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# BICARTESIAN COHERENCE REVISITED

*Abstract*. A survey is given of results about coherence for categories with finite products and coproducts. For these results, which were published previously by the authors in several places, some formulations and proofs are here corrected, and matters are updated. The categories investigated in this paper formalize equality of proofs in classical and intuitionistic conjunctive-disjunctive logic without distribution of conjunction over disjunction.

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## 1. Coherence

Categorists call *coherence* what logicians would probably call *completeness*. This is, roughly speaking, the question whether we have assumed for a particular brand of categories all the equations between arrows we should have assumed. Completeness need not be understood here as completeness with respect to models. We may have also a syntactical notion of completeness—something like the Post completeness of the classical propositional calculus—but often some sort of model-theoretical completeness is implicit in coherence questions. Matters are made more complicated by the fact that categorists do not like to talk about syntax, and do not perceive the problem as being one of finding a match between syntax and semantics. They do not talk of formal systems, axioms and models.

Moreover, questions that logicians would consider to be questions of *decidability*, which is of course not the same as completeness, are involved in what categorists call coherence. A coherence problem often involves the question of deciding whether two terms designate the same arrow, i.e. whether a diagram of arrows commutes. Coherence is understood mostly as solving this problem, which we call the *commuting problem*, in [22] (see p. 117, which mentions [20] and [21] as the origin of this understanding). The commuting problem seems to be involved also in the understanding of coherence of [17, Section 10].

Completeness and decidability, though distinct, are not foreign to each other. A completeness proof with respect to a manageable model may provide, more or less immediately, tools to solve decision problems. For example, the completeness proof for the classical propositional calculus with respect to the two-element Boolean algebra provides immediately a decision procedure for theoremhood.

The simplest coherence questions are those where it is intended that all arrows of the same type should be equal, i.e. where the category envisaged is a preorder. The oldest coherence problem is of that kind. This problem has to do with monoidal categories, and was solved by Mac Lane in [23]. The monoidal category freely generated by a set of objects is a preorder. So Mac Lane could claim that showing coherence is showing that "all diagrams commute".

In cases where coherence amounts to showing preorder, i.e. showing that from a given set of equations, assumed as axioms, we can derive all equations (provided the equated terms are of the same type), from a logical point of view we have to do with *axiomatizability*. We want to show that a decidable set of axioms (and we wish this set to be as simple as possible, preferably given by a finite number of axiom schemata) delivers all the intended equations. If preorder is intended, then all equations are intended. Axiomatizability is in general connected with logical questions of completeness, and a standard logical notion of completeness is completeness of a set of axioms. Where all diagrams should commute, coherence does not seem to be a question of model-theoretical completeness, but even in such cases it may be conceived that the model involved is a discrete category.

Categorists are interested in axiomatizations that permit extensions. These extensions are in a new language, with new axioms, and such extensions of the axioms of monoidal categories need not yield preorders any more. Categorists are also interested, when they look for axiomatizations, in finding the combinatorial building blocks of the matter. The axioms are such building blocks, as in knot theory the Reidemeister moves are the combinatorial building blocks of knot and link equivalence (see [3, Chapter 1], or any other textbook in knot theory).

In Mac Lane's second coherence result of [23], which has to do with symmetric monoidal categories, it is not intended that all equations between arrows of the same type should hold. What Mac Lane does can be described in logical terms in the following manner. On the one hand, he has an axiomatization, and, on the other hand, he has a model category where arrows are permutations; then he shows that his axiomatization is complete with respect to this model. It is no wonder that his coherence problem reduces to the completeness problem for the usual axiomatization of symmetric groups.

Algebraists do not speak of axiomatizations, but of *presentations by generators and relations*. The axiomatizations we envisage are purely equational axiomatizations, as in algebraic varieties. Such were the axiomatizations of [23]. Categories are algebras with partial operations, and we are interested in the equational theories of these algebras.

In Mac Lane's coherence results for monoidal and symmetric monoidal categories one has to deal only with natural isomorphisms. However, in the coherence result for symmetric monoidal closed categories of [19] there are already natural and dinatural transformations that are not isomorphisms.

A natural transformation is tied to a relation between the argument-places of the functor in the source and the argument-places of the functor in the target. This relation corresponds to a relation between occurrences of letters in formulae, and in composing natural transformations we compose these relations. With dinatural transformations the matter is more complicated, and composition poses particular problems (see [24]). In this paper we deal with natural transformations. Our general notion of coherence does not, however, presuppose naturality and dinaturality.

Our notion of a coherence result is one that covers Mac Lane's and Kelly's coherence results mentioned above, but it is more general. We call coherence a result that tells us that there is a faithful functor $G$ from a category $\mathcal{S}$ freely generated in a certain class of categories to a "manageable" category $\mathcal{M}$. This calls for some explanation.

It is desirable, though perhaps not absolutely necessary, that the functor $G$ be *structure-preserving*, which means that it preserves structure at least up to isomorphism. In all coherence results we will consider here, the functor $G$ will preserve structure strictly, i.e. "on the nose". The categories $\mathcal{S}$ and $\mathcal{M}$ will be in the same class of categories, and $G$ will be obtained by extending in a unique way a map from the generators of $\mathcal{S}$ into $\mathcal{M}$.

The category $\mathcal{M}$ is *manageable* when equations of arrows, i.e. commuting diagrams of arrows, are easier to consider in it than in $\mathcal{S}$. The best is if the commuting problem is obviously decidable in $\mathcal{M}$, while it was not obvious that it is such in $\mathcal{S}$.

With our approach to coherence we are oriented towards solving the commuting problem. This should be stressed because other authors may give a more prominent place to other problems. We have used on purpose the not very precise term "manageable" for the category $\mathcal{M}$ to leave room for modifications of our notion of coherence, which would be oriented towards solving another problem than the commuting problem.

In this paper, the manageable category $\mathcal{M}$ will be the category *Rel* with arrows being relations between occurrences of letters in formulae. In [14] and elsewhere we have taken *Rel* to be the category of relations between finite ordinals, which is not essentially different from what we do in this paper. The previous category *Rel* is the skeleton of the new one. We have mentioned above the connection between *Rel* and natural transformations. The commuting problem in *Rel* is obviously decidable.

The freely generated category $\mathcal{S}$ will be the bicartesian category, i.e. category with all finite products and coproducts, freely generated by a set of objects, or a related category of that kind. The generating set of objects may be conceived as a discrete category. In our understanding of coherence, replacing this discrete generating category by an arbitrary category would prevent us to solve coherence—simply because the commuting problem in the arbitrary generating category may be undecidable. Far from having more general, stronger, results if the generating category is arbitrary, we may end up by having no result at all.

The categories $\mathcal{S}$ in this paper are built ultimately out of *syntactic* material, as logical systems are built. Categorists are not inclined to formulate their coherence results in the way we do—in particular, they do not deal often with syntactically built categories. If, however, more involved and more abstract formulations of coherence that may be found in the literature (for early references on this matter see [18]) have practical consequences for solving the commuting problem, our way of formulating coherence has these consequences as well.

That there is a faithful structure-preserving functor $G$ from the syntactical category $\mathcal{S}$ to the manageable category $\mathcal{M}$ means that for all arrows $f$ and $g$ of $\mathcal{S}$ with the same source and the same target we have

$$f = g \text{ in } \mathcal{S} \text{ iff } Gf = Gg \text{ in } \mathcal{M}.$$

The direction from left to right in this equivalence is contained in the functoriality of $G$, while the direction from right to left is faithfulness proper.

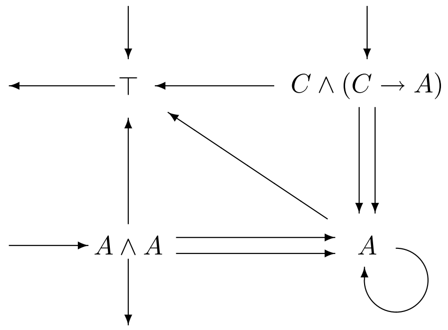
If $\mathcal{S}$ is conceived as a syntactical system, while $\mathcal{M}$ is a model, the faithfulness equivalence we have just stated is like a completeness result in logic. The left-to-right direction, i.e. functoriality, is soundness, while the right-to-left direction, i.e. faithfulness, is completeness proper.

If $G$ happens to be one-one on objects, then we obtain that $\mathcal{S}$ is isomorphic to a subcategory of $\mathcal{M}$—namely, its image under $G$ in $\mathcal{M}$. We will have such a situation in this paper, where $G$ will be identity on objects.

In this paper we will separate coherence results involving terminal objects and initial objects from those not involving them. These objects cause difficulties, and the statements and proofs of the coherence results gain by having these difficulties kept apart.

## 2. Coherence and proof theory

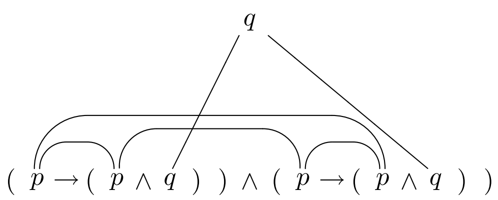
If one envisages a deductive system as a graph whose nodes are formulae:



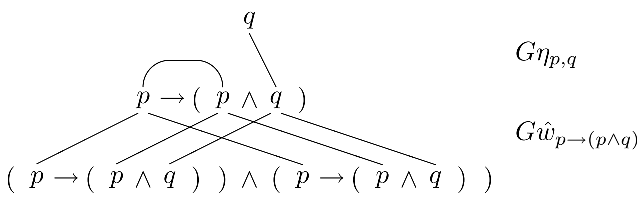

and whose arrows are derivations from the sources understood as premises to the targets understood as conclusions, then equality of derivations usually transforms this deductive system into a category of a particular brand. This category has a structure induced by the connectives of the deductive system. Although equality of derivation is dictated by logical concerns, usually the categories we end up with are of a kind that categorists have already introduced for their own reasons. The prime example here is given by the deductive system for the conjunction-implication fragment of intuitionistic propositional logic. After derivations in this deductive system are equated according to ideas about normalization of derivations that stem from Gentzen, one obtains the cartesian closed category $\mathcal{K}$ freely generated by a set of propositional letters (see [22] for the notion of cartesian closed category).

Equality of proofs in intuitionistic logic has not led up to now to a coherence result—a coherence theorem is not forthcoming for cartesian closed categories. If we take that the model category $\mathcal{M}$ is a category whose arrows are graphs like the graphs of [19], then we do not have a faithful functor $G$ from the free cartesian closed category $\mathcal{K}$ to $\mathcal{M}$. We will now explain why $G$ is not even a functor.

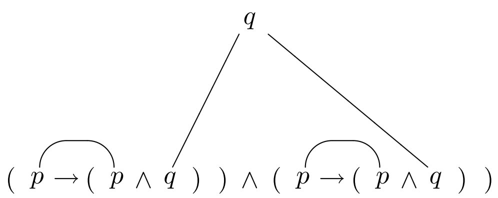
If $\eta_{p,q}$ is the canonical arrow from $q$ to $p \to (p \wedge q)$, where $A \to B$ and $A \wedge B$ stand for $B^A$ and $A \times B$ respectively, while $\hat{w}_A$ is the diagonal arrow from $A$ to $A \wedge A$, then $G(\hat{w}_{p\to(p\wedge q)} \circ \eta_{p,q})$:




which is obtained from




is different from $G((\eta_{p,q} \wedge \eta_{p,q}) \circ \hat{w}_q)$:

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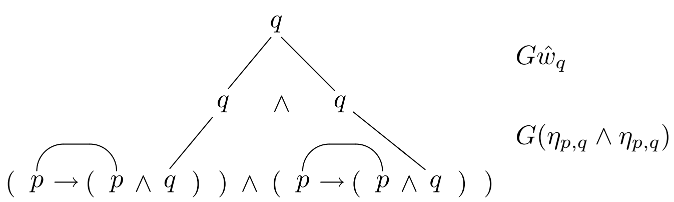
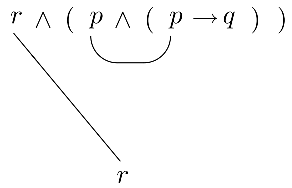



So, if $\hat{w}$ is a natural transformation, then $G$ is not a functor. The naturality of $\hat{w}$, and other arrows of that kind, tied to structural rules ($\hat{w}$ is tied to contraction, and $\hat{k}^1$ below to thinning), is desirable because it corresponds to the permuting of these rules in a cut-elimination or normalization procedure.

