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ALGEBRO-GEOMETRIC INTEGRATION IN CLASSICAL AND STATISTICAL MECHANICS

CONTENTS

Introduction

This article is an enlarged version of the talk given by the author on the Meeting on Mathematical Methods in Models of Mechanics, organized by Serbian Academy of Sciences and Arts in its Novi Sad Branch in October 2003.

We devoted the talk to the 10th anniversary of the Seminar on Mathematical Methods in Mechanics, which is being hold in the Mathematical Institute SANU. Main scientific topics in the focus of the Seminar in that period (1993–2003) are geometry of integrable dynamical systems, connections with complex algebro-geometric and finite-zone integration methods, applications in models of classical, quantum and statistical mechanics etc. Here, we are going to give a brief review of classical results and modern research streams in these areas, as well as the original results.

The article is organized as follows. The next two sections contain necessary notions and statements from algebraic geometry and integrable dynamical systems — in Section 1 we list basic definitions related to the theory of integrable systems, while Section 2 is a brief introduction to the theory of Riemann surfaces. In order to keep the presentation reasonably short, we intensively assume two references published in last few years in Belgrade [50, 26], and we refer readers to them for details and clarifications regarding algebraic geometry and Poisson structures. Let us emphasize that these mathematical techniques are the main tools for the research performed in the framework of the Seminar on Mathematical Methods in Mechanics.

In Section 3 we give a concise review of classical and modern results concerning the motion of the rigid body about the fixed point. In Section 4, the original results concerning a generalization of the classical Hess–Appel'rot rigid body system and its integration in both classical and algebro-geometric ways are presented [22, 23].

In Section 5 we return again to classical subjects, presenting Poncelet theorem on closed polygonal lines inscribed in one and circumscribed about another conic in the plane and Cayley's condition that describe analytically such polygons. In Section 6, billiards as an important class of dynamical systems are introduced. In Section 7 , we present the original results – the generalization of the Cayley's condition related to elliptical billiards in the space of arbitrary finite dimension [27, 28]. Section 8 is aimed to present the author's results on separable potential perturbations of integrable billiard systems [16, 17]. The last Section 9 is devoted to exactly solvable models in Statistical Mechanics and problems of algebro-geometric classification of the solutions of the Quantum Yang–Baxter equation. Some of the author's results, obtained in the general framework of Krichever's approach based on vacuum curvrs and vectors (see [46, 18, 19, 20]), are presented.

1. Poisson structures and completely integrable systems

The algebra $C^\infty(M)$ of smooth functions on a symplectic manifold (M, ω) admits a binary operation $\{f, g\} := \omega(X_f, X_g)$, where X_f and X_g are Hamiltonian vector fields defined by Hamiltonians f and g. Its basic properties are

- bilinearity;
- antisymmetricity: $\{f, g\} = -\{g, f\}$.
- the Jacobi identity: $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$.
- the Leibnitz rule: $\{f, gh\} = \{f, g\}h + g\{f, h\}$.

A more general class of manifolds are Poisson manifolds.

Definition 1. *Poisson algebra* is a commutative algebra with an antisymmetric bilinear operation $\{\cdot, \cdot\}$ satisfying the Jacobi identity and the Leibnitz rule. A manifold M is a Poisson manifold if there is an operation $\{\cdot, \cdot\}$ giving to $C^\infty(M)$ a structure of a Poisson algebra.

Let H be a smooth function on a Poisson manifold M. Then the dynamical system $\dot{x} = \{x, H\}$ is a Hamiltonian system with the Hamiltonian function H. A function F which is constant along the trajectories of the system is called a *first integral*. For a Hamiltonian system with the Hamiltonian function H, a function F is a first integral if and only if $\{H, F\} = 0$.

Let us recall that for functions F, H for which $\{H, F\} \equiv 0$ we say that they are *in involution*. Specially, since $\{H, H\} = 0$, the Hamiltonian function itself is a first integral for the Hamiltonian system. The following fundamental theorem describes the topological structure of flows of an important class of Hamiltonian systems.

Theorem 1 (Liouville–Arnol'd)**.** *Let M be a symplectic manifold and assume $n = \frac{1}{2}\dim(M)$ functions in involution $F_1, \dots, F_n : M \to \mathbb{R}$ are given.*

Denote $c := (c_1, \dots, c_n) \in \mathbb{R}^n$ and $M_c = \{x \in M \mid F_k(x) = c_k\}$. If the functions $F_1, \dots, F_n$ are independent on M_c, then:

1. *M_c is a smooth manifold, invariant with respect to the Hamiltonian diffeomorphism generated by functions F_k.*
2. *If the manifold M_c is compact and connected, then it is diffeomorphic to a torus $\mathbb{T}^n = (\mathbb{S}^1)^n$.*
3. *There exist coordinates $(\varphi_1, \dots, \varphi_n) \in \mathbb{T}^n$ such that the Hamiltonian equations with the Hamiltonian F_1 have the form $\dot{\varphi}_1 = \varpi_{1,c}, \dots, \dot{\varphi}_n = \varpi_{n,c}$ linearizing the flow.*

Definition 2. A Hamiltonian system (M^{2n}, ω, H) that has n independent first integrals in involution is *completely integrable in Liouville sense*.

All Hamiltonian systems with one degree of freedom are obvious examples of completely integrable systems. Starting with two degrees of freedom, the situation is not simple at all any more.

Example 1. The problem of geodesics on the surfaces of revolution in $\mathbb{R}^3$ is completely integrable.

Example 2. The problem of geodesics on ellipsoid in $\mathbb{E}^n$ is completely integrable, as a consequence of the Jacobi–Chasles theorem.

Completely integrable systems have, according to Theorem 1, very regular dynamics. However, they are very rare. Although, for any such a system, there exist action-angle coordinates where this system could be explicitly integrated, the construction of those coordinates is not explicit. Thus, in the theory of completely integrable systems two basic and usually difficult questions exist:

- *For a given system to show that it is completely integrable*;
- *For a given completely integrable system to perform explicit integration.*

For the systems given in the first two examples, integration is done by methods of separation of variables of Hamilton–Jacobi equation. After 1967 and discovery of infinite-dimensional completely integrable systems, such as Korteweg – de Vries equation, new techniques of solving such problems were found. These techniques are based on the inverse scattering methods, and some additional analytical, algebraic or algebro-geometric theories are used.

2. Riemann surfaces, a brief introduction

Theorem 2. *The next three definitions of genus are equivalent:*

- $g = \frac{1}{2}\dim H^1_{DR}(\Sigma)$
- $g = \dim \Omega^1(\Sigma) = \dim \check{H}^0(\Sigma; \Omega^1)$
- $g = \dim \check{H}^1(\Sigma; \mathcal{O})$.

Thus, on a Riemann surface of a genus g, there exist exactly g linearly independent holomorphic differentials. Let as consider now a case of elliptic and hyper-elliptic curves.

Example 3. On a hyper-elliptic curve of genus g given by the equation

$$y^2 = P_{2g+1}(x),$$

one basis of holomorphic differentials consists of $\omega_i = \dfrac{z^{i-1}}{P_{2g+1}(z)}dz$, $i = 1, \dots, g$. (For $g = 1$ the case of elliptic curves is included.)

Theorem 3 (Riemann–Roch)**.** *Let D be a divisor on compact Riemann surface Σ of genus g. Then $\check{H}^0(\Sigma; \mathcal{O}_D)$ and $\check{H}^1(\Sigma; \mathcal{O}_D)$ are finitely dimensional vector spaces and*

$$\dim \check{H}^0(\Sigma; \mathcal{O}_D) - \dim \check{H}^1(\Sigma; \mathcal{O}_D) = 1 - g + \deg(D).$$

Definition 3. A divisor D satisfying $l(K - D) = 0$ is called *nonspecial.* Otherwise, a divisor D is *special*, and the number $l(K - D)$ is called *the index of speciality.*

From Poincaré–Hopf theorem, it follows

Proposition 1. *On a curve Γ of genus g, the degree of canonical divisor K_Γ is* $\deg K_\Gamma = 2g - 2$. *Any divisor D of degree greater than $2g - 2$ is nonspecial.*

Definition 4. A Riemann surface Γ of genus greater then 1 is *hyper-elliptic* if there exists a holomorphic two-sheeted covering $\pi : \Gamma \to \mathbb{CP}^1$.

Example 4. If a Riemann surface is hyper-elliptic in the sense of the last definition, then it represents a normalization of a curve given by the equation

$$y^2 = P_{2g+2}(x), \quad g \geq 2.$$

Two-sheeted covering π induces an involution σ on the hyper-elliptic curve Γ. If the curve is defined by the last equation, then the involution is given by the formula $\sigma(x, y) = (x, -y)$, and the set B of fixed points of the involution is in one to one correspondence with the set $B' = \{x_1, \dots, x_{2g+2}\}$ of zeroes of the polynomial $P_{2g+2} = a \prod(x - x_i)$.

Exercise 1. Let P, Q be two arbitrary points on the hyper-elliptic curve Γ. Prove that the divisors $P + \sigma(P)$ and $Q + \sigma(Q)$ are equivalent.

The class of divisors $P + \sigma(P)$ we denote by L. It does not depend, according to the last Exercise, on the choice of the point P. Let $T \subset B$ be a subset with even cardinality. We use the following notation:

$$e_T = \sum_{P_i \in T} P_i - \frac{|T|}{2} L.$$

Exercise 2. Prove:

- $2e_T = 0$.
- $e_{T_1} + e_{T_2} = e_{T_1 \Delta T_2}$, where Δ denotes the symmetric set difference.
- $e_{T_1} = e_{T_2}$ if and only if $T_1 = T_2$ or $T_1 = B \setminus T_2$.
- On hyper-elliptic curve Γ of genus g, it holds $K_\Gamma = (g-1)L$.

2.1. Matrix of periods of a Riemann surface. Suppose Γ is a given, compact, nonsingular Riemann surface of genus g. Denote by $(a_1, \dots, a_g, b_1, \dots, b_g)$ a basis of homologies $H_1(\Gamma, \mathbb{Z})$, which is *canonical*, i.e., such that

$$a_i \circ a_j = b_i \circ b_j = 0, \ a_i \circ b_j = \delta_{ij}, \quad i, j = 1, \dots, g.$$

Denote by $\tilde{\Gamma}$ the fundamental $4g$-angle, with edges $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$. The surface Γ can be realized by gluing the edges of $\tilde{\Gamma}$.

Let ω, ω' be closed differential on Γ, and let

$$A_i = \int_{a_i} \omega, \quad B_i = \int_{b_i} \omega, \quad A'_i = \int_{a_i} \omega', \quad B'_i = \int_{b_i} \omega',$$

for $i = 1, \dots, g$ be their periods on canonical basis of cycles. Then

$$\iint_\Gamma \omega \wedge \omega' = \sum_{i=1}^{g} (A_i B'_i - A'_i B_i).$$

Let us fix a basis of holomorphic differentials $[\omega_1, \dots, \omega_g]$ such that

$$\int_{a_j} \omega_k = 2\pi i \delta_{jk}, \quad j, k = 1, \dots, g.$$

For a basis normalized in that way, denote by B_{jk} the matrix of b-periods:

$$B_{jk} = \int_{b_j} \omega_k, \quad j,k = 1,\dots,g.$$

Definition 5. The matrix B_{jk} is called period matrix of a Riemann surface Γ.

