

**Marko Nedeljkov and Stevan Pilipović**

**GENERALIZED FUNCTION ALGEBRAS  
AND PDEs WITH SINGULARITIES.  
A SURVEY**

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## Introduction

The aim of this survey article is to explain basic ideas of generalized function algebras (Part I), to illustrate the analysis of second order PDEs (Part II) and first order hyperbolic systems (Part III) with singular coefficients and singular data.

Part I deals with the basic definitions of Colombeau type spaces and algebras, while Parts II and III present main statements of our investigations on PDEs of quoted types. Especially, complete proofs on solving elliptic linear equations with singular coefficients and singular data are given.

Colombeau had constructed his well-known algebras by purely algebraic methods, [9, 10]. Since then, algebras of Colombeau generalized numbers and functions became a very useful framework for linear problems with singularities and especially for nonlinear problems. Here we refer to monographs [3, 11, 29, 31, 68, 72] for the so-called Colombeau approach and for another approach, we refer to monographs [87, 88].

Many linear and nonlinear problems with irregular data or irregular coefficients, have been successfully analyzed by the mean of appropriate approximations through nets of  $C^\infty$  functions which fits into Colombeau algebra  $\mathcal{G}$  of generalized functions. We extend the references of this article in order to emphasize a part of large literature related to linear and nonlinear equations in the framework of generalized function algebras.

For the general theory of Colombeau type generalized functions, beside the quoted monographs, we refer to [29, 30, 46, 48, 49, 50] for generalized functions on manifolds, to [4, 5, 17, 18, 21, 62, 83, 98] for embeddings of different function spaces and to [19, 24, 25, 63, 89, 90] for the topology in Colombeau generalized function algebras and spaces.

Linear equations and pseudodifferential operators were studied in [23, 34, 35, 52, 66, 82], while local and microlocal analysis within Colombeau type algebras were studied in [37, 38, 55, 56, 76, 78, 91].

Concerning nonlinear equations, not mentioned in this paper, one can see also [3, 15, 20, 27, 28, 36, 47, 64, 92].

We give in Part I a general concept of extending the theory of locally convex spaces and algebras to the corresponding Colombeau type spaces and algebras. Since we are mainly interested in equations, we give several constructions which are related to Sobolev and Hölder spaces of functions. Also, for the later use of a class of stochastic equations, we recall the definition of a generalized stochastic process. In the last subsection of Part II, we introduce generalized semigroups of operators which will be used in the analysis of a semilinear heat equation with singularities.

We present in Part II, Subsection 1.1 our method of solving various classes of second order elliptic linear differential equations with strong singularities in the framework of generalized function algebras. We start with the Whitney type definitions of generalized functions on closed sets. Then, with all the details, we construct and discuss corresponding boundary value problems in relation to the classical results. The results of this subsection are not published before.

A quasilinear Dirichlet problem for uniformly elliptic equations whose coefficients have the lack of regularity assumptions and with singular boundary conditions is considered in Subsection 1.2 [84]. In our setting of a problem, we replace an equation  $\operatorname{div} \mathbf{A}(Du) = 0$  with a net of equations with regular coefficients and a singular boundary condition with an appropriate regularized net of boundary conditions.

In Subsection 2.1 is considered a semilinear wave equations [60] in space dimension  $n \leq 9$  with singular data and various types of nonlinearities. In general, a nonlinear term is regularized with respect to a small parameter  $\varepsilon$  such that it becomes globally Lipschitz for each  $\varepsilon$ . A net of solutions to a net of Cauchy problems obtained in this way determines a generalized solution. For certain growth conditions on a nonlinear term the equation is uniquely solved without regularizations. Note, in certain cases, a solution to the regularized equation is also a solution to the non-regularized one.

Subsection 2.2 deals with some classes of wave equations with stochastic perturbations as singularities.

We study in Section 3 [67] a class of heat equations with singularities extending the use of semigroups to some classes of PDE's with singular coefficients. The general idea is simple and it lies in the core of a construction of generalized functions. Regularized PDE, in fact a net of equations, is solved with an appropriate net of semigroups. The solution obtained in this way will represent a generalized function. For this reason, we will use different variants of Colombeau-like generalized function algebras. By the use of semigroups related to the Schrödinger operator  $\Delta - V$  we present the results concerning the semilinear heat equation with singular Cauchy data.

Part III is devoted to solving a class of Riemann problems to one-dimensional  $2 \times 2$  conservation law systems which do not always admit a classical entropy solution consisting of elementary waves. Two new solution types, delta and singular shock waves, could appear in such a situation. We shall use two solutions concepts for describing them.

Also, we shall briefly describe what might happen when such a wave interacts with the wave of the same type or with some other elementary wave.

We present in the Appendix a very general construction of generalized function Colombeau type algebras through a purely topological description of such algebras. We will show that such algebras fit very well in the general theory of the well known sequence spaces forming appropriate algebras [17].

## PART I: BASIC DEFINITIONS

### 1. Different algebras and spaces of generalized functions

**1.1. Extensions over locally convex spaces and algebras.** This subsection contains special constructions of Colombeau type algebras. One can find a more abstract general approach in [17] as well as in the Appendix.

Let  $E$  be a vector space on  $\mathbb{C}$  with an increasing sequence of seminorms  $\mu_n$ ,  $n \in \mathbb{N}$ . The space of moderate nets of  $\mathcal{E}_M(E)$ , respectively, of null nets of  $\mathcal{N}(E)$ , is constituted by nets  $R_\varepsilon \in E^{(0,1]}$  with the properties

$$(\forall n \in \mathbb{N})(\exists a \in \mathbb{R})(\mu_n(R_\varepsilon) = \mathcal{O}(\varepsilon^a)),$$

$$\text{respectively, } (\forall n \in \mathbb{N})(\forall b \in \mathbb{R})(\mu_n(R_\varepsilon) = \mathcal{O}(\varepsilon^b)).$$

( $\mathcal{O}$  is the Landau symbol.) The quotient space  $\mathcal{G}(E) = \mathcal{E}_M(E)/\mathcal{N}(E)$  with elements  $[F_\varepsilon], [G_\varepsilon], \dots$ , (equivalence classes are denoted by  $[\cdot]$ ) is called the Colombeau extension of  $E$ . Putting  $v_n(R_\varepsilon) = \sup\{a; \mu_n(R_\varepsilon) = \mathcal{O}(\varepsilon^a)\}$  and  $e_n((R_\varepsilon)\varepsilon, (S_\varepsilon)\varepsilon) = \exp(-v_n(R_\varepsilon - S_\varepsilon))$ ,  $n \in \mathbb{N}$ , we obtain that  $(e_n)_n$  is a sequence of ultra-pseudometrics defining the ultra-metric topology (sharp topology) on  $\mathcal{G}(E)$ .

If  $E = \mathbb{C}$  (or  $E = \mathbb{R}$ ) and the seminorms are equal to the absolute value, then the corresponding spaces are  $\mathcal{E}_0, \mathcal{N}_0$ ;  $\mathcal{E}_0$  is an algebra and  $\mathcal{N}_0$  is an ideal and, as a quotient, one obtains Colombeau algebra of generalized complex numbers  $\bar{\mathbb{C}} = \mathcal{E}_0/\mathcal{N}_0$  (or  $\bar{\mathbb{R}}$ ). These algebras are not fields as expected at the first sight, but rings. If a set  $O$  is open in  $\mathbb{R}^n$  and  $E = C^\infty(O)$  is endowed with the usual sequence of seminorms (this is Schwartz space  $\mathcal{E}(O)$ ),

$$\mu_{K_\nu, \nu}(\phi) = \sup_{x \in K_\nu, |\alpha| \leq \nu} |\phi^{(\alpha)}(x)|, \nu \in \mathbb{N}_0,$$

where  $(K_\nu)_\nu$  is an increasing sequence of compact sets that exhausts  $O$ , then the above definition gives Colombeau simplified algebra  $\mathcal{G}(O) = \mathcal{E}_M(O)/\mathcal{N}(O)$  [9], [72]. Its elements are called generalized functions and we keep this name for elements of any spaces or algebras constructed as extensions of some functional space  $E$ .

Then the embedding of compactly supported Schwartz distributions (elements of  $\mathcal{E}'(O)$ ) is made through the convolution with a net of mollifiers  $\phi_\varepsilon = \varepsilon^{-n} \phi(\cdot/\varepsilon)$  constructed by a rapidly decreasing function  $\phi \in \mathcal{S}(\mathbb{R}^n)$  with the properties  $\int \phi(t) dt = 1$ ,  $\int t^m \phi(t) dt = 0$ ,  $m \in \mathbb{N}^n$ . The embedding is given by  $f \mapsto [f * \phi_\varepsilon|_O]$ . By the sheaf properties of  $\mathcal{D}'(O)$  and  $\mathcal{G}(O)$ , this embedding is extended to  $\mathcal{D}'(O)$ .

Besides a cited paper, one can also look in [9] for Colombeau's original approach.

**1.2. Colombeau generalized functions with uniform bounds.** In general, for an open set  $O \subset \mathbb{R}^n$ , denote by  $C_b^\infty(O)$  the algebra of smooth functions on  $O$  bounded together with all their derivatives.

We shall briefly repeat some definitions of Colombeau algebra given in [77]. Denote  $\mathbb{R}_+^2 := \mathbb{R} \times (0, \infty)$ ,  $\overline{\mathbb{R}}_+^2 := \mathbb{R} \times [0, \infty)$ . Let  $C_b^\infty(\mathbb{R}_+^2)$  be a set of all functions  $u \in C^\infty(\mathbb{R}_+^2)$  satisfying  $u|_{\mathbb{R} \times (0, T)} \in C_b^\infty(\mathbb{R} \times (0, T))$  for every  $T > 0$ . Let us remark that every element of  $C_b^\infty(\mathbb{R}_+^2)$  has a smooth extension up to the line  $\{t = 0\}$ , i.e.,  $C_b^\infty(\mathbb{R}_+^2) = C_b^\infty(\overline{\mathbb{R}}_+^2)$ . This is also true for  $C_b^\infty(\mathbb{R}_+^2)$ .

We will give explicit definitions and one can easily make the description of these spaces and algebras as in Subsection 1.1.  $\mathcal{E}_{M,g}(\mathbb{R}_+^2)$  is the set of all maps  $G : (0, 1) \times \mathbb{R}_+^2 \rightarrow \mathbb{R}$ ,  $(\varepsilon, x, t) \mapsto G_\varepsilon(x, t)$ ,  $G_\varepsilon \in C_b^\infty(\mathbb{R}_+^2)$  for every  $\varepsilon \in (0, 1)$  satisfying: for every  $(\alpha, \beta) \in \mathbb{N}_0^2$  and  $T > 0$ , there exists  $N \in \mathbb{N}$  such that

$$\sup_{(x,t) \in \mathbb{R} \times (0, T)} |\partial_x^\alpha \partial_t^\beta G_\varepsilon(x, t)| = \mathcal{O}(\varepsilon^{-N}), \text{ as } \varepsilon \rightarrow 0.$$

$\mathcal{N}_g(\mathbb{R}_+^2)$  is the set of all  $G_\varepsilon \in \mathcal{E}_{M,g}(\mathbb{R}_+^2)$  satisfying: for every  $(\alpha, \beta) \in \mathbb{N}_0^2$ ,  $a \in \mathbb{R}$  and  $T > 0$

$$\sup_{(x,t) \in \mathbb{R} \times (0, T)} |\partial_x^\alpha \partial_t^\beta G_\varepsilon(x, t)| = \mathcal{O}(\varepsilon^a), \text{ as } \varepsilon \rightarrow 0.$$

Clearly,  $\mathcal{N}_g(\mathbb{R}_+^2)$  is an ideal of the multiplicative differential algebra  $\mathcal{E}_{M,g}(\mathbb{R}_+^2)$ . Thus one defines the multiplicative differential algebra  $\mathcal{G}_g(\mathbb{R}_+^2)$  of generalized functions by  $\mathcal{G}_g(\mathbb{R}_+^2) = \mathcal{E}_{M,g}(\mathbb{R}_+^2) / \mathcal{N}_g(\mathbb{R}_+^2)$ . All operations in  $\mathcal{G}_g(\mathbb{R}_+^2)$  are defined by the corresponding ones in  $\mathcal{E}_{M,g}(\mathbb{R}_+^2)$ .

If one uses  $C_b^\infty(O)$  instead of  $C_b^\infty(\mathbb{R}_+^2)$ , for an open connected set  $O \subset \mathbb{R}^n$ , one obtains  $\mathcal{E}_{M,g}(O)$ ,  $\mathcal{N}_g(O)$  and consequently, the space of generalized functions on a real line,  $\mathcal{G}_g(O)$ .

Additionally, if functions from  $\mathcal{E}_{M,g}(\mathbb{R})$  and  $\mathcal{N}_g(\mathbb{R})$  are substituted with reals, one obtains the ring  $\mathcal{E}_{M,0}$  and its ideal  $\mathcal{N}_0$ , respectively. Thus, the ring of generalized real numbers is defined by  $\overline{\mathbb{R}} = \mathcal{E}_{M,0} / \mathcal{N}_0$ .

In the sequel,  $G$  denotes an element (equivalence class) in  $\mathcal{G}_g(O)$  defined by  $G_\varepsilon \in \mathcal{E}_{M,g}(O)$ .

Since  $C_b^\infty(\mathbb{R}_+^2) = C_b^\infty(\overline{\mathbb{R}}_+^2)$ , a restriction of a generalized function to  $\{t = 0\}$  is defined in the following way. For given  $G \in \mathcal{G}_g(\mathbb{R}_+^2)$ , its restriction  $G|_{t=0} \in \mathcal{G}_g(\mathbb{R})$  is the class determined by a function  $G_\varepsilon(x, 0) \in \mathcal{E}_{M,g}(\mathbb{R})$ . In the same way as above,  $G(x - ct) \in \mathcal{G}_g(\mathbb{R})$  is defined by  $G_\varepsilon(x - ct) \in \mathcal{E}_{M,g}(\mathbb{R})$ .

If  $G \in \mathcal{G}_g$  and  $f$  is a smooth function polynomially bounded together with all its derivatives, then one can easily show that the composition  $f(G)$ , defined by a representative  $f(G_\varepsilon)$ ,  $G \in \mathcal{G}_g$  makes sense. It means that  $f(G_\varepsilon) \in \mathcal{E}_{M,g}$  if  $G_\varepsilon \in \mathcal{E}_{M,g}$ , and  $f(G_\varepsilon) - f(H_\varepsilon) \in \mathcal{N}_g$  if  $G_\varepsilon - H_\varepsilon \in \mathcal{N}_g$ .

The equality in the space of the generalized functions  $\mathcal{G}_g$  is not appropriate for conservation laws as one can see in [72]. A generalized function  $G \in \mathcal{G}_g(O)$  is said to be associated with  $u \in \mathcal{D}'(O)$ ,  $G \approx u$ , if for some (and hence every) representative  $G_\varepsilon$  of  $G$ ,  $G_\varepsilon \rightarrow u$  in  $\mathcal{D}'(O)$  as  $\varepsilon \rightarrow 0$ . Two generalized functions  $G$  and  $H$  are said

to be associated,  $G \approx H$ , if  $G - H \approx 0$ . One can easily verify that the association is linear and an equivalence relation.

A generalized function  $G \in \mathcal{G}_g(O)$  is pointwisely non-negative if for every  $x \in O$ ,  $G(x) \geq 0$ , i.e., there exists  $Z_\varepsilon \in \mathcal{N}_0$  such that  $G_\varepsilon(x) \geq Z_\varepsilon$ , for  $\varepsilon$  small enough.

A generalized function  $G \in \mathcal{G}_g(O)$  is distributionally non-negative if for every  $\psi \in C_0^\infty(O)$ ,  $\int_O G_\varepsilon(x)\psi(x) \geq 0$ , for  $\varepsilon$  small enough.

Let  $u \in \mathcal{D}'_{L^\infty}(\mathbb{R})$ . Let  $\mathcal{A}_0$  be the set of all functions  $\phi \in \mathcal{D}(\mathbb{R})$  satisfying  $\phi(x) \geq 0$ ,  $x \in \mathbb{R}$ ,  $\int \phi(x) dx = 1$  and  $\text{supp } \phi \subset [-1, 1]$ . Let  $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(x/\varepsilon)$ ,  $x \in \mathbb{R}$ . Then

$$\iota_\phi : u \mapsto \text{class of } u * \phi_\varepsilon$$

defines a mapping of  $\mathcal{D}'_{L^\infty}(\mathbb{R})$  into  $\mathcal{G}_g(\mathbb{R})$ . It is clear that  $\iota_\phi$  commutes with the derivation. Also,  $\iota_\phi(\delta)$  is a class defined by a delta net  $\phi_\varepsilon$ .

**1.3. Generalized function algebras over Hölder spaces.** Let  $O$  be a bounded open set in  $\mathbb{R}^n$  and  $\alpha \in (-1, 1)$ . Recall [26, p. 94], a domain  $O$  and its boundary are of class  $C^{k,\alpha}$ , for  $0 \leq \alpha \leq 1$ , if at each point  $x_0 \in \partial O$  there is a ball  $B = B_{x_0}$  and a bijection  $\psi : B \rightarrow D$  such that  $\psi(B \cap O) \subset \mathbb{R}_+^n$ ,  $\psi(B \cap \partial O) \subset \partial \mathbb{R}_+^n$ , and  $\psi \in C^{k,\alpha}(B)$ ,  $\psi^{-1} \in C^{k,\alpha}(D)$ . A domain  $O$  has a boundary portion  $T \in \partial O$  of class  $C^{k,\alpha}$  if at each point  $x_0 \in T$  there is a ball  $B_{x_0}$  in which the above conditions are satisfied and  $B \cap \partial O \subset T$ .

We will consider the Colombeau extensions in cases  $E = C^{k,\alpha}(\bar{O})$ ,  $k \in \mathbb{N}$  and  $E = C^\infty(\bar{O})$ . We will use the norms

$$\begin{aligned} |f|_{k,O} &= \sup\{|f^{(p)}(x)|; |p| \leq k, x \in O\}, \\ |f|_{k,\alpha,O} &= |f|_{k,O} + [f]_{k,\alpha,O}, \quad k \in \mathbb{N}_0, \end{aligned}$$

where, for  $f \in C^\infty(\bar{O})$ ,  $k \in \mathbb{N}_0$ ,

$$[f]_{k,\alpha,O} = \sup \left\{ \frac{|f^{(p)}(x) - f^{(p)}(y)|}{|x - y|^\alpha}; (x, y) \in O, x \neq y, |p| = k \right\}.$$

The completion of  $C^\infty(\bar{O})$  with respect to the norm  $|\cdot|_{k,\alpha,O}$  defines  $E_k = C^{k,\alpha}(\bar{O})$ ,  $k \in \mathbb{N}$ . Recall, if  $k + \alpha < k' + \alpha'$ , then the embedding of  $C^{k,\alpha}(\bar{O})$  into  $C^{k',\alpha'}(\bar{O})$  is a compact linear operator.

Note that the sequences of norms  $\|\cdot\|_{k,\alpha}$ ,  $k \in \mathbb{N}$  and  $\|\cdot\|_k$ ,  $k \in \mathbb{N}$  define the same uniform structure on  $C^\infty(\bar{O})$  as the usual one.

In case  $E = C^\infty(\bar{O})$ , we need one more construction. Let  $G_\varepsilon$  be a net in  $C^{0,\alpha}(\bar{O})$  such that

$$G_\varepsilon \in C^{k,\alpha}(\bar{O}), \quad \varepsilon < \varepsilon_k, \quad k \in \mathbb{N},$$

where  $\varepsilon_k$  strictly decreases to zero.

Two such nets are in relation,  $G_\varepsilon \sim R_\varepsilon$ , if

$$G_\varepsilon = R_\varepsilon, \quad \varepsilon < \varepsilon_0, \text{ for some } \varepsilon_0 \in (0, 1).$$

This is an equivalence relation and with the corresponding classes, elements in  $C^{0,\alpha}(\bar{O})/\sim$ , we define spaces  $\mathcal{E}_M[E]$ ,  $\mathcal{N}[E]$ . Then we define the corresponding Colombeau type space  $\mathcal{G}[E] = \mathcal{E}_M[E]/\mathcal{N}[E]$ . Note that there exists a canonical

isomorphism of  $\mathcal{G}[E]$  onto  $\mathcal{G}(E)$  if  $E = C^\infty(\bar{O})$ . In case  $E = C^{k,\alpha}(\bar{O})$ , we have  $\mathcal{G}(E) = \mathcal{G}[E]$ .

**1.4. Colombeau–Sobolev type spaces and algebras.** Although we have introduced two general concepts of constructions of generalized function algebras, still the flexibility of Colombeau main idea enable us to construct some other types of spaces and algebras useful in the analysis of problems with singularities.

Let  $O$  be an open, connected subset of  $\mathbb{R}^n$  with a smooth boundary. Let  $H^{r,s}(O)$  be Sobolev space of functions in  $L^s(O)$  with all distributional derivatives of order  $|\alpha| \leq r$  belonging to  $L^s(O)$ , equipped with the usual norm. In case  $s = 2$ , we simply write  $H^r(O)$ . One can find Colombeau type algebras  $\mathcal{G}_{L^p,L^q}$  in [7] and [60]. We shall describe the special case of the last one,  $\mathcal{G}_{L^2,L^2}$  space, denoted by  $\mathcal{G}_{2,2}$  in the present paper.

$\mathcal{E}_{2,2}([0,T] \times \mathbb{R}^n)$  is the algebra of all  $G_\varepsilon \in \mathcal{E}([0,T] \times \mathbb{R}^n)$  with the property that for all  $T > 0$  and  $\alpha \in \mathbb{N}_0^n$  there exists  $N \in \mathbb{N}$  such that

$$\|\partial^\alpha G_\varepsilon\|_{L^2([0,T] \times \mathbb{R}^n)} = \mathcal{O}(\varepsilon^{-N}).$$

We say that  $\|\partial^\alpha G_\varepsilon\|_{L^2}$  is moderate or that it has a moderate bound.

$\mathcal{N}_{2,2}([0,T] \times \mathbb{R}^n)$  is the algebra of all  $G_\varepsilon \in \mathcal{E}([0,T] \times \mathbb{R}^n)$  with the property that for all  $T > 0$ ,  $\alpha \in \mathbb{N}_0^n$  and  $a \in \mathbb{R}$

$$\|\partial^\alpha G_\varepsilon\|_{L^2([0,T] \times \mathbb{R}^n)} = \mathcal{O}(\varepsilon^a).$$

We say that  $\|\partial^\alpha G_\varepsilon\|_{L^2}$  is negligible.

As above, we define

$$\mathcal{G}_{2,2}([0,T] \times \mathbb{R}^n) = \mathcal{E}_{2,2}([0,T] \times \mathbb{R}^n) / \mathcal{N}_{2,2}([0,T] \times \mathbb{R}^n).$$

One can similarly define spaces  $\mathcal{E}_{2,2}(\mathbb{R}^n)$ ,  $\mathcal{N}_{2,2}(\mathbb{R}^n)$  and  $\mathcal{G}_{2,2}(\mathbb{R}^n)$  but independently of time variable  $t$ .

Let  $Q$  denote  $[0,T] \times O$  or  $O$ . The proof that  $\mathcal{N}_{2,2}(Q)$  is an ideal of  $\mathcal{E}_{2,2}(Q)$  is given in [7]. Sobolev embedding theorems give that  $\mathcal{E}_{2,2}(Q) \subset \mathcal{E}_g(Q)$  and  $\mathcal{N}_{2,2}(Q) \subset \mathcal{N}_g(Q)$ . Thus there exists a canonical mapping  $\mathcal{G}_{2,2}(Q) \rightarrow \mathcal{G}_g(Q)$ . Also, this means that in  $\mathcal{G}_{2,2}(Q)$  instead of  $L^2$ -norm on the strip  $[0,T] \times \mathbb{R}^n$  one can use  $L^\infty$ -norm on  $[0,T]$  and  $L^2$ -norm on  $\mathbb{R}^n$  and vice versa.

**1.5. Generalized stochastic processes.** At the beginning we recall some basic facts from classical stochastic analysis. Let  $(\Omega, \Sigma, \mu)$  be a probability space. A weakly measurable mapping

$$X : \Omega \rightarrow \mathcal{D}'(\mathbb{R}^d)$$

is called a generalized stochastic process on  $\mathbb{R}^d$ .

For each fixed function  $\varphi \in \mathcal{D}(\mathbb{R}^d)$ , the mapping  $\Omega \rightarrow \mathbb{R}$  defined by

$$\omega \rightarrow \langle X(\omega), \varphi \rangle$$

is a random variable.



The space of generalized stochastic processes will be denoted by  $\mathcal{D}'_{\Omega}(\mathbb{R}^d)$ . The characteristic functional of a process  $X$  is

$$C_X(\varphi) = \int e^{i\langle X(\omega), \varphi \rangle} d\mu(\omega), \quad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

Take probability space to be the space of tempered distributions  $\Omega = \mathcal{S}'(\mathbb{R}^d)$  and  $\Sigma$  to be the Borel  $\sigma$ -algebra generated by the weak topology. Then there is a unique probability measure  $\mu$  on  $(\Omega, \Sigma)$  such that

$$\int e^{i\langle X(\omega), \varphi \rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\varphi\|_{L^2(\mathbb{R}^d)}^2}, \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

It is a well known result following from the Bochner–Minlos theorem (see [39], for example). White noise process  $\dot{W} : \Omega \rightarrow \mathcal{D}'(\mathbb{R}^d)$  is the identity mapping

$$\langle \dot{W}(\omega), \varphi \rangle = \langle \omega, \varphi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

It is a generalized Gaussian process with mean zero and variance

$$D(\dot{W}(\varphi)) = E(\dot{W}(\varphi)^2) = \|\varphi\|_{L^2(\mathbb{R}^d)}^2,$$

where  $E$  denotes expectation.

**Definition 1.**  $\mathcal{G}_g$ -Colombeau random generalized function on a probability space  $(\Omega, \Sigma, \mu)$  is a mapping  $U : \Omega \rightarrow \mathcal{G}_g(Q)$  such that there exists a function  $U : (0, 1) \times Q \times \Omega \rightarrow \mathbb{R}$  with the following properties:

- 1) For fixed  $\varepsilon \in (0, 1)$ ,  $(x, \omega) \rightarrow U(\varepsilon, x, \omega)$  is jointly measurable in  $Q \times \Omega$ .
- 2)  $\varepsilon \rightarrow U(\varepsilon, \cdot, \omega)$  belongs to  $\mathcal{E}_g(Q)$  almost surely in  $\omega \in \Omega$ , and it is a representative of  $U(\omega)$ .

By  $\mathcal{G}_g^{\Omega}(Q)$  we denote the algebra of  $\mathcal{G}_g$ -Colombeau random generalized functions on  $\Omega$ . A family of  $\mathcal{G}_g$ -Colombeau random generalized functions is called  $\mathcal{G}_g$ -Colombeau generalized stochastic process.

**Definition 2.**  $\mathcal{G}_{2,2}$ -Colombeau random generalized function on a probability space  $(\Omega, \Sigma, \mu)$  is a mapping  $U : \Omega \rightarrow \mathcal{G}_{2,2}(Q)$  such that there exists a function  $U : (0, 1) \times Q \times \Omega \rightarrow \mathbb{R}$  with the following properties:

- 1) For fixed  $\varepsilon \in (0, 1)$ ,  $(x, \omega) \rightarrow U(\varepsilon, x, \omega)$  is jointly measurable in  $Q \times \Omega$ .
- 2)  $\varepsilon \rightarrow U(\varepsilon, \cdot, \omega)$  belongs to  $\mathcal{E}_{2,2}(Q)$  almost surely in  $\omega \in \Omega$ , and it is a representative of  $U(\omega)$ .

By  $\mathcal{G}_{2,2}^{\Omega}(Q)$  we denote the algebra of  $\mathcal{G}_{2,2}$ -Colombeau random generalized functions on  $\Omega$ . A family of  $\mathcal{G}_{2,2}$ -Colombeau random generalized functions is called  $\mathcal{G}_{2,2}$ -Colombeau generalized stochastic process.

As usual, the variable  $\varepsilon$  will be written as a subindex

**1.6. Vector valued Colombeau type spaces.** We will make some necessary modifications to define and use generalized semigroups. The main difference comparing with the previous section is that one does not need all derivatives of a representative. This will open new possibilities for applications, but also make a work with them more complicated (spaces of such generalized functions are not algebras in general).

