of LS's. Combined usage of LS's is not justified in terms of time efficiency, but it is justified in terms of increased success rate. Higher mutation rate in this setup leads to better time efficiency and higher success rate, except for a few less complex problem instances.

4. Some variants of the logic LPP_2

The lack of compactness in the presence of a finitary axiomatization might cause a logical problem: there are consistent sets of formulas that have no model. Example 4 contains such a set for LPP_2. One way to avoid consistency of unsatisfiable sets is to employ infinitary logic as we do above. On the other hand, the lack of compactness motivates also investigations of models in which probabilities have a fixed finite range in which case a finitary axiomatization does not imply the above problem any more. In this section we present three logics inspired by the idea of restricting the range of probability measures. In the first logic (denoted $LPP_2^{\mathrm{Fr}(n)}$) we give a finitary sound and complete axiomatization with respect to a class of models with measures which have a fixed finite range of the form $\{0, 1/n, 2/n, \dots, n-1/n, 1\}$. Then we introduce another logic (denoted $LPP_2^{A,\omega_1,\mathrm{Fin}}$) in which the assumption about the range of the measure is relaxed, and we consider the class of all probabilistic models whose measures have arbitrary finite ranges (without the requirement that the range is fixed in advance). Finally, we analyze the logic LPP_2^S. It involves a rule that enables us to syntactically define the range of the probability function which will appear in the interpretation.

4.1. Logic $LPP_2^{\mathrm{Fr}(n)}$. Let n be a fixed positive integer, and Range $= \{0, 1/n, \dots, (n-1)/n, 1\}$. If $s \in [0, 1)$, then s^+ denotes $\min\{r \in \text{Range} : s < r\}$. If $s \in (0, 1]$, $s^- = \max\{r \in \text{Range} : s > r\}$. The most of the notions defined in Section 3 are also used for the logic $LPP_2^{\mathrm{Fr}(n)}$. The main, but important, differences are:

- in Definition 1-the finitely additive measure μ maps the algebra H to Range and

- in Definition 6-proofs are finite sequences of formulas.

Note that $LPP_2^{\mathrm{Fr}(n)}$-models are given relatively to n, and that different choices of n produce different logics. The axiomatic system $Ax_{LPP_2^{\mathrm{Fr}(n)}}$ contains all the axioms from the system Ax_{LPP_2}, and the inference rules 1 and 2 (but note Rule 3), as well as the following new axiom:

(7) $P_{>s}\alpha \to P_{\geqslant s^+}\alpha$

Since the only infinitary inference rule from Ax_{LPP_2} (Rule 3) is not included in $Ax_{LPP_2^{\mathrm{Fr}(n)}}$, it is a finitary axiomatic system. Nevertheless, many statements from the previous section still hold. The next lemma states that Axiom 7 implies that the range of measures must be the set Range.

Lemma 22. *Let α be a sentence. Then:*

(1) $\vdash P_{<r}\alpha \to P_{\leqslant r^-}\alpha$,
(2) $\vdash P_{>r}\alpha \leftrightarrow P_{\geqslant r^+}\alpha$,
(3) $\vdash P_{\leqslant r^-}\alpha \leftrightarrow P_{<r}\alpha$,
(4) $\vdash \bigvee_{s\in\mathrm{Range}} P_{=s}\alpha$,
(5) $\vdash \underline{\vee}_{s\in\mathrm{Range}} P_{=s}\alpha$, *where $\underline{\vee}$ denotes the exclusive disjunction.*

Proof. (1) The considered formula is equivalent to Axiom 7 because $P_{>r}\alpha = \neg P_{\leqslant r}\alpha = \neg P_{\geqslant 1-r}\neg\alpha = P_{<1-r}\neg\alpha$, and $P_{\geqslant r^+}\alpha = P_{\geqslant 1-(1-r^+)}\alpha = P_{\leqslant 1-r^+}\neg\alpha = P_{\leqslant(1-r)^-}\neg\alpha$.

(2) The formula is obtained from the axioms 7 and 3′.

(3) The formula is obtained from Axiom 3, and Lemma 22(1).

(4) From Axiom 2' $P_{\leqslant 1}\alpha$ $(= \neg P_{>1}\alpha)$, we have $\vdash (P_{\geqslant 1}\alpha \vee \neg P_{\geqslant 1}\alpha) \wedge \neg P_{>1}\alpha$. Thus,

$$\vdash (P_{\geqslant 1}\alpha \wedge \neg P_{>1}\alpha) \vee (\neg P_{\geqslant 1} \wedge \neg P_{>1}\alpha).$$

From $P_{\geqslant 1}\alpha \wedge \neg P_{>1}\alpha = P_{=1}\alpha$, and $\vdash P_{<1}\alpha \to P_{\leqslant 1}\alpha$, we have $\vdash P_{=1}\alpha \vee P_{<1}\alpha$. From $\vdash P_{<1}\alpha \leftrightarrow ((P_{\geqslant 1^-}\alpha \vee \neg P_{\geqslant 1^-}\alpha) \wedge P_{<1}\alpha)$, $\vdash (P_{\geqslant s}\alpha \to P_{\geqslant s^-}\alpha) \leftrightarrow (P_{<s^-}\alpha \to P_{<s}\alpha)$, we have

$$\vdash P_{<1}\alpha \leftrightarrow ((P_{\geqslant 1^-}\alpha \wedge \neg P_{>1^-}\alpha) \vee (P_{<1^-}\alpha \wedge P_{<1}\alpha)), \text{ and } \vdash P_{=1}\alpha \vee P_{=1^-}\alpha \vee P_{<1^-}\alpha.$$

In such a way we obtain $\vdash (\bigvee_{s\in\mathrm{Range}} P_{=s}\alpha) \vee P_{<0}\alpha$. Since $\vdash \neg P_{<0}\alpha$, we finally have $\vdash \bigvee_{s\in\mathrm{Range}} P_{=s}\alpha$.

(5) From $P_{=r}\alpha = P_{\geqslant r}\alpha \wedge \neg P_{>r}\alpha$, and the axiom 3, we have $\vdash P_{=r}\alpha \to \neg P_{=s}\alpha$, for $s > r$. Similarly, by the axiom 3′, we have $\vdash P_{=r}\alpha \to \neg P_{=s}\alpha$, for $s < r$. It follows that $\vdash P_{=r}\alpha \to \neg P_{=s}\alpha$, for $r \neq s$, and $\vdash \underline{\vee}_{s\in\mathrm{Range}} P_{=s}\alpha$. □

The completeness proofs for the classes:

$$LPP_{2,\mathrm{Meas}}^{\mathrm{Fr}(n)}, \quad LPP_{2,\mathrm{Meas,All}}^{\mathrm{Fr}(n)}, \quad LPP_{2,\mathrm{Meas},\sigma}^{\mathrm{Fr}(n)} \quad \text{and} \quad LPP_{2,\mathrm{Meas,Neat}}^{\mathrm{Fr}(n)}$$

are similar to the corresponding proofs from the previous section. In the sequel we sketch this proof and emphasize some modified steps.

We begin as in the statements 8, 9 and 10. In the counterpart of Theorem 13 we do not use the step 2b of the construction of a maximal consistent set, but otherwise follow the corresponding proof. Then, the statements 12(1)–12(6) obviously hold, while Lemma 12(7) needs some explanation. By Lemam 22(5), the supremum s of the set $\{r : P_{\geqslant r}\alpha \in T\}$ must be in the set Range. Also, for that s, it must be $P_{\geqslant s}\alpha \in T$, where T is the considered maximal consistent set. Thus, Lemma 12(7) holds. A canonical model $\mathbf{M}_T = \langle W, H, \mu, v\rangle$ is introduced as above. Note that in the counterpart of Theorem 14 for every formula $\alpha \in \mathrm{For}_C$, $\sup\{r : P_{\geqslant r}\alpha \in \mathcal{T}\}$ is the same as $\max\{r : P_{\geqslant r}\alpha \in \mathcal{T}, r \in \mathrm{Range}\}$, because the set Range is finite. Theorems 15– 18 can be now proved similarly as it is done above.

Theorem 23 announces a property that does not hold for the systems considered in the previous section. Another difference between logics from this and the previous sections is illustrated in Example 24.

Theorem 23 (Compactness theorem for $LPP_2^{\mathrm{FR}(n)}$)**.** *Let L be any class of models considered in this section and T be a set of formulas. If every finite subset of T is L-satisfiable, then T is L-satisfiable.*

Proof. If T is not L-satisfiable, then it is not $Ax_{LPP_2^{\mathrm{FR}(n)}}$-consistent. It follows that $T \vdash \bot$. Since the axiomatic system $Ax_{LPP_2^{\mathrm{FR}(n)}}$ is finitary one, there must be a finite set $T' \subset T$ such that $T' \vdash \bot$. It is a contradiction because every finite subset of T is both L-satisfiable and $Ax_{LPP_2^{\mathrm{FR}(n)}}$-consistent. $\square$

Example 24. For every positive integer n and Range defined as above, it is easy to construct an $LPP_{2,\mathrm{Meas}}$-model $\mathbf{M} = \langle W, H, \mu, v\rangle$ which does not satisfy that Axiom 7. For example, let $n = 3$, and $p \in \phi$:

- $W = \{w_1, w_2\}$,
- H is the power set of W
- $\mu(w_1) = 1/2$, $\mu(w_2) = 1/2$ and
- $v(w_1, p) = \mathrm{true}$, $v(w_2, p) = \mathrm{false}$.

Since $\mu([p]_{\mathbf{M}}) = 1/2$, obviously $\mathbf{M} \models P_{>1/3}p$, and $\mathbf{M} \not\models P_{\geqslant 2/3}p$, so the instance $P_{>1/3}p \to P_{\geqslant 2/3}p$ of Axiom 7 does not hold in $\mathbf{M}$. $\square$

Finally, decidability of the satisfiability problem for the classes of models considered in this section can be proved similarly as Theorem 19. Only, note that the measures of atoms must be in the set Range. Since that set is always finite, there are only finitely many possibilities for such distributions, and decidability easily follows.

4.2. Logic $LPP_2^{A,\omega_1,\mathrm{Fin}}$. In Section 4.1 the considered measures have a fixed finite range. Using ideas from [98], that assumption is relaxed, and we prove the completeness theorem with respect to the class of all probabilistic models whose measures have arbitrary finite ranges (without the requirement that the range is fixed in advance). In the sequel some notions from [4] are used.

Let A be a countable admissible set and $\omega \in A$. We use $LPP_2^{A,\omega_1,\mathrm{Fin}}$ to denote our logic. The language of $LPP_2^{A,\omega_1,\mathrm{Fin}}$ is a subset of A. It is the classical propositional language ($\neg$, $\wedge$, $\vee$) augmented by a list of unary probabilistic operators of the form $P_{\geqslant s}$, for every $s \in [0, 1] \cap A$. An important characteristic of $LPP_2^{A,\omega_1,\mathrm{Fin}}$ is that the conjunction symbol and the disjunction symbol may be applied to *finite or* *countable sets of probability formulas.* It means that if $G \in A$ is a set of formulas of $LPP_2^{A,\omega_1,\mathrm{Fin}}$, then: $\bigwedge_{B\in G}\Phi$ and $\bigvee_{B\in G}\Phi$ and are also $LPP_2^{A,\omega_1,\mathrm{Fin}}$-formulas (but note that all formulas from the set G must be from For_P). For an $LPP_2^{A,\omega_1,\mathrm{Fin}}$-formula Φ, the formula $\Phi\neg$ is obtained by moving a negation inside the formula Φ over the classical connectives. For example, $(\bigwedge_{\Phi\in G}\Phi)\neg$ denotes $\bigvee_{\Phi\in G}\neg\Phi$, and similarly for the other classical connectives.

Here we consider a particular subclass of the class $LPP_{2,\mathrm{Meas}}$ of all measurable probabilistic models. We denote it $LPP_{2,\mathrm{Meas}}^{A,\omega_1,\mathrm{Fin}}$,and it contains all measurable models whose measures have finite ranges. The satisfaction relation $\models$ generalizes the corresponding relation from Definition 2. The new cases are related to infinitary formulas:

- if G is a finite or countable set of For_P-formulas, $\mathbf{M}\models\bigwedge G$ iff for every $B\in G$, $\mathbf{M}\models B$, and
- if G is a finite or countable set of For_P-formulas, $\mathbf{M}\models\bigvee G$ iff there is some $b\in G$ so that $\mathbf{M}\models B$.

The axiomatic system $Ax_{LPP_2^{A,\omega_1,\mathrm{Fin}}}$ contains all the axioms and rules from the system Ax_{LPP_2}, and also the following new axioms:

(7) $(\neg\Phi)\leftrightarrow(\Phi\neg)$
(8) $(\bigwedge_{B\in G}B)\to C$, $C\in G$, $G\in A$, G is a set of probability formulas
(9) $\bigvee_{c>0}\bigwedge_{\alpha\in G}(P_{>0}\alpha\to P_{>c}\alpha)$, $G\in A$, G is a set of classical propositional formulas

and the rule

(4) From $B\to C$, for all $C\in G$, infer $B\to\bigwedge_{C\in G}C$, G is a set of probability formulas

introduced in [53]. In the completeness proof a result[4] from [9] and the weak-strong model construction from [98] will be used. A weak model is an $LPP_{2,\mathrm{Meas}}^{A,\omega_1,\mathrm{Fin}}$-model defined above.

Theorem 25. *An $LPP_2^{A,\omega_1,\mathrm{Fin}}$-formula Φ is consistent iff it is satisfiable in a weak model in which every $LPP_2^{A,\omega_1,\mathrm{Fin}}$-theorem is true.*

Proof. The simpler direction follows from the soundness of the axiomatic system. For the other direction, let $A_1, A_2, \ldots$ be an enumeration of all $LPP_2^{A,\omega_1,\mathrm{Fin}}$-$\mathrm{For}_P$-formulas. We modify the construction from Theorem 13:

(1) $T_0=\{\Phi\}\cup Cn_C(\Phi)\cup\{P_{\geqslant 1}\alpha:\alpha\in Cn_C(\Phi)\}$
(2) for every $i\geqslant 0$,
  (a) if $A_i=\bigvee_{B\in G}B$ and $T_i\cup\{A_i\}$ is consistent, then for some $B\in G$, $T_{i+1}=T_i\cup\{A_i\}\cup\{B\}$ such that T_{i+1} is consistent, otherwise
  (b) if $T_i\cup\{A_i\}$ is consistent, then $T_{i+1}=T_i\cup\{A_i\}$, otherwise
  (c) if A_i is of the form $\beta\to P_{\geqslant s}\gamma$, and $T_i\cup\{A_i\}$ is not consistent, then $T_{i+1}=T_i\cup\{\neg A_i,\beta\to\neg P_{\geqslant s-\frac{1}{n}}\gamma\}$, for some positive integer n, so that T_{i+1} is consistent, otherwise

[4]**Theorem 3.2.10 from** [9] If μ is a finitely additive measure and there is a real number $c\in(0,1)$ such that $\mu(\theta)>c$, whenever $\mu(\theta)\neq 0$, then μ has a finite range.$\square$

(d) $T_{i+1} = T_i \cup \{\neg A_i\}$.

(3) $\mathcal{T} = \bigcup_{i=0}^{\infty} T_i$.

We can show that every T_i obtained by the new step in the construction (the step 2a) is also consistent. To prove that, suppose that $T_i \cup \{A_i\}$ is consistent, where $A = \bigvee_{B \in G} B$, but that for every $B \in G$, the set $T_{i+1} = T_i \cup \{A_i\} \cup \{B\}$ is not consistent. It means that

- $T_i \cup \{A_i\} \cup \{B\} \vdash \bot$, for every $B \in G$
- $T_i \cup \{A_i\} \vdash \neg B$, for every $B \in G$
- $T_i \cup \{A_i\} \vdash \bigwedge_{B \in G} \neg B$, by Rule 4
- $T_i \cup \{A_i\} \vdash \neg \bigvee_{B \in G} B$, by Axiom 7

which contradicts consistency of $T_i \cup \{A_i\}$. Then, we can follow the completeness prof for $LPP_{2,\mathrm{Meas}}$, and construct the canonical model $\mathbf{M}_\Phi$. The axioms guarantee that $\mathbf{M}_\Phi$ is a weak model in which every $LPP_2^{A,\omega_1,\mathrm{Fin}}$-theorem is true, and that $\mathbf{M}_\Phi \models \varphi$ iff $\varphi \in \mathcal{T}$. □

Note that, although in a weak model (since Axiom 9 holds) for every For_C-formula α the following condition is fulfilled:

$$\text{if } \mathbf{M} \models P_{>0}\alpha \text{ then } \mathbf{M} \models P_{>c}\alpha. \tag{6}$$

it may be the case that there is no single $c > 0$ such that the condition (6) holds for all formulas. Thus, we will now construct the corresponding strong model, i.e., a weak model $\mathbf{M}$ which satisfies that there is a $c > 0$ such that for every For_C-formula α the condition (6) holds. By Theorem 3.2.10 from [9] (see Footnote 4), measures from a strong model have finite ranges, and the model belongs to the $LPP_{2,\mathrm{Meas}}^{A,\omega_1,\mathrm{Fin}}$-class.

Theorem 26. *An $LPP_2^{A,\omega_1,\mathrm{Fin}}$-formula Φ is consistent iff it is satisfiable in a strong model in which every $LPP_2^{A,\omega_1,\mathrm{Fin}}$-theorem is true.*

Proof. Again, the simpler direction follows from the soundness of the axiomatic system. To prove another part of the statement we consider a language L_A containing:

- the following three kinds of variables:
  - variables for sets (X, Y, Z,...),
  - variables for elements (x, y, z,...),
  - variables for reals from $[0,1]$ (r, s,...), and
  - variables for positive reals greater than 1 (u,v,...)
- the predicates: $\leqslant$ for reals, $V(u,u)$, $E(x,X)$ and $\mu(X,r)$,
- a set constant symbol W_α for every $LPP_2^{A,\omega_1,\mathrm{Fin}}$-$\mathrm{For}_C$-formula α,
- a constant symbol r' for every real number $r \in [0,1] \cap A$, and
- two function symbols for additions and multiplications for reals.

The intended meaning of $E(x,X)$ is $x \in X$, $V(u,u)$ means that a formula Φ with the Gödel-number u (denoted $gb(\Phi) = u$) holds in the model, while $\mu(X,r)$ can be understood as "r is the measure of X". We use $\mu(X) \geqslant r$ to denote $(\exists s)(s \geqslant r \wedge \mu(X,s))$, and $V(\Phi)$ to denote $V(gb(\Phi), gb(\Phi))$.

We define a theory T of $L_{\omega_1\omega} \cap A$ which contains the following formulas:

(1) $(\forall X)(\forall Y)((\forall x)(E(x,X) \leftrightarrow E(x,Y)) \leftrightarrow X = Y)$
(2) $(\forall x)(E(x,W_{\alpha\wedge\beta}) \leftrightarrow (E(x,W_\alpha) \wedge E(x,W_\beta)))$ for every $\alpha \wedge \beta \in \mathrm{For}_C$
(3) $(\forall x)(E(x,W_{\neg\alpha}) \leftrightarrow \neg E(x,W_\alpha))$, for every $\alpha \in \mathrm{For}_C$
(4) $(\forall x)(E(x,W_{p\vee\neg p})$
(5) $V(\alpha) \leftrightarrow W_\alpha = W_{p\vee\neg p}$, for every $\alpha \in \mathrm{For}_C$
(6) $V(P_{\geqslant s}\alpha) \leftrightarrow \mu(W_\alpha) > s$, for every $\alpha \in \mathrm{For}_C$
(7) $V(\bigwedge_{B\in G} B) \leftrightarrow \bigwedge_{B\in G} V(B)$, for every set of probability formulas $G \in A$
(8) $V(\neg B) \leftrightarrow \neg V(B)$, for every $LPP_2^{A,\omega_1,\mathrm{Fin}}$-$\mathrm{For}_P$-formula B
(9) $(\forall X)(\exists_1 r)\mu(X,r)$
(10) $(\forall X)(\forall Y)((\mu(X,r) \wedge \mu(Y,s) \wedge \neg(\exists y)(E(y,X) \wedge E(y,Y))) \rightarrow$
$\rightarrow (\exists Z)((\forall y)((E(y,X) \vee E(y,Y)) \leftrightarrow E(y,Z)) \wedge \mu(Z, r+s)))$
(11) $(\forall X)((\forall y)E(y,X) \rightarrow \mu(X,1))$
(12) $(\exists r > 0)(\forall X)(\mu(X) > 0 \rightarrow \mu(X) > r)$
(13) Axioms for Archimedian fields for real numbers
(14) $(\forall x)E(x,W_\Psi)$ where Ψ is an axiom of $LPP_2^{A,\omega_1,\mathrm{Fin}}$
(15) $(\exists x)E(x,W_\Phi)$ where Φ is the formula from the formulation of the statement.

Let a standard model for L_A be $\langle W,H,F,V,E,\mu,+,*,\leqslant,W_\alpha,r\rangle_{\alpha\in\mathrm{For}_C, r\in F}$, where $H \subset 2^W$, $F = F' \cap [0,1]$, $F' \subset R$ a field, $V \subset R\times R$, $E \subset W \times H$, $\mu : H \to F$, $+,* : F^2 \to F$, $\leqslant \subset F^2$, and $W_\alpha \in H$.

Let $\mathbf{M} = \langle W,H,\mu,v\rangle$ be a weak model for $LPP_2^{A,\omega_1,\mathrm{Fin}}$. If we define $W_\alpha = \bigcup_{w\in W}[\alpha]_w$, and $H = \{W_\alpha : \alpha \in \mathrm{For}_C\}$, it easy to show that M can be transformed to a standard model. On the other hand, if Ψ is a consistent $LPP_2^{A,\omega_1,\mathrm{Fin}}$-formula, then there is a weak model in which it is satisfied, and consequently there is a standard model in which $V(\Psi)$ holds.

Let $T_0 \subset T$, $T_0 \in A$. Since Axiom 9 holds in the weak model M it follows that every T_0 has a model. Hence, by the Barwise compactness theorem, T has a model $\mathbf{M}' = \langle W,H,F,V,E,\mu,+,*,\leqslant,W_\alpha,r\rangle_{\alpha\in\mathrm{For}_C, r\in F}$. We define a strong model $\mathbf{M}'' = \langle W,H,\mu,v\rangle$ such that the following holds:

- for every $w \in W$, $v(w,p) = \text{true}$ iff $w \in W_p$ for every primitive proposition p,
- $H = \{W_\alpha : \alpha \in \mathrm{For}_C\}$,
- $\mu(X) = r$ iff $\mu(X,r)$ holds in $\mathbf{M}'$.