Proposition 2 (Riemann bilinear relations)**.** *For the period matrix B_{jk} of a Riemann surface, it holds:*

- *The matrix B is symmetric.*
- *The matrix B has a negatively defined real part.*

Definition 6. A matrix B is called Riemannian matrix, if it satisfies properties of the last proposition. The set of such $g \times g$ matrices is called the Sigel half-plane and is denoted by $\mathcal{H}_g$.

Thus, every period matrix of a Riemann surface is a Riemannian matrix. The converse question is highly nontrivial: *Which Riemannian matrices are period matrices of some Riemann surface?* This classical and very important problem of XIX century algebraic geometry is known as *the Riemann–Shöttke problem* and it was open for more than a century. It was solved quite recently, in the middle of 1980's, using the techniques of *the soliton theory*, Japanese mathematician Shiota proved the so-called *Novikov's conjecture.* We will tell something more about this at the and of this Section.

2.2. Jacobian of a Riemann surface. The Abel map.

Denote the standard basis of $\mathbb{C}^g$ by $e = [e_1,\dots,e_g]$, $(e_i)_k = \delta_{ik}$.

Exercise 3. Let B be a Riemannian matrix. Then $2g$ vectors $e_1,\dots,e_g, Be_1,\dots,Be_g$ are linearly independent over $\mathbb{R}$.

Let us consider an integer-valued lattice Λ_B in $\mathbb{C}^g$ generated by the vectors $2\pi i e_j$, Be_k, $k,j = 1,\dots,g$:

$$\Lambda_B : \quad 2\pi i M + BN, \quad M,N \in \mathbb{Z}^g.$$

Then $2g$-dimensional torus $\mathbb{T}^{2g} = \mathbb{T}(B) = \mathbb{C}^g/\Lambda_B$ defines a g-dimensional *Abel variety*, a g-dimensional complex torus.

Definition 7. If a matrix B is a period matrix of some Riemann surface Γ of genus g, then $\mathbb{T}(B)$ is called the Jacobian variety of a surface Γ, denoted by $\mathbb{T}(B) = \mathrm{Jac}(\Gamma)$.

Let a compact, smooth Riemann surface Γ of genus g be given with some canonical basis of homologies (a,b) and with corresponding normalized basis of holomorphic differentials $[\omega_1,\dots,\omega_g]$. Choosing an arbitrary point P_0 on Γ, let us consider g *Abel integrals*

$$u_i(P) = \int_{P_0}^{P} \omega_i, \quad i = 1,\dots,g,$$

assuming one and the same integration path every time.

Together with holomorphic differentials, known also as *Abel differentials of the first kind*, meromorphic differentials play important role as well.

Definition 8. The Abel differentials of the second kind $\omega_P^{(n)}$ are meromorphic differentials with a unique pole at a point P of order $n+1$, locally represented by

$$\omega_P^{(n)} = \frac{dz}{z^{n+1}} + \cdots .$$

The Abel differentials of the third kind ω_{PQ} are determined by unique simple poles P, Q with residua $+1, -1$.

These differentials are uniquely determined by the conditions:

$$\int_{a_i} \omega_P^{(n)} = 0, \quad \int_{a_i} \omega_{PQ} = 0, \quad i = 1, \dots, g.$$

Exercise 4. Prove the following formulae

$$\int_{b_i} \omega_P^{(n)} = \frac{1}{n!} \frac{d^{n-1} f_i(Q)}{dz^{n-1}}, \quad i = 1, \dots, g, \ n \in \mathbb{N}, \tag{1}$$

$$\int_{b_i} \omega_{PQ} = \int_Q^P \omega_i, \quad i = 1, \dots, g, \tag{2}$$

where $\omega_i = f_i(z)\, dz$ locally represents basic holomorphic differential around a point Q.

Exercise 5. Given four arbitrary points on a Riemann surface, prove:

$$\int_{Q_1}^{Q_2} \omega_{Q_3 Q_4} = \int_{Q_3}^{Q_4} \omega_{Q_1 Q_2}.$$

Exercise 6. Prove that the formula

$$\mathcal{A}(P) = (u_1(P), \dots, u_g(P)) \tag{3}$$

defines a mapping $\mathcal{A} :\ \Gamma \to \mathrm{Jac}(\Gamma)$.

Definition 9. The mapping $\mathcal{A} :\ \Gamma \to \mathrm{Jac}(\Gamma)$ defined by formula (3) is called the Abel mapping.

The natural question is *wether given points $P_1, \dots, P_n$ and $Q_1, \dots, Q_n$ represent a divisor of zeroes and poles of some meromorphic function on a surface* Γ. The answer is given in the following

Theorem 4 (Abel)**.** *Given points $P_1, \dots, P_n$ and $Q_1, \dots, Q_n$ form divisors of zeroes and poles of a meromorphic function on a Riemann surface* Γ *if and only if the relation*

$$\sum_{i=1}^n \mathcal{A}(P_i) = \sum_{i=1}^n \mathcal{A}(Q_i)$$

takes place on the Jacobian $\mathrm{Jac}(\Gamma)$.

2.3. Riemann theta-function. An important tool is introduced by the following

Definition 10. Given an arbitrary $g \times g$ Riemann matrix B, $B \in \mathcal{H}_g$. The Riemann theta-function $\theta(z, B)$ is defined by the series:

$$\theta(z,B) = \sum_{n\in\mathbb{Z}^g} \exp\big((Bn,n) + (n,z)\big). \tag{4}$$

Proposition 3. *The series* (4) *converges uniformly and absolutely on every compact subset of* $\mathbb{C} \times \mathcal{H}_g$ *and it defines a holomorphic function.*

Proposition 4. *The following periodic relations are valid:*

$$\theta(z + 2\pi i e_k, B) = \theta(z,B), \quad k = 1,\dots,g,$$
$$\theta(z + Be_k, B) = \exp(-B_{kk}/2 - z_k)\,\theta(z,B), \quad k = 1,\dots,g.$$

Similarly, Riemann theta-functions with characteristics can be introduced for arbitrary real vectors $a, b \in \mathbb{R}^g$:

$$\theta[2a,2b](z) = \exp\Big\{\frac{1}{2}(Ba,a) + (z + 2\pi i b, a)\Big\}\theta(z + 2\pi i b + Ba).$$

2.4. The Jacobi inversion problem and the Riemann theorem about zeroes of a theta-function. Starting from the case of genus 2 Riemann surfaces, there is no sense to invert fixed Abel integral. The following system

$$\zeta_1 = \int_{P_0}^{P_1} \frac{dz}{\sqrt{P_5(z)}} + \int_{P_0}^{P_2} \frac{dz}{\sqrt{P_5(z)}},$$
$$\zeta_2 = \int_{P_0}^{P_1} \frac{zdz}{\sqrt{P_5(z)}} + \int_{P_0}^{P_2} \frac{zdz}{\sqrt{P_5(z)}},$$

we are going to consider in the next section, in connection with the Kowalevski case of rigid body motion. The problem is to determine points P_1, P_2 as functions of given values ζ_1, ζ_2. Observing symmetric appearance of points P_1 and P_2 in the above formulae, the problem can be reduced to find expressions of symmetric functions of P_1, P_2, through ζ_1, ζ_2. Historically, it was Jacobi who solved this problem in genus two case.

For an arbitrary genus, corresponding general Jacobi problem of inversion was formulated and solved by Riemann.

Given an arbitrary, smooth Riemann surface Γ of genus g, with a fixed canonical basis of homology cycles and corresponding basis of holomorphic differentials. By using the Abel mapping, we define

$$\mathcal{A}^n : \ S^n(\Gamma) \to \mathrm{Jac}(\Gamma), \quad \mathcal{A}^n(P_1,\dots,P_n) = \sum_{i=1}^{n} \mathcal{A}(P_i),$$

where $S^n(X)$ denotes symmetric n-th degree of a set X.

Proposition 5. *Let a nonspecial divisor* $D = P_1 + \cdots + P_g$ *be given; then in a neighborhood of the point* $\mathcal{A}^g(P_1,\dots,P_g) \in \mathrm{Jac}(\Gamma)$ *the mapping* $\mathcal{A}^g$ *is invertible.*

In the general case, the divisor $D = P_1 + \cdots + P_g$ is nonspecial. Thus, the inverse of the mapping $\mathcal{A}^g$ is defined almost everywhere. To find explicitly the inverse, Riemann essentially used theta-functions. Let us present some of their basic properties, necessary for the solution of the Jacobi inversion problem.

Suppose a vector $f \in \mathbb{C}^g$ be given. Consider the function $F(P) = \theta(\mathcal{A}(P) - f)$, where $\theta(z) = \theta(z, B)$ is the theta-function of the surface Γ. Function F is well defined and analytic on the fundamental $4g$-angle $\tilde{\Gamma}$, and for almost all f it is not identically equal zero.

Proposition 6. *If the function F is not identically zero, then it has exactly g zeroes in $\tilde{\Gamma}$.*

Definition 11. A vector $\mathcal{K} = (K_1, \dots, K_g)$, where

$$K_j = \frac{2\pi i + B_{jj}}{2} - \frac{1}{2\pi i} \sum_{l \neq j} \left(\int_{a_l} \omega_l(P) \int_{P_0}^{P} \omega_j \right), \quad j = 1, \dots, g,$$

is called the vector of Riemann constants.

Proposition 7. *If a function F is not identically zero and if $P_1, \dots, P_g$ are its zeroes, then $\mathcal{A}^g(P_1, \dots, P_g) = f - \mathcal{K}$.*

Theorem 5 (Riemann)**.** *Given a vector f such that $F(P) = \theta(\mathcal{A}(P) - \mathcal{K} - f)$ is not identically zero. Then:*

- *the function F has exactly g zeroes $P_1, \dots, P_g$, giving the solution of the Jacobi problem $u_i(P_1) + \cdots + u_i(P_g) = f_i$, $i = 1, \dots, g$.*
- *The divisor $P_1 + \cdots + P_g$ is nonspecial.*

The set of zeroes of the theta-function defined on the Jacobian of the Riemann surface Γ is called the theta divisor or the Θ-divisor of the Riemann surface, denoted also by Θ_Γ.

2.5. The Baker–Akhiezer function.

In the theory of integrable systems an important role plays the notion of the Baker–Akhiezer function.

Definition 12. Given n points $P_1, \dots, P_n$ on a Riemann surface of genus g, with local parameters k_i^{-1}, $i = 1, \dots, n$, $k_i^{-1}(P_i) = 0$, n polynomials $q_i(k)$ and a nonspecial divisor D, then n-point Baker–Akhiezer function ψ corresponding to the data, is

- meromorphic on $\Gamma \setminus \{P_1, \dots, P_n\}$;
- for its divisor it holds $(\psi) + D \geq 0$;
- when P tends to P_i, the function $\psi(P) \exp(-q_i(k_i(P)))$ is analytical.