**Definition 3.**  $\mathcal{E}_{C^1, H^2}([0, T] : \mathbb{R}^n)$  (respectively  $\mathcal{N}_{C^1, H^2}([0, T] : \mathbb{R}^n)$ ),  $T > 0$ , is the vector space of nets  $G_\varepsilon$  of functions

$$G_\varepsilon \in C^0([0, T] : H^2(\mathbb{R}^n)) \cap C^1((0, T) : L^2(\mathbb{R}^n)), \quad \varepsilon \in (0, 1)$$

with the property: for every  $T_1 \in (0, T)$  there exists  $a \in \mathbb{R}$ , (respectively, for every  $a \in \mathbb{R}$ ) such that

$$(1) \quad \max \left\{ \sup_{t \in [0, T]} \|G_\varepsilon(t)\|_{H^2}, \sup_{t \in [T_1, T]} \|\partial_t G_\varepsilon(t)\|_{L^2} \right\} = \mathcal{O}(\varepsilon^a), \quad \text{as } \varepsilon \rightarrow 0.$$

The quotient space  $\mathcal{G}_{C^1, H^2}([0, T] : \mathbb{R}^n) = \frac{\mathcal{E}_{C^1, H^2}([0, T] : \mathbb{R}^n)}{\mathcal{N}_{C^1, H^2}([0, T] : \mathbb{R}^n)}$  is a Colombeau type vector space.

Dropping the conditions on  $\partial_t G_\varepsilon$  in (1) we obtain spaces  $\mathcal{E}_{C^0, H^2}([0, T] : \mathbb{R}^n)$ ,  $\mathcal{N}_{C^0, H^2}([0, T] : \mathbb{R}^n)$  and  $\mathcal{G}_{C^0, H^2}([0, T] : \mathbb{R}^n)$ .

By Sobolev lemma we have

**Lemma 4.** *If  $n \leq 3$ , then  $\mathcal{E}_{C^1, H^2}([0, T] : \mathbb{R}^n)$  is an algebra with the multiplication and  $\mathcal{N}_{C^1, H^2}([0, T] : \mathbb{R}^n)$  is an ideal of  $\mathcal{E}_{C^1, H^2}([0, T] : \mathbb{R}^n)$ . Therefore,  $\mathcal{G}_{C^1, H^2}([0, T] : \mathbb{R}^n)$  is an algebra with the multiplication. The same holds for  $\mathcal{E}_{C^0, H^2}([0, T] : \mathbb{R}^n)$ ,  $\mathcal{N}_{C^0, H^2}([0, T] : \mathbb{R}^n)$  and  $\mathcal{G}_{C^0, H^2}([0, T] : \mathbb{R}^n)$ .*

Substituting  $H^2$ -norm with  $L^2$ -norm in (1) we obtain vector spaces

$$\mathcal{E}_{C^1, L^2}([0, T] : \mathbb{R}^n), \quad \mathcal{N}_{C^1, L^2}([0, T] : \mathbb{R}^n) \quad \text{and} \quad \mathcal{G}_{C^1, L^2}([0, T] : \mathbb{R}^n).$$

Canonical mapping  $\iota_{L^2} : \mathcal{G}_{C^1, H^2}([0, T] : \mathbb{R}^n) \rightarrow \mathcal{G}_{C^1, L^2}([0, T] : \mathbb{R}^n)$  is defined by  $\iota_{L^2}(G) = H$ , where  $H = [G_\varepsilon]$  and  $G_\varepsilon$  is a representative of  $G$ .

Space  $\mathcal{G}_{H^2}(\mathbb{R}^n)$  is defined in a similar way as  $\mathcal{G}_{C^1, H^2}(\mathbb{R}^n)$ , but with representatives independent of time variable  $t$ . This space is also an algebra in case  $n \leq 3$ . Let us give more details for space  $\mathcal{G}_{H^2, \infty}([0, T] : \mathbb{R}^n)$ .

$\mathcal{E}_{H^2, \infty}(\mathbb{R}^n)$ , (respectively,  $\mathcal{N}_{H^2, \infty}(\mathbb{R}^n)$ ) is the space of nets  $G_\varepsilon$  of functions  $G_\varepsilon \in H^{2, \infty}(\mathbb{R}^n)$ ,  $\varepsilon \in (0, 1)$ , with the property: there exists  $a \in \mathbb{R}$  (respectively, for every  $a \in \mathbb{R}$ ) such that

$$\|G_\varepsilon\|_{H^{2, \infty}(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^a), \quad \text{as } \varepsilon \rightarrow 0.$$

Both spaces are algebras with the usual multiplication and  $\mathcal{N}_{H^2, \infty}(\mathbb{R}^n)$  is an ideal. Colombeau type algebra is defined by

$$\mathcal{G}_{H^2, \infty}(\mathbb{R}^n) = \frac{\mathcal{E}_{H^2, \infty}(\mathbb{R}^n)}{\mathcal{N}_{H^2, \infty}(\mathbb{R}^n)}.$$

## 2. Generalized semigroups

The definitions and assertions are from [67].

Let  $(E, \|\cdot\|)$  be a Banach space and let  $\mathcal{L}(E)$  be the space of all linear continuous mappings  $E \rightarrow E$ .

**Definition 5.**  $\mathcal{SE}_M([0, \infty) : \mathcal{L}(E))$  is the space of nets  $S_\varepsilon$  of continuous mappings  $S_\varepsilon : [0, \infty) \rightarrow \mathcal{L}(E)$ ,  $\varepsilon \in (0, 1)$ , with the properties  $S_\varepsilon(0) = I$ ,  $\varepsilon \in (0, 1)$ , and that for every  $T > 0$  there exists  $a \in \mathbb{R}$  such that

$$(2) \quad \sup_{t \in [0, T]} \|S_\varepsilon(t)\| = \mathcal{O}(\varepsilon^a), \quad \text{as } \varepsilon \rightarrow 0.$$

$\mathcal{SN}([0, \infty) : \mathcal{L}(E))$  is the space of nets  $N_\varepsilon$  of continuous mappings  $N_\varepsilon : [0, \infty) \rightarrow \mathcal{L}(E)$ ,  $\varepsilon \in (0, 1)$ , with the properties:

$$(a) \quad \text{For every } b \in \mathbb{R} \text{ and } T > 0, \quad \sup_{t \in [0, T]} \|N_\varepsilon(t)\| = \mathcal{O}(\varepsilon^b), \quad \text{as } \varepsilon \rightarrow 0.$$

(b) There exists a net  $H_\varepsilon$  in  $\mathcal{L}(E)$  such that  $\lim_{t \rightarrow 0} \frac{N_\varepsilon(t)}{t}x = H_\varepsilon x$ , for every  $x \in E$  and  $\varepsilon$  small enough, and for every  $b > 0$ ,  $\|H_\varepsilon\| = \mathcal{O}(\varepsilon^b)$ , as  $\varepsilon \rightarrow 0$ .

**Proposition 6.**  $\mathcal{SE}_M([0, \infty) : \mathcal{L}(E))$  is an algebra with respect to composition and  $\mathcal{SN}([0, \infty) : \mathcal{L}(E))$  is an ideal of  $\mathcal{SE}_M([0, \infty) : \mathcal{L}(E))$ .

Now we define Colombeau type algebra as the factor algebra

$$\mathcal{SG}([0, \infty) : \mathcal{L}(E)) = \frac{\mathcal{SE}_M([0, \infty) : \mathcal{L}(E))}{\mathcal{SN}([0, \infty) : \mathcal{L}(E))}.$$

Elements of  $\mathcal{SG}([0, \infty) : \mathcal{L}(E))$  will be denoted by  $S = [S_\varepsilon]$ , where  $S_\varepsilon$  is a representative of the above class.

**Definition 7.**  $S \in \mathcal{SG}([0, \infty) : \mathcal{L}(E))$  is called a Colombeau  $C_0$ -semigroup if it has a representative  $S_\varepsilon$  such that, for some  $\varepsilon_0 > 0$ ,  $S_\varepsilon$  is a  $C_0$ -semigroup, for every  $\varepsilon < \varepsilon_0$ .

In the sequel we will use only representatives  $S_\varepsilon$  of a Colombeau  $C_0$ -semigroup  $S$  which are  $C_0$ -semigroups, for  $\varepsilon$  small enough.

**Proposition 8.** Let  $S_\varepsilon$  and  $\tilde{S}_\varepsilon$  be representatives of a Colombeau  $C_0$ -semigroup  $S$ , with the infinitesimal generators  $A_\varepsilon$ ,  $\varepsilon < \varepsilon_0$ , and  $\tilde{A}_\varepsilon$ ,  $\varepsilon < \tilde{\varepsilon}_0$ , respectively, where  $\varepsilon_0$  and  $\tilde{\varepsilon}_0$  correspond (in the sense of Definition 7) to  $S_\varepsilon$  and  $\tilde{S}_\varepsilon$ , respectively. Then  $D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$ , for every  $\varepsilon < \bar{\varepsilon}_0 = \min\{\varepsilon_0, \tilde{\varepsilon}_0\}$  and  $A_\varepsilon - \tilde{A}_\varepsilon$  can be extended to be an element of  $\mathcal{L}(E)$ , denoted again by  $A_\varepsilon - \tilde{A}_\varepsilon$ . Moreover, for every  $a \in \mathbb{R}$ ,  $\|A_\varepsilon - \tilde{A}_\varepsilon\| = \mathcal{O}(\varepsilon^a)$ , as  $\varepsilon \rightarrow 0$ .

Now we define the infinitesimal generator of a Colombeau  $C_0$ -semigroup  $S$ . Denote by  $\mathcal{A}$  the set of pairs  $(A_\varepsilon, D(A_\varepsilon))$  where  $A_\varepsilon$  is a closed linear operator on  $E$  with dense domain  $D(A_\varepsilon) \subset E$ , for every  $\varepsilon \in (0, 1)$ . We introduce an equivalence relation in  $\mathcal{A}$  by:  $(A_\varepsilon, D(A_\varepsilon)) \sim (\tilde{A}_\varepsilon, D(\tilde{A}_\varepsilon))$  if there exists  $\varepsilon_0 \in (0, 1)$  such that  $D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$ , for every  $\varepsilon < \varepsilon_0$ , and for every  $a \in \mathbb{R}$  there exists  $C > 0$  such that, for  $x \in D(A_\varepsilon)$ ,  $\|(A_\varepsilon - \tilde{A}_\varepsilon)x\| \leq C\varepsilon^a\|x\|$ , as  $\varepsilon \rightarrow 0$ .

Since  $A_\varepsilon$  has a dense domain in  $E$ ,  $A_\varepsilon - \tilde{A}_\varepsilon$  can be extended to be an operator in  $\mathcal{L}(E)$  satisfying  $\|A_\varepsilon - \tilde{A}_\varepsilon\| = \mathcal{O}(\varepsilon^a)$ ,  $\varepsilon \rightarrow 0$ , for every  $a \in \mathbb{R}$ .

We denote by  $A$  the corresponding element of the quotient space  $\mathcal{A}/\sim$ . Due to Proposition 8, the following definition makes sense.

**Definition 9.**  $A \in \mathcal{A}/\sim$  is the infinitesimal generator of a Colombeau  $C_0$ -semigroup  $S$  if there exists a representative  $A_\varepsilon$  of  $A$  such that  $A_\varepsilon$  is the infinitesimal generator of  $S_\varepsilon$ , for  $\varepsilon$  small enough.

We collect some obvious properties in the following proposition (cf. [79]).

**Proposition 10.** *Let  $S$  be a Colombeau  $C_0$ -semigroup with the infinitesimal generator  $A$ . Then there exists  $\varepsilon_0 \in (0, 1)$  such that:*

(a) *Mapping  $t \mapsto S_\varepsilon(t)x : [0, \infty) \rightarrow E$  is continuous for every  $x \in E$  and  $\varepsilon < \varepsilon_0$ .*

(b) 
$$\lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} S_\varepsilon(s)x \, ds = S_\varepsilon(t)x, \quad \varepsilon < \varepsilon_0, \quad x \in E.$$

(c) 
$$\int_0^t S_\varepsilon(s)x \, ds \in D(A_\varepsilon), \quad \varepsilon < \varepsilon_0, \quad x \in E.$$

(d) *For every  $x \in D(A_\varepsilon)$  and  $t \geq 0$ ,  $S_\varepsilon(t)x \in D(A_\varepsilon)$  and*

$$\frac{d}{dt} S_\varepsilon(t)x = A_\varepsilon S_\varepsilon(t)x = S_\varepsilon(t)A_\varepsilon x, \quad \varepsilon < \varepsilon_0.$$

(e) *Let  $S_\varepsilon$  and  $\tilde{S}_\varepsilon$  be representatives of Colombeau  $C_0$ -semigroup  $S$ , with infinitesimal generators  $A_\varepsilon$  and  $\tilde{A}_\varepsilon$ ,  $\varepsilon < \varepsilon_0$ , respectively. Then, for every  $a \in \mathbb{R}$  and  $t \geq 0$*

$$\left\| \frac{d}{dt} S_\varepsilon(t) - \tilde{A}_\varepsilon S_\varepsilon(t) \right\| = \mathcal{O}(\varepsilon^a), \quad \text{as } \varepsilon \rightarrow 0.$$

(f) *For every  $x \in D(A_\varepsilon)$  and every  $t, s \geq 0$ ,*

$$S_\varepsilon(t)x - S_\varepsilon(s)x = \int_s^t S_\varepsilon(\tau)A_\varepsilon x \, d\tau = \int_s^t A_\varepsilon S_\varepsilon(\tau)x \, d\tau, \quad \varepsilon < \varepsilon_0.$$

**Theorem 11.** *Let  $S$  and  $\tilde{S}$  be two Colombeau  $C_0$ -semigroups with infinitesimal generators  $A$  and  $B$ , respectively. If  $A = B$ , then  $S = \tilde{S}$ .*

**Example 12.** *Semigroups of Schrödinger-type operators.* Let  $V \in \mathcal{G}_{W^{2,\infty}}(\mathbb{R}^n)$  be of logarithmic type. Then differential operators  $A_\varepsilon u = (\Delta - V_\varepsilon)u$ ,  $u \in W^2(\mathbb{R}^n)$ ,  $\varepsilon < 1$ , are infinitesimal generators of  $C_0$ -semigroups  $S_\varepsilon$ ,  $\varepsilon < 1$ , and  $S_\varepsilon$  is a representative of a generalized  $C_0$ -semigroup  $S \in \mathcal{SG}([0, \infty) : \mathcal{L}(L^2(\mathbb{R}^n)))$ .

Let  $\varepsilon < 1$ . Operator  $A_\varepsilon$  is the infinitesimal generator of the corresponding  $C_0$ -semigroup  $S_\varepsilon : [0, \infty) \rightarrow \mathcal{L}(L^2(\mathbb{R}^n))$  defined by the Feynman–Kac formula:

$$S_\varepsilon(t)\psi(x) = \int_O \exp\left(-\int_0^t V_\varepsilon(\omega(s)) \, ds\right) \psi(\omega(t)) \, d\mu_x(\omega), \quad t \geq 0, \quad x \in \mathbb{R}^n,$$

for  $\psi \in L^2(\mathbb{R}^n)$ , where  $O = \prod_{t \in [0, \infty)} \overline{\mathbb{R}^n}$  and  $\mu_x$  is the Wiener measure concentrated at  $x \in \mathbb{R}^n$  (cf. [86] or [95]).

The assumption on  $V$  implies that there exists  $C > 0$  such that

$$\begin{aligned} |S_\varepsilon(t)\psi(x)| &\leq \exp\left(t \sup_{s \in \mathbb{R}^n} |V_\varepsilon(s)|\right) \int_O |\psi(\omega(t))| d\mu_x(\omega) \\ &= \varepsilon^{-Ct} (4\pi t)^{-n/2} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4t}\right) |\psi(y)| dy, \end{aligned}$$

for every  $t > 0$ ,  $x \in \mathbb{R}^n$  and  $\varepsilon < 1$ . Recall that the heat kernel is given by

$$E_n(t, x) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{x^2}{4t}\right), \quad t > 0, x \in \mathbb{R}^n,$$

and its  $L^1(\mathbb{R}^n)$ -norm equals 1 for every  $t > 0$ . By the Young inequality,

$$|S_\varepsilon(t)\psi| \leq e^{-Ct} \|E_n(t, \cdot)\|_{L^1(\mathbb{R}^n)} \|\psi\|_{L^2(\mathbb{R}^n)}, \quad t > 0, \varepsilon < 1.$$

Therefore, there exists  $C_0 > 0$  such that  $\sup_{t \in [0, T]} \|S_\varepsilon(t)\psi\|_{L^2} \leq C_0 \varepsilon^{-CT} \|\psi\|_{L^2}$ ,  $\varepsilon < 1$ , for every  $T$ , i.e.,  $S_\varepsilon(t)$ ,  $t \in [0, T]$ , satisfies relation (2) and

$$S = [S_\varepsilon] \in \mathcal{SG}([0, \infty) : \mathcal{L}(L^2(\mathbb{R}^n))).$$

**Remark 13.** We refer to [70] for an approach to generalized semigroup theory based on uniformly continuous classical semigroups.

## PART II: SECOND ORDER EQUATIONS

### 3. Elliptic PDEs

**3.1. Linear elliptic PDE.** We will consider elliptic boundary value problems with very singular boundary data and coefficients. Because of that the solutions are considered in a large space of generalized functions and moreover, the concept of being a solution is adequately extended.

**3.1.1. Introduction.** The restriction of a generalized function on  $A \subset \Omega$  is defined by the restriction of a representative. Recall, the support of  $G \in \mathcal{G}(\Omega)$  is the complement of the largest open subset of  $\Omega$  where  $G$  is the zero generalized function. The space of all compactly supported generalized functions is denoted by  $\mathcal{G}_c(\Omega)$ .

In the sequel we use the notation  $(A)_{-\gamma} = \{x \in A : \text{dist}(x, \partial A) \geq \gamma\}$  and  $\phi_\gamma = \gamma^{-n} \phi(\cdot/\gamma)$ ,  $\gamma > 0$ , where  $\phi = \phi_1(x_1) \cdots \phi_n(x_n) \in C_0^\infty(\mathbb{R}^n)$ ,  $\int \phi_i(x) dx = 1$ ,  $i = 1, \dots, n$ . Note that  $\phi_\gamma$ ,  $\gamma > 0$  is a delta net. For the sake of simplicity, let us assume that  $\phi$  is a radially symmetric, positive function in the open unit ball and that  $\phi$  is supported by the closed unit ball in  $\mathbb{R}^n$ . Let  $\psi_\varepsilon = \mathbf{1}_{(\Omega)_{2\varepsilon}} * \phi_\varepsilon$ . We define an inclusion  $\iota$  of  $\mathcal{D}'(\Omega)$  into  $\mathcal{G}(\Omega)$  in the following way. If  $g \in \mathcal{D}'(\Omega)$ , then  $\iota(g) = G$  is represented by  $G_\varepsilon = (g \cdot \psi_\varepsilon) * \phi_\varepsilon$ .

Let  $\mathbb{D}$  be some space of test functions. We say that  $G_1, G_2 \in \mathcal{G}(\Omega)$  are equal in  $\mathbb{D}$ -sense,  $G_1 \stackrel{\mathbb{D}}{=} G_2$ , if  $\langle G_1, \varphi \rangle = \langle G_2, \varphi \rangle$  in  $\overline{\mathbb{C}}$  for every  $\varphi \in \mathbb{D}$ . Similarly, a generalized function  $G_1$  and  $g \in \mathcal{D}'$  equals in  $\mathbb{D}$ -sense,  $G_1 \stackrel{\mathbb{D}}{=} g$ , if  $\langle G_1, \varphi \rangle = \langle g, \varphi \rangle$  for every  $\varphi \in \mathbb{D}$ . Usually,  $\mathbb{D}$  denotes Sobolev space  $H^m$  or  $H_0^m$ .

The  $s$ -association ( $\approx_s$ ),  $s \geq 0$  of generalized complex numbers is defined as follows. For  $G \in \overline{\mathbb{C}}$ ,  $G \approx_s 0$  means that  $G$  has a representative  $G_\varepsilon$  such that  $G_\varepsilon = o(\varepsilon^s)$  as  $\varepsilon \rightarrow 0$ . If  $G_1, G_2$  are in  $\mathcal{G}(\Omega)$ , then  $G_1 \stackrel{\mathbb{D}}{\approx}_s G_2$  if  $\langle G_1 - G_2, \varphi \rangle \approx_s 0$  for every  $\varphi \in \mathbb{D}$ . If  $s = 0$ , then the notation  $\dots$ -association is often used instead of  $\dots$ -0-association. As in the previous case one can define association between a generalized function in  $\mathcal{G}$  and a generalized function in  $\mathcal{D}'$ .

We will recall the definition of the space of generalized functions on a closed set (cf. [5]) in order to give a meaning to the Dirichlet problem in  $\mathcal{E}_M$  and thus in  $\mathcal{G}$ .

Let  $X$  be a non-void subset of  $\mathbb{R}^n$  and  $\{G_\varepsilon^\alpha, \alpha \in \mathbb{N}_0^n\}$  be a family of mappings  $G_\varepsilon^\alpha : (0, 1) \times X \rightarrow \mathbb{C}$ . Denote by  $\mathcal{E}_{W,M}(X)$  the vector space of families  $\{G_\varepsilon^\alpha, \alpha \in \mathbb{N}_0^n\}$  which satisfy the following conditions:

(a)  $\{G_\varepsilon^\alpha, \alpha \in \mathbb{N}_0^n\}$  has a locally moderate growth when  $\varepsilon \rightarrow 0$ . This means that for every  $\alpha \in \mathbb{N}_0^n$  and  $x_0 \in X$  there exist a neighborhood  $V$  of  $x_0$ ,  $N \in \mathbb{R}$ ,  $C > 0$  and  $\eta > 0$  such that  $|G_\varepsilon^\alpha(x)| \leq C\varepsilon^{-N}$ ,  $x \in V \cap X$ ,  $\varepsilon \in (0, \eta)$ .

(b) There exists  $\eta > 0$  such that the family  $\{X \ni x \mapsto G_\varepsilon^\alpha(x), \varepsilon < \eta, \alpha \in \mathbb{N}_0^n\}$  satisfies requirements defining Whitney's  $C^\infty$ -function on  $X$ , that is for every  $m \in \mathbb{N}$ ,  $\alpha \in \mathbb{N}_0^n$ ,  $|\alpha| \leq m$  and  $x_0 \in X$  there exist a neighborhood  $V$  of  $x_0$  and  $c_\varepsilon > 0$  such that

$$(3) \quad \left| G_\varepsilon^\alpha(x) - \sum_{|\beta| \leq m-|\alpha|} \frac{(x-x')^\beta G_\varepsilon^{\alpha+\beta}(x')}{\beta!} \right| \leq c_\varepsilon |x-x'|^{m-|\alpha|-1},$$

for every  $x, x' \in V$ ,  $\varepsilon \in (0, \eta)$ .

(c) Constants  $c_\varepsilon$  are locally bounded above by  $c\varepsilon^{-N}$  as  $\varepsilon \rightarrow 0$ . More precisely, for every  $m \in \mathbb{N}$ ,  $\alpha \in \mathbb{N}_0^n$ ,  $|\alpha| \leq m$  and  $x_0 \in X$  there exist a neighborhood  $V$  of  $x_0$ ,  $N \in \mathbb{R}$ ,  $C > 0$  and  $\eta > 0$  such that (3) holds with  $c_\varepsilon = C\varepsilon^{-N}$ .

The ideal  $\mathcal{N}_W(X)$  of  $\mathcal{E}_{W,M}(X)$  is the set of those  $\{G_\varepsilon^\alpha, \alpha \in \mathbb{N}_0^n\}$  which satisfy: for every  $\alpha \in \mathbb{N}_0^n$ , and  $x_0 \in X$  there exists a neighborhood  $V$  of  $x_0$  such that for every  $q > 0$  there exist  $C > 0$  and  $\eta > 0$  such that  $|G_\varepsilon^\alpha(x)| \leq C\varepsilon^q$ ,  $x \in V \cap X$ ,  $\varepsilon \in (0, \eta)$ .

Put  $\mathcal{G}_W(X) = \mathcal{E}_{W,M}(X)/\mathcal{N}_W(X)$ . Clearly, if  $G \in \mathcal{G}(\Omega)$ , where  $\Omega$  is an open set containing  $X$ , then  $\{D^\alpha G_\varepsilon|_X, \alpha \in \mathbb{N}_0^n\} \in \mathcal{E}_{W,M}$  defines the restriction  $G|_X \in \mathcal{G}_W$ .

**Theorem 14.** [5] *Let  $X$  be a closed subset of  $\mathbb{R}^n$ . Then the restriction map  $\mathcal{G}(\mathbb{R}^n) \rightarrow \mathcal{G}_W(X)$  is surjective. In particular, for given  $\{G_\varepsilon^\alpha, \alpha \in \mathbb{N}_0^n\} \in \mathcal{E}_{W,M}(X)$  there exists  $F_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)$  such that  $\{D^\alpha F_\varepsilon|_X - G_\varepsilon^\alpha, \alpha \in \mathbb{N}_0^n\} \in \mathcal{N}_W(X)$ .*

**3.1.2. Generalized Dirichlet problem.** A differential operator of the form  $P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ , where  $a_\alpha \in \mathcal{G}(\mathbb{R}^n)$ , is called a generalized differential operator. A representative of  $P(x, D)$  is given by  $P_\varepsilon(x, D) = \sum_{|\alpha| \leq m} a_{\alpha, \varepsilon}(x) D^\alpha$ , where  $a_{\alpha, \varepsilon} \in \mathcal{E}_M(\mathbb{R}^n)$ , is a representative of  $a_\alpha$ ,  $|\alpha| \leq m$ . Note that if  $b_{\alpha, \varepsilon}$  is another representative of  $a_\alpha$ ,  $|\alpha| \leq m$ , then

$$\sum_{|\alpha| \leq m} a_{\alpha, \varepsilon}(x) D^\alpha G_\varepsilon - \sum_{|\alpha| \leq m} b_{\alpha, \varepsilon}(x) D^\alpha G_\varepsilon \in \mathcal{N}(\mathbb{R}^n), G_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n).$$

Let  $O$  be a bounded open set in  $\mathbb{R}^n$ ,  $H \in \mathcal{G}(\mathbb{R}^n)$  and let  $F \in \mathcal{G}_W(\partial O)$  be defined by a family  $\{F_\varepsilon^\alpha, \alpha \in \mathbb{N}_0^n\}$ . Consider the following boundary value problem

$$(4) \quad P(x, D)G \stackrel{\mathbb{D}_0^m(O)}{\approx} H, \text{ in } O, G|_{\partial O} = F \quad (\mathbb{D}_0^m(O) = H^m(O) \cap H_0^{m-1}(O)).$$

Theorem 14 implies that there exists  $\tilde{F} \in \mathcal{G}(\mathbb{R}^n)$  such that  $\tilde{F}|_{\partial O} = F$ . Let  $V = P(x, D)\tilde{F}$  and  $U$  be a solution to the problem

$$P(x, D)U \stackrel{\mathbb{D}_0^m(O)}{\approx} H - V \text{ in } O, U|_{\partial O} = 0.$$

Then  $G = U + \tilde{F}$  is a solution to (4).

So, in the sequel we shall consider the following problem

$$(5) \quad P(x, D)G \stackrel{\mathbb{D}_0^m(O)}{\approx} H \text{ in } O, G|_{\partial O} = 0,$$

in terms of representatives,

$$\lim_{\varepsilon \rightarrow 0} \int (P_\varepsilon(x, D)G_\varepsilon(x) - H_\varepsilon(x))\psi(x) dx = 0, \quad \psi \in \mathbb{D}_0^m(O)$$

$$\{D^\alpha G_\varepsilon|_{\partial O}, \alpha \in \mathbb{N}_0^m\} \in \mathcal{N}_W(\partial O).$$

**Theorem 15.** *With the assumptions given above, for every  $s \geq 0$  there exists a solution  $G \in \mathcal{G}(\mathbb{R}^n)$  to (5) in  $\mathbb{D}_0^m(O)$ -s-associated sense.*

*Proof.* Let  $P_\varepsilon^*(x, D) = \sum_{|\alpha| \leq m} \tilde{a}_{\alpha, \varepsilon}(x)D^\alpha$  be the adjoint operator to  $P_\varepsilon(x, D)$ . Since  $\tilde{a}_{\alpha, \varepsilon} \in \mathcal{E}_M(\mathbb{R}^n)$ , there exist  $\tilde{N}_1 > 0$  and  $C_1 > 0$  such that

$$B_\varepsilon = \max \{|\nabla \tilde{a}_{\alpha, \varepsilon}(x)|, |\tilde{a}_{\alpha, \varepsilon}(x)|, x \in \bar{O}, |\alpha| \leq m\} \leq C_1 \varepsilon^{-\tilde{N}_1},$$

for  $\varepsilon$  small enough, say  $\varepsilon < \eta_1$ .

