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GENERALIZED FUNCTIONS IN SOLVING LINEAR MATHEMATICAL MODELS IN MECHANICS

CONTENTS

0. Introduction

The aim of this paper is to consider the necessity of introducing the generalized functions for the construction and solving mathematical models.

Mathematical models in mechanics have been usually given by a partial differential equation with some boundary and initial conditions.

With regards to the construction of a mathematical model the following remarks are worthy of mention:

First we have to catch sight and then to select the basic elements of the situation (of the object) we wish to model. Consequently, a mathematical model is only an approximation of the object to which it corresponds.

Or to put in another, more pessimistic consideration: All models are wrong, some models are "useful" [30]. But there are several requirements that mathematical models must satisfy in order to be "useful". Structural stability of the model is probably the most important requirement. Also, because of the approximate value of a model, it is natural to expect that if we can find a family of solutions to the model equation and if there exists a subfamily which is convergent, then the limit has also to be a solution. The difficulty lies in finding a topology not overly restrictive but such that the found limit has a meaning for the treated object.

That is one of the sources of the "weak" and "generalized" solutions to mathematical models which will be used in this paper, as well.

Many authors have pointed at shortcomings of the classical analysis with regards to the solving partial differential equations. L. Hörmander [27] illustrated them by the equation of the vibrating string

$$\frac{\partial^2}{\partial x^2}\, v(x,t) - \frac{\partial^2}{\partial t^2}\, v(x,t) = 0.$$

Its classical solution has been given by $v(x,t) = f(x+t) + g(x-t)$, where f and g are arbitrary functions with continuous second derivatives. In his opinion the limits of sequences of such solutions have also to be taken as solutions (Laplace operator has just this property).

He continues with such a consideration for the nonhomogeneous equation

$$\frac{\partial^2}{\partial x^2}\, v(x,t) - \frac{\partial^2}{\partial t^2}\, v(x,t) = F(x,t),$$

where $F(x,t)$ is continuous and equals zero outside a bounded set. If F has also continuous first partial derivatives, then the cited nonhomogeneous equation has a classical solution

$$v(x,t) = -\frac{1}{2} \iint\limits_{\tau-t+|x-\xi|<0} F(\xi,\tau)\, d\xi\, d\tau.$$

In case that $F(x,t)$ is only continuous, the found solution $v(x,t)$ has continuous first partial derivatives and has to be admitted as solution, as well. Such solutions are called "weak solutions".

Secondly, partial differential equations have been given by partial derivatives which are very restrictive operations in usual topology in $\mathbb{R}^n$ (in classical analysis) and have not to be continuous. The first systematic elaborated idea to overcome these shortcomings of the classical derivatives has been given by S. L. Sobolev (cf. [57]). He started from the space $\mathbf{L}^p(\Omega)$, $p \geqslant 1$, where Ω is an open set in $\mathbb{R}^n$. Let $\varphi, \psi \in \mathcal{C}^m(\overline{\Omega})$, $\operatorname{supp}\psi = K$, K compact set in Ω. Then

$$\int\limits_{\Omega} \Big[\varphi(x) \frac{\partial^m \psi(x)}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} + (-1)^{m+1} \psi(x)\, \frac{\partial^m \varphi(x)}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}}\Big]\, dx = 0, \qquad m_1 + \cdots + m_n = m.$$

If we know only that $\varphi \in \mathbf{L}^p(\Omega)$, $p \geqslant 1$, and that there exists $\omega_{m_1,\dots,m_n} \in \mathbf{L}_{\mathrm{loc}}(\Omega)$ such that

$$\int\limits_{\Omega} \Big[\varphi(x) \frac{\partial^m \psi(x)}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} + (-1)^{m+1} \psi(x) \omega_{m_1,\dots,m_n}(x)\Big]\, dx = 0$$

for every ψ with the cited properties, then $\omega_{m_1,\dots,m_n}$ is defined as Sobolev's generalized derivative

$$\frac{\partial^m \varphi(x)}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} \overset{\text{def}}{=} \omega_{m_1,\dots,m_n}(x).$$

This is the basic idea for the theory of Sobolev's spaces which are very useful in the theory of partial differential equations.

Schwartz's distributions (cf. [56]) generalize Sobolev's idea and represent a theory which gives impressive results in the theory of partial differential equations. To every locally integrable function it corresponds in a unique way a distribution. Every distribution has all partial derivatives which are continuous operators. The space $\mathcal{D}'$ of distributions is the least extension of the space of continuous functions in which all elements have all partial derivatives. Moreover, derivatives are continuous operators. Consequently, if we have a convergent sequence or a convergent filter with the countable basis of the filter (cf. [56, I, p. 53]) as solution to a linear partial differential equation in $\mathcal{D}'$, then the limit of this sequence or of this filter is also a solution to this equation.

To this day many spaces of generalized functions have been elaborated (cf. [13], [18], [20], [24], [31], [32], [40], [47], [53], [56]) which can be useful in considering mathematical models. Not only to find a generalized solution to a model, but also to improve the classical methods for solving them. In this sense the integral transforms of generalized functions have an important role.

A very significant fact is that the spaces of generalized functions have not only been used to solve a mathematical models, but also in the construction of models.

Some elements and relations in the theoretical physics can be defined only by using generalized functions. Let us mention first of all the Dirac δ-"function". The quantum field theory is an impressive example of a theory which uses generalized functions to express some phenomena from physics (cf. [15], [16], [29], [65]).

The utility of mathematics for many problems of science and society is increasingly evident. However we can not neglect some doubt in this linking. Namely, mathematics pretends to claims of absolute certainty by means of mathematical proofs. But this certainty is paid for by logical disconnection from empirical reality. One can find cited the following Einstein sentence (cf. [12]): "As far as the properties of mathematics refer to reality, they are not certain and as far as they are certain, they do not refer to reality".