Theorem 6. [33] *Given a nonspecial divisor D of degree N. Then the dimension of the space of Baker–Akhiezer function is $N - g + 1$.*

Example 5. If $N = g$, then the Baker–Akhiezer function ψ is determind uniquely up to a scalar factor. It is given by the formula

$$\psi(P) = a \exp\left(\sum_{j=1}^{n} \int_Q^P \Omega_{q_j}\right) \frac{\theta\Big(\mathcal{A}(P) + \sum\limits_{j=1}^{n} U^{(q_j)} - \mathcal{A}(D) - \mathcal{K}\Big)}{\theta\left(\mathcal{A}(P) - \mathcal{A}(D) - \mathcal{K}\right)},$$

where Ω_{q_j} are Abel differentials of the second order, with a principle part around P_j of the form $dq_j(k_j(P))$ normalized by the condition of annulation of the a periods; $2\pi i U^{(q_j)}$ are the vectors of their b-periods.

2.6. Riemann–Shöttke problem and Novikov's conjecture. We saw that every period matrix of a Riemann surface is a Riemannian matrix. The converse question *which Riemannian matrices are period matrices of some Riemann surface* is classical and very important problem of XIX century algebraic geometry known as *the Riemann–Shöttke problem.* It was solved quite recently, in the middle of 1980's, using the techniques of the Baker–Akhiezer functions and the soliton theory, through the so-called *Novikov's conjecture.* (see [32])

It was known after Krichever (see [33] and references therein) that there exist certain theta-function formulae associated with period matrices which give solutions of the *Kadomtsev–Petviashvili (KP)* equation from the soliton theory

$$(u_t + uu_x + u_{xxx})_x + u_{yy} = 0.$$

The Novikov conjecture is a converse statement that a Riemannian matrix is a period matrix *only if it gives a solution of KP equation through the Krichever formulae.*

In a weak form the Novikov conjecture has been proven by Dubrovin in 1981 [32]. The complete solution of Novikov's conjecture and the Riemann–Shöttke problem was done by Shiota in 1986 [60, 57]. The highlight of Shiota's proof was use of a notion of *tau-function* introduced by Sato school few years before, giving opportunity to involve simultaneously the whole hierarchy of integrable systems associated with the KP equation.

3. Rotations of a heavy rigid body about a fixed point

Let us consider rotations of a rigid body about a fixed point O, under the gravitational field. Motion of the rigid body is represented in two coordinate systems: *the fixed* $Oxyz$, and *the moving frame* $OXYZ$, which is attached to the body.

Traditionally, vectors in the fixed frame are denoted by small letters, and in the moving frame by capital letters. The vector $\Omega(t) = (p, q, r)$ will denote *angular velocity in the moving frame* and velocity V of a point Q is $V = \Omega \times Q$. Now the kinetic momentum G becomes $G = \iiint_\sigma Q \times (\Omega \times Q)\, dm = J(\Omega)$, where the operator J is symmetric and called inertia tensor of a rigid body.

The operator J defines quadratic form which gives *the ellipsoid of inertia* of the body $(JX, X) = 1$. The ellipsoid describes the mass distribution in the body. Choosing the basis $e = [i, j, k]$ where the operator J is diagonal, we get $[J]_e =$

$I = \text{diag}(A, B, C)$. These three numbers A, B, C, the principal momenta of inertia, which describe the mass distribution, together with the coordinates of the mass center $\chi = (x_0, y_0, z_0)$, give complete description of the dynamical properties of the rigid body. (Instead of A, B, C we will also use $I_1,\ I_2,\ I_3$ as a notation for the principal momenta.)

In the same basis the vector of kinetic momentum becomes

$$G = Api + Bqj + Crk.$$

Denote by $\Gamma = (\gamma, \gamma', \gamma'')$ coordinates of the vertical orth in the moving frame. Gravitational force acts in direction of Γ, and assuming $mg = 1$, we get $L = \chi \times \Gamma$, where L is the principal momentum of forces. From the equation $\dot{G} = L$, the first group of the Euler–Poisson equations follow:

$$\dot{M} = M \times \Omega + \chi \times \Gamma, \tag{5}$$

where $M = I\Omega$.

The second group of Euler–Poisson equations follow from the fact that the vector Γ is fixed in the space:

$$\dot{\Gamma} = \Gamma \times \Omega. \tag{6}$$

The equations (5) and (6) are six differential equations of motion on Ω and Γ as functions of time.

3.1. The first integrals of motion. Integrable cases. The Euler–Poisson equations always have three first integrals of motion:

$$F_1 = \frac{1}{2}\langle I\Omega, \Omega\rangle + \langle \Gamma, \chi\rangle \quad \text{(energy integral)},$$
$$F_2 = \langle \Gamma, \Gamma\rangle\ (= 1), \quad F_3 = \langle I\Omega, \Gamma\rangle.$$

The Euler case (1751). It is defined by the condition $\chi = 0$. The additional first integral is $F_4 = \langle M, M\rangle$.

The Lagrange case (1788). This case is defined by the conditions $A = B$ and $\chi = (0, 0, z_0)$. So, the ellipsoid of inertia is symmetric, and mass-center is placed on the symmetry axis. Additional first integral, linear in impulses, is $F_4 = M_3$.

The Kowalevski case. It is well known that Kowalevski, in her celebrated 1889 paper [45], starting with a careful analysis of the solutions of the Euler and the Lagrange case of rigid-body motion, formulated a problem *to describe the parameters* (A, B, C, x_0, y_0, z_0)*, for which the Euler–Poisson equations have a general solution in the form of uniform functions with only moving poles as singularities.* Here $I = \text{diag}(A, B, C)$ represents the inertia operator, and $\chi = (x_0, y_0, z_0)$ is the center of mass of the rigid body.

Then, in §1 of [45], some necessary conditions were formulated and a new case was discovered, now known as Kowalevski case, as a unique possible beside the cases of Euler and Lagrange: $A = B = 2C$, $\chi = (x_0, 0, 0)$. Additional first integral found by Kowalevski is of the fourth degree in impulses

$$F_4 = \Big(\Omega_1^2 - \Omega_2^2 + \frac{x_0}{I_3}\Gamma_1\Big)^2 + \Big(2\Omega_1\Omega_2 + \frac{x_0}{I_3}\Gamma_2\Big)^2.$$

The integration of the Kowalevski case. The problem of Kowalevski of a motion of a rigid body about a fixed point, can be reduced to the solution of the system

$$\dot{s}_1 = \frac{\sqrt{P_5(s_1)}}{(s_1 - s_2)}, \quad \dot{s}_2 = \frac{\sqrt{P_5(s_2)}}{(s_2 - s_1)}, \tag{7}$$

where s_i are so-called *Kowalevski variables.*

However, considering the situation where all momenta of inertia are different, Kowalevski came to the relation analogue to the following (see [39]):

$$x_0\sqrt{A(B-C)} + y_0\sqrt{B(C-A)} + z_0\sqrt{C(A-B)} = 0.$$

And she concluded that it should be $x_0 = y_0 = z_0$ in such a case, giving the Euler case.

But, it was Appel'rot who noticed in the beginning of 1890's, that the last relation admits one more case, not mentioned by Kowalevski:

$$x_0\sqrt{A(B-C)} + z_0\sqrt{C(A-B)} = 0, \quad y_0 = 0,$$

under the assumption $A > B > C$. Such systems were considered also by Hess, even before Appel'rot, in 1890. But such intriguing position corresponding to the Kowalevski paper, made the Hess–Appel'rot systems very attractive for leading Russian mathematicians from the end of XIX century. After few years, Nekrasov and Lyapunov provided new arguments and they demonstrated that the Hess–Appel'rot systems did not satisfy the condition investigated by Kowalevski, which means that conclusion of §1 of [45] *was correct.*

A few years ago, we constructed a Lax representation for it (see [22]). We provided the Lax representation for all new systems, generalizing the Lax pair from [22]. It appeared that new systems belong to the class of *isoholomorphic systems.* This class of systems was introduced and studied in [23], in connection with the Lagrange bitop.

Such systems have specific distribution of zeroes in Lax matrices. Therefore standard integration technics of [31, 1] cannot be applied directly. Its integration requires more detailed analysis of geometry of the Prym varieties and it is based on Mumford's relation on theta-divisors of unramified double coverings.

The L operator, a quadratic polynomial in λ of the form $\lambda^2 C + \lambda M + \Gamma$, in the case $n = 4$ satisfies the condition $L_{12} = L_{21} = L_{34} = L_{43} = 0$. Such a situation, explicitly excluded by Adler–van Moerbeke (see [1, Theorem 1]) and implicitly by Dubrovin (see [31, Lemma 5 and Corollary]) has been studied for the first time in [23].

Study of the spectral curve and the Baker–Akhiezer function for the four-dimensional Hess–Appel'rot systems (see [24, 25]) shows that, similarly to [23], dynamics of the system is related to certain Prym variety Π. It is connected to the evolution of divisors of some meromorphic differentials Ω^i_j. From the condition on zeroes of the Lax matrix, it follows that differentials $\Omega^1_2,\ \Omega^2_1,\ \Omega^3_4,\ \Omega^4_3$ are *holomorphic* during the whole evolution. Compatibility of this requirement with dynamics is based on Mumford's relation $\Pi^- \subset \Theta$, (see [23]), where Π^- is a translation of the Prym variety Π.

Classical Hess–Appel'rot system. Let $J_1 < J_2 < J_3$ and $\chi = (x_0, y_0, z_0)$. Hess in [42] and Appel'rot in [4] found that if the inertia momenta and the radius vector of center of masses satisfy the conditions

$$y_0 = 0, \quad x_0\sqrt{J_2 - J_1} + z_0\sqrt{J_3 - J_2} = 0 \tag{8}$$

then, the surface

$$F_4 = M_1 x_0 + M_3 z_0 = 0 \tag{9}$$

is invariant. Integration of such system, using classical techniques can be found in [39]. In [22], an L-A pair for the Hess–Appel'rot system is constructed:

$$\begin{aligned} &\dot{L}(\lambda) = [L(\lambda), A(\lambda)],\\ &L(\lambda) = \lambda^2 C + \lambda M + \Gamma, \quad A(\lambda) = \lambda\chi + \Omega, \quad C = \frac{J_1 + J_3}{J_1 J_3}\chi, \end{aligned}$$

where the skew-symmetric matrices represent the vectors denoted by the same letter. Also, the basic steps in algebro-geometric integration procedure are given.

The Zhukovskiĭ geometric interpretation of the conditions (8), (9) (see [67, 49]) Let us consider the ellipsoid

$$\frac{M_1^2}{J_1} + \frac{M_2^2}{J_2} + \frac{M_3^2}{J_3} = 1,$$

and the plane containing the middle axis and intersecting the ellipsoid through a circle. Denote by l corresponding normal to the plane, which passes through the fixed point O. Then the conditions (8), (9) mean that the center of masses lies on the line l.