Let  $\varepsilon \in (0, \eta_1)$  be given. Let  $\Pi_\varepsilon$  be a cube  $\{x : |x_i| \leq b, i = 1, \dots, n\}$ , which contains  $O$ . Put  $N_\varepsilon = B_\varepsilon b \varepsilon^{-q}/2$ , where  $q$  will be determined later, and divide  $\Pi_\varepsilon$  by hyperplanes

$$x_i = bk/N_\varepsilon, \quad i = 1, \dots, n, \quad k = 0, \pm 1, \dots, \pm(N_\varepsilon - 1), \quad N_\varepsilon \in \mathbb{N},$$

into  $(2N_\varepsilon)^n$  cubes  $\Pi_{j, \varepsilon}$ ,  $j = 1, \dots, (2N_\varepsilon)^n$ . These cubes can be renumerated such that  $\Pi_j$ ,  $j = 1, \dots, J_\varepsilon$  cover  $\bar{O}$  and denote  $O_{j, \varepsilon} = O \cap \Pi_{j, \varepsilon}$ . Then  $J_\varepsilon = \mathcal{O}(\varepsilon^{-n(q+\tilde{N}_1)})$  as  $\varepsilon \rightarrow 0$ .

Denote by  $X_{j, \varepsilon}$  the center of  $\Pi_{j, \varepsilon}$  and  $\tilde{A}_{\alpha, j, \varepsilon} = \tilde{a}_{\alpha, \varepsilon}(X_{j, \varepsilon})$ . For  $\varepsilon$  small enough, let  $\{\psi_{j, \varepsilon}\}$  be a partition of the unity defined in the following way.

$$\tilde{\psi}_{j, \varepsilon} = \mathbf{1}_{O_{j, \varepsilon}} * \phi_{\varepsilon^d}, \quad \psi_{j, \varepsilon} = \frac{\tilde{\psi}_{j, \varepsilon}}{\sum_{j=1}^{J_\varepsilon} \tilde{\psi}_{j, \varepsilon}}, \quad j = 1, \dots, J_\varepsilon.$$

Note that  $\psi_{j, \varepsilon} \equiv 1$  on  $\tilde{K}_{j, \varepsilon} \subset O_{j, \varepsilon}$  and  $\text{mes}(\text{supp } \psi_{j, \varepsilon} \setminus \tilde{K}_{j, \varepsilon}) \leq C_0 \varepsilon^d$ , where  $C_0$  does not depend on  $j$  and  $d$  to be chosen later. Moreover,

$$\sup_{|\alpha| \leq m, j \leq J_\varepsilon} \|\tilde{a}_{\alpha, \varepsilon} - \tilde{A}_{\alpha, j, \varepsilon}\|_{L^\infty(O_{j, \varepsilon})} \leq B_\varepsilon \frac{b}{2N_\varepsilon} = \varepsilon^q.$$

Since  $H_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)$ , there exist  $N_2 > 0$  and  $C_2 > 0$  such that

$$\|H_\varepsilon \psi_{j, \varepsilon}\|_{L^\infty(\bar{O})} \leq \|H_\varepsilon\|_{L^\infty(\bar{O})} \leq C_2 \varepsilon^{-N_2}, \quad j = 1, \dots, J_\varepsilon,$$

for  $\varepsilon$  small enough. Denote  $H_{j, \varepsilon} := H_\varepsilon \psi_{j, \varepsilon}$ .

Let  $\tilde{G}_{j, \varepsilon} \in \mathcal{E}_M(\mathbb{R}^n)$  be a solution to

$$P_{j, \varepsilon}(D)\tilde{G}_{j, \varepsilon} = \sum_{|\alpha| \leq m} A_{\alpha, j, \varepsilon} D^\alpha \tilde{G}_{j, \varepsilon} = H_{j, \varepsilon}, \quad j \leq J_\varepsilon,$$

which exists by Theorem 1 in [82], where  $P_{j, \varepsilon}(D)$  is the adjoint operator for  $P_{j, \varepsilon}^*(D) = \sum_{|\alpha| \leq m} \tilde{A}_{\alpha, j, \varepsilon} D^\alpha$ . Put  $G_{j, \varepsilon}(x) = \tilde{G}_{j, \varepsilon}(x) \kappa_{j, \varepsilon}(x) \xi_\varepsilon(x)$ , where

$$\kappa_{j, \varepsilon} = \mathbf{1}_{(\tilde{K}_{j, \varepsilon})_{-\varepsilon^d/2}} * \phi_{\varepsilon^d/2}, \quad \xi_\varepsilon = \mathbf{1}_{(O)_{-3\varepsilon^d/4}} * \phi_{\varepsilon^d/4}, \quad j \leq J_\varepsilon.$$



Let  $K_{j,\varepsilon} = \{x \in O_{j,\varepsilon} : G_{j,\varepsilon}(x) = \tilde{G}_{j,\varepsilon}(x)\}$ ,  $j = 1, \dots, J_\varepsilon$ . Obviously,

$$\sup_{j \leq J_\varepsilon} \text{mes}(O_{j,\varepsilon} \setminus K_{j,\varepsilon}) = O(\varepsilon^{nd}) \text{ as } \varepsilon \rightarrow 0.$$

By inspecting the proof of Theorem 1 in [82] one can see that there exist  $C_0, M > 0$  and  $\tilde{N}_0 > 0$ , which depend only on  $H$  and  $a_\alpha$  such that

$$\sup_{j \leq J_\varepsilon} \|\tilde{G}_{j,\varepsilon}\|_{L^\infty(\bar{O})} \leq C_0 \varepsilon^{-M-\tilde{N}_0},$$

for  $\varepsilon$  small enough.

One can easily see that  $\text{supp}(G_{i,\varepsilon}) \cap \text{supp}(G_{j,\varepsilon}) = \emptyset$ ,  $i \neq j$  and  $G_{j,\varepsilon} \in \mathcal{E}_M(\bar{O})$ .

Let  $G_\varepsilon = \sum_{j=1}^{J_\varepsilon} G_{j,\varepsilon}$  and let  $\psi$  be an arbitrary function in  $\mathbb{D}_0^m(O) = H^m(O) \cap H_0^{m-1}(O)$ . Then

$$\int_O \psi(x) P_\varepsilon(x, D) \sum_{j=1}^{J_\varepsilon} G_{j,\varepsilon}(x) - \int_O \psi(x) H_\varepsilon(x) dx = I_1 + I_2,$$

where

$$I_1 = \int_O \psi(x) P_\varepsilon(x, D) \sum_{j=1}^{J_\varepsilon} G_{j,\varepsilon}(x) - \int_O \psi(x) \sum_{j=1}^{J_\varepsilon} P_{j,\varepsilon}(D) G_{j,\varepsilon}(x) dx,$$

$$I_2 = \int_O \psi(x) \sum_{j=1}^{J_\varepsilon} P_{j,\varepsilon}(D) G_{j,\varepsilon}(x) - \int_O \psi(x) H_\varepsilon(x) dx.$$

By using  $H_\varepsilon = \sum_{j=1}^{J_\varepsilon} H_{j,\varepsilon}$ , and  $P_{j,\varepsilon}(D) \tilde{G}_{j,\varepsilon} = H_{j,\varepsilon}$ ,  $j \leq J_\varepsilon$ , and passing to the adjoint operators, we have

$$\begin{aligned} |I_2| &= \left| \sum_{j=1}^{J_\varepsilon} \int_O P_{j,\varepsilon}^*(D) \psi(x) (\tilde{G}_{j,\varepsilon}(x) - G_{j,\varepsilon}(x)) dx \right| \\ &\leq \sum_{j=1}^{J_\varepsilon} \int_O |P_{j,\varepsilon}^*(D) \psi(x)| \cdot |\tilde{G}_{j,\varepsilon}(x)| \cdot |1 - \kappa_{j,\varepsilon}(x) \xi_\varepsilon(x)| dx \\ &\leq \sum_{j=1}^{J_\varepsilon} \sum_{|\alpha| \leq m} \|\tilde{a}_{\alpha,\varepsilon}\|_{L^\infty(\bar{O})} \int_O |D^\alpha \psi(x)| \cdot |\tilde{G}_{j,\varepsilon}(x)| \cdot |1 - \kappa_{j,\varepsilon}(x) \xi_\varepsilon(x)| dx. \end{aligned}$$

Using Cauchy–Schwartz inequality and bounds for  $a_{\alpha,\varepsilon}$ ,  $|\alpha| \leq m$  it follows

$$\begin{aligned} |I_2| &\leq C_1 \varepsilon^{-\tilde{N}_1} \|\psi\|_{H^m(O)} \sum_{j=1}^{J_\varepsilon} \left( \int_{O_{j,\varepsilon}} |\tilde{G}_{j,\varepsilon}(x)|^2 |1 - \kappa_{j,\varepsilon}(x) \xi_\varepsilon(x)|^2 dx \right)^{1/2} \\ &\leq C_1 \varepsilon^{-\tilde{N}_1} \|\psi\|_{H^m(O)} \|\tilde{G}_{j,\varepsilon}\|_{L^\infty(\bar{O})} \sum_{j=1}^{J_\varepsilon} (\text{mes}(O_{j,\varepsilon} \setminus K_{j,\varepsilon}))^{1/2} \\ &= \mathcal{O}(\varepsilon^{d/2 - \tilde{N}_1 - \tilde{N}_0 - M - n(q + \tilde{N}_1)}), \quad \varepsilon \rightarrow 0. \end{aligned}$$

This implies that  $|I_2| \rightarrow 0$  as  $\varepsilon \rightarrow 0$  if  $d > 2(\tilde{N}_1 + \tilde{N}_0 + M + n(q + \tilde{N}_1))$ . Thus, the first constant  $d$  is now determined. Further on,

$$\begin{aligned}
|I_1| &\leq \sum_{j=1}^{J_\varepsilon} \left| \int_{K_{j,\varepsilon}} \psi(x)(P_\varepsilon(x, D)G_{j,\varepsilon}(x) - P_{j,\varepsilon}(D)G_{j,\varepsilon}(x)) dx \right| \\
&\quad + \sum_{j=1}^{J_\varepsilon} \left| \int_{O_{j,\varepsilon} \setminus K_{j,\varepsilon}} \psi(x)(P_\varepsilon(x, D)G_{j,\varepsilon}(x) - P_{j,\varepsilon}(D)G_{j,\varepsilon}(x)) dx \right| \\
&\leq \sum_{j=1}^{J_\varepsilon} \left| \int_{K_{j,\varepsilon}} G_{j,\varepsilon}(x)(P_\varepsilon^*(x, D)\psi(x) - P_{j,\varepsilon}^*(D)\psi(x)) dx \right| \\
&\quad + \sum_{j=1}^{J_\varepsilon} \left| \int_{O_{j,\varepsilon} \setminus K_{j,\varepsilon}} G_{j,\varepsilon}(x)(P_\varepsilon^*(x, D)\psi(x) - P_{j,\varepsilon}^*(D)\psi(x)) dx \right| \\
&\leq \sum_{j=1}^{J_\varepsilon} \int_{K_{j,\varepsilon}} |G_{j,\varepsilon}(x)| \sum_{|\alpha| \leq m} |\tilde{a}_{\alpha,\varepsilon}(x) - A_{\alpha,j,\varepsilon}| \cdot |D^\alpha \psi(x)| dx \\
&\quad + \sum_{|\alpha| \leq m} J_\varepsilon \sup_{j \leq J_\varepsilon} \int_{O_{j,\varepsilon} \setminus K_{j,\varepsilon}} |G_{j,\varepsilon}(x)| \cdot \|\tilde{a}_{\alpha,\varepsilon}\|_{L^\infty(\bar{O})} |D^\alpha \psi(x)| dx \\
&\leq \varepsilon^q \sum_{|\alpha| \leq m} \int_O |G_{j,\varepsilon}(x)| \cdot |D^\alpha \psi(x)| dx \\
&\quad + J_\varepsilon \sum_{|\alpha| \leq m} \|\tilde{a}_{\alpha,\varepsilon}\|_{L^\infty(\bar{O})} \sup_{j \leq J_\varepsilon} \left( \int_{O_{j,\varepsilon} \setminus K_{j,\varepsilon}} |G_{j,\varepsilon}(x)|^2 \right)^{1/2} \|D^\alpha \psi\|_{L^2(O)} \\
&\leq \varepsilon^q \|G_\varepsilon\|_{L^2(O)} \|\psi\|_{H^m(O)} \\
&\quad + J_\varepsilon \sum_{|\alpha| \leq m} \|\tilde{a}_{\alpha,\varepsilon}\|_{L^\infty(\bar{O})} \sup_{j \leq J_\varepsilon} \|\psi\|_{H^m(O)} \|G_{j,\varepsilon}\|_{L^\infty(\bar{O})} (\text{mes}(O_{j,\varepsilon} \setminus K_{j,\varepsilon}))^{1/2} \\
&\leq \varepsilon^q \|G_{j,\varepsilon}\|_{L^\infty(\bar{O})} (\text{mes}(O))^{1/2} \|\psi\|_{H^m(O)} \\
&\quad + C\varepsilon^{d/2-n(q+\tilde{N}_1)} \|\psi\|_{H^m(O)} \sum_{|\alpha| \leq m} \|\tilde{a}_{\alpha,\varepsilon}\|_{L^\infty(\bar{O})} \sup_{j \leq J_\varepsilon} \|G_{j,\varepsilon}\|_{L^\infty(\bar{O})} \\
&\leq \mathcal{O}(\varepsilon^{q-M-\tilde{N}_0}) + \mathcal{O}(\varepsilon^{d/2-n(q+\tilde{N}_1)-\tilde{N}_1-M-\tilde{N}_0}), \quad \varepsilon \rightarrow 0.
\end{aligned}$$

This implies that  $|I_1| \rightarrow 0$  as  $\varepsilon \rightarrow 0$  if  $d > 2(n(q + \tilde{N}_1) + \tilde{N}_1 + M + \tilde{N}_0)$  and  $q > M + \tilde{N}_0$ . More precisely,  $|I_1|$  and  $|I_2|$  are  $\mathbb{D}_0^m$ - $s$ -associated with zero if one chooses  $q > M + \tilde{N}_0 + s$ , and then  $d > 2(n(q + \tilde{N}_1) + \tilde{N}_1 + M + \tilde{N}_0) + s$ . This proves the theorem.  $\square$

**Remark 16.** Denote by  $\mathbb{D}_0^{m,\tilde{\alpha},t}(O)$  the space of nets with elements  $\Psi_\varepsilon \in C^{m,\tilde{\alpha}}(\bar{O}) \cap H_0^{m-1}(O)$ ,  $\varepsilon \in (0, 1)$  such that  $\|D^\alpha \Psi_\varepsilon\|_{L^\infty(\bar{O})} = \mathcal{O}(\varepsilon^{-t})$ ,  $\varepsilon \rightarrow 0$ ,  $|\alpha| \leq m$ . Then

$$\|\Psi_\varepsilon\|_{H^m(O)} \leq \sum_{|\alpha| \leq m} (\text{mes}(O))^{1/2} \|D^\alpha \Psi_\varepsilon\|_{L^\infty(\overline{O})} = \mathcal{O}(\varepsilon^{-t}), \quad \varepsilon \rightarrow 0.$$

If  $q > M + \tilde{N}_0 + s + t$  and  $d > 2(n(q + \tilde{N}_1) + \tilde{N}_1 + M + \tilde{N}_0 + t) + s$ , then

$$|\langle P_\varepsilon(x, D)G_\varepsilon, \Psi_\varepsilon \rangle - \langle H_\varepsilon, \Psi_\varepsilon \rangle| = o(\varepsilon^s), \quad \text{as } \varepsilon \rightarrow 0, \quad \Psi_\varepsilon \in \mathbb{D}_0^{m,t}(O),$$

i.e., for every  $s > 0$  there exists a solution to

$$P(x, D)G \stackrel{\mathbb{D}_0^{m,t}(O)}{\approx_s} H \text{ in } O, \quad G|_{\partial O} = 0.$$

This result will be used in the following two theorems. The assumption  $\Psi_\varepsilon \in H_0^{m-1}(O)$  is crucial in the construction of the solution. This will restrict applications of the above theorem to strictly elliptic problem of order greater than two.

### 3.1.3. Applications. Let

$$(6) \quad L = \sum_{i,j=1}^n a_{ij}(x)D_iD_j + \sum_{i=1}^n b_i(x)D_i + c(x)$$

be a differential operator with real coefficients such that

$$(7) \quad a_{ij}, b_i, c \in C^1(\overline{\Omega}), \quad c \leq 0, \quad a_{ij} = a_{ji}, \quad i, j = 1, \dots, n.$$

Assume that

$$(8) \quad \Omega \text{ is bounded and } \partial\Omega \text{ is of } C^\infty \text{ class.}$$

**Remark 17.** For a method used in this and the following section, the regularization of coefficients of a differential operator and a function  $h$  is needed. Since  $C^\infty(\overline{\Omega})$  is not dense in  $C^{\tilde{\alpha}}(\overline{\Omega}) = C^{0,\tilde{\alpha}}(\overline{\Omega})$ ,  $\tilde{\alpha} \in (0, 1)$ , (cf. [97, Remark 2 in 4.5.1]) we suppose that the coefficients and  $h$  are in  $C^1(\overline{\Omega})$ .

Assume that there exist  $\lambda > 0$  and  $\Lambda > 0$  such that

$$(9) \quad \lambda|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n.$$

By Theorem 6.14 and (6.42) in [26], we have

$$(10) \quad \text{If } h \in C^{\tilde{\alpha}}(\overline{\Omega}), \text{ there exists a unique solution } g \in C^{2,\tilde{\alpha}}(\overline{\Omega}) \text{ to Dirichlet problem } Lg = h \text{ in } \Omega, \quad g = 0 \text{ on } \partial\Omega \text{ and}$$

$$(11) \quad \|g\|_{C^{2,\tilde{\alpha}}(\overline{\Omega})} \leq C\|h\|_{C^{\tilde{\alpha}}(\overline{\Omega})}, \quad \text{for some } C > 0.$$

Let  $h \in C^1(\overline{\Omega})$ ,  $h = 0$  outside of  $\overline{\Omega}$  and  $H_\varepsilon = h * \check{\phi}_\varepsilon$  ( $\phi$  is a radially symmetric function in  $C_0^\infty(\mathbb{R}^n)$   $\int \phi(x) dx = 1$ ,  $\check{\phi}_\varepsilon = \varepsilon^{-n}\phi(-\cdot/\varepsilon)$ .) Consider the Dirichlet problems

$$(12) \quad Lg = h \text{ in } \Omega, \quad g|_{\partial\Omega} = 0,$$

$$(13) \quad LZ_\varepsilon = H_\varepsilon \text{ in } \Omega, \quad Z_\varepsilon|_{\partial\Omega} = 0, \quad \text{for fixed } \varepsilon \in (0, 1),$$

$$(14) \quad LG \stackrel{H^2(\Omega) \cap H_0^1(\Omega)}{\approx} H \text{ in } \Omega, \quad G|_{\partial\Omega} = 0.$$

The last equation means that

$$(15) \quad \lim_{\varepsilon \rightarrow 0} \int (LG_\varepsilon - H_\varepsilon)\psi \, dx = 0, \text{ for every } \psi \in H^2(\Omega) \cap H_0^1(\Omega), G|_{\partial\Omega} \in \mathcal{N}_W(\partial\Omega),$$

in terms of representatives. Let  $g$ ,  $Z_\varepsilon$  and  $G_\varepsilon$  be solutions to (12), (13) and (15), respectively, where solutions  $g$  and  $Z_\varepsilon$  exist by (10) and  $G_\varepsilon$  is the solution given in Theorem 15.

**Theorem 18.** *Let  $L$  and  $\Omega$  satisfy assumptions given above. Moreover, assume that  $L$  has coefficients which are smooth on  $\bar{\Omega}$ . The generalized solution  $G$  to (14) constructed in Theorem 15 is  $C^{\tilde{\alpha}}(\bar{\Omega}) \cap H_0^1(\Omega)$ -associated,  $\tilde{\alpha} > 0$ , with the classical solution  $g$  to (12).*

*Proof.* By [97], Theorem 1 in 4.5.2,  $C^{\tilde{\alpha}}(\bar{\Omega}) = [C^0(\bar{\Omega}), C^1(\bar{\Omega})]_{\tilde{\alpha}}$ . ( $[\cdot, \cdot]_{\tilde{\alpha}}$  denotes an interpolation space.) This implies [97, (7) in 4.5.2],

$$(16) \quad \|f\|_{C^{\tilde{\alpha}}(\bar{\Omega})} \leq \|f\|_{C^0(\bar{\Omega})}^{1-\tilde{\alpha}} \|f\|_{C^1(\bar{\Omega})}^{\tilde{\alpha}}, \quad f \in C^1(\bar{\Omega}).$$

and in the special case,

$$\|h - H_\varepsilon\|_{C^{\tilde{\alpha}}(\bar{\Omega})} \leq \|h - H_\varepsilon\|_{C^0(\bar{\Omega})}^{1-\tilde{\alpha}} \|h - H_\varepsilon\|_{C^1(\bar{\Omega})}^{\tilde{\alpha}} \rightarrow 0, \quad \varepsilon \rightarrow 0,$$

where  $h$  and  $H_\varepsilon$  are the functions from (12) and (15) respectively.

The above inequality, boundedness of  $\Omega$  and (11) imply  $\|g - Z_\varepsilon\|_{H^1(\Omega)} \rightarrow 0$ , as  $\varepsilon \rightarrow 0$  and

$$\int (g(x) - Z_\varepsilon(x))\theta(x) \, dx \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0, \quad \theta \in C^{\tilde{\alpha}}(\bar{\Omega}) \cap H_0^1(\Omega).$$

We have to prove that

$$\int_{\Omega} (G_\varepsilon(x) - Z_\varepsilon(x))\theta(x) \, dx \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

The boundary value problem  $L^*\psi = \theta$  in  $\Omega$ ,  $\psi|_{\partial\Omega} = 0$  has a solution in  $C^{2,\tilde{\alpha}}(\bar{\Omega}) \cap H_0^1(\Omega)$ . This follows from (10) since the adjoint operator  $L^*$  satisfies the assumptions given in this paragraph. We have

$$\begin{aligned} \int_{\Omega} G_\varepsilon(x)\theta(x) \, dx &= \int_{\Omega} G_\varepsilon(x)L^*\psi(x) \, dx = \int_{\Omega} LG_\varepsilon(x)\psi(x) \, dx, \\ \int_{\Omega} Z_\varepsilon(x)\theta(x) \, dx &= \int_{\Omega} Z_\varepsilon(x)L^*\psi(x) \, dx, \quad \psi \in C^{2,\tilde{\alpha}}(\bar{\Omega}) \cap H_0^1(\Omega). \end{aligned}$$

By Theorem 15

$$\int_{\Omega} (G_\varepsilon(x) - Z_\varepsilon(x))\theta(x) \, dx = \int_{\Omega} (LG_\varepsilon(x) - H_\varepsilon(x))\psi(x) \, dx \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

This proves the theorem.  $\square$

We continue to consider  $L$  of the form (6) which satisfies (7), (8) and (9). Let  $\bar{\Omega}$  satisfies the cone property [97, Definition in 4.2.3] in addition. Let

$$\tilde{L}_\varepsilon = \sum_{i,j=1}^n a_{ij,\varepsilon}(x)D_iD_j + \sum_{i=1}^n b_{i,\varepsilon}(x)D_i + c_\varepsilon(x)$$

be regularized operator for  $L$ , where  $(\mathbf{1}_\Omega a_{ij}) * \check{\phi}_\varepsilon(x) = a_{ij,\varepsilon}(x)$ ,  $(\mathbf{1}_\Omega b_i) * \check{\phi}_\varepsilon(x) = b_{i,\varepsilon}(x)$ ,  $(\mathbf{1}_\Omega c) * \check{\phi}_\varepsilon(x) = c_\varepsilon(x)$ ,  $x \in \mathbb{R}$ ,  $\check{\phi}_\varepsilon(x) = \phi(-x)$ . Clearly,  $c_\varepsilon(y) \leq 0$ ,  $y \in \Omega$ . Since  $a_{ij,\varepsilon} \rightarrow a_{ij}$ ,  $b_{i,\varepsilon} \rightarrow b_i$  and  $c_\varepsilon \rightarrow c$  in  $C^0(\bar{\Omega})$  and  $C^1(\bar{\Omega})$  as  $\varepsilon \rightarrow 0$ , one gets convergence in  $C^{\tilde{\alpha}}(\bar{\Omega})$ , too.

Multiplying  $a_{ij}$  in (9) by  $\mathbf{1}_\Omega$ , then multiplying all the members of (9) by  $\phi_\varepsilon(y-x)$  and by integrating over  $\mathbb{R}^n$ , we obtain

$$\lambda|\xi|^2 \int_{\Omega} \phi_\varepsilon(y-x) dy \leq \sum_{i,j=1}^n a_{ij,\varepsilon}(x) \xi_i \xi_j \leq \Lambda|\xi|^2 \int_{\Omega} \phi_\varepsilon(y-x) dy \leq \Lambda|\xi|^2,$$

where  $y \in \Omega$ ,  $\xi \in \mathbb{R}^n$ ,  $\varepsilon < \varepsilon_0$ . By using radial symmetry and non-negativity of the function  $\phi$  and the cone property of  $\bar{\Omega}$ ,

$$\int_{\Omega} \phi_\varepsilon(y-x) dy \geq \int_{\Gamma_{x,h}} \phi_\varepsilon(y-x) dy \geq \tilde{D} > 0,$$

where  $\Gamma_{x,h}$  is a cone with the vertex at zero and height  $h$ , and  $y \in x + \Gamma_{x,h} \subset \Omega$ . The constant  $\tilde{D}$  does not depend on  $\varepsilon$  and we have

$$\tilde{D}\lambda|\xi|^2 \leq \sum_{i,j=1}^n a_{ij,\varepsilon}(x) \xi_i \xi_j \leq \Lambda|\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n, \quad \varepsilon \leq \varepsilon_0.$$

**Theorem 19.** *Let  $L$  satisfy given assumptions. Then the generalized solution  $G$  to*

$$\tilde{L}G \stackrel{H^2(\Omega) \cap H_0^1(\Omega)}{\approx} H \text{ in } \Omega, \quad G|_{\partial\Omega} = 0,$$

constructed in Theorem 15, is  $C^{\tilde{\alpha}}(\bar{\Omega}) \cap H_0^1(\Omega)$ -associated to the classical solution  $g \in H^{2,\tilde{\alpha}}(\bar{\Omega})$  to (12).