Dually, if $\varepsilon_{p,q}$ is the canonical arrow from $p \wedge (p \to q)$ to $q$, and $\hat{k}^1_{A,B}$ is the first projection from $A \wedge B$ to $A$, then $G(\hat{k}^1_{r,q} \circ (\mathbf{1}_r \wedge \varepsilon_{p,q}))$:

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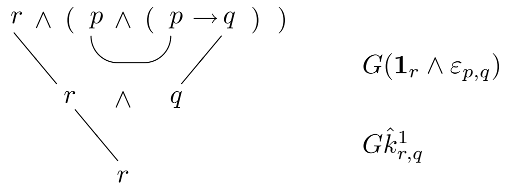
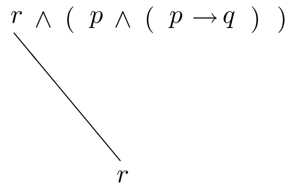



is different from $G\hat{k}^1_{r,p\wedge(p\to q)}$:




So, if $\hat{k}^1$ is a natural transformation, then $G$ is not a functor. The faithfulness of $G$ fails because of a counterexample in [27], involving a natural number object in *Set* and the successor function. This does not exclude that with a more sophisticated model category $\mathcal{M}$ we might still be able to obtain coherence for cartesian closed categories (for an attempt along these lines see [25]).

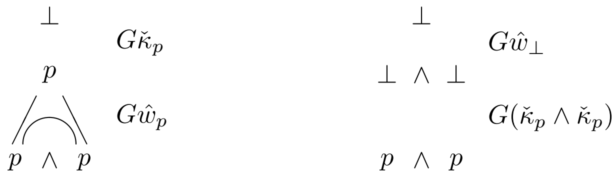
Equality of proofs in classical logic may, however, lead to coherence with respect to model categories that catch up to a point the idea of *generality* of proofs. Such is in particular the category *Rel* mentioned in the preceding section, whose arrows are relations between occurrences of propositional letters in the premises and conclusions. The idea that generality of proofs may serve as a criterion for identity of proofs stems from Lambek's pioneering papers in categorial proof theory of the late 1960s (see [22] for references). This criterion says, roughly, that two derivations represent the same proof when their generalizations with respect to diversification of variables (without changing the rules of inference) produce derivations with the same source and target, up to a renaming of variables.

Although coherence with respect to *Rel* is related to generality, it is not exactly that. The question is should $G\hat{w}_p$ be the relation in the left one or in the right one of the following two diagrams:


The second option, induced by dealing with equivalence relations, or by connecting all letters that must remain the same in generalizing proofs (see [12] and [13]), would lead to abolishing the naturality of $\hat{w}$. For example, in the following instance of the naturality equation for $\hat{w}$:

$$\hat{w}_p \circ \check{\kappa}_p = (\check{\kappa}_p \wedge \check{\kappa}_p) \circ \hat{w}_\perp$$

for $\check{\kappa}_p$ being the unique arrow from the initial object $\perp$ to $p$, we do not have that $G(\hat{w}_p \circ \check{\kappa}_p)$ is equal to $G((\check{\kappa}_p \wedge \check{\kappa}_p) \circ \hat{w}_\perp)$:




We obtain similarly that $\kappa$ cannot be natural.

It is shown in [14] that coherence with respect to the model category *Rel* could justify plausibly equality of derivations in various systems of propositional logic, including classical propositional logic. The goal of that book was to explore the limits of coherence with respect to the model category *Rel*. This does not exclude that other coherence results may involve other model categories, and, in particular, with a model category different from *Rel*, classical propositional logic may induce a different notion of Boolean category than the one introduced in Chapter 14 of [14]. That notion of Boolean category was not motivated *a priori*, but was dictated by coherence with respect to *Rel*. The definition of that notion was however not given via coherence, but via an equational axiomatization. We take such definitions as being proper axiomatic definitions.

We could easily define nonaxiomatically a notion of Boolean category with respect to graphs of the Kelly–Mac Lane kind (see [19]). Equality of graphs would dictate what arrows are equal. In this notion, conjunction would not be a product, because the diagonal arrows and the projections would not make natural transformations (see above), and, analogously, disjunction would not be a coproduct (cf. [14, Section 14.3]) The resulting notion of Boolean category would not be trivial—the freely generated categories of that kind would not be preorders—, but its nonaxiomatic definition would be trivial. There might exist a nontrivial equational axiomatic definition of this notion. Finding such a definition is an open problem.

We are looking for nontrivial axiomatic definitions because such definitions give information about the combinatorial building blocks of our notions, as Reidemeister moves give information about the combinatorial building blocks of knot equivalence. Our axiomatic equational definition of Boolean category in [14] is of the nontrivial, combinatorially informative, kind. Coherence of these Boolean categories with respect to *Rel* is a theorem, whose proof in [14] requires considerable effort.

Another analogous example is provided by the notion of monoidal category, which was introduced in a not entirely axiomatic way, via coherence, by Bénabou in [2], and in the axiomatic way, such as we favour, by Mac Lane in [23]. For Bénabou, coherence is built into the definition, and for Mac Lane it is a theorem. One could analogously define the theorems of classical propositional logic as being the tautologies (this is done, for example, in [4, Sections 1.2–3]), in which case completeness would not be a theorem, but would be built into the definition.

In this paper we prove coherence for categories that formalize equality of proofs in classical and intuitionistic conjunctive-disjunctive logic without distribution of conjunction over disjunction. This fragment of logic also covers the additive connectives of linear and other substructural logics (where distribution anyway should not be assumed). When to this fragment we add the true and absurd propositional constants matters become more complicated, and we do not how to prove unrestricted coherence in all cases.

## 3. Lattice categories

In the remaining sections of this paper we deal with coherence with respect to *Rel* for categories with a double cartesian structure, i.e. with finite products and finite coproducts. We take this as a categorification of the notion of lattice. As before, we distinguish cases with and without special objects, which are here the empty product and the empty coproduct, i.e. the terminal and initial objects. Categories with all finite products and coproducts, including the empty ones, are usually called *bicartesian* categories (see [22]). Categories with all nonempty finite products and coproducts are called *lattice* categories in [14]. The results presented here are adapted from [9], [11], the revised version of [10] and [14, Chapter 9].

We pay particular attention to questions of maximality, i.e. to the impossibility of extending our axioms without collapse into preorder, and hence triviality. This maximality is a kind of syntactical completeness. (The sections on maximality improve upon results reported in [9], [11] and [10], and are taken over from [14, Chapter 9].)

Our techniques are partly based on a composition elimination for conjunctive logic, related to normalization in natural deduction, and on a simple composition elimination for conjunctive-disjunctive logic, implicit in Gentzen's cut elimination.

We define now the category **L** built out of syntactic material. The objects of the category **L** are the formulae of the propositional language $\mathcal{L}$, generated out of a set of infinitely many propositional letters, for which we use $p$, $q$, $r, \ldots$, sometimes with indices, with the binary connectives $\wedge$ and $\vee$, for which we use $\xi$. For formulae we use $A$, $B$, $C, \ldots$, sometimes with indices.

To define the arrows of **L**, we define first inductively a set of expressions called the *arrow terms* of **L**. Every arrow term will have a *type*, which is an ordered pair of formulae of $\mathcal{L}_\wedge$. We write $f : A \vdash B$ when the arrow term $f$ is of type $(A, B)$. Here $A$ is the *source*, and $B$ the *target* of $f$. For arrow terms we use $f$, $g$, $h, \ldots$, sometimes with indices. Intuitively, the arrow term $f$ is the code of a derivation of the conclusion $B$ from the premise $A$ (which explains why we write $\vdash$ instead of $\rightarrow$).

For all formulae $A$, $B$ and $C$ of $\mathcal{L}$ the following *primitive arrow terms*:

$$\mathbf{1}_A : A \vdash A,$$

$$\hat{w}_A : A \vdash A \wedge A, \qquad \check{w}_A : A \vee A \vdash A,$$

$$\hat{k}^i_{A_1,A_2} : A_1 \wedge A_2 \vdash A_i, \quad \check{k}^i_{A_1,A_2} : A_i \vdash A_1 \vee A_2, \qquad \text{for } i \in \{1,2\},$$

are arrow terms. (Intuitively, these are the axioms of our logic with the codes of their trivial derivations.)

Next we have the following inductive clauses:

if $f : A \vdash B$ and $g : B \vdash C$ are arrow terms,
then $(g \circ f) : A \vdash C$ is an arrow term;
if $f_1 : A_1 \vdash B_1$ and $f_2 : A_2 \vdash B_2$ are arrow terms,
then $(f_1 \xi f_2) : A_1 \xi A_2 \vdash B_1 \xi B_2$ is an arrow term.

(Intuitively, the operations on arrow terms $\circ$ and $\xi$ are codes of the rules of inference of our logic.) This defines the arrow terms of **L**. As we do usually with formulae, we will omit the outermost parentheses of arrow terms.

We stipulate first that all the instances of $f = f$ and of the following equations are equations of **L**:

*categorial equations*:

$(cat\ 1)$ $\quad f \circ \mathbf{1}_A = \mathbf{1}_B \circ f = f : A \vdash B,$

$(cat\ 2)$ $\quad h \circ (g \circ f) = (h \circ g) \circ f,$

*bifunctorial equations*:

$(\xi\ 1)$ $\quad \mathbf{1}_A \,\xi\, \mathbf{1}_B = \mathbf{1}_{A \xi B}$

$(\xi\ 2)$ $\quad (g_1 \circ f_1) \,\xi\, (g_2 \circ f_2) = (g_1 \,\xi\, g_2) \circ (f_1 \,\xi\, f_2),$

*naturality equations*: for $f : A \vdash B$ and $f_i : A_i \vdash B_i$, where $i \in \{1,2\}$,

$(\hat{w}\ nat)$ $\quad (f \wedge f) \circ \hat{w}_A = \hat{w}_B \circ f,$

$(\check{w}\ nat)$ $\quad f \circ \check{w}_A = \check{w}_B \circ (f \vee f),$

$(\hat{k}^i\ nat)$ $\quad f_i \circ \hat{k}^i_{A_1,A_2} = \hat{k}^i_{B_1,B_2} \circ (f_1 \wedge f_2),$

$(\check{k}^i\ nat)$ $\quad (f_1 \vee f_2) \circ \check{k}^i_{A_1,A_2} = \check{k}^i_{B_1,B_2} \circ f_i,$

*triangular equations*: for $i \in \{1,2\}$,

$(\hat{w}\hat{k})$ $\quad \hat{k}^i_{A,A} \circ \hat{w}_A = \mathbf{1}_A,$

$(\check{w}\check{k})$ $\quad \check{w}_A \circ \check{k}^i_{A,A} = \mathbf{1}_A,$

$(\hat{w}\hat{k}\hat{k})$ $\quad (\hat{k}^1_{A,B} \wedge \hat{k}^2_{A,B}) \circ \hat{w}_{A \wedge B} = \mathbf{1}_{A \wedge B},$

$(\check{w}\check{k}\check{k})$ $\quad \check{w}_{A \vee B} \circ (\check{k}^1_{A,B} \vee \check{k}^2_{A,B}) = \mathbf{1}_{A \vee B}.$

This concludes the list of axiomatic equations stipulated for **L**. To define all the equations of **L** it remains only to say that the set of these equations is closed under symmetry and transitivity of equality and under the rules

$$(\circ\ cong)\quad \frac{f = f' \qquad g = g'}{g \circ f = g' \circ f'} \qquad\qquad (\xi\ cong)\quad \frac{f_1 = f_1' \qquad f_2 = f_2'}{f_1\,\xi\, f_2 = f_1'\,\xi\, f_2'}$$

On the arrow terms of **L** we impose the equations of **L**. This means that an arrow of **L** is an equivalence class of arrow terms of **L** defined with respect to the smallest equivalence relation such that the equations of **L** are satisfied (see [14, Section 2.3], for details).

The kind of category for which **L** is the one freely generated out of the set of propositional letters (which may be understood as a discrete category) we call *lattice* category (see [14, Section 9.4], for a precise definition). Usually, such categories would be called categories with finite nonempty products and coproducts. The objects of a lattice category that is a partial order make a lattice.

## 4. The functor $G$

The objects of the category *Rel* are the objects of **L**, i.e. the formulae of $\mathcal{L}$. An arrow $R\colon A \vdash B$ of *Rel* is a set of ordered pairs $(x, y)$ such that $x$ is an occurrence of a propositional letter in the formula $A$ and $y$ is an occurrence of a propositional letter in the formula $B$; in other words, arrows are binary relations between the sets of occurrences of propositional letters in formulae. We write either $(x, y) \in R$ or $xRy$, as usual. In this category, $\mathbf{1}_A\colon A \vdash A$ is the identity relation, i.e. identity function, that assigns to every occurrence of a propositional letter in $A$ that same occurrence. In $\mathcal{L}$ there are no formulae in which no propositional letter occurs, but where we have such formulae (as in the language $\mathcal{L}_{\top,\bot}$ considered later in this paper), the empty set of ordered pairs corresponds to $\mathbf{1}_A\colon A \vdash A$ if no propositional letter occurs in $A$. The empty relation is the identity relation on the empty set.