Having this interpretation in mind, we choose the basis of moving frame such that the third axis is l, the second one is directed as the middle axis of ellipsoid, and the first one is chosen according to the orientation of the orthogonal frame. In this basis (see [9]), particular integral (9) becomes $F_4 = M_3 = 0$, the matrix J obtains the form:

$$J = \begin{pmatrix} J_1 & 0 & J_{13} \\ 0 & J_1 & 0 \\ J_{13} & 0 & J_3 \end{pmatrix},$$

and $\chi = (0, 0, z_0)$. This will serve us as a motivation for the definition of the four-dimensional Hess–Appel'rot system.

4. The definition of Lagrange bitop and its basic properties

The equations of motion of a heavy n-dimensional rigid body fixed at a point in the moving frame are:

$$\dot{M} = [M, \Omega] + [\Gamma, \chi], \quad \dot{\Gamma} = [\Gamma, \Omega], \tag{10}$$

where the moving frame is such that the matrix I is diagonal in it, $\mathrm{diag}(I_1, \dots, I_n)$. Here $M_{ij} = (I_i + I_j)\Omega_{ij} \in \mathrm{so}(n)$ is the kinetic momentum, $\Omega \in \mathrm{so}(n)$ is the angular velocity, $\chi \in \mathrm{so}(n)$ is a given constant matrix (describing a generalized center of the mass), $\Gamma \in \mathrm{so}(n)$. Then $I_i + I_j$ are the principal inertia momenta. These equations are on the semidirect product $\mathrm{so}(n) \times \mathrm{so}(n)$ and they were introduced in [59].

We are going to consider a four-dimensional case of these equations defined by

$$\begin{matrix} I_1 = I_2 = a \\ I_3 = I_4 = b \end{matrix} \quad \text{and} \quad \chi = \begin{pmatrix} 0 & \chi_{12} & 0 & 0 \\ -\chi_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \chi_{34} \\ 0 & 0 & -\chi_{34} & 0 \end{pmatrix} \tag{11}$$

with the conditions $a \neq b$, $\chi_{12}, \chi_{34} \neq 0$, $|\chi_{12}| \neq |\chi_{34}|$. We will call this system *the Lagrange bitop.*

Proposition 8. [22] *The equations of motion* (10) *under the conditions* (11) *have an L-A pair representation* $\dot{L}(\lambda) = [L(\lambda), A(\lambda)]$*, where*

$$L(\lambda) = \lambda^2 C + \lambda M + \Gamma, \quad A(\lambda) = \lambda\chi + \Omega, \tag{12}$$

and $C = (a + b)\chi$.

One can observe that both leading terms in the operators L and A (matrices C and χ) are skewsymmetric, while in [31, 32, 34, 51, 8] one is always symmetric and another one is skewsymmetric.

Before analyzing the spectral properties of the matrices $L(\lambda)$, we will change the coordinates in order to diagonalize the matrix C. In this new basis the matrices $L(\lambda)$ have the form $\tilde{L}(\lambda) = U^{-1} L(\lambda) U$,

$$\tilde{L}(\lambda) = \begin{pmatrix} -i\Delta_{34} & 0 & -\beta_3^* - i\beta_4^* & i\beta_3 - \beta_4 \\ 0 & i\Delta_{34} & -i\beta_3^* - \beta_4^* & -\beta_3 + i\beta_4 \\ \beta_3 - i\beta_4 & -i\beta_3 + \beta_4 & -i\Delta_{12} & 0 \\ i\beta_3^* + \beta_4^* & \beta_3^* + i\beta_4^* & 0 & i\Delta_{12} \end{pmatrix},$$

where $\Delta_{12} = \lambda^2 C_{12} + \lambda M_{12} + \Gamma_{12}$, $\Delta_{34} = \lambda^2 C_{34} + \lambda M_{34} + \Gamma_{34}$, and

$$\beta_3 = x_3 + \lambda y_3, \quad x_3 = \frac{1}{2}(\Gamma_{13} + i\Gamma_{23}),$$

$$\beta_4 = x_4 + \lambda y_4, \quad x_4 = \frac{1}{2}(\Gamma_{14} + i\Gamma_{24}),$$

$$\beta_3^* = \bar{x}_3 + \lambda \bar{y}_3, \quad y_3 = \frac{1}{2}(M_{13} + iM_{23}),$$

$$\beta_4^* = \bar{x}_4 + \lambda \bar{y}_4, \quad y_4 = \frac{1}{2}(M_{14} + iM_{24}).$$

The spectral polynomial $p(\lambda,\mu)=\det(\tilde{L}(\lambda)-\mu\cdot 1)$ has the form
$$p(\lambda,\mu)=\mu^4+P(\lambda)\mu^2+[Q(\lambda)]^2,$$
where
$$P(\lambda)=\Delta_{12}^2+\Delta_{34}^2+4\beta_3\beta_3^*+4\beta_4\beta_4^*,\quad Q(\lambda)=\Delta_{12}\Delta_{34}+2i(\beta_3^*\beta_4-\beta_3\beta_4^*).$$
We can rewrite it in terms of M_{ij} and Γ_{ij}:
$$P(\lambda)=A\lambda^4+B\lambda^3+D\lambda^2+E\lambda+F,\quad Q(\lambda)=G\lambda^4+H\lambda^3+I\lambda^2+J\lambda+K.$$
Their coefficients
$$\begin{aligned}
A&=C_{12}^2+C_{34}^2=\langle C_+,C_+\rangle+\langle C_-,C_-\rangle,\\
B&=2C_{34}M_{34}+2C_{12}M_{12}=2\left(\langle C_+,M_+\rangle+\langle C_-,M_-\rangle\right),\\
D&=M_{13}^2+M_{14}^2+M_{23}^2+M_{12}^2+M_{34}^2+2C_{12}\Gamma_{12}+2C_{34}\Gamma_{34}\\
&=\langle M_+,M_+\rangle+\langle M_-,M_-\rangle+2\left(\langle C_+,\Gamma_+\rangle+\langle C_-,\Gamma_-\rangle\right),\\
E&=2\Gamma_{12}M_{12}+2\Gamma_{13}M_{13}+2\Gamma_{14}M_{14}+2\Gamma_{23}M_{23}+2\Gamma_{24}M_{24}+2\Gamma_{34}M_{34}\\
&=2\left(\langle \Gamma_+,M_+\rangle+\langle \Gamma_-,M_-\rangle\right),\\
F&=\Gamma_{12}^2+\Gamma_{13}^2+\Gamma_{14}^2+\Gamma_{23}^2+\Gamma_{24}^2+\Gamma_{34}^2=\langle\Gamma_+,\Gamma_+\rangle+\langle\Gamma_-,\Gamma_-\rangle,\\
G&=C_{12}C_{34}=\langle C_+,C_-\rangle,\\
H&=C_{34}M_{12}+C_{12}M_{34}=\langle C_+,M_-\rangle+\langle C_-,M_+\rangle,\\
I&=C_{34}\Gamma_{12}+\Gamma_{34}C_{12}+M_{12}M_{34}+M_{23}M_{14}-M_{13}M_{24}\\
&=\langle C_+,\Gamma_-\rangle+\langle C_-,\Gamma_+\rangle+\langle M_+,M_-\rangle,\\
J&=M_{34}\Gamma_{12}+M_{12}\Gamma_{34}+M_{14}\Gamma_{23}+M_{23}\Gamma_{14}-\Gamma_{13}M_{24}-\Gamma_{24}M_{13}\\
&=\langle M_+,\Gamma_-\rangle+\langle M_-,\Gamma_+\rangle,\\
K&=\Gamma_{34}\Gamma_{12}+\Gamma_{23}\Gamma_{14}-\Gamma_{13}\Gamma_{24}=\langle\Gamma_+,\Gamma_-\rangle.
\end{aligned}\tag{13}$$
are integrals of motion of the system (10), (11). We used two vectors $M_+,M_-\in R^3$ which correspond to $M_{ij}\in\mathrm{so}(4)$ according to
$$(M_+,M_-)\to\begin{pmatrix}0&-M_+^3&M_+^2&-M_-^1\\ M_+^3&0&-M_+^1&-M_-^2\\ -M_+^2&M_+^1&0&-M_-^3\\ M_-^1&M_-^2&M_-^3&0\end{pmatrix}$$
Here M_+^j are the j-th coordinates of the vector M_+. The system (10), (11) is Hamiltonian with the Hamiltonian function
$$\mathcal{H}=\frac{1}{2}(M_{13}\Omega_{13}+M_{14}\Omega_{14}+M_{23}\Omega_{23}+M_{12}\Omega_{12}+M_{34}\Omega_{34})+\chi_{12}\Gamma_{12}+\chi_{34}\Gamma_{34}$$
The algebra $\mathrm{so}(4)\times\mathrm{so}(4)$ is 12 dimensional. The general orbits of the coadjoint action are 8 dimensional. According to [59], the Casimir functions are coefficients of $\lambda^0,\lambda,\lambda^4$ in the polynomials $[\det\tilde{L}(\lambda)]^{1/2}$ and $-\frac{1}{2}\operatorname{Tr}(\tilde{L}(\lambda))^2$.

Since
$$[\det\tilde{L}(\lambda)]^{1/2}=G\lambda^4+H\lambda^3+I\lambda^2+J\lambda+K,\quad -\frac{1}{2}\operatorname{Tr}(\tilde{L}(\lambda))^2=A\lambda^4+E\lambda+F,$$

the Casimir functions are J, K, E, F. Nontrivial integrals of motion are B, D, H, I. They are in involution. Nontrivial integrals of motion are B, D, H, I are independent in the case $\chi_{12} \neq \pm\chi_{34}$. When $|\chi_{12}| = |\chi_{34}|$, then $2H = B$ or $2H = -B$ and there are only 3 independent integrals in involution. So we have

Proposition 9. [22] *For $|\chi_{12}| \neq |\chi_{34}|$, the system* (10), (11) *is completely integrable in the Liouville sense.*

There are two families of integrable Euler–Poisson equations introduced by Ratiu in [59]. *The generalized symmetric case* is defined by the conditions

$$I_1 = \cdots = I_n, \quad \chi \text{ arbitrary};$$

and *the generalized Lagrange case* which is defined by

$$I_1 = I_2 = a, \quad I_3 = \cdots = I_n = b, \quad \chi_{ij} = 0 \text{ if } (i,j) \notin \{(1,2),(2,1)\}.$$

The system (10), (11) does not fall in any of those families and together with them it makes the complete list of systems with the L operator of the form

$$L(\lambda) = \lambda^2 C + \lambda M + \Gamma.$$

Proposition 10. [22] *If $\chi_{12} \neq 0$, then the Euler–Poisson equations* (10) *could be written in the form* (12) *(with arbitrary C) if and only if the equations* (10) *describe the generalized symmetric case, the generalized Lagrange case or the Lagrange bitop, including the case $\chi_{12} = \pm\chi_{34}$.*

One can compare this with [63, Theorem 15, ch. 53]. The proofs of the Propositions 8–10 can be found in [22].