*Proof.* Let  $\theta \in C^{\tilde{\alpha}}(\bar{\Omega}) \cap H_0^1(\Omega)$  be given. Then (10) and (11) applied to  $L_\varepsilon$  imply that for every  $\varepsilon \in (0, 1)$  the solution  $\Psi_\varepsilon$  to  $\tilde{L}_\varepsilon^* \Psi_\varepsilon = \theta$  in  $\Omega$ ,  $\Psi_\varepsilon|_{\partial\Omega} = 0$  belongs to  $C^{2,\tilde{\alpha}}(\bar{\Omega}) \cap H_0^1(\Omega)$  and

$$\sup_{|\beta| \leq 2, x \in \Omega} |D^\beta \Psi_\varepsilon(x)| \leq C \left( \sup_{x \in \Omega} |\theta(x)| + \sup_{x,y \in \bar{\Omega}} \frac{|\theta(x) - \theta(y)|}{|x-y|^{\tilde{\alpha}}} \right) = \|\theta\|_{C^{\tilde{\alpha}}(\bar{\Omega})}, \quad \varepsilon \in (0, 1),$$

where  $C > 0$  depends only on  $\lambda$ ,  $\Lambda$  and the diameter of the set  $\Omega$ . By the remark after Theorem 15 we have that for every  $s > 0$  there exists  $G_\varepsilon$  such that

$$\langle \tilde{L}_\varepsilon G_\varepsilon, \Psi_\varepsilon \rangle - \langle H_\varepsilon, \Psi_\varepsilon \rangle = o(\varepsilon^s), \quad \varepsilon \rightarrow 0.$$

Denote by  $Z_\varepsilon$  a smooth solution to  $\tilde{L}_\varepsilon Z_\varepsilon = H_\varepsilon$  in  $\Omega$ ,  $Z_\varepsilon|_{\partial\Omega} = 0$ ,  $\varepsilon \in (0, 1)$  is fixed. Then

$$\begin{aligned} (17) \quad & \int_{\Omega} (G_\varepsilon(x) - Z_\varepsilon(x))\theta(x) dx = \int_{\Omega} (G_\varepsilon(x) - Z_\varepsilon(x))L_\varepsilon^* \Psi_\varepsilon(x) dx \\ & = \int_{\Omega} (\tilde{L}_\varepsilon G_\varepsilon(x) - \tilde{L}_\varepsilon Z_\varepsilon(x))\Psi_\varepsilon(x) dx = \langle \tilde{L}_\varepsilon G_\varepsilon - H_\varepsilon, \Psi_\varepsilon \rangle = o(\varepsilon^s), \quad \varepsilon \rightarrow 0. \end{aligned}$$

Let  $\tilde{H}_\varepsilon = \tilde{L}_\varepsilon g$ , where  $g \in C^{2,\tilde{\alpha}}(\bar{\Omega})$  is the solution to (12). Inequality (11) and (16) imply

$$\begin{aligned} \|(\tilde{L}_\varepsilon - L)g\|_{C^{\tilde{\alpha}}(\bar{\Omega})} &\leq \sum_{i=1}^n \sum_{j=1}^n \|a_{ij,\varepsilon} - a_{ij}\|_{C^{\tilde{\alpha}}(\bar{\Omega})} \|D_i D_j g\|_{L^\infty(\Omega)} \\ &\quad + \sum_{i=1}^n \|b_{i,\varepsilon} - b_i\|_{C^{\tilde{\alpha}}(\bar{\Omega})} \|D_i g\|_{L^\infty(\Omega)} + \|c_\varepsilon - c\|_{C^{\tilde{\alpha}}(\bar{\Omega})} \|g\|_{L^\infty(\Omega)} \rightarrow 0, \end{aligned}$$

as  $\varepsilon \rightarrow 0$ . This implies that  $\tilde{H}_\varepsilon - h \xrightarrow{C^{\tilde{\alpha}}(\bar{\Omega})} 0$  as  $\varepsilon \rightarrow 0$ , i.e.,  $\tilde{H}_\varepsilon - H_\varepsilon \xrightarrow{C^{\tilde{\alpha}}(\bar{\Omega})} 0$  (since  $h \in C^1(\Omega)$ ). Finally, (10) implies  $\|Z_\varepsilon - g\|_{C^{2,\tilde{\alpha}}(\bar{\Omega})} \leq C \|\tilde{H}_\varepsilon - H_\varepsilon\|_{C^{\tilde{\alpha}}(\bar{\Omega})} \rightarrow 0$ . This and (17) prove the theorem.  $\square$

### 3.1.4. A class of generalized elliptic differential operators of order $2m$ .

Consider a family of equations

$$(18) \quad \begin{aligned} P_\varepsilon(x, D)G_\varepsilon &= \sum_{|\alpha| \leq 2m} a_{\alpha,\varepsilon}(x) D^\alpha G_\varepsilon(x) = H_\varepsilon(x), \quad x \in \Omega, \\ \{D^\alpha G_\varepsilon|_{\partial\Omega}, \alpha \in \mathbb{N}_0^n\} &\in \mathcal{N}_W(\partial\Omega), \end{aligned}$$

where:

1.  $\Omega$  is an open bounded set with a smooth boundary  $\partial\Omega$ .
2.  $a_{\alpha,\varepsilon} \in \mathcal{E}_M(\mathbb{R}^n)$  is complex valued,  $|\alpha| \leq 2m$ ,  $H_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)$ .
3. For every  $\varepsilon \in (0, \varepsilon_0)$ ,  $P_\varepsilon(x, D)$  is uniformly and strongly elliptic (cf. [96, Ch. 36, (36.3)]) and moreover, there exist  $C_0 > 0$  and  $p_0 \geq 0$  such that

$$C_0 \varepsilon^{p_0} \|u\|_{H^m(\Omega)}^2 \leq \operatorname{Re} \langle P_\varepsilon(x, D)u, u \rangle_{L^2(\Omega)}, \quad u \in C_0^\infty, \quad \varepsilon < \varepsilon_0.$$

Then, for every fixed  $\varepsilon < \varepsilon_0$ ,

$$(19) \quad \begin{aligned} P_\varepsilon(x, D) : H_0^m(\Omega) &\rightarrow H^{-m}(\Omega) \text{ is a surjective isomorphism} \\ \text{and the solution to (18)} &\text{ satisfies } \|G_\varepsilon\|_{H^m} \leq C_0^{-1} \varepsilon^{-p_0} \|H_\varepsilon\|_{H^{-m}(\Omega)}. \end{aligned}$$

(cf. [96, Theorem 36.2, Lemma 23.1])

The second assumption means that there exist  $\nu_1, \nu_2 > 0$  such that

$$(20) \quad \sup_{|\alpha| \leq m} \|a_{\alpha,\varepsilon}\|_{L^\infty(\bar{\Omega})} = \mathcal{O}(\varepsilon^{-\nu_1}),$$

$$(21) \quad \|H_\varepsilon\|_{L^\infty(\bar{\Omega})} = \mathcal{O}(\varepsilon^{-\nu_2}), \quad \text{as } \varepsilon \rightarrow 0.$$

Note that (21) implies  $\|H_\varepsilon\|_{H^{-m}(\Omega)} = \mathcal{O}(\varepsilon^{-\nu_2})$ , as  $\varepsilon \rightarrow 0$ .

**Theorem 20.** *Let  $P_\varepsilon(x, D)$ ,  $H_\varepsilon$  and  $\Omega$  satisfy the conditions given above. Then:*

(a) *For every  $s \geq 0$  there exists a solution  $G_{s,\varepsilon} \in \mathcal{E}_M(\mathbb{R}^n)$  to (18) in the  $H_0^{2m}(\Omega)$ - $s$ -associated sense, i.e.,*

$$\langle P_\varepsilon(x, D)G_{s,\varepsilon} - H_\varepsilon, \psi \rangle = o(\varepsilon^s), \quad \varepsilon \rightarrow 0, \quad \psi \in H_0^{2m}(\Omega)$$

*and  $\{D^\alpha G_{s,\varepsilon}|_{\partial\Omega} = 0, \alpha \in \mathbb{N}_0^n\} \in \mathcal{N}_W(\partial\Omega)$ . The solution constructed in the proof will be called  $s$ -solution.*

(b) Let  $T_{s,\varepsilon}$  be an  $s$ -solution to  $P_\varepsilon(x, D)T_{s,\varepsilon} = R_\varepsilon$ , where  $R_\varepsilon$  satisfies (21). Moreover, assume that

$$\varepsilon^{-(p_0+s_0m)} \|R_\varepsilon - H_\varepsilon\|_{H^{-m}(\Omega)} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

where  $s_0 > \nu_1 + \nu_2 + p_0 + s$ . Then  $T_{s,\varepsilon}$  and  $G_{s,\varepsilon}$  are  $H^{-m}(\Omega)$ -0-associated. Specially, if  $R_\varepsilon$  and  $H_\varepsilon$  are in the same class of  $\mathcal{G}(\Omega)$ , then the appropriate solutions are  $H^{-m}(\Omega)$ - $\tilde{s}$ -associated for arbitrary  $\tilde{s} > 0$ . This is a kind of the uniqueness of solution to (18).

(c) Let  $P(x, D)$  have  $C^\infty(\mathbb{R}^n)$  coefficients which satisfy condition 3 with  $p_0 = 0$  and let  $C_0$  do not depend on  $\varepsilon$ . Assume that  $H \in H^{-m}(\Omega)$  is of the form  $H = \sum_{|\alpha| \leq m} D^\alpha f_\alpha$ , where  $f_\alpha \in L^2(\Omega)$ ,  $f_\alpha(x) = 0$  for  $x \notin \Omega$ ,  $|\alpha| \leq m$ . Let

(i)  $U \in H_0^m(\Omega)$  be the solution to  $P(x, D)U = H$ ,  $U|_{\partial\Omega} = 0$  (which exists by the Lax-Milgram Lemma) and

(ii)  $G_{s,\varepsilon}$  be the  $s$ -solution to

$$(22) \quad P(x, D)G_{s,\varepsilon} = H_\varepsilon \text{ in } \Omega, \quad G_\varepsilon|_{\partial\Omega} = 0,$$

in  $H_0^{2m}$ - $s$ -associated sense, where  $H_\varepsilon = H * \phi_{\varepsilon^d}$  for an appropriate  $d = d(s) > 0$ .

Then  $U \stackrel{H^{-m}(\Omega)}{\approx} G_0$ .

(d) Denote by  $G_\varepsilon$  the solution in  $\mathbb{D}_0^m(\Omega)$ -0-associated sense to (22) constructed in the proof of Theorem 15 and by  $G_{0,\varepsilon}$  the solution to (22) in  $H_0^{2m}(\Omega)$ -0-associated sense. Then  $G_\varepsilon \stackrel{\mathbb{D}_P(\Omega)}{\approx} G_{0,\varepsilon}$ , where  $\mathbb{D}_P(\Omega)$  is the set of all nets  $\Psi_\varepsilon$  in  $H_0^0(\Omega)$  such that there exist a function  $\psi \in H_0^{2m}(\Omega)$  and  $\eta > 0$  such that  $\Psi_\varepsilon = P_\varepsilon^* \psi$ , for every  $\varepsilon < \eta$ .

*Proof.* (a) Recall, (19) implies that for every fixed  $\varepsilon < \varepsilon_0$  there exists a solution  $g_\varepsilon \in H_0^m(\Omega)$  to equation  $P_\varepsilon(x, D)g_\varepsilon = H_\varepsilon$ , in  $\Omega$ , that is

$$(23) \quad \langle P_\varepsilon(x, D)g_\varepsilon, \psi \rangle = \langle H_\varepsilon, \psi \rangle, \quad \psi \in H_0^m(\Omega).$$

By ellipticity of  $P_\varepsilon(x, D)$  for every fixed  $\varepsilon$ , the solution  $g_\varepsilon$  to (23) is in  $C^\infty(\Omega)$ . Let us prove that  $g_\varepsilon \in \mathcal{E}_M(\Omega)$ . Let  $D_i$  be an arbitrary derivative of the first order. Then

$$D_i(P_\varepsilon(x, D)g_\varepsilon(x)) = P_\varepsilon(x, D)D_i g_\varepsilon(x) + \tilde{P}_\varepsilon(x, D)g_\varepsilon(x),$$

where  $\tilde{P}_\varepsilon(x, D) = \sum_{|\alpha| \leq 2m} D_i a_{\alpha,\varepsilon}(x) D^\alpha g_\varepsilon(x)$ . Integration by parts implies

$$\begin{aligned} \|\tilde{P}_\varepsilon(\cdot, D)g_\varepsilon\|_{H^{-m}(\Omega)} &= \sup_{\|\phi\|_{H^m(\Omega)} \leq 1} \left| \int_\Omega \tilde{P}_\varepsilon(x, D)g_\varepsilon(x) \phi(x) dx \right| \\ &\leq \sup_{\|\phi\|_{H^m(\Omega)} \leq 1} \int_\Omega \left| \sum_{|\alpha| \leq m} b_{\alpha,\varepsilon}(x) D^\alpha g_\varepsilon(x) \right| \cdot \left| \sum_{|\beta| \leq m} c_{\beta,\varepsilon} D^\beta \phi(x) \right| dx, \end{aligned}$$

where  $b_{\alpha,\varepsilon}, c_{\beta,\varepsilon} \in \mathcal{E}_M(\mathbb{R}^n)$ ,  $|\alpha| \leq m$ ,  $|\beta| \leq m$ . Since (19) and (23) imply

$$\|g_\varepsilon\|_{H^m(\Omega)} = \mathcal{O}(\varepsilon^{-(p_0+\nu_2)}), \quad \varepsilon \rightarrow 0,$$

then

$$\|\tilde{P}_\varepsilon(\cdot, D)g_\varepsilon\|_{H^{-m}(\Omega)} = \mathcal{O}(\varepsilon^{-\nu_3}), \quad \varepsilon \rightarrow 0,$$

and

$$P_\varepsilon(x, D)D_i g_\varepsilon(x) = D_i H_\varepsilon(x) - \tilde{P}_\varepsilon(x, D)g_\varepsilon(x) \in H^{-m} \cap \mathcal{E}_M(\mathbb{R}^n)$$

for a suitable  $\nu_3 > 0$ . Thus, (19) implies  $\|D_i g_\varepsilon\|_{H^m(\Omega)} = \mathcal{O}(\varepsilon^{-\nu_4})$ ,  $\varepsilon \rightarrow 0$ , for some  $\nu_4 > 0$ . By induction with the respect to the orders of derivatives, it follows that  $g_\varepsilon \in \mathcal{E}_M(\Omega)$ . Put

$$(24) \quad \kappa_\varepsilon = \mathbf{1}_{(\Omega)_{-3\varepsilon^{s_0/4}}} * \phi_{\varepsilon^{s_0/4}},$$

where  $s_0$  will be determined later. Note that  $\text{mes}(\Omega \setminus (\Omega)_{-\varepsilon^{s_0}}) = \mathcal{O}(\varepsilon^{s_0})$  as  $\varepsilon \rightarrow 0$ . Define  $G_{s,\varepsilon} = g_\varepsilon \kappa_\varepsilon$ . Then  $G_{s,\varepsilon} \in \mathcal{E}_M(\mathbb{R}^n)$  since  $g_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)$ .

For arbitrary  $\psi \in H_0^{2m}$ ,

$$\begin{aligned} |I_\varepsilon| &= |\langle P_\varepsilon(x, D)G_{s,\varepsilon} - H_\varepsilon, \psi \rangle| = |\langle G_{s,\varepsilon} - H_\varepsilon, P_\varepsilon^*(x, D)\psi \rangle| \\ &\leq \int_\Omega |g_\varepsilon(x)(1 - \kappa_\varepsilon(x))| \cdot |P_\varepsilon^*(x, D)\psi(x)| dx \\ &\leq 2 \sup_{|\alpha| \leq m} \|\tilde{a}_{\alpha,\varepsilon}\|_{L^\infty(\overline{\Omega})} \|\psi\|_{H^{2m}(\Omega)} \|g_\varepsilon\|_{L^2(\Omega)} (\text{mes}(\Omega \setminus (\Omega)_{-\varepsilon^{s_0}}))^{1/2} \\ &= \mathcal{O}(\varepsilon^{s_0 - (\nu_1 + \nu_2 + p_0)}), \quad \varepsilon \rightarrow 0. \end{aligned}$$

The assertion follows by choosing  $s_0 > \nu_1 + \nu_2 + p_0 + s$ .

(b) Let  $t_\varepsilon$  be the solution to (23) when  $H_\varepsilon$  is replaced by  $R_\varepsilon$  and  $T_\varepsilon = t_\varepsilon \kappa_\varepsilon$ , where  $\kappa_\varepsilon$  is given by (24). Then, (19) implies

$$\begin{aligned} \|G_{s,\varepsilon} - T_{s,\varepsilon}\|_{H^m(\Omega)} &= \|(g_{s,\varepsilon} - t_{s,\varepsilon})\kappa_\varepsilon\|_{H^m(\Omega)} \\ &\leq \sup_{|\alpha| \leq m} \|g_{s,\varepsilon} - t_{s,\varepsilon}\|_{H^m(\Omega)} \|D^\alpha \kappa_\varepsilon\|_{L^\infty(\Omega)} \\ &\leq C_0^{-1} \varepsilon^{-p_0} \varepsilon^{-s_0 m} \|F_\varepsilon - R_\varepsilon\|_{H^{-m}(\Omega)}. \end{aligned}$$

Now, for  $\psi \in H^{-m}(\Omega)$

$$|\langle G_{s,\varepsilon} - T_{s,\varepsilon}, \psi \rangle| \leq C \|G_{s,\varepsilon} - T_{s,\varepsilon}\|_{H^m(\Omega)} \|\psi\|_{H^{-m}(\Omega)} \leq C_1 \varepsilon^{-p_0 - s_0 m} \|F_\varepsilon - R_\varepsilon\|_{H^{-m}(\Omega)}$$

and the proof follows.

(c) The assertion is a direct consequence of (b).

(d) Let  $\Psi_\varepsilon \in \mathbb{D}_P(\Omega)$  and  $G_\varepsilon$  be the solution constructed in Theorem 15. By the definition of  $\mathbb{D}_P(\Omega)$  there exists  $\psi_1 \in H_0^{2m}(\Omega)$  such that  $P_\varepsilon^*(x, D)\psi_1 = \Psi_\varepsilon$  for every  $\varepsilon < \eta$ . Then

$$\begin{aligned} \int_\Omega (G_\varepsilon - G_{0,\varepsilon})(x) \Psi_\varepsilon(x) dx &= \int_\Omega (G_\varepsilon - G_{0,\varepsilon})(x) P_\varepsilon^*(x, D)\psi_1(x) dx \\ &= \int_\Omega P_\varepsilon(x, D)(G_\varepsilon - G_{0,\varepsilon})(x) \psi_1(x) dx \rightarrow 0, \quad \varepsilon \rightarrow 0. \end{aligned}$$

This proves (d). □



**3.2. Quasilinear elliptic PDE.** First we give a simple example in order to illustrate our approach to a class of quasilinear elliptic PDE.

**Example 21.** Let  $\delta_{(0,1)}(x_1, x_2) = \delta(x_1)\delta(x_2 - 1)$ ,  $(x_1, x_2) \in \mathbb{R}^2$  be the delta distribution concentrated at  $(0, 1)$  and  $B_1$  be the ball with the radius 1 and center  $(0, 0)$ . Define  $\delta_{(0,1)}|_{\partial B_1}$  by  $\langle \delta_{(0,1)}|_{\partial B_1}, \phi \rangle = \phi(0, 1)$ , where  $\phi$  is a smooth function on the circle  $\partial B_1$ . Consider a Dirichlet problem formally written as

$$\Delta u(x) = 0, \quad x \in B_1 \subset \mathbb{R}^2, \quad u|_{\partial B_1} = \delta_{(0,1)}|_{\partial B_1}.$$

We approximate  $\delta(x_1)\delta(x_2 - 1)$  by a net  $\frac{1}{\varepsilon^2}\phi\left(\frac{x_1}{\varepsilon}\right)\phi\left(\frac{x_2 - 1}{\varepsilon}\right)$ ,  $(x_1, x_2) \in \mathbb{R}^2$ ,  $\varepsilon \in (0, 1)$ , with the properties  $\phi \in C_0^\infty(\mathbb{R})$ ,  $\int \phi = 1$  and  $\text{supp } \phi \in [-1, 1]$  (the net of mollifiers). Then, we replace the given problem with the family of problems

$$\begin{aligned} \Delta u_\varepsilon(x) &= 0, \quad x \in B_1 \subset \mathbb{R}^2, \\ u_\varepsilon|_{\partial B_1} &= \frac{1}{\varepsilon^2}\phi\left(\frac{x_1}{\varepsilon}\right)\phi\left(\frac{\sqrt{1-x_1^2}-1}{\varepsilon}\right), \quad |x_1| < \varepsilon \\ &\text{and zero on the rest of the boundary.} \end{aligned}$$

Using the Poisson formula we obtain a family of solutions  $U_\varepsilon$  of corresponding classical solutions.

Assume that  $\phi(0) = 0$ . Then, in the sense of the weak convergence in  $\mathcal{D}'(B_1)$ ,

$$U_\varepsilon(x) \rightarrow f(x) = C_\phi \frac{|x|^2 - 1}{x_1^2 + (x_2 - 1)^2} \quad \text{as } \varepsilon \rightarrow 0, \quad \text{where } C_\phi = -\frac{1}{4\pi}\phi'(0) \int_{-1}^1 u^2\phi(u) du.$$

Note that  $f$  is a solution to  $\Delta u = 0$  in  $B_1$ . Moreover, for any point  $(x_{10}, x_{20}) \in \partial B_1 \setminus (0, 1)$

$$\lim_{(x_1, x_2) \rightarrow (x_{10}, x_{20})} \frac{x_1^2 + x_2^2 - 1}{x_1^2 + x_2^2 - 2x_2 + 1} = 0 \quad ((x_1, x_2) \in B_1)$$

and, for  $\theta = \pi/2$ ,  $\lim_{r \rightarrow 1^-} \frac{r^2 - 1}{r^2 - 2r + 1} = -\infty$ . This shows the ‘‘blow up’’ of a solution at  $(0, 1)$ .

Let  $\mathcal{Q}_\varepsilon$  be a net of elliptic nonlinear operators of divergent type of the form

$$(25) \quad \mathcal{Q}_\varepsilon(u) = \text{div } \mathbf{A}_\varepsilon(Du) = a_\varepsilon^{i,j}(Du)D_{i,j}u, \quad \varepsilon < 1,$$

where  $a_\varepsilon^{i,j}(p) = D_{p_i}A_\varepsilon^j(p)$ , or, in case  $n = 2$ , let  $\mathcal{Q}_\varepsilon$  be a net of elliptic nonlinear operators of the form

$$(26) \quad \mathcal{Q}_\varepsilon(u) = a_\varepsilon^{i,j}(x, u, Du)D_{i,j}u, \quad u \in C^\infty(\bar{O}).$$

We assume that  $a_\varepsilon^{i,j}$ ,  $\varepsilon \in (0, 1)$  are smooth functions on  $O$ . If  $\lambda_\varepsilon$  and  $\Lambda_\varepsilon$  denote respectively the minimum and maximum eigenvalues, then we have

$$\begin{aligned} 0 < \lambda_\varepsilon(x, t, p)|\xi|^2 &\leq a_\varepsilon^{i,j}(x, t, p)\xi_i\xi_j \leq \Lambda_\varepsilon(x, t, p)|\xi|^2, \\ p &\in \mathbb{R}^n, \quad \xi \in \mathbb{R}^n \setminus \{0\}, \quad x \in O, \quad t \in \mathbb{R}, \quad \varepsilon < \varepsilon_0. \end{aligned}$$

Assume additionally:

$$(27) \quad (\forall d \in \mathbb{N}_0^n)(\exists l_d \in \mathbb{R})(\exists a_d \in \mathbb{R})$$

$$\sup \left\{ \frac{|\partial_x^d a_\varepsilon^{i,j}(x, t, p)|}{(1 + |t| + |p|)^{a_d}}; x \in \bar{O}, t \in \mathbb{R}, p \in \mathbb{R}^n \right\} = \mathcal{O}(\varepsilon^{l_d}).$$

$$(28) \quad (\exists C > 0)(\exists \mu > 0)(\exists b \in \mathbb{R})$$

$$\frac{\varepsilon^\mu}{C}(1 + |t| + |p|)^b \leq \lambda_\varepsilon(x, t, p) \leq \Lambda_\varepsilon(x, t, p) \leq \frac{C}{\varepsilon^\mu}(1 + |t| + |p|)^b,$$

$$p \in \mathbb{R}^n, x \in \bar{O}, t \in \mathbb{R}, \varepsilon < \varepsilon_0.$$

In the case when the net  $\mathcal{Q}_\varepsilon$  is of the form (26) then  $n = 2$ , and if it is of the divergent form (25), then we exclude variables  $x$  and  $t$  in the conditions given above. Note that condition (28) implies  $\Lambda_\varepsilon/\lambda_\varepsilon \leq C^2/\varepsilon^{2\mu}$ ,  $\varepsilon < \varepsilon_0$ . We will consider this net in the framework of  $\mathcal{G}_{C^k, \alpha}$ . In this case we will use the notation  $\mathcal{F} = \mathcal{E}_{C^k, \alpha}$ . With the given properties  $\mathcal{Q}_\varepsilon$  is called the net of uniformly elliptic moderate continuous operators.

**Example 22.** (i) All the examples given in [26, pp. 260–262], for  $n = 2$  can serve as examples in our framework but now with singular boundary conditions.

(ii) Consider in  $\mathbb{R}^3$  the operator

$$Q(x, u, Du) = \left( 1 + \sum_{i=1}^3 \delta(D_i) \right) \Delta u \quad (\delta \text{ is the delta distribution}).$$

With the regularization of  $\delta = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \phi(\frac{\cdot}{\varepsilon})$ , we have

$$Q_\varepsilon(x, u, Du) = \left( \frac{1}{\varepsilon} \phi\left(\frac{D_1 u}{\varepsilon}\right) + \frac{1}{\varepsilon} \phi\left(\frac{D_2 u}{\varepsilon}\right) \frac{1}{\varepsilon} \phi\left(\frac{D_3 u}{\varepsilon}\right) + 1 \right) \Delta u.$$

( $\phi$  is a compactly supported smooth function with the integral equals 1.) Then

$$\lambda_\varepsilon = 1 \text{ and } \Lambda_\varepsilon = \left( \frac{1}{\varepsilon} \phi\left(\frac{p_1}{\varepsilon}\right) + \frac{1}{\varepsilon} \phi\left(\frac{p_2}{\varepsilon}\right) \frac{1}{\varepsilon} \phi\left(\frac{p_3}{\varepsilon}\right) + 1 \right).$$

This operator is of the form (25) for which all the assumptions given above hold.

We need a “slope condition” adapted to the setting of Colombeau theory.

**Definition 23.** Let  $E = C^{k, \alpha}(\bar{O})$  for some  $k \in \mathbb{N}$  (cf. 1.1 and  $\mathcal{G}_{C^k, \alpha}$ ),  $\psi_\varepsilon \in \mathcal{F} = \mathcal{E}_{C^k, \alpha}$  and  $\Gamma_\varepsilon = \{(x, z_\varepsilon), x \in \partial O, z_\varepsilon = \psi_\varepsilon(x)\}$ . The boundary  $\partial O$  satisfies a moderate slope condition if for any  $P_\varepsilon \in \Gamma_\varepsilon$  there exist hyperplanes  $\pi_{\varepsilon, P_\varepsilon}^+$  and  $\pi_{\varepsilon, P_\varepsilon}^-$  defined by  $z_\varepsilon = \pi_{\varepsilon, P_\varepsilon}^+(x)$  and  $z_\varepsilon = \pi_{\varepsilon, P_\varepsilon}^-(x)$  such that

$$\pi_{\varepsilon, P_\varepsilon}^-(x) \leq \psi_\varepsilon(x) \leq \pi_{\varepsilon, P_\varepsilon}^+(x), x \in \partial O, \varepsilon < \varepsilon_0$$

and such that for some  $K > 0$  and some  $m \in \mathbb{R}$ ,

$$\sup\{|D\pi_{\varepsilon, P_\varepsilon}^+(x)|, |D\pi_{\varepsilon, P_\varepsilon}^-(x)|; x \in \partial O, P_\varepsilon \in \Gamma_\varepsilon\} \leq K\varepsilon^m, \varepsilon < \varepsilon_0.$$

**Proposition 24.** Let  $\mathcal{Q}_\varepsilon$  be a net of uniformly elliptic operators of the form (25) or (26) with  $a_\varepsilon^{i,j} \in C^{k+1}(\bar{O})$  ( $k \in \mathbb{N}$ ) satisfying (27) with  $d \leq k + 1$  and (28).

Let  $E = C^{k+2,\alpha}(\bar{O})$ ,  $\psi_\varepsilon \in \mathcal{E}_{C^{k+2,\alpha}}$ , where  $\partial O$  is of  $C^{k+2,\alpha}$  class and it satisfies a moderate slope condition with  $\psi_\varepsilon$ . Then, there exists  $U_\varepsilon \in \mathcal{E}_{C^{k+2,\alpha}}$  such that

$$\mathcal{Q}_\varepsilon(U_\varepsilon) = 0, U_\varepsilon|_{\partial O} = \psi_\varepsilon, \varepsilon < 1.$$

This theorem implies the solvability in  $\mathcal{G}_{C^{k,\alpha}}$ . The process of regularization of equation  $\operatorname{div} \mathbf{A}(Du) = 0$ ,  $u|_{\partial O} = \psi$  with singular coefficients and singular data leads to the approximated net of solutions by the mean of previous theorem.