So in considerations mathematical models we have two extreme positions:

First, if a solution to the constructed mathematical model is not quite mathematically rigorous, but none the less leads to an excellent conformity with experimental observation, then one can consider such solutions valued by nature, if not by mathematics.

Second, one may choose to recognize mathematical models and their solutions if and only if the model is based on classical foundations and solutions have been obtained in absolute mathematical rigorousness.

In this paper we shall work with generalized solutions which are:

- well-defined;
- obtained in a mathematically correct way which allows to see why their introduction is necessary;
- solutions of linear mathematical models arising from mechanics and which claim can be validated by natural conditions;
- elements of spaces acceptable to the specialists working in mechanics.
- a pointer to the very abstract possibilities of the today's cutting–edge mathematics.

The paper is divided into three parts. In the first we repeat some definitions and results from spaces of generalized functions we need subsequently. In the second part we give constructions of some interesting new mathematical models in mechanics. In the third we solve the constructed models illustrating the possibilities of methods which have been offered by generalized functions in solving mathematical models in mechanics. We have not insisted on complete mathematical proofs if they were overly large and if they can be found in the published papers cited.

1. Spaces of generalized functions

In this paper we use the space of distributions $\mathcal{D}'$ with some subspaces and the space of hyperfunctions $\mathcal{B}$.

1.1. The space of distributions.

1.1.1. *Definitions and notation.* We repeat some definitions and facts that we need in our exposition. There are now a lot of books in which one can find spaces of distributions elaborated in different volumes. We cite only some, we use (cf. [24], [56], [66]). If the cited result is not well-known, then we give the proof, as well.

Let Ω denote an open subset of $\mathbb{R}^n$ (Ω can be $\mathbb{R}^n$ on the whole). The *support of a function* φ ($\operatorname{supp}\varphi$) defined on Ω is the closure in Ω of the set $\{x \in \Omega;\ \varphi(x) \neq 0\}$. The space $\mathcal{D}(\Omega)$ is the space $\{\varphi \in \mathcal{C}^\infty(\mathbb{R}^n);\ \operatorname{supp}\varphi \subset \Omega\}$. A sequence $\{\varphi_j\} \subset \mathcal{D}(\Omega)$ converges in $\mathcal{D}(\Omega)$ to zero if and only if there exists a compact set $K \subset \Omega$ such that:

1. $\operatorname{supp}\varphi_j \subset K$, $j \in \mathbb{N}$;
2. for every $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n \equiv \mathbb{N}_0^n$, $\varphi_j^{(\alpha)} \to 0$ uniformly on K;

$$\varphi_j^{(\alpha)} = \Big(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}\Big)\varphi_j.$$

$\mathcal{D}'(\Omega)$ is the space of all continuous linear functionals on $\mathcal{D}(\Omega)$. It is called *the space of distributions* on Ω. The value of a distribution f at a function $\varphi \in \mathcal{D}(\Omega)$ will be denoted by $\langle f, \varphi\rangle$.

Every locally integrable function f on Ω defines the *regular distribution* $[f]$, by $\langle [f], \varphi\rangle = \int_\Omega f(x)\varphi(x)\,dx$, $\varphi \in \mathcal{D}(\Omega)$. Two functions $f, g \in \mathbf{L}^1_{\mathrm{loc}}(\Omega)$ define the same distribution $[f] = [g]$ on Ω if and only if $f = g$ a.e. on Ω.

Suppose that $u_x \in \mathcal{D}'(\mathbb{R}^n)$, $v_y \in \mathcal{D}'(\mathbb{R}^m)$. By

$$\langle w, \varphi\rangle = \big\langle u_x, \langle v_y, \varphi(x, y)\rangle\big\rangle = \big\langle v_y, \langle u_x, \varphi(x, y)\rangle\big\rangle$$

is defined the distribution $w \in \mathcal{D}'(\mathbb{R}^{n+m})$, where $\varphi \in \mathcal{D}(\mathbb{R}^{n+m})$ and x, y denote variables in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. The distribution w is called *tensor product* of the distributions u_x and v_y; one writes $w = u_x \otimes v_y$.

Let u_x and v_y belong to $\mathcal{D}'(\mathbb{R}^n)$. If there exists a distribution $z \in \mathcal{D}'(\mathbb{R}^n)$ defined by $\langle z, \varphi\rangle = \langle u_x \otimes v_y,\ \varphi(x + y)\rangle$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then z is called the *convolution*of u_x and v_y and is denoted by $u_x * v_y$.

From the properties of convolution we mention only: if $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $u \in \mathcal{D}'(\mathbb{R}^n)$, then $\varphi * u \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $\varphi * u = u * \varphi = \langle u_x, \varphi(y - x)\rangle$.

Let $D^m u$ denote the m-th derivative in the sense of distributions (see Section 1.1.2), then $D^m\delta * u = D^m u$, $m = (m_1, \dots, m_n) \in \mathbb{N}_0^n$.

An important subspace of $\mathcal{D}'(\mathbb{R}^n)$ is the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$. Let us define it. By $\mathcal{S}(\mathbb{R}^n)$ we denote the *space of rapidly decreasing functions* φ with the property that for every pair of multi-indices $\alpha, \beta \in \mathbb{N}_0^n$, $\sup\limits_{x\in\mathbb{R}^n} |x^\alpha \varphi^{(\beta)}(x)| < \infty$.