The $L(\lambda)$ matrix is a quadratic polynomial in the spectral parameter λ with matrix coefficients. The general theories describing the isospectral deformations for polynomials with matrix coefficients were developed by Dubrovin [31, 32] in the middle of 70's and by Adler, van Moerbeke [1] few years later. Dubrovin's approach was based on the Baker–Akhiezer function. Both approaches were applied in rigid body problems (see [51, 1] respectively).

But, as it was shown in [23], non of these two theories can be directly applied in cases like this. Necessary modifications were suggested in [23], where the procedure of algebro-geometric integration was presented. It is based on some nontrivial facts from the theory of Prym varieties, such as the Mumford relation on theta divisors of unramified double coverings and the Mumford–Dalalyan theory (see [23, 55, 56, 13, 61, 62]).

Here we are going to follow closely the procedure from [23], with necessary changes, calculations and comments. As usual, we start with the spectral curve $\Gamma : \det(\tilde{L}(\lambda) - \mu \cdot 1) = 0$. So, we have

$$\Gamma : \mu^4 + \mu^2(\Delta_{12}^2 + \Delta_{34}^2 + 4\beta_3\beta_3^* + 4\beta_4\beta_4^*) + [\Delta_{12}\Delta_{34} + 2i(\beta_3^*\beta_4 - \beta_3\beta_4^*)]^2 = 0.$$

There is an involution $\sigma : (\lambda, \mu) \to (\lambda, -\mu)$ on the curve Γ, which corresponds to the skew-symmetry of the matrix $L(\lambda)$. Denote the factor-curve by $\Gamma_1 = \Gamma/\sigma$.

Lemma 1. *The curve Γ_1 is a smooth hyperelliptic curve of the genus $g(\Gamma_1) = 3$. The arithmetic genus of the curve Γ is $g_a(\Gamma) = 9$.*

Proof. The curve: $\Gamma_1 : u^2 + P(\lambda)u + [Q(\lambda)]^2 = 0$, is hyperelliptic, and its equation in the canonical forme is $u_1^2 = [P(\lambda)]^2/4 - [Q(\lambda)]^2$, where $u_1 = u + P(\lambda)/2$. Since $[P(\lambda)]^2/4 - [Q(\lambda)]^2$ is a polynomial of the degree 8, the genus of the curve Γ_1 is $g(\Gamma_1) = 3$. The curve Γ is a double covering of Γ_1, and the ramification divisor is of the degree 8. According to the Riemann–Hurwitz formula, the arithmetic genus of Γ is $g_a(\Gamma) = 9$.

Lemma 2. *The spectral curve Γ has four ordinary double points S_i, $i = 1, \dots, 4$. The genus of its normalization $\tilde{\Gamma}$ is five.*

Lemma 3. *The singular points S_i of the curve Γ are fixed points of the involution σ. The involution σ exchanges the two branches of Γ at S_i.*

Together with the curve Γ_1, one can consider curves $\mathcal{C}_1$ and $\mathcal{C}_2$ defined by the equations

$$\mathcal{C}_1 : v^2 = P(\lambda)/2 + Q(\lambda), \quad \mathcal{C}_2 : v^2 = P(\lambda)/2 - Q(\lambda).$$

Since the curve Γ_1 is hyperelliptic, in a study of the Prym variety Π the Mumford –Dalalyan theory can be applied (see [28, 24, 10]). Thus, using the previous Lemma, we come to

Theorem 7. a) *The Prymian Π is isomorphic to the product of the curves E_i:*

$$\Pi = \mathrm{Jac}(\mathcal{C}_1) \times \mathrm{Jac}(\mathcal{C}_2).$$

b) *The curve $\tilde{\Gamma}$ is the desingularization of $\Gamma_1 \times_{\mathbb{P}^1} \mathcal{C}_2$ and $\mathcal{C}_1 \times_{\mathbb{P}^1} \Gamma_1$.*
c) *The canonical polarization divisor Ξ of Π satisfies*

$$\Xi = E_1 \times \Theta_2 + \Theta_1 \times E_2,$$

where Θ_i is the theta-divisor of E_i.

4.1. Equally splitting double hyperelliptic coverings. According to the Mumford –Dalalyan theory (see [56, 13, 61]), double unramified coverings over a hyperelliptic curve $y^2 = P_{2g+2}(x)$ of genus g are in the correspondence with the divisions of the set of the zeroes of the polynomial P_{2g+2} on two disjoint nonempty subsets with even number of elements. We will consider those coverings which correspond to the divisions on subsets with *equal number of elements* and we can call them *equallysplitting*, since the Prym variety splits then as a sum of two varieties of equal dimension.

Now, let us consider with the fixed operator A from (12) the whole hierarchy of systems defined by the Lax equations

$$\dot{L}_B^{(N)} = [L_B^{(N)}, A], \quad L_B^{(N)}(\lambda) = \lambda^N B + \lambda^{N-1} M_1 + \cdots + M_N.$$

So $L_B^{(N)}(\lambda)$ is a polynomial in λ of degree $N \geq 2$, and the matrix B is proportional to the matrix χ: $B = d\chi$.

Generalizing the situation from the subection above, we see that the spectral curve Γ_N is a singular curve of the form

$$p_N(\lambda, \mu) = \mu^4 + P_N(\lambda)\mu^2 + [Q_N(\lambda)]^2 = 0,$$

where the polynomials P_N, Q_N have degree $\deg P_N = \deg Q_N = 2N$. So, its normalization is a double covering over the hyperelliptic curve

$$\mu_1^2 = P_N^2(\lambda)/4 - Q_N^2(\lambda)$$

of genus $g_N = 2N - 1$. This covering corresponds to the division of the set of zeroes on subsets of zeroes of the polynomials $P_N/2 - Q_N$ and $P_N/2 + Q_N$. This is an equallysplitting covering under the assumption $|\chi_{12}| \neq |\chi_{34}|$ we fixed at the beginning. It is easy to see that all equallysplitting coverings can be realized in such a way. So we have

Theorem 8. *The Lagrange bitop hierarchy realizes all equallysplitting coverings over the hyperelliptic curves of odd genus.*

5. The Poncelet theorem and Cayley's type conditions

The following integrable mechanical system is well known: motion of a free particle within an ellipsoid in the Euclidean space of any dimension d. On the boundary, the particle obeys the billiard law. Integrability of the system is related to classical geometrical properties of elliptical billiards: the Chasles, Poncelet and Cayley theorems. According to the Chasles theorem [5] every line in this space is tangent to $d-1$ quadrics confocal to the outer ellipsoid. Even more, all segments of the particle's trajectory are tangent to the same $d-1$ quadrics [5, 53]. The Poncelet theorem [58, 6, 47, 12, 11] put some light on closed billiard trajectories: *there exists a closed trajectory with $d-1$ given confocal caustics if and only if infinitely many such trajectories exist, and all of them have the same period.* Since the periodicity of a billiard trajectory depends only on its caustic surfaces, it is a natural question to find an analytical connection between them and corresponding period.

Cayley found [10] an analytical condition for caustic conics in the Euclidean plane case. Algebro-geometric proof of Cayley's theorem from Griffiths and Harris paper [41] is going to be presented now.

Given two ellipses $C(x) = 0$ and $D(x) = 0$ in the plane. From a given point a of the first ellipse, there exist two tangents $t_1,\ t_2$ on the second conic. These tangents intersect the first one, beside the point a, also at the points $b_1,\ b_2$ respectively. *The Chasles correspondence* relates the points b_1 and b_2 to the point a.

Theorem 9 (Poncelet)**.** *Given a polygon inscribed in one of the conics and subscribed around the another. Then there exist infinitely many such polygons; every point of the first ellipse is a vertex of one of them. All those polygons have the same number of edges.*

Next question is to find an analytical condition to determine weather for two given conics there exist n-tangle inscribed in one of them and subscribed around the another one. Such a condition was established by Cayley.

Theorem 10 (Cayley)**.** *There exist an n-tangle inscribed in D and subscribed around C if and only if*

$$\begin{vmatrix} C_3 & C_4 & \dots & C_{p+1} \\ C_4 & C_5 & \dots & C_{p+2} \\ \dots & \dots & \dots & \dots \\ C_{p+1} & C_{p+2} & \dots & C_{2p-1} \end{vmatrix} = 0; \quad \begin{vmatrix} C_2 & C_3 & \dots & C_{p+1} \\ C_3 & C_4 & \dots & C_{p+2} \\ \dots & \dots & \dots & \dots \\ C_{p+1} & C_{p+2} & \dots & C_{2p} \end{vmatrix} = 0,$$

where in the first case $n = 2p$, and $n = 2p + 1$ in the second. The matrix elements are determined from the development: $\sqrt{C + \lambda D} = A + B\lambda + C_2\lambda^2 + C_3\lambda^3 + \cdots$.

A mechanical interpretation of these theorems will be done in the next Section.

There are several proofs of these theorems. All of them are based on the theory of elliptic curves and functions.

Let C and D be two conics in $\mathbb{CP}^2$, intersecting at four points x_0, x_1, x_2, x_3. The dual conic D^* consists of tangents on D. Let us consider a configuration

$$E = \{(x, \xi) \mid x \in \xi\} \subset C \times D^*.$$

Then, E is a Riemann surface with two involutions $i(x, \xi) = (x', \xi)$, $i'(x', \xi) = (x', \xi')$. Their composition $j = i' \circ i$ is given by $j(x, \xi) = (x', \xi')$.

Thus, the Poncelet construction, starting with $p = (x, \xi)$ gives a polygon with n edges if and only if $j^n(p) = p$.

A mapping $E \to \mathbb{C} : (x, \xi) \mapsto x$ is two-sheeted covering of a Riemann sphere $\mathbb{CP}^1$, with four ramification points x_0, x_1, x_2, x_3. Applying the Hurvitz formula, we get $\chi(E) = 2\chi(P^1) - 4 = 0$, i.e., E is an elliptic curve.

One can chose (x_0, ξ_0) as neutral element of the group of the elliptic curve E, and denote $p = (\bar{x}, \bar{\xi}) = j(x_0, \xi_0)$.

To prove the Poncelet theorem, one has to show that:

the condition $j^n(p) = p$ does not depend on choice of the point p.

It follows from the next theorem.