## 4. Hyperbolic PDEs

**4.1. Semilinear wave equation.** In our approach we connect two areas: the  $L^2$ -theory for the nonlinear wave equation

$$(29) \quad \partial_t^2 u - \Delta u + g(u) = 0, g(0) = 0, u = u(x, t), x \in \mathbb{R}^n, t \geq 0,$$

with the initial data

$$u(x, 0) = a(x), u_t(x, 0) = b(x), x \in \mathbb{R}^n,$$

involving energy estimates and the theory of generalized functions where nonlinear operations makes sense for a large collection of singular objects.

Concerning  $g$ , if it is not globally Lipschitz, then it is substituted by a net of globally Lipschitz functions  $g_\varepsilon(u)$ . Then the obtained net of equations, called regularized equation, is solved for each fixed  $\varepsilon$ .

In some cases  $g$  is not regularized and the growth conditions on  $g$  are involved for the existence and uniqueness of a solution similarly as in the classical theory.

We use here the algebra  $\mathcal{G}_{\infty, L^2}([0, T] \times \mathbb{R}^n)$ . Also we use the notation  $\mathcal{F} = \mathcal{E}_{L^2}([0, T] \times \mathbb{R}^n)$ . Consider a family of equations in  $\mathcal{E}_{\infty, L^2}([0, T] \times \mathbb{R}^n)$

$$(30) \quad (\partial_t^2 - \Delta)G_\varepsilon = -g(G_\varepsilon), G_\varepsilon|_{t=0} = A_\varepsilon, \partial_t G_\varepsilon|_{t=0} = B_\varepsilon, \varepsilon \in (0, 1),$$

where  $A_\varepsilon, B_\varepsilon \in \mathcal{E}_{\infty, L^2}(\mathbb{R}^n)$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  is smooth, polynomially bounded together with all its derivatives and  $g(0) = 0$ .

Equation (30), with the regularization  $g_\varepsilon$  instead of  $g$  is called the regularized equation for (29).

**Proposition 25.** a) Let  $n \leq 5$ . Then there exists a regularized net  $g_\varepsilon$  such that for every  $T > 0$  there exists a unique solution to (30) in  $\mathcal{G}_{\infty, L^2}([0, T] \times \mathbb{R}^n)$ .

b) Let  $n = 6$  and let  $\|A_\varepsilon\|_{H^{3,2}}$  and  $\|B_\varepsilon\|_{H^{2,2}}$  be bounded by  $(\log(\log(\varepsilon^{-1})))^s$ , as  $\varepsilon \rightarrow 0$ , where  $s < 1$ . Then there exists a regularized net  $g_\varepsilon$  such that for every  $T > 0$  there exists a unique solution to (30) in  $\mathcal{G}_{\infty, L^2}([0, T] \times \mathbb{R}^n)$ .

**Remark 26.** Let  $n = 7$ . In order to obtain the existence of a unique solution with the moderate growth of all its derivatives, we need that  $H^{3,2}$ -norms of initial data are bounded by  $\underbrace{\log(\log \dots (\log \varepsilon^{-1}) \dots)}_q$  with respect to  $\varepsilon$  for some  $s$  and  $q$ .

This follows from [80, Theorem 4.8]. Cases  $n = 8, 9$  can be handled out using the procedure and Lemmas 2.1–2.20 in the same paper as well as a composition of the logarithmic function sufficiently many times.

The proof of quoted theorem for  $n = 3$  implies the next corollary.

**Corollary 27.** *Let  $n = 3$ ,  $g(y)$  be globally Lipschitz and its first derivative be polynomially bounded. Then for every  $T > 0$  there exists a solution to (30) in  $\mathcal{G}_{\infty, L^2}([0, T] \times \mathbb{R}^n)$ .*

**Remark 28.** If  $g(y)$  is globally Lipschitz, for  $n = 4, 5, 6$ , we need to assume appropriate conditions for the first and second derivatives of  $g$ . If  $n = 7, 8, 9$ , then the assumptions of corollary are more complicated.

For a non-regularized wave equation, the following is true.

**Proposition 29.** *Equation*

$$(\partial_t^2 - \Delta)G = -G^3, \quad G|_{t=0} = A, \quad \partial_t G|_{t=0} = B,$$

where  $A, B \in \mathcal{G}_{\infty, L^2}(\mathbb{R}^3)$ , has a unique solution in  $\mathcal{G}_{\infty, L^2}([0, T] \times \mathbb{R}^3)$  for every  $T > 0$  if there exist representatives of initial data such that

$$\|(\nabla^2 A_\varepsilon, \nabla B_\varepsilon)\|_{L^2} = o((\log \varepsilon^{-1})^{1/2}).$$

The presented results are from [60]. One can look also in [10, 74] for some other results concerning the wave equation.

**4.2. Stochastic wave equations.** We will present the main results from the paper [69] concerning the different stochastic wave equations. Before that, we have to define Colombeau generalized processes like in [75].

Consider the problem

$$(31) \quad (\partial_t^2 - \partial_x^2)U + F(U) \cdot S = 0,$$

$$(32) \quad U|_{t=0} = A, \quad \partial_t U|_{t=0} = B,$$

where  $A$  and  $B$  are  $\mathcal{G}_{2,2}$ -Colombeau generalized stochastic processes on  $\mathbb{R}$ , that is,  $A, B \in \mathcal{G}_{2,2}^\Omega(\mathbb{R})$ , and  $S \in \mathcal{G}_{2,2}^\Omega([0, T] \times \mathbb{R})$  is  $\mathcal{G}_{2,2}$ -Colombeau generalized stochastic process on  $\mathbb{R}^2$  with compact support. We suppose that the function  $F$  is smooth, polynomially bounded together with all its derivatives and that  $F(0) = 0$ . We look for a solution  $U \in \mathcal{G}_{2,2}^\Omega([0, T] \times \mathbb{R})$ . We substitute  $F$  by a family of smooth functions  $F_\varepsilon$ ,  $\varepsilon \in (0, 1)$ , which is called the regularization of  $F$ . This is done in the following way. We choose the smooth function  $F_\varepsilon$  with the property that there exists a net  $a_\varepsilon$  such that for every  $\alpha \in \mathbb{N}_0$  there exist  $\varepsilon_0 \in (0, 1)$  and  $m^\alpha \in \mathbb{N}$  such that

$$F_\varepsilon(y) = F(y), \text{ for } |y| \leq a_\varepsilon, \quad \varepsilon < \varepsilon_0 \\ \|D^\alpha F_\varepsilon(y)\|_{L^\infty} = \mathcal{O}(a_\varepsilon^{m^\alpha}).$$

In the sequel we shall denote  $m = \sup_{|\alpha| \leq 1} m^\alpha$ .

Denote by  $\tilde{F} = [F_\varepsilon]$ , where  $F_\varepsilon \in \mathcal{E}_{2,2}^\Omega([0, T] \times \mathbb{R})$  has properties as above. Then, instead of non-regularized equation (31)–(32), we consider the regularized one

$$(33) \quad (\partial_t^2 - \partial_x^2)U + \tilde{F}(U) \cdot S = 0,$$

$$(34) \quad U|_{\{t=0\}} = A, \quad \partial_t U|_{\{t=0\}} = B,$$

where  $S = [S_\varepsilon] \in \mathcal{G}_{2,2}^\Omega([0, T] \times \mathbb{R})$  and  $A, B \in \mathcal{G}_{2,2}^\Omega(\mathbb{R})$ .

Note that for  $U_\varepsilon, V_\varepsilon \in \mathcal{E}_{2,2}^\Omega([0, T] \times \mathbb{R})$  such that  $U_\varepsilon - V_\varepsilon \in \mathcal{N}_{2,2}^\Omega([0, T] \times \mathbb{R})$ , we have that  $\tilde{F}(U_\varepsilon) - \tilde{F}(V_\varepsilon) \in \mathcal{N}_{2,2}^\Omega([0, T] \times \mathbb{R})$ .

First, one can see the connection of regularized and non-regularized one dimensional wave equation in the following theorem.

**Theorem 30.** *Let  $G$ , a primitive function of  $F$ , be nonnegative and  $G(0) = 0$ . Let Colombeau generalized stochastic process  $S \in \mathcal{G}_{2,2}^\Omega(\mathbb{R})$  be nonnegative and depend only on the variable  $x$ , i.e., there exists a representative  $S_\varepsilon$  of  $S$  such that  $S_\varepsilon(x) \geq 0$ , for all  $\varepsilon$  small enough and  $x \in \mathbb{R}$ . Suppose that  $\|(B_\varepsilon, \partial_x A_\varepsilon)\|_{L^2} = o(a_\varepsilon)$ , as  $\varepsilon \rightarrow 0$ , where  $a_\varepsilon$  is the corresponding net used in regularization of function  $F$ . Then, for every  $T > 0$ , the solution to the regularized equation (33)–(34) is also the solution to the non-regularized equation (31)–(32).*

**4.2.1. Cubic wave equation with nonnegative stochastic process.** We consider the problem

$$(35) \quad (\partial_t^2 - \Delta)U + U^3 \cdot S = 0,$$

$$(36) \quad U|_{\{t=0\}} = A, \quad \partial_t U|_{\{t=0\}} = B,$$

where we suppose that  $A, B \in \mathcal{G}_{2,2}^\Omega(\mathbb{R}^3)$  are  $\mathcal{G}_{2,2}$ -Colombeau generalized stochastic processes such that

$$(37) \quad \|(B_\varepsilon, \nabla A_\varepsilon)\|_{L^2} = o((\log \varepsilon^{-1})^{1/4}),$$

and  $S \in \mathcal{G}_g^\Omega(\mathbb{R}^3)$  is nonnegative  $\mathcal{G}_g$ -Colombeau generalized stochastic process which depends only on variable  $x$  and such that

$$(38) \quad \|S_\varepsilon\|_{L^\infty} = o((\log \varepsilon^{-1})^{1/2}).$$

**Theorem 31.** *Let stochastic processes  $A, B \in \mathcal{G}_{2,2}^\Omega(\mathbb{R}^3)$  have representatives which satisfy condition (37) and  $S \in \mathcal{G}_g^\Omega(\mathbb{R}^3)$  be nonnegative stochastic process that depends only on variable  $x$  and has a representative which satisfies (38). Then, for every  $T > 0$ , problem (35),(36) has a unique solution almost surely in  $\mathcal{G}_{2,2}^\Omega([0, T] \times \mathbb{R}^3)$ .*

**Remark 32.** If a Colombeau stochastic generalized process  $S$  is a image of a generalized stochastic process, then one can use a regularization which ensures estimate (38). This remark could be added after each further assertion when we need estimates on a stochastic term.

**4.2.2. Cubic wave equation with multiplicative stochastic process.** We consider the problem

$$(39) \quad (\partial_t^2 - \Delta)U + U \cdot S + U^3 = 0,$$

$$(40) \quad U|_{\{t=0\}} = A, \quad \partial_t U|_{\{t=0\}} = B,$$

where stochastic processes  $A, B \in \mathcal{G}_{2,2}^\Omega(\mathbb{R}^3)$  are such that

$$(41) \quad \|(B_\varepsilon, \nabla A_\varepsilon)\|_{L^2} = o((\log \varepsilon^{-1})^{1/2}),$$

and  $S \in \mathcal{G}_g^\Omega([0, T] \times \mathbb{R}^3)$  is such that

$$(42) \quad \|S_\varepsilon\|_{L^\infty} = o(\log(\log \varepsilon^{-1})^{1/2T}).$$

**Theorem 33.** *Let  $\mathcal{G}_{2,2}$ -Colombeau generalized stochastic processes  $A, B \in \mathcal{G}_{2,2}^\Omega(\mathbb{R}^3)$  satisfy condition (41) and  $\mathcal{G}_g$ -Colombeau stochastic process  $S \in \mathcal{G}_g^\Omega([0, T] \times \mathbb{R}^3)$  satisfy (42). Then, for every  $T > 0$ , problem (39)–(40) has a unique solution almost surely in  $\mathcal{G}_{2,2}^\Omega([0, T] \times \mathbb{R}^3)$ .*

#### 4.2.3. Cubic Klein–Gordon equation with additive stochastic process.

We consider the problem

$$(43) \quad (\partial_t^2 - \Delta)U + U + U^3 + S = 0,$$

$$(44) \quad U|_{\{t=0\}} = A, \quad \partial_t U|_{\{t=0\}} = B,$$

where stochastic processes  $A, B \in \mathcal{G}_{2,2}^\Omega(\mathbb{R}^3)$  satisfy

$$(45) \quad \|(B_\varepsilon, \nabla A_\varepsilon)\|_{L^2} = o((\log \varepsilon^{-1})^{1/2}),$$

and  $S \in \mathcal{G}_{2,2}^\Omega([0, T] \times \mathbb{R}^3)$  is such that

$$(46) \quad \|S_\varepsilon\|_{L^\infty} = o((\log \varepsilon^{-1})^{1/2})$$

$$(47) \quad S_\varepsilon \text{ has a compact support.}$$

**Theorem 34.** *Let  $\mathcal{G}_{2,2}$ -Colombeau generalized stochastic processes  $A, B \in \mathcal{G}_{2,2}^\Omega(\mathbb{R}^3)$  and  $S \in \mathcal{G}_{2,2}^\Omega([0, T] \times \mathbb{R}^3)$  satisfy conditions (45) and (46)–(47), respectively. Then, for  $T > 0$ , the problem (43)–(44) has a unique solution almost surely in  $\mathcal{G}_{2,2}^\Omega([0, T] \times \mathbb{R}^3)$ .*

The literature concerning the stochastic wave equation and generalized processes is quite rich. One can look in papers [1] or [75], for example.

## 5. Semilinear parabolic PDE

Two types of equations in generalized functions algebra,  $\mathcal{G}_{C^1, H^2}([0, T] : \mathbb{R}^n)$  (given below). The first one is a Cauchy problem

$$(\partial_t - \Delta)U + VU = 0, \quad U(0, x) = U_0(x),$$

where potential  $V$  is a singular distribution, for example the delta distribution or a linear combination of its derivatives. It will be presented here.

The second type is a nonlinear Cauchy problem

$$(\partial_t - \Delta)U + VU = f(t, U), \quad U(0, x) = U_0(x),$$

where  $f$  satisfies certain conditions.

In both types of equations  $U_0$  is an element of Colombeau-type space,  $\mathcal{G}_{H^2}(\mathbb{R}^n)$ . This involves singular data, embedded singular distributions, for example of the form  $U_0 = \sum_{i=0}^2 f_i^{(i)}$ ,  $f_i \in L^2$ ,  $i = 0, 1, 2$ , again the important standpoint of our approach.

We will present the use of generalized  $C_0$ -semigroups in solving a class of heat equations with singular potentials and singular data. First note that the multiplication of elements  $G \in \mathcal{G}_{H^2, \infty}(\mathbb{R}^n)$  and  $H \in \mathcal{G}_{C^1, H^2}([0, T] : \mathbb{R}^n)$  gives an element in  $\mathcal{G}_{C^1, H^2}([0, T] : \mathbb{R}^n)$ . Indeed, if  $G_\varepsilon \in \mathcal{E}_{H^2, \infty}(\mathbb{R}^n)$  and  $H_\varepsilon \in \mathcal{E}_{C^1, H^2}([0, T] : \mathbb{R}^n)$  then  $G_\varepsilon H_\varepsilon \in \mathcal{E}_{C^1, H^2}([0, T] : \mathbb{R}^n)$ .

Similarly, if  $G_\varepsilon \in \mathcal{N}_{H^2, \infty}(\mathbb{R}^n)$  or  $H_\varepsilon \in \mathcal{N}_{C^1, H^2}([0, T] : \mathbb{R}^n)$ , then  $G_\varepsilon H_\varepsilon \in \mathcal{N}_{C^1, H^2}([0, T] : \mathbb{R}^n)$ . Thus, multiplication of potential  $V \in \mathcal{G}_{H^2, \infty}(\mathbb{R}^n)$  and a function  $U \in \mathcal{G}_{C^1, H^2}([0, T] : \mathbb{R}^n)$  which is expected to be a solution to equation

$$\partial_t U = (\Delta - V)U, \quad U(0, x) = U_0(x),$$

makes sense.

**Definition 35.** Let  $A$  be represented by a net  $A_\varepsilon, \varepsilon \in (0, 1)$ , of linear operators with the common domain  $H^2(\mathbb{R}^n)$  and with ranges in  $L^2(\mathbb{R}^n)$ . A generalized function  $G \in \mathcal{G}_{C^1, H^2}([0, T] : \mathbb{R}^n)$ ,  $T > 0$ , is said to be a solution to equation  $\partial_t G = AG$  if

$$\sup_{t \in [0, T]} \|\partial_t G_\varepsilon(t, \cdot) - A_\varepsilon G_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^a), \quad \text{for every } a \in \mathbb{R}.$$

**5.0.4. General potential.** We will consider in this subsection singular potentials, elements of  $\mathcal{G}_{H^2, \infty}(\mathbb{R}^n)$ . Especially when the potential is a power of the delta generalized function.

**Theorem 36.** Let  $V \in \mathcal{G}_{H^2, \infty}(\mathbb{R}^n)$  be of logarithmic type,  $U_0 = [U_{0\varepsilon}] \in \mathcal{G}_{H^2}(\mathbb{R}^n)$  and  $[S_\varepsilon]$  be defined as in Example 12. Let  $T > 0$ . Then  $U = SU_0 \in \mathcal{G}_{C^1, H^2}([0, T] : \mathbb{R}^n)$  ( $U_\varepsilon(t, x) = S_\varepsilon(t)U_{0\varepsilon}(x)$ ,  $\varepsilon < 1$ ) is the unique solution to equation

$$(48) \quad \partial_t U(t, x) - \Delta U(t, x) + V(x)U(t, x) = 0, \quad U(0, x) = U_0(x).$$

in the sense of Definition 35.

Note that in our construction of a solution to (48) the perturbations with elements in  $\mathcal{N}_{H^2, \infty}$  null-nets do not effect the solution. More precisely, if  $V_\varepsilon$  is substituted by  $V_\varepsilon + R_\varepsilon$ ,  $R_\varepsilon \in \mathcal{N}_{H^2, \infty}$ , in (48), we have the same generalized solution.

**5.0.5. Powers of the generalized delta function as a potential.** Let  $\phi_\varepsilon$  be a net of mollifiers

$$(49) \quad \phi_\varepsilon = \varepsilon^{-n} \phi(\cdot/\varepsilon), \quad \varepsilon \in (0, 1),$$

where  $\phi \in C_0^\infty(\mathbb{R}^n)$ ,  $\int \phi(x) dx = 1$  and  $\phi(x) \geq 0$ ,  $x \in \mathbb{R}^n$ . It represents the generalized delta function  $\delta = [\phi_\varepsilon] \in \mathcal{G}(\mathbb{R}^n)$ .

Different  $\phi_\varepsilon$ 's (with the prescribed properties on  $\phi$ ) define different infinitesimal generators. Let us show this. Put  $A_\varepsilon = \Delta - \phi_\varepsilon$  and  $\tilde{A}_\varepsilon = \Delta - \tilde{\phi}_\varepsilon$ ,  $\varepsilon < 1$ . The equality of infinitesimal generators would imply that

$$\|(A_\varepsilon - \tilde{A}_\varepsilon)u\|_{L^2}^2 = \varepsilon^{-2n} \int_{\mathbb{R}^n} |\phi(y) - \tilde{\phi}(y)|^2 |u(\varepsilon y)|^2 dt \leq C_a \varepsilon^a \|u\|_{L^2}^2, \quad \varepsilon < 1$$

for every  $a > 0$  (and corresponding  $C_a > 0$ ). Thus, it follows that  $\phi = \tilde{\phi}$ .

Let  $m \in \mathbb{N}$ . We will use the  $\delta^m = [\phi_\varepsilon^m]_{m \in \mathbb{N}}$  as the definition of  $m$ -th power of  $\delta \in \mathcal{G}(\mathbb{R}^n)$ . Let  $A_{\varepsilon,m}u = (\Delta - \phi_\varepsilon^m)u$ ,  $u \in H^2(\mathbb{R}^n)$ ,  $\varepsilon < 1$ .  $A_{\varepsilon,m}$  is the infinitesimal generator of the semigroup

$$S_{\varepsilon,m} : [0, \infty) \rightarrow \mathcal{L}(L^2(\mathbb{R}^n)), \quad S_\varepsilon(t) = \exp((\Delta - \phi_\varepsilon^m)t), \quad t \geq 0 \text{ (cf. [79])}.$$

It follows that  $S_{\varepsilon,m}$  is a representative of a generalized  $C_0$ -semigroup  $S \in \mathcal{LG}([0, \infty) : \mathcal{L}(L^2(\mathbb{R}^n)))$ .

We know that  $S_{\varepsilon,m}\psi$ ,  $\varepsilon < 1$  and  $\psi \in L^2(\mathbb{R}^n)$ , given by

$$S_\varepsilon(t)\psi(x) = \int_O \exp\left(-\int_0^t \phi_\varepsilon^m(\omega(s))ds\right) \psi(\omega(t))d\mu_x(\omega), \quad x \in \mathbb{R}^n, \quad t \geq 0.$$

Since  $\phi_\varepsilon(x) \geq 0$ ,  $x \in \mathbb{R}^n$ ,  $\varepsilon < 1$ , it follows that the set  $\{S_{\varepsilon,m} : \varepsilon \in (0, 1), t \geq 0\}$  is bounded in  $\mathcal{L}(L^2(\mathbb{R}^n))$  (not only moderate). Thus (31) holds for  $S_{\varepsilon,m}$ .

Our goal is to prove the following theorem, where the assumption  $n \geq 2$  is crucial.

**Theorem 37.** *Let  $n \geq 2$ ,  $m \in \mathbb{N}$ ,  $T > 0$  and  $U_0 \in H^2(\mathbb{R}^n)$ . Then*

$$(50) \quad U_{\varepsilon,m}(t, x) = \int_O \exp\left(-\int_0^t \varepsilon^{-mn} \phi^m\left(\frac{\omega(s)}{\varepsilon}\right) ds\right) U_0(\omega(t)) d\mu_x(\omega),$$

$$x \in \mathbb{R}^n, \quad t \geq 0, \quad \varepsilon < 1$$

defines a representative of a solution  $U \in \mathcal{G}_{C^1, H^2}([0, T] : \mathbb{R}^n)$  to the equation

$$\partial_t U(t, x) - \Delta U(t, x) + \delta^m(x)U(t, x) = 0, \quad U(0, x) = U_0(x).$$

The solution is unique in the sense of Definition 35.

Moreover net (50) converges to  $\tilde{U}(t, \cdot) = e^{-\Delta t}U_0(\cdot)$  uniformly on compact sets of  $\mathbb{R}^n$  of  $H^2(\mathbb{R}^n)$ , for every  $t \geq 0$ .

We need several notions and properties of  $n$ -dimensional Brownian motions,  $n \geq 2$ . Recall that the hitting time  $\tau_A$  of a subset  $A$  of  $\mathbb{R}^n$  is defined by  $\tau_A = \inf\{t > 0 : \omega(t) \in A\}$  ( $\tau_A = \infty$  if  $\omega(t) \notin A$  for all  $t > 0$ ). We refer to [85, Ch. 1 Sec. 2], for the elementary properties of hitting times. Recall, a Borel set  $A$  is said to be polar if  $\mu_x(\{\omega \in O : \omega(t) \in A \text{ for some } t < \infty\}) = 0$ . We will use the fact that every one-point set is polar for  $n \geq 2$ . This is not true for  $n = 1$  and that is the essential reason for different results in the cases  $n \geq 2$  and  $n = 1$ .

Let  $B_\varepsilon = \{x \in \mathbb{R}^n : \|x\| \leq \varepsilon\}$ ,  $B = B_1$ . Take  $\varepsilon \in (0, 1)$ ,  $t > 0$  and define

$$W_{B_\varepsilon}(t) = \{\tau_{B_\varepsilon} < t\} = \{\omega : \text{there exists } 0 < s < t, \omega(s) \in B_\varepsilon\},$$

$$W_{B_\varepsilon} = \bigcup_{t>0} W_{B_\varepsilon}(t).$$

Clearly,  $W_{B_\varepsilon}(t) \subset W_{B_\varepsilon}(s)$ ,  $0 < t \leq s$ . Note that

$$W_{B_\varepsilon}(s) \setminus W_{B_\varepsilon}(t) = \{t \leq \tau_{B_\varepsilon} < s\}, \quad 0 < t < s$$

and

$$(51) \quad W_{B_\varepsilon}(s) \setminus W_{B_\varepsilon}(t) \subset W_{B_\varepsilon} \setminus W_{B_\varepsilon}(t) \subset \{t - 1 < \tau_{B_\varepsilon}\},$$

for  $s > t > 1$ .



Choose an increasing sequence  $(t_m)_m$  such that  $t_{m+1} > t_m + 1$ , and (51) holds for every  $m \in \mathbb{N}$ .

**Lemma 38.** 1) For every compact subset  $K$  of  $\mathbb{R}^n$  and  $\varepsilon < 1$ , there exists a constant  $C_\varepsilon > 0$  such that  $\mu_x(W_{B_\varepsilon}) \leq C_\varepsilon$ . 2)  $\limsup_{\varepsilon \rightarrow 0} \sup_{x \in K} \mu_x(W_{B_\varepsilon}) = 0$ .

Powers of the generalized delta function,  $\delta^\alpha$ ,  $\alpha \in (0, 1)$ , are defined in this paper by

$$(52) \quad \delta^\alpha = [(\phi_\varepsilon)^\alpha * \phi_\varepsilon], \quad \varepsilon \in (0, 1).$$

The reason for introducing (52) is simple: When  $\alpha \in (0, 1)$ , the function  $\phi_\varepsilon^\alpha$ ,  $\varepsilon < 1$  is not smooth, in general. Note, generalized function  $[\phi_\varepsilon * \phi_\varepsilon]$  is only associated with the generalized delta function  $\delta = [\phi_\varepsilon]$ .

Since one-point sets are not polar for  $n = 1$ , we could not use the same arguments as in the case  $n \geq 2$ . Note that functions in  $H^2(\mathbb{R})$  are continuous and bounded.

**Proposition 39.** Let  $\alpha \in (0, 1)$ ,  $T > 0$  and  $U_0 \in H^2(\mathbb{R})$ . Then by

$$(53) \quad U_\varepsilon(t, x) = \int_O \exp\left(-\int_0^t (\phi_\varepsilon)^\alpha * \phi_\varepsilon(\omega(s)) ds\right) U_0(\omega(t)) d\mu_x(\omega)$$

$$t > 0, \quad x \in \mathbb{R}, \quad \varepsilon < 1,$$

is defined a representative of a solution  $U(t, x) \in \mathcal{G}_{C^1, H^2}([0, T] \times \mathbb{R})$  to equation

$$\partial_t U(t, x) - \Delta U(t, x) + \delta^\alpha(x)U(t, x) = 0, \quad U(0, x) = U_0(x).$$

The solution is unique in the sense of Definition 35.

Net (53) has a subsequence  $(U_{\varepsilon_\nu, \alpha}(t, x))_{\nu \in \mathbb{N}}$ , converging to  $\tilde{U}(t, x) = e^{-\Delta t}U_0(x)$ ,  $t \geq 0$ ,  $x \in \mathbb{R}$  in the weak topology of  $L^2([0, T] \times \mathbb{R})$ .

**Example 40.** Assume  $n \geq 2$ ,  $T > 0$ ,  $V \in H^{1, \infty}(\mathbb{R}^n)$ , and  $f \in C^1([0, \infty) \times \mathbb{R}^n)$  satisfies  $f(s, 0) = 0$ ,  $s \in \mathbb{R}$  and  $|f(s, y_1) - f(s, y_2)| \leq C|y_1 - y_2|$ .

Let  $U_0(x) = \delta(x)$ ,  $x \in \mathbb{R}^n$ , i.e.,  $U_{0\varepsilon} = \phi_\varepsilon$ ,  $\varepsilon < 1$  (cf. (49)). Then for fixed  $\varepsilon < 1$ ,

$$\partial_t U_\varepsilon(t, x) = (\Delta_x - V(x))U_\varepsilon(t, x) + f(t, U_\varepsilon(t, x)), \quad U_\varepsilon(0, x) = \phi_\varepsilon,$$

has a unique classical solution  $U_\varepsilon$  in  $C^0([0, T], L^1(\mathbb{R}^n)) \cap C^1((0, T), L^1(\mathbb{R}^n))$  and  $U_\varepsilon(t, x) \in H^{2, 1}(\mathbb{R}^n)$  for every  $t > 0$ . Again we have  $U_\varepsilon(t, x) \in C^0((0, T) : H^2(\mathbb{R}^n))$ ,  $\varepsilon < 1$ . We will show that there exists a sequence  $(U_{\varepsilon_\nu})_{\nu \in \mathbb{N}}$  converging to some  $U \in L^q_{\text{loc}}((0, T), \mathbb{R}^n)$ ,  $1 \leq q < n/(n-1)$ , in  $L^q_{\text{loc}}((0, T), \mathbb{R}^n)$  such that  $\partial_t U = (\Delta - V)U$  in  $\mathcal{D}'((0, T), \mathbb{R}^n)$ .