The space of linear continuous functional on $\mathcal{S}(\mathbb{R}^n)$ is called the *space of tempered distributions* and is denoted by $\mathcal{S}'(\mathbb{R}^n)$. Let Γ denote the closed, convex and acute cone and $C = \operatorname{int}\Gamma$. Let K be a compact set in $\mathbb{R}^n$. By $\mathcal{S}'(\Gamma + K)$ is denoted the space of tempered distributions with supports in the closed set $\Gamma + K \subset \mathbb{R}^n$. Then $\mathcal{S}'(\Gamma+)$ is defined by

$$\mathcal{S}'(\Gamma+) = \bigcup_{K\subset\mathbb{R}^n} \mathcal{S}'(\Gamma + K).$$

The set $\mathcal{S}'(\Gamma+)$ forms an algebra that is associative and commutative if for the operation of multiplication one takes the convolution, denoted by $*$.

1.1.2. *Derivatives of a distribution.* Let D^{α_i} denote the α_i-th derivative in x_i of a distribution. It is defined as

$$\langle D^{\alpha_i} f, \varphi\rangle = \left\langle (-1)^{\alpha_i} f, \varphi^{(\alpha_i)}\right\rangle,\ \ f \in \mathcal{D}'(\Omega),\ \varphi \in \mathcal{D}(\Omega).$$

Then for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $D^\alpha f = D^{\alpha_1} \cdots D^{\alpha_n} f$.

We list some properties of the *derivatives of distributions*:

1. Every distribution has all derivatives D^{α_i} and $D^{\alpha_i} D^{\alpha_j} = D^{\alpha_j} D^{\alpha_i}$, $i, j = 1, \ldots, n$.

2. The differentiation of distributions is a linear and continuous mapping $\mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$.

3. In particular, every regular distribution has derivatives of any order. In this sense every locally integrable function has distributional derivatives. The derivative of a regular distribution has not to be regular distribution.

4. If $F \in \mathcal{C}^\alpha(\Omega)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, then $D^\alpha[F] = [F^{(\alpha)}]$. Moreover, if $a \in \mathcal{C}^\infty(\Omega)$, then $aD^\alpha[F] = [aF^{(\alpha)}]$.

5. If $F, G \subset \mathcal{C}(\Omega)$ and $D_{x_i}[F] = [G]$, then there exists $F^{(1)}_{x_i}$ and $F^{(1)}_{x_i} = G$, $i \in (1, \ldots, n)$.

6. Let η denote the function

$$\eta(x) = \begin{cases} 0, & |x| \geqslant 1, \\ \exp(|x|^2 - 1)^{-1}, & |x| < 1,\ |x|^2 = x_1^2 + \cdots + x_n^2 \end{cases}$$

and let $c = \int_{\mathbb{R}^n} f(x)\,dx$, $\omega_k(x) = c^{-1} k^n f(kx)$, $\Omega_{1/k} = \{x \in \mathbb{R}^n,\ d(x, \Omega) \leqslant 1/k\}$, $d(x, \Omega) = \inf_{y \in \Omega} |x - y|$. If $f, g \in \mathbf{L}^p(\Omega)$, $1 \leqslant p < \infty$ and $D^\alpha[f] = [g]$, then for $\Omega_1 \subset \Omega$ and $(\Omega_1)_{1/k} \subset \Omega$, $k \geqslant k_0$, $\|(f * \delta_n)^{(\alpha)} - g\|_{\mathbf{L}^p(\Omega_1)} \to 0$, $n \to \infty$, where $\delta_n = \omega_{1/n} \subset \mathcal{D}(\Omega)$ and $*$ is the sign of convolution.

7. Some properties which can be useful in solving differential and partial differential equations.

If $\Omega \subset \mathbb{R}$, $u \in \mathcal{D}'(\Omega)$ and $D^m u(x) + f_{m-1}(x) D^{m-1} u(x) + \cdots + f_0(x) u(x) = F(x)$, where $f_i \in \mathcal{C}^\infty(\Omega)$, $i = 0, 1, \ldots, m-1$, and $F \in \mathcal{C}^p(\Omega)$, $p \in \mathbb{N}_0$, then the solution u is defined by a function belonging to $\mathcal{C}^{m+p}(\Omega)$ and represents the classical solution.

Let $\{\delta_n\}_{n \in \mathbb{N}}$, $\operatorname{supp} \delta_n$, $n \in \mathbb{N}$, belong to the compact set $K \subset \mathbb{R}^n$ and $[\delta_n] \to \delta$ in $\mathcal{D}'(\mathbb{R}^n)$; let also L be a linear differential operator with constant coefficients. Then every solution $u \in \mathcal{D}'(\mathbb{R}^n)$ to $L(u) = 0$ is a limit in $\mathcal{D}'(\mathbb{R}^n)$ of a sequence $\{u_j\}_{j \in \mathbb{N}}$ of classical solutions to $L(u) = 0$. The sequence $\{u_j\}$ can be $u_j = u * \delta_j$ (cf. [64]).

8. Derivatives of a regular distribution

8.1. One dimensional case. Let $f \in \mathcal{C}^{(p)}((-\infty, b))$, $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and H_a be a function such that $H_a(x) = 0$, $-\infty < x < a < b$; $H_a(x) = 1$, $0 \leqslant a \leqslant x < b$. Denote by $[H_a f]$ the regular distribution defined by $H_a f$. Hence, $[H_a f] \in \mathcal{D}'((-\infty, b))$, $\operatorname{supp}[H_a f] \subset [a, b)$ or $[H_a f] \in \mathcal{D}'([a, b))$, as well as $\mathcal{D}'([a, b)) = \{T \in \mathcal{D}'(-\infty, b); \operatorname{supp} T \subset [a, b)\}$. By $[f_a^{(p)}]$, $p \in \mathbb{N}$, we denote the distribution defined by the function $f_a^{(p)}$ equals to $f^{(p)}(x)$, $x \in (a, b)$ and equals zero for $x \in (-\infty, a)$ and is not defined for $x = a$.