Theorem 11. *The Poncelet construction with arbitrary initial condition $q = (x, \xi)$ leads to a closed n-tangle if and only if $np = 0$ on the elliptic curve E.*

Suppose a pencil of conics containing the points x_0, x_1, x_2, x_3 is done by D_t : $tC(x) + D(x) = 0$. The determinant $\det(tC + D)$ is a polynomial of third degree in t, with roots t_1, t_2, t_3 different from zero. For $t \neq t_i$, we construct a tangent on D_t which contains x_0. Let $x(t)$ be the second intersecting point of this tangent with the conic C. The values $t = t_i$ are mapped to x_i, and $t = \infty$ to x_0, since $D_\infty = C$. In this way, we have proved the following

Proposition 11. *The elliptic curve E is birationally equivalent to the Riemann surface of an algebraic function $\sqrt{\det(tC + D)}$ with the origin corresponding to the point $t = \infty$ and with the point $p = (\bar{x}, \bar{\xi})$ corresponding to one of two points over $t = 0$.*

Now, the Cayley condition can be derived from the previous results, by using the following

Proposition 12. *Given an elliptic curve $E : y^2 = (x - a)(x - b)(x - c)$, with a, b, c mutually different, not equal to zero. Suppose the point corresponding to $x = \infty$ is* *chosen to be neutral on E and suppose p is one of two points which correspond to $x = 0$. Then, p is of a finite order n if and only if*

$$\begin{vmatrix} C_3 & C_4 & \dots & C_{m+1} \\ C_4 & C_5 & \dots & C_{m+2} \\ \dots & \dots & \dots & \dots \\ C_{m+1} & C_{m+2} & \dots & C_{2m-1} \end{vmatrix} = 0, \qquad \begin{vmatrix} C_2 & C_3 & \dots & C_{m+1} \\ C_3 & C_4 & \dots & C_{m+2} \\ \dots & \dots & \dots & \dots \\ C_{m+1} & C_{m+2} & \dots & C_{2m} \end{vmatrix} = 0,$$

for $n = 2m$ in the first, and for $n = 2m + 1$ in the second case, where matrix elements are defined by

$$\sqrt{(x-a)(x-b)(x-c)} = A + Bx + C_2x^2 + C_3x^3 + \cdots .$$

The generalization of Cayley's theorem for arbitrary finite dimension is established by Dragović and Radnović [27, 28, 29, 30]. This generalization was done in [27, 28] by use of the Veselov–Moser discrete quadratic $L - A$ pair for the classical Heisenberg magnetic model [54].

The integrability of elliptical billiard systems in the Lobachevsky space was proved by Veselov in [64]. There, Veselov used discrete linear $L - A$ pair, which is quite different from the one used in the Euclidean case.

6. Basic notions on billiard systems

Let (Q, g) be a d-dimensional Riemannian manifold and let $D \subset Q$ be a domain with a smooth boundary Γ. Let $\pi : T^*Q \to Q$ be a natural projection and let g^{-1} be the contravariant metric on the cotangent bundle, in coordinates

$$|p| = \sqrt{g^{-1}(p,p)} = \sqrt{g^{ij}p_ip_j}, \quad p \in T^*_xQ.$$

Consider the *reflection mapping* $r : \pi^{-1}\Gamma \to \pi^{-1}\Gamma$, $p_- \mapsto p_+$, which associates the covector $p_+ \in T^*_xQ$, $x \in \Gamma$ to a covector $p_- \in T^*_xQ$ such that the following conditions hold:

$$|p_+| = |p_-|, \quad p_+ - p_- \perp \Gamma.$$

A *billiard* in D is a dynamical system with the phase space $M = T^*D$ whose trajectories are geodesics given by the Hamiltonian equations

$$\dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p}, \quad H(p,x) = \frac{1}{2}g_x^{-1}(p,p),$$

reflected at points $x \in \Gamma$ according to the billiard law: $r(p_-) = p_+$. Here p_- and p_+ denote the momenta before and after the reflection. If some potential force field $V(x)$ is added than the system is described with the same reflection law and Hamiltonian equations with the Hamiltonian $H(p,x) = \frac{1}{2}g_x^{-1}(p,p) + V(x)$.

A function $f : T^*Q \to \mathbb{R}$ is an *integral* of the billiard system if it commutes with the Hamiltonian ($\{f, H\} = 0$) and does not change under the reflection ($f(x,p) = f(x, r(p))$, $x \in \Gamma$). The billiard is *completely integrable in the sense of Birkhoff* if it has d integrals polynomial in the momenta, which are in involution, and almost everywhere independent (see [47]).

The classical integrable examples, with smooth boundary, are billiards inside ellipsoids on the Euclidean and hyperbolic spaces and spheres, with integrals quadratic in the velocities [47]. These systems can be also considered as discrete integrable systems [54]. The explicit integrations in terms of theta-functions are performed by Veselov, Moser and Fedorov (see [54], [36]).

7. Periodical trajectories of elliptical billiards in $\mathbb{R}^d$

In this section, first, we are going to list the main steps of algebro-geometric integration of the elliptic billiard, following [54]. Then, the connection between periodic billiard trajectories and points of finite order on the corresponding hyperelliptic curve will be established and the Cayley-type conditions will be derived, as they were obtained in [27, 28].

7.1. XYZ Model and Isospectral Curves.

Following [54], the billiard system will be considered as a system with the discrete time. Using its integration procedure, the connection between periodic billiard trajectories and points of finite order on the corresponding hyperelliptic curve will be established.

Elliptical Billiard as a Mechanical System with the Discrete Time. Let the ellipsoid in $\mathbb{R}^d$ be given by $(Ax, x) = 1$. We can assume that A is a diagonal matrix, with different eigenvalues. The billiard motion within the ellipsoid is determined by the following equations:

$$x_{k+1} - x_k = \mu_k y_{k+1}$$
$$y_{k+1} - y_k = \nu_k A x_k,$$

where

$$\mu_k = -\frac{2(Ay_{k+1}, x_k)}{(Ay_{k+1}, y_{k+1})}, \quad \nu_k = -\frac{2(Ax_k, y_k)}{(Ax_k, Ax_k)}.$$

Here x_k is a sequence of points of billiard bounces, while $y_k = \dfrac{x_k - x_{k-1}}{|x_k - x_{k-1}|}$ are the momenta.

Connection between Billiard and XYZ Model. To the billiard system with the discrete time, Heisenberg XYZ model can be joined, in the way described by Veselov and Moser in [54] and which is going to be presented here.

Consider the mapping $\varphi : (x, y) \mapsto (x', y')$ given by

$$x'_k = Jy_{k+1} = J(y_k + \nu_k A x_k), \quad y'_k = -J^{-1}x_k, \quad J = A^{-\frac{1}{2}}.$$

Notice that the dynamics of φ contains the billiard dynamics:

$$x''_k = Jy'_{k+1} = -x_{k+1}, \quad y''_k = -J^{-1}x'_k = -y_{k+1},$$

and define the sequence $(\bar{x}_k, \bar{y}_k)$:

$$(\bar{x}_0, \bar{y}_0) := (x_0, y_0), \quad (\bar{x}_{k+1}, \bar{y}_{k+1}) := \varphi(\bar{x}_k, \bar{y}_k),$$

which obeys the following relations:

$$\bar{x}_{k+1} = J\bar{y}_k + \nu_k J^{-1}\bar{x}_k, \quad \bar{y}_{k+1} = -J^{-1}\bar{x}_k,$$

where the parameter ν_k is such that $|\bar{y}_k| = 1$, $(A\bar{x}_k, \bar{x}_x) = 1$. This can be rewritten in the following way:

$$\bar{x}_{k+1} + \bar{x}_{k-1} = \nu_k J^{-1}\bar{x}_k.$$

Now, for the sequence $q_k := J^{-1}\bar{x}_k$, we have:

$$q_{k+1} + q_{k-1} = \nu_k J^{-1} q_k, \quad |q_k| = 1.$$

These equations represent the equations of the discrete Heisenberg XYZ system.

Theorem 12. [54] *Let $(\bar{x}_k, \bar{y}_k)$ be the sequence connected with elliptical billiard in the described way. Then $q_k = J^{-1}\bar{x}_k$ is a solution of the discrete Heisenberg system.*

Conversely, if q_k is a solution to the Heisenberg system, then the sequence $x_k = (-1)^k J q_{2k}$ is a trajectory of the discrete billiard within an ellipsoid.

Integration of the Discrete Heisenberg XYZ System. Usual scheme of algebro-geometric integration contains the following [54]. First, the sequence $L_k(\lambda)$ of matrix polynomials has to be determined, together with a factorization

$$L(\lambda) = B(\lambda)C(\lambda) \mapsto C(\lambda)B(\lambda) = B'(\lambda)C'(\lambda) = L'(\lambda),$$

such that the dynamics $L \mapsto L'$ corresponds to the dynamics of the system q_k. For each problem, finding this sequence of matrices requires a separate search and a mathematician with the excellent intuition. All matrices L_k are mutually similar, and they determine the same *isospectral curve* $\Gamma : \det(L(\lambda) - \mu I) = 0$. The factorization $L_k = B_k C_k$ gives splitting of spectrum of L_k. Denote by ψ_k the corresponding eigenvectors. Consider these vectors as meromorphic functions on Γ and denote their pole divisors by D_k.

The sequence of divisors is linear on the Jacobian of the isospectral curve, and this enables us to find, conversely, eigenfunctions ψ_k, then matrices L_k, and, finally, the sequence (q_k).

Now, integration of the discrete XYZ system by this method will be shortly presented. Details of the procedure can be found in [54].

The equations of discrete XYZ model are equivalent to the isospectral deformation:

$$L_{k+1}(\lambda) = A_k(\lambda) L_k(\lambda) A_k^{-1}(\lambda),$$

where

$$L_k(\lambda) = J^2 + \lambda q_{k-1} \wedge J q_k - \lambda^2 q_{k-1} \otimes q_{k-1}, \quad A_k(\lambda) = J - \lambda q_k \otimes q_{k-1}.$$

The equation of the isospectral curve $\Gamma : \det(L(\lambda) - \mu I) = 0$ can be written in the following form:

$$\nu^2 = \prod_{i=1}^{d-1}(\mu - \mu_i)\prod_{j=1}^{d}(\mu - J_j^2), \tag{14}$$

where $\nu = \lambda \prod_{i=1}^{d-1}(\mu - \mu_i)$ and $\mu_1, \dots, \mu_{d-1}$ are zeroes of the function:

$$\phi_\mu(x, Jy) = \sum_{i=1}^{d} \frac{F_i(x, y)}{\mu - J_i^2},$$

$$F_i = x_i^2 + \sum_{j \neq i} \frac{(x \wedge Jy)_{ij}^2}{J_i^2 - J_j^2}, \quad x = q_{k-1}, \quad y = q_k.$$

It can be proved that $\mu_1, \dots, \mu_{d-1}$ are parameters of the caustics corresponding to the billiard trajectory [53]. Another way for obtaining the same conclusion is to calculate them directly by taking the first segment of the billiard trajectory to be parallel to a coordinate axe.

If eigenvectors ψ_k of matrices $L_k(\lambda)$ are known, it is possible to determine uniquely members of the sequence q_k. Let D_k be the divisor of poles of function ψ_k on curve Γ. Then [54]:

$$D_{k+1} = D_k + P_\infty - P_0,$$

where P_∞ is the point corresponding to the value $\mu = \infty$ and P_0 to $\mu = 0$, $\lambda = (q_k, J^{-1} q_{k+1})^{-1}$.

7.2. Characterization of Periodical Billiard Trajectories.

In the next lemmae, we establish a connection between periodic billiard sequences q_k and periodic divisors D_k.