Since (15) holds in $\mathbf{M}'$, $\mathbf{M}'' \models \Phi$. $\square$

Completeness also holds for Σ_1 definable theories, but we it is possible to show that it cannot be generalized to arbitrary theories.

4.3. Logic LPP_2^S. Another generalization of the logic $LPP_2^{A,\omega_1,\mathrm{Fin}}$ contains an infinitary rule which enables us to syntactically define the range S of the probability function which appears in the interpretation:

- From $A \to P_{\neq s}\alpha$, for every $s \in S$, infer $A \to \bot$.

However, we will skip all technical details here and discuss another logic which extends LPP_2^S in Subsection 9.1.

5. Logic LPP_1

In this section we will present the logic LPP_1 which extends LPP_2 so that iterations of the probabilistic operators are allowed. For example, $\alpha \wedge P_{\geqslant s} P_{\geqslant r}\beta$ is a formula of LPP_1. In that way we can express statements about higher order probabilities and mix classical and probabilistic formulas. More formally, the set For_{LPP_1} of formulas is the smallest set containing primitive propositions, and closed under formation rules: if α and β are formulas, then $P_{\geqslant s}\alpha$, $\neg\alpha$ and $\alpha \wedge \beta$ are formulas. The formulas from the set For_{LPP_1} will be denoted by α, β,...

The corresponding semantics can be given as follows:

Definition 27. An LPP_1*-model* is a structure $\mathbf{M} = \langle W, \mathrm{Prob}, v\rangle$ where:

- W is a nonempty set of objects called worlds,
- Prob is a probability assignment which assigns to every $w \in W$ a probability space, such that $\mathrm{Prob}(w) = \langle W(w), H(w), \mu(w)\rangle$, where:
  - $W(w)$ is a non empty subset of W,
  - $H(w)$ is an algebra of subsets of $W(w)$ and
  - $\mu(w) : H(w) \to [0,1]$ is a finitely additive probability measure.
- v assigns to every $w \in W$ a two-valued evaluation of the primitive propositions, i.e., for every $w \in W$, $v(w) : \phi \to \{\text{true}, \text{false}\}$.

Note that, in contrast to Definition 1, there are as many probability spaces (in a model $\mathbf{M} = \langle W, \mathrm{Prob}, v\rangle$) as the worlds (in the set W), i.e., for every world w there is a particular $\langle W(w), \mathrm{Prob}(w), \mu(w)\rangle$. As a consequence, the satisfiability relation is now defined between worlds and formulas:

Definition 28. *The satisfiability relation* $\models$ fulfills the following conditions for every LPP_1-model $\mathbf{M} = \langle W, \mathrm{Prob}, v\rangle$ and every world $w \in W$:

- if $p \in \phi$ is a primitive proposition, $\mathbf{M}, w \models \alpha$ iff $v(w)(p) = \text{true}$,
- $\mathbf{M}, w \models \neg\alpha$ iff $\mathbf{M}, w \not\models \alpha$,
- $\mathbf{M}, w \models \alpha \wedge \beta$ iff $\mathbf{M}, w \models \alpha$ and $\mathbf{M}, w \models \beta$, and
- $\mathbf{M}, w \models P_{\geqslant s}\alpha$ iff $\mu(w)([\alpha]_{\mathbf{M},w}) \geqslant s$,

where $[\alpha]_{\mathbf{M},w}$ denotes the set $\{u \in W(w) : \mathbf{M}, u \models \alpha\}$. We will omit $\mathbf{M}$ from $\mathbf{M}, w \models \alpha$ and write $w \models \alpha$ if $\mathbf{M}$ is clear from the context. Similarly, we will write $[\alpha]_w$ instead of $[\alpha]_{\mathbf{M},w}$.

Similarly as above, we consider measurable models only. An LPP_1-model $\mathbf{M} = \langle W, \mathrm{Prob}, v\rangle$ is *measurable* if for every $w \in W$ the set $H(w) = \{[\alpha]_w : \alpha \in \mathrm{For}_{LPP_1}\}$. $LPP_{1,\mathrm{Meas}}$ denotes the class of all measurable LPP_1-models.

Definition 29. A formula $\alpha \in \mathrm{For}_{LPP_1}$ is *satisfiable* if there is a world w in an $LPP_{1,\mathrm{Meas}}$-model $\mathbf{M}$ such that $w \models \alpha$; α is *valid* if it is satisfied in each world in each $LPP_{1,\mathrm{Meas}}$-model. A set T of formulas is satisfiable if there is a world w in an $LPP_{1,\mathrm{Meas}}$-model $\mathbf{M}$ such that $w \models \alpha$ for every $\alpha \in T$.

5.0.1. Axiomatization, completeness, decidability. It is interesting that a sound and complete axiomatization with respect to the mention class $LPP_{1,\mathrm{Meas}}$ can be given by the axiomatic system Ax_{LPP_2} from Section 3. Of course, instances of axiom schemata must obey the syntactical rules that hold in this section. However, the notions of deducibility and consistency introduced in the definitions 6 and 7 must be changed.

Definition 30. A formula α is *deducible from a set* T of formulas ($T \vdash \alpha$) if there is an at most denumerable sequence of formulas α_0, α_1,..., α, such that every α_i is an axiom or a formula from the set T, or it is derived from the preceding formulas by an inference rule, with the exception that Rule 2 can be applied to the theorems only. If $\emptyset \vdash \alpha$, we say that α is a theorem ($\vdash \alpha$).

Definition 31. A set T of formulas is *inconsistent* if $T \vdash \alpha$, for every formula α, otherwise it is *consistent*. Equivalently, T is inconsistent iff $T \vdash \bot$. A set T of formulas is *maximal* if for every formula α either $\alpha \in T$ or $\neg\alpha \in T$.

Now, the restriction from Definition 30 that Rule 2 can be applied to the theorems only guarantees that Deduction theorem for LPP_1 holds. Also, the counterparts of the statements 10–13 can be proved in the same way as above. The canonical model $\mathbf{M} = \langle W, \mathrm{Prob}, v\rangle$ can be defined such that:

- $W = \{w : w \text{ is a maximal consistent set of formulas}\}$,
- for every primitive proposition $p \in \phi$, and every $w \in W$, $v(w)(p) = \top$ iff $p \in w$, and
- for every $w \in W$, $\mathrm{Prob}(w) = \langle W(w), H(w), \mu(w)\rangle$ is defined as follows:
  - $W(w) = W$,
  - $H(w) = \{\{u : u \in W, \alpha \in u\} : \alpha \in \mathrm{For}_{LPP_1}\}$, and
  - $\mu(w)(\{u : u \in W, \alpha \in u\}) = \sup\{s : P_{\geqslant s}\alpha \in w\}$.

Similarly as above, we can prove that for every formula α and every world w, $\alpha \in w$ iff $w \models \alpha$. It follows that:

- for all α and w, $[\alpha]_w = \{u : u \in W, \alpha \in u\}$,
- for all w, $\mathrm{Prob}(w) = \langle W(w), H(w), \mu(w), \rangle$ is a probability space,
- the canonical model $\mathbf{M}$ is an $LPP_{1,\mathrm{Meas}}$-model, and
- every consistent set of formulas is satisfiable in some world from $\mathbf{M}$,

i.e., we obtain the extended completeness theorem for the class $LPP_{1,\mathrm{Meas}}$. Furthermore, reasoning as in the sections 3.4 and 4, we can prove completeness for the following classes of models $LPP_{1,\mathrm{Meas,All}}$, $LPP_{1,\mathrm{Meas},\sigma}$ and $LPP_{1,\mathrm{Meas,Neat}}$, and logics $LPP_1^{\mathrm{Fr}(n)}$, $LPP_1^{A,\omega_1,\mathrm{Fin}}$ and LPP_1^S.

Decidability and complexity of the satisfiability problem for the class $LPP_{1,\mathrm{Meas}}$ are analyzed in the sequel of this section.

Theorem 32. *If a formula α is satisfiable, then it is satisfiable in an $LPP_{1,\mathrm{Meas}}$-model with a finite number of worlds. The number of worlds in that model is at most 2^k, where k denotes the number of subformulas of α.*

Proof. Suppose that α holds in a world of an $LPP_{1,\mathrm{Meas}}$-model $\mathbf{M} = \langle W, \mathrm{Prob}, v\rangle$. Let $\mathrm{Subf}(\alpha)$ denote the set of all subformulas of α, and $k = |\mathrm{Subf}(\alpha)|$. Let $\approx$ denote the equivalence relation over W^2, such that $w \approx u$ iff for every $\beta \in \mathrm{Subf}(\alpha)$, $w \models \beta$ iff $u \models \beta$. The quotient set $W_{/\approx}$ is finite. From every class C_i we choose an element and denote it w_i. We consider the model $\mathbf{M}^* = \langle W^*, \mathrm{Prob}^*, v^*\rangle$, where:

- $W^* = \{w_i\}$,
- Prob^* is defined as follows:
  - $W^*(w_i) = \{w_j \in W^* : (\exists u \in C_{w_j}) u \in W(w_i)\}$
  - $H^*(w_i)$ is the powerset of $W^*(w_i)$,
  - $\mu^*(w_i)(w_j) = \mu(w_i)(C_{w_j})$, and for any $D \subset H^*(w_i)$, $\mu^*(w_i)(D) = \sum_{w_j \in D} \mu^*(w_i)(w_j)$,
- $v^*(w_i)(p) = v(w_i)(p)$, for every primitive proposition $p \in \phi$.

It is easy to show that $\mathbf{M}^*$ is an $LPP_{1,\mathrm{Meas}}$-model. For example, for every w_i, $\mu^*(w_i)$ is a finitely additive probability measure, since

$$\mu^*(w_i)(W^*(w_i)) = \sum_{w_j \in W^*(w_i)} \mu^*(w_i)(w_j) = \sum_{C_{w_j} \in W_{/\approx}} \mu^*(w_i)(C_{w_j}) = 1.$$

We can now show that for every $\beta \in \mathrm{Subf}(\alpha)$, β is satisfiable in $\mathbf{M}$ iff it is satisfiable in $\mathbf{M}^*$. If $\beta \in \phi$, $\mathbf{M}, w \models \beta$ iff for $w_i \in C_w$, $\mathbf{M}, w_i \models \beta$ iff $\mathbf{M}^*, w_i \models \beta$. The cases related to $\wedge$ and $\neg$ can be proved as usual. Finally, let $\beta = P_{\geqslant s}\gamma$. Then, $\mathbf{M}, w \models P_{\geqslant s}\gamma$ iff for $w_i \in C_w$, $\mathbf{M}, w_i \models P_{\geqslant s}\gamma$ iff

$$s \leqslant \mu(w_i)([\gamma]_{\mathbf{M},w_i}) = \sum_{C_u : \mathbf{M}, u \models \gamma} \mu(w_i)(C_u) = \sum_{C_u : \mathbf{M}^*, u \models \gamma} \mu^*(w_i)(C_u) = \mu^*(w_i)([\gamma]_{\mathbf{M}^*,w_i})$$

iff $\mathbf{M}^*, w_i \models P_{\geqslant s}\gamma$.

Finally, it is clear that the number of different classes in $W_{/\approx}$ is at most 2^k, and the same holds for the number of worlds in $\mathbf{M}^*$. □

Theorem 33 (Decidability theorem). *The logic LPP_1 is decidable.*

Proof. As it is noted above, a formula α is $LPP_{1,\mathrm{Meas}}$-satisfiable iff it is satisfiable in an $LPP_{1,\mathrm{Meas}}$-model with at most 2^k worlds, where k denotes the number of subformulas of α. Observe that it does not necessary imply decidability of the satisfiability problem for the class $LPP_{1,\mathrm{Meas}}$ because there are infinitely many such models. Nevertheless, the next procedure decides the satisfiability problem. The procedure is applied for every such $l \leqslant 2^k$.

Let $\mathrm{Subf}(\alpha) = \{\beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_m\}$, and $k = n + m$. In every world w from $\mathbf{M}$ exactly one of the formulas of the form

$$\delta_w = \beta_1 \wedge \ldots \wedge \beta_n \wedge \neg\gamma_1 \wedge \ldots \wedge \neg\gamma_m$$

holds. For every $l \leqslant 2^k$ we will consider l formulas of the above form. The chosen formulas are not necessarily different, but at least one of the formulas must contain the examined formula α. Using probabilistic constraints (i.e., formulas of the form $P_{\geqslant s}\beta$, $\neg P_{\geqslant s}\beta$) from the formulas we shall examine whether there is an $LPP_{1,\mathrm{Meas}}$-model $\mathbf{M}$ with l worlds such that for some world w from the model $w \models \alpha$. We do not try to determine probabilities precisely. Rather, we just check whether there are probabilities such that probabilistic constraints are satisfied in the corresponding world. To do that, for every world w_i, $i < l$, we consider a system of linear equalities and inequalities of the form (we write $\beta \in \delta_w$ to denote that β occurs positively in the top conjunction of δ_w, i.e., if δ_w can be seen as $\bigwedge_i \delta_i$, then for some i, $\beta = \delta_i$):

$$(7) \qquad \begin{aligned} \sum_{j=1}^{l} \mu(w_i)(w_j) &= 1 \\ \mu(w_i)(w_j) &\geqslant 0, \quad \text{for every world } w_j \\ \sum_{w_j:\beta\in\delta_{w_j}} \mu(w_i)(w_j) &\geqslant s, \quad \text{for every } P_{\geqslant s}\beta \in \delta_{w_i} \\ \sum_{w_j:\beta\in\delta_{w_j}} \mu(w_i)(w_j) &< s, \quad \text{for every } \neg P_{\geqslant s}\beta \in \delta_{w_i} \end{aligned}$$

The first two rows correspond to the general constraints: the probability of the set of all worlds must be 1, while the probability of every measurable set of worlds must be nonnegative. The last two rows correspond to the probabilistic constraints, because

$$\sum_{w_j:\beta\in\delta_{w_j}} \mu(w_i)(w_j) = \mu(w_i)([\beta]_{w_i}).$$

Such a system is solvable iff there is a probability $\mu(w_i)$ satisfying all probabilistic constraints that appear in δ_{w_i}. Note that there are finitely many such systems that can be solved in a finite number of steps.

If the above test is positively solved there is an $LPP_{1,\text{Meas}}$-model in which every world $w_i \models \delta_{w_i}$. Since α belongs to at least one of the formulas δ_{w_i}, we have that α is satisfiable. If the test fails, and there is another possibility of choosing l and/ot the set of l formulas δ_w, we continue with the procedure, otherwise we conclude that α is not satisfiable.

It is easy to see that the procedure terminates in a finite number of steps. Thus, the satisfiability problem for the class $LPP_{1,\text{Meas}}$ is decidable. Since $\models \alpha$ iff $\neg\alpha$ is not satisfiable, the $LPP_{1,\text{Meas}}$-validity problem is also decidable. □

The satisfiability problem for the class $LPP_{1,\text{Meas}}$ is in PSPACE, while NP is the lower bound of the complexity. The former statement is a consequence of the PSPACE-completeness of a more expressive logic from [28], while the later statement follows from the fact that the logic LPP_2 can be seen as a sublogic of LPP_1.

6. Some extensions of the probabilistic language

In this section we will analyze some possible extensions of the considered probabilistic language. The first extension contains probabilistic operators of the form Q_F with the intended meaning "the probability belongs to the set F". The next extension allows reasoning about qualitative probabilities. Finally, we mention a logic introduces in [27] in which linear combinations of probabilities can be expressed. All extensions will be considered in the framework of the logic LPP_2, but analogue analysis can be performed for the other above presented logics.

6.1. Probability operators of the form Q_F. We will use $LPP_{2,P,Q,O}$ to denote a probability logic which depends on a recursive family O of recursive subsets of S in a manner which will be explained below, while P and Q in the index means that two kinds of probabilistic operators will be used. More precisely, the language of $LPP_{2,P,Q,O}$ extends the LPP_2-language with a list of unary probabilistic operators of the form Q_F, where $F \in O$. For example, the set $\mathrm{For}_{LPP_{2,P,Q,O}}$ of formulas contains $Q_F\alpha \to \neg P_{\geqslant s}\beta$. Note that every particular choice of the family O of sets produces a different probability language, a different set of probability formulas and a distinct $LPP_{2,P,Q,O}$-logic.

To give semantics to formulas, we use the class $LPP_{2,\mathrm{Meas}}$ of measurable LPP_2-models, and the corresponding satisfiability relation (from Definition 2) with additional requirement that:

- $\mathbf{M} \models Q_F\alpha$ iff $\mu([\alpha]) \in F$, for every $F \in O$

which covers the case of the new operators. Note that $\neg Q_F\alpha$ is not equivalent to $Q_{[0,1]\setminus F}\alpha$ because $[0,1] \setminus F \notin O$, and the later is not a well formed formula.

It is obvious, using the semantics of $P_{\geqslant s}$ and Q_F-operators, that for a set $F = \{f_1, f_2, \ldots\} \in O$, $Q_F\alpha \leftrightarrow \bigvee_{f_i \in F} P_{=f_i}\alpha$. But, if the set F is not finite, the right side of this equality is an infinitary disjunction which does not belong to the set $\mathrm{For}_{LPP_{2,P,Q,O}}$ of formulas. Similarly for the formula $P_{\geqslant s}\alpha \leftrightarrow Q_{[s,1]}\alpha$, where s is a rational number from $[0,1)$, the formula $Q_{[s,1]}\alpha \notin \mathrm{For}_{LPP_{2,P,Q,O}}$. More formally:

Definition 34. Let $\Phi, \Psi \in \mathrm{For}_{LPP_{2,P,Q,O}}$. The set $\mathrm{Mod}(\Phi) = \{\mathbf{M} \in LPP_{2,\mathrm{Meas}} : M \models \Phi\}$ consists of all $LPP_{2,\mathrm{Meas}}$-models of the Φ. Φ is *definable from* Ψ if $\mathrm{Mod}(\Phi) = \mathrm{Mod}(\Psi)$.

The above discussion suggests that in a general case neither the $P_{\geqslant .}$-operators are definable from the $Q_.$-operators (i.e., some formulas on the language $\{\neg, \wedge, P_{\geqslant .}\}$ are not definable from the formulas on the language $\{\neg, \wedge, Q_.\}$), nor are the $Q_.$-operators definable from the $P_{\geqslant .}$-operators. The next theorems formalize these conclusions.

Theorem 35. *Let O be a recursive family of recursive rational subsets of $[0,1]$, $F \in O$ an infinite set, and $LPP_{2,P,Q,O}$ the corresponding logic. For an arbitrary primitive proposition $p \in \phi$, there is no probabilistic formula A on the sublanguage $\{\neg, \wedge, P_{\geqslant .}\}$ such that $Q_F p$ is definable from α.*

Proof. Suppose that there is a formula A on the language $\{\neg, \wedge, P_{\geqslant .}\}$ such that $\mathrm{Mod}(Q_F p) = \{\langle W, H, \mu, v\rangle : \mu([p]) \in F\} = \mathrm{Mod}(A)$. Recall that A is satisfiable iff at least a system from the set of all linear systems that correspond to $DNF(A)$ is satisfiable. Let a_t's be the atoms of A and y_t's be the corresponding measures. The solutions of any of those systems must satisfy $\sum_{a_t \in DNF(p)} y_t \in F$. But, the solutions of the systems are of the following form: $y_t \in (r,s)$, $y_t \in [r,s)$, $y_t \in (r,s]$, and $y_t \in [r,s]$. Such sets of solutions cannot produce the infinite, but denumerable set F as it is required. Hence, $Q_F p$ is not definable over A. □

Theorem 36. *Let O be a recursive family of recursive rational subsets of $[0,1]$, $LPP_{2,P,Q,O}$ the corresponding logic, and $s \in S \setminus \{1\}$. For an arbitrary primitive* *proposition $p \in \phi$, there is no probabilistic formula A on the sublanguage $\{\neg, \wedge, Q_{\cdot}\}$ such that $P_{\geqslant s}p$ is definable from A.*

Proof. Suppose that there is a formula A on the language $\{\neg, \wedge, Q_{\cdot}\}$ such that $\mathrm{Mod}(P_{\geqslant s}p) = \mathrm{Mod}(A)$. The models of A are exactly those that satisfy $\mu[p] \geqslant s$. But, similarly as above, the set of values for $\mu[p]$ produced by $\mathrm{Mod}(A)$ can be either denumerable, or its complement is denumerable. Hence, $P_{\geqslant s}p$ cannot be definable over A. □

Example 37. Formulas with the new probabilistic operators are suitable for reasoning about discrete sample spaces. For example, consider an experiment which consists of tossing a fair coin an arbitrary, but finite number of times. Then, $Q_F\alpha$ holds in this model, where α means that only heads (i.e., no tails) is observed in the experiment, and F denotes the set $\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\}$. Since Q_F is not definable over the probability language $\{\neg, \wedge, P_{\geqslant \cdot}\}$, this sentence cannot be described in the probability logics used so far.

6.1.1. Expressiveness of $LPP_{2,P,Q,O}$-logics. As it is noted above, every particular choice of the family of sets O produces a different $LPP_{2,P,Q,O}$-logic. In this section we describe a relation of "being more expressive" between these logics. The fact that the corresponding hierarchy has no upper bounds, is a good reason for introducing many probabilistic logics with new type of probability operators, since no single probabilistic logic covers all contexts. The choice of particular logic depends on the particular situation that we wish to formalize.

Definition 38. Let F be a rational subset of $[0, 1]$. The *quasi complement* of F is a set $1 - F = \{1 - f : f \in F\}$.

Example 39. If $F = \{\frac{1}{2^i} : i = 1, 2, \ldots\}$, then, following Definition 38, $1 - F = \{\frac{2^i - 1}{2^i} : i = 1, 2, \ldots\}$.

It is easy to see that the quasi complement has the following properties:

- $1 - (F \cap G) = (1 - F) \cap (1 - G)$,
- $1 - (F \cup G) = (1 - F) \cup (1 - G)$,
- $1 - (F \smallsetminus G) = (1 - F) \smallsetminus (1 - G)$ and
- $1 - (1 - F) = F$.