Since the function $(H_a f)^{(k)}$ has in general a discontinuity of the first kind in $x = a$, $k = 0, 1, \ldots, p$, by the well-known formula (cf. [56])

$$D^p[H_a f] = [f_a^{(p)}] + f^{(p-1)}(a)\delta(x-a) + \cdots + f(a)\delta^{(p-1)}(x-a)$$
$$= [f_a^{(p)}] + R_{p,a}(f) = [H_a f^{(p)}] + R_{p,a}(f), \tag{1.1}$$

where $D^p[H_a f]$ is the derivative of order p in the sense of distributions, and (cf. [56])

$$R_{p,a}(f) = f^{(p-1)}(a)\delta(x-a) + \cdots + f(a)\delta^{(p-1)}(x-a).$$

Definition 1.1. [60] Let α be a positive real number such that $m-1 < \alpha < m$ for a fixed $m \in N$. The α-th fractional derivative of a function $f \in \mathcal{C}([0,\infty))$ is defined by

$$f^{(\alpha)}(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^m}{dx^m} \int_0^x f(x-t)t^{m-1-\alpha}dt, \quad x > 0,$$

if this derivative exists.

Proposition 1.1. *Let α be a real number such that $0 < \alpha < 1$ and let $f \in \mathcal{C}((0,b))$, f bounded on $[0,\varepsilon]$, $\varepsilon > 0$, or, more generally, let $|f(x)| \leqslant Mx^{-(\beta-\alpha)}$, $0 < x < \varepsilon$, for an $\varepsilon > 0$, $0 < \alpha < \beta < 1$. Then:*

$$\left[f^{(\alpha)}(x)|_{(0,b)}\right] = \frac{1}{\Gamma(1-\alpha)} D_x \left[H_0(x) \int_0^x f(x-t)t^{-\alpha}dt\right].$$

Proof. By (1.1)

$$\left[f^{(\alpha)}(x)|_{(0,b)}\right] = \frac{1}{\Gamma(1-\alpha)} D_x \left[H_0(x) \int_0^x f(x-t)t^{-\alpha}dt\right]$$
$$- \frac{1}{\Gamma(1-\alpha)} \lim_{x\to 0^+} \int_0^x f(x-t)t^{-\alpha}dt$$
$$= \frac{1}{\Gamma(1-\alpha)} D_x \left[H_0(x) \int_0^x f(x-t)t^{-\alpha}dt\right]. \qquad \square$$

We have to prove that $\lim_{x\to 0} \int_0^x f(x-t)t^{-\alpha}dt = 0$. Let $x > 0$. Then we have

$$\int_0^x (x-t)^{-(\beta-\alpha)}t^{-\alpha}dt = \frac{2-\beta}{(1-(\beta-\alpha))x} \int_0^x t^{-\alpha}(x-t)^{1-(\beta-\alpha)}dt$$
$$= \frac{(2-\beta)x^{1-(\beta-\alpha)}}{(1-(\beta-\alpha))x} \int_0^x t^{-\alpha}\Big(1-\frac{t}{x}\Big)^{1-(\beta-\alpha)}dt.$$

Since $0 \leqslant t < x$, $\left|(1-t/x)^{1-(\beta-\alpha)}\right| < 1$,

$$\left|\int_0^x (x-t)^{-(\beta-\alpha)} t^{-\alpha} dt\right| \leqslant \frac{(2-\beta)x^{1-\beta}}{(1-(\beta-\alpha))(1-\alpha)} \to 0, \quad x \to 0^+.$$

We denote $\left[f^{(\alpha)}(x)|_{(0,b)}\right]$ by $D^\alpha[H_0 f]$.

8.2. The n-dimensional case. We use the following notation: $P = \prod_{i=1}^n [a_i, b_i)$, $0 \leqslant a_i < b_i$, $i = 1, \dots, n$; $\Omega = \overline{\mathbb{R}}_-^n + P$, then $P \subset \Omega$; $H_a^n(x) = H_{a_1}(x_1) \cdots H_{a_n}(x_n)$, $H_{a_i}(x_i) = 1$, $a_i \leqslant x_i < b_i$; $H_{a_i}(x_i) = 0$, $x_i < a_i$, $i = 1, \dots, n$. Let f be a function with continuous partial derivatives on Ω; $[H_a^n f]$ is the distribution, defined by $H_a^n f$, belonging to $\mathcal{D}'(\Omega)$ and to $\mathcal{D}'(P)$, as well. Finally, $\left(\partial^p f/\partial x_i^p\right)_{a_i}$ is the function equal to $\partial^p f/\partial x_i^p$ on the $\operatorname{int} P \cup \{x, x_j = a_j,\ j \neq i\}$, and equal to zero on $\Omega \setminus P$, but is not defined for $x_i = a_i$.

Proposition 1.2. *With the notation as above, we have*

$$D_{x_i}^p [H_a^n f] = \left[H_a^n \Big(\frac{\partial^p}{\partial x_i^p} f\Big)_{a_i}\right] + R_{p,a_i}(f), \quad p \in \mathbb{N}, \tag{1.2}$$

where

$$R_{p,a_i}(f) = \left[H_a^n \frac{\partial^{p-1}}{\partial x_i^{p-1}} f(x)|_{x_i=a_i}\right] \times \delta(x_i - a_i) + \cdots + \left[H_a^n f(x)|_{x_i=a_i}\right] \times \delta^{(p-1)}(x_i - a_i). \tag{1.3}$$

Proof. The method of the proof is just the same as for (1.1). □

Proposition 1.3. *With the notation as in Proposition* 1.2*, we have*

$$\begin{aligned} D_{x_j}^q D_{x_i}^p [H_a^n f] &= \left[H_a^n \frac{\partial^q}{\partial x_j^q}\Big(\Big(\frac{\partial^p}{\partial x_i^p} f\Big)_{a_i}\Big)_{a_j}\right] \\ &+ \left[H_a^n \frac{\partial^{q-1}}{\partial x_j^{q-1}}\Big(\frac{\partial^p}{\partial x_i^p} f\Big)_{a_i}(x)|_{x_j=a_j}\right] \times \delta(x_j - a_j) \\ &+ \left[H_a^n \Big(\frac{\partial^p}{\partial x_i^p} f\Big)_{a_i}(x)|_{x_j=a_j}\right] \times \delta^{(q-1)}(x_j - a_j) + D_{x_j}^q R_{p,a_i}(f). \end{aligned}$$

Proof. We have only to apply $D_{x_j}^q$ to (1.2). □

Remark. To realize $D_{x_j}^q R_{p,a_i}$ we have to use (1.3).