**Remark 41.** The classical theory of semigroups is used here as a tool for finding generalized solutions to a nonlinear heat equations. One can find different approach in [12], [33] or [100].

## PART III: HYPERBOLIC SYSTEMS

### 6. Semilinear hyperbolic systems

Let

$$(54) \quad (\partial_t + \Lambda(x, t)\partial_x)y(x, t) = F(x, t, y(x, t)), \quad y(x, 0) = A(x)$$

be a semilinear hyperbolic system, where  $\Lambda$  is a real diagonal matrix and a mapping  $y \mapsto F(x, t, y)$  is in  $\mathcal{O}_M(\mathbb{C}^n)$  with uniform bounds for  $(x, t) \in K \subset \subset \mathbb{R}^2$ . In [72] a generalized solution to (54) is constructed when  $A$  is an arbitrary generalized function and  $F$  has a bounded gradient with respect to  $y$  for  $(x, t) \in K \subset \subset \mathbb{R}^2$ .

Here,  $F$  is substituted by  $F_{h(\varepsilon)}$  which has a bounded gradient with respect to  $y$  for every fixed  $\varepsilon$  and converges pointwisely to  $F$  as  $\varepsilon \rightarrow 0$ . Our aim is to find a generalized solution to

$$(55) \quad (\partial_t + \Lambda(x, t)\partial_x)Y(x, t) = F_{h(\varepsilon)}(x, t, Y(x, t)), \quad Y(x, 0) = A(x).$$

We fix a decreasing function  $h : (0, 1) \rightarrow (0, \infty)$  such that  $h(\varepsilon) = \mathcal{O}((\log \varepsilon^{-1})^{1/2})$ ,  $h(\varepsilon) \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ .

Denote by  $B_r$  the cube  $|x| \leq r$ ,  $|t| \leq r$ ,  $|y| \leq r$ , where  $y = (u_1, v_1, \dots, u_n, v_n)$ . Let  $\varepsilon_i$  be a decreasing sequence of positive numbers such that  $h(\varepsilon_{i+1}) = i$ ,  $i \in \mathbb{N}$ . This implies that  $h(\varepsilon) \geq i - 1$  if  $\varepsilon < \varepsilon_i$ . Let

$$S_i = B_i \cap \{(x, t, u, v), |F(x, t, u, v)| \leq i - 1\} \\ \cap \{(x, t, u, v), |\nabla_{u,v} F(x, t, u, v)| \leq i - 1\}, \quad i \in \mathbb{N}.$$

Let  $\kappa_i$  be the characteristic function of  $S_i$ ,  $i \in \mathbb{N}$ . Put

$$\kappa_{h(\varepsilon)} = (\kappa_i * \phi_{1/h(\varepsilon)}), \quad \varepsilon \in [\varepsilon_{i+1}, \varepsilon_i], \quad i \in \mathbb{N}, \\ F_{h(\varepsilon)}^k = F^k \kappa_{h(\varepsilon)}, \quad \varepsilon \in (0, \varepsilon_1), \quad k \in \{1, \dots, n\}.$$

Then there exists a constant  $C = C(C_0) > 0$  and  $\varepsilon_1 > 0$  such that

$$\|F_{h(\varepsilon)}\|_{L^\infty(\mathbb{R}^{2+2n})} \leq Ch(\varepsilon) \\ \|\nabla_{u,v} F_{h(\varepsilon)}\|_{L^\infty(\mathbb{R}^{2+2n})} \leq Ch(\varepsilon)^2, \quad \varepsilon \in (0, \varepsilon_1).$$

**Definition 42.**  $G = (G_1, \dots, G_n) \in (\mathcal{G}(\mathbb{R}^2))^n$  is a solution to (55) if any of its representative  $G_\varepsilon$  satisfies the system

$$(56) \quad (\partial_t + \Lambda(x, t)\partial_x)G_\varepsilon(x, t) = F_{h(\varepsilon)}(x, t, G_\varepsilon(x, t)) + d_{1,\varepsilon}(x, t), \\ G_\varepsilon(x, 0) = A_\varepsilon(x) + d_{2,\varepsilon}(x),$$

where  $A_\varepsilon \in (\mathcal{E}_M(\mathbb{R}))^n$  is a representative for some  $d_{2,\varepsilon} \in (\mathcal{N}(\mathbb{R}))^n$ , and  $d_{1,\varepsilon} \in (\mathcal{N}(\mathbb{R}^2))^n$ . We call (55) and (56) the h-regularized system.

**Theorem 43.** *Assume that every component of the mapping  $y \mapsto F(x, t, y)$  belongs to  $\mathcal{O}_M(\mathbb{C}^n)$  and has uniform bounds for  $(x, t) \in K \Subset \mathbb{R}^2$ . Then the h-regularized system (55) has a unique solution in  $(\mathcal{G}(\mathbb{R}^2))^n$  whenever the initial data is in  $(\mathcal{G}(\mathbb{R}))^n$ .*

The proof follows by using the method of characteristics and the fixed point theorem.

**Theorem 44.** *Let the initial data  $A$  in (54) belong to  $(C(\mathbb{R}))^n$ .*

(a) *The solution  $G_h$  to the regularized system (55) is  $L^\infty$ -associated with the continuous local solution  $g$  to (54) in  $K_{T_0}$ , for some  $T_0 > 0$ .*

(b) *Assume that (54) is globally well posed. Then the solution  $G_h$  to (55) is  $L^\infty$ -associated with the continuous solution  $g$  to (54) on each  $K_T$ .*

**Remark 45.** If for every compact set  $K \subset \mathbb{R}^2$  there exists  $C > 0$  such that

$$\sup_{(x,t) \in K, y \in \mathbb{C}^n} |F(x, t, y)| \leq C \quad \text{or} \quad \sup_{(x,t) \in K, y \in \mathbb{C}^n} |\nabla_y F(x, t, y)| \leq C,$$

then system (54) is globally well posed.

The presented assertions are from [65]. One can look also in [71] or [2]. Multi-dimensional case is done in [52], initial-boundary problem is a topic of paper [43], while some nice results in special cases can be found in papers [14] and [73].

## 7. Systems of conservation laws

**7.1. Introduction.** For an  $n \times n$  hyperbolic system ( $n$  real eigenvalues) in one space dimension

$$(57) \quad U_t + f(U)_x = 0, \quad U : O \subset \mathbb{R}^2 \rightarrow \mathbb{R}^n, \quad f : \mathbb{R}^n \rightarrow \mathbb{R}$$

with Riemann initial data

$$(58) \quad U|_{t=0} = \begin{cases} U_0, & x < 0, \\ U_1, & x > 0, \end{cases} \quad U_0, U_1 \text{ are real vectors,}$$

there exists a unique entropy solution, provided  $\|U_1 - U_0\|_{L^\infty}$  small enough (Lax in 50's (see [53], for example)). The classical solution to the above Riemann problem consists of shock, rarefaction waves and contact discontinuities. (If  $n$  real eigenvalues are all different, then system (57) is called strictly hyperbolic.) Also, methods for solving an arbitrary Cauchy problem (Glimm scheme, wave front tracking algorithm (Di Perna for  $2 \times 2$  system (70's), Bressan et al (90's) for  $n \times n$  system), see [8] are based on the fact that the total variation of the initial data is small enough.

So, the first reason for introducing solutions containing Dirac  $\delta$  distribution is a possible managing of the system with "large" initial total variation. The second reason for doing this is that some systems of conservation laws, perturbed by a "viscosity" matrix,  $U_t + f(U)_x = \varepsilon A(U)U_{xx}$  have solutions which limit contains

terms with  $\delta$  distribution, as  $\varepsilon \rightarrow 0$  ([72] and [41]). The third reason is that if one perturb Riemann data by smooth functions

$$U|_{t=0} = U_{0,\varepsilon} \rightarrow \begin{cases} U_0, & x < 0 \\ U_1, & x > 0 \end{cases}, \text{ as } \varepsilon \rightarrow 0,$$

local smooth solution has not only gradient catastrophe (the case of shock waves), but also  $L^\infty$  catastrophe (the  $L^\infty$  norm of the smooth solution goes to infinity in a finite time).

The aim of this section is threefold:

1. We shall give two solution concepts where  $\delta$  function appears (delta and singular shock waves), see [57], [58] or [61].

2. Describe when it is possible to find such solutions (delta and delta singular locus, see [58] for these notions).

3. Give some results of interaction of delta and singular shock waves with other types of elementary waves in some special cases (see [59] or [61]).

At the end, we shall present some of numerous open problems concerning the above topics.

**7.2. Some examples.** Dirac  $\delta$  distribution, as a part of a viscosity limit for solutions to some systems of conservation laws was numerically observed in [51]. In papers [72] and [41] is proved that the viscosity limit of some Riemann data for the system

$$\begin{aligned} u_t + (u^2/2)_x &= 0 \\ v_t + (uv)_x &= 0 \end{aligned}$$

contain  $\delta$  distribution. In [42], the arbitrary Riemann problem for the system

$$(59) \quad \begin{aligned} u_t + (u^2 - v)_x &= 0 \\ v_t + (u^3/3 - u)_x &= 0 \end{aligned}$$

which is a modified model of spreading ion acoustic waves is solved. For some initial data ( $u_1$  is in the area  $Q_7$  at Figure 1), the solution (in approximating sense) is given by

$$\begin{aligned} u_\varepsilon(x, t) &= G_\varepsilon(x - ct) + a\sqrt{\frac{t}{\varepsilon}}\rho\left(x - c\frac{t}{\varepsilon}\right), \\ v_\varepsilon(x, t) &= H_\varepsilon(x - ct) + a^2\frac{t}{\varepsilon}\rho^2\left(x - c\frac{t}{\varepsilon}\right), \end{aligned}$$

where  $G_\varepsilon$  and  $H_\varepsilon$  converge to appropriate step functions defined by the Riemann initial data,  $\rho_\varepsilon^2(\cdot) := \varepsilon^{-1}\rho^2(\cdot/\varepsilon)$ , where  $\rho \in C_0^\infty$ ,  $\int \rho^2 = 1$ , converges to the delta distribution and  $\rho_\varepsilon^i$  converges to zero in  $\mathcal{D}'$  as  $\varepsilon \rightarrow 0$ ,  $i = 1, 3$ .

Pressureless gas dynamics model

$$(60) \quad \begin{aligned} u_t + (uv)_x &= 0 \\ (uv)_t + (uv^2)_x &= 0 \end{aligned}$$

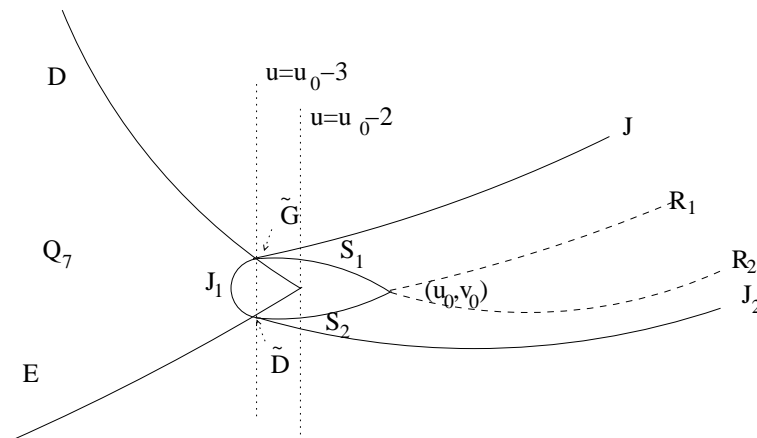


FIGURE 1. Singular shock wave area

can be transformed (after the elimination of the variable  $u_t$  from the second equation) into

$$\begin{aligned} u_t + (u^2)_x &= 0 \\ v_t + (uv)_x &= 0 \end{aligned}$$

for which the Riemann problem is solved in [94]. Some of the solutions contains  $\delta$  distribution as a term. The authors used Dafermos–Di Perna viscosity limit method (viscous term is given by  $\varepsilon t u_{xx}$ , so the viscous approximation allows self-similar solutions), and the results are justified by assuming that  $u \in L^\infty$ ,  $v$  is a Borel measure, and  $u$  has an appropriate value at line of discontinuity.

The same ideas are used in [99] for the systems of the form

$$\begin{aligned} u_t + (f(u)v)_x &= 0 \\ (uv)_t + (f(u)v^2)_x &= 0. \end{aligned}$$

In [40], a measure theoretic solution to the above system is given.

In [32], a solution to an arbitrary Riemann problem for the system

$$(61) \quad \begin{aligned} u_t + (u^2/2)_x &= 0 \\ v_t + ((u-1)v)_x &= 0 \end{aligned}$$

which is a very simplified MHD-model, by using Vol’pert idea of multiplication of functions with bounded total variation and distributions. In some cases, the solution contains a term with  $\delta$  distribution.

Finally, we shall mention the paper [22], where the author found viscosity limits to the Riemann problem

$$\begin{aligned} u_t + (f(u))_x &= 0 \\ (uv)_t + (g(u)v)_x &= 0 \end{aligned}$$

(under mild assumptions on functions  $f$  and  $g$ ). Again, some limits contain  $\delta$  distribution as a term.

**7.3. Solution concepts.** In the sequel we shall restrict ourselves to  $2 \times 2$  systems, i.e., system (57) we shall write in the following form

$$(62) \quad \begin{aligned} u_t + (f_1(u)v + f_2(u))_x &= 0 \\ v_t + (g_1(u)v + g_2(u))_x &= 0, \end{aligned}$$

where  $f_1, f_2, g_1, g_2$  are smooth functions, polynomially bounded with all its derivatives. One can see that the above system can be substituted by more general one

$$\begin{aligned} (\tilde{f}_1(u)v + \tilde{f}_2(u))_t + (f_1(u)v + f_2(u))_x &= 0 \\ (\tilde{g}_1(u)v + \tilde{g}_2(u))_t + (g_1(u)v + g_2(u))_x &= 0, \end{aligned}$$

without any major change in statements and concepts given here.

As it was written in the introduction, we shall present two solution concepts which are suitable for multiplication of distributions (in fact,  $\delta$  distribution with a discontinuous function). The first one is based on the Colombeau generalized function space introduced in [77]. The second one is based on splitting  $\delta$  distribution into two parts, which are divided by a discontinuity line.

**7.3.1. First solution concept.** We shall use Colombeau space  $\mathcal{G}_g$  in this section. Let us start with a simple lemma often used in the rest of it.

**Lemma 46.** *The generalized function defined by the representative  $\phi_\varepsilon(x - ct) \in \mathcal{E}_{M,g}(\mathbb{R}_+^2)$ ,  $\phi \in \mathcal{A}_0$ ,  $c \in \mathbb{R}$ , is associated with  $\delta(x - ct) \in \mathcal{D}'(\mathbb{R}_+^2)$ .*

*Proof.* Let  $\psi \in C_0^\infty(\mathbb{R}_+^\infty)$  and

$$I_\varepsilon := \iint \varepsilon^{-1} \phi((x - ct)/\varepsilon) \psi(x, t) \, dx \, dt.$$

Changing the variables  $(x - ct)/\varepsilon \mapsto y$ ,  $t \mapsto s$ , using the Lebesgue dominated convergence theorem and the properties of the functions from  $\mathcal{A}_0$  gives

$$\begin{aligned} I_\varepsilon &= \iint \phi(y) \psi(\varepsilon y + cs, s) \, dy \, ds \\ &\rightarrow \int \left( \int \phi(y) \, dy \right) \psi(cs, s) \, ds = \int \psi(cs, s) \, ds, \text{ as } \varepsilon \rightarrow 0. \quad \square \end{aligned}$$

The step functions, mapped by  $\iota$  into  $\mathcal{G}_g(\mathbb{R})$ , belong to the following important class of generalized functions.  $G \in \mathcal{G}_g(O)$  is said to be of a bounded type if

$$\sup_{x \in O} |G_\varepsilon(x)| = \mathcal{O}(1) \text{ as } \varepsilon \rightarrow 0,$$

for every  $T > 0$ .

**Definition 47.** (a)  $G \in \mathcal{G}(\mathbb{R})$  is said to be a generalized step function with value  $(y_0, y_1)$  if it is of bounded type and

$$G_\varepsilon(y) = \begin{cases} y_0, & y < -\varepsilon \\ y_1, & y > \varepsilon \end{cases}$$

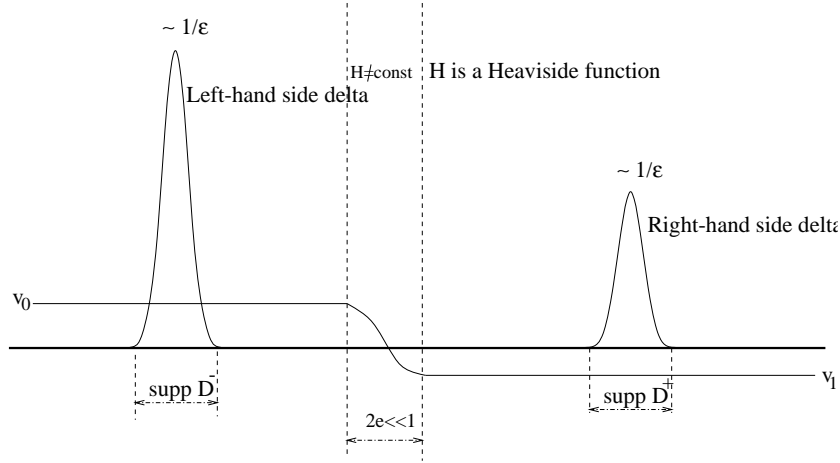


FIGURE 2. Delta shock wave

Denote  $[G] := y_1 - y_0$ .

(b)  $D \in \mathcal{G}_g(\mathbb{R})$  is said to be generalized slitted delta function (S $\delta$ -function for short) with value  $(\alpha_0, \alpha_1)$  if  $D = \alpha_0 D^- + \alpha_1 D^+$ , where  $\alpha_0 + \alpha_1 = 1$  and  $D^\pm \in \mathcal{G}_g(\mathbb{R})$  are associated to delta distribution and  $D^- G \approx y_0 \delta$  and  $D^+ G \approx y_1 \delta$ , for any generalized step function  $G$  with value  $(y_0, y_1)$ .

**Remark 48.** One can give fixed representatives for a generalized split delta function in the following way

$$D_\varepsilon^\pm(y) := \frac{1}{\varepsilon} \phi\left(\frac{y - (\pm 2\varepsilon)}{\varepsilon}\right), \phi \in \mathcal{A}_0.$$

Note that  $D_\varepsilon^\pm$  are in fact shifted model delta nets.

**Lemma 49.** *If  $G$  is a generalized step function with value  $(y_0, y_1)$  and  $D$  is an S $\delta$ -function with value  $(\alpha_0, \alpha_1)$ , then the following hold.*

(i)  $f(G)$  is a generalized step function with value  $(f(y_0), f(y_1))$ , where  $f$  is a smooth function.

(ii)  $G \cdot D \approx (y_0 \alpha_0 + y_1 \alpha_1) \delta$ .

*Proof.* The proof is a straightforward consequence of the definitions. □

**Remark 50.** The support property of S $\delta$ -function ensures the uniqueness in the association sense of its product with a generalized step function. This was done in order to deal with conservation law systems given in a general form. Of course that some other choices can be more efficient in specific cases (see [11] and literature there, for example).

The generalized initial data for our system are now generalized step functions  $G$  and  $H$  with values  $(u_0, u_1)$  and  $(v_0, v_1)$ , respectively. One can see that the inclusion

by  $\iota_\phi$  of a classical step function gives a generalized step function in the sense of Definition 47 (a) for every  $\phi \in \mathcal{A}_0$ .

**Definition 51.**  $(U, V) \in (\mathcal{G}(\mathbb{R}_+^2))^2$  is called delta shock wave solution to Riemann problem (62, 58) if (a) and (b) hold:

$$(a) \quad \begin{aligned} U_t + (f_1(U)V + f_2(U, V))_x &\approx 0 \\ V_t + (g_1(U)V + g_2(U, V))_x &\approx 0. \\ U|_{t=0} &= G, V|_{t=0} = H. \end{aligned}$$

$$(b) \quad U(x, t) = G(x - ct), \text{ and } V(x, t) = H(x - ct) + s(t)(\beta_0 D^-(x - ct) + \beta_1 D^+(x - ct)).$$

Here,  $G$  and  $H$  are generalized step functions,  $f_i, g_i, i = 1, 2$ , are smooth functions polynomially bounded together with all their derivatives,  $f_2$  and  $g_2$  are also sublinearly bounded with respect to  $V$ ,  $c \in \mathbb{R}$  is a speed of the shock wave,  $s \in C^1([0, \infty))$ ,  $s(0) = 0$ ,  $\beta_0 + \beta_1 = 1$ , and  $D = \beta_0 D^-(x - ct) + \beta_1 D^+(x - ct)$  is an  $S\delta$ -function.

The function  $s(t)\beta_0$  is called the left-handed strength of the wave, and  $s(t)\beta_1$  is called the right-handed strength of the wave. Its sum  $(s(t))$  is called the strength of delta or singular shock wave.

**Definition 52.** A generalized function  $d \in \mathcal{G}_g(\mathbb{R})$  is said to be  $m$ -singular delta function ( $m$ SD-function for short) with value  $(\beta_0, \beta_1)$  if  $d = \beta_0 d^- + \beta_1 d^+$ ,  $d^\pm \in \mathcal{G}_g(\mathbb{R})$ ,  $(d^\pm)^i \approx 0, i \in \{1, \dots, m-1\}$ ,  $(d^\pm)^m \approx \delta$ ,  $(d^-)^m G \approx y_0 \delta$  and  $(d^+)^m G \approx y_1 \delta$ , for each generalized step function  $G$  with value  $(y_0, y_1)$ .

Let  $m$  be an odd positive integer. A generalized function  $d \in \mathcal{G}_g(\mathbb{R})$  is said to be  $m'$ -singular delta function ( $m'$ SD-function for short) with value  $(\beta_0, \beta_1)$  if  $d = \beta_0 d^- + \beta_1 d^+$ ,  $d^\pm \in \mathcal{G}_g(\mathbb{R})$ ,  $(d^\pm)^i \approx 0, i \in \{1, \dots, m-2, m\}$ ,  $(d^\pm)^{m-1} \approx \delta$ ,  $(d^-)^{m-1} G \approx y_0 \delta$  and  $(d^+)^{m-1} G \approx y_1 \delta$ , for each generalized step function  $G$  with value  $(y_0, y_1)$ . (That implies  $\beta_0^m + \beta_1^m = 1$ .)

An  $S\delta$ -function  $D$  and an  $m$ SD-function (or an  $m'$ SD-function)  $d$  are said to be compatible if  $d^m D \approx 0$  (or  $d^{m-1} D \approx 0$ ).

**Remark 53.** One can construct such functions in a similar way as an  $S\delta$ -function, with  $\text{supp } d_\varepsilon^- \subset (-\infty, \varepsilon)$ ,  $\text{supp } d_\varepsilon^+ \subset (\varepsilon, \infty)$ . Compatibility conditions can be achieved by demanding that  $D$  and  $d$  have disjoint supports for  $\varepsilon$  small enough, for example.

The definition of  $m'$ SD-function  $d$  implies  $Gd^m \approx 0$  if  $G$  is a generalized step function.

Now we shall give the definition of singular shock wave and a useful lemma.

**Definition 54.**  $(U, V) \in (\mathcal{G}(\mathbb{R}_+^2))^2$  is called singular shock wave solution to Riemann problem (1-3) if (a) and (b) hold:

$$(a) \quad \begin{aligned} U_t + (f_1(U)V + f_2(U, V))_x &\approx 0 \\ V_t + (g_1(U)V + g_2(U, V))_x &\approx 0. \\ U|_{t=0} &= G, V|_{t=0} = H. \end{aligned}$$



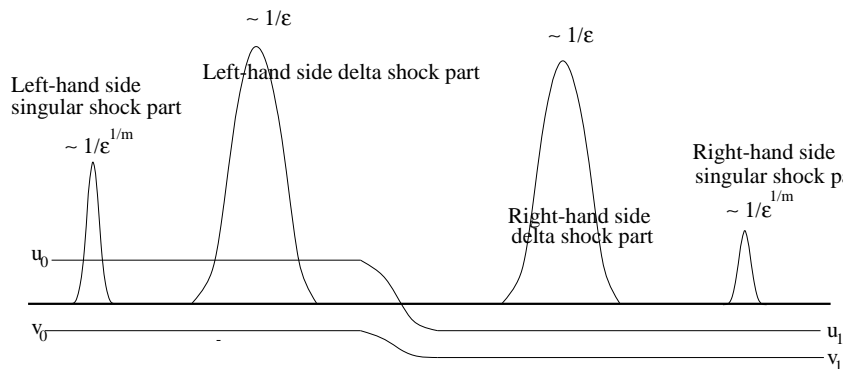


FIGURE 3.  $m$  singular shock wave

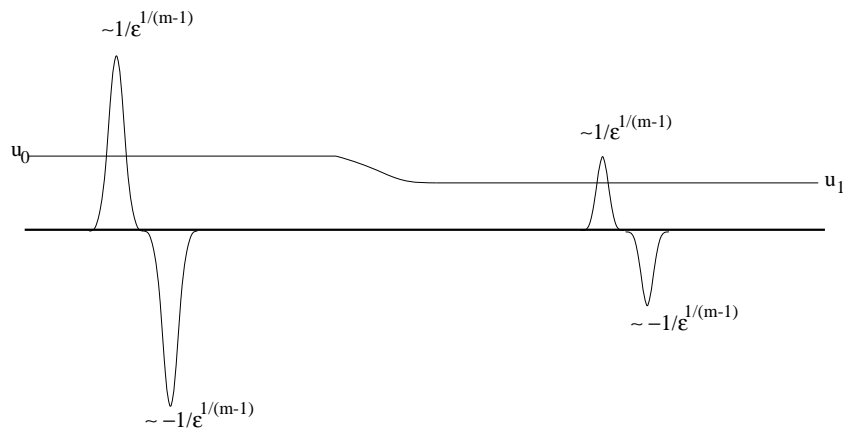


FIGURE 4.  $m'$  part of a singular shock wave

$$(b) \quad \begin{aligned} U(x, t) &= G(x - ct) + s_1(t)(\alpha_0 d^-(x - ct) + \alpha_1 d^+(x - ct)), \\ V(x, t) &= H(x - ct) + s_2(t)(\beta_0 D^-(x - ct) + \beta_1 D^+(x - ct)) \\ &\quad + s_3(t)(\gamma_0 d^-(x - ct) + \gamma_1 d^+(x - ct)). \end{aligned}$$

Here  $G$  and  $H$  are generalized step functions,  $f_i, g_i, i = 1, 2$ , are polynomials of the degree at most  $m$   $c \in \mathbb{R}$  is a speed of the shock,  $s, s_1, s_2 \in C^1([0, \infty))$ ,  $s_1(0) = s_2(0) = s_3(0) = 0$ ,  $D$  is an  $S\delta$ -function, as before, and  $d_j$  are  $m$ SD or  $m'$ SD-function,  $j = 1, 2$ .

Here, the strength of a singular shock wave is  $s_2(t)$ , and the left-and right-hand sided strengths are defined as in the case of delta shock wave.

From Definition 52 it follows that  $\alpha_0^k + \alpha_1^k = \beta_0^k + \beta_1^k = \gamma_0 + \gamma_1 = 1$ , where  $k = m$  in the case of  $m$ SD-and  $k = m - 1$  in the case of  $m'$ SD-functions.