These properties, as well as the properties of $\cup, \cap$ and $\smallsetminus$, guarantee that an arbitrary expression on the language $\{\cup, \cap, \smallsetminus, 1-\}$ can be rewritten in a normal form as a finite union of finite intersections of differences between sets and quasi complements of sets.

Definition 40. Let O_1 and O_2 be recursive families of recursive rational subsets of $[0, 1]$. Let $F_1 \in O_1$. F_1 is *representable* in O_2 if it is equal to a finite union of finite intersections of sets, differences between sets and quasi complements of sets from O_2 and sets $[r, s]$, $[r, s)$, $(r, s]$ and (r, s), where r and s are rational numbers from $[0, 1]$. The family of sets O_1 is *representable* in O_2 if each set $F_1 \in O_1$ is representable in O_2.

Example 41. Let us consider a positive integer $k > 0$, the sets

$F_1 = \{\frac{1}{2^i} : i = k, k+1, \dots\} \cup \{\frac{3^i-1}{3^i} : i = k, k+1, \dots\}$,
$F_2 = \{\frac{1}{2^i} : i = 1, 2, \dots\}$,
$F_3 = \{\frac{1}{3^i} : i = 1, 2, \dots\}$,
and the family $O_2 = \{F_2, F_3\}$. By Definition 40, F_1 is representable in O_2 because

$$F_1 = \big(F_2 \cap [0, 1/2^k]\big) \cup \big((1 - F_3) \cap \big[(3^k - 1)/3^k, 1\big]\big).$$

On the other hand, the set $F_4 = \{1/2^{2i} : i = 1, 2, \dots\}$ is not representable in O_2.

Theorem 42. *Let O_1 and O_2 be recursive families of recursive rational subsets of $[0,1]$. Let $F_1 \in O_1$ be representable in O_2. For an arbitrary formula $\alpha \in \mathrm{For}_C$, there is a formula $\phi \in \mathrm{For}_{LPP_{2,P,Q,O_2}}$ such that $\mathrm{Mod}(Q_{F_1}\alpha) = \mathrm{Mod}(\phi)$, i.e., $Q_{F_1}\alpha$ and ϕ have the same models.*

Proof. Suppose that $F_1 = \bigcup_{i=1}^m \bigcap_{j=1}^{k_i} F_{i,j}$ (for the meaning of $F_{i,j}$ see below). It is easy to see that for an arbitrary formula $\alpha \in \mathrm{For}_C$ we have (in the $LPP_{2,P,Q,O_1 \cup O_2}$):

$$\vdash Q_{F_1}\alpha \leftrightarrow \bigvee_{i=1}^{m} \bigwedge_{j=1}^{k_i} R_{F_{i,j}}\alpha$$

where

$$R_{F_{i,j}}\alpha = \begin{cases} P_{\geqslant s}\alpha \wedge P_{\leqslant r}\alpha, & \text{if } F_{i,j} = [s,r] \\ P_{\geqslant s}\alpha \wedge P_{< r}\alpha, & \text{if } F_{i,j} = [s,r) \\ P_{> s}\alpha \wedge P_{\leqslant r}\alpha, & \text{if } F_{i,j} = (s,r] \\ P_{> s}\alpha \wedge P_{< r}\alpha, & \text{if } F_{i,j} = (s,r) \\ Q_{F_{i,j}}\alpha, & \text{if } F_{i,j} \in O_2 \\ Q_{F'_{i,j}}\neg\alpha, & \text{if } F_{i,j} = 1 - F'_{i,j}, F'_{i,j} \in O_2 \\ Q_{F'_{i,j}}\alpha \wedge \neg Q_{F''_{i,j}}\alpha, & \text{if } F_{i,j} = F'_{i,j} \setminus F''_{i,j}, F'_{i,j}, F''_{i,j} \in O_2 \\ Q_{F'_{i,j}}\alpha \wedge \neg(P_{\geqslant s}\alpha \wedge P_{\leqslant r}\alpha), & \text{if } F_{i,j} = F'_{i,j} \setminus [s,r], F'_{i,j} \in O_2 \\ Q_{F'_{i,j}}\alpha \wedge \neg(P_{\geqslant s}\alpha \wedge P_{< r}\alpha), & \text{if } F_{i,j} = F'_{i,j} \setminus [s,r), F'_{i,j} \in O_2 \\ Q_{F'_{i,j}}\alpha \wedge \neg(P_{> s}\alpha \wedge P_{\leqslant r}\alpha), & \text{if } F_{i,j} = F'_{i,j} \setminus (s,r], F'_{i,j} \in O_2 \\ Q_{F'_{i,j}}\alpha \wedge \neg(P_{> s}\alpha \wedge P_{< r}\alpha), & \text{if } F_{i,j} = F'_{i,j} \setminus (s,r), F'_{i,j} \in O_2 \end{cases}$$

Formula $\bigvee_{i=1}^m \bigwedge_{j=1}^{k_i} R_{F_{i,j}}\alpha$ belongs to LPP_{2,P,Q,O_2}, and

$$\mathrm{Mod}(Q_{F_1}\alpha) = \mathrm{Mod}\Big(\bigvee_{i=1}^{m} \bigwedge_{j=1}^{k_i} R_{F_{i,j}}\alpha\Big). \qquad \square$$

Definition 43. Let O_1 and O_2 be recursive families of recursive rational subsets of $[0,1]$, and L_1 and L_2 be the corresponding $LPP_{2,P,Q,O}$-logics. The logic L_2 is *more expressive* than the logic L_1 ($L_1 \leqslant L_2$) if for every formula $\phi \in \mathrm{For}_{LPP_{2,P,Q,O_1}}$ there is a formula $\psi \in \mathrm{For}_{LPP_{2,P,Q,O_2}}$ such that $\mathrm{Mod}(\phi) = \mathrm{Mod}(\psi)$.

Theorem 44. *Let O_1 and O_2 be recursive families of recursive rational subsets of $[0,1]$, and L_1 and L_2 be the corresponding $LPP_{2,P,Q,O}$-logics. The family O_1 is representable in the family O_2 iff $L_1 \leqslant L_2$.*

Proof. ($\Rightarrow$) Let $A \in \mathrm{For}_{LPP_{2,P,Q,O_1}}$. A is equivalent to

$$DNF(A) = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{k_i} X^{i,j}(p_1, \ldots, p_n),$$

where every $X^{i,j}$ can be from the set $\{P_{\geqslant s_{i,j}}, P_{<s_{i,j}}, Q_{F_{i,j}}, \neg Q_{F_{i,j}}\}$. Furthermore, $\mathrm{Mod}(A) = \bigcup_{i=1}^{m} \bigcap_{j=1}^{k_i} \mathrm{Mod}(X^{i,j}(p_1, \ldots, p_n))$. Let us consider the case where $X^{i,j} = Q_{F_{i,j}}$. By the hypothesis the set $F_{i,j}$ is representable in O_2. Using the theorem 42 there is a formula $B_{i,j} \in \mathrm{For}_{LPP_{2,P,Q,O_2}}$ such that $\mathrm{Mod}(X^{i,j}(p_1, \ldots, p_n)) = \mathrm{Mod}(B_{i,j})$, and similarly for $X^{i,j} = \neg Q_{F_{i,j}}$, whilst the cases where $X^{i,j} = P_{\geqslant s_{i,j}}$, or $X^{i,j} = P_{<s_{i,j}}$ are both expressible in the logics L_1 and L_2. Hence, there is a formula $B \in \mathrm{For}_{LPP_{2,P,Q,O_2}}$ such that $\mathrm{Mod}(A) = \mathrm{Mod}(B)$.

($\Leftarrow$) To avoid repetition of similar arguments, in the sequel of this proof we will use $Q_{\{f\}}$ instead of $P_{=f}$. By the hypothesis, for every primitive proposition $p \in \phi$, and every $F_1 \in O_1$ there is a formula $\Phi \in \mathrm{For}_{LPP_{2,P,Q,O_2}}$ so that $\mathrm{Mod}(Q_{F_1}p) = \mathrm{Mod}(\Phi)$. If F_1 is an empty set, or a finite set, the formula $Q_{F_1}p \leftrightarrow \bigvee_{f \in F_1} Q_{\{f\}}p$ is a theorem (an empty disjunction is a contradiction), and $F_1 = \bigcup_{f \in F_1}[f, f]$ is representable in O_2.

We can show that, if $F_1 = \{f_1, f_2, \ldots\}$ is an infinite set of rational numbers from $[0, 1]$, the formula Φ cannot be propositional. Suppose that $B \in \mathrm{For}_C$. Then, the following cases must be distinguished:

- if $\Phi \to \neg p$ and $\Phi \to p$ are not theorems, consider the model $\mathbf{M} = \langle \{w_1, w_2\}, 2^{\{w_1,w_2\}}, \mu, v\rangle$ such that $\mu(\{w_1\}) = q$, $\mu(\{w_2\}) = 1 - q$, where q is an irrational number, $v(w_1)(p) = v(w_1)(\beta) = \top$, and $v(w_2)(\neg p) = v(w_2)(\beta) = \top$; since $\mu([p]) = q$, it follows that $\mathbf{M} \in \mathrm{Mod}(\Phi)$ and $\mathbf{M} \notin \mathrm{Mod}(Q_{F_1}p)$, a contradiction,
- if $\Phi \to \neg p$ is not a theorem, whilst $\Phi \to p$ is a theorem, consider an $s \in F_1 \setminus \{0\}$, and the model $\mathbf{M} = \langle \{w_1, w_2\}, 2^{\{w_1,w_2\}}, \mu, v\rangle$ such that $\mu(\{w_1\}) = s$, $\mu(\{w_2\}) = 1 - s$, $v(w_1)(p) = v(w_1)(\neg\Phi) = \top$, and $v(w_2)(\neg p) = v(w_2)(\neg\Phi) = \top$; since $\mu([p]) = s$, it follows that $\mathbf{M} \notin \mathrm{Mod}(\Phi)$ and $\mathbf{M} \in \mathrm{Mod}(Q_{F_1}p)$, a contradiction, and
- if $\Phi \to \neg p$ is a theorem, consider an $s \in F_1 \setminus \{0\}$, and the model $\mathbf{M} = \langle \{w_1, w_2\}, 2^{\{w_1,w_2\}}, \mu, v\rangle$ such that $\mu(\{w_1\}) = s$, $\mu(\{w_2\}) = 1 - s$, $v(w_1)(p) = v(w_1)(\neg\Phi) = \top$, and $v(w_2)(\neg p) = v(w_2)(\Phi) = \top$; since $\mu([p]) = s$, it follows that $\mathbf{M} \notin \mathrm{Mod}(\Phi)$ and $\mathbf{M} \in \mathrm{Mod}(Q_{F_1}p)$, a contradiction.

Hence, $\Phi \in \mathrm{For}_{LPP_{2,P,Q,O_2}} \setminus \mathrm{For}_C$. Let the disjunctive normal form of Φ be $DNF(\Phi) = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{k_i} X^{i,j}(p_1, \ldots, p_n)$ such that all $\bigwedge_{j=1}^{k_i} X^{i,j}(p_1, \ldots, p_n)$ are consistent. Since $\Phi \leftrightarrow (\Phi \wedge P_{\geqslant 0}p)$ is a valid formula, we can suppose that the primitive proposition p appears in Φ. Let p be p_1, and $a_1, \ldots, a_{2^n}$ be the list of all atoms of Φ ordered such that $a_i = p \wedge \ldots$, for $i = 1, \ldots, 2^{n-1}$, and $a_i = \neg p \wedge \ldots$, for $i = 2^{n-1} + 1, \ldots, 2^n$. Let $y_1, \ldots, y_{2^n}$ denote the atoms' measures. All the $LPP_{2,\mathrm{Meas}}$-models can be seen as points $\langle s_1, s_2, \ldots, s_{2^n}\rangle$ in the 2^n-dimensional space E, such that ith coordinate corresponds to y_i, for all $i = 1, \ldots, 2^n$. Since

$\mathrm{Mod}(Q_{F_1}p_1) = \mathrm{Mod}(\Phi)$, we have for every $y \in [0,1]$:

$\langle y, 0, \ldots, 0, 1-y, 0, \ldots, 0\rangle \in \mathrm{Mod}(Q_{F_1}p_1)$ iff $\langle y, 0, \ldots, 0, 1-y, 0, \ldots, 0\rangle \in \mathrm{Mod}(\Phi)$, where the entry $1-y$ is in the $2^{n-1}+1$'st position. Thus,

$$y \in F_1 \text{ iff } y \in \bigcup_{i=1}^{m} \bigcap_{j=1}^{k_i} \big\{y \mid \langle y, 0, \ldots, 0, 1-y, 0, \ldots, 0\rangle \in M(X^{i,j}(p_1, \ldots, p_n))\big\},$$

and by straightforward inspection of equalities, inequalities and constraints that can appear in the systems corresponding to the disjuncts from $DNF(\Phi)$, the set

$$F_1 = \big\{y \mid \langle y, 0, \ldots, 0, 1-y, 0, \ldots, 0\rangle \in \mathrm{Mod}(X^{i,j}(p_1, \ldots, p_n))\big\},$$

is representable in the family O_2.

Since every $F_1 \in O_1$ is representable in the family O_2, the family O_1 is representable in O_2. □

Theorem 44 correlates the relations of "being more expressive" between the $LPP_{2,P,Q,O}$-logics, and "being representable in" between the corresponding families of sets. In the sequel we investigate the later relation having in mind the former one. The relation "being more expressive" describes the hierarchy of expressiveness of the $LPP_{2,P,Q,O}$-logics.

Definition 45. Let O be a recursive family of rational subsets of $[0,1]$. The family of all rational subsets of $[0,1]$ that are representable in O is denoted by $\overline{O}$.

It is easy to see, using Definition 40, that a family $\overline{O}$ is closed under finite union, finite intersection, quasi complement and difference of sets. Each family $\overline{O}$ contains all finite rational subsets of $[0,1]$. Since the operations of union and intersection satisfy the commutative, associative, absorption and distributive laws, every family $\overline{O}$ with the standard set operations is a distributive lattice. Note that, if complement of a set F is understood as $[0,1] \setminus F$, $\overline{O}$ is not a Boolean algebra since $[0,1] \setminus F \not\subset S$. On the other hand, if $S \in \overline{O}$, and complement is understood as $S \setminus F$, $\overline{O}$ becomes a Boolean algebra.

Definition 46. Let O_1 and O_2 be recursive families of rational subsets of $[0,1]$. The binary relation $\sim$ is defined such that $O_1 \sim O_2$ iff $\overline{O_1} = \overline{O_2}$.

The relation $\sim$ is an equivalence relation on the set $\mathcal{O}$ of all recursive families of rational subsets of $[0,1]$. We use $\mathcal{O}_{/\sim}$ to denote the corresponding quotient set. Each equivalence class $o \in \mathcal{O}_{/\sim}$ contains a unique maximal family O_o such that $O_o = \overline{O}$. For such an equivalence class o and the corresponding family O_o we say that O_o represents o. Let the set $\{O_o : O_o \text{ represents } o \in \mathcal{O}_{/\sim}\}$ be denoted by $\mathcal{O}^*$. Clearly, O^* is countable.

Definition 47. Let O_1 and O_2 be different families from $\mathcal{O}^*$. Then $O_1 < O_2$ iff O_1 is representable in O_2.

Theorem 48. *Let O_1 and O_2 be different families from $\mathcal{O}^*$. Then $O_1 < O_2$ iff $\overline{O_1} \subset \overline{O_2}$.*

Proof. The statement is an immediate consequence of the corresponding definitions. □

Theorem 49. *The structure* $(\mathcal{O}^*, <)$ *is a lattice.*

Proof. Since $\subset$ is a partial ordering, by Theorem 48, the relation $<$ defined on $\mathcal{O}^*$ is a partial ordering, too. Moreover, any two elements of $(\mathcal{O}^*, <)$ posses both the least upper bound, and the greatest lower bound. Suppose $O_1, O_2 \in \mathcal{O}^*$. Let $O_3 = \overline{O_1 \cup O_2}$. Obviously, $O_1 < O_3$, and $O_2 < O_3$. Suppose that there is an $O_4 \in \mathcal{O}^*$, such that $O_1 < O_4$ and $O_2 < O_4$. But then, by Theorem 48, $O_1 \subset O_4$, $O_2 \subset O_4$, and $O_1 \cup O_2 \subset O_4$. It follows that $O_3 < O_4$. Hence, $\overline{O_1 \cup O_2}$ is the least upper bound of $\{O_1, O_2\}$. Similarly, the greatest lower bound of $\{O_1, O_2\}$ is $\overline{O_1 \cap O_2}$. Since $(\mathcal{O}^*, <)$ is a partially ordered set such that any two elements posses both a least upper bound, and a greatest lower bound, it is a lattice. □

The meet $(\cdot)$ and join $(+)$ operations can be defined as usual:

$$O_1 \cdot O_2 = \overline{O_1 \cap O_2}, \quad \text{and} \quad O_1 + O_2 = \overline{O_1 \cup O_2}.$$

Since every set that is representable both in O_1 and in O_2, is representable in $O_1 \cap O_2$, we have $\overline{O_1 \cap O_2} = O_1 \cap O_2$, and $O_1 \cdot O_2 = O_1 \cap O_2$. On the other hand, note that the join operation and the set union do not coincide, because for some $O_1, O_2 \in \mathcal{O}^*$, it can be $O_1 \cup O_2 \neq \overline{O_1 \cup O_2}$.

Theorem 50. *The lattice* $(\mathcal{O}^*, <)$ *is not a modular.*

Proof. We can find a counter example for the modularity law: if $O_2 < O_1$, then $(O_1 \cdot (O_2 + O_3)) = (O_2 + (O_1 \cdot O_3))$. Let $Prim = \{k_1, k_2, \ldots\}$ denote the set of all prime numbers. Then, consider the sets: $F_1 = \{\frac{1}{2^i} : i = 1, 2, \ldots\}$, $F_2 = \{\frac{1}{2^{k_i}} : i = 1, 2, \ldots\}$, and $F_3 = F_1 \setminus \{\frac{1}{2^{k_{2i-1}}} : i = 1, 2, \ldots\}$, and the families $O_1, O_2, O_3 \in \mathcal{O}^*$, such that $O_1 = \overline{\{F_1, F_2\}}$, $O_2 = \overline{\{F_2\}}$, and $O_2 = \overline{\{F_3\}}$. Obviously, $O_2 \subset O_1$, and $O_2 < O_1$. Since $F_1 = F_2 \cup F_3$, F_1 is representable in $O_2 + O_3$, and also in $O_1 \cdot (O_2 + O_3)$. On the other hand, F_1 is neither representable in O_2 nor in O_3. Thus, F_1 is not representable in $O_2 + (O_1 \cdot O_3)$, and the modularity law does not hold. □

Theorem 51. $\overline{\emptyset}$ *is the smallest element of* $(\mathcal{O}^*, <)$.

Proof. $\overline{\emptyset}$ contains all the finite rational subsets of $[0, 1]$ only. Since an arbitrary $O \in \mathcal{O}^*$ contains these sets, $\overline{\emptyset} \subset O$ and $\overline{\emptyset} < O$. □

Let $F_1 = \{r_0, r_1, \ldots\}$ be a rational subset of $[0, 1]$ with only one accumulation point. Let $O_1 = \overline{\{F_1\}}$, $O_2 \in \mathcal{O}^*$, and $O_2 < O_1$. Note that a set $F_2 \in O_2$ can be either a finite set, or an infinite set such that symmetric difference of either F_1 and F_2, $(F_1 \setminus F_2) \cup (F_2 \setminus F_1)$, or $1 - F_1$ and F_2 is finite. If all the sets from O_2 are finite, then $O_2 = 0$. Suppose that there is an infinite set $F_2 \in O_2$ that is representable in O_1. F_2 differs from F_1 (or $1 - F_1$) in finitely many elements. It follows that F_1 is representable in O_2, $O_1 < O_2$, and $O_1 = O_2$. Hence, O_1 is an atom of $(\mathcal{O}^*, <)$. Suppose that a family $O \in \mathcal{O}^*$ contains a set F with finitely many accumulation points. For every $F_1 \subset F$ with only one accumulation point, and $O_1 = \overline{\{F_1\}}$ holds $O_1 < O$. Finally, let us consider a family which contains a set with infinitely many accumulation points. Suppose that a set F_0 is dense in $(a_0, b_0) \subset [0,1]$, and $O_0 = \overline{\{F_0\}}$. We can obtain two sequences $a_0 < a_1 < a_2 < \cdots$ and $b_0 > b_1 > b_2 > \cdots$ such that $a_i < b_j$ for every i and j, a sentence of sets $F_0 \supset F_1 \supset F_2 \supset \cdots$ that are dense in $(a_1, b_1) \subset [0,1]$, $(a_2, b_2) \subset [0,1], \cdots$, respectively, and an infinite sentence of families $O_1 = \overline{\{F_1\}}$, $O_2 = \overline{\{F_2\}}, \ldots$, such that $0 < \cdots < O_2 < O_1 < O_0$. Obviously, there is no atom in this sequence.

In particular, we have the following theorems:

Theorem 52. *A necessary and sufficient condition that an $O \in \mathcal{O}^*$ be an atom is that $O = \overline{\{F\}}$, where F is a set with only one accumulation point. The lattice $(\mathcal{O}^*, <)$ is non-atomic.*

Theorem 53. *There is no greatest element in $(\mathcal{O}^*, <)$. Consequently, the lattice $\mathcal{O}^*$ is σ-incomplete.*

Proof. Since the family of all recursive subsets of S is not recursive, for each recursive family O of recursive subsets of S there is a recursive $F \subseteq S$ non-representable by O. Hence, there is no greatest element in $\mathcal{O}^*$. Furthermore, σ-incompleteness is an immediate consequence of the fact that $\mathcal{O}^*$ is a countable ordering without upper bounds. $\square$

Thus, we can define a hierarchy of the $LPP_{2,P,Q,O}$-logics, so that a logic L_1 is less expressive that a logic L_2 ($L_1 < L_2$) iff the corresponding families O_1 and O_2 of rational subsets of $[0,1]$ satisfy a similar requirement ($O_1 < O_2$). The hierarchy of the probability logics is isomorphic to $(\mathcal{O}^*, <)$. Thus, the probability logic LPP_2 is on the lowest level in the hierarchy of the $LPP_{2,P,Q,O}$-logics and corresponds to the 0-element of $(\mathcal{O}^*, <)$.