We illustrate the use of Proposition 1.3 by calculating the following expressions $D_{x_2} D_{x_1}[H_a^2 f]$, $D_{x_1}^2 D_{x_2}^2 [H_a^2 f]$ and $D_{x_1}^\alpha D_{x_2}^2 [H_a^2 f]$.

1) $D_{x_2} D_{x_1}[H_a^2 f]$. Let us start with the first derivatives.

$$D_{x_1}[H_a^2 f] = \left[H_a^2 \Big(\frac{\partial}{\partial x_1} f\Big)_{a_1}\right] + \delta(x_1 - a_1) \times [H_{a_2}(x_2) f(a_1, x_2)],$$

$$D_{x_2}[H_a^2 f] = \left[H_a^2 \Big(\frac{\partial}{\partial x_2} f\Big)_{a_2}\right] + [H_{a_1}(x_1) f(x_1, a_2)] \times \delta(x_2 - a_2),$$

$$D_{x_2}D_{x_1}[H_a^2 f] = D_{x_2}\Big[H_a^2\Big(\frac{\partial}{\partial x_1}f\Big)_{a_1}\Big] + \delta(x_1-a_1)\times D_{x_2}[H_{a_2}(x_2)f(a_1,x_2)]$$
$$= \Big[H_a^2\Big(\frac{\partial^2}{\partial x_1\partial x_1}f\Big)_{a_1,a_2}\Big] + D_{x_1}[H_{a_1}(x_1)f(x_1,a_2)]\times\delta(x_2-a_2)$$
$$-f(a_1,a_2)\delta(x_1-a_1)\times\delta(x_2-a_2) + \delta(x_1-a_1)\times D_{x_2}[H_{a_2}(x_2)f(a_1x_2)],$$

where $\Big(\frac{\partial^2}{\partial x_2\partial x_1}f\Big)_{a_1,a_2} = \frac{\partial^2}{\partial x_2\partial x_1}f(x,y),\ (x,y)\in(a_1,b_1)\times(a_2,b_2)$.

Remark. a) This formula is derived by supposing that:

$$\Big(\frac{\partial f}{\partial x_1}f\Big)_{a_1}(x_1,x_2)\Big|_{x_2=a_2} = \Big(\frac{\partial}{\partial x_1}f(x_1,a_2)\Big)_{a_1}$$
$$\Big(\frac{\partial f}{\partial x_2}f\Big)_{a_2}(x_1,x_2)\Big|_{x_1=a_2} = \Big(\frac{\partial}{\partial x_2}f(a_1,x_2)\Big)_{a_2}.$$

b) It follows that $D_{x_1}D_{x_2}[H_a^2 f] = D_{x_2}D_{x_1}[H_a^2 f]$.

2) $D_{x_1}^2 D_{x_2}^2[H_a^2 f]$. By a similar procedure as in 1) we have

$$D_{x_1}^2 D_{x_2}^2[H_a^2 f] = \Big[H_a^2\Big(\frac{\partial^4}{\partial x_1^2\partial x_2^2}f\Big)_{a_1,a_2}\Big] + D_{x_1}^2[H_{a_1}f(x_1,a_2)]\times\delta^{(1)}(x_2-a_2)$$
$$+\delta^{(1)}(x_1-a_1)\times D_{x_2}^2[H_{a_2}f(a_1,x_2)] + D_{x_1}^2\Big[H_{a_1}\frac{\partial}{\partial x_2}f(x_1,a_2)\Big]\times\delta(x_2-a_2)$$
$$+\delta(x_1-a_1)\times D_{x_2}^2\Big[H_{a_2}\frac{\partial}{\partial x_1}f(a_1,x_2)\Big] - f(a_1,a_2)\big(\delta^{(1)}(x_1-a_1)\times\delta^{(1)}(x_2-a_2)\big)$$
$$-\frac{\partial}{\partial x_2}f(a_1,a_2)\big(\delta^{(1)}(x_1-a_1)\times\delta(x_2-a_2)\big) - \frac{\partial}{\partial x_1}f(a_1,a_2)\big(\delta(x_1-a_1)\times\delta^{(1)}(x_2-a_2)\big)$$
$$-\frac{\partial^2}{\partial x_1\partial x_2}f(a_1,a_2)\big(\delta(x_1-a_1)\times\delta(x_2-a_2)\big).$$

3) $D_{x_1}^\alpha D_{x_2}^2[H_a^2 f]$, $a_1=0$, $b_1=\infty$ and θ is Heaviside's function.

$$\Big[D_{x_1}^\alpha\Big(\frac{\partial^2}{\partial x_2^2}f\Big)_{a_2}\Big] = \frac{1}{\Gamma(1-\alpha)}D_{x_1}\Big[H_a^2\Big(\Big(\frac{\partial^2}{\partial x_2^2}f\Big)_{a_2} *_{x_1}\theta(x_1)x_1^{-\alpha}\Big)\Big]$$
$$= \frac{1}{\Gamma(1-\alpha)}D_{x_1}D_{x_2}^2\big[\big(H_a^2 f *_{x_1}\theta(x_1)x_1^{-\alpha}\big)\big]$$
$$-\frac{1}{\Gamma(1-\alpha)}D_{x_1}\Big[\Big(H_a^2\frac{\partial}{\partial x_2}f(x_1,x_2)\big|_{x_2=a_2} *_{x_1}\theta(x_1)x_1^{-\alpha}\Big)\Big]\times\delta(x_2)$$
$$-\frac{1}{\Gamma(1-\alpha)}D_{x_1}\big[\big(H_a^2 f(x_1,x_2)\big|_{x_2=a_2} *_{x_1}\theta(x_1)x_1^{-\alpha}\big)\big]\times\delta^{(1)}(x_2).$$

9. If $u,v\in\mathcal{D}'(\mathbb{R}^n)$ and $u*v$ exists, then for $m=(m_1,\ldots,m_n)\in\mathbb{N}_0^n$ one has $D^m(u*v) = (D^m u*v) = (u*D^m v)$.