**Lemma 55.** a) Let  $d \in \mathcal{G}_g(\mathbb{R})$  be an  $m$ SD-function with value  $(\beta_0, \beta_1)$ ,  $\beta_0^m + \beta_1^m = 1$ ,  $G \in \mathcal{G}_g(\mathbb{R})$  generalized step function with value  $(y_0, y_1)$ ,  $s \in C^1(\mathbb{R}_+)$ ,  $s(0) = 0$ , and  $\Gamma(y) = \sum_{i=0}^m a_i y^i$  be a real valued polynomial. Then

$$\begin{aligned} & \Gamma(G(x - ct)) + s(t)d(x - ct) \\ & \approx \Gamma(G(x - ct)) + a_m s^m(t) (\beta_0^m (d^-)^m (x - ct) + \beta_1^m (d^+)^m (x - ct)) \\ & \approx \Gamma(G(x - ct)) + a_m s^m(t) \delta(x - ct). \end{aligned}$$

b) Let  $d \in \mathcal{G}_g(\mathbb{R})$  be an  $m'$ SD-function with value  $(\beta_0, \beta_1)$ ,  $\beta_0^{m-1} + \beta_1^{m-1} = 1$ , while  $G$ ,  $s$  and  $\Gamma$  are as above. Then

$$\begin{aligned} & \Gamma(G(x - ct)) + s(t)d(x - ct) \\ & \approx \Gamma(G(x - ct)) + a_{m-1} s^{m-1}(t) (\beta_0^{m-1} (d^-)^{m-1} (x - ct) + \beta_1^{m-1} (d^+)^{m-1} (x - ct)) \\ & \quad + m a_m s^{m-1}(t) (\beta_0^{m-1} y_0 (d^-)^{m-1} (x - ct) + \beta_1^{m-1} y_1 (d^+)^{m-1} (x - ct)) \\ & \approx \Gamma(G(x - ct)) + a_{m-1} s^{m-1}(t) \delta(x - ct) \\ & \quad + m a_m s^{m-1}(t) (\beta_0^{m-1} y_0 + \beta_1^{m-1} y_1) \delta(x - ct). \end{aligned}$$

**7.3.2. Second solution concept.** We shall now briefly describe the second solution concept we are using. Suppose  $\overline{R_+^2}$  is divided into finitely disjoint open sets  $O_i \neq \emptyset$ ,  $i = 1, \dots, n$  with piecewise smooth boundary curves  $\Gamma_i$ ,  $i = 1, \dots, m$ , that is  $O_i \cap O_j = \emptyset$ ,  $\bigcup_{i=1}^n \overline{O}_i = \overline{R_+^2}$  where  $\overline{O}_i$  denotes the closure of  $O_i$ . Let  $C(\overline{O}_i)$  be the space of bounded and continuous real-valued functions on  $\overline{O}_i$ , equipped with the  $L^\infty$ -norm. Let  $\mathcal{M}(\overline{O}_i)$ , be the space of measures on  $\overline{O}_i$ .

We consider the spaces  $C_\Gamma = \prod_{i=1}^n C(\overline{O}_i)$ ,  $\mathcal{M}_\Gamma = \prod_{i=1}^n \mathcal{M}(\overline{O}_i)$ . The product of an element  $G = (G_1, \dots, G_n) \in C_\Gamma$  and  $D = (D_1, \dots, D_n) \in \mathcal{M}_\Gamma$  is defined as an element  $D \cdot G = (D_1 G_1, \dots, D_n G_n) \in \mathcal{M}_\Gamma$ , where each component is defined as the usual product of a continuous function and a measure.

Every measure on  $\overline{O}_i$  can be viewed as a measure on  $\overline{\mathbb{R}_+^2}$  with support in  $\overline{O}_i$ . This way we obtain a mapping

$$\begin{aligned} m &: \mathcal{M}_\Gamma \rightarrow \mathcal{M}(\overline{\mathbb{R}_+^2}) \\ m &: D \mapsto D_1 + D_2 + \dots + D_n. \end{aligned}$$

A typical example is obtained when  $\overline{\mathbb{R}_+^2}$  is divided into two regions  $O_1, O_2$  by a piecewise smooth curve  $x = \gamma(t)$ . The delta function  $\delta(x - \gamma(t)) \in \mathcal{M}(\overline{\mathbb{R}_+^2})$  along the line  $x = \gamma(t)$  can be split in a non unique way into a left-hand side  $D^- \in \mathcal{M}(\overline{O}_1)$  and the right-hand component  $D^+ \in \mathcal{M}(\overline{O}_2)$  such that

$$\delta(x - \gamma(t)) = \alpha_0(t) D^- + \alpha_1(t) D^+ = m(\alpha_0(t) D^- + \alpha_1(t) D^+)$$

with  $\alpha_0(t) + \alpha_1(t) = 1$ . A solution of the form

$$(63) \quad \begin{aligned} u(x, t) &= G(x - ct) \\ v(x, t) &= H(x - ct) + s(t)(\alpha_0 D^- + \alpha_1 D^+) \end{aligned}$$

is called delta shock wave.

The solution concept which allows to incorporate such two sided delta functions as well as shock waves is modeled along the lines of the classical weak solution concept and proceeds as follows:

Step 1: Perform all nonlinear operations of functions in the space  $C_\Gamma$ .

Step 2: Perform multiplications with measures in the space  $\mathcal{M}_\Gamma$ .

Step 3: Map the space  $\mathcal{M}_\Gamma$  into  $\mathcal{M}(\overline{\mathbb{R}_+^2})$  by means of the map  $m$  and embed it into the space of distributions.

Step 4: Perform the differentiation in the sense of distributions and require that the equation is satisfied in this sense.

Note that in the case of absence of a measure part (Step 2), this is the precisely the concept of a weak solution to equations in divergence form.

**7.4. Existence theorems.** Delta locus is the set of all points  $U_1 = (u_1, v_1) \in \mathbb{R}^2$  such that there exists a delta shock solution to system (62, 58). Singular delta locus is the set of all points  $U_1 \in \mathbb{R}^2$  such that there exists a singular shock solution to system (62), (58). In this case  $f_1, f_2, g_1$  and  $g_2$  have to be polynomials.

As one can see, these definitions are just a simple analogue to the definition of Hugoniot locus.

**7.4.1. Existence theorems using the first solution concept.** We shall present (without proofs) two theorems about delta locus and singular delta locus from [58].

**Theorem 56.** a) *Let  $f_1 \not\equiv \text{const}$ . Then a delta shock wave solution to (62), (58) exists if  $u_0 \neq u_1, f_1(u_0) \neq f_1(u_1)$  and*

$$(64) \quad \begin{aligned} c &= \frac{f_1(u_1)v_1 + f_2(u_1, v_1) - f_1(u_0)v_0 - f_2(u_0, v_0)}{u_1 - u_0} \\ &= \frac{g_1(u_0)f_1(u_1) - g_1(u_1)f_1(u_0)}{f_1(u_1) - f_1(u_0)}, \end{aligned}$$

where  $c$  is the velocity of the delta shock. The set of all points  $(u_1, v_1)$  such that (64) holds is the delta locus of the system (for the point  $(u_0, v_0)$ ).

b) *If  $f_1(u_0) = f_1(u_1) = 0$  (specially, if  $f_1 \equiv 0$ ) and  $g_1 \not\equiv \text{const}$ , then the delta locus is the set of all points  $(u_1, v_1)$  such that  $g_1(u_0) \neq g_1(u_1)$ .*

c) *If  $f_1 \equiv 0$  and  $g_1 \equiv b \in \mathbb{R}$ , then the delta locus is the set of all points  $(u_1, v_1)$  such that  $b(u_1 - u_0) = f_2(u_1) - f_2(u_0)$ .*

The main point of the proof is to express  $c$  from the first equation of the system and then substitute this value of  $c$  into the second one. Then, one has to find appropriate  $s(t) = \sigma t$ , and other coefficients to “compensate” so called Rankine–Hugoniot deficit in the second equation and eliminate terms associated to  $\delta'$ . The exact proof contains a lot of details, so we shall omit it. One has to use the definitions and lemmas above.

For the second theorem, we have to make same assumption and give notation. Suppose that the maximal degree of all polynomials in the fluxes equals  $m$ . Let

$$f_1(y) = \sum_{i=0}^m a_{1,i}y^i, \quad f_2(y) = \sum_{i=0}^m a_{2,i}y^i, \quad g_1(y) = \sum_{i=0}^m b_{1,i}y^i, \quad g_2(y) = \sum_{i=0}^m b_{2,i}y^i.$$

The following lemma is obvious consequence of the given definitions.

**Lemma 57.** a) Let  $d \in \mathcal{G}_g(\mathbb{R})$  be an  $m$ SD-function with value  $(\beta_0, \beta_1)$ ,  $\beta_0^m + \beta_1^m = 1$ ,  $G \in \mathcal{G}_g(\mathbb{R})$  generalized step function with value  $(y_0, y_1)$ ,  $s \in C^1(\mathbb{R}_+)$ ,  $s(0) = 0$ , and let  $\Gamma(y) = \sum_{i=0}^m a_i y^i$  be a real valued polynomial. Then

$$\begin{aligned} & \Gamma(G(x-ct)) + s(t)d(x-ct) \\ & \approx \Gamma(G(x-ct)) + a_m s^m(t) (\beta_0^m (d^-)^m(x-ct) + \beta_1^m (d^+)^m(x-ct)) \\ & \approx \Gamma(G(x-ct)) + a_m s^m(t) \delta(x-ct). \end{aligned}$$

b) Let  $d \in \mathcal{G}_g(\mathbb{R})$  be an  $m'$ SD-function with value  $(\beta_0, \beta_1)$ ,  $\beta_0^{m-1} + \beta_1^{m-1} = 1$ , while  $G$ ,  $s$  and  $\Gamma$  are as above. Then

$$\begin{aligned} & \Gamma(G(x-ct) + s(t)d(x-ct)) \\ & \approx \Gamma(G(x-ct)) + a_{m-1} s^{m-1}(t) (\beta_0^{m-1} (d^-)^{m-1}(x-ct) + \beta_1^{m-1} (d^+)^{m-1}(x-ct)) \\ & \quad + m a_m s^{m-1}(t) (\beta_0^{m-1} y_0 (d^-)^{m-1}(x-ct) + \beta_1^{m-1} y_1 (d^+)^{m-1}(x-ct)) \\ & \approx \Gamma(G(x-ct)) + a_{m-1} s^{m-1}(t) \delta(x-ct) \\ & \quad + m a_m s^{m-1}(t) (\beta_0^{m-1} y_0 + \beta_1^{m-1} y_1) \delta(x-ct). \end{aligned}$$

Now, we are in position to state the main theorem in this subsection.

**Theorem 58.** Let  $G(x-ct)$  and  $H(x-ct)$  be generalized step functions with the speed  $c$  and values  $(u_0, u_1)$  and  $(v_0, v_1)$ , respectively. Let

$\alpha = c(v_1 - v_0) - (g_1(u_1)v_1 + g_2(u_1) - g_1(u_0)v_0 - g_2(u_0)) = c[H] - [g_1(G)H + g_2(G)]$  be Rankine–Hugoniot deficit.

A singular shock wave solution exists if one of the following two assertions are true:

(i) There exists a solution  $(\alpha_0, y_0) \in \mathbb{R}^2$  to the system

$$\begin{aligned} \alpha[f_1(G)]\alpha_0 + \sigma[H]a_{1,m}y_0 &= \sigma(v_1 a_{1,m} + a_{2,m}) + \alpha f_1(u_1) \\ \alpha[g_1(G)]\alpha_0 + \sigma[H]b_{1,m}y_0 &= \sigma(v_1 b_{1,m} + b_{2,m}) + \alpha(g_1(u_1) - c), \end{aligned}$$

for some  $\sigma \in \mathbb{R} \setminus \{0\}$ . If  $m$  is an even number, then  $\sigma$  also has to be positive and  $y_0 \in [0, 1]$ .

(ii)  $m$  is an odd number and there exists a solution  $(\alpha_0, y_0) \in \mathbb{R} \times \mathbb{R}_+$  to the system

$$\begin{aligned} \alpha[f_1(G)]\alpha_0 + \sigma(a_{1,m-1}[G] + m a_{1,m}[GH] + m a_{2,m}[G])y_0 \\ &= \sigma(a_{1,m-1}v_1 + m a_{1,m}u_1v_1 + m a_{2,m}u_1 + a_{2,m-1}) + \alpha f_1(u_1) \\ \alpha[g_1(G)]\alpha_0 + \sigma(b_{1,m-1}[G] + m b_{1,m}[GH] + m b_{2,m}[G])y_0 \\ &= \sigma(b_{1,m-1}v_1 + m b_{1,m}u_1v_1 + m b_{2,m}u_1 + b_{2,m-1}) + \alpha(g_1(u_1) - c) \end{aligned}$$

for some  $\sigma \in \mathbb{R}_+$ .

The speed of the singular shock waves is always given by

$$c = \frac{f_1(u_1)v_1 + f_2(u_1) - f_1(u_0)v_0 - f_2(u_0)}{u_1 - u_0} = \frac{[f_1(G)H + f_2(G)]}{[G]}$$

Again, we shall omit the proof because it is even longer than the proof of the previous theorem, and the idea is almost the same. The main difference is that now  $d^\pm$  plays significant role in constructing a solution, even they are zeros in distributional sense.

**Remark 59.** By using the first solution concept, all results from the introduction are recovered, except (60). System (60) does not have delta shock wave solution, but singular shock wave solution, where the density of the gas is distributionally greater or equal to zero. This singular shock solution gives the same distribution limit as the original one, obtained by using the measure theoretical and vanishing viscosity in method given in [99] and [40].

**Remark 60.** Let us note that the delta locus is just a curve in  $\mathbb{R}_+^2$  and the delta singular locus is an area, in general. This has deep consequence: it is not easy to solve arbitrary Riemann problem without using a singular delta shock locus. The usual delta shock wave can not be followed or can not follow any of the elementary waves, if the system (57) is strictly hyperbolic. But the combination of 1-rarefaction and singular shock wave or singular shock and 2-rarefaction wave is quite possible, even in this case.

**7.4.2. Existence theorems using the second solution concept.** In the paper [57], the following theorem is proved.

**Theorem 61.** *A point  $(u_1, v_1)$  is in the delta locus of a point  $(u_0, v_0)$  for the Riemann problem (62, 58) if  $f_2$  and  $g_2$  do not depend on  $v$  and the following holds:*

$$(a) \quad g_1(u_0) \neq g_1(u_1), \quad (b) \quad f_1(u_0) \frac{k_1(g_1(u_1) - c)}{g_1(u_1) - g_1(u_0)} = f_1(u_1) \frac{k_1(g_1(u_0) - c)}{g_1(u_1) - g_1(u_0)},$$

where  $k_1 = c[G] - [f_1(G)H + f_2(G)]$ , and  $c$  is a speed of the delta shock wave.

We shall prove the theorem, just to show the simplicity of this solution concept.

*Proof.* Let us denote by  $s_0(t) = s(t)\beta_0$  and  $s_1(t) = s(t)\beta_1$ . The substitution of functions in (63) into (62)–(58) and the use of the Rankine–Hugoniot conditions gives the following equation

$$\begin{aligned} & (-c[G] + [f_1(G)H + f_2(G)])\delta(x-ct) + (f_1(s_0(t)u_0)\delta^-(x-ct) + f_1(s_1(t)u_1)\delta^+(x-ct))_x \\ & = (-c[G] + [f_1(G)H + f_2(G)])\delta(x-ct) + (f_1(s_0(t)u_0) + f_1(s_1(t)u_1))\delta'(x-ct) = 0. \end{aligned}$$

Suppose that  $u_0 \neq u_1$ . From the above equation, one obtains the value of the speed  $c$  and the coupling equations for  $s_0$  and  $s_1$ :

$$(65) \quad \begin{aligned} c &= [f_1(G)H + f_2(G)]/[G] \\ s_0(t)f_1(u_0) + s_1(t)f_1(u_1) &= 0. \end{aligned}$$

Doing the same for the second equation, one obtains

$$(66) \quad \begin{aligned} & -c[H] + (s_0(t) + s_1(t))'\delta(x-ct) - c(s_0(t) + s_1(t))\delta'(x-ct) \\ & + [g_1(G)H + g_2(G)]\delta(x-ct) + (s_0(t)g_1(u_0) + s_1(t)g_1(u_1))\delta'(x-ct) = 0. \end{aligned}$$

Since  $c$  is already determined,

$$(s_0(t) + s_1(t))' = c[H] - [g_1(G)H + g_2(G)], \text{ i.e., } s_0(t) + s_1(t) = k_1 t,$$

and  $k_1$  is called Rankine–Hugoniot deficit [42]. Now, one obtains the following system of equations for  $s_0$  and  $s_1$ :

$$\begin{aligned} (g_1(u_0) - c)s_0(t) + (g_1(u_1) - c)s_1(t) &= 0 \\ s_0(t) + s_1(t) &= k_1 t. \end{aligned}$$

If  $g_1(u_0) = g_1(u_1)$ , then  $k_1 = 0$ , i.e., there is no delta shock wave solution. Otherwise,

$$s_0(t) = \frac{k_1(g_1(u_1) - c)}{g_1(u_1) - g_1(u_0)}, \quad s_1(t) = \frac{k_1(c - g_1(u_0))}{g_1(u_1) - g_1(u_0)}$$

are determined. Using these values and the second equation in (65), one gets the assertion of the theorem.

Now, let  $u_0 = u_1$ . Then, from the above equations, one can see that  $k_1 = 0$  and there is no delta shock wave solution to (62)–(58).  $\square$

**Remark 62.** Again, the solutions admitting delta shock wave, mentioned in introduction has also the same solution in this sense, except (60).

**7.4.3. Admissibility conditions.** A delta or singular shock wave is said to be admissible wave (entropy one) if it is overcompressive, i.e.  $\lambda_i(u_0) \geq c \geq \lambda_i(u_1)$ ,  $i = 1, \dots, n$ , where  $\lambda_i$  is  $i$ -th characteristics for the system  $u_t + \nabla f(u)u_x = 0$ , which is equivalent to (62) for smooth solutions, and  $c$  is a speed of the delta or singular shock wave.

If the  $i$ -th characteristic field is linearly degenerate, i.e.,  $\nabla \lambda_i \cdot r_i = 0$ , where  $r_i$  is  $i$ -th eigenvector for the matrix  $\nabla f$ , then the delta or singular shock wave with singular support on the  $i$ -th characteristics is admissible and called delta or singular delta discontinuity (one can see [61] for a precise definition).

**7.5. Intersection of delta or singular shock waves with themselves and other elementary waves.** The main idea is simple: if delta or singular shock wave interacts with some (delta or singular) shock wave at the point  $(x_0, t_0)$ , one has to solve the new initial data problem

$$(67) \quad u|_{t=t_0} = \begin{cases} u_0, & x < x_0, \\ u_2, & x > x_0; \end{cases} \quad v|_{t=t_0} = \begin{cases} v_0, & x < x_0, \\ v_2, & x > x_0, \end{cases} + \gamma \delta_{(x_0, t_0)}.$$

Before considering interaction of delta or singular shock wave with a rarefaction wave, in order to see what result is expected one can decompose the rarefaction wave into a large family of approximate, but non-entropy physical waves.

In the case of interaction of delta or singular shock wave with rarefaction or shock wave,  $\gamma$  is a strength of an initial delta or singular shock wave. If delta or singular shock waves interacts mutually, then  $\gamma$  is sum of strengths of these waves.

**Definition 63.** The set of points  $(u_2, v_2)$  for which there exists a solution to (62), (67) in a form of delta (singular) shock wave is called  $\gamma$ -second delta (singular delta) locus.

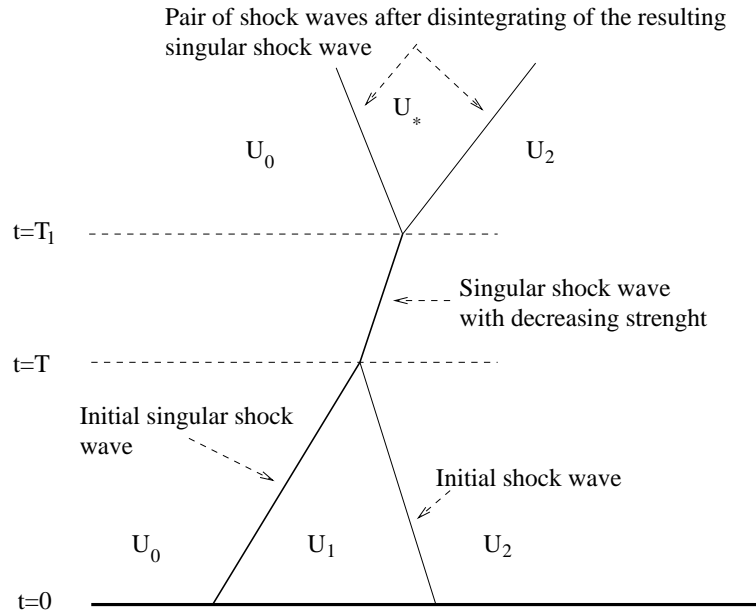


FIGURE 5. Splitting of singular shock wave

**7.5.1. Some results for the first solution concept.** A second singular locus is not easy to write in a simple form for general case (62), and we shall skip it here. Let us just mention that a  $\gamma$ -second delta (singular delta) locus contains as its subset delta (singular delta) locus for  $\gamma > 0$ , as it was shown in [59].

The major problem is intersection of a delta or singular shock wave and rarefaction wave due to continuous integration of delta function and some continuous function (rarefaction fan). One has to solve an ordinary differential equation and consider the admissibility condition afterward. Due to this fact, the singular support of singular shock wave is a curve, not a straight line as before. In special cases it can be done more easily. For example, for system (59) a complete analysis of interaction for singular shock waves and any other elementary wave or another singular shock wave is done (see [59]).

We shall present one phenomena obtained in the cited paper. For an usual Riemann problem, a strength of singular shock wave increases (linearly) with time. During a interaction with rarefaction wave it can decrease with time. If the strength reaches zero, then the singular shock wave decouples into two ordinary shock waves (see Figure 5, where “initial shock wave” is non-admissible one—it is a result of rarefaction wave approximation with a fan of such shock waves)

**7.5.2. Some results for the second solution concept.** In contrast to previous case, there exist a theorem describing the second delta locus in a simple form.

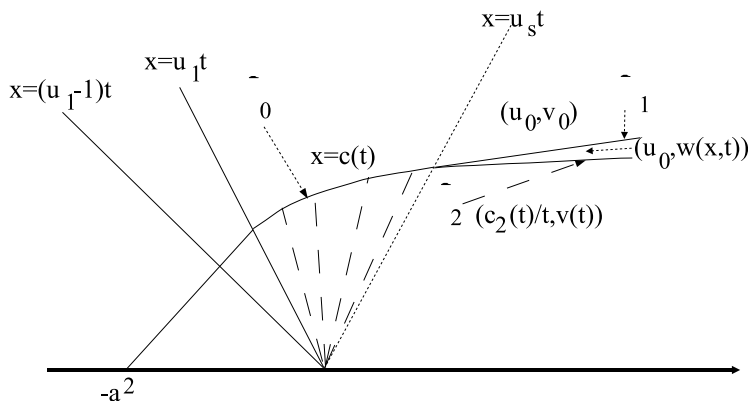


FIGURE 6. Delta contact discontinuity and unbounded part of solution

**Theorem 64.** *A point  $(u_2, v_2)$  is in the second delta locus of the point  $(u_0, v_0)$  if one of the following is true.*

(a)  $f_1 \not\equiv \text{const}$  and

$$\frac{f_1(u_2)v_2 + f_1(u_2) - f_1(u_0)v_0 - f_1(u_0)}{u_2 - u_0} = \frac{g_1(u_0)f_1(u_2) - g_1(u_2)f_1(u_0)}{f_1(u_2) - f_1(u_0)},$$

(b)  $f_1 \equiv 0$  and  $g_1(u_0) \neq g_1(u_2)$

(c) *If  $(u_2, v_2)$  is in a Hugoniot locus of the point  $(u_0, v_0)$ .*

But, complete analysis is done only for system (61) so far (up to our knowledge). Again, we obtained few new interesting things. The first one is an existence of delta contact discontinuity (which is possible only if a given system is not genuinely nonlinear). And the second one is that we start with piecewise constant function, the solution can be unbounded in a region with Lebesgue measure greater than zero. That is, a part of solution (after intersection of a delta shock wave and rarefaction wave) is  $L_{\text{loc}}^\infty$  function (going to infinity as  $1/\sqrt{y}$  as  $y \rightarrow 0$ ). One can see illustration in Figure 6: the function  $w$  is unbounded, the line 1 denotes the delta contact discontinuity curve, while the line 2 denotes a shock wave curve. We shall demonstrate how a delta shock curve  $x = c(t)$  can be found during the intersection of delta shock wave and rarefaction wave in the case of system (61).

The function  $x = c(t)$  has to satisfy the following ordinary differential equation:

$$(68) \quad -c'(t) \left( \frac{c(t)}{t} - u_0 \right) + \frac{1}{2} \left( \left( \frac{c(t)}{t} \right)^2 - u_0^2 \right) = 0, \quad c(t_0) = x_0,$$

which has the unique solution  $c(t) = u_0 t - a \sqrt{2(u_0 - u_1)t}$ ,  $t \geq t_0$ . This example also shows why the intersection problem depends highly on a system in question: The equation (68) should be explicitly solved, which is not always possible.



But the main problems and interesting phenomena of this intersection appears when the delta shock wave is no longer overcompressive during the interaction. One can found complete results in [61].

**7.6. Numerical verification.** Let us give an example for a possible approach in numerical verification of a delta or singular shock wave. Using the first solution method in [58] one can find a singular shock wave solution to (60) for some Riemann data which converges to the measure valued solution described in [40] or [99]. After a “natural” change of variables  $uv \mapsto w$  one gets the following system in evolution form

$$(69) \quad \begin{aligned} u_t + w_x &= 0 \\ w_t + (w^2/u)_x &= 0 \end{aligned}$$

which makes sense because  $u$  is the density, and there is no vacuum state in this case. Transformed system do not permit measure theoretical results for some initial data, since square of  $w$  appears in the flux function. But, using Colombeau generalized functions, it has the same (up to association relation) solutions for all initial data as the original one.

For the system (69) after mollifying the initial data in a usual way (convolution with a delta model net), one can try to use finite volume scheme (modified Godunov scheme, see [54]) together with moving mesh method [93]. This was done in [16]. Obtained solution resembles the solution given in [58] (obtained by the first solution method).

**Remark 65.** The word “resembles” in the above context means that the numerical speed of a singular shock wave is arbitrarily near the theoretical one, and the masses delta function part of singular shock wave are linearly growing with respect to time, as expected.

Conservation law systems and generalized functions are subject of a large number of papers. We shall refer to book [11], where one can find a further reference and many nice examples for Colombeau generalized function approach.

**7.7. Open problems.** As it was announced in the beginning of this part, now we shall present some of numerous open problems. More dimensional cases are totally excluded from the list bellow, because the number and form of problems in this case is quite large and vogue.

- (i) Uniqueness in some sense (*No results so far*).
- (ii) Avoiding not wanted delta or singular delta shock waves (*Overcompressibility condition is not enough*).
- (iii) Overcome linearity in one variable (*There are some results using Colombeau generalized functions*).
- (iv) General interactions of these new singularities (*Probably, the solution highly depends on particular systems*).
- (v) How delta shock waves can be followed (or follow) rarefaction wave, as singular shock waves do (*No results so far*).

## 8. Appendix

**8.1. Algebras of weighted sequence spaces.** In this appendix we give another approach to the Colombeau type algebras which is related to the topological structure of certain exponentially weighted sequence spaces. All these classes of algebras are simply determined by the (locally convex) space  $E$ , and a sequence of weights  $r : \mathbb{N} \rightarrow \mathbb{R}_+$  (or sequence of sequences) which serves to construct an ultrametric on the sequence space  $E^{\mathbb{N}}$ . The sequence  $r = (r_n)_n$  is assumed to be decreasing to zero. This implies that sequence spaces under consideration ( $\subset E^{\mathbb{N}}$ ) contain as a subspace  $E \sim * \text{diag } E^{\mathbb{N}}$  and that they induce the discrete topology on  $E$ . This is well-known for the sharp topology for Colombeau type algebras. But our analysis implies that if one has a Colombeau type algebra containing the Dirac delta distribution  $\delta$  as an embedded Colombeau generalized function, then the topology induced on the basic space must be discrete. This is an analogous result to the Schwartz's "impossibility result" concerning the product of distributions. construction of Colombeau type algebras.