6.1.2. Complete axiomatization. Let us consider a fixed recursive family O of recursive subsets of S and the corresponding $LPP_{2,P,Q,O}$-logic. The axiomatic system $Ax_{LPP_{2,P,Q,O}}$ extends the system Ax_{LPP_2} with the following axiom:

(7) $P_{=s}\alpha \to Q_F\alpha$, where $F \in O$ and $s \in F$

and the inference rule:

(4) From $P_{=s}\alpha \Rightarrow \phi$, for all $s \in F$, infer $Q_F\alpha \Rightarrow \phi$.

As an illustration we give a list of useful theorems of $Ax_{LPP_{2,P,Q,O}}$:

Theorem 54. *If all the mentioned formulas belong to the set* $\mathrm{For}_{LPP_{2,P,Q,O}}$, *the following holds in the corresponding $LPP_{2,P,Q,O}$-logic:*

(1) $\vdash Q_F\alpha \to Q_G\alpha$, *for* $F \subset G$
(2) $\vdash (Q_F\alpha \wedge Q_G\alpha) \leftrightarrow Q_{F\cap G}\alpha$
(3) $\vdash (Q_F\alpha \vee Q_G\alpha) \leftrightarrow Q_{F\cup G}\alpha$
(4) $\vdash (Q_F\alpha \wedge P_{\geqslant s}\alpha) \leftrightarrow Q_{[s,1]\cap F}\alpha$, *and similar for* $P_{\geq s}\alpha$, $P_{\leq s}\alpha$, $P_{\leqslant s}\alpha$
(5) $\vdash Q_F\alpha \leftrightarrow Q_{1-F}\neg\alpha$, *where* $1 - F = \{1 - f : f \in F\}$
(6) $\vdash (Q_F\alpha \wedge \neg Q_G\alpha) \leftrightarrow Q_{F\smallsetminus G}\alpha$

Proof. Let us consider the case (1). If $F, G \in O$:

$\vdash P_{=s}\alpha \to Q_G\alpha$ for every $s \in F \subset G$, by Axiom 7
$\vdash Q_F\alpha \to Q_G\alpha$, by Rule 4.

The other statements follow similarly. □

The completeness proof for $Ax_{LPP_{2,P,Q,O}}$ follows the ideas from the corresponding proof from Section 3.4.

6.1.3. Decidability. In Section 3.5 we proved decidability of the LPP_2 logic which can be seen as an $LPP_{2,P,Q,O}$-logic with the empty family O. The proof involves a reduction of a formula to a system of linear (in)equalities. A look on this method indicates that the similar procedure might be applied for an arbitrary $LPP_{2,P,Q,O}$-logic. However, since there are also the operators of the form Q_F, instead of the system (5), we have to consider linear systems of the following form:

$$
\begin{aligned}
&\sum_{i=1}^{2^n} y_i = 1 \\
&y_i \geqslant 0, \quad \text{for } i = 1, \dots, 2^n \\
&\sum_{a_t \in X^1(p_1,\dots,p_n) \in D} y_t \begin{cases} \geqslant s_1 & \text{if } X^1 = P_{\geqslant s_1} \\ < s_1 & \text{if } X^1 = P_{<s_1} \\ \in F_1 & \text{if } X^1 = Q_{F_1} \\ \notin F_1 & \text{if } X^1 = \neg Q_{F_1} \end{cases} \\
&\dots \\
&\sum_{a_t \in X^k(p_1,\dots,p_n) \in D} y_t \begin{cases} \geqslant s_k & \text{if } X^k = P_{\geqslant s_k} \\ < s_k & \text{if } X^k = P_{<s_k} \\ \in F_k & \text{if } X^1 = Q_{F_k} \\ \notin F_k & \text{if } X^1 = \neg Q_{F_k} \end{cases}
\end{aligned}
\tag{8}
$$

An obvious statement holds:

Theorem 55. *An $LPP_{2,P,Q,O}$-logic is decidable iff for every probabilistic formula $A \in \mathrm{For}_{LPP_{2,P,Q,O}} \setminus \mathrm{For}_C$ there is at least one disjunct from $DNF(A)$ such that the corresponding system* (8) *is solvable.*

The requirement from Theorem 55 is very strong. For example, consider the system

$$
\begin{aligned}
y_1 + y_2 &= 1 \\
y_i &\geqslant 0, \text{ for } i = 1, 2 \\
y_1 &\geqslant s \\
y_1 &\in F
\end{aligned}
$$

obtained from the formula $P_{\geqslant s}p \wedge Q_F p$. The system is solvable only if $F \cap [s, 1] \neq \emptyset$ is decidable, and this depends on the set F. If F is a codomain of a suitable rational-valued function, the system can be solved, but, in the general case, decidability of the set F does not imply that either the system is solvable or that the $LPP_{2,P,Q,O}$-logic is decidable. However, there are recursive families O such that the corresponding probabilistic logics are decidable. A trivial example of this kind is any recursive $O \subseteq [S]^{<\omega}$, where $[S]^{<\omega}$ is the family of all finite subsets of S. A nontrivial example of a decidable logics concerns the logic which is characterized by the family O such that each $F \in O$ is definable (with rational parameters) in the language of ordered groups.

6.2. Qualitative probabilities. Reasoning about qualitative probabilities is one of the most common cases of qualitative reasoning. Here we offer the first strongly complete formalization of the notion of qualitative probability within the framework of probabilistic logic. We obtain the language of the corresponding logic (denoted $LPP_{2,\preceq}$) by extending the LPP_2-language with an additional binary operator $\preceq$, such that for some For_C-formulas α and β, $\alpha \preceq \beta$ means "β is at least probable as α". Similarly as in Section 6.1, we use the class $LPP_{2,\mathrm{Meas}}$ of measurable LPP_2-models, and the corresponding satisfiability relation (from Definition 2) with another additional requirement that:

- If $\alpha, \beta \in \mathrm{For}_C$, $\mathbf{M} \models \alpha \preceq \beta$ iff $\mu([\alpha]) \leqslant \mu([\beta])$,

The axiomatic system $Ax_{LPP_{2,\preceq}}$ extends the system Ax_{LPP_2} with the following axioms:

(7) $(P_{\leqslant s}\alpha \wedge P_{\geqslant s}\beta) \to \alpha \preceq \beta$
(8) $(\alpha \preceq \beta \wedge P_{\geqslant s}\alpha) \to P_{\geqslant s}\beta$,

and the inference rule:

(4) From $A \to (P_{\geqslant s}\alpha \to P_{\geqslant s}\beta)$ for every $s \in S$, infer $A \to \alpha \preceq \beta$.

The next theorem gives us some useful properties of the probability operator $\preceq$:

Theorem 56. *Suppose that T is a set of formulas and that α, β, $\gamma \in \mathrm{For}_C$. Then the following holds:*

(1) $T \vdash \alpha \preceq \beta$ *if and only if* $T \vdash P_{\geqslant s}(\alpha) \to P_{\geqslant s}(\beta)$ *for all* $s \in S$;
(2) $\vdash \alpha \preceq \beta \vee \beta \preceq \alpha$;
(3) $\vdash (\alpha \preceq \beta \wedge \beta \preceq \gamma) \to \alpha \preceq \gamma$;
(4) $\vdash \alpha \preceq \alpha$;
(5) *If* $T \vdash P_{\geqslant 1}(\alpha \to \beta)$ *then* $T \vdash \alpha \preceq \beta$;
(6) *If* $T \vdash \alpha \to \beta$ *then* $T \vdash \alpha \preceq \beta$.

Proof. Since (5) directly follows from (1), (4) from (2), and (6) is a consequence of (5) and Rule 2, we will prove only the first three statements.

(1) Suppose that $T \vdash \alpha \preceq \beta$. By the axioms 1 and 8 we have that

$$T \vdash \alpha \preceq \beta \to (P_{\geqslant s}(\alpha) \to P_{\geqslant s}(\beta)).$$

Now applying Rule 1 we obtain that $T \vdash P_{\geqslant s}(\alpha) \to P_{\geqslant s}(\beta)$. Conversely, suppose that $T \vdash P_{\geqslant s}(\alpha) \to P_{\geqslant s}(\beta)$ for each $s \in S$. Then by Axiom 1

$$T \vdash P_{\geqslant 0}(\alpha) \to (P_{\geqslant s}(\alpha) \to P_{\geqslant s}(\beta))$$

for each $s \in S$. Applying Rule 4 we deduce that $T \vdash P_{\geqslant 0}(\alpha) \to \alpha \preceq \beta$. Finally, since $T \vdash P_{\geqslant 0}(\alpha)$ (Axiom 2), by Rule 1 we conclude that $T \vdash \alpha \preceq \beta$.

(2) First let us observe that Axiom 7 is equivalent to

$$\neg(\alpha \preceq \beta) \to (P_{\geqslant s}(\beta) \to P_{>s}(\alpha)).$$

Since $\vdash P_{>s}(\alpha) \to P_{\geqslant s}(\alpha)$, we have that $\vdash \neg(\alpha \preceq \beta) \to (P_{\geqslant s}(\beta) \to P_{\geqslant s}(\alpha))$ for every $s \in S$, so by Rule 4 we obtain $\vdash \neg(\alpha \preceq \beta) \to \beta \preceq \alpha$, which is equivalent to $\vdash \alpha \preceq \beta \vee \beta \preceq \alpha$.

(3) According to Deduction theorem, it is sufficient to prove that

$$\alpha \preceq \beta, \beta \preceq \gamma \vdash \alpha \preceq \gamma.$$

Since $\alpha \preceq \beta \vdash P_{\geqslant s}(\alpha) \to P_{\geqslant s}(\beta)$ and $\beta \preceq \gamma \vdash P_{\geqslant s}(\beta) \to P_{\geqslant s}(\gamma)$, we have that

$$\alpha \preceq \beta, \beta \preceq \gamma \vdash P_{\geqslant s}(\alpha) \to P_{\geqslant s}(\gamma).$$

This holds for all $s \in S$, so applying the statement (1) from this theorem, we obtain that $\alpha \preceq \beta, \beta \preceq \gamma \vdash \alpha \preceq \gamma$. ☐

The corresponding completeness proof follows the same steps as above for LPP_2. Also, decidability can be proved in the same way as in Section 3.5 since the only new type of formulas ($\alpha \preceq \beta$) can be reduced to an inequality of the form:

$$\sum_{a_k \in \mathrm{CDNF}(\alpha)} \mu([a_k]) \leqslant \sum_{a_k \in \mathrm{CDNF}(\beta)} \mu([a_k]).$$

It is also interesting that, if we add the qualitative probability operator to the logic $LPP_2^{\mathrm{Fr}(n)}$, due to the fact that the set Range (which denotes the range of the considered probability functions) is finite, $\alpha \preceq \beta$ can be seen as an abbreviation of the formula $\bigwedge_{s \in \mathrm{Range}}(P_{\geqslant s}\alpha \to P_{\geqslant s}\beta)$. Thus, the notion of the qualitative probability is definable in $LPP_2^{\mathrm{Fr}(n)}$, and the logics $LPP_2^{\mathrm{Fr}(n)}$ and $LPP_{2,\preceq}^{\mathrm{Fr}(n)}$ coincide (in the sense that the later one is a conservative extension of the former logic).

7. First order probability logics

This section is devoted to a probabilistic extension of first order classical logic. In this case interleaving of the probabilistic operators and the classical quantifiers is important, especially when we compare first order probability logics to first order modal logics. Thus, to avoid repetition and contrary to Section 3, we will start here with the logic $LFOP_1$, the first order counterpart of the propositional probability logic LPP_1.

7.1. Syntax. The language of the $LFOP_1$-logic is an extension of the classical first order language. It is a countable set which contains for each non negative integer k, k-ary relation symbols $P_0^k, P_1^k, \dots$, and k-ary function symbols $F_0^k, F_1^k, \dots$, and the logical symbols $\wedge$, and $\neg$, quantifier $\forall$, a list of unary probability operators $P_{\geqslant s}$ for every rational number $s \in [0,1]$, variables $x, y, z, \dots$, and parentheses.

The notions of existential quantifier, arity of a functional or a relational symbol, term, atomic formula, bound and free variables, sentence, and a term free for a variable in a formula can be defined as usual, while the set For_{LFOP_1} of formulas is the smallest set containing atomic formulas and closed under formation rules: if α and β are formulas, then $\neg\alpha$, $P_{\geqslant s}\alpha$, $\alpha \wedge \beta$ and $(\forall x)\alpha$ are formulas.

Example 57. An example of a formula is:

$$P_{\geqslant s}(\forall x)P_1^1(x) \to P_3^2(y, F_0^0) \wedge P_{\geqslant r}P_{\geqslant t}P_1^1(F_1^0).$$

$\alpha(x_1, \ldots, x_m)$ indicates that free variables of the formula α form a subset of $\{x_1, \ldots, x_m\}$. If t is a term free for x in α, then $\alpha(t/x)$ denotes the result of substituting in α the term t for all free occurrences of x. We will also use the shorter form $\alpha(t)$ to denote the same substitution.

7.2. Semantics. The models we will use are similar to $LPP_{1,\mathrm{Meas}}$-models with an important difference that worlds of models are now classical first order models. More formally:

Definition 58. An $LFOP_1$*-model* is a structure $\mathbf{M} = \langle W, H, \mu, v\rangle$ where:

- W is a non empty set of objects called worlds,
- D associates a non empty domain $D(w)$ with every world $w \in W$,
- I associates an interpretation $I(w)$ with every world $w \in W$ such that:
  - $I(w)(F_i^k)$ is a function from $D(w)^k$ to $D(w)$, for all i, and k,
  - $I(w)(P_i^k)$ is a relation over $D(w)^k$, for all i, and k.
- Prob is a probability assignment which assigns to every $w \in W$ a probability space, such that $\mathrm{Prob}(w) = \langle W(w), H(w), \mu(w)\rangle$, where:
  - $W(w)$ is a non empty subset of W,
  - $H(w)$ is an algebra of subsets of $W(w)$ and
  - $\mu(w) : H(w) \to [0, 1]$ is a finitely additive probability measure.

The next definitions reflect the mentioned fact that worlds in $LFOP_1$-models are classical first order models.

Definition 59. Let $\mathbf{M} = \langle W, D, I, \mathrm{Prob}\rangle$ be an $LFOP_1$-model. A*variable valuation* v assigns some element of the corresponding domain to every world w and every variable x, i.e., $v(w)(x) \in D(w)$. If $w \in W$, $d \in D(w)$, and v is a valuation, then $v_w[d/x]$ is a valuation like v except that $v_w[d/x](w)(x) = d$.

Definition 60. For an $LFOP_1$-model $\mathbf{M} = \langle W, D, I, \mathrm{Prob}\rangle$ and a valuation v the*value of a term t* (denoted by $I(w)(t)_v$) is:

- if t is a variable x, then $I(w)(x)_v = v(w)(x)$, and
- if $t = F_i^m(t_1, \ldots, t_m)$, then
  $I(w)(t)_v = I(w)(F_i^m)(I(w)(t_1)_v, \ldots, I(w)(t_m)_v)$.

Definition 61. The truth value of a formula α in a world $w \in W$ for a given $LFOP_1$-model $\mathbf{M} = \langle W, D, I, \mathrm{Prob}\rangle$, and a valuation v (denoted by $I(w)(\alpha)_v$) is:

- if $\alpha = P_i^m(t_1, \ldots, t_m)$, then $I(w)(\alpha)_v = \top$ if $\langle I(w)(t_1)_v, \ldots, I(w)(t_m)_v\rangle \in I(w)(P_i^m)$, otherwise $I(w)(\alpha)_v = \bot$,
- if $\alpha = \neg\beta$, then $I(w)(\alpha)_v = \top$ if $I(w)(\beta)_v = \bot$, otherwise $I(w)(\alpha)_v = \bot$,
- if $\alpha = P_{\geqslant s}\beta$, then $I(w)(\alpha)_v = \top$ if $\mu(w)\{u \in W(w) : I(u)(\beta)_v = \top\} \geqslant s$, otherwise $I(w)(\alpha)_v = \bot$,

- if $\alpha = \beta \wedge \gamma$, then $I(w)(\alpha)_v = \top$ if $I(w)(\beta)_v = \top$, and $I(w)(\gamma)_v = \top$, otherwise $I(w)(\alpha)_v = \bot$, and
- if $\alpha = (\forall x)\beta$, then $I(w)(\alpha)_v = \top$ if for every $d \in D$, $I(w)(\beta)_{v_w[d/x]} = \top$, otherwise $I(w)(\alpha)_v = \bot$.

Definition 62. A formula *holds in a world* w from an $LFOP_1$-model $\mathbf{M} = \langle W, D, I, \mathrm{Prob}\rangle$ (denoted by $(M, w) \models \alpha$) if for every valuation v, $I(w)(\alpha)_v = \top$. If $d \in D(w)$, we will use $(\mathbf{M}, w) \models \alpha(d)$ to denote that for every valuation v, $I(w)(\alpha(x))_{v_w[d/x]} = \top$.

A sentence α is*satisfiable* if there is a world w in an $LFOP_1$-model $\mathbf{M}$ such that $(\mathbf{M}, w) \models \alpha$. A set T of sentences is satisfiable if there is a world w in an $LFOP_1$-model $\mathbf{M}$ such that for every $\alpha \in T$, $(\mathbf{M}, w) \models \alpha$.

A sentence α is*valid* if for every $LFOP_1$-model $\mathbf{M} = \langle W, D, I, \mathrm{Prob}\rangle$ and every world $w \in W$, $(\mathbf{M}, w) \models \alpha$.

In the sequel we will consider a class of all $LFOP_1$-models that satisfy:

- all the worlds from a model have the same domain, i.e., for all $v, w \in W$, $D(v) = D(w)$,
- for every sentence α, and every world w from a model $\mathbf{M}$ the set $\{u \in W(w) : I(u)(\alpha)_v = \top\}$ of all worlds from $W(w)$ that satisfy α is measurable, and
- the terms are rigid, i.e., for every model their meanings are the same in all worlds.

We use $LFOP_{1,\mathrm{Meas}}$ to denote that class of all fixed domain measurable models with rigid terms.

Example 63. Let us consider the formula $P_{\geqslant s}P_1^1(x)$, and suppose that for an $LFOP_{1,\mathrm{Meas}}$-model $\mathbf{M} = \langle W, D, I, \mathrm{Prob}\rangle$, $w \in W$, $(\mathbf{M}, w) \models P_{\geqslant s}P_1^1(x)$. By Definition 62, this holds iff for every valuation v, $I(w)(P_{\geqslant s}P_1^1(x))_v = \top$ iff $(\mathbf{M}, w) \models (\forall x)P_{\geqslant s}P_1^1(x)$.

On the other hand, as we will show in Example 64, the satisfiability of the formula $P_{\geqslant s}P_1^1(x)$ does not imply the satisfiability of $P_{\geqslant s}(\forall x)P_1^1(x)$. The example assures an already existing impression that, although probability and modal logics are closely related, modal necessity (denoted by $\Box$) is a stronger notion than probability necessity (probability one, $P_{\geqslant 1}$).

Example 64. Let us consider, the well known Barcan formula of the first order modal logic:

$$\mathbf{BF}\quad (\forall x)\Box\alpha(x) \rightarrow \Box(\forall x)\alpha(x)$$

It is proved that **BF** holds in the class of all first order fixed domain modal models, and that it is independent from the other first order modal axioms. However, the behavior of the reminiscence of this formula:

$$\mathbf{BF}(s)\quad (\forall x)P_{\geqslant s}\alpha(x) \rightarrow P_{\geqslant s}(\forall x)\alpha(x)$$

is quite different.

If $s = 0$, $\mathbf{BF}(0)$ is valid, because $P_{\geqslant 0}(\forall x)\alpha(x)$ always holds since probability functions are nonnegative. So, suppose that $s > 0$. Let us consider the $LFOP_{1,\mathrm{Meas}}$-model $\mathbf{M}$ such that:

- $W = \{w_1, w_2, w_3, w_4\}$
- $D = \{d_1, d_2\}$,
- $(\mathbf{M}, w_2) \models P_1^1(d_1)$, $(\mathbf{M}, w_2) \not\models P_1^1(d_2)$, $(\mathbf{M}, w_3) \models P_1^1(d_1)$, $(\mathbf{M}, w_3) \models P_1^1(d_2)$, $(\mathbf{M}, w_4) \not\models P_1^1(d_1)$, $(\mathbf{M}, w_4) \models P_1^1(d_2)$,
- $\mu(w_1)(w_2) = \frac{1}{n}$, $\mu(w_1)(w_3) = s - \frac{1}{n}$ $\mu(w_1)(w_4) = \frac{1}{n}$

It is easy to see that $(\mathbf{M}, w_1) \models (\forall x)P_{\geqslant s}P_1^1(x)$, because

$$\mu(w_1)(\{w : w \models P_1^1(d_1)\}) = \mu(w_1)(\{w_2, w_3\}) = s,$$
$$\mu(w_1)(\{w : w \models P_1^1(d_2)\}) = \mu(w_1)(\{w_3, w_4\}) = s.$$

On the other hand, $(\mathbf{M}, w_1) \not\models (\forall x)P_1^1(x)$, $(\mathbf{M}, w_2) \not\models (\forall x)P_1^1(x)$, and $(\mathbf{M}, w_4) \not\models (\forall x)P_1^1(x)$, whilst $(\mathbf{M}, w_3) \models (\forall x)P_1^1(x)$. Since $\mu(w_1)(\{w_3\}) = s - \frac{1}{n}$, $(\mathbf{M}, w_1) \not\models P_{\geqslant s}(\forall x)P_1^1(x)$, and for $s > 0$, $(\mathbf{M}, w_1) \not\models \mathbf{BF(}s\mathbf{)}$.