1.1.3. *The convergence of a sequence of distributions.* A sequence $\{u_n\}_{n\in\mathbb{N}} \in \mathcal{D}'(\Omega)$ is called convergent to $u \in \mathcal{D}'(\Omega)$ if for every $\varphi \in \mathcal{D}(\Omega)$ the limit $\lim_{n\to\infty} \langle u_n, \varphi\rangle = \langle u, \varphi\rangle$ exists and is finite.

If the sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbf{L}^1_{\mathrm{loc}}(\Omega)$ converges in $\mathbf{L}^1_{\mathrm{loc}}(\Omega)$ to zero, then the sequence $\{[u_n]\}_{n\in\mathbb{N}} \subset \mathcal{D}'(\Omega)$ converges to zero in $\mathcal{D}'(\Omega)$, as well. In particular, the space $\mathcal{C}(\Omega)$ can replace $\mathbf{L}^1_{\mathrm{loc}}(\Omega)$ in this statement (cf. [56]).

1.1.4. *Distributional-valued functions.* Let $\Omega_x \subset \mathbb{R}^n$ and $\Omega_t \subset \mathbb{R}^m$ be open sets. We define the function w on Ω_x with values in $\mathcal{D}'(\Omega_t)$; $w : \Omega_x \ni x \to w(x) \in \mathcal{D}'(\Omega_t)$. Such a function w is called *distributional-valued function.* A distributional-valued function w defined on $\Omega_x \subset \mathbb{R}$ is of the class $\mathcal{C}^1$ if the limit

$$\lim_{\varepsilon\to 0}\Big\langle \frac{1}{\varepsilon}\big((w(x+\varepsilon\xi) - w(x)), \varphi\big)\Big\rangle$$

exists for every $\varphi \in \mathcal{D}(\Omega_t)$ where x and $x + \varepsilon\xi$ belong to Ω_x, i.e.

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\big(w(x+\varepsilon\xi) - w(x)\big) \text{ exists in } \mathcal{D}'(\Omega_t).$$

We put by definition that in $\mathcal{D}'(\Omega_t)$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\big(w(x+\varepsilon\xi) - w(x)\big) = w^{(1)}_{x_j}(x).$$

Repeating p times this procedure, we obtain the *distributional-valued function of class* $\mathcal{C}^p$ (cf. [64]).

1.1.5. *The Laplace transform of distributions.* To define the *Laplace transform* (in short LT) of distributions we start with the Laplace transform of tempered distributions. The notion and definitions we will use were given in 1.1.

If $\Gamma + K$ is convex, as it will be in our case, then the LT of $f \in \mathcal{S}'(\Gamma+)$ is defined by

$$\hat{f}(z) = \mathcal{L}(f)(z) = \langle f(t), e^{-zt}\rangle,\ z \in C + i\mathbb{R}^n,$$

where $t = (t_1, \dots, t_n)$, $z = (z_1, \dots, z_n)$, $zt = z_1t_1 + \dots + z_nt_n$ and $C = \operatorname{int}\Gamma$. It is one to one operation.

For the properties of so defined LT one can consult [66]. We shall cite only those used in the sequel:

1) $\mathcal{L}\Big(\dfrac{\partial^m}{\partial t_i^m} f\Big)(z) = (z_i)^m \mathcal{L}(f)(z)$.
2) If $f \in \mathcal{S}'(\Gamma+)$ and $g \in \mathcal{S}'(\Gamma+)$, then $\mathcal{L}(f \times g)(z, s) = \mathcal{L}(f)(z)\mathcal{L}(g)(s)$, $z \in C + i\mathbb{R}^n$, $s \in C + i\mathbb{R}^n$.
3) If $f, g \in \mathcal{S}'(\Gamma+)$, then $f * g \in \mathcal{S}'(\Gamma+)$ and $\mathcal{L}(f * g)(z) = \mathcal{L}(f)(z)\mathcal{L}(g)(z)$, $z \in C + i\mathbb{R}^n$.
4) If $f \in L_{\mathrm{loc}}([0,\infty))$ and is bounded in a neighborhood of zero, $0 < \beta < 1$, $n = 1$, then $\mathcal{L}(f^{(\beta)})(z) = z^\beta \mathcal{L}(f)(z)$.
5) $\mathcal{L}(\delta(t - t_0))(z) = e^{-zt_0}$.
6) $\mathcal{L}(f)(z + a) = \mathcal{L}(e^{-at} f)(z)$, $\operatorname{Re} a > 0$.