For $R_1 : A \vdash B$ and $R_2 : B \vdash C$, the set of ordered pairs of the composition $R_2 \circ R_1\colon A \vdash C$ is $\{(x, y) \mid \exists z(xR_1z \textit{ and } zR_2y)\}$. Let $x_j(A)$ be the $j$-th occurrence of a propositional letter in $A$ counting from the left, and let $|A|$ be the number of occurrences of propositional letters in $A$ (so $1 \leqslant j \leqslant |A|$). For $R_i : A_i \vdash B_i$, with $i \in \{1, 2\}$, the set of ordered pairs of $R_1\,\xi\, R_2\colon A_1\,\xi\, A_2 \vdash B_1\,\xi\, B_2$, for $\xi \in \{\wedge, \vee\}$, is the disjoint union of the following two sets:

$$\{(x_j(A_1\,\xi\, A_2), x_k(B_1\,\xi\, B_2)) \mid (x_j(A_1), x_k(B_1)) \in R_1\},$$

$$\{(x_{j+|A_1|}(A_1\,\xi\, A_2), x_{k+|B_1|}(B_1\,\xi\, B_2)) \mid (x_j(A_2), x_k(B_2)) \in R_2\}.$$

With the operation on objects that corresponds to the binary connective $\xi$, this operation $\xi$ on arrows gives a biendofunctor in *Rel*.

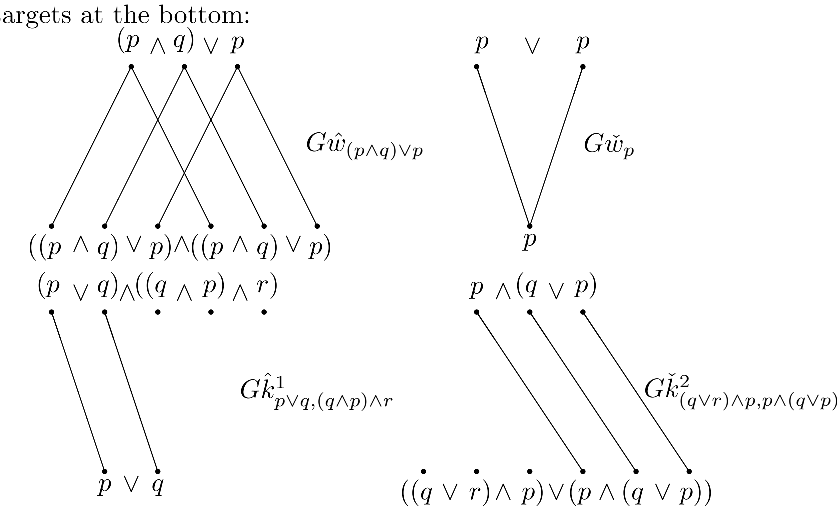
In *Rel* we have the relations $G\hat{w}_A : A \vdash A \wedge A$, $G\check{w}_A : A \vee A \vdash A$, $G\hat{k}^i_{A_1,A_2} : A_1 \wedge A_2 \vdash A_i$, and $G\check{k}^i_{A_1,A_2} : A_i \vdash A_1 \vee A_2$, for $i \in \{1, 2\}$, whose sets of ordered pairs are defined as follows:

$(x_j(A), x_k(A \wedge A)) \in G\hat{w}_A$ iff $(x_k(A \vee A), x_j(A)) \in G\check{w}_A$ iff $j \equiv k \pmod{|A|}$;

$(x_j(A_1 \wedge A_2), x_k(A_1)) \in G\hat{k}^1_{A_1,A_2}$ iff $(x_k(A_1), x_j(A_1 \vee A_2)) \in G\check{k}^1_{A_1,A_2}$ iff $j = k$;

$(x_j(A_1 \wedge A_2), x_k(A_2)) \in G\hat{k}^2_{A_1,A_2}$ iff $(x_k(A_2), x_j(A_1 \vee A_2)) \in G\check{k}^2_{A_1,A_2}$ iff $j = k+|A_1|$.

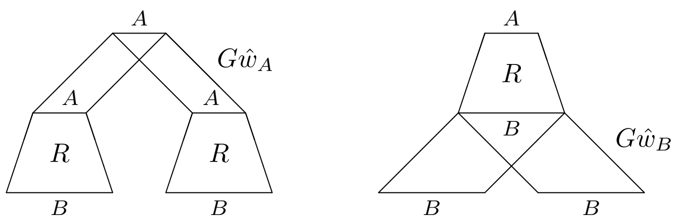
It is not difficult to check that all these arrows of *Rel* give rise to natural transformations. This is clear from the graphical representation of relations in *Rel*. Here are a few examples of such graphical representations, with sources written at the top and targets at the bottom:




For $R\colon A \vdash B$, the naturality equation

$$(R \wedge R) \circ G\hat{w}_A = G\hat{w}_B \circ R,$$

which corresponds to the equation ($\check{w}$ *nat*) of the preceding section, and which we take as an example, is justified in the following manner via graphs:




We can now define a functor $G$ from the category $\mathbf{L}$ to the category *Rel*. On objects we have that $GA$ is $A$. We have defined $G$ above on the primitive arrow terms of $\mathbf{L}$, and we have

$$G(f \,\xi\, g) = Gf \,\xi\, Gg,$$
$$G(g \circ f) = Gg \circ Gf.$$

To ascertain that this defines a functor from $\mathbf{L}$ to *Rel*, it remains to check that if $f = g$ in $\mathbf{L}$, then $Gf = Gg$ in *Rel*, which we do by induction on the length of the derivation of $f = g$ in $\mathbf{L}$.

It is easy to check by induction that if for $f\colon A \vdash B$ we have $(x_j(A), x_k(B)) \in Gf$, then $x_j(A)$ and $x_k(B)$ are occurrences of the same propositional letter.

Our first task in this paper is to show that the functor $G$ from $\mathbf{L}$ to *Rel* is faithful. We call this result *Lattice Coherence*, and we say that $\mathbf{L}$ is *coherent*. Since $G$ is identity on objects, this means that $\mathbf{L}$ is isomorphic to a subcategory of *Rel*.

It is clear that if $\mathbf{L}$ is coherent in the sense just specified, then it is decidable whether arrow terms of $\mathbf{L}$ are equal in $\mathbf{L}$. In logical terms, one would say that the coherence of $\mathbf{L}$ implies the decidability of the equational system used to define $\mathbf{L}$. This is because equality of arrows is clearly decidable in *Rel*. So coherence here implies a solution to the commuting problem.

## 5. Coherence for lattice categories

We define by induction a set of terms for the arrows of $\mathbf{L}$ that we call *Gentzen terms*. The identity arrow terms $\mathbf{1}_A$ are Gentzen terms, and we assume that Gentzen terms are closed under the following operations on arrow terms, besides the operation $\circ$, where $=_{dn}$ is read "denotes":

$$\frac{f_1 \colon C \vdash A_1 \qquad f_2 \colon C \vdash A_2}{\langle f_1, f_2\rangle =_{dn} (f_1 \wedge f_2) \circ \hat{w}_C \colon C \vdash A_1 \wedge A_2}$$

$$\frac{g_i \colon A_i \vdash C}{\hat{K}^i_{A_{3-i}} g_i =_{dn} g_i \circ \hat{k}^i_{A_1,A_2} \colon A_1 \wedge A_2 \vdash C}$$

$$\frac{g_1 \colon A_1 \vdash C \qquad g_2 \colon A_2 \vdash C}{[g_1, g_2] =_{dn} \check{w}_C \circ (g_1 \vee g_2) \colon A_1 \vee A_2 \vdash C}$$

$$\frac{f_i \colon C \vdash A_i}{\check{K}^i_{A_{3-i}} f_i =_{dn} \check{k}^i_{A_1,A_2} \circ f_i \colon C \vdash A_1 \vee A_2}$$

It is easy to verify that the following equations hold for Gentzen terms (these equations can serve for an alternative formulation of $\mathbf{L}$):

| | | | |
|---|---|---|---|
| $(\hat{K}1)$ | $g \circ \hat{K}^i_A f = \hat{K}^i_A (g \circ f)$, | $(\check{K}1)$ | $\check{K}^i_A g \circ f = \check{K}^i_A (g \circ f)$, |
| $(\hat{K}2)$ | $\hat{K}^i_A g \circ \langle f_1, f_2\rangle = g \circ f_i$, | $(\check{K}2)$ | $[g_1, g_2] \circ \check{K}^i_A f = g_i \circ f$, |
| $(\hat{K}3)$ | $\langle g_1, g_2\rangle \circ f = \langle g_1 \circ f, g_2 \circ f\rangle$, | $(\check{K}3)$ | $g \circ [f_1, f_2] = [g \circ f_1, g \circ f_2]$, |
| $(\hat{K}4)$ | $\mathbf{1}_{A\wedge B} = \langle \hat{K}^1_B \mathbf{1}_A, \hat{K}^2_A \mathbf{1}_B\rangle$, | $(\check{K}4)$ | $\mathbf{1}_{A\vee B} = [\check{K}^1_B \mathbf{1}_A, \check{K}^2_A \mathbf{1}_B]$, |
| $(\hat{K}5)$ | $\hat{K}^i_D \langle f_1, f_2\rangle = \langle \hat{K}^i_D f_1, \hat{K}^i_D f_2\rangle$, | $(\check{K}5)$ | $\check{K}^i_D [g_1, g_2] = [\check{K}^i_D g_1, \check{K}^i_D g_2]$, |

$$(\hat{K}\check{K}) \qquad \hat{K}^i_C \check{K}^j_D h = \check{K}^j_D \hat{K}^i_C h,$$

with appropriate types assigned to $f$, $g$, $f_i$ and $g_i$.

It is very easy to show that for every arrow term of $\mathbf{L}$ there is a Gentzen term denoting the same arrow. We can prove the following theorem for $\mathbf{L}$.

**Composition Elimination.** *For every arrow term $h$ there is a composition-free Gentzen term $h'$ such that $h = h'$.*

*Proof.* We find first a Gentzen term denoting the same arrow as $h$. Take a subterm $g \circ f$ of this Gentzen term such that both $f$ and $g$ are composition-free. We call such a subterm a *topmost cut*. We show that $g \circ f$ is equal either to a composition-free Gentzen term or to a Gentzen term all of whose compositions occur in topmost cuts of strictly smaller length than the length of $g \circ f$. The possibility of eliminating composition in topmost cuts, and hence every composition, follows by induction on the length of topmost cuts.

The cases where $f$ or $g$ are $\mathbf{1}_A$ are taken care of by $(cat\ 1)$; the cases where $f$ is $\hat{K}^i_A f'$ are taken care of by $(\hat{K}1)$; and the case where $g$ is $\langle g_1, g_2 \rangle$ is taken care of by $(\hat{K}3)$.

We have next cases dual to the last two, where $g$ is $\check{K}^i_A g'$, which is taken care of by $(\check{K}1)$, and where $f$ is $[f_1, f_2]$, which is taken care of by $(\check{K}3)$. In the remaining cases, if $f$ is $\langle f_1, f_2 \rangle$, then $g$ is either of a form already covered by cases above, or $g$ is $\hat{K}^i_A g'$, and we apply $(\hat{K}2)$. Finally, if $f$ is $\check{K}^i_A f'$, then $g$ is either of a form already covered by cases above, or $g$ is $[g_1, g_2]$, and we apply $(\check{K}2)$. ⊣

Note that we use only the equations $(\hat{K}1)$–$(\hat{K}3)$ and $(\check{K}1)$–$(\check{K}3)$ in this proof (which is taken over from [11], Section 3). We can then prove the following lemma for **L**.

**Invertibility Lemma for $\wedge$.** *Let $f\colon A_1 \wedge A_2 \vdash B$ be a Gentzen term. If for every $(x, y) \in Gf$ we have that $x$ is in $A_1$, then $f$ is equal to a Gentzen term of the form $\hat{K}^1_{A_2} f'$, and if for every $(x, y) \in Gf$ we have that $x$ is in $A_2$, then $f$ is equal to a Gentzen term of the form $\hat{K}^2_{A_1} f'$.*

*Proof.* By Composition Elimination for **L**, we can assume that $f$ is composition-free, and then we proceed by induction on the length of the target $B$ (or on the length of $f$). If $B$ is a letter, then $f$ must be equal in **L** to an arrow term of the form $\hat{K}^i_{A_{3-i}} f'$. The condition on $Gf$ dictates whether $i$ here is 1 or 2.