7.3. A sound and complete axiomatic system. The axiomatic system Ax_{LFOP_1} is a combination of a classical first order axiomatization and the probabilistic axioms introduced in Section 3. It involves the following axiom schemas:

(1) all the axioms of the classical propositional logic
(2) $(\forall x)(\alpha \to \beta) \to (\alpha \to (\forall x)\beta)$, where x is not free in α
(3) $(\forall x)\alpha(x) \to \alpha(t/x)$, where $\alpha(t/x)$ is obtained by substituting all free occurrences of x in $\alpha(x)$ by the term t which is free for x in $\alpha(x)$
(4) $P_{\geqslant 0}\alpha$
(5) $P_{\leqslant r}\alpha \to P_{<s}\alpha$, $s > r$
(6) $P_{<s}\alpha \to P_{\leqslant s}\alpha$
(7) $(P_{\geqslant r}\alpha \wedge P_{\geqslant s}\beta \wedge P_{\geqslant 1}(\neg\alpha \vee \neg\beta)) \to P_{\geqslant \min(1,r+s)}(\alpha \vee \beta)$
(8) $(P_{\leqslant r}\alpha \wedge P_{<s}\beta) \to P_{<r+s}(\alpha \vee \beta)$, $r + s \leqslant 1$

and inference rules:

(1) From α and $\alpha \to \beta$ infer β.
(2) From α infer $(\forall x)\alpha$
(3) From α infer $P_{\geqslant 1}\alpha$.
(4) From $\beta \to P_{\geqslant s - \frac{1}{k}}\alpha$, for every integer $k \geqslant \frac{1}{s}$, infer $\beta \to P_{\geqslant s}\alpha$.

We use the notions of deducibility and consistency introduced in the definitions 30 and 31 from Section 5. The theorems 65 and 66 show that Ax_{LFOP_1} characterizes the set of all $LFOP_{1,\mathrm{Meas}}$-valid sentences.

Theorem 65 (Soundness theorem)**.** *The axiomatic system Ax_{LFOP_1} is sound with respect to the $LFOP_{1,\mathrm{Meas}}$ class of models.*

Proof. Let α' be an instance of a classical propositional axiom α obtained by substituting propositional letters by formulas. Suppose that the formula α' is not valid, i.e., that for some world w from a model $\mathbf{M}$, and a valuation v, $I(w)(\alpha')_v = \bot$. It follows that we can find a classical propositional valuation ρ such that $\rho(\alpha) = \bot$, a contradiction. Let $\mathbf{M} = \langle W, D, I, \mathrm{Prob}\rangle$ and $w \in W$ such that $(\mathbf{M}, w) \models (\forall x)\alpha(x)$.

It means that $I(w)((\forall x)\alpha(x))_v = \top$ for every valuation v. Among these valuations there must be one (denoted v') which assigns to x the value $d = I(w)(t)_v$. For this valuation $I(w)(\alpha(x))_{v'} = \top$. Since $I(w)(\alpha(x))_{v'} = I(w)(\alpha(t/x))_v$, we have $I(w)(\alpha(t/x))_v = \top$ for every valuation. Thus, every instance of Axiom 3 is valid. Note that the assumptions about fixed domains and rigidness of terms are crucial. If it is not the case, and $\alpha(t/x)$ is of the form $P_{\geqslant s}\beta(t/x)$, the term t refers to objects in other worlds (different from w). It can have a consequence that $I(w)(\alpha(t/x))_v = \bot$. The axioms 4– 8 concern the properties of measures from $LFOP_{1,\mathrm{Meas}}$-models and obviously hold in every model. The inference rules 1 and 2 are validity-preserving for the same reason as in the classical first order logic. Consider Rule 3 and suppose that a formula α is valid. It must hold in every world from every $LFOP_{1,\mathrm{Meas}}$-model. Thus, for every model $\mathbf{M} = \langle W, D, I, \mathrm{Prob}\rangle$, and $w \in W$, the sets $\{u \in W(w) : (\mathbf{M}, u) \models \alpha\}$ and $W(w)$ coincide. Since $\mu(w)(W(w)) = 1$, it follows that $(\mathbf{M}, w) \models P_{\geqslant 1}\alpha$. Rule 4 preserves validity because of the properties of the set of rational numbers. □

Theorem 66 (Extended completeness theorem for $LFOP_{1,\mathrm{Meas}}$)**.** *The axiomatic system Ax_{LFOP_1} is sound with respect to the $LFOP_{1,\mathrm{Meas}}$ class of models.*

Proof. The completeness proof follows the same ideas as above, for example as in Section 5. The main new step is that, since we work with first-order formulas, we have a special kind of maximal consistent sets called saturated sets. A set T of formulas is saturated if it is maximal consistent and satisfies:

- if $\neg(\forall x)\alpha(x) \in T$, then for some term t, $\neg\alpha(t) \in T$.

We can prove a counterpart of Theorem 13, where the new step:

- if the set T_{i+1} is obtained by adding a formula of the form $\neg(\forall x)\beta(x)$ to the set T_i, then for some $c \in C$, $\neg\beta(c)$ is also added to T_{i+1}, so that T_{i+1} is consistent,

guarantees that every consistent set of sentences can be extended to a saturated set (C is a countably infinite set of new constant symbols). Then, the canonical model $\mathbf{M} = \langle W, D, I, \mathrm{Prob}\rangle$ can be defined in the following way:

- W is the set of all saturated sets,
- D is the set of all variable-free terms,
- for every $w \in W$, $I(w)$ is an interpretation such that:
  - for every function symbol F_i^m, $I(w)(F_i^m)$ is a function from D^m to D such that for all variable-free terms $t_1, \ldots, t_m$ in $\overline{\mathcal{L}}$, $F_i^m : \langle t_1, \ldots, t_m\rangle \to F_i^m(t_1, \ldots, t_m)$, and
  - for every relation symbol P_i^m, $I(w)(P_i^m) = \{\langle t_1, \ldots, t_m\rangle$ for all variable-free terms $t_1, \ldots, t_m \in \overline{\mathcal{L}} : P_i^m(t_1, \ldots, t_m) \in w\}$.
- for every $w \in W$, $\mathrm{Prob}(w) = \langle W(w), H(w), \mu(w)\rangle$ such that:
  - $W(w) = W$,
  - $H(w)$ is a class of sets $[\alpha] = \{w \in W : \alpha \in w\}$, for every sentence α, and
  - for every set $A \in H(w)$, $\mu(w)(A) = \sup_s\{P_{\geqslant s}\alpha \in w\}$.

and the rest of the proof is standard. □

The same arguments as in Section 3.4 can be used to prove completeness of Ax_{LFOP_1} with respect to the classes: $LPFOP_{1,\text{Meas},\text{All}}$, $LFOP_{1,\text{Meas},\text{Neat}}$ and $LFOP_{1,\text{Meas},\sigma}$, while the modifications similar to the ones from the previous sections will be appropriate for logics $LFOP_1^{\text{Fr}(n)}$, $LFOP_1^{A,\omega_1,\text{Fin}}$, $LFOP_1^S$, and the logic $LFOP_2$ (the first order probability logic without iterations) and its variants.

7.4. Decidability. The logic $LFOP_1$ and its variants contain classical first order logic. Thus, they are undecidable. The monadic fragments of the considered systems are undecidable, too. To show that, we can use the procedure due to Saul Kripke [48, 62] and consider a translation of classical first order formulas that contain only one binary relation symbol P^2 to monadic probability formulas such that a classical first order formula is valid if and only if its translation is a valid probability formula. The original translation replaces every expression of the form $P^2(t_1, t_2)$ in a classical formula by $\lozenge(P_1^1(t_1) \wedge P_2^1(t_2))$, and instead of the modal formula we can use its probabilistic counterpart $P_{\geqslant 0}(P_1^1(t_1) \wedge P_2^1(t_2))$. Since the fragment of the classical first order logic with a single binary relation symbol is not decidable, the same holds for the monadic fragments of the first order probability logics with iterations of probability operators. However, it is interesting that the monadic fragment of $LFOP_2$ is decidable.

8. Probabilistic logics with the non-classical base

Let us use the term*the basic logic* for a logic from which we start building a probability logic. So far, we have used only classical (propositional or first order) logic as the basic logic, and it might be usefull to provide some motivation for other possible choices. The most important reason to change the basic logic, from our point of view, might be the very nature of classical logic. Namely, it is basically the logic of mathematics conceived as pertaining to some outside (Platonic) reality. On this conception, statements are either true or false and forever so (truth is independent of time and place), there is no room for modality (maybe, possibly,. . .) or value judgment. It is not surprising that the resulting logic will have some consequences which seem rather odd in real-life situations and this issue has been debated throughout the last century, often under the heading "paradoxes of the material implication". In this section we will address those issues, and consider two logics: in the first one we will use intuitionistic logic as the basic logic, while in the second we will start from a temporal base. However, we do not argue that either "classical" or any of "non-classical" probabilistic logics is the unique logic for modelling probabilistic reasoning. Our view is more pragmatic: we believe that there are real-life situations in which the former approach could be appropriate, but the same holds for the later one.

8.1. An intuitionistic probability logic. Intuitionistic logic arises quite naturally from a conception of mathematics as a human endeavor not pertaining to some outside reality. Since the statements of mathematics are not about something which exists out there, they cannot be true or false but only proved or disproved. This leaves another category of statements, those which are as yet undetermined. Thus intuitionistic logic may be viewed as the logic of the growth of human knowledge (as opposed to the classical logic which we may regard as the logic of the static Platonic universe of mathematical objects). Thanks to this, intuitionistic logic has less consequences which would seem rather unintuitive in a real-life situation (e.g., $(p \to q) \vee (q \to p)$ and $(p \to (q \vee r)) \to ((p \to q) \vee (p \to r))$ are not intuitionistic theorems, i.e., there are models in which they are false). In reality, there is the fact that the intuitionistic logic might be the least popular non-classical logic among the practitioners of artificial intelligence and computer science in general. However, for those comfortable with the ubiquitous S4-modal logic and uncomfortable with intuitionism, we should emphasize that these two logics are practically the same: their models are the same, while the Gödel translation enables us to interpret syntax. Furthermore, as we shall show in the Remark at the end of this section, intuitionistic logic arises naturally whenever we deal with possible worlds semantics. In any case, starting with intuitionistic logic, we naturally have, besides proved statements (probability is 1) and disproved statements (probability is 0), undetermined statements whose probability should range between 0 and 1. This is more obvious if we consider a Kripke model in which we can assign a probability to a formula on the basis of the number of possible worlds in which it is true. In our approach the probabilistic operators have the classical treatment. As a justification, we may say that once we determine the probability of an uncertain proposition α, it should be either greater or equal to some $s \in [0,1]$ or not, so it is not unreasonable to assume $P_{\geqslant s}\alpha \vee \neg P_{\geqslant s}\alpha$ (even if we reject $\alpha \vee \neg\alpha$).

We use LPP_2^I to denote the corresponding intuitionistic probability logic. At the propositional level, the language contains the connectives $\neg$, $\wedge$, $\vee$ and $\to$, while on the probabilistic level we have two lists of unary probabilistic operators $(P_{\geqslant s})_{s\in S}$, and $(P_{\leqslant s})_{s\in S}$, and the connectives $\neg$ and $\wedge$. Note that, since we have the intuitionistic base:

- at the propositional level, the propositional connectives are independent, and
- at the probabilistic level, the probabilistic operators $P_{\geqslant \cdot}$, and $P_{\leqslant \cdot}$ are independent, but $\vee$ and $\to$ can be defined from $\neg$ and $\wedge$.

Similarly as for the logic LPP_2, we do not allow iterations of probabilistic operators, and define the sets For_I of propositional formulas, For_P of probabilistic formulas, and $\mathrm{For}_{LPP_2^I}$ of all formulas, as in Section 3.1.

8.1.1. Semantics. Corresponding to the structure of the set $\mathrm{For}_{LPP_2^I}$, there are two levels in the definition of models. At the first level there is the notion of intuitionistic Kripke models [63], while probability comes in the picture at the second level.

Definition 67. An *intuitionistic Kripke* model for the language For_I is a structure $\langle W, \leqslant, v\rangle$ where:

- $\langle W, \leqslant\rangle$ is a partially ordered set of possible worlds which is a tree, and
- v is a valuation function, i.e., v maps the set W into the powerset $\mathcal{P}(\phi)$, which satisfies the condition: for all w, $w' \in W$, $w \leqslant w'$implies $v(w) \subseteq v(w')$.

The last requirement from Definition 67 allows that v does not determine the status of some primitive propositions from ϕ in some worlds. In each Kripke model we define the forcing relation $\Vdash \subset W \times \mathrm{For}_I$ by the following definition:

Definition 68. Let $\langle W, \leqslant, v\rangle$ be an intuitionistic Kripke model. *The forcing relation* $\Vdash$ is defined by the following conditions for every $w \in W$, α, $\beta \in \mathrm{For}_I$:

- if $\alpha \in \phi$, $w \Vdash \alpha$ iff $\alpha \in v(w)$,
- $w \Vdash \alpha \wedge \beta$ iff $w \Vdash \alpha$ and $w \Vdash \beta$,
- $w \Vdash \alpha \vee \beta$ iff $w \Vdash \alpha$ or $w \Vdash \beta$,
- $w \Vdash \alpha \to \beta$ iff for every $w' \in W$ if $w \leqslant w'$ then $w' \nVdash \alpha$ or $w' \Vdash \beta$, and
- $w \Vdash \neg\alpha$ iff for every $w' \in W$ if $w \leqslant w'$ then $w' \nVdash \alpha$.

We read $w \Vdash \alpha$ as "*w forces α*" or "α is true in the world w". *Validity in the intuitionistic Kripke model* $\langle W, \leqslant, v\rangle$ is defined by $\langle W, \leqslant, v\rangle \vDash \alpha$ iff $(\forall w \in W) w \Vdash \alpha$. A formula α is *valid* ($\vDash \alpha$) if it is valid in every intuitionistic Kripke model.

Let $\mathbf{M}_I = \langle W, \leqslant, v\rangle$ be an intuitionistic Kripke model. We use $[\alpha]_{\mathbf{M}_I}$ (or shortly $[\alpha]$ if $\mathbf{M}_I$ is clear from the context) to denote $\{w \in W : w \Vdash \alpha\}$ for every $\alpha \in \mathrm{For}_I$. The family $H_I = \{[\alpha]_{\mathbf{M}_I} : \alpha \in \mathrm{For}_I\}$ is a Heyting algebra with operations:

$$[\alpha]\cup[\beta] = [\alpha\vee\beta], \quad [\alpha]\cap[\beta] = [\alpha\wedge\beta], \quad [\alpha] \Rightarrow [\beta] = [\alpha \to \beta], \quad \text{and} \quad \sim [\alpha] = [\neg\alpha].$$

Thus, H_I is a lattice on W, but it may be not closed under complementation.

Definition 69. A*measurable probabilistic model* is a structure $\mathbf{M} = \langle W, \leqslant, v, H, \mu\rangle$ where:

- $\mathbf{M}_I = \langle W, \leqslant, v\rangle$ is an intuitionistic Kripke model,
- H is an algebra on W containing $H_I = \{[\alpha]_{\mathbf{M}_I} : \alpha \in \mathrm{For}_I\}$,
- $\mu : H \to [0, 1]$ is a finitely additive probability.

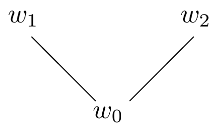
Note that H contains all sets of the form $W \smallsetminus [\alpha]_{\mathbf{M}_I}$, even if for some $\alpha \in \mathrm{For}_I$ it may be that $W \smallsetminus [\alpha]_{\mathbf{M}_I} \neq [\neg\alpha]_{\mathbf{M}_I}$. The fact that $[\neg\alpha]$ does not have to contain the complement of $[\alpha]$ is the reason why we need both $P_{\geqslant s}$ and $P_{\leqslant s}$ operators since $P_{\leqslant s}\alpha$ will not imply $P_{\geqslant 1-s}\neg\alpha$.

We use $LPP^I_{2,\mathrm{Meas}}$ to denote the class of all measurable probabilistic models.

Definition 70. *The satisfiability relation* $\vDash$ is defined by the following conditions for every $LPP^I_{2,\mathrm{Meas}}$-model $\mathbf{M} = \langle W, \leqslant, v, H, \mu\rangle$:

- if $\alpha \in \mathrm{For}_I$, $\mathbf{M} \vDash \alpha$ if $(\forall w \in W) w \Vdash \alpha$,
- $\mathbf{M} \vDash P_{\geqslant s}\alpha$ if $\mu([\alpha]) \geqslant s$,
- $\mathbf{M} \vDash P_{\leqslant s}\alpha$ if $\mu([\alpha]) \leqslant s$,
- if $A \in \mathrm{For}_P$, $\mathbf{M} \vDash \neg A$ if $\mathbf{M} \vDash A$ does not hold, and
- if $A, B \in \mathrm{For}_P$, $\mathbf{M} \vDash A \wedge B$ if $\mathbf{M} \vDash A$, and $\mathbf{M} \vDash B$.

Definition 71. A formula $\Phi \in \mathrm{For}_{LPP^I_2}$ is*satisfiable* if there is a $LPP^I_{2,\mathrm{Meas}}$-model $\mathbf{M}$ such that $\mathbf{M} \vDash \Phi$; Φ is*valid* if for every $LPP^I_{2,\mathrm{Meas}}$-model $\mathbf{M}$, $\mathbf{M} \vDash \Phi$; a set of formulas is *satisfiable* if there is an $LPP^I_{2,\mathrm{Meas}}$-model $\mathbf{M}$ such that for every formula Φ from the set, $\mathbf{M} \vDash \Phi$.



$$v(w_2) = \{q\} \quad \mu(\{w_2\}) = 1/3$$
$$v(w_1) = \{p\} \quad \mu(\{w_1\}) = 1/3$$
$$v(w_0) = \emptyset \quad \mu(\{w_0\}) = 1/3$$

FIGURE 2. A tree-like probabilistic model

8.1.2. Examples. In this section we consider some consequences of probabilistic reasoning which is based on classical logic and which can be avoided using probabilistic logic based on intuitionistic logic.

Example 72. It is well known that $\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$ is a classical tautology, called De Morgan's law which is not an intuitionistic tautology. Still, even if we believe that it is impossible to have your cake and eat it, we do not believe that it is impossible to have your cake and we also do not believe that it is impossible to eat your cake. More formally, we would like to have $P_{\geqslant 1}\neg(p \wedge q)$, but also $P_{\leqslant \epsilon}\neg p$ and $P_{\leqslant \epsilon}\neg q$ for come small ϵ, which is impossible with classical logic.

Example 73. Consider the classical tautology $(p \rightarrow q) \vee (q \rightarrow p)$ and probability logic based on classical logic. Since tautologies have probability equal to 1, $P_{\geqslant 1}((p \rightarrow q) \vee (q \rightarrow p))$ is valid. Let us now take a real-life situation, where p and q mean "it rains" and "the sprinkler is on", respectively. It is clear that the sprinkler should not be on when it rains, i.e., that $p \rightarrow q$ should have low probability, say less than ϵ ($P_{\leqslant \epsilon}(p \rightarrow q)$). Since probability is additive, the measure of the union of two sets is less or equal than the sum of the measures of those sets. Thus, the probability of $q \rightarrow p$ has to be high. In other words, we get that it is very probable that it will rain whenever the sprinkler is on ($P_{\geqslant 1-\epsilon}(q \rightarrow p)$). If we were designing a controller for the sprinkler, this certainly would not be a desirable consequence.

On the other hand, $(p \rightarrow q) \vee (q \rightarrow p)$ is not an intuitionistic tautology. Consider the model from Figure 2. Recall that $p \rightarrow q$ being false in a Kripke model means that there is at least one possible world in which it is raining but the sprinkler is off. It is easy to see that $w_1 \Vdash q \rightarrow p$, $w_1 \nVdash p \rightarrow q$ $w_2 \Vdash p \rightarrow q$, $w_2 \nVdash q \rightarrow p$, $w_0 \nVdash (p \rightarrow q) \vee (q \rightarrow p)$, $\mu([(p \rightarrow q) \vee (q \rightarrow p)]_M) = 2/3$, and $M \nvDash P_{\geqslant 1}((p \rightarrow q) \vee (q \rightarrow p))$. Thus, the above consequence, that with high probability sprinkler causes raining, does not follow any more.

Note also that we can construct a model in which both $p \rightarrow q$ and $q \rightarrow p$ will have very low probability, say less than $1/n$, by simply adding $n-3$ linearly ordered new worlds below w_0 in M, and having $\mu(w) = 1/n$, $w \in W$. In the same model we have $P_{\geqslant 1}\neg(p \wedge q)$, $P_{\leqslant 1/n}\neg p$ and $P_{\leqslant 1/n}\neg q$, demonstrating the point of the previous example.

Example 74. Consider the classical (but not intuitionistic) tautology $(p \rightarrow (q \vee r)) \rightarrow ((p \rightarrow q) \vee (p \rightarrow r))$. Starting with classical logic makes $P_{\geqslant 1}((p \rightarrow (q \vee r)) \rightarrow ((p \rightarrow q) \vee (p \rightarrow r)))$ valid in probabilistic logic. If we take now p to be a description of our knowledge, q to be the P=NP-hypothesis, and r its negation, we obtain that $P_{\geqslant 1}((p \rightarrow (q \vee r))$ since $q \vee r$ is $q \vee \neg q$. It follows that $P_{\geqslant 1}((p \rightarrow q) \vee (p \rightarrow r))$, which either means that our knowledge is inconsistent or that there is a considerable probability of at least one of the sentences "The P=NP-hypothesis follows from our current knowledge", "The negation of P=NP-hypothesis follows from our current knowledge", which is not very much likely. Again, since the above propositional formula is not intuitionisticaly valid, there is no such conclusion in the "intitionistic" probability logic.

8.1.3. Axiomatization, completeness, decidability. An axiomatization that characterizes the set of all $LPP^I_{2,\mathrm{Meas}}$-valid formulas can be obtain by combining:

- any propositional intuitionistic axiomatization for For_I,
- any classical propositional axiomatization for For_P and
- probabilistic axioms and rules from Section 3.3

with the proviso that in this framework the probabilistic operators $P_{\geqslant \cdot}$, and $P_{\leqslant \cdot}$ are independent, so for example, Axiom 3 from the system Ax_{LPP_2} should be rewritten in the form: $P_{\geqslant 1-r}\neg\alpha \to \neg P_{\geqslant s}\alpha$ for $s > r$. We skip the corresponding completeness proof, but the proof of decidability for LPP^I_2 contains more new details and we give it in the next theorems. Let $A \in \mathrm{For}_P$ and $\mathrm{Subf}_I(A) = \{\alpha \in \mathrm{For}_I : \alpha \text{is a subformula of} A\}$. Let $|A|$ and $|\mathrm{Subf}_I(A)|$ denote the length of A, and the number of formulas in $|\mathrm{Subf}_I(A)|$, respectively. Obviously, $|\mathrm{Subf}_I(A)| \leqslant |A|$.