Lemma 4. [27] *Sequence of divisors D_k is n-periodic if and only if the sequence q_k is also periodic with the period n or $q_{k+n} = -q_k$ for all k.*

Lemma 5. [27] *The billiard is, up to the central symmetry, periodic with the period n if and only if the divisor sequence D_k joined to the corresponding Heisenberg XYZ system is also periodic, with the period $2n$.*

Applying the previous lemma, we obtain the main statement of this section:

Theorem 13. [28] *A condition on a billiard trajectory inside ellipsoid $\mathcal{Q}_0$ in $\mathbb{R}^d$, with non-degenerate caustics $\mathcal{Q}_{\mu_1}, \dots, \mathcal{Q}_{\mu_{d-1}}$, to be periodic, up to the central symmetry, with the period $n \geq d$ is:*

$$\operatorname{rank} \begin{pmatrix} B_{n+1} & B_n & \dots & B_{d+1} \\ B_{n+2} & B_{n+1} & \dots & B_{d+2} \\ \dots & \dots & \dots & \dots \\ B_{2n-1} & B_{2n-2} & \dots & B_{n+d-1} \end{pmatrix} < n - d + 1,$$

where $\sqrt{(x-\mu_1)\cdots(x-\mu_{d-1})(x-a_1)\cdots(x-a_d)} = B_0 + B_1 x + B_2 x^2 + \cdots$.

Cases of Singular Isospectral Curve. When all $a_1, \dots, a_d, \mu_1, \dots, \mu_{d-1}$ are mutually different, then the isospectral curve has no singularities in the affine part. However, singularities appear in the following three cases and their combinations:

(i) $a_i = \mu_j$ for some i, j. The isospectral curve (14) decomposes into a rational and a hyperelliptic curve. Geometrically, this means that the caustic corresponding to μ_i degenerates into the hyperplane $x_i = 0$. The billiard trajectory can be asymptotically tending to that hyperplane (and therefore cannot be periodic), or completely placed in this hyperplane. Therefore, closed trajectories appear when they are placed in a coordinate hyperplane. Such a motion can be discussed like in the case of dimension $d - 1$.

(ii) $a_i = a_j$ for some $i \neq j$. The boundary $\mathcal{Q}_0$ is symmetric.

(iii) $\mu_i = \mu_j$ for some $i \neq j$. The billiard trajectory is placed on the corresponding confocal quadric hyper-surface.

In the cases (ii) and (iii) the isospectral curve Γ is a hyperelliptic curve with singularities. In spite of their different geometrical nature, they both need the same analysis of the condition $2nP_0 \sim 2nE$ for the singular curve (14).

Aa a consequence of the Theorem 13, it can be applied not only for the case of the regular isospectral curve, but in the cases (ii) and (iii), too. Therefore, the following interesting property holds.

Theorem 14. *If the billiard trajectory within an ellipsoid in d-dimensional Eucledean space is periodic, up to the central symmetry, with the period $n < d$, then it is placed in one of the n-dimensional planes of symmetry of the ellipsoid.*

Proof. This follows immediately from Theorem 13 and the fact that the section of a confocal family of quadrics with a coordinate hyperplane is again a confocal family. □

This property can be seen easily for $d = 3$.

Example 6. Consider the billiard motion in an ellipsoid in the 3-dimensional space, with $\mu_1 = \mu_2$, when the segments of the trajectory are placed on generatrices of the corresponding one-folded hyperboloid, confocal to the ellipsoid. If there existed a periodic trajectory with period $n = d = 3$, the three bounces would have been coplanar, and the intersection of that plane and the quadric would have consisted of three lines, which is impossible. It is obvious that any periodic trajectory with period $n = 2$ is placed along one of the axes of the ellipsoid. So, there is no periodic trajectories contained in a confocal quadric surface, with period less or equal to 3.

8. Separable perturbations of integrable billiards

Appell introduced four families of hypergeometric functions of two variables in 1880's. Soon, he applied them in a solution of the Tisserand problem in the celestial mechanics. The Appell functions have several other applications, for example in the theory of algebraic equations, algebraic surfaces... The aim of this paper is to point out the relationship between the Appell functions F_4 and another subject from classical mechanics-separability of variables in the Hamilton-Jacobi equations.

The equation

$$\lambda V_{xy} + 3\,(yV_x - xV_y) + (y^2 - x^2)V_{xy} + xy\,(V_{xx} - V_{yy}) = 0, \tag{15}$$

appeared in Kozlov's paper [44] as a condition on the function $V = V(x, y)$ to be an integrable perturbation of certain type for billiard systems inside an ellipse

$$\frac{x^2}{A} + \frac{y^2}{B} = 1, \quad \lambda = A - B. \tag{16}$$

This equation is a special case of the Bertrand–Darboux equation [7, 14, 66]

$$(V_{yy} - V_{xx})(-2axy - b'y - bx + c_1) + 2V_{xy}(ay^2 - ax^2 + by - b'x + c - c') \\ + V_x(6ay + 3b) + V_y(-6ax - 3b') = 0.$$

It corresponds to the choice $a = -1/2$, $b = b' = c_1 = 0$, $c - c' = -\lambda/2$. The Bertrand–Darboux equation represents the necessary and sufficient condition for a natural mechanical system with two degrees of freedom

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y)$$

to be separable in elliptical coordinates or some of their degenerations.

Solutions of the equation (15) in the form of the Laurent polynomials in x, y were described in [16, 17]. The starting observation of this paper, that such solutions are simply related to the well-known hypergeometric functions of the Appell type is presented. Such a relation automatically gives a wider class of solutions of the equation (15)– new potentials are obtained for non-integer parameters. But what is more important, it shows the existence of a connection between separability of classical systems on one hand, and the theory of hypergeometric functions on the other one. Basic references for the Appell functions are [2, 3, 65]. Further, in section 3, similar formulae for potential perturbations for the Jacobi problem for geodesics on an ellipsoid from [16].

In the case of more than two degrees of freedom, the natural generalization for the equation (15) is the system:

The system

$$(a_i - a_r)^{-1}\left(x_i^2 V_{rs} - x_i x_r V_{is}\right) = (a_i - a_s)^{-1}\left(x_i^2 V_{rs} - x_i x_s V_{ir}\right) \quad i \neq r \neq s \neq i;$$

$$(a_i - a_r)^{-1} x_i x_r \left(V_{ii} - V_{rr}\right) - \sum_{j \neq i,r} (a_i - a_j)^{-1} x_i x_j V_{jr} \\ + V_{ir}\left[\sum_{j \neq i,r} (a_i - a_j)^{-1} x_j^2 + (a_r - a_i)^{-1}(x_i^2 - x_r^2)\right] \\ + V_{ir} + 3(a_i - a_r)^{-1}\left(x_r V_i - x_i V_r\right) = 0, \quad i \neq r,$$

where $V_i = \partial V/\partial x_i$, of $(n-1)\binom{n}{2}$ equations was formulated in [52] for arbitrary number of degrees of freedom n. In [52] the generalization of the Bertrand–Darboux theorem is proved. According to that theorem, the solutions of the system are potentials separable in generalized elliptic coordinates.

Some deeper explanation of the connection between the separability in elliptic coordinates and the Appell hypergeometric functions is not known yet.

8.1. Basic notations. The function F_4 is one of the four hypergeometric functions in two variables introduced by Appell [2, 3] and defined as a series:

$$F_4(a, b, c, d; x, y) = \sum \frac{(a)_{m+n}(b)_{m+n}}{(c)_m (d)_n} \frac{x^m}{m!} \frac{y^n}{n!},$$

where $(a)_n$ is the standard Pochhammer symbol:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\dots(a+n-1), \quad (a)_0 = 1,$$

(For example $m! = (1)_m$.)

The series F_4 is convergent for $\sqrt{x}+\sqrt{y} \le 1$. The functions F_4 can be analytically continued to the solutions of the equations:

$$\begin{aligned} x(1-x)\frac{\partial^2 F}{\partial x^2} - y^2\frac{\partial^2 F}{\partial y^2} - 2xy\frac{\partial^2 F}{\partial x \partial y} + [c-(a+b+1)x]\frac{\partial F}{\partial x} \\ -(a+b+1)y\frac{\partial F}{\partial y} - abF = 0, \\ y(1-y)\frac{\partial^2 F}{\partial y^2} - x^2\frac{\partial^2 F}{\partial x^2} - 2xy\frac{\partial^2 F}{\partial x \partial y} + [c'-(a+b+1)y]\frac{\partial F}{\partial y} \\ -(a+b+1)x\frac{\partial F}{\partial x} - abF = 0, \end{aligned}$$

8.2. Billiard inside an ellipse and its separable perturbations. Following [44, 15, 16] we will start with a billiard system which describes a particle moving freely within an ellipse (2). At the boundary we assume elastic reflections with equal impact and reflection angles. This system is completely integrable and it has an additional integral

$$K_1 = \frac{\dot{x}^2}{A} + \frac{\dot{y}^2}{B} - \frac{(\dot{x}y - \dot{y}x)^2}{AB}.$$

We are interested in a potential perturbations $V = V(x,y)$ such that the perturbed system has an integral $\tilde{K}_1$ of the form $\tilde{K}_1 = K_1 + k_1(x,y)$, where $k_1 = k_1(x,y)$ depends only on coordinates. This specific condition leads to the equation (15) on V (see [44]).

In [15, 16] the Laurent polynomial solutions of the equation (15) were given. The basic set of solutions consists of the functions

$$V_k = \sum_{i=0}^{k-2}(-1)^i \sum_{s=1}^{k-i-1} U_{kis}(x,y,\lambda) + y^{-2k}, \quad k \in N,$$

$$W_k = \sum_{i=0}^{k-2}\sum_{s=1}^{k-i-1} (-1)^s U_{kis}(y,x,\lambda) + x^{-2k}, \quad k \in N,$$

where

$$U_{kis} = \binom{s+i-1}{i} \frac{[1-(k-i)][2-(k-i)]\dots[s-(k-i)]}{\lambda^{s+i} s!} x^{2s} y^{-2k+2i}.$$

Now, we are going to rewrite the above formulae:

$$V_k = \sum_{i=0}^{k-2}(-1)^i \sum_{s=1}^{k-i-1} U_{kis}(x,y,\lambda) + y^{-2k}, \quad k \in N$$

$$= \sum_{i=0}^{k-2}(-1)^i \sum_{s=1}^{k-i-1} \frac{\Gamma(s+i)\Gamma(s+i-k+1)}{\Gamma(i+1)\Gamma(s)\Gamma(i-k+1)\Gamma(s+1)} \frac{x^{2s}y^{2(i-k)}}{\lambda^{s+i}} + y^{-2k}$$
$$= \frac{1}{y^{2k}}\left((1-k)\sum_{i=0}^{k-2}\sum_{s=1}^{k-i-1} \frac{(1)_{s+i-1}(2-k)_{s+i-1}}{i!(1)_{s-1}s!(1-k)_i} \frac{x^{2s}}{\lambda^s} \frac{(-y^2)^i}{\lambda^i} + 1\right)$$
$$= \frac{1}{y^{2k}}\left((1-k)\frac{x^2}{\lambda}\sum_{i=0}^{k-2}\sum_{s=0}^{k-i-2} \frac{(1)_{s+i}(2-k)_{s+i}}{(2)_s(1-k)_i} \frac{(x^2)^s}{s!\lambda^s} \frac{(-y^2)^i}{i!\lambda^i} + 1\right)$$
$$= \frac{1}{\tilde{y}^k\lambda^k}\big((1-k)\tilde{x}F_4(1;\ 2-k;\ 2,\ 1-k,\ \tilde{x},\ -\tilde{y}) + 1\big),$$

where $\tilde{x} = x^2/\lambda$, $\tilde{y} = -y^2/\lambda$, and F_4 is the Appell function. We have just obtained a simple formula which expresses the potentials V_k, from [16], for $k \in N$ through the Appell functions. (The scalar coefficient λ^{-k} is not essential and we will not write it any more). We can use this formula to spread the family of solutions of the equation (15) out of the set of the Laurent polynomials. We obtain new solutions of the equation (15) if we let the parameter k in the last formula to be arbitrary, not only a natural number.