If $B$ is $B_1 \wedge B_2$ and $f$ is not of the form $\hat{K}^i_{A_{3-i}} f'$, then $f$ must be of the form $\langle f_1, f_2 \rangle$ (the condition on $Gf$ precludes that $f$ be an identity arrow term). We apply the induction hypothesis to $f_1\colon A_1 \wedge A_2 \vdash B_1$ and $f_2\colon A_1 \wedge A_2 \vdash B_2$, and use the equation $(\hat{K}5)$.

If $B$ is $B_1 \vee B_2$ and $f$ is not of the form $\hat{K}^i_{A_{3-i}} f'$, then $f$ must be of the form $\check{K}^j_{B_{3-j}} g$, for $j \in \{1, 2\}$. We apply the induction hypothesis to $g\colon A_1 \wedge A_2 \vdash B_i$, and use the following instance of the equation $(\hat{K}\check{K})$:

$$\check{K}^j_{B_{3-j}} \hat{K}^i_{A_{3-i}} g' = \hat{K}^i_{A_{3-i}} \check{K}^j_{B_{3-j}} g'. \qquad ⊣$$

We have a dual Invertibility Lemma for $\vee$. We can then prove the following result of [11, Section 4].

**Lattice Coherence.** *The functor $G$ from* **L** *to* Rel *is faithful.*

*Proof.* Suppose $f, g\colon A \vdash B$ are arrow terms of **L** and $Gf = Gg$. We proceed by induction on the sum of the lengths of $A$ and $B$ to show that $f = g$ in **L**. If $A$ and $B$ are both letters, then we conclude by Composition Elimination for **L** that an arrow term of $\mathbf{L}$ of the type $A \vdash B$ exists iff $A$ and $B$ are the same letter $p$, and we must have $f = g = \mathbf{1}_p$ in $\mathbf{L}$. Note that we do not need here the assumption $Gf = Gg$.

If $B$ is $B_1 \wedge B_2$, then for $i \in \{1,2\}$ we have that $\hat{k}^i_{B_1,B_2} \circ f$ and $\hat{k}^i_{B_1,B_2} \circ g$ are of type $A \vdash B_i$. We also have

$$G(\hat{k}^i_{B_1,B_2} \circ f) = G\hat{k}^i_{B_1,B_2} \circ Gf = G\hat{k}^i_{B_1,B_2} \circ Gg = G(\hat{k}^i_{B_1,B_2} \circ g),$$

whence, by the induction hypothesis, we have $\hat{k}^i_{B_1,B_2} \circ f = \hat{k}^i_{B_1,B_2} \circ g$ in $\mathbf{L}$. Then we infer

$$\langle \hat{k}^1_{B_1,B_2} \circ f, \hat{k}^2_{B_1,B_2} \circ f \rangle = \langle \hat{k}^1_{B_1,B_2} \circ g, \hat{k}^2_{B_1,B_2} \circ g \rangle,$$

from which $f = g$ follows with the help of the equations $(\hat{K}3)$ and $(\hat{K}4)$. We proceed analogously if $A$ is $A_1 \vee A_2$.

Suppose now that $A$ is $A_1 \wedge A_2$ or a letter, and $B$ is $B_1 \vee B_2$ or a letter, but $A$ and $B$ are not both letters. Then by Composition Elimination for $\mathbf{L}$ we have that $f$ is equal in $\mathbf{L}$ to an arrow term of $\mathbf{L}$ that is either of the form $f' \circ \hat{k}^i_{A_1,A_2}$ or of the form $\check{k}^i_{B_1,B_2} \circ f'$. Suppose $f = f' \circ \hat{k}^1_{A_1,A_2}$. Then for every $(x,y) \in Gf$ we have $x \in GA_1$.

By the Invertibility Lemma for $\wedge$, it follows that $g$ is equal in $\mathbf{L}$ to an arrow term of the form $g' \circ \hat{k}^1_{A_1,A_2}$. From $Gf = Gg$ we can infer easily that $Gf' = Gg'$, and so by the induction hypothesis $f' = g'$, and hence $f = g$.

We reason analogously when $f = f' \circ \hat{k}^2_{A_1,A_2}$. If $f = \check{k}^i_{B_1,B_2} \circ f'$, then again we reason analogously, applying the Invertibility Lemma for $\vee$. $\dashv$

This proof of Lattice Coherence is simpler than a proof given in [11]. In the course of that previous proof one has also coherence results for two auxiliary categories related to $\mathbf{L}$. We will need these categories later, but we do not need these coherence results. For the sake of completeness, however, we record them here too.

Let $\hat{\mathbf{L}}_\vee$ be the category defined as $\mathbf{L}$ with the difference that the primitive arrow terms $\check{w}$ and $\check{k}^i$ are excluded, as well as the equations involving them. The Gentzen formulation of $\hat{\mathbf{L}}_\vee$ is obtained by taking the operation $\vee$ on arrow terms instead of the operations $[\ ,\ ]$ and $\check{K}^i$.

The category $\check{\mathbf{L}}_\wedge$ is isomorphic to $\hat{\mathbf{L}}_\vee^{op}$. In $\check{\mathbf{L}}_\wedge$, the $\wedge$ and $\vee$ of $\hat{\mathbf{L}}_\vee$ are interchanged.

One can easily prove Composition Elimination for $\hat{\mathbf{L}}_\vee$ (and hence also for $\check{\mathbf{L}}_\wedge$) by abbreviating the proof of Composition Elimination for $\mathbf{L}$ above. For $\hat{\mathbf{L}}_\vee$ we do not have the cases where $f$ is $[f_1, f_2]$ or $\check{K}^i_A f'$, but $f$ can be $f_1 \vee f_2$. Then, if $g$ is not of a form already covered by the proof above, it must be $g_1 \vee g_2$, and we apply the bifunctorial equation $(\vee 2)$.

A composition-free arrow term of $\hat{\mathbf{L}}_\vee$ may be reduced to a unique normal form, which can then be used to demonstrate coherence for $\hat{\mathbf{L}}_\vee$, i.e. the fact that the functor $G$ from $\hat{\mathbf{L}}_\vee$ to *Rel* is faithful (see [11, Section 4]).

## 6. Coherence for sesquicartesian categories

We define now the category $\mathbf{L}_{\top,\bot}$, whose definition extends the definition of $\mathbf{L}$ with the terminal object $\top$ and the initial object $\bot$, i.e. nullary product and coproduct. The objects of this category are the formulae of the propositional language $\mathcal{L}_{\top,\bot}$, generated out of a set of infinitely many propositional letters with the binary connectives $\wedge$ and $\vee$, and the nullary connectives, i.e. propositional constants, $\top$ and $\bot$.

The arrow terms of $\mathbf{L}_{\top,\bot}$ are defined as the arrow terms of $\mathbf{L}$ save that for every object $A$ we have the additional primitive arrow terms

$$\hat{\kappa}_A : A \vdash \top, \qquad \check{\kappa}_A : \bot \vdash A,$$

and for all arrow terms $f : A \vdash \top$ and $g : \bot \vdash A$ we have the additional axiomatic equations

$$(\hat{\kappa}) \quad \hat{\kappa}_A = f, \qquad (\check{\kappa}) \quad \check{\kappa}_A = g,$$

$$(\hat{k}\bot) \quad \hat{k}^1_{\bot,\bot} = \hat{k}^2_{\bot,\bot}, \qquad (\check{k}\top) \quad \check{k}^1_{\top,\top} = \check{k}^2_{\top,\top}.$$

It is easy to see that with the help of the last two equations we obtain that the pairs

$$\begin{aligned} \hat{k}^1_{\bot,\bot} &= \hat{k}^2_{\bot,\bot} : \bot \wedge \bot \vdash \bot \quad \text{and} \quad \check{\kappa}_{\bot\wedge\bot} = \hat{w}_\bot : \bot \vdash \bot \wedge \bot, \\ \check{k}^1_{\top,\top} &= \check{k}^2_{\top,\top} : \top \vdash \top \vee \top \quad \text{and} \quad \hat{\kappa}_{\top\vee\top} = \check{w}_\top : \top \vee \top \vdash \top \end{aligned}$$

are inverses of each other. This shows that every letterless formula of $\mathcal{L}_{\top,\bot}$ is isomorphic in $\mathbf{L}_{\top,\bot}$ either to $\top$ or to $\bot$.

The kind of category for which $\mathbf{L}_{\top,\bot}$ is the one freely generated out of the set of propositional letters we call *dicartesian* category. The objects of a dicartesian category that is a partial order make a lattice with top and bottom.

By omitting the equations $(\hat{k}\bot)$ and $(\check{k}\top)$ in the definition of $\mathbf{L}_{\top,\bot}$ we would obtain the *bicartesian* category freely generated by the set of propositional letters (cf. [22, Section I.8]). Dicartesian categories were considered under the name *coherent bicartesian* categories in the printed version of [10].

We previously believed wrongly that we have proved coherence for dicartesian, alias coherent bicartesian, categories. Lemma 5.1 of the printed version of [10] is however not correct. We prove here only a restricted coherence result for dicartesian categories. A study of equality of arrows in bicartesian categories may be found in [5].

Suppose that in the definition of $\mathbf{L}_{\top,\bot}$ we omit one of $\top$ and $\bot$ from the language, and we omit all the arrow terms and equations involving the omitted nullary connective. When we omit $\top$, we obtain the category $\mathbf{L}_\bot$, and when we omit $\bot$, we obtain the category $\mathbf{L}_\top$. It is clear that $\mathbf{L}_\bot$ is isomorphic to $\mathbf{L}^{op}_\top$. In the printed version of [10] the categories for which $\mathbf{L}_\bot$ is the one freely generated by the set of propositional letters were called *coherent sesquicartesian* categories. We call them now just *sesquicartesian* categories.

The category *Set*, whose objects are sets and whose arrows are functions, with cartesian product $\times$ as $\wedge$, disjoint union $+$ as $\vee$, a singleton set $\{*\}$ as $\top$ and the empty set $\emptyset$ as $\bot$, is a bicartesian category, but not a dicartesian category. It is, however, a sesquicartesian category in the $\mathbf{L}_\bot$ sense, but not in the $\mathbf{L}_\top$ sense. This is because in *Set* we have that $\emptyset \times \emptyset$ is equal to $\emptyset$, but $\{*\} + \{*\}$ is not isomorphic to $\{*\}$.

To define the functor $G$ from $\mathbf{L}_{\top,\bot}$ to *Rel* we assume that the objects of *Rel* are the formulae of $\mathcal{L}_{\top,\bot}$. Everything else in the definition of *Rel* remains unchanged; in particular, the arrows are sets of ordered pairs of occurrences of propositional *letters* (no propositional constant is involved). In the definition of the functor $G$ we stipulate that for $G\hat{\kappa}_A$ and $G\check{\kappa}_A$ we have the empty set of ordered pairs. This serves also for the definition of the functors $G$ from $\mathbf{L}_\bot$ and $\mathbf{L}_\top$ to *Rel*.

We can establish unrestricted coherence for sesquicartesian categories, with a proof taken over from the revised version of [10], which we will present below. (This proof differs from the proof in the printed version of [10], which relied also on Lemma 5.1, and is not correct.) It is obtained by enlarging the proof of Lattice Coherence.

The Gentzen formulation of $\mathbf{L}_{\top,\bot}$ is obtained like that of $\mathbf{L}$ save that we have in addition the primitive Gentzen terms $\hat{\kappa}_A : A \vdash \top$ and $\check{\kappa}_A : \bot \vdash A$. For Gentzen terms we have as additional equations, besides $(\hat{\kappa})$ and $(\check{\kappa})$, the following equations:

$$(\hat{K}\bot) \qquad \hat{K}^1_\bot \mathbf{1}_\bot = \hat{K}^2_\bot \mathbf{1}_\bot,$$

$$(\check{K}\top) \qquad \check{K}^1_\top \mathbf{1}_\top = \check{K}^2_\top \mathbf{1}_\top,$$

which amount to $(\hat{k}\bot)$ and $(\check{k}\top)$.

We can prove Composition Elimination for $\mathbf{L}_{\top,\bot}$ by enlarging the proof for $\mathbf{L}$. We have as new cases first those where $f$ is $\check{\kappa}_A$ or $g$ is $\hat{\kappa}_A$, which are taken care of by the equations $(\check{\kappa})$ and $(\hat{\kappa})$. The following case remains. If $f$ is $\hat{\kappa}_A$, then $g$ is of a form covered by cases already dealt with. Note that we do not need the equations $(\hat{K}\bot)$ and $(\check{K}\top)$ for this proof (so that we have also Composition Elimination for the free bicartesian category).