7) If $f \in \mathbf{L}_{\rm loc}(\mathbb{R}^n_+)$ and $|f(x)| \leqslant Me^{qx}$, $x \geqslant x_0 > 0$, then $f(x)e^{-qx} \in \mathcal{S}'(\overline{\mathbb{R}}^n_+)$ and

$$\int\limits_{\mathbb{R}^n_+} e^{-(z+q)t} f(t)\,dt = \int\limits_{\mathbb{R}^n_+} e^{-zt}e^{-qt} f(t)\,dt = \mathcal{L}(e^{-qt}f)(z).$$

Let $\mathcal{H}_a^{(\alpha,\beta)}(C)$, $\alpha \geqslant 0$, $\beta \geqslant 0$, $a \geqslant 0$, denote the sets of holomorphic functions on $C + i\mathbb{R}^n$ which satisfy the following growth condition:

$$|f(z)| \leqslant Me^{a|x|}(1+|z|^2)^{\alpha/2}(1+d^{-\beta}(x,\partial C)), \quad z = x+iy \in C + i\mathbb{R}^n,$$

where ∂C is the boundary of C and $d(x, \partial C)$ is the distance between x and ∂C. We set

$$\mathcal{H}_a(C) = \bigcup_{\alpha,\beta \geqslant 0} \mathcal{H}_a^{(\alpha,\beta)}(C) \quad \text{and} \quad \mathcal{H}_+(C) = \bigcup_{a \geqslant 0} \mathcal{H}_a(C).$$

Proposition 1.4. [66, p. 191] *The algebras $H_+(C)$ and $S'(C^*+)$ and also their subalgebras $H_0(C)$ and $S'(C^*)$ are isomorphic. This isomorphism is accomplished via the LT.* ($C^* = \{t \in R^n;\ tx = t_1x_1 + \cdots + t_nx_n \geqslant 0,\ \forall x \in C\}$).

A property of the defined LT which can be used in a practical way is the following:

– Let P be the set $\prod_{i=1}^n [a_i, b_i)$, $0 \leqslant a_i < b_i$, $i = 1, \ldots, n$. Then $\overline{P}$ is compact. Since $\overline{\mathbb{R}}^n_+$ is a closed convex and acute cone, $\mathcal{S}'(\overline{\mathbb{R}}^n_+ + \overline{P})$ is well defined (see 1.1.1).

– Let $f \in \mathcal{S}'(\overline{\mathbb{R}}^n_+ + \overline{P})$. The LT of f, $\mathcal{L}(f)$, can be obtained by subsequent applications of the LT-s $\mathcal{L}_1(f), \ldots, \mathcal{L}_n(f)$, $\mathcal{L}(f) = \mathcal{L}_1(f) \circ \cdots \circ \mathcal{L}_n(f)$.

– If $\sigma \geqslant 0$, $f \in \mathcal{S}'(C^*+)$ and $g = e^{\sigma t} f$, then by definition $\mathcal{L}(g)(s) = \langle f(t), e^{-(s-\sigma)t}\rangle$, $\operatorname{Re} s > \sigma$.

– Let $F(s)$ be a function holomorphic for $\operatorname{Re} s > \sigma$. The function $F(\xi + \sigma)$ is holomorphic for $\operatorname{Re}\xi > 0$. If $F(\xi+\sigma) \in \mathcal{H}(\mathbb{R}_+)$, then there exists $f \in \mathcal{S}'(\overline{\mathbb{R}_+})$ such that $\mathcal{L}(e^{\sigma t} f)(s) = F(s)$.

H. Komatsu defined the Laplace transform for any hyperfunction (cf. [33]). The same idea we use to define the Laplace transform for a large class of distributions.

Let $\mathcal{A}$ be the vector space:

$$\mathcal{A} = \big\{T \in e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+ + \overline{P});\ \operatorname{supp} T \subset \{(\overline{\mathbb{R}}^n_+ + \overline{P}) \smallsetminus P\}\big\},\ \omega \in \mathbb{R},$$

where $e^{\omega t} = e^{\omega t_1} \cdots e^{\omega t_n}$. $\mathcal{A}$ is a subspace of $e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+ + \overline{P})$. We can define an equivalence relation in $e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+ + \overline{P})$ by $f \sim g \Longleftrightarrow f - g \in \mathcal{A}$. Let $\mathcal{B}$ denote

$$\mathcal{B} = e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+ + \overline{P})/\mathcal{A}, \quad b \in \mathcal{B} \Longleftrightarrow b = \operatorname{class}(T) \equiv cl(T), \quad T \in e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+ + \overline{P}).$$

Definition 1.2. [60] Let $\mathcal{D}'(P)$ denote the space of distributions defined on P. Then

$$\mathcal{D}'_\omega(P) = \big\{T \in \mathcal{D}'(P); \exists \overline{T} \in e^{\omega t}\mathcal{S}'\big((\overline{\mathbb{R}}^n_+ + \overline{P})\big), \overline{T}|_P = T\big\},$$

where $\overline{T}|_P$ is the restriction of $\overline{T}$ on P. Since $\mathcal{D}'$ is not a flabby sheaf, $\mathcal{D}'_\omega(P) \neq \mathcal{D}'(P)$.

Proposition 1.5. *$\mathcal{D}'_\omega(P)$ is algebraically isomorphic to $\mathcal{B}$.*

Proof. If $T \in \mathcal{D}'_\omega(P)$, then there exists $\overline{T} \in e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+ + \overline{P})$ such that $\overline{T}|_P = T$. We can define the mapping $\lambda : \mathcal{D}'_\omega(P) \to \mathcal{B}$, for $T \in \mathcal{D}'_\omega(P)$, $\lambda(T) = cl(\overline{T}) \in \mathcal{B}$. The inverse mapping λ^{-1} exists and $\lambda^{-1}(cl(\overline{T})) = \overline{T}|_P = T \in \mathcal{D}'(P)$. T does not depend on the chosen element from $cl(\overline{T})$. If we take an other representative T_1 of the $cl(\overline{T})$, then $T_1 = \overline{T} + S$, $S \in \mathcal{A}$. Then $T_1|_P = \overline{T}|_P$. Now it is easily seen that λ is an algebraic isomorphism of two vector spaces. □

Definition 1.3. The LT of elements in $D'_\omega(P)$ is defined by

$$\mathcal{L}(\mathcal{D}'_\omega(P)) = \mathcal{L}(e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+ + \overline{P}))/\mathcal{L}(\mathcal{A}).$$

If $T \in D'_\omega(P)$, then $L(T) = cl(L\overline{T})$, where $\overline{T}$ is such that $\overline{T}|_P = T$.