Theorem 75. *A probabilistic formula $A \in \mathrm{For}_P$ is satisfiable iff it is satisfiable in a finite probabilistic model containing at most $2^{|A|^2}$ worlds.*

Proof. Let $\mathbf{M} = \langle W, \leqslant, v, H, \mu\rangle$, and $\mathbf{M} \models A$. For every $w \in W$, we use $\mathrm{Subf}_I(w)$ to denote the set of all formulas from $\mathrm{Subf}_I(A)$ forced in w, i.e., $\mathrm{Subf}_I(w) = \{\alpha \in \mathrm{Subf}_I(A) : w \Vdash \alpha\}$.

In the sequel we will follow the idea from [121, Theorem 5.3.4], and select some of the worlds from W to construct a finite model $\mathbf{M}^*$ satisfying A.

Let w_0 be the least element from W. We define the worlds of $\mathbf{M}^*$ (indexed by finite sequence) in the following way:

- $u_{\langle\rangle} = w_0$, where $\langle\rangle$ denotes the empty sequence,
- given u_σ let $u_{\sigma*\langle 1\rangle},\dots,u_{\sigma*\langle k\rangle}$ be the maximal set of worlds $w^{(1)},\dots,w^{(k)}$ from W such that for every $i, j \in \{1,\dots,k\}$:
  - $u_\sigma \leqslant w^{(i)}$,
  - $\mathrm{Subf}_I(u_\sigma) \neq \mathrm{Subf}_I(w^{(i)})$,
  - if $u_\sigma \leqslant w \leqslant w^{(i)}$, then either $\mathrm{Subf}_I(u_\sigma) = \mathrm{Subf}_I(w)$ or $\mathrm{Subf}_I(w) = \mathrm{Subf}_I(w^{(i)})$, and
  - if $i \neq j$, then $\mathrm{Subf}_I(w^{(i)}) \neq \mathrm{Subf}_I(w^{(j)})$.

Let $W^* = \{\sigma : u_\sigma \text{ is defined}\}$, $\leqslant^*$ be the usual ordering of finite sequences, and for all $\sigma \in W^*$, and $\alpha \in \mathrm{Var}$, $\alpha \in v^*(\sigma)$ iff $\alpha \in v(u_\sigma)$.

Using the induction on complexity of formulas we can prove that for every $\alpha \in \mathrm{Subf}_I(A)$ and every $\sigma \in W^*$, $\sigma \Vdash \alpha$ in $\langle W^*, \leqslant^*, v^*\rangle$ iff $u_\sigma \Vdash \alpha$ in $\langle W, \leqslant, v\rangle$. If $\alpha \in \mathrm{Var}$, the statement holds by the definition of v^*. Let $\alpha = \beta \to \gamma$. Suppose that $\sigma \nVdash \beta \to \gamma$. Then there is some $\rho \in W^*$ such that $\sigma \leqslant^* \rho$, $\rho \Vdash \beta$ and $\rho \nVdash \gamma$. By the induction hypothesis, $u_\rho \Vdash \beta$ and $u_\rho \nVdash \gamma$, $u_\sigma \leqslant u_\rho$, and $u_\sigma \nVdash \beta \to \gamma$. On the other hand, suppose that $u_\sigma \nVdash \beta \to \gamma$. Then, there are two possibilities. First, if $u_\sigma \Vdash \beta$ it must be $u_\sigma \nVdash \gamma$, and by the induction hypothesis $\sigma \Vdash \beta$ and $\sigma \nVdash \gamma$, which means that $\sigma \nVdash \beta \to \gamma$. In the second case there is some $w \in W$ such that $u_\sigma \leqslant w$, $w \Vdash \beta$, and $w \nVdash \gamma$. Since $u_\sigma \nVdash \beta$, obviously $\mathrm{Subf}_I(u_\sigma) \neq \mathrm{Subf}_I(w)$. According to the above construction, there must be some $u_\theta \in W$ such that $u_\sigma \leqslant u_\theta \leqslant w$, $\sigma \leqslant^* \theta$, $u_\theta \Vdash \beta$, and $u_\theta \nVdash \gamma$. By the induction hypothesis, $\theta \Vdash \beta$, $\theta \nVdash \gamma$, and $\sigma \nVdash \beta \to \gamma$. The other cases follow similarly.

Let μ' be the finitely additive probability defined on $\{\{w \in W^* : w \Vdash \alpha\} : \alpha \in \mathrm{Subf}_I(A)\}$ such that $\mu'(\{w \in W^* : w \Vdash \alpha\}) = \mu([\alpha]_M)$. Since for every $\alpha \in \mathrm{Subf}_I(A)$, $[\alpha]_M \neq \emptyset$ iff $\{w \in W^* : w \Vdash \alpha\} \neq \emptyset$, is easy to see that μ' is correctly defined. Let $\mathbf{M}^* = \langle W^*, \leqslant^*, v^*, H^*, \mu^* \rangle$ be the probabilistic model such that H^* is the smallest algebra on W^* containing family $\{[\alpha]_{\mathbf{M}^*} : \alpha \in \mathrm{For}_I\}$, while μ^* is a finitely additive probability on H^* which is an extension of μ'. Note that it follows from Theorem 2 that such an extension always exists. Since probabilities of For_I-subformulas of A remain the same, $\mathbf{M}^* \vDash A$.

Finally, note that the set W^* is finite because every world has at most $2^{|\mathrm{Subf}_I(A)|}$ immediate successors and every branch contains at most $|\mathrm{Subf}_I(A)|$ worlds. Thus $|W^*| \leqslant (2^{|\mathrm{Subf}_I(A)|})^{|\mathrm{Subf}_I(A)|} \leqslant 2^{|A|^2}$. $\square$

Theorem 76. *The satisfiability problem for probabilistic formulas is decidable.*

Proof. It follows from Theorem 75 that A is satisfiable iff it is satisfiable in a probabilistic model with at most $k_A = 2^{|A|^2}$ worlds. Thus, we can check satisfiability following ideas from Theorem 33: for every l, $1 \leqslant l \leqslant k_A$, there is only finitely many intuitionistic models with different valuations with respect to the set of propositional letters that occur in A. For every such intuitionistic model $\mathbf{M}_I = \langle W, \leqslant, v \rangle$ we can find the algebra H generated by the set $\{[\alpha]_{\mathbf{M}_I} : \alpha \in \mathrm{Subf}_I(A)\}$, and consider a linear system similar to the system (7). As there is a finite number of models and linear systems we have to check, and since linear programming problem is decidable, the same holds for the considered satisfiability problem. $\square$

8.1.4. Remark. We will show here that even if we start with classical logic, possible-worlds semantics naturally produces intuitionistic logic. It turns out that intuitionistic implication will coincide with conditional probability when probability is equal to 1.

Let us start with a standard possible-world model $M = \langle W, H, \mu, v \rangle$. We may define a pre-order (reflexive and transitive relation) R on W by: uRw iff for every primitive proposition p, $v(u,p) = \text{true}$ implies $v(w,p) = \text{true}$. From this we may obtain a partial order in the usual way. First we introduce an equivalence relation $\sim$ defined by: $u \sim w$ iff uRw and wRu, and then we split W into equivalence classes: $C_u = \{w : u \sim w\}$. Now we may pick a selection $W' \subset W$ of representatives of equivalence classes (one for each class). So we have $(\forall u \in W)(\exists w \in W')(u \sim w)$. Obviously, R induces a partial order $\leqslant$ on W' such that $u \leqslant w$ iff uRw. Now we have a Kripke model with a partial ordering relation on worlds $\langle W', \leqslant, v \rangle$ which makes it a model for intuitionistic logic. Namely, we may define (semantically) a new propositional connective $\to$ by: $w \vDash \alpha \to \beta$ iff $(\forall w' \geqslant w)(w' \vDash \alpha$ implies

$w' \vDash \beta$). We may also define a new, intuitionistic, negation by: $-\alpha = \alpha \to \bot$. Therefore, even if we start with classical logic, when we come to models, we have an intuitionistic implication built in.

The interest in intuitionistic implication, besides the arguments proposed at the start of this section, comes from the fact that conditional probability, which is often used as the proper form of entailment in the context of probability logic, coincides in a sense with the intuitionistic implication, as can be seen from the following theorem.

Theorem 77. $\mu(\alpha \to \beta) = 1$ *iff* $\dfrac{\mu(\alpha \wedge \beta)}{\mu(\alpha)} = 1$.

However, this symmetry holds only in the case when probability is equal to 1. It is possible to construct models in which conditional probability is high while the probability of (intuitionistic) implication is low and vice versa. The reason is that, despite the fact that both operators are define globally (and not locally, in each world) the definitions are quite different. Conditional probability considers (i.e., counts) only worlds in which α is true, while intuitionistic implication takes into account also their predecessors. We may say, in a sense, that conditional probability disregards the development of events and regards only the final stages (with regard to the validity of α), i.e., the analysis starts with the worlds in which α is true and disregards the previous stages in which α may be "not yet true". Existence of long time-lines which end with worlds in which α is not true adds to the probability of $\alpha \to \beta$, while it is irrelevant for the conditional probability. On the other hand, a long sequence which has an ending in which α is true and β is not, reduces considerably the probability of $\alpha \to \beta$, while it may, in the presence of a relevant number of worlds in which both α and β are true, be insignificant for conditional probability.

8.2. A discrete linear-time probabilistic logic. In this section we describe a way in which probabilistic reasoning can be enriched with some temporal features. The temporal part of the logics is a standard discrete linear-time logic LTL [119], where the flow of time is isomorphic to natural numbers, i.e., each moment of time has a unique possible future, while the corresponding language contains the "next" operator ($\bigcirc$) and the reflexive strong "until" operator (U), (the operators "sometime" F and "always" G are definable: ($\mathrm{F}\alpha = \top \mathrm{U} \alpha$ and $\mathrm{G}\alpha = \neg \mathrm{F} \neg \alpha$). Similarly as in Section 7, nesting of the probabilistic and temporal operators is important and we will start from the logic LPP_1. In our logic, denoted LPP_1^{LTL}, the probabilistic operators quantify events along a single time line. It allows us to express sentences such as "(according to the current set of information) the probability that, sometime in the future, α is true is at least s". And, as the knowledge can evolve during the time, the probability of α might change too. Note that, since the operators "sometime" and "always" can be seen as the existential and universal quantifiers over time instants, the probabilistic operators give more refined quantitative characterization of sets of time instants definable by formulas. We may try to motivate the proposed semantics in the following way.

Example 78. A suitable representation of all possible outcomes of an infinite sequence of probabilistic experiments (let us say that experiments A and B are permanently repeated resulting in a or $\neg a$, and b or $\neg b$, respectively) could be an inifinite tree, where every branch corresponds to a possible realization of the sequence of the experiments, and every time instant is described in the form $\pm a, \pm b$ depending on obtaining (or not obtaining) a and b in the corresponding experiment. We might be interested in probabilistic properties that hold for all branches. In that case we can reason about an arbitrary branch and need ability to express probabilities of events along it, for example that the probability of the event a is at least s, or some more complicated conditions, like that in every time instant, if the probability of a is less than r, then b must hold forever.

The set $\mathrm{For}_{LPP_1^{LTL}}$ of formulas is defined inductively as the smallest set containing primitive propositions and closed under formation rules: if α and β are formulas, then $\neg\alpha$, $\bigcirc\alpha$, $P_{\geqslant s}\alpha$, for every $s \in S$, $\alpha \wedge \beta$, and $\alpha \mathrm{U} \beta$ are formulas. We will use the following notational definition: $\bigcirc^0\alpha = \alpha$, and $\bigcirc^{i+1}\alpha = \bigcirc \bigcirc^i \alpha$ for $i \geqslant 0$. If $T = \{\alpha_1, \alpha_2, \ldots\}$ is a set of formulas, then $\bigcirc T$ denotes $\{\bigcirc\alpha_1, \bigcirc\alpha_2, \ldots\}$.

Example 79. An example of a formula is $(\bigcirc P_{\geqslant r} p \wedge \mathrm{F} P_{<s}(p \rightarrow q)) \rightarrow \mathrm{G} P_{=t} q$ which can be read as "if the probability of p in the next moment is at least r and sometime in the future q follows from p with the probability less than s, then the probability of q will always be equal to t."

8.2.1. Semantics. The semantics for LPP_1^{LTL} is a Kripke-style one using sequences of natural numbers as frames.

Definition 80. An LPP_1^{LTL}*-model* is a structure $\mathbf{M} = \langle W, \mathrm{Prob}, v\rangle$ where:

- $W = \{w_0, w_1, \ldots\}$ is a sequence of *time instants*,
- Prob is a probability assignment which assigns to every $w \in W$ a probability space, such that $\mathrm{Prob}(w) = \langle W(w), H(w), \mu(w)\rangle$, where:
  - $W(w) = \{w_j : j \geqslant i\}$,
  - $H(w)$ is an algebra of subsets of $W(w)$ and
  - $\mu(w) : H(w) \rightarrow [0, 1]$ is a finitely additive probability measure.
- v assigns to every $w \in W$ a two-valued evaluation of the primitive propositions, i.e., for every $w \in W$, $v(w) : \phi \rightarrow \{\mathrm{true}, \mathrm{false}\}$.

Definition 81. Let $\mathbf{M} = \langle W, \mathrm{Prob}, v\rangle$ be a LPP_1^{LTL}-model, $i \in \omega$ and α be a formula. The *satisfiability relation* $\models$ is inductively defined as follows:

- if $p \in \phi$ is a primitive proposition, $w_i \models p$ if $v(w_i)(p) = \mathrm{true}$,
- $w_i \models \neg\alpha$ if $w_i \not\models \alpha$,
- $w_i \models P_{\geqslant s}\alpha$ if $\mu(w_i)(\{w_{i+j}, j \geqslant 0 : w_{i+j} \models \alpha\}) \geqslant s$,
- $w_i \models \bigcirc\alpha$ if $w_{i+1} \models \alpha$,
- $w_i \models \alpha \wedge \beta$ if $w_i \models \alpha$ and $w_i \models \beta$.
- $w_i \models \alpha \mathrm{U} \beta$ if there is an integer $j \geqslant 0$ such that $w_{i+j} \models \beta$, and for every k such that $0 \leqslant k < j$, $w_{i+k} \models \alpha$.

We concern a reflexive, strong version of the until operator, i.e., if $\alpha \mathrm{U} \beta$ holds in a time instant, β must eventually hold. In the above definition the future includes the present, so that:

- $w_i \models \mathrm{F}\alpha$ if there is $j \geqslant 0$ such that $w_{i+j} \models \alpha$, and
- $w_i \models \mathrm{G}\alpha$ if for every $j \geqslant 0$, $w_{i+j} \models \alpha$.

Also, the present time instant is included when the probability of formulas are considered. All the presented results then can be proved with essentially no change if we use the temporal and probabilistic operators referring to the strict future that does not concern the present.

Again, we will consider*measurable* models only, i.e., the class $LPP_{1,\mathrm{Meas}}^{LTL}$ of all LPP_1^{LTL}-models such that for every $w_i \in W$ the set $H(w_i)) = \{[\alpha]_{w_i} : \alpha \in \mathrm{For}_{LPP_1^{LTL}}\}$, where $[\alpha]_{w_i} = \{w_{i+j} : j \geqslant 0, w_{i+j} \models \alpha\}$.

The notions of *satisfiable* and *valid* formulas and *satisfiable sets* of formulas are defined as in Section 5.

8.2.2. Axiomatization. An axiomatization $Ax_{LPP_1^{LTL}}$ that characterizes the set of all $LPP_{1,\mathrm{Meas}}^{LTL}$-valid formulas extends the system Ax_{LPP_2} (having in mind that instances of the axiom schemas and rules must obey the syntactical rules for LPP_1^{LTL}) with the following axiom schemas:

(7) $\bigcirc(\alpha \to \beta) \to (\bigcirc\alpha \to \bigcirc\beta)$
(8) $\neg \bigcirc \alpha \leftrightarrow \bigcirc\neg\alpha$
(9) $\alpha \mathrm{U}\beta \leftrightarrow \beta \vee (\alpha \wedge \bigcirc(\alpha \mathrm{U}\beta))$
(10) $\alpha \mathrm{U}\beta \to F\beta$
(11) $\mathrm{G}\alpha \to P_{\geqslant 1}\alpha$

while the inference rules should be rewritten in the following form:

(1) from α and $\alpha \to \beta$ infer β
(2) from α infer $\bigcirc\alpha$
(3) from $\beta \to \bigcirc^i\alpha$ for all $i \geqslant 0$, infer $\beta \to G\alpha$
(4) from $\beta \to \bigcirc^m P_{\geqslant s-\frac{1}{k}}\alpha$, for any $m \geqslant 0$, and for every $k \geqslant \frac{1}{s}$, infer $\beta \to \bigcirc^m P_{\geqslant s}\alpha$.

The main novelty in $Ax_{LPP_1^{LTL}}$ concerns axioms about temporal reasoning (the axioms 7 and 8 are the usual axioms for the next operator $\bigcirc$, as well as the axioms 9 and 10 for the until operator) and mixing of probabilistic and temporal reasoning (Axiom 11). There are two infinitary inference rules: 3 and 4. The former one characterizes the always operator.

In this framework we can use the definitions 30 and 31 of deduction and consistency.

Note that, similarly to the probabilistic logics, compactness does not hold for LTL. For example, every finite subset of the set $\{\mathrm{F}^n p : n \text{is a positive integer}\} \cup \{\mathrm{FG}\neg p\}$ is satisfiable, while the set itself is not. So, the temporal part of $Ax_{LPP_1^{LTL}}$ offers possibility to prove extended completeness which cannot be proved using finitary means.

Modifications of $Ax_{LPP_1^{LTL}}$ according to ideas presented in the previous sections could produce the corresponding axiomatic systems for a first order logic for reasoning about discrete linear time and probability, a temporal probabilistic logic with probabilistic functions with a fixed finite range, etc. Also, we can specify additional relationships between the flow of time and the probability measures by adding new axioms:

Example 82. The formula $\neg\alpha \to (P_{\geqslant s}\alpha \to \bigcirc P_{\geqslant s}\alpha)$, considered as an additional axiom scheme, characterizes models with the property that if a formula does not hold in a time instant, then in the next time instant its probability will be not decreased.

8.2.3. Completeness and decidability. The proof of extended completeness again follows the ideas given in the previous sections, so we only outline the main new details.

Theorem 83 (Extended completeness theorem for $LPP_{1,\mathrm{Meas}}^{LTL}$)**.** *A set T of formulas is $Ax_{LPP_1^{LTL}}$-consistent iff it is $LPP_{1,\mathrm{Meas}}^{LTL}$-satisfiable.*

Proof. We start with Deduction theorem. For example, assume that $T, \alpha \vdash \beta \to \mathrm{G}\beta'$ is obtained by Rule 3. Then:

(1) $T, \alpha \vdash \beta \to \bigcirc^i\beta'$, for $i \geqslant 0$,
(2) $T \vdash \alpha \to (\beta \to \bigcirc^i\beta')$, for $i \geqslant 0$, by the induction hypothesis,
(3) $T \vdash (\alpha \wedge \beta) \to \bigcirc^i\beta'$, for $i \geqslant 0$,
(4) $T \vdash (\alpha \wedge \beta) \to \mathrm{G}\beta'$, by Rule 3,
(5) $T \vdash \alpha \to (\beta \to \mathrm{G}\beta')$.

The axioms and rules imply some auxiliary statements (T denotes a consistent set of formulas):

(1) $\vdash \mathrm{G}\alpha \leftrightarrow \alpha \wedge \bigcirc\mathrm{G}\alpha$,
(2) $\vdash \mathrm{G}\bigcirc\alpha \leftrightarrow \bigcirc\mathrm{G}\alpha$,
(3) $\vdash (\bigcirc\alpha \to \bigcirc\beta) \to \bigcirc(\alpha \to \beta)$,
(4) $\vdash \bigcirc(\alpha \wedge \beta) \leftrightarrow (\bigcirc\alpha \wedge \bigcirc\beta)$,
(5) $\vdash \bigcirc(\alpha \vee \beta) \leftrightarrow (\bigcirc\alpha \vee \bigcirc\beta)$,
(6) $\mathrm{G}\alpha \vdash \bigcirc^i\alpha$ for every $i \geqslant 0$,
(7) if $\vdash \alpha$, then $\vdash \mathrm{G}\alpha$,
(8) if $T \vdash \alpha$, where T is a set of formulas, then $\bigcirc T \vdash \bigcirc\alpha$.
(9) for $j \geqslant 0$, $\bigcirc^j\beta, \bigcirc^0\alpha, \dots, \bigcirc^{j-1}\alpha \vdash \alpha\mathrm{U}\beta$,
(10) For any formula α, either $T \cup \{\alpha\}$ is consistent or $T \cup \{\neg\alpha\}$ is consistent.
(11) If $\neg(\alpha \to \mathrm{G}\beta) \in T$, then there is $j_0 \geqslant 0$ such that $T \cup \{\alpha \to \neg\bigcirc^{j_0}\beta\}$ is consistent.
(12) If $\neg(\alpha \to \bigcirc^m P_{\geqslant s}\beta) \in T$, then there is $j_0 > \frac{1}{s}$ such that $T \cup \{\alpha \to \neg\bigcirc^m P_{\geqslant s-\frac{1}{j_0}}\beta\}$ is consistent.

For example, the statement (9) follows in the following way. Assume $\vdash \alpha$. By application of Rule 2, we get $\vdash \bigcirc^k\alpha$, for every $k \in \omega$. We obtain $\vdash \mathrm{G}\alpha$ by Rule 3. From Axiom 11 and by application of Modus Ponens, we have $\vdash P_{\geqslant 1}\alpha$.

Then we can show that every consistent set T of formulas can be extended to a maximal consistent set. Let α_0, α_1,... be an enumeration of all formulas. A maximal consistent extension $\mathcal{T}$ of T can be obtained as follows:

(1) $T_0 = T$.
(2) For every $i \geqslant 0$ if $T_i \cup \{\alpha_i\}$ is consistent, then $T_{i+1} = T_i \cup \{\alpha_i\}$. Otherwise, if α_i is of the form $\gamma \rightarrow \mathrm{G}\beta$, then $T_{i+1} = T_i \cup \{\neg\alpha_i, \gamma \rightarrow \neg \bigcirc^{j_0} \beta\}$ for some $j_0 \geqslant 0$ such that T_{i+1} is consistent. Otherwise, α_i is of the form $\gamma \rightarrow \bigcirc^m P_{\geqslant s}\beta$, then $T_{i+1} = T_i \cup \{\neg\alpha_i, \gamma \rightarrow \neg \bigcirc^m P_{\geqslant s - \frac{1}{j_0}}\beta\}$ for some $j_0 > 0$ such that T_{i+1} is consistent. Otherwise, $T_{i+1} = T_i \cup \{\neg\alpha_i\}$.
(3) $\mathcal{T} = \bigcup_{i=0}^{\infty} T_i$.