In order to simplify the construction, we will consider sequences  $(f_n)_{n \in \mathbb{N}}$  instead of nets  $f_\varepsilon$ . The passage from one to another concept is simple: with  $\varepsilon = 1/n$  and reversely. Consider a semi-normed algebra  $(E, p)$  such that  $p(ab) \leq p(a)p(b)$ ,  $a, b \in E$  and a sequence  $r \in \mathbb{R}_+^{\mathbb{N}}$  decreasing to zero. Define for  $f \in E^{\mathbb{N}}$

$$\|f\|_{p,r} := \limsup_{n \rightarrow \infty} p(f_n)^{r_n}.$$

This is well defined for any  $f \in E^{\mathbb{N}}$ , with values in  $\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{\infty\}$ . With this definition, let  $\mathcal{F}_{p,r} = \{f \in E^{\mathbb{N}} : \|f\|_{p,r} < \infty\}$ ,  $\mathcal{K}_{p,r} = \{f \in E^{\mathbb{N}} : \|f\|_{p,r} = 0\}$ . Then the following holds:

**Proposition 66.** (a) *The function  $d_{p,r} : \mathcal{F}_{p,r} \times \mathcal{F}_{p,r} \rightarrow \mathbb{R}_+$ ,  $(f, g) \mapsto \|f - g\|_{p,r}$ , is an ultrapseudometric on  $\mathcal{F}_{p,r}$ .*

(b)  *$\mathcal{F}_{p,r}$  is a subalgebra of  $E^{\mathbb{N}}$ , and  $\mathcal{K}_{p,r}$  is an ideal of  $\mathcal{F}_{p,r}$ ; thus  $\mathcal{G}_{p,r} := \mathcal{F}_{p,r}/\mathcal{K}_{p,r}$  is an algebra.*

(c)  *$\tilde{d}_{p,r} : \mathcal{G}_{p,r} \times \mathcal{G}_{p,r} \rightarrow \mathbb{R}_+$ ,  $(F, G) \mapsto d_{p,r}(f, g)$ , is an ultrametric on  $\mathcal{G}_{p,r}$ , where  $f \in F$ ,  $g \in G$  are any representatives of the classes  $F = f + \mathcal{K}_{p,r}$  resp.  $G = g + \mathcal{K}_{p,r}$ .*

(d)  *$\mathcal{G}_{p,r} = \mathcal{F}_{p,r}/\mathcal{K}_{p,r}$  is a topological algebra, the quotient topology being the same than the topology induced by the ultrametric  $\tilde{d}_{p,r}$ .*

We give the construction of generalized constants. For this,  $E$  will be the underlying field  $\mathbb{R}$  or  $\mathbb{C}$ , and  $p = |\cdot|$  the absolute value. For  $r = 1/\log$ , we get the ring of Colombeau's numbers  $\overline{\mathbb{C}}$  (and  $\overline{\mathbb{R}}$ ). Let  $r_n = \frac{1}{\log n}$ ,  $n \geq 2$ .

Colombeau's algebras of generalized constants represented by sequences with polynomial growth modulo sequences of more than polynomial decrease, because

$$\begin{aligned} \limsup |x_n|^{1/\log n} < \infty &\iff \exists C : \limsup |x_n|^{1/\log n} = C \\ &\iff \exists B, \exists n_0, \forall n > n_0 : |x_n| \leq B^{\log n} = n^{\log B} \\ &\iff \exists \gamma : |x_n| = o(n^\gamma). \end{aligned}$$

If we put,  $\limsup = 0$  (for the ideal) then the corresponding  $C$  above equals zero and thus  $\forall B > 0$  resp.  $\forall \gamma$  we have  $|x_n| = o(n^\gamma)$ .

Consider now Hölder type spaces  $E = C^{k,\alpha}(\bar{O})$  (cf. [26]),  $\alpha \in (0, 1]$  and  $k \in \mathbb{N}_0$  (with  $|\cdot|_{k,\alpha}$ -norm It is a Banach space and we can apply the same construction with  $p = \|\cdot\|_{k,\alpha}$ . The corresponding Colombeau type algebra is defined by  $\mathcal{G}_{C^{k,\alpha}} := \mathcal{F}/\mathcal{K}$ , where

$$\begin{aligned} \mathcal{F} &:= \left\{ u \in (C^{k,\alpha}(\bar{O}))^{\mathbb{N}} \mid \limsup \|u_n\|_{s,\infty}^{\frac{1}{\log n}} < \infty \right\}, \\ \mathcal{K} &:= \left\{ u \in (C^{k,\alpha}(\bar{O}))^{\mathbb{N}} \mid \limsup \|u_n\|_{s,\infty}^{\frac{1}{\log n}} = 0 \right\}. \end{aligned}$$

This algebra will be used for the analysis of elliptic equation in Part II.

**8.1.1. Constructions with locally convex vector spaces.** Consider now an algebra  $E$  which is a locally convex vector space on  $\mathbb{C}$ , equipped with an arbitrary set of seminorms  $p \in \mathcal{P}$  determining its locally convex structure. Assume that

$$\forall p \in \mathcal{P}, \exists \bar{p} \in \mathcal{P}, C \in \mathbb{R}_+ : \forall x, y \in E : p(xy) \leq C \bar{p}(x) \bar{p}(y).$$

Let

$$\begin{aligned} \mathcal{F}_{\mathcal{P},r} &= \left\{ f \in E^{\mathbb{N}} \mid \forall p \in \mathcal{P} : \|f\|_{p,r} < \infty \right\}, \\ \mathcal{K}_{\mathcal{P},r} &= \left\{ f \in E^{\mathbb{N}} \mid \forall p \in \mathcal{P} : \|f\|_{p,r} = 0 \right\}. \end{aligned}$$

Then the following holds:

**Proposition 67.** (a) For every  $p \in \mathcal{P}$ ,  $d_{p,r} : E^{\mathbb{N}} \times E^{\mathbb{N}} \rightarrow \overline{\mathbb{R}}_+$ ,  $(f, g) \mapsto \|f - g\|_{p,r}$ , is an ultrapseudometric on  $\mathcal{F}_{\mathcal{P},r}$ .

(b)  $\mathcal{F}_{\mathcal{P},r}$  is a (sub-)algebra of  $E^{\mathbb{N}}$ , and  $\mathcal{K}_{\mathcal{P},r}$  is an ideal of  $\mathcal{F}_{\mathcal{P},r}$ .

(c)  $\mathcal{G}_{\mathcal{P},r} := \mathcal{F}_{\mathcal{P},r}/\mathcal{K}_{\mathcal{P},r}$  is an algebra.

(d) For every  $p \in \mathcal{P}$ ,  $\tilde{d}_{p,r} : \mathcal{G}_{\mathcal{P},r} \times \mathcal{G}_{\mathcal{P},r} \rightarrow \overline{\mathbb{R}}_+$ ,  $(F, G) \mapsto d_{p,r}(f, g)$  is an ultrametric on  $\mathcal{G}_{\mathcal{P},r}$ , where  $f, g$  are any representatives of the classes  $F = f + \mathcal{K}_{\mathcal{P},r}$  resp.  $G = g + \mathcal{K}_{\mathcal{P},r}$ .

(e)  $\mathcal{G}_{\mathcal{P},r} := \mathcal{F}_{\mathcal{P},r}/\mathcal{K}_{\mathcal{P},r}$  is a topological algebra, the quotient topology being the same than the topology induced by the family of ultrametries  $\{\tilde{d}_{p,r}\}_{p \in \mathcal{P}}$ .

**Example 68.** Let  $E = C^\infty(O)$ ,  $\mathcal{P} = \{p_\nu\}_{\nu \in \mathbb{N}}$  with  $p_\nu(f) := \sup_{|\alpha| \leq \nu, |x| \leq \nu} |f^{(\alpha)}(x)|$ , and  $r = 1/\log$ . Then,  $\mathcal{G}_{\mathcal{P},r} = \mathcal{F}_{\mathcal{P},r}/\mathcal{K}_{\mathcal{P},r}$  with

$$\begin{aligned} \mathcal{F}_{\mathcal{P},r} &= \left\{ (f_n)_n \in C^\infty(O)^{\mathbb{N}} \mid \forall \nu \in \mathbb{N} : \limsup_{n \rightarrow \infty} p_\nu(f_n)^{1/\log n} < \infty \right\}, \\ \mathcal{K}_{\mathcal{P},r} &= \left\{ (f_n)_n \in C^\infty(O)^{\mathbb{N}} \mid \forall \nu \in \mathbb{N} : \limsup_{n \rightarrow \infty} p_\nu(f_n)^{1/\log n} = 0 \right\}. \end{aligned}$$

we obtain the simplified Colombeau algebra  $\mathcal{G}_s$ .

So called full Colombeau algebra  $\mathcal{G}$  is related to a more delicate procedure and it is omitted. We only note that the embedding of Schwartz distributions and of smooth functions into  $\mathcal{G}$  is well known. Also it is well known that the multiplication of smooth function embedded into  $\mathcal{G}$  is the usual multiplication.

**Example 69.** The following example is also of interest. Take  $E = \mathcal{D}_{L^p}(O)$ ,  $p > 1$ ,  $\mathcal{P} = \{p_\nu\}_{\nu \in \mathbb{N}}$  with  $p_\nu(f) := \sup_{|\alpha| \leq \nu} \|f^{(\alpha)}\|_{L^p}$ , and  $r = 1/\log$ . Then,  $\mathcal{G}_{L^p} = \mathcal{F}_{\mathcal{P},r}/\mathcal{K}_{\mathcal{P},r}$  with

$$\begin{aligned}\mathcal{F}_{\mathcal{P},r} &= \left\{ (f_n)_n \in \mathcal{D}_{L^p}(O)^\mathbb{N} \mid \forall \nu \in \mathbb{N} : \limsup_{n \rightarrow \infty} p_\nu(f_n)^{1/\log n} < \infty \right\}, \\ \mathcal{K}_{\mathcal{P},r} &= \left\{ (f_n)_n \in \mathcal{D}_{L^p}(O)^\mathbb{N} \mid \forall \nu \in \mathbb{N} : \limsup_{n \rightarrow \infty} p_\nu(f_n)^{1/\log n} = 0 \right\}.\end{aligned}$$

is Colombeau type algebra used for the investigations of wave and heat equation.

**8.1.2. Projective and inductive limits.** *Projective limit.* The construction that follows leads to algebras of generalized ultradistributions of Beurling and Roumieu type. We will give only the general concepts of the construction.

Let  $(E_\nu^\mu, p_\nu^\mu)_{\mu, \nu \in \mathbb{N}}$  be a family of semi-normed algebras over  $\mathbb{C}$ , such that

$$\forall \mu, \nu \in \mathbb{N} : E_{\nu+1}^\mu \hookrightarrow E_\nu^\mu, \quad E_\nu^{\mu+1} \hookrightarrow E_\nu^\mu,$$

where  $\hookrightarrow$  means continuously embedded. This implies that there exist constants  $C_\nu^\mu, \tilde{C}_\nu^\mu \in \mathbb{R}_+$  such that

$$\forall \mu, \nu \in \mathbb{N} : p_\nu^\mu \leq C_\nu^\mu p_{\nu+1}^\mu, \quad p_\nu^\mu \leq \tilde{C}_\nu^\mu p_\nu^{\mu+1},$$

but without loss of generality one can take  $C_\nu^\mu, \tilde{C}_\nu^\mu = 1$ ,  $\forall \mu, \nu \in \mathbb{N}$ . Then let

$$\overleftarrow{E} := \text{proj lim}_{\mu \rightarrow \infty} \overleftarrow{E}^\mu = \text{proj lim}_{\mu \rightarrow \infty} \text{proj lim}_{\nu \rightarrow \infty} E_\nu^\mu = \text{proj lim}_{\nu \rightarrow \infty} E_\nu^\nu.$$

Define

$$\begin{aligned}\overleftarrow{\mathcal{F}}_{p,r} &= \left\{ f \in \overleftarrow{E}^\mathbb{N} \mid \forall \mu, \nu \in \mathbb{N} : \|f\|_{p_\nu^\mu, r} < \infty \right\}, \\ \overleftarrow{\mathcal{K}}_{p,r} &= \left\{ f \in \overleftarrow{E}^\mathbb{N} \mid \forall \mu, \nu \in \mathbb{N} : \|f\|_{p_\nu^\mu, r} = 0 \right\}.\end{aligned}$$

(Here  $p \equiv ((p_\nu^\mu)_\nu)^\mu$  stands (on the l.h.s.) for the whole family of seminorms.) Then Proposition 67 holds, with the slight changes of notations introduced above.

*Inductive limit* Consider now a family  $(E_\nu^\mu, p_\nu^\mu)_{\mu, \nu \in \mathbb{N}}$  of semi-normed spaces over  $\mathbb{C}$ , such that

$$(70) \quad \forall \mu, \nu \in \mathbb{N} : E_\nu^\mu \hookrightarrow E_{\nu+1}^\mu, \quad E_\nu^{\mu+1} \hookrightarrow E_\nu^\mu.$$

This implies that there exist constants  $C_\nu^\mu, \tilde{C}_\nu^\mu \in \mathbb{R}_+$  such that

$$\forall \mu, \nu \in \mathbb{N} : p_{\nu+1}^\mu \leq C_\nu^\mu p_\nu^\mu, \quad p_\nu^\mu \leq \tilde{C}_\nu^\mu p_\nu^{\mu+1},$$

but again one can assume  $C_\nu^\mu, \tilde{C}_\nu^\mu = 1$ ,  $\forall \mu, \nu \in \mathbb{N}$ . Now let

$$\forall \mu \in \mathbb{N} : \overrightarrow{E}^\mu = \text{*ind lim}_{\nu \rightarrow \infty} E_\nu^\mu.$$

Assume that for every  $\mu, \nu', \nu'' \in \mathbb{N}$  there exist  $\nu \in \mathbb{N}$  and  $C > 0$  such that

$$p_\nu^\mu(fg) \leq C p_{\nu'}^\mu(f) p_{\nu''}^\mu(g), \quad f \in E_{\nu'}^\mu, \quad g \in E_{\nu''}^\mu.$$

Note that (70) implies that  $\forall \mu \in \mathbb{N} : \overrightarrow{E}^{\mu+1} \hookrightarrow \overrightarrow{E}^\mu$ . Now let

$$\overrightarrow{E} := \text{proj lim}_{\mu \rightarrow \infty} \overrightarrow{E}^\mu = \text{proj lim}_{\mu \rightarrow \infty} \text{*ind lim}_{\nu \rightarrow \infty} E_\nu^\mu,$$

and define

$$\begin{aligned}\overrightarrow{\mathcal{F}}_{p,r} &:= \left\{ f \in \overrightarrow{E}^{\mathbb{N}} \mid \forall \mu \in \mathbb{N}, \exists \nu \in \mathbb{N} : f \in (E_{\nu}^{\mu})^{\mathbb{N}} \wedge \|f\|_{p_{\nu}^{\mu},r} < \infty \right\}, \\ \overrightarrow{\mathcal{K}}_{p,r} &:= \left\{ f \in \overrightarrow{E}^{\mathbb{N}} \mid \forall \mu \in \mathbb{N}, \exists \nu \in \mathbb{N} : f \in (E_{\nu}^{\mu})^{\mathbb{N}} \wedge \|f\|_{p_{\nu}^{\mu},r} = 0 \right\}.\end{aligned}$$

**Proposition 70.** (i) Writing  $\overleftrightarrow{\cdot}$  for both,  $\overleftarrow{\cdot}$  or  $\overrightarrow{\cdot}$ , we have that  $\overleftrightarrow{\mathcal{F}}_{p,r}$  is an algebra and  $\overleftrightarrow{\mathcal{K}}_{p,r}$  is an ideal of  $\overleftrightarrow{\mathcal{F}}_{p,r}$ ; thus,  $\overleftrightarrow{\mathcal{G}}_{p,r} := \overleftrightarrow{\mathcal{F}}_{p,r} / \overleftrightarrow{\mathcal{K}}_{p,r}$  is an algebra.

(ii) For every  $\mu, \nu \in \mathbb{N}$ ,  $d_{p_{\nu}^{\mu}} : (E_{\nu}^{\mu})^{\mathbb{N}} \times (E_{\nu}^{\mu})^{\mathbb{N}} \rightarrow \overline{\mathbb{R}}_{+}$  defined by  $d_{p_{\nu}^{\mu}}(f, g) = \|f - g\|_{p_{\nu}^{\mu},r}$  is an ultrapseudometric on  $(E_{\nu}^{\mu})^{\mathbb{N}}$ . Moreover  $(d_{p_{\nu}^{\mu}})_{\mu,\nu}$  induces a topological algebra structure on  $\overleftrightarrow{\mathcal{F}}_{p,r}$  (since  $d_{p_{\nu}^{\mu}}(0, f \cdot g) \leq d_{p_{\nu}^{\mu}}(0, f)d_{p_{\nu}^{\mu}}(0, g)$ ) such that the intersection of neighborhoods of zero equals  $\overleftrightarrow{\mathcal{K}}_{p,r}$ .

(iii) From (ii),  $\overleftrightarrow{\mathcal{G}}_{p,r} = \overleftrightarrow{\mathcal{F}}_{p,r} / \overleftrightarrow{\mathcal{K}}_{p,r}$  becomes a topological algebra which topology can be defined by the family of ultrametries  $(\tilde{d}_{p_{\nu}^{\mu}})_{\mu,\nu}$ , where  $\tilde{d}_{p_{\nu}^{\mu}}([f], [g]) = d_{p_{\nu}^{\mu}}(f, g)$ ,  $[h]$  stands for the class of  $h$ .

(iv) If  $\tau_{\mu}$  denotes the inductive limit topology on  $\mathcal{F}_{p,r}^{\mu} = \bigcup_{\nu \in \mathbb{N}} ((E_{\nu}^{\mu})^{\mathbb{N}}, d_{\mu,\nu})$ ,  $\mu \in \mathbb{N}$ , then  $\overrightarrow{\mathcal{F}}_{p,r}$  is a topological algebra for the projective limit topology of the family  $(\mathcal{F}_{p,r}^{\mu}, \tau_{\mu})_{\mu}$ .

$(\tilde{E}_{\nu}^{\mu})^{\mathbb{N}}$ , consists of elements  $f \in (E_{\nu}^{\mu})^{\mathbb{N}}$  with finite  $d_{\mu,\nu}(f)$ .

Without assuming completeness of  $E$ , it holds:

**Proposition 71.** (i)  $\overleftarrow{\mathcal{F}}_{p,r}$  is complete.

(ii) If for all  $\mu \in \mathbb{N}$ , a subset of  $\overrightarrow{\mathcal{F}}_{p,r}^{\mu}$  is bounded if and only if it is a bounded subset of  $(E_{\nu}^{\mu})^{\mathbb{N}}$  for some  $\nu \in \mathbb{N}$ , then  $\overrightarrow{\mathcal{F}}_{p,r}$  is sequentially complete.

**8.1.3. Comments on the Schwartz' impossibility result.** In the definition of sequence spaces  $\overrightarrow{\mathcal{F}}_{p,r}$  resp.  $\overleftarrow{\mathcal{F}}_{p,r}$ , we assumed  $r_n \searrow 0$  as  $n \rightarrow \infty$ . Clearly, one could consider sequence spaces of the same type with  $r_n$  only bounded, or even  $r_n \rightarrow \infty$ . In the former case ( $r_n$  bounded), the space  $\overleftrightarrow{\mathcal{F}}_{p,r}$  (where  $\overleftrightarrow{\cdot}$  stands for  $\overleftarrow{\cdot}$  or  $\overrightarrow{\cdot}$ ) contains  $*\text{diag } E^{\mathbb{N}}$  topology, via the embedding  $\overleftrightarrow{E} \ni f \mapsto (f)_n \in \overleftrightarrow{E}^{\mathbb{N}}$ . In the second case (when  $r_n \rightarrow \infty$ ), this embedding is not possible.

In the case we consider ( $r_n \rightarrow 0$ ), the induced topology on  $\overleftrightarrow{E}$  is a discrete topology. But this is necessarily so, since we want to include “divergent” sequences in  $\overleftrightarrow{\mathcal{F}}_{p,r}$ .

In order to have an appropriate topological algebra containing “ $\delta$ ”, we must have that our generalized topological algebra induces a discrete topology on the original algebra  $\overleftrightarrow{E}$ . This conclusion is in analogy to Schwartz' impossibility statement for multiplication of distributions.

**8.1.4. Sequences of scales.** We can consider a sequence  $(r^m)_m$  of positive sequences  $(r_n^m)_n$  such that

$$\forall m, n \in \mathbb{N} : r_{n+1}^m \leq r_n^m; \quad \lim_{n \rightarrow \infty} r_n^m = 0.$$

In addition to this, we request either of the following conditions:

$$\forall m, n \in \mathbb{N} : r_n^{m+1} \geq r_n^m \quad \text{or} \quad \forall m, n \in \mathbb{N} : r_n^{m+1} \leq r_n^m.$$

Then let, in the first resp. second case :

$$\begin{aligned} \overleftarrow{\mathcal{F}}_{p,r} &= \bigcap_{m \in \mathbb{N}} \overleftarrow{\mathcal{F}}_{p,r^m}, & \overleftarrow{\mathcal{K}}_{p,r} &= \bigcup_{m \in \mathbb{N}} \overleftarrow{\mathcal{K}}_{p,r^m} \\ \text{resp. } \overleftarrow{\mathcal{F}}_{p,r} &= \bigcup_{m \in \mathbb{N}} \overleftarrow{\mathcal{F}}_{p,r^m}, & \overleftarrow{\mathcal{K}}_{p,r} &= \bigcap_{m \in \mathbb{N}} \overleftarrow{\mathcal{K}}_{p,r^m} \end{aligned}$$

(where again  $p = (p_\nu)_{\nu, \mu}$ ). Then again,  $\overleftarrow{\mathcal{G}}_{p,r} := \overleftarrow{\mathcal{F}}_{p,r} / \overleftarrow{\mathcal{K}}_{p,r}$  will be an algebra.

The following example cover so called Egorov's type algebra.  $r_n^m = \begin{cases} 1, & \text{if } n \leq m \\ 0, & \text{if } n > m \end{cases}$  gives the Egorov-type algebras, where the "subalgebra" contains everything and the ideal contains only stationary null sequences.

**8.1.5. General remarks on embeddings of duals.** Under mild assumptions on  $\overleftarrow{E}$ , we can show that our algebras of (classes of) sequences contains elements of the strong dual space  $\overleftarrow{E}'$ . Let  $C^0(\mathbb{R}^s)$  be the space of continuous functions with projective topology given by sup norms on the balls of radius  $\nu \in \mathbb{N}^*$ ,  $p_\nu(f) = \sup\{|f(x)|; |x| \leq \nu\}$ . We shall assume in the sequel that  $\overleftarrow{E}$  is a dense subspace of  $C^0(\mathbb{R}^s)$  and the inclusion mapping  $\overleftarrow{E} \rightarrow C^0(\mathbb{R}^s)$  is continuous. Then, we have the following

**Proposition 72.** (i)  $\delta : \overleftarrow{E} \rightarrow \mathbb{C}$ ,  $\delta(\phi) := \phi(0)$  is an element of  $\overleftarrow{E}'$ .

(ii) Let  $\overleftarrow{E}$  be sequentially weakly dense in  $\overleftarrow{E}'$ . Then, a sequence  $(\delta_n)_n \in E \cap (C^0)'$  with the property  $\exists \eta, \theta > 0 : \forall n \in \mathbb{N} : \sup_{|x| > \theta} |\delta_n(x)| < \eta$ , converging weakly to  $\delta$ , cannot be bounded in  $\overleftarrow{E}$ .

Thus, the appropriate choice of the sequence  $r$  appeared to be important to have at least  $\delta$  embedded into the corresponding algebra. It can be chosen such that:

In  $\overleftarrow{E}$  case, for every  $\mu, \nu \in \mathbb{N}$

$$\limsup_{n \rightarrow \infty} p_\nu^\mu(\delta_n)^{r_n} = A_\nu^\mu \quad \text{and} \quad \exists \mu_0, \nu_0 : A_{\nu_0}^{\mu_0} \neq 0.$$

In  $\overrightarrow{E}$  case, for every  $\mu \in \mathbb{N}$  exists  $\nu \in \mathbb{N}$  such that the above limit holds.

**8.2. Association.** The notion of a weak limit or of a weak solutions is transferred to generalized function algebras to various notions of associations. Thus their importance is underlined through the applications to nonlinear equations or linear one with singularities.

General concept:  $\mathcal{J} - X$ -association. The  $\mathcal{J} - X$ -association of elements  $F, G \in \mathcal{G} = \mathcal{F}/\mathcal{K}$  is defined in terms of an additive subgroup  $\mathcal{J}$  of  $\mathcal{F}$  containing the ideal  $\mathcal{K}$ , and a set  $X$  of generalized numbers, by

$$F \underset{\mathcal{J}, X}{\approx} G \iff \forall x \in X : x \cdot (F - G) \in \mathcal{J}/\mathcal{K}.$$

As  $\mathcal{J}$  is not an ideal, the association is not compatible with the multiplication in  $\mathcal{F}$  (not even by generalized numbers, only by elements of  $E$ ). However, in the case of differential algebras,  $\mathcal{J}$  is usually chosen such that  $\approx$  is stable under differentiation. If the set  $X$  contains only number 1, then we simply write  $F \underset{\mathcal{J}}{\approx} G \iff F - G \in \mathcal{J}/\mathcal{K}$ .

For example, consider  $N = \{x \in \mathbb{C}^{\mathbb{N}} \mid \lim x_n = 0\}$ , the set of null sequences. This gives usual association of generalized numbers,

$$[x] \sim [y] \iff [x] \underset{N}{\approx} [y] \iff x_n - y_n \rightarrow 0$$

which is well defined because all elements of the ideal tend to zero.

Strong  $s$ -association. is defined for  $s \in \mathbb{R}_+$  by

$$F \overset{s}{\simeq} G \iff F \underset{\mathcal{J}_{\mathcal{P},r}^{(s)}}{\approx} G \text{ with } \mathcal{J}_{\mathcal{P},r}^{(s)} = \{f \in \mathcal{F} \mid \forall p \in \mathcal{P} : \|f\|_{p,r} < e^{-s}\}.$$

For  $s = 0$ , we write  $F \underset{\mathcal{J}^{(s)}}{\simeq} G$  and simply call them strongly associated. On the other hand,  $F \overset{s}{\simeq} G$  for all  $s \geq 0$  implies  $F = G$ .

Weak associations. The following types of associations are defined in terms of a duality product<sup>1</sup>  $\langle \cdot, \cdot \rangle : \overleftarrow{E} \times \mathbf{D} \rightarrow \mathbb{C}$ , and

$$\mathcal{J} = \mathcal{J}_M = \{f \in \overleftarrow{E}^{\mathbb{N}} \mid \forall \psi \in \mathbf{D} : (\langle f_n, \psi \rangle)_n \in M\}.$$

where  $M$  is some additive subgroup of  $\mathbb{C}^{\mathbb{N}}$ .

$s - \mathbf{D}'$ -association is defined by  $F \overset{s}{\simeq} G \iff F \underset{\mathcal{J}_{N, X_s}}{\approx} G$  with  $X_s = \{[(e^{s/r_n})_n]\}$  for  $s \in \mathbb{R}$ .

**Example 73.** In the case of Colombeau's algebra this has already been considered (with  $\mathbf{D} = \mathcal{D}$ ): For  $s = 0$  we get the so-called weak association  $[f] \approx [g] \iff f_n - g_n \rightarrow 0$  in  $\mathcal{D}'$ . For  $s \neq 0$ ,  $[f] \overset{s}{\simeq} [g] \iff n^s(f_n - g_n) \rightarrow 0$  in  $\mathcal{D}'$ . In the case of ultradistributions, we take  $\mathbf{D} = \mathcal{D}^{(m)}$  and  $e^{s/r_n} = \exp[s n^{\frac{1}{m'-1}}]$  for Beurling case, and analogous definitions in the Roumieu case.

*Weak  $s$ -association* is defined by  $F \overset{s}{\simeq} G \iff F \underset{\mathcal{J}_I}{\approx} G$  where  $I = \mathcal{J}_{|\cdot|,r,s}$  for any  $s \in \mathbb{R}$ . For  $s = 0$ , we write  $F \overset{sw}{\simeq} G$  and call  $F$  and  $G$  *strong-weak associated*.

**Remark 74.** Weak  $s$ -association implies  $s - \mathbf{D}'$ -association, but conversely  $s - \mathbf{D}'$ -association only implies weak  $s'$ -association with  $s' < s$ .

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<sup>1</sup> $\mathbf{D}$  stands for a test function space such that  $E \hookrightarrow \mathbf{D}'$ .

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