Let the category $\hat{\mathbf{L}}_{\vee,\top,\bot}$ be defined like the category $\hat{\mathbf{L}}_\vee$ save that it involves also $\hat{\kappa}$ and the equations $(\hat{\kappa})$ and $(\hat{k}\bot)$, and let the category $\check{\mathbf{L}}_{\wedge,\top,\bot}$ be defined like the category $\check{\mathbf{L}}_\wedge$ save that it involves also $\check{\kappa}$ and the equations $(\check{\kappa})$ and $(\check{k}\top)$. Composition Elimination is provable for $\hat{\mathbf{L}}_{\vee,\top,\bot}$ and $\check{\mathbf{L}}_{\wedge,\top,\bot}$ by abbreviating the proof of Composition Elimination for $\mathbf{L}_{\top,\bot}$, in the same way as we abbreviated the proof of Composition Elimination for $\mathbf{L}$ in order to obtain Composition Elimination for $\hat{\mathbf{L}}_\vee$.

An arrow term of $\mathbf{L}_{\top,\bot}$ is in *standard form* when it is of the form $g \circ f$ for $f$ an arrow term $\hat{\mathbf{L}}_{\vee,\top,\bot}$ and $g$ an arrow term of $\check{\mathbf{L}}_{\wedge,\top,\bot}$. We can then prove the following.

**Standard-Form Lemma.** *Every arrow term of* $\mathbf{L}_{\top,\bot}$ *is equal in* $\mathbf{L}_{\top,\bot}$ *to an arrow term in standard form.*

*Proof.* By categorial and bifunctorial equations, we may assume that we deal with a factorized arrow term $f$ none of whose factors is a complex identity (i.e., $f$ is a big composition of composition-free arrow terms none of which is equal to an identity arrow; see [14, Sections 2.6–7], for precise definitions of these notions) and every factor of $f$ is either an arrow term of $\hat{\mathbf{L}}_{\vee,\top,\bot}$, and then we call it a $\wedge$-*factor*, or an arrow term of $\check{\mathbf{L}}_{\wedge,\top,\bot}$, when we call it a $\vee$-*factor*.

Suppose $f : B \vdash C$ is a $\wedge$-factor and $g : A \vdash B$ is a $\vee$-factor. We show by induction on the length of $f \circ g$ that in $\mathbf{L}_{\top,\bot}$

$$(*) \quad f \circ g = g' \circ f' \quad \text{or} \quad f \circ g = f' \quad \text{or} \quad f \circ g = g'$$

for $f'$ a $\wedge$-factor and $g'$ a $\vee$-factor.

We will consider various cases for $f$. In all such cases, if $g$ is $\check{w}_B$, then we use $(\check{w}\ nat)$. If $f$ is $\hat{w}_B$, then we use $(\hat{w}\ nat)$. If $f$ is $\hat{k}^i_{D,E}$ and $g$ is $g_1 \wedge g_2$, then we use $(\hat{k}^i\ nat)$. If $f$ is $f_1 \wedge f_2$ and $g$ is $g_1 \wedge g_2$, then we use bifunctorial and categorial equations and the induction hypothesis.

If $f$ is $f_1 \vee f_2$, then we have the following cases. If $g$ is $\check{k}^i_{B_1,B_2}$, then we use $(\check{k}^i\ nat)$. If $g$ is $g_1 \vee g_2$, then we use bifunctorial and categorial equations and the induction hypothesis.

Finally, cases where $f$ is $\hat{\kappa}_B$ or $g$ is $\check{\kappa}_B$ are taken care of by the equations $(\hat{\kappa})$ and $(\check{\kappa})$. This proves $(*)$, and it is clear that $(*)$ is sufficient to prove the lemma. ⊣

We can also prove Composition Elimination and an analogue of the Standard-Form Lemma for $\mathbf{L}_{\bot}$. Next we have the following lemmata for $\mathbf{L}_{\top,\bot}$ and $\mathbf{L}_{\bot}$.

**Lemma 1.** *If for $f, g\colon A \vdash B$ either $A$ or $B$ is isomorphic to $\top$ or $\bot$, then $f = g$.*

*Proof.* If $A$ is isomorphic to $\bot$ or $B$ is isomorphic to $\top$, then the matter is trivial. Suppose $i\colon B \vdash \bot$ is an isomorphism. Then from

$$\hat{k}^1_{\bot,\bot} \circ \langle i \circ f, i \circ g\rangle = \hat{k}^2_{\bot,\bot} \circ \langle i \circ f, i \circ g\rangle$$

we obtain $i \circ f = i \circ g$, which yields $f = g$. We proceed analogously if $A$ is isomorphic to $\top$. ⊣

**Lemma 2.** *If for $f, g\colon A \vdash B$ we have $Gf = Gg = \emptyset$, then $f = g$.*

*Proof.* This proof depends on the Standard-Form Lemma above. We write down $f$ in the standard form $f_2 \circ f_1$ for $f_1 : A \vdash C$ and $g$ in the standard form $g_2 \circ g_1$ for $g_1 : A \vdash D$. Since $\check{k}^i$ and $\check{\kappa}$ do not occur in $f_1$, for every occurrence $z$ of a propositional letter in $C$ we have an occurrence $x$ of that propositional letter in $A$ such that $(x, z) \in Gf_1$, and since $\hat{k}^i$ and $\hat{\kappa}$ do not occur in $f_2$, for every occurrence $z$ of a propositional letter in $C$ we have an occurrence $y$ of that propositional letter in $B$ such that $(z, y) \in Gf_2$. So if $C$ were not letterless, then $Gf$ would not be empty. We conclude analogously that $D$, as well as $C$, is a letterless formula.

If both $C$ and $D$ are isomorphic to $\top$ or $\bot$, then we have an isomorphism $i : C \vdash D$, and $f = f_2 \circ i^{-1} \circ i \circ f_1$. By Lemma 1, we have $i \circ f_1 = g_1$ and $f_2 \circ i^{-1} = g_2$, from which $f = g$ follows. If $i\colon C \vdash \bot$ and $j : \top \vdash D$ are isomorphisms, then by Lemma 1 we have

$$f_2 \circ f_1 = g_2 \circ j \circ \hat{\kappa}_{\bot} \circ i \circ f_1 = g_2 \circ g_1,$$

and so $f = g$. (Note that $\hat{\kappa}_{\bot} = \check{\kappa}_{\top}$.) ⊣

We can then prove the following.

**Sesquicartesian Coherence.** *The functor* $G$ *from* $\mathbf{L}_\bot$ *to* Rel *is faithful.*

*Proof.* We have Lemma 2 for the case when $Gf = Gg = \emptyset$. When $Gf = Gg \neq \emptyset$, we proceed as in the proof of Lattice Coherence, appealing if need there is to Lemma 2, until we reach the case when $A$ is $A_1 \wedge A_2$ or a letter, and $B$ is $B_1 \vee B_2$ or a letter, but $A$ and $B$ are not both letters. In that case, by Composition Elimination, the arrow term $f$ is equal in $\mathbf{L}_\bot$ either to an arrow term of the form $f' \circ \hat{k}^i_{A_1,A_2}$, or to an arrow term of the form $\check{k}^i_{B_1,B_2} \circ f'$. Suppose $f = f' \circ \hat{k}^1_{A_1,A_2}$. Then for every $(x,y) \in Gf$ we have that $x$ is in $A_1$. (We reason analogously when $f = f' \circ \hat{k}^2_{A_1,A_2}$.)

By Composition Elimination too, $g$ is equal in $\mathbf{L}_\bot$ either to an arrow term of the form $g' \circ \hat{k}^i_{A_1,A_2}$, or to an arrow term of the form $\check{k}^i_{B_1,B_2} \circ g'$. In the first case we must have $g = g' \circ \hat{k}^1_{A_1,A_2}$, because $Gg = G(f' \circ \hat{k}^1_{A_1,A_2}) \neq \emptyset$, and then we apply the induction hypothesis to derive $f' = g'$ from $Gf' = Gg'$. Hence $f = g$ in $\mathbf{L}_\bot$.

Suppose $g = \check{k}^1_{B_1,B_2} \circ g'$. (We reason analogously when $g = \check{k}^2_{B_1,B_2} \circ g'$.) Let $f'' \colon A_1 \vdash B_1 \vee B_2''$ be the substitution instance of $f' \colon A_1 \vdash B_1 \vee B_2$ obtained by replacing every occurrence of propositional letter in $B_2$ by $\bot$. There is an isomorphism $i \colon B_2'' \vdash \bot$, and $f''$ exists because in $Gf$, which is equal to $G(\check{k}^1_{B_1,B_2} \circ g')$, there is no pair $(x,y)$ with $y$ in $B_2$. So we have an arrow $f''' \colon A_1 \vdash B_1$, which we define as $[\mathbf{1}_{B_1}, \check{\kappa}_{B_1}] \circ (\mathbf{1}_{B_1} \vee i) \circ f''$. It is easy to verify that $G(\check{k}^1_{B_1,B_2} \circ f''') = Gf'$, and that $G(f''' \circ \hat{k}^1_{A_1,A_2}) = Gg'$. By the induction hypothesis, we obtain $\check{k}^1_{B_1,B_2} \circ f''' = f'$ and $f''' \circ \hat{k}^1_{A_1,A_2} = g'$, from which we derive $f = g$. We reason analogously when $f = \check{k}^i_{B_1,B_2} \circ f'$. ⊣

From Sesquicartesian Coherence we infer coherence for $\mathbf{L}_\top$, which is isomorphic to $\mathbf{L}_\bot^{op}$.

## 7. Restricted coherence for dicartesian categories

For dicartesian categories we can prove easily a simple restricted coherence result, which was sufficient for the needs of [14]. A more general, but still restricted, coherence result with respect to *Rel*, falling short of full coherence, may be found in the revised version of [10, Section 7]. We present first the simple restricted coherence result, and will deal with the more general restricted coherence result later on.

We define inductively formulae of $\mathcal{L}_{\top,\bot}$ in *disjunctive normal form* (*dnf*): every $\vee$-free formula is in *dnf*, and if $A$ and $B$ are both in *dnf*, then $A \vee B$ is in *dnf*. We define dually formulae of $\mathcal{L}_{\top,\bot}$ in *conjunctive normal form* (*cnf*): every $\wedge$-free formula is in *cnf*, and if $A$ and $B$ are both in *cnf*, then $A \wedge B$ is in *cnf*.

**Restricted Dicartesian Coherence.** *Let* $f, g \colon A \vdash B$ *be arrow terms of* $\mathbf{L}_{\top,\bot}$ *such that* $A$ *is in* dnf *and* $B$ *in* cnf. *If* $Gf = Gg$, *then* $f = g$ *in* $\mathbf{L}_{\top,\bot}$.

*Proof.* If $Gf = Gg = \emptyset$, then we apply Lemma 2. If $Gf = Gg \neq \emptyset$, then we proceed as in the proof of Lattice Coherence, by induction on the sum of the lengths of $A$ and $B$, appealing if need there is to Lemma 2, until we reach the case when $A$ is

$A_1 \wedge A_2$ or a letter, and $B$ is $B_1 \vee B_2$ or a letter, but $A$ and $B$ are not both letters. In that case there is no occurrence of $\vee$ in $A$ and no occurrence of $\wedge$ in $B$. We then rely on the composition-free form of $f$ and $g$ in $\mathbf{L}_{\top,\bot}$ and on the equation $(\hat{K}\check{K})$. ⊣

To improve upon this result we need the following lemma for $\mathbf{L}_{\top,\bot}$, and the definitions that follow. This lemma is analogous up to a point to the Invertibility Lemma for $\vee$.

**Lemma 3.** *Let $f : A \vdash B_1 \vee B_2$ be a Gentzen term such that $Gf \neq \emptyset$ and $\vee$ does not occur in $A$. If for every $(x, y) \in Gf$ we have that $y$ is in $B_1$, then there is a Gentzen term $g : A \vdash B_1$ such that $Gf = G\check{K}^1_{B_2} g$.*

*Proof.* We proceed by induction on the length of $A$. Suppose $f$ is a composition-free Gentzen term. If $A$ is a propositional letter, then by the assumption on $Gf$ we have that $f$ is of the form $\check{K}^1_{B_2} f'$, and we can take that $g$ is $f'$.