For a maximal consistent extension $\mathcal{T}$ of a consistent set T of formulas we define the canonical model $\mathbf{M}_T = \langle W, \mathrm{Prob}, v\rangle$ such that:

- $W = w_0, w_1, \ldots$, $w_0 = \mathcal{T}$, and for $i > 0$, $w_i = \{\alpha : \bigcirc\alpha \in w_{i-1}\}$,
- for $i \geqslant 0$, $\mathrm{Prob}(w_i) = \langle W(w_i), H(w_i), \mu(w_i)\rangle$ is defined as follows:
  - $W(w_i) = \{w_{i+j} : j \geqslant 0\}$,
  - $H(w_i) = \{\{w_{i+j} : j \geqslant 0, \alpha \in w_{i+j}\}\}$,
  - for $\mu(w_i)(\{w_{i+j} : j \geqslant 0, \alpha \in w_{i+j}\}) = \sup_s\{P_{\geqslant s}\alpha \in w_i\}$,
- for every primitive proposition $p \in \phi$, and every $w_i \in W$, $v(w_i)(p) = \top$ iff $p \in w_i$.

First of all, we can prove that for every $i \geqslant 0$, w_i is a maximal consistent set. By hypothesis, w_0 is maximal and consistent. Suppose that w_{i+1} is not maximal. There is a formula α such that $\{\alpha, \neg\alpha\} \cap w_{i+1} = \emptyset$. Consequently, $\{\bigcirc\alpha, \bigcirc\neg\alpha\} \cap w_i = \emptyset$. We obtain that $\{\bigcirc\alpha, \neg\bigcirc\alpha\} \cap w_i = \emptyset$ which is in contradiction with the maximality of w_i. Suppose that w_{i+1} is not consistent, i.e., that $w_{i+1} \vdash \alpha \wedge \neg\alpha$. Then, $w_i \vdash \bigcirc(\alpha \wedge \neg\alpha)$, and $w_i \vdash \bigcirc\alpha \wedge \neg \bigcirc \alpha$ which is in contradiction with consistency of w_i.

Then, similarly as in the previous sections, we can show that $\mathbf{M}_T$ is an $LPP_{1,\mathrm{Meas}}^{LTL}$-model such that for all w_i and α, $\alpha \in w_i$ iff $w_i \models \alpha$. For example, if $\alpha = \bigcirc\beta$, we have $w_i \models \alpha$ iff $w_{i+1} \models \beta$ iff $\beta \in w_{i+1}$ iff $\alpha \in w_i$ (by the construction of w_{i+1}). □

For the previously presented logics as the first step in the proofs of their decidability we have used some kind of the filtration technique which helps as to show that every formula is satisfiable iff it is satisfiable in a finite model. The problem is that the filtration cannot be used here since the $LPP_{1,\mathrm{Meas}}^{LTL}$-models are (by their definition) infinite. However, we can show (following the ideas presented in [119]) that a formula is satisfiable if and only if it is satisfiable in an model such that the sequence of time instants of the model has a finite initial sequence of time instants followed by another finite sequence of time instants which permanently repeats and in that way forms the rest of the whole time-line. The lengths of both sequences are bounded by functions of the size of the considered formula. The full proof of decidability and complexity of the $LPP_{1,\mathrm{Meas}}^{LTL}$-satisfiability problem can be found in [91]. As it is rather long, we give only the corresponding main statements:

Theorem 84. *Every $LPP_{1,\mathrm{Meas}}^{LTL}$-satisfiable formula α is satisfiable in a model with the starting sequence of time instants, followed by the sequence of time instants* *which permanently repeats. The length of the former sequence is* $\leqslant 2^{2|\alpha|}+1$*, and the length of the later sequence is* $\leqslant (2^{|\alpha|}+1)\times 2^{|\alpha|}$*, where* $|\alpha|$ *denotes the length of* α*.*

Theorem 85 (Decidability and complexity for LPP_1^{LTL})**.** *The* LPP_1^{LTL} *is decidable. The* $LPP_{1,\mathrm{Meas}}^{LTL}$*-satisfiability problem is* PSPACE*-hard and in non-deterministic exponential time.*

9. Logics with conditional probability

An important reason to consider conditional probability logics is given in [80]. It is argued there that conditional probability offers a more natural generalization of the rule "if α, then β" than probability of implication. Namely, if α has a low non-negative probability and $\neg\alpha\wedge\beta$ is very likely to happen, then "the probability of $\alpha\to\beta$" could be very high (since $\alpha\to\beta$ holds whenever α is false) and does not properly reflect the meaning of the rule, while on the other hand, "the conditional probability of β given α" is more appropriate.

Also, it turns out that a specific kind of conditional probability (with a nonarchimedean range) is useful in modelling default reasoning. We start this section by presenting a logic (denoted $LPCP_2^{S,\approx}$) which formalizes such conditional probability and represents approximate probabilistic knowledge, but in a similar way axiomatizations could be given to ordinary $[0,1]$-real-valued conditional probability. $LPCP_2^{S,\approx}$ can be seen as a generalization of the logic LPP_2^S from Section 4.3.

In the second part of the section we introduce another logic $LPCP_2^{\mathrm{Chr}}$ which axiomatizes so-called de Finetti's view of conditional probability [20]. In that approach conditional probability is seen as more primitive concept than unconditional probability, in contrast to Kolmogorov's definition where conditional probability is defined via unconditional probability. Conditional probability in the sense of de Finetti can be defined using a structure $\langle W,H,\mu\rangle$, where W is a non empty set, H is an algebra of subsets of W, and $\mu: H\times H^0\to[0,1]$, $H^0=H\smallsetminus\{\emptyset\}$, is a (coherent) conditional probability satisfying:

- $\mu(A,A)=1$, for every $A\in H^0$,
- $\mu(\cdot,A)$ is a finitely additive probability on H for any given $A\in H^0$,
- $\mu(C\cap B,A)=\mu(B,A)\cdot\mu(C,B\cap A)$, for all $C\in H$ and A, B, $A\cap B\in H^0$.

Note that $\mu(A,B)$ has a meaning with the only condition that B is different from the impossible event.

9.1. A logic with approximate conditional probabilities. In this subsection, we use notions defined in Section 3, and only emphasize the main novelties. Let S be the unit interval of the Hardy field $Q[\epsilon]$. $Q[\epsilon]$ is a recursive nonarchimedean field which contains all rational functions of a fixed positive infinitesimal ϵ which belongs to a nonstandard elementary extension ${}^*R$ of the standard real numbers [56, 117]. An element ϵ of ${}^*R$ is an infinitesimal if $|\epsilon|<\frac{1}{n}$ for every natural number n. $Q[\epsilon]$ contains all rational numbers. Let $Q[0,1]$ denote the set of rational numbers from $[0,1]$.

The language of $LPCP_2^{S,\approx}$, beside the set ϕ of primitive propositions and Boolean connectives $\neg$ and $\wedge$, contains binary probabilistic operators:

$$(CP_{\leqslant s})_{s\in S}, \quad (CP_{\geqslant s})_{s\in S}, \quad \text{and} \quad (CP_{\approx r})_{r\in Q[0,1]}.$$

If $\alpha,\beta \in \mathrm{For}_C$ and $s \in S$, $r \in Q[0,1]$ then $CP_{\geqslant s}(\alpha,\beta)$, $CP_{\leqslant s}(\alpha,\beta)$ and $CP_{\approx r}(\alpha,\beta)$ are *basic probability formulas* with the intended meaning "the conditional probability of α given β is at least (at most) s and, approximately r". The set For_P of probabilistic propositional formulas is the smallest set containing all basic probability formulas and closed under Boolean formation rules (so, there are no iterations of the probabilistic operators). The set of formulas $\mathrm{For}_{LPCP_2^{S,\approx}}$ is $\mathrm{For}_C \cup \mathrm{For}_P$. Also:

- $CP_{<s}(\alpha,\beta)$ denotes $\neg CP_{\geqslant s}(\alpha,\beta)$ for $\alpha,\beta \in \mathrm{For}_C$, $s \in S$,
- $CP_{>s}(\alpha,\beta)$ denotes $\neg CP_{\leqslant s}(\alpha,\beta)$ for $\alpha,\beta \in \mathrm{For}_C$, $s \in S$,
- $CP_{=s}(\alpha,\beta)$ denotes $CP_{\geqslant s}(\alpha,\beta) \wedge CP_{\leqslant s}(\alpha,\beta)$ for $\alpha,\beta \in \mathrm{For}_C$, $s \in S$ and
- $P_{\rho\, s}\alpha$ denotes $CP_{\rho\, s}(\alpha,\top)$ for $\alpha \in \mathrm{For}_C$ and $\rho \in \{\geqslant,\leqslant,>,<,=,\approx\}$.

It should be noted that $CP_{\geqslant}$ and $CP_{\leqslant}$ are not interdefinable since the appropriate equivalence breaks down when the probability of the condition is 0.

9.1.1. Semantics. We consider the class $LPCP^{S,\approx}_{2,\mathrm{Meas},\mathrm{Neat}}$ of all measurable neat $LPCP_2^{S,\approx}$-models, which can be defined in the same way as the class $LPP_{2,\mathrm{Meas},\mathrm{Neat}}$ from Section 3.2, with the important difference that:

- μ is an S-valued finitely additive measure, i.e., $\mu : H \to S$.

The neatness condition is used to make our models a subclass of ${}^*R$-probabilistic models of [61, 66]. This facilitates the explanation of a possible application of $LPCP_2^{S,\approx}$ to default reasoning. All the results can be also proved for the class of measurable (but not necessarily neat) $LPCP_2^{S,\approx}$-models.

Definition 86. *The satisfiability relation* $\models \subseteq LPCP^{S,\approx}_{2,\mathrm{Meas},\mathrm{Neat}} \times \mathrm{For}_{LPCP_2^{S,\approx}}$ fulfills the following conditions for every $LPCP^{S,\approx}_{2,\mathrm{Meas},\mathrm{Neat}}$-model $\mathbf{M} = \langle W,H,\mu,v\rangle$:

(1) if $\alpha \in \mathrm{For}_C$, $\mathbf{M} \models \alpha$ if $(\forall w \in W)v(w)(\alpha) = \text{true}$,
(2) $\mathbf{M} \models CP_{\leqslant s}(\alpha,\beta)$ if either $\mu([\beta]_{\mathbf{M}}) = 0$ and $s = 1$ or $\mu([\beta]_{\mathbf{M}}) > 0$ and $\frac{\mu([\alpha\wedge\beta]_{\mathbf{M}})}{\mu([\beta]_{\mathbf{M}})} \leqslant s$,
(3) $\mathbf{M} \models CP_{\geqslant s}(\alpha,\beta)$ if either $\mu([\beta]_{\mathbf{M}}) = 0$ or $\mu([\beta]_{\mathbf{M}}) > 0$ and $\frac{\mu([\alpha\wedge\beta]_{\mathbf{M}})}{\mu([\beta]_{\mathbf{M}})} \geqslant s$,
(4) $\mathbf{M} \models CP_{\approx r}(\alpha,\beta)$ if either $\mu([\beta]_{\mathbf{M}}) = 0$ and $r = 1$ or $\mu([\beta]_{\mathbf{M}}) > 0$ and for every positive integer n, $\frac{\mu([\alpha\wedge\beta]_{\mathbf{M}})}{\mu([\beta]_{\mathbf{M}})} \in [\max\{0, r-1/n\}, \min\{1, r+1/n\}]$.
(5) if $A \in \mathrm{For}_P^S$, $\mathbf{M} \models \neg A$ if $M \not\models A$,
(6) if $A,B \in \mathrm{For}_P^S$, $\mathbf{M} \models A \wedge B$ if $\mathbf{M} \models A$ and $\mathbf{M} \models B$.

Condition 3 is formulated on the useful assumption that the conditional probability is by default 1, whenever the condition has the probability 0. Also, note that the condition 4 is equivalent to saying that the conditional probability equals $r - \epsilon_i$ (or $r + \epsilon_i$) for some infinitesimal $\epsilon_i \in S$. It is easy to see that the defined operators will behave as expected, e.g., $\mathbf{M} \models P_{<s}\alpha$ iff $\mu([\alpha]_{\mathbf{M}}) < s$.

9.1.2. Axiomatization and completeness. The axiomatic system $Ax_{LPCP_2^{S,\approx}}$ which characterizes the set of all $LPCP^{S,\approx}_{2,\mathrm{Meas},\mathrm{Neat}}$-valid formulas contains the following set of axiom schemata:

(1) all For_C-instances of classical propositional tautologies
(2) all For_P-instances of classical propositional tautologies
(3) $CP_{\geqslant 0}(\alpha,\beta)$
(4) $CP_{\leqslant s}(\alpha,\beta) \to CP_{<t}(\alpha,\beta)$, $t > s$
(5) $CP_{<s}(\alpha,\beta) \to CP_{\leqslant s}(\alpha,\beta)$
(6) $P_{\geqslant 1}(\alpha \leftrightarrow \beta) \to (P_{=s}\alpha \to P_{=s}\beta)$
(7) $P_{\leqslant s}\alpha \leftrightarrow P_{\geqslant 1-s}\neg\alpha$
(8) $(P_{=s}\alpha \wedge P_{=t}\beta \wedge P_{\geqslant 1}\neg(\alpha\wedge\beta)) \to P_{=\min(1,s+t)}(\alpha \vee \beta)$
(9) $P_{=0}\beta \to CP_{=1}(\alpha,\beta)$
(10) $(P_{=t}\beta \wedge P_{=s}(\alpha\wedge\beta)) \to CP_{=s/t}(\alpha,\beta)$, $t \neq 0$
(11) $CP_{\approx r}(\alpha,\beta) \to CP_{\geqslant r_1}(\alpha,\beta)$, for every rational $r_1 \in [0,r)$
(12) $CP_{\approx r}(\alpha,\beta) \to CP_{\leqslant r_1}(\alpha,\beta)$, for every rational $r_1 \in (r,1]$

and inference rules:

(1) From φ and $\varphi \to \psi$ infer ψ.
(2) If $\alpha \in \mathrm{For}_C$, from α infer $P_{\geqslant 1}\alpha$.
(3) From $A \to P_{\neq s}\alpha$, for every $s \in S$, infer $A \to \bot$.
(4) For every $r \in Q[0,1]$, from $A \to CP_{\geqslant r-1/n}(\alpha,\beta)$, for every integer $n \geqslant 1/r$, and $A \to CP_{\leqslant r+1/n}(\alpha,\beta)$ for every integer $n \geqslant 1/(1-r)$, infer $A \to CP_{\approx r}(\alpha,\beta)$.

It is easy to see (just put $\top$ instead of β) that the axioms 3–5 generalize the corresponding axioms from the system Ax_{LPP_2}. Axiom 9 conforms with the useful practice of assuming conditional probability to be 1, whenever the condition has the probability 0. Axiom 10 expresses the standard definition of conditional probability, while the axioms 11 and 12 and Rule 4 describe the relationship between the standard conditional probability and the conditional probability infinitesimally close to some rational $r \in Q[0,1]$. The rules 3 and 4 are infinitary. Rule 3 guarantees that the probability of a formula belongs to the set S.

A useful, but straightforward theorem is:

Theorem 87. *Let $\alpha,\beta \in \mathrm{For}_C$. Then:*

(1) $\vdash CP_{\geqslant t}(\alpha,\beta) \to CP_{\geqslant s}(\alpha,\beta)$, $t > s$
(2) $\vdash CP_{\leqslant t}(\alpha,\beta) \to CP_{\leqslant s}(\alpha,\beta)$, $t < s$
(3) $\vdash CP_{=t}(\alpha,\beta) \to \neg CP_{=s}(\alpha,\beta)$, $t \neq s$
(4) $\vdash CP_{=t}(\alpha,\beta) \to \neg CP_{\geqslant s}(\alpha,\beta)$, $t < s$
(5) $\vdash CP_{=t}(\alpha,\beta) \to \neg CP_{\leqslant s}(\alpha,\beta)$, $t > s$
(6) $\vdash CP_{=r}(\alpha,\beta) \to CP_{\approx r}(\alpha,\beta)$, $r \in Q[0,1]$.
(7) $\vdash CP_{\approx r_1}(\alpha,\beta) \to \neg CP_{\approx r_2}(\alpha,\beta)$, *for* $r_1, r_2 \in Q[0,1]$, $r_1 \neq r_2$.
(8) $\vdash P_{=0}\beta \to \neg CP_{\leqslant s}(\alpha,\beta)$, *for* $s < 1$.
(9) $\vdash P_{\leqslant 1}\alpha$.

Note that, by restricting β to $\top$, we obtain analogous statements for unconditional probabilities.

The main novelty in the completeness proof follows concerns the construction of a maximal consistent extensions of a consistent set. Following notations from Theorem 13, now the construction is:

(1) $T_0 = T \cup Cn_C(T) \cup \{P_{\geqslant 1}\alpha : \alpha \in Cn_C(T)\}$
(2) for every $i \geqslant 0$,
  (a) if $T_{2i} \cup \{A_i\}$ is consistent, then $T_{2i+1} = T_{2i} \cup \{A_i\}$;
  (b) otherwise, if $T_{2i} \cup \{A_i\}$ is not consistent, we have:
    (i) if A_i is of the form $A \to CP_{\approx r}(\alpha, \beta)$, then $T_{2i+1} = T_{2i} \cup \{\neg A_i, A \to \neg CP_{\geqslant r-1/n}(\alpha, \beta)\}$, or $T_{2i+1} = T_{2i} \cup \{\neg A_i, A \to \neg CP_{\leqslant r+1/n}(\alpha, \beta)\}$, for some integer n, where n is chosen such that T_{2i+1} is consistent (we prove that this is possible below);
    (ii) otherwise, $T_{2i+1} = T_{2i} \cup \{\neg A_i\}$,
(3) for every $i \geqslant 0$, $T_{2i+2} = T_{2i+1} \cup \{P_{=s}\alpha_i\}$, where s is chosen to be an arbitrary element of S such that T_{2i+2} is consistent (we prove that this is possible below),
(4) for every $i \geqslant 0$, if T_i is enlarged by a formula of the form $P_{=0}\alpha$, add $\neg\alpha$ to $T_i \cup \{P_{=0}\alpha\}$ as well.
(5) $\mathcal{T} = \bigcup_{i=0}^{\infty} T_i$.

Let us consider the step (3) of the construction, and suppose that for every $s \in S$, $T_{2i+1} \cup \{P_{=s}\alpha_i\}$ is not consistent. Let $T_{2i+1} = T_0 \cup T^+_{2i+1}$, where T^+_{2i+1} denotes the set of all formulas $B \in \mathrm{For}_P$ that were added to T_0 in the previous steps of the construction. Then the following contradicts consistency of T_{2i+1}:

(1) $T_0, T^+_{2i+1}, P_{=s}\alpha_i \vdash \bot$, for every $s \in S$, by the hypothesis
(2) $T_0, T^+_{2i+1} \vdash \neg P_{=s}\alpha_i$, for every $s \in S$, by Deduction theorem
(3) $T_0 \vdash (\bigwedge_{B \in T^+_{2i+1}} B) \to \neg P_{=s}\alpha_i$, for every $s \in S$, by Deduction theorem
(4) $T_0 \vdash (\bigwedge_{B \in T^+_{2i+1}} B) \to \bot$, by Rule 3
(5) $T_{2i+1} \vdash \bot$,

The set $\mathcal{T}$ satisfies:

(1) There is exactly one $s \in S$ such that $P_{=s}\alpha \in \mathcal{T}$.
(2) There is exactly one $s \in S$ such that $CP_{=s}(\alpha, \beta) \in \mathcal{T}$.
(3) If $CP_{\geqslant s}(\alpha, \beta) \in \mathcal{T}$, there is $r \in S$ such that $r \geqslant s$ and $CP_{=r}(\alpha, \beta) \in \mathcal{T}$.
(4) If $CP_{\leqslant s}(\alpha, \beta) \in \mathcal{T}$, there is $r \in S$ such that $r \leqslant s$ and $CP_{=r}(\alpha, \beta) \in \mathcal{T}$.
(5) If $CP_{\approx r_1}(\alpha, \beta) \in \mathcal{T}$ and $r_2 \in Q[0,1] \setminus \{r_1\}$, then $CP_{\approx r_2}(\alpha, \beta) \notin \mathcal{T}$

As an example, let us consider the statement (1). According to Theorem 87 (3), if $P_{=s}\alpha \in \mathcal{T}$, then for every $t \neq s$, $P_{=t}\alpha \notin \mathcal{T}$. On the other hand, if for every $s \in S$, $\neg P_{=s}\alpha \in \mathcal{T}$, then $\mathcal{T} \vdash \neg P_{=s}\alpha$ for every $s \in S$. By Rule 3, $\mathcal{T} \vdash \bot$ which contradicts consistency of $\mathcal{T}$. Thus, for every $\alpha \in \mathrm{For}_C$, there is exactly one $s \in S$ such that $P_{=s}\alpha \in \mathcal{T}$. Finally, the corresponding canonical model $\mathbf{M}_T$ can be defined as in Section 3.4, and we have:

Theorem 88 (Extended completeness theorem for $LPCP^{S,\approx}_{2,\mathrm{Meas},\mathrm{Neat}}$). *A set T of formulas is $Ax_{LPCP^{S,\approx}_2}$-consistent iff it is $LPCP^{S,\approx}_{2,\mathrm{Meas},\mathrm{Neat}}$-satisfiable.*

9.1.3. Decidability. The proof of decidability of $LPCP_2^{S,\approx}$ [114] is rather long, and, similarly as in Section 8.2, we will omit it here. The proof contains a reduction of the the $LPCP^{S,\approx}_{2,\mathrm{Meas},\mathrm{Neat}}$-satisfiability to linear programming problem, as in Section 3.5. However, note that Section 3.5 deals with the standard real-valued probabilities, while in $LPCP_2^{S,\approx}$ the range of probabilities is recursive and contains non-standard values, and there are operators of the form $CP_{\approx r}$ that do not appear above. Thus, in the reduction we have to eliminate the $CP_{\approx .}$-operators and to try to solve linear systems in an extension of $Q[\epsilon]$. The next example contains an illustration of the technique from [114].