Let $V(x,y) = \sum a_{nm}x^n y^m$. Then the equation (15) reduces to

$$\lambda nma_{n,m} = (n+m)(ma_{n-2,m} - na_{n,m-2}).$$

If one of the indices, for example the first one, belongs to Z, then V does not have essential singularities. Put $a_{0,-2\gamma} = 1$, where γ is not necessary an integer.

Let us define

$$a_{\underbrace{2s+2}_{n},\underbrace{2i-2\gamma}_{m}} = \frac{(-1)^i(1)_{s+i}(2-\gamma)_{s+i}}{(2)_s(1-\gamma)_i s! i! \lambda^{s+i}}.$$

and denote

$$V_\gamma = \tilde{y}^{-\gamma}\left((1-\gamma)\tilde{x}F_4(1, 2-\gamma, 2, 1-\gamma, \tilde{x}, \tilde{y}) + 1\right). \tag{17}$$

Then we have

Theorem 15. *Every function V_γ given with* (17) *and $\gamma \in C$ is a solution of the equation* (15)*).*

The theorem gives new potentials for noninteger γ.

Mechanical interpretation. With $\gamma \in R^-$ and the coefficient multiplying V_γ positive, we have potential barrier along x-axis. We can consider billiard motion in upper half plane. Then we can assume that a cut is done along negative part of y-axis, in order to get unique-valued real function as a potential.

8.3. The Jacobi problem for geodesics on an ellipsoid.

The Jacobi problem for the geodesics on an ellipsoid

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = 1$$

has an additional integral

$$K_1 = \Big(\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2}\Big)\Big(\frac{\dot{x}^2}{A} + \frac{\dot{y}^2}{B} + \frac{\dot{z}^2}{C}\Big).$$

Potential perturbations $V = V(x, y, z)$ such that perturbed systems have integrals of the form $\tilde{K}_1 = K_1 + k(x, y, z)$ satisfy the following system (see [16])

$$\begin{aligned}
&\left(\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2}\right) V_{xy}\frac{A-B}{AB} - 3\frac{y}{B^2}\frac{V_x}{A} + 3\frac{x}{A^2}\frac{V_y}{B} + \left(\frac{x^2}{A^3} - \frac{y^2}{B^3}\right)V_{xy}\\
&\qquad + \frac{xy}{AB}\left(\frac{V_{yy}}{A} - \frac{V_{xx}}{B}\right) + \frac{zx}{CA^2}V_{zy} - \frac{zy}{CB^2}V_{zx} = 0\\
&\left(\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2}\right) V_{yz}\frac{B-C}{BC} - 3\frac{z}{C^2}\frac{V_y}{B} + 3\frac{y}{B^2}\frac{V_z}{C} + \left(\frac{y^2}{B^3} - \frac{z^2}{C^3}\right)V_{yz}\\
&\qquad + \frac{yz}{BC}\left(\frac{V_{zz}}{B} - \frac{V_{yy}}{C}\right) + \frac{xy}{AB^2}V_{xz} - \frac{xz}{AC^2}V_{xy} = 0\\
&\left(\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2}\right) V_{zx}\frac{C-A}{AC} - 3\frac{x}{A^2}\frac{V_z}{C} + 3\frac{z}{C^2}\frac{V_x}{A} + \left(\frac{z^2}{C^3} - \frac{x^2}{A^3}\right)V_{zx}\\
&\qquad + \frac{xz}{AC}\left(\frac{V_{xx}}{C} - \frac{V_{zz}}{A}\right) + \frac{zy}{BC^2}V_{xy} - \frac{yx}{BA^2}V_{yz} = 0
\end{aligned} \tag{18}$$

The last system (18) replaces the equation (15) in this problem. Solutions of the system in the Laurent polynomial form were found in [16]. We can transform them in the following way.

$$\begin{aligned}
V_{l_0}(x,y,z) &= \sum_{0\le k\le s, k+c\le l_0} (-1)^s \binom{s+k-1}{k} (x^2)^{-l_0+k}(y^2)^s(z^2)^{l_0-(k+s)-1}\\
&\quad\times \frac{C^{s+k}(C-A)^s(C-B)^k 2^{k+s}(-l_0+1)\dots(-l_0+(k+s))}{B^k A^s (B-A)^{k+s} 2^s 2^k s!(-l_0+1)\dots(-l_0+k)} (z^2)^{l_0-(k+s)-1}\\
&= \sum \frac{(s+k-1)!(-l_0+1)(-l_0+2)_{s+k-1}(z^2)^{l_0}}{k!(s-1)!s!(-l_0+1)_k (x^2)^{l_0}} \left[\frac{x^2C(A-C)}{z^2(B-A)A}\right]^s \left[\frac{y^2C(C-B)}{z^2(B-A)B}\right]^k\\
&= (-l_0+1)\left(\frac{z^2}{x^2}\right)^{l_0} \sum \frac{(1)_{s+k-1}(-l_0+2)_{s+k-1}}{(2)_{s-1}(-l_0+1)_k}\hat{x}^s\hat{y}^k\\
&= (-l_0+1)\left(\frac{z^2}{x^2}\right)^{l_0} F_4(1; -l_0+2; 2, -l_0+1, \hat{x}, \hat{y}),
\end{aligned}$$

where

$$\frac{x^2C(A-C)}{z^2(B-A)A} = \hat{x}, \quad \frac{y^2C(C-B)}{z^2(B-A)B} = \hat{y}$$

In the above formulae l_0 is an integer. We have the straightforward generalization:

Theorem 16. *For every $\gamma \in C$ the function*

$$V_\gamma = (-\gamma+1)\left(\frac{z^2}{x^2}\right)^\gamma F_4(1; -\gamma+2; 2, -\gamma+1, \hat{x}, \hat{y}),$$

is a solution of the system (18).

9. Algebro-geometric approach to the quantum Yang–Baxter equation

One of the central objects in mathematical physics in last 25 years is the R matrix, or the solution $R(t,h)$ of the quantum Yang–Baxter equation

$$R^{12}(t_1-t_2,h)R^{13}(t_1,h)R^{'23}(t_2,h)=R^{23}(t_2,h)(R^{13}(t_1,h)R^{12}(t_1-t_2,h).$$

Here t is so called *spectral parameter* and h is *Planck constant.* If the h dependence satisfies the quasi-classical property $R=I+hr+O(h^2)$ the classical r-matrix r satisfies the classical Yang–Baxter equation. Classification of the solutions of the classical Yang–Baxter equation was done by Belavin and Drinfeld in 1982. The problem of classification of the quantum R matrices is still open. Some results have been obtained in the basic 4×4 case (see [46, 18, 19, 20]).

Krichever in [46] applied the idea of "finite-gap" integration to the theory of the Yang equation

$$R^{12}L^{13}L^{'23}=L^{'23}L^{13}R^{12}.$$

The principal objects that are considered are $2n\times 2n$ matrices L, understood as 2×2 matrices whose elements are $n\times n$ matrices; $L=l^{i\alpha}_{j\beta}$ is considered as a linear operator in the tensor product $C^n\otimes C^2$. The theorem from [46] uniquely characterizes them by the following spectral data:

(1) the vacuum vectors, i.e., vectors of the form $X\otimes U$, which L maps to vectors of the same form $Y\otimes V$, where $X,Y\in C^n$ and $U,V\in C^2$;
(2) the vacuum curve $\Gamma: P(u,v)=\det L=0$, where $L^i_j=V^\beta L^{i\alpha}_{j\beta}U_\alpha$, $(V^\beta)=(1,-v)$, $X_n=Y_n=U_2=V_2=1$; $U_1=u$, $V_1=v$;
(3) the divisors of the vector-valued functions $X(u,v)$, $Y(u,v)$, $U(u,v)$, $V(u,v)$, which are meromorphic on the curve Γ. But the Krichever method used in [46, 18, 19, 20, 21] works with even-dimensional matrices. Here we want to discus the case of odd-dimensional matrices considering the case of 9×9 matrices. We introduce the notion of *vacuum locus* as an analogue of the vacuum curve. We also show that a vacuum locus could be a finite set for some of the solutions of the quantum Yang–Baxter equation.

Now, the matrices $L=l^{i\alpha}_{j\beta}$ are considered as a linear operator in the tensor product $C^3\otimes C^3$. The same is for matrices R. As before, we want to parametrize the vacuum vectors, i.e., vectors of the form $X\otimes U$, which L maps to vectors of the same form $Y\otimes V$, where $X,Y,U,V\in C^3$. Assume the notation:

$$U^t=(u_1,u_2,1),\ V^t=(v_1,v_2,1),\ \tilde V_1=(1,0,-v_1),\ \tilde V_2=(0,1,-v_2).$$

The vacuum locus is the set which parametrizes the vacuum vectors.
Lemma-Definition *The affine part of the vacuum locus is the set of* $(u_1,u_2,v_1,v_2)\in C^4$ *such that*

$$P(u_1,u_2,v_1,v_2):=\det L(\lambda)=0$$

identically in λ*, where* $L^i_j(\lambda)=(\tilde V_1+\lambda\tilde V_2)^\beta L^{i\alpha}_{j\beta}U_\alpha$.

The lemma follows from the fact that if two regular matrix binomials of the first degree are equivalent then they are strictly equivalent (see [37]). The condition

$\det L(\lambda) = 0$ identically in λ gives four equations in C^4 since $\det L(\lambda)$ is a polynomial of the third degree in λ. So, for the general matrix L, the set $P(u_1, u_2, v_1, v_2)$ is a finite subset of C^4. The working hypothesis among the specialists was that in a case of the solutions of the quantum Yang–Baxter equation which depend on spectral parameter, there should be an algebraic curve which parametrizes some of the vacuum vectors. However, even in the case of the solutions of the Yang–Baxter equation it is possible that the vacuum locus is a finite set. This can be proved for the famous Izergin–Korepin 9×9 R-matrix (see [43]).

Proposition 13. *The vacuum locus for the Izergin–Korepin R-matrix is a finite set.*

The structure of this set is still not clear. In order to apply some of the Krichever ideas such set should have a subset which satisfies two conditions:

- it is closed for the composition of relations properly defined;
- it is big enough to give a possibility to reconstruct matrices R, L, L' and their products.

This could lead to the construction of the solutions of the Yang-Baxter equation in which spectral parameter belongs to some discrete group.

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