If $A$ is not a propositional letter and $f$ is not of the form $\check{K}^1_{B_2} f'$ (by the assumption on $Gf$, the Gentzen term $f$ cannot be of the form $\check{K}^2_{B_1} f'$), then, since $\vee$ does not occur in $A$, we have that $f$ is of the form $\hat{K}^i_{A''} f'$ for $f' : A' \vdash B_1 \vee B_2$. Note that $Gf' \neq \emptyset$ and $\vee$ does not occur in $A'$. Since for every $(x, y)$ in $Gf'$ we have that $y$ is in $B_1$, we may apply the induction hypothesis to $f'$ and obtain $g' : A' \vdash B_1$ such that $Gf' = G\check{K}^1_{B_2} g'$. By relying on the equation $(\hat{K}\check{K})$, we can take that $g$ is $\hat{K}^i_{A''} g'$. ⊣

A formula $C$ of $\mathcal{L}_{\top,\bot}$ is called a *contradiction* when there is in $\mathbf{L}_{\top,\bot}$ an arrow of the type $C \vdash \bot$. For every formula that is not a contradiction there is a substitution instance isomorphic to $\top$. Suppose $C$ is not a contradiction, and let $C^\top$ be obtained from $C$ by substituting $\top$ for every propositional letter. If $C^\top$ were not isomorphic to $\top$, then since every letterless formula of $\mathcal{L}_{\top,\bot}$ is isomorphic in $\mathbf{L}_{\top,\bot}$ either to $\top$ or to $\bot$, we would have an isomorphism $i : C^\top \vdash \bot$. Since there is obviously an arrow $u : C \vdash C^\top$ formed by using $\hat{\kappa}_p$, we would have $i \circ u : C \vdash \bot$, and $C$ would be a contradiction.

A formula $C$ of $\mathcal{L}_{\top,\bot}$ is called a *tautology* when there is in $\mathbf{L}_{\top,\bot}$ an arrow of the type $\top \vdash C$. For every formula that is not a tautology there is a substitution instance isomorphic to $\bot$. (This is shown analogously to what we had in the preceding paragraph.)

A formula of $\mathcal{L}_{\top,\bot}$ is called $\bot$-*normal* when for every subformula $D \wedge C$ or $C \wedge D$ of it with $C$ a contradiction, there is no occurrence of $\vee$ in $D$. A formula of $\mathcal{L}_{\top,\bot}$ is called $\top$-*normal* when for every subformula $D \vee C$ or $C \vee D$ of it with $C$ a tautology, there is no occurrence of $\wedge$ in $D$.

We can now formulate our second partial coherence result for dicartesian categories.

**Restricted Dicartesian Coherence II.** *If $f, g : A \vdash B$ are terms of $\mathbf{L}_{\top,\bot}$ such that $Gf = Gg$ and either $A$ is $\bot$-normal or $B$ is $\top$-normal, then $f = g$ in $\mathbf{L}_{\top,\bot}$.*

*Proof.* Suppose $A$ is $\bot$-normal. Lemma 2 covers the case when $Gf = Gg = \emptyset$. So we assume $Gf = Gg \neq \emptyset$, and proceed as in the proof of Sesquicartesian Coherence by induction on the sum of the lengths of $A$ and $B$. The basis of this induction and the cases when $A$ is of the form $A_1 \vee A_2$ or $B$ is of the form $B_1 \wedge B_2$ are settled as in the proof of Sesquicartesian Coherence.

Suppose $A$ is $A_1 \wedge A_2$ or a propositional letter and $B$ is $B_1 \vee B_2$ or a propositional letter, but $A$ and $B$ are not both propositional letters. (The cases when $A$ or $B$ is a constant object are excluded by the assumption that $Gf = Gg \neq \emptyset$.) We proceed then as in the proof of Sesquicartesian Coherence until we reach the case when $f = f' \circ \hat{k}^1_{A_1,A_2}$ and $g = \check{k}^1_{B_1,B_2} \circ g'$.

Suppose $A_2$ is not a contradiction. Then there is an instance $A_2^\top$ of $A_2$ and an isomorphism $i : \top \vdash A_2^\top$. (To obtain $A_2^\top$ we substitute $\top$ for every letter in $A_2$.) Let $g'' : A_1 \wedge A_2^\top \vdash B_1$ be the substitution instance of $g' : A_1 \wedge A_2 \vdash B_1$ obtained by replacing every occurrence of propositional letter in $A_2$ by $\top$. Such a term exists because in $Gg$, which is equal to $G(f' \circ \hat{k}^1_{A_1,A_2})$, there is no pair $(x, y)$ with $x$ in $A_2$.

So we have an arrow $g''' = g'' \circ (\mathbf{1}_{A_1} \wedge i) \circ \langle \mathbf{1}_{A_1}, \hat{\kappa}_{A_1} \rangle : A_1 \vdash B_1$. It is easy to verify that $G(\check{k}^1_{B_1,B_2} \circ g''') = Gf'$ and that $G(g''' \circ \hat{k}^1_{A_1,A_2}) = Gg'$. By the induction hypothesis we obtain $\check{k}^1_{B_1,B_2} \circ g''' = f'$ and $g''' \circ \hat{k}^1_{A_1,A_2} = g'$, from which we derive $f = g$.

Suppose $A_2$ is a contradiction. Then by the assumption that $A$ is $\bot$-normal we have that $\vee$ does not occur in $A_1$. We may apply Lemma 3 to $f' : A_1 \vdash B_1 \vee B_2$ to obtain $f''' : A_1 \vdash B_1$ such that $Gf' = G(\check{k}^1_{B_1,B_2} \circ f''')$. It is easy to verify that then $Gg' = G(f''' \circ \hat{k}^1_{A_1,A_2})$, and we may proceed as in the proof of Sesquicartesian Coherence.

We proceed analogously when $B$ is $\top$-normal, relying on a lemma dual to Lemma 3. $\dashv$

Consider the following definitions:

$$A^0_\bot = A \wedge \bot, \qquad A^{n+1}_\bot = (A^n_\bot \vee \top) \wedge \bot,$$
$$f^0_\bot = f \wedge \mathbf{1}_\bot, \qquad f^{n+1}_\bot = (f^n_\bot \vee \mathbf{1}_\top) \wedge \mathbf{1}_\bot,$$
$$A^0_\top = A \vee \top, \qquad A^{n+1}_\top = (A^n_\top \wedge \bot) \vee \top,$$
$$f^0_\top = f \vee \mathbf{1}_\top, \qquad f^{n+1}_\top = (f^n_\top \wedge \mathbf{1}_\bot) \vee \mathbf{1}_\top.$$

Then for $f^n$ being

$$(\check{k}^1_{A,\top} \wedge \mathbf{1}_\bot)^n_\top \circ \hat{k}^1_{(A\wedge\bot)^n_\top,\bot} : A^{n+1}_\bot \vdash A^{n+1}_\top$$

and $g^n$ being

$$\check{k}^1_{(A\vee\top)^n_\bot,\top} \circ (\hat{k}^1_{A,\bot} \vee \mathbf{1}_\top)^n_\bot : A^{n+1}_\bot \vdash A^{n+1}_\top$$

we have $Gf^n = Gg^n$, but we suppose that $f^n = g^n$ does not hold in $\mathbf{L}_{\top,\bot}$. The equation $f^0 = g^0$ is

$$((\check{k}^1_{A,\top} \wedge \mathbf{1}_\bot) \vee \mathbf{1}_\top) \circ \hat{k}^1_{(A\wedge\bot)\vee\top,\bot} = \check{k}^1_{(A\vee\top)\wedge\bot,\top} \circ (\hat{k}^1_{A,\bot} \vee \mathbf{1}_\top) \wedge \mathbf{1}_\bot :$$
$$((A \wedge \bot) \vee \top) \wedge \bot \vdash ((A \vee \top) \wedge \bot) \vee \top.$$

Note that $A_{\bot}^{n+1}$ is not $\bot$-normal, and $A_{\top}^{n+1}$ is not $\top$-normal.

We don't know whether it is sufficient to add to $\mathbf{L}_{\top,\bot}$ the equations $f^n = g^n$ for every $n \geqslant 0$ in order to obtain full coherence for the resulting category.

As a corollary of Restricted Dicartesian Coherence II, we obtain that if $f, g : A \vdash B$ are terms of $\mathbf{L}_{\top,\bot}$ such that $Gf = Gg$, while $A$ and $B$ are isomorphic either to formulae of $\mathcal{L}$ (i.e. to formulae in which $\top$ and $\bot$ do not occur) or to letterless formulae, then $f = g$ in $\mathbf{L}_{\top,\bot}$. This corollary is analogous to the restricted coherence result for symmetric monoidal closed categories of Kelly and Mac Lane in [19] (see [15, Section 3.1]).

## 8. Maximality

A syntactically built category such as $\mathbf{L}$ and $\mathbf{L}_{\top,\bot}$ is called *maximal* when adding any new axiomatic equation between arrow terms of this category yields a category that is a preorder. The new axiomatic equation is supposed to be closed under substitution for propositional letters, as the equations of $\mathbf{L}$ and $\mathbf{L}_{\top,\bot}$ were. (This notion of maximality for syntactical categories is defined more precisely in [14, Section 9.3].) Maximality is an interesting property when the initial category, like $\mathbf{L}$ and $\mathbf{L}_{\top,\bot}$ here, is not itself a preorder. We will deal in subsequent sections with maximality for $\mathbf{L}$ and $\mathbf{L}_{\top,\bot}$.

The maximality property above is analogous to the property of usual formulations of the classical propositional calculus called *Post completeness*. That this calculus is Post complete means that if we add to it any new axiom-schema in the language of the calculus, then we can prove every formula. An analogue of Böhm's Theorem in the typed lambda calculus implies, similarly, that the typed lambda calculus cannot be extended without falling into triviality, i.e. without every equation (between terms of the same type) becoming derivable (see [26], [8] and references therein; see [1, Section 10.4], for Böhm's Theorem in the untyped lambda calculus).

Let us now consider several examples of common algebraic structures with analogous maximality properties. First, we have that semilattices are maximal in the following sense.

Let $a$ and $b$ be terms made exclusively of variables and of a binary operation $\cdot$, which we interpret as meet or join. That the equation $a = b$ holds in a semilattice $S$ means that *every* instance of $a = b$ obtained by substituting names of elements of $S$ for variables holds in $S$. Suppose $a = b$ does not hold in a free semilattice $S_F$ (so it is not the case that $a = b$ holds in every semilattice). Hence there must be an instance of $a = b$ obtained by substituting names of elements of $S_F$ for variables such that this instance does not hold in $S_F$. It is easy to conclude that in $a = b$ there must be at least two variables, and that $S_F$ must have at least two free generators. Then every semilattice in which $a = b$ holds is trivial—namely, it has a single element.

Here is a short proof of that. If $a = b$ does not hold in $S_F$, then there must be a variable $x$ in one of $a$ and $b$ that is not in the other. Then from $a = b$, by substituting $y$ for every variable in $a$ and $b$ different from $x$, and by applying the semilattice equations, we infer either $x = y$ or $x \cdot y = y$. If we have $x = y$, we are done, and, if we have $x \cdot y = y$, then we have also $y \cdot x = x$, and hence $x = y$.

Semilattices with unit, distributive lattices, distributive lattices with top and bottom, and Boolean algebras are maximal in the same sense. The equations $a = b$ in question are equations between terms made exclusively of variables and the operations of the kind of algebra we envisage: semilattices with unit, distributive lattices, etc. That such an equation holds in a particular structure means, as above, that every substitution instance of it holds. However, the number of variables in $a = b$ and the number of generators of the free structure mentioned need not always be at least two.

If we deal with semilattices with unit $\mathbf{1}$, then $a = b$ must have at least one variable, and the free semilattice with unit must have at least one free generator. We substitute $\mathbf{1}$ for every variable in $a$ and $b$ different from $x$ in order to obtain $x = \mathbf{1}$, and hence triviality. So semilattices with unit are maximal in the same sense.

The same sort of maximality can be proven for distributive lattices, whose operations are $\wedge$ and $\vee$, which we call conjunction and disjunction, respectively. Then every term made of $\wedge$, $\vee$ and variables is equal to a term in disjunctive normal form (i.e. a multiple disjunction of multiple conjunctions of variables; see the preceding section for a precise definition), and to a term in conjunctive normal form (i.e. a multiple conjunction of multiple disjunctions of variables; see the preceding section). These normal forms are not unique. If $a = b$, in which we must have at least two variables, does not hold in a free distributive lattice $D_F$ with at least two free generators, then either $a \leqslant b$ or $b \leqslant a$ does not hold in $D_F$. Suppose $a \leqslant b$ does not hold in $D_F$. Let $a'$ be a disjunctive normal form of $a$, and let $b'$ be a conjunctive normal form of $b$. So $a' \leqslant b'$ does not hold in $D_F$. From that we infer that for a disjunct $a''$ of $a'$ and for a conjunct $b''$ of $b'$ we do not have $a'' \leqslant b''$ in $D_F$. This means that there is no variable in common in $a''$ and $b''$; otherwise, the conjunction of variables $a''$ would be lesser than or equal in $D_F$ to the disjunction of variables $b''$. If in a distributive lattice $a = b$ holds, then $a'' \leqslant b''$ holds too, and hence, by substitution, we obtain $x \leqslant y$. So $x = y$.