Example 89. Let us consider the formula $A = C \wedge ((D \vee B) \to (D \wedge B))$, where B, C and D denote $CP_{\approx 0}(q, \top)$, $CP_{\approx 1}(\neg p \wedge \neg q, \neg q)$ and $CP_{\approx 0.4}(p \wedge q, q)$, respectively. The set of atoms, $At(A)$, contains $a_1 = p \wedge q$, $a_2 = p \wedge \neg q$, $a_3 = \neg p \wedge q$ and $a_4 = \neg p \wedge \neg q$. Let x_i denote the measure of atom a_i. The formula A is equivalent to $(B \wedge C \wedge D) \vee (\neg B \wedge C \wedge \neg D)$. We start with the first conjunct $B \wedge C \wedge D$ and suppose that the measures of q and $\neg q$ are greater than zero, i.e., that $x_1 + x_3 > 0$, and $x_2 + x_4 > 0$. $B \wedge C \wedge D$ is satisfiable iff the same holds for the following system:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 = 1, &\quad x_i \geqslant 0 \text{ for } i = 1, 4 \\ x_1 + x_3 > 0 &\quad x_2 + x_4 > 0 \\ x_1 + x_3 &\approx 0 \\ x_2/(x_2 + x_4) &\approx 1 \\ x_1/(x_1 + x_3) &\approx 0.4 \end{aligned}$$

which is equivalent to

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 = 1, &\quad x_i \geqslant 0 \text{ for } i = 1, 4 \\ x_1 + x_3 > 0 &\quad x_2 + x_4 > 0 \\ 0 < x_1 + x_3 &< n_1\epsilon \\ x_4/(x_2 + x_4) &< 1/n_2 \\ 0.4 - n_3\epsilon < x_1/(x_1 + x_3) &< 0.4 + n_3\epsilon \end{aligned}$$

for some n_1, n_2, $n_3 \in N$. If we replace n_1, n_2, n_3 by their maximum denoted by n, we obtain an equivalent system. Since $\approx$ does not appear in the last system, Fourier-Motzkin elimination can be performed in the standard way. The procedure finishes with the true condition $(1 - n\epsilon)/n < 1$ which means that the considered formula is satisfiable.

9.1.4. Modelling default reasoning. The central notion in the field of default reasoning is the notion of default rules. A default rule, which can be seen as a sentence of the form "if α, then generally β", can be written as[5] $\alpha \rightarrowtail \beta$. A default base Δ is a set of default rules. Default reasoning is described in terms of the corresponding consequence relation $\mid\!\sim$, i.e., we are interested in determining the set

[5] Note that the other authors use different symbols ($\to$, $\mid\!\sim$, for example) to denote the "default implication". In the present setting those symbols may cause confusion, so we prefer to introduce a new symbol here.

of defaults that are the consequences of a default base. Then, if α is a description of our knowledge and $\Delta|\!\sim \alpha \rightarrowtail \beta$, we (plausibly) conclude that β is the case. There are a number of papers which describe $|\!\sim$ in terms of classes of models and the corresponding satisfiability relations $\models$ such that $\Delta|\!\sim \alpha \rightarrowtail \beta$ if for every model M satisfying Δ, $M \models \alpha \rightarrowtail \beta$. In [61, 66] a set of properties which form a core of default reasoning, called the system P, and the corresponding deduction relation $\vdash_P$ were proposed. The system P is based on the following axiom and rules ($\models$ denotes classical validity):

- $\alpha \rightarrowtail \alpha$ (Reflexivity)
- from $\models \alpha \leftrightarrow \alpha'$ and $\alpha \rightarrowtail \beta$, infer $\alpha' \rightarrowtail \beta$ (Left logical equivalence)
- from $\alpha \rightarrowtail \beta$ and $\alpha \rightarrowtail \gamma$, infer $\alpha \rightarrowtail \beta \wedge \gamma$ (And)
- from $\alpha \rightarrowtail \gamma$ and $\beta \rightarrowtail \gamma$, infer $\alpha \vee \beta \rightarrowtail \gamma$ (Or)
- from $\alpha \rightarrowtail \beta$ and $\alpha \rightarrowtail \gamma$, infer $\alpha \wedge \beta \rightarrowtail \gamma$ (Cautious monotonicity).

Then, for a default base Δ, $\Delta \vdash_P \alpha \rightarrowtail \beta$ if $\alpha \rightarrowtail \beta$ is deducible from Δ using the above axiom and rules. Default consequence relation was also described in terms of preferential models, and it was proved that the system P is sound and complete with respect to the class of all such models:

Theorem 90. [61, Theorem 5.18] $\Delta|\!\sim \alpha \rightarrowtail \beta$ *with respect to the class of all preferential models if and only if* $\Delta \vdash_P \alpha \rightarrowtail \beta$.

The same holds for a special proper subclass of the class of preferential models, the so-called rational models, also considered in [66]. These two classes are not distinguishable using the language of defaults. It turns out that many other approaches to default reasoning are characterized by P. For example, in [66] a family of nonstandard ($^*R$) probabilistic models characterizing $\vdash_P$ was proposed. An $^*R$-probabilistic model can be defined in a similar way as $LPCP^{S,\approx}_{2,\mathrm{Meas},\mathrm{Neat}}$-models, with the exception that $\mu : H \to R^*$. A default $\alpha \rightarrowtail \beta$ holds in an $^*R$-probabilistic model if either the probability of α is 0 or the conditional probability of β given α is infinitesimally close to 1.

We can use $CP_{\approx 1}(\beta, \alpha)$ to syntactically represent the default $\alpha \rightarrowtail \beta$. In the sequel, we will use $\alpha \rightarrowtail \beta$ both in the original context of the system P and to denote the corresponding translation $CP_{\approx 1}(\beta, \alpha)$. In the case of a finite default base our approach produces the same result as the other mentioned approaches, namely it is equivalent to P.

Theorem 91. *For every finite default base* Δ *and for every default* $\alpha \rightarrowtail \beta$

$$\Delta \vdash_P \alpha \rightarrowtail \beta \ \text{ iff } \ \Delta \vdash_{Ax_{LPCP_2^{S,\approx}}} \alpha \rightarrowtail \beta.$$

Theorem 91 cannot be generalized to an arbitrary default base Δ, as it is illustrated by the following example.

Example 92. It is proved in [66, Lemma 2.7] that the infinite set of defaults $T = \{p_i \rightarrowtail p_{i+1}, p_{i+1} \rightarrowtail \neg p_i\}$, where p_i's are propositional letters for every integer $i \geqslant 0$, has only non well-founded preferential models (a preferential model containing an infinite descending chain of states) in which $p_0 \not\rightarrowtail \bot$, i.e., p_0 is consistent. It means that $T \nvdash_P p_0 \rightarrowtail \bot$. On the other hand, $T \vdash_{Ax_{LPCP_2^{S,\approx}}} p_0 \rightarrowtail \bot$ since the following holds. Let an $LPCP^{S,\approx}_{2,\mathrm{Meas},\mathrm{Neat}}$-model $\mathbf{M} = \langle W, H, \mu, v\rangle$ satisfy the set T. If $\mu([p_i]) = 0$, for some $i > 0$, then it must be $\mu([p_0]) = 0$, and $\mathbf{M} \vDash p_0 \rightarrowtail \bot$. Thus, suppose that $\mu([p_i]) \neq 0$, for every $i > 0$. Then, for every $i \geqslant 0$: $\frac{\mu([p_i \wedge p_{i+1}])}{\mu([p_i])} \approx 1$ and $\frac{\mu([\neg p_i \wedge p_{i+1}])}{\mu([p_{i+1}])} \approx 1$, i.e., $\frac{\mu([p_i \wedge p_{i+1}])}{\mu([p_i])} = 1 - \epsilon_1$ and $\frac{\mu([\neg p_i \wedge p_{i+1}])}{\mu([p_{i+1}])} = 1 - \epsilon_2$, for some infinitesimals ϵ_1 and ϵ_2. A simple calculation shows that which means that $\mu([p_i]) \leqslant \epsilon_0 \mu([p_{i+1}])$ for some infinitesimal ϵ_0. Since, for some c and k, $\epsilon_0 \leqslant c\epsilon^k$, it follows that for every $i > 0$, $0 \leqslant \mu([p_0]) \leqslant \epsilon^i$. Since $\mu([p_0]) \in S$ and there is no positive element of S with such property, it follows that $\mu([p_0]) = 0$, $[p_0] = \emptyset$ and $M \vDash p_0 \rightarrowtail \bot$. Since $\mathbf{M}$ is an arbitrary $LPCP^{S,\approx}_{2,\mathrm{Meas},\mathrm{Neat}}$-model, $T \vdash_{LPCP_2^{S,\approx}} p_0 \rightarrowtail \bot$.

Note that the above proof of $\mu([p_0]) = 0$, does not hold in the case when the range of the probability is the unit interval of ${}^*R$ because ${}^*R$ is ω_1-saturated (which means that the intersection of any countable decreasing sequence of nonempty internal sets must be nonempty). As a consequence, thanks to the restricted ranges of probabilities that are allowed in $LPCP^{S,\approx}_{2,\mathrm{Meas},\mathrm{Neat}}$-class of models, our system goes beyond the system P, when we consider infinite default bases.

$LPCP_2^{S,\approx}$ is rich enough not only to express formulas that represents defaults but also to describe more: probabilities of formulas, negations of defaults, combinations of defaults with the other (probabilistic) formulas etc. Let us now consider some situations where these possibilities allow us to obtain more conclusions than in the framework of the language of defaults.

Example 93. The translation of rational monotonicity, $((\alpha \rightarrowtail \beta) \wedge \neg(\alpha \rightarrowtail \neg\gamma)) \rightarrow ((\alpha \wedge \gamma) \rightarrowtail \beta)$, is $LPCP^{S,\approx}_{2,\mathrm{Meas},\mathrm{Neat}}$-valid since rational monotonicity is satisfied in every ${}^*R$-probabilistic model, and $LPCP^{S,\approx}_{2,\mathrm{Meas},\mathrm{Neat}}$ is a subclass of that class of models. The same holds for the formula $\neg(\mathrm{true} \rightarrowtail \mathrm{false})$ corresponding to another property called normality in [31].

Note that in this example we use negated defaults that are not expressible in P.

Example 94. Let the default base consist of the following two defaults $s \rightarrowtail b$ and $s \rightarrowtail t$, where s, b and t means Swedes, blond and tall, respectively [6]. Because of the inheritance blocking problem, in some systems (for example in P) it is not possible to conclude that Swedes who are not tall are blond ($(s \wedge \neg t) \rightarrowtail b$). Since our system and P coincide if the default base is finite, the same holds in our framework. In fact, there are some $LPCP^{S,\approx}_{2,\mathrm{Meas},\mathrm{Neat}}$-models in which the previous formula is not satisfied. Avoiding a discussion of intuitive acceptability of the above conclusion, we point out that by adding some additional assumptions ($CP_{=1-\epsilon}(t, s)$ and $CP_{=1-\epsilon^2}(b, s)$) to the default base we can entail that conclusion too. First, note that the assumptions are compatible with defaults $s \rightarrowtail t$ and $s \rightarrowtail b$. Then, an easy calculation shows that $\frac{P(s \wedge \neg t)}{P(s)} = \frac{P(s) - P(s \wedge t)}{P(s)} = \frac{P(s) - P(s) + P(s)\epsilon}{P(s)} = \epsilon$, and similarly

$\frac{P(s\wedge\neg b)}{P(s)} = \epsilon^2$. Finally, we can estimate the conditional probability of b given $s\wedge\neg t$:

$$\frac{P(s\wedge\neg t\wedge b)}{P(s\wedge\neg t)} = \frac{P(s\wedge\neg t) - P(s\wedge\neg t\wedge\neg b)}{P(s\wedge\neg t)} \geqslant \frac{\epsilon P(s) - \epsilon^2 P(s)}{\epsilon P(s)} = 1-\epsilon.$$

It follows that $(s\wedge\neg t) \rightarrowtail b$.

9.2. A logic with coherent conditional probabilities. In this subsection S again denotes the unit interval of rational numbers. The set For_P contains*basic probability formulas* of the form $CP_{\geqslant s}(\alpha,\beta)$ and their boolean combinations.

Definition 95. An $LPCP_2^{\mathrm{Chr}}$*-model* is a structure $\mathbf{M} = \langle W, H, \mu, v\rangle$ where:

- W is a nonempty set,
- H is an algebra of subsets of W, $H^0 = H \setminus \{\emptyset\}$
- $\mu : H\times H^0 \to [0,1]$, is a conditional probability satisfying
- $\mu(A,A) = 1$, for every $A\in H^0$,
  - $\mu(\cdot, A)$ is a finitely additive probability on H for any given $A\in H^0$, and
  - $\mu(C\cap B, A) = \mu(B,A)\cdot\mu(C, B\cap A)$, for all $C\in H$ and $A, B, A\cap B \in H^0$.
- $v : W\times\phi \to \{\mathrm{true}, \mathrm{false}\}$ provides for each world $w\in W$ a two-valued evaluation of the primitive proposition.

Let $LPCP^{\mathrm{Chr}}_{2,\mathrm{Meas}}$ denote the class of all measurable $LPCP_2^{\mathrm{Chr}}$-models.

The axiomatic system $Ax_{LPCP_2^{\mathrm{Chr}}}$ which characterizes the set of all $LPCP^{\mathrm{Chr}}_{2,\mathrm{Meas}}$-valid formulas contains the following axiom schemata:

(1) all instances of the classical propositional tautologies,
(2) $CP_{\geqslant 0}(\alpha,\beta)$,
(3) $CP_{\leqslant r}(\alpha,\beta) \to CP_{<s}(\alpha,\beta)$, $s>r$,
(4) $CP_{<s}(\alpha,\beta) \to CP_{\leqslant s}(\alpha,\beta)$,
(5) $(CP_{\geqslant r}(\alpha,\gamma) \wedge CP_{\geqslant s}(\beta,\gamma) \wedge CP_{\geqslant 1}(\neg\alpha\vee\neg\beta,\gamma)) \to CP_{\geqslant \min\{1,r+s\}}(\alpha\vee\beta,\gamma)$,
(6) $(CP_{\leqslant r}(\alpha,\gamma) \wedge CP_{<s}(\beta,\gamma)) \to CP_{<r+s}(\alpha\vee\beta,\gamma)$, $r+s<1$,
(7) $CP_{\geqslant s}(\alpha,\gamma) \wedge CP_{\geqslant r}(\beta,\alpha\wedge\gamma) \to CP_{\geqslant s\cdot r}(\alpha\wedge\beta,\gamma)$,
(8) $CP_{\geqslant 1}(\beta,\gamma) \wedge CP_{\geqslant s}(\alpha\wedge\beta,\gamma) \to CP_{\geqslant s}(\alpha,\beta\wedge\gamma)$,

and inference rules:

(1) from α and $\alpha\to\beta$ infer β,
(2) from $\alpha\to\beta$ infer $CP_{\geqslant 1}(\beta,\alpha)$,
(3) from $A \to (CP_{\geqslant t}(\beta,\gamma) \to CP_{\geqslant s\cdot t}(\alpha\wedge\beta,\gamma))$, for every rational number t from $(0,1)$, infer $A \to CP_{\geqslant s}(\alpha,\gamma\wedge\beta)$.

Note that Rule 3 is the only infinitary rule in $Ax_{LPCP_2^{\mathrm{Chr}}}$. It corresponds to the last part in the definition of conditional probability. The proof of

Theorem 96 (Extended completeness theorem for $LPCP^{\mathrm{Chr}}_{2,\mathrm{Meas}}$)**.** *A set T of formulas is $Ax_{LPCP_2^{\mathrm{Chr}}}$-consistent iff it is $LPCP^{\mathrm{Chr}}_{2,\mathrm{Meas}}$-satisfiable.*

follows the main steps from the previous sections, while by reducing the $LPCP^{\mathrm{Chr}}_{2,\mathrm{Meas}}$ satisfiability problem to the problem of checking coherence of conditional probability assessments which is decidable [20], we have that

Theorem 97. *The logic $LPCP_2^{\mathrm{Chr}}$ is decidable.*

The next example contains two formulas that illustrate some peculiarities of $LPCP_2^{\mathrm{Chr}}$.

Example 98. The formula $A = CP_{=0}(\alpha,\beta) \to CP_{>0}(\beta,\top)$ is not $LPCP_2^{\mathrm{Chr}}$-valid. Let us consider the following $LPCP_{2,\mathrm{Meas}}^{\mathrm{Chr}}$-model $\mathbf{M}$:

- $W = \{w_1, w_2\}$,
- $H = 2^W$,
- $\mu(\{w_1\}, W) = 0$, $\mu(\{w_2\}, W) = 1$, $\mu(\{w_1\}, \{w_1\}) = 1$, $\mu(\{w_2\}, \{w_1\}) = 0$, $\mu(\{w_1\}, \{w_2\}) = 0$, $\mu(\{w_2\}, \{w_2\}) = 1$,
- $v(w_1, p) = v(w_2, p) = v(w_1, q) = \text{true}$, $v(w_2, q) = \text{false}$.

In this model $\mu([p],[q]) = \mu(\{w_1, w_2\}, \{w_1\}) = 1$, $\mu([\neg p],[q]) = \mu(\emptyset, \{w_1\}) = 0$, and $\mu([q],[\top]) = \mu(\{w_1\}, W) = 0$. It means that $\mathbf{M} \models CP_{=0}(\neg p, q) \wedge CP_{=0}(q,\top)$, and A is not $LPCP_{2,\mathrm{Meas}}^{\mathrm{Chr}}$-valid.

The formula $B = CP_{=0}(\beta,\top) \wedge CP_{\geqslant \frac{1}{2}}(\alpha,\beta) \to CP_{\leqslant \frac{1}{2}}(\neg\alpha,\beta)$ is $LPCP_2^{\mathrm{Chr}}$-valid since $\mu([\cdot],[\beta])$ is finitely additive probability measure.

Note that both formulas from Example 98 have the opposite behavior when we use the Kolmogorov's approach to conditional probability (with the very often and useful assumption that the conditional probability of α given β is 1, if the probability of β is 0), i.e., A is valid, while B is not.

10. Related work

As we mentioned in Section 2, a lot of recent interest in probability logic was initiated by [79] in which Nillson gave a procedure for probabilistic entailment which, given probabilities of premises, could calculate bounds on probabilities of the derived sentences. The Nillson's approach was semantic and stimulated some authors to provide axiomatizations and decision procedures for the logic. In the same year Gaifman published a paper [35] which studied higher order probabilities and connections with modal logics.

In [27] Fagin, Halpern and Megiddo presented a propositional logic with real-valued probabilities in which higher level probabilities were not allowed (the logic was similar to LPP_2). The language of that logic allowed basic probabilistic formulas of the form $a_1 w(\alpha_1) + \cdots + a_n w(\alpha_n) \geqslant s$, where a_i's and s are rational numbers, α_i's classical propositional formulas, and $w(\alpha_i)$'s denote probabilities of α_i's. Probabilistic formulas are boolean combinations of basic probabilistic formulas. The corresponding class of models was $LPP_{2,\mathrm{Meas}}$. A finitary axiomatic system for the logic was given. Since the compactness theorem does not hold for their logic, the authors were able to prove only the simple completeness. As we mentioned above, the paper contains a proof of decidability and complexity of the logic. Models that are not measurable were also considered there. Dropping the measurability requirement made things more complicated. In that case inner and outer measures should be used since the finite additivity does not hold for the considered models. Finally, conditional probabilities were also discussed. To obtain a complete axiomatization, the authors used the machinery of the theory of real closed fields. We note that our syntax can be extended in a straightforward manner, such that the set of well formed formulas and the related results from [27] can be exactly obtained. The papers [28, 44] of the same authors introduced a probabilistic extension of the modal logic of knowledge which is similar to LPP_1.

The papers [30, 125] presented logics with probability functions that have a fixed finite range, similar to the logic $LPP_2^{\mathrm{Fr}(n)}$.

Frisch and Haddawy presented in [32] an incomplete iteration procedure which computes increasingly narrow probability intervals. The procedure can be stopped at any time yielding partial information about the probability of sentences, and allowing one to make a tradeoff between precision and computational time. Computational aspects of probabilistic logics were also discussed in [36]. The paper [52] showed that it is possible to apply a very efficient numerical method of column generation to solve the $LPP_{2,\mathrm{Meas}}$-satisfiability problem.

First order probability logics were discussed in [1, 43]. It was shown that the set of valid formulas of the considered logic (which was similar to $LFOP_1$) is not recursively enumerable. Thus, no finitary axiomatization is possible.

In [38] a propositional logic which can be used for reasoning about probabilistic processes was presented. Besides all differences between our logic and that one, in [38] an idea to prove completeness using an infinitary rule was used similarly as in our approach.

A rule similar to Rule 3 from the axiomatic system $Ax_{LPCP_2^{S,\approx}}$ was given in [3] by Alechina. The main difference is that her rule was restricted to rationals only.

A sound first order axiomatization (which is not complete) for a logic which formalized probabilistic temporal reasoning was given in [39]. This system differs from our LPP_1^{LTL} since time intervals and a branching structure of time were considered there.

In [16, 17] Boričić and Rašković extended Heyting propositional logic by probabilistic operators. Since predicates "at lest r" and "at most r" are not mutually expressible in that context, both types of operators $P_{\geqslant r}$ and $P_{\leqslant r}$ were present in the corresponding language. Marchioni and Godo presented in [73] a modal fuzzy logic approach to model probabilistic reasoning in the sense of De Finetti. Also, in that logic, Łukasiewicz implication can be used to express comparative statements. Conditional probabilities were combined with default reasoning in a semantically based approach in [2, 70].

Uncertain reasoning is also interesting in the framework of economy. For example, an axiomatization for so-called type spaces (a notion that plays the role of probabilistic models in our paper) within the framework of probabilistic logic was given in [45]. The proposed axiomatization was simply complete with respect to the introduced semantics. A strongly complete infinitary axiomatization for type spaces is given in [77]. The main difference between that system and our approach is that infinitary formulas are allowed in [77]. As a consequence, that logic is undecidable, due to the cardinality argument.

Finally, for more comprehensive list of the papers on probability logics the reader could consult [94].

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