For distributive lattices with top $\top$ and bottom $\bot$, we proceed analogously via disjunctive and conjunctive normal form. Here $a = b$ may be even without variables, and the free structure may have even an empty set of free generators. The additional cases to consider are when in $a'' \leqslant b''$ we have that $a''$ is $\top$ and $b''$ is $\bot$. In any case, we obtain $\top \leqslant \bot$, and hence our structure is trivial.

The same sort of maximality can be proven for Boolean algebras, i.e. complemented distributive lattices. Boolean algebras must have top and bottom. In a disjunctive normal form now the disjuncts are conjunctions of variables $x$ or terms $\bar{x}$, where $\bar{\ }$ is complementation, or the disjunctive normal form is just $\top$ or $\bot$; analogously for conjunctive normal forms. Then we proceed as for distributive lattices with an equation $a = b$ that may be even without variables, until we reach that $a'' \leqslant b''$, which does not hold in a free Boolean algebra $B_F$, whose set of free generators may be even empty, holds in our Boolean algebra. If $x$ is a conjunct of $a''$, then in $b''$ we cannot have a disjunct $x$; but we may have a disjunct $\bar{x}$. The same holds for the conjuncts $\bar{x}$ of $a''$. It is excluded that both $x$ and $\bar{x}$ are conjuncts of $a''$, or disjuncts of $b''$; otherwise, $a'' \leqslant b''$ would hold in $B_F$. Then for every conjunct $x$ in $a''$ and every disjunct $\bar{y}$ in $b''$ we substitute $\top$ for $x$ and $y$, and for every other variable we substitute $\bot$. In any case, we obtain $\top \leqslant \bot$, and hence our Boolean algebra is trivial. This is essentially the proof of Post completeness for the classical propositional calculus, due to Bernays and Hilbert (see [28, Section 2.4], and [16, Section I.13]), from which we can infer the ordinary completeness of this calculus with respect to valuations in the two-element Boolean algebra—namely, with respect to truth tables—and also completeness with respect to any nontrivial model.

As examples of common algebraic structures that are not maximal in the sense above, we have semigroups, commutative semigroups, lattices, and many others. What is maximal for semilattices and is not maximal for lattices is the equational theory of the structures in question. The equational theory of semilattices cannot be extended without falling into triviality, while the equational theory of lattices can be extended with the distributive law, for example.

The notions of maximality envisaged in this section were extreme (or should we say "maximal"), in the sense that we envisaged collapsing only into preorder. For semilattices, distributive lattices, etc., this is also preorder for a one-object category. We may, however, envisage relativizing our notion of maximality by replacing preorder with a weaker property, such that structures possessing it are trivial, but not so trivial (cf. [7, Section 4.11]). We will encounter maximality in such a relative sense in the last section.

As an example of relative maximality in a common algebraic structure we can take symmetric groups. Consider the standard axioms for the symmetric group $\mathcal{S}_n$, where $n \geqslant 2$, with the generators $s_i$, for $i \in \{1, \ldots, n-1\}$, corresponding to transpositions of immediate neighbours (see [6, Section 6.2]). If to $\mathcal{S}_n$ for $n \geqslant 5$ we add an equation $a = \mathbf{1}$ where $a$ is built exclusively of the generators $s_i$ of $\mathcal{S}_n$ with composition, and $a = \mathbf{1}$ does not hold in $\mathcal{S}_n$, then we can derive $s_i = s_j$. This does not mean that the resulting structure will be a one-element structure, i.e. the trivial one-element group. It will be such if $a$ is an odd permutation, and if $a$ is an even permutation, then we will obtain a two-element structure, which is $\mathcal{S}_2$. This can be inferred from facts about the normal subgroups of $\mathcal{S}_n$. Simple groups are maximal in the nonrelative sense, envisaged above for semilattices.

## 9. Maximality of lattice categories

We will show in this section that $\mathbf{L}$ is maximal in the sense specified at the beginning of the preceding section; namely, in the interesting way. (We take over this result from [11, Section 5], and [14, Section 9.5].)

Suppose $A$ and $B$ are formulae of $\mathcal{L}$ in which only $p$ occurs as a letter. If for some arrow terms $f_1, f_2 : A \vdash B$ of $\mathbf{L}$ we have $Gf_1 \neq Gf_2$, then for some $x$ in $A$ and some $y$ in $B$ we have $(x, y) \in Gf_1$ and $(x, y) \notin Gf_2$, or vice versa. Suppose $(x, y) \in Gf_1$ and $(x, y) \notin Gf_2$.

For every subformula $C$ of $A$ and every formula $D$ let $A^C_D$ be the formula obtained from $A$ by replacing the particular occurrence of the formula $C$ in $A$ by $D$. It can be shown that for every subformula $A_1 \vee A_2$ of $A$ we have an arrow term $h\colon A^{A_1\vee A_2}_{A_j} \vdash A$ of $\mathbf{L}$, built by using $\check{k}^j_{A_1,A_2}$, such that there is an $x'$ in $A^{A_1\vee A_2}_{A_j}$ for which $(x', x) \in Gh$. Hence, for such an $h$, we have $(x', y) \in G(f_1 \circ h)$ and $(x', y) \notin G(f_2 \circ h)$.

We compose $f_i$ repeatedly with such arrow terms until we obtain the arrow terms $f'_i\colon p \wedge \ldots \wedge p \vdash B$ of $\mathbf{L}$ such that parentheses are somehow associated in $p \wedge \ldots \wedge p$ and for some $z$ in $(p \wedge \ldots \wedge p)$ we have $(z, y) \in Gf'_1$ and $(z, y) \notin Gf'_2$. The formula $p \wedge \ldots \wedge p$ may also be only $p$. We may further compose $f'_i$ with other arrow terms of $\mathbf{L}$ in order to obtain the arrow terms $f''_i$ of type $p \wedge A' \vdash B$ or $p \vdash B$ such that $A'$ is of the form $p \wedge \ldots \wedge p$ with parentheses somehow associated. Let us use 0 to denote the first occurrence of a propositional letter in a formula, counting from the left. So we have $(0, y) \in Gf''_1$ but $(0, y) \notin Gf''_2$.

By working dually on $B$ we obtain the arrow terms $f'''_i$ of $\mathbf{L}$ of type $p\wedge A' \vdash p\vee B'$, for $A'$ of the form $p \wedge \ldots \wedge p$ and $B'$ of the form $p \vee \ldots \vee p$, or of type $p \wedge A' \vdash p$, or of type $p \vdash p \vee B'$, such that $(0, 0) \in Gf'''_1$ and $(0, 0) \notin Gf'''_2$. (We cannot obtain that $f'''_1$ and $f'''_2$ are of type $p \vdash p$, since, otherwise, by Composition Elimination for $\mathbf{L}$, $f'''_2$ would not exist.)

There is an arrow term $h^\wedge\colon p \vdash p \wedge \ldots \wedge p$ of $\mathbf{L}$ defined by using $\hat{w}$ such that for every $x \in G(p \wedge \ldots \wedge p)$ we have $(0, x) \in Gh^\wedge$. We define analogously with the help of $\check{w}$ an arrow term $h^\vee\colon p \vee \ldots \vee p \vdash p$ of $\mathbf{L}$ such that for every $x$ in $p \vee \ldots \vee p$ we have $(x, 0) \in Gh^\vee$. The arrow terms $h^\wedge$ and $h^\vee$ may be $\mathbf{1}_p\colon p \vdash p$.

If $f'''_i$ is of type $p \wedge A' \vdash p \vee B'$, let $f^\dagger_i\colon p \wedge p \vdash p \vee p$ be defined by

$$f^\dagger_i =_{df} (\mathbf{1}_p \vee h^\vee) \circ f'''_i \circ (\mathbf{1}_p \vee h^\wedge).$$

By Composition Elimination for $\mathbf{L}$, we have that $Gf^\dagger_i$ must be a singleton. Let us use 1 to denote the second occurrence of a propositional letter in a formula, counting from the left. If $(1, 0)$ or $(1, 1)$ belongs to $Gf^\dagger_2$, then for $f^*_i : p \wedge p \vdash p$ defined as $\check{w}_p \circ f^\dagger_i$ we have $(0, 0) \in Gf^*_1$ and $(0, 0) \notin Gf^*_2$. If $(0, 1)$ or $(1, 1)$ belongs to $Gf^\dagger_2$, then for $f^*_i\colon p \vdash p \vee p$ defined as $f^\dagger_i \circ \hat{w}_p$ we have $(0, 0) \in Gf^*_1$ and $(0, 0) \notin Gf^*_2$.

If $f'''_i$ is of type $p \wedge A' \vdash p$, then for $f^*_i : p \wedge p \vdash p$ defined as $f'''_i \circ (\mathbf{1}_p \vee h^\wedge)$ we have $(0, 0) \in Gf^*_1$ and $(0, 0) \notin Gf^*_2$.

If $f'''_i$ is of type $p \vdash p\vee B'$, then for $f^*_i\colon p \vdash p\vee p$ defined as $(\mathbf{1}_p \vee h^\vee) \circ f'''_i$ we have $(0, 0) \in Gf^*_1$ and $(0, 0) \notin Gf^*_2$. In all that we have by Composition Elimination for $\mathbf{L}$ that $Gf^*_i$ must be a singleton.

In cases where $f^*_i$ is of type $p \wedge p \vdash p$, by Composition Elimination for $\mathbf{L}$, by the conditions on $Gf^*_1$ and $Gf^*_2$, and by the functoriality of $G$, we obtain in $\mathbf{L}$ the equation $f^*_i = \hat{k}^i_{p,p}$. (This follows from Lattice Coherence too.) So in $\mathbf{L}$ extended with $f_1 = f_2$ we can derive the equation

$$(\hat{k}\hat{k}) \quad \hat{k}^1_{p,p} = \hat{k}^2_{p,p}.$$

In cases where $f^*_i$ is of type $p \vdash p \vee p$, we conclude analogously that we have in $\mathbf{L}$ the equation $f^*_i = \check{k}^i_{p,p}$, and so in $\mathbf{L}$ extended with $f_1 = f_2$ we can derive

$$(\check{k}\check{k}) \quad \check{k}^1_{p,p} \;=\; \check{k}^2_{p,p}.$$

If either of $(\hat{k}\hat{k})$ and $(\check{k}\check{k})$ holds in a lattice category $\mathcal{A}$, then $\mathcal{A}$ is a preorder.

It remains to remark that if for some arrow terms $g_1$ and $g_2$ of $\mathbf{L}$ of the same type we have that $g_1 = g_2$ does not hold for $\mathbf{L}$, then by Lattice Coherence we have $Gg_1 \neq Gg_2$. If we take the substitution instances $g_1'$ of $g_1$ and $g_2'$ of $g_2$ obtained by replacing every letter by a single letter $p$, then we obtain again $Gg_1' \neq Gg_2'$. If $g_1 = g_2$ holds in a lattice category $\mathcal{A}$, then $g_1' = g_2'$ holds too, and $\mathcal{A}$ is a preorder, as we have shown above. This concludes the proof of maximality for $\mathbf{L}$. (In the original presentation of this proof in [11, Section 5], there are some slight inaccuracies in the definition of $f_i^*$.)

## 10. Relative maximality of dicartesian categories

The category $\mathbf{L}_{\top,\bot}$ is not maximal in the sense in which $\mathbf{L}$ is. This is shown by the following counterexample.

Let $Set_*$ be the category whose objects are sets with a distinguished element $*$, and whose arrows are $*$-preserving functions $f$ between these sets; namely, $f(*) = *$. This category is isomorphic to the category of sets with partial functions. The following definitions serve to show that $Set_*$ is a category in which we can interpret the objects and arrow terms of $\mathbf{L}_{\top,\bot}$:

$$\mathrm{I} = \{*\}, \qquad a' = \{(x,*) \mid x \in a - I\}, \qquad b'' = \{(*,y) \mid y \in b - \mathrm{I}\},$$

$$\begin{aligned} a \otimes b \;&=\; ((a-\mathrm{I}) \wedge (b-\mathrm{I})) \cup \mathrm{I},\\ a \boxtimes b \;&=\; (a \otimes b) \cup a' \cup b'',\\ a \boxplus b \;&=\; a' \cup b'' \cup \mathrm{I}. \end{aligned}$$

Note that $a \boxtimes b$ is isomorphic in $Set$ to the cartesian product $a \times b$; the element $*$ of $a \boxtimes b$ corresponds to the element $(*,*)$ of $a \times b$.

The functions $\hat{k}^i_{a_1,a_2} : a_1 \boxtimes a_2 \to a_i$, for $i \in \{1,2\}$, are defined by

$$\hat{k}^i_{a_1,a_2}(x_1,x_2) = x_i, \qquad \hat{k}^i_{a_1,a_2}(*) = *;$$

for $f_i\colon c \to a_i$, the function $\langle f_1, f_2\rangle\colon c \to a_1 \boxtimes a_2$ is defined by

$$\langle f_1, f_2\rangle(z) = \begin{cases} (f_1(z), f_2(z)) & \text{if } f_1(z) \neq * \text{ or } f_2(z) \neq * \\ * & \text{if } f_1(z) = f_2(z) = *; \end{cases}$$

and the function $\hat{\kappa}_a : a \to \mathrm{I}$ is defined by $\hat{\kappa}_a(x) = *$. Having in mind the isomorphism between $a \boxtimes b$ and $a \times b$ mentioned above, the functions $\hat{k}^i_{a_1,a_2} : a_1 \boxtimes a_2 \to a_i$ correspond to the projection functions, while $\langle \_ , \_ \rangle$ corresponds to the usual pairing operation on functions.

The functions $\check{k}^i_{a_1,a_2} : a_i \to a_1 \boxplus a_2$ are defined by

$$\begin{aligned} &\check{k}^1_{a_1,a_2}(x) = (x,*), \quad \check{k}^2_{a_1,a_2}(x) = (*,x), \quad \text{for } x \neq *,\\ &\check{k}^i_{a_1,a_2}(*) = *; \end{aligned}$$

for $g_i\colon a_i \to c$, the function $[g_1, g_2]\colon a_1 \boxplus a_2 \to c$ is defined by

$$\begin{aligned}&[g_1, g_2](x_1, x_2) = g_i(x_i), \ \text{for } x_i \neq *,\\ &[g_1, g_2](*) = *;\end{aligned}$$

finally, the function $\check{\kappa}_a : \mathrm{I} \to a$ is defined by $\check{\kappa}_a(*) = *$.

If we take that $\wedge$ is $\boxtimes$ and $\vee$ is $\boxplus$ then it can be checked in a straightforward manner that $Set_*$ and $Set_*$ without I are lattice categories, and if in $Set_*$ we take further that both $\top$ and $\bot$ are I, then $Set_*$ is a dicartesian category.

Consider now the category $Set_*^{\emptyset}$, which is obtained by adding to $Set_*$ the empty set $\emptyset$ as a new object, and the empty functions $\emptyset_a : \emptyset \to a$ as new arrows. The identity arrow $\mathbf{1}_\emptyset$ is $\emptyset_\emptyset$. For $Set_*^{\emptyset}$, we enlarge the definitions above by

$$\begin{aligned}\emptyset \boxtimes a &= a \boxtimes \emptyset = \emptyset,\\ \emptyset \boxplus a &= a \boxplus \emptyset = a,\\ \hat{k}^i_{a_1,a_2} &= \emptyset_{a_i}, \ \text{for } a_1 = \emptyset \text{ or } a_2 = \emptyset,\\ \langle \emptyset_{a_1}, \emptyset_{a_2}\rangle &= \emptyset_{a_1 \boxtimes a_2},\\ \hat{\kappa}_\emptyset &= \emptyset_{\mathrm{I}},\\ \check{k}^i_{a_1,a_2} &= \emptyset_{a_1 \boxplus a_2}, \ \text{for } a_i = \emptyset,\\ [f_1, \emptyset_c] &= f_1, \qquad\qquad [\emptyset_c, f_2] = f_2,\end{aligned}$$

and define now the function $\check{\kappa}_a : \emptyset \to a$ by $\check{\kappa}_a = \emptyset_a$. Then it can be checked that $Set_*^{\emptyset}$ where $\wedge$ is $\boxtimes$ and $\vee$ is $\boxplus$ as before, while $\top$ is I and $\bot$ is $\emptyset$, is a dicartesian category too.

In $\mathbf{L}_{\top,\bot}$ the equation $\hat{k}^1_{p,\bot} = \check{\kappa}_p \circ \hat{k}^2_{p,\bot}$ does not hold, because $G\hat{k}^1_{p,\bot} \neq \emptyset$ and $G(\check{\kappa}_p \circ \hat{k}^2_{p,\bot}) = \emptyset$, but in $Set_*^{\emptyset}$ this equation holds, because both sides are equal to $\emptyset_\emptyset$. Since $Set_*^{\emptyset}$ is not a preorder, we can conclude that $\mathbf{L}_{\top,\bot}$ is not maximal.

Although this maximality fails, the category $\mathbf{L}_{\top,\bot}$ may be shown maximal in a relative sense. This relative maximality result, which we are going to demonstrate now, says that every dicartesian category that satisfies an equation $f = g$ between arrow terms of $\mathbf{L}_{\top,\bot}$ such that $Gf \neq Gg$ (which implies that $f = g$ is not in $\mathbf{L}_{\top,\bot}$) satisfies also some particular equations. These equations do not give preorder in general, but a kind of "contextual" preorder. Moreover, when $\mathbf{L}_{\top,\bot}$ is extended with some of these equations we obtain a maximal category.

If for some arrow terms $f_1, f_2 : A \vdash B$ of $\mathbf{L}_{\top,\bot}$ we have $Gf_1 \neq Gf_2$, then for some $x$ in $A$ and some $y$ in $B$ we have $(x, y) \in Gf_1$ and $(x, y) \notin Gf_2$, or vice versa. Suppose $(x, y) \in Gf_1$ and $(x, y) \notin Gf_2$. Suppose $x$ is an occurrence of $p$, so that $y$ must be an occurrence of $p$ too.

Let $A'$ be the formula obtained from the formula $A$ by replacing $x$ by $p \wedge \bot$, and every other occurrence of letter or $\top$ by $\bot$. Dually, let $B'$ be the formula obtained from $B$ by replacing $y$ by $p \vee \top$, and every other occurrence of letter or $\bot$ by $\top$. Let us use 0, as in the preceding section, to denote the first occurrence of a propositional letter in a formula, counting from the left. Then it can be shown that there is an arrow term $h^A : A' \vdash A$ of $\mathbf{L}_{\top,\bot}$ such that $Gh^A = \{(0, x)\}$, and an arrow term $h^B : B \vdash B'$ of $\mathbf{L}_{\top,\bot}$ such that $Gh^B = \{(y, 0)\}$. We build $h^A$ with

$\hat{k}^1_{p,\bot} : p \wedge \bot \vdash p$ and instances of $\check{\kappa}_C : \bot \vdash C$, with the help of the operations $\wedge$ and $\vee$ on arrow terms. Analogously, $h^B$ is built with $\check{k}^1_{p,\top} : p \vdash p \vee \top$ and instances of $\hat{\kappa}_C : C \vdash \top$. It can also be shown that there are arrow terms $j^A : p \wedge \bot \vdash A'$ and $j^B : B' \vdash p \vee \top$ of $\mathbf{L}_{\top,\bot}$ such that $Gj^A = Gj^B = \{(0,0)\}$. These arrow terms stand for isomorphisms of $\mathbf{L}_{\top,\bot}$.

Then it is clear that for $f'_i$ being

$$j^B \circ h^B \circ f_i \circ h^A \circ j^A : p \wedge \bot \vdash p \vee \top,$$

with $i \in \{1,2\}$, we have $Gf'_1 = \{(0,0)\}$, while $Gf'_2 = \emptyset$. Hence, by Composition Elimination for $\mathbf{L}_{\top,\bot}$ and by the functoriality of $G$, we obtain in $\mathbf{L}_{\top,\bot}$ the equations

$$\begin{aligned} f'_1 &= \check{k}^1_{p,\top} \circ \hat{k}^1_{p,\bot}, \\ f'_2 &= \check{\kappa}_{p\vee\top} \circ \hat{k}^2_{p,\bot} \;=\; \check{k}^2_{p,\top} \circ \hat{\kappa}_{p\wedge\bot}. \end{aligned}$$

(This follows from Restricted Dicartesian Coherence too.) If we write $\mathbf{0}_{\bot,\top}$ for $\hat{\kappa}_\bot$, which is equal to $\check{\kappa}_\top$ in $\mathbf{L}_{\top,\bot}$, then in $\mathbf{L}_{\top,\bot}$ we have

$$f'_2 = \check{k}^2_{p,\top} \circ \mathbf{0}_{\bot,\top} \circ \hat{k}^2_{p,\bot}.$$

So in $\mathbf{L}_{\top,\bot}$ extended with $f_1 = f_2$ we can derive

$$(\hat{k}\check{k}) \quad \check{k}^1_{p,\top} \circ \hat{k}^1_{p,\bot} \;=\; \check{k}^2_{p,\top} \circ \mathbf{0}_{\bot,\top} \circ \hat{k}^2_{p,\bot}.$$

The equation

$$(\hat{k}\check{\kappa}) \quad \hat{k}^1_{p,\bot} \;=\; \check{\kappa}_p \circ \hat{k}^2_{p,\bot},$$

which holds in $Set^{\emptyset}_*$, and which we have used above for showing the nonmaximality of $\mathbf{L}_{\top,\bot}$, clearly yields $(\hat{k}\check{k})$, which hence holds in $Set^{\emptyset}_*$, and which hence we could have also used for showing this nonmaximality.

If we refine the procedure above by building $A'$ and $B'$ out of $A$ and $B$ more carefully, then in some cases we could derive $(\hat{k}\check{\kappa})$ or its dual

$$(\check{k}\hat{\kappa}) \quad \check{k}^1_{p,\top} \;=\; \check{k}^2_{p,\top} \circ \hat{\kappa}_p$$

instead of $(\hat{k}\check{k})$. We do not replace $x$ by $p \wedge \bot$ in building $A'$, and we can proceed more selectively with other occurrences of letters and $\top$ in $A$, in order to obtain an $A'$ isomorphic to $p$ if possible. We can proceed analogously when we build $B'$ out of $B$ to obtain a $B'$ isomorphic to $p$ if possible.

Note that we have the following:

$$\begin{aligned} \check{\kappa}_{p\wedge\bot} \circ \hat{k}^2_{p,\bot} &= \langle \check{\kappa}_p, \mathbf{1}_\bot \rangle \circ \hat{k}^2_{p,\bot} \\ &= \langle \hat{k}^1_{p,\bot}, \hat{k}^2_{p,\bot} \rangle, \text{ with } (\hat{k}\check{\kappa}), \\ &= \mathbf{1}_{p\wedge\bot}. \end{aligned}$$

In the other direction, it is clear that the equation derived yields $(\hat{k}\check{\kappa})$. So with $(\hat{k}\check{\kappa})$ we have that $C \wedge \bot$ and $\bot$ are isomorphic, and, analogously, with $(\check{k}\hat{\kappa})$ we have that $C \vee \top$ and $\top$ are isomorphic. It can be shown that the natural logical category defined as $\mathbf{L}_{\top,\bot}$ save that we assume in addition both $(\hat{k}\check{\kappa})$ and $(\check{k}\hat{\kappa})$ is maximal. (This is achieved by eliminating letterless subformulae from $C$ and $D$ in

$g_1, g_2 : C \vdash D$ such that $Gg_1 \neq Gg_2$, and falling upon the argument used for the maximality of **L** in the preceding section.)

If $f : a \vdash b$ is any arrow of a dicartesian category $\mathcal{A}$ and $(\hat{k}\check{k})$ holds in $\mathcal{A}$, then we have in $\mathcal{A}$

$$\begin{aligned} \check{k}^1_{b,\top} \circ f \circ \hat{k}^1_{a,\bot} &= \check{k}^1_{b,\top} \circ \hat{k}^1_{b,\bot} \circ (f \wedge \mathbf{1}_\bot) \\ &= \check{k}^2_{b,\top} \circ \mathbf{0}_{\bot,\top} \circ \hat{k}^2_{a,\bot}, \end{aligned}$$

and hence for $f, g : a \vdash b$ we have in $\mathcal{A}$

$$(\hat{k}\check{k}fg) \quad \check{k}^1_{b,\top} \circ f \circ \hat{k}^1_{a,\bot} = \check{k}^1_{b,\top} \circ g \circ \hat{k}^1_{a,\bot}.$$

So, although $\mathbf{L}_{\top,\bot}$ is not maximal, it is maximal in the relative sense that every dicartesian category that satisfies an equation $f = g$ between arrow terms of $\mathbf{L}_{\top,\bot}$ such that $Gf \neq Gg$ satisfies also $(\hat{k}\check{k})$ and $(\hat{k}\check{k}fg)$. Some of these dicartesian categories may satisfy more than just $(\hat{k}\check{k})$ and $(\hat{k}\check{k}fg)$. They may satisfy $(\hat{k}\check{\kappa})$ or $(\check{k}\hat{\kappa})$, which yields

$$f \circ \hat{k}^1_{a,\bot} = g \circ \hat{k}^1_{a,\bot} \qquad \text{or} \qquad \check{k}^1_{b,\top} \circ f = \check{k}^1_{b,\top} \circ g,$$

and some may be preorders.

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