Remark. Let H_P be the function $H_P(t) = 1$, $t \in P$, $H(t) = 0$, $t \in \mathbb{R}^n \setminus P$. Then:

a) If $f \in \mathbf{L}_{\text{loc}}(\overline{\mathbb{R}}_+)$, then the regular distribution $[H_P f]$ defined by $H_P f$ belongs to $\mathcal{D}'_\omega(P)$ and f has the LT in the sense of Definition 1.3.

b) If $f \in e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}_+ + \overline{P}_+)$ and $g \in \mathcal{A}$, then $f * g \in \mathcal{A}$, as well.

1.1.6. *Extension of a distribution.* We know that there exist distributions defined on an open set Ω which can not be extended to an open set $\Omega_1 \supset \overline{\Omega}$. This is a consequence that $\mathcal{D}'$ is not a flabby shief. There are theorems which give the conditions for the extendability. We cite one such theorem we use in the sequel:

Proposition 1.6. [64] *Let T be a distribution on a bounded open set $\Omega \subset \mathbb{R}^n$ and let $\Omega_1 \supset \overline{\Omega}$. Then T is extendable to Ω_1 if and only if there exist constants C and $k \in N_0$ satisfying $|\langle T, \varphi\rangle| \leqslant C \sum_{|\alpha|\leqslant k} \lim_{x\in\mathbb{R}^n} |\varphi^{(\alpha)}(x)|$ for $\varphi \in \mathcal{D}(\Omega)$.*

1.2. The space of hyperfunctions.

1.2.1. *Notation and definitions.* The space of hyperfunctions was introduced by M. Sato (cf. [52], [53]) in 1958. By H. Komatsu's opinion ([32]), the idea of hyperfunctions has been employed most successfully since a long time ago. He cited some examples from mathematics and physics, to prove it.

The theory of hyperfunctions in many variables calls for deep results in algebraic topology (cf. [32], [53]). But if one restricts oneself to the one dimensional case, this theory is of easier access. Fortunately we need only this theory of one variable.

Let Ω be an open set in $\mathbb{R}$ and V an open set in $\mathbb{C}$ containing Ω as a relatively closed set (Ω is a closed subset of V). Let $\mathcal{O}(V)$ denote the space of *holomorphic functions* on V. Then *hyperfunctions* on Ω are by definition the elements in the quotient space $\mathcal{B}(\Omega) = \mathcal{O}(V \setminus \Omega)/\mathcal{O}(V)$. If $F \in \mathcal{O}(V \setminus \Omega)$, then we denote by $[F]$ the class of F; F is called a *defining function* of the hyperfunction $[F]$.

The definition of $\mathcal{B}(\Omega)$ does not depend on the choice of the complex neighborhood of V.

$\mathcal{B}$ is a flabby sheaf. Consequently, if Ω_1 is an open subset of Ω, then any hyperfunction $f \in \mathcal{B}(\Omega_1)$ can be extended to an $\tilde{f} \in \mathcal{B}(\Omega)$. This is a very important property of $\mathcal{B}$. Distributions have not this property. That is the reason for Definition 1.2.

$\mathcal{B}(\Omega)$ contains $\mathcal{C}(\Omega)$, $\mathbf{L}_{\mathrm{loc}}(\Omega)$, $\mathcal{D}'(\Omega)$, the space of real analytic functions on Ω, ultradistributions on $\Omega \ldots$. One can find in [32] what conditions has to satisfy the defining function F of an hyperfunction $f = [F]$ so that f belongs to some subspaces of $\mathcal{B}(\Omega)$.

Let $\Omega = (-\infty, b)$ and $-\infty < a < b$; then the space of hyperfunctions with support in $[a, b)$ is $\mathcal{B}_{[a,b)} = \mathcal{O}\big(\mathbb{C}_{x<b} \smallsetminus [a,b)\big)/\mathcal{O}(\mathbb{C}_{x<b})$, where $\mathbb{C}_{x<b} = \{z \in \mathbb{C};\ \operatorname{Re} z < b\}$.

1.2.2. *The space of Laplace hyperfunctions and their Laplace transform.* Let $\mathbf{O}$ be the radial compactification of the complex plane and V an open set in $\mathbf{O}$. $\mathcal{O}^{\exp}(V)$ the space of functions F on V such that F is holomorphic on $\mathbb{C} \cap V$ and on each compact set $K \subset V$, $|F(z)| \leqslant Ce^{H|z|}$, $z \in K \cap \mathbb{C}$, with constants H and C. The space $\mathcal{B}^{\exp}_{[a,\infty]}$ of Laplace hyperfunctions with support in $[a,\infty]$ is defined by

$$\mathcal{B}^{\exp}_{[a,\infty]} = \mathcal{O}^{\exp}\big(\mathbf{O} \smallsetminus [a,\infty]\big)/\mathcal{O}^{\exp}(\mathbf{O}).$$

An $f \in \mathcal{B}^{\exp}_{[a,\infty]}$ is represented by $F \in \mathcal{O}^{\exp}\big(\mathbf{O} \smallsetminus [a,\infty]\big)$, $f = [F] = \{F + G;\ G \in \mathcal{O}^{\exp}(\mathbf{O})\}$. The *Laplace transform* $\tilde{\mathcal{L}}f(\xi)$ of an $f = [F] \in \mathcal{B}^{\exp}_{[a,\infty]}$ is defined by

$$\tilde{\mathcal{L}}f(\xi) = \int_L e^{-\xi z} F(z)\, dz \in \tilde{\mathcal{L}}\mathcal{B}^{\exp}_{[a,\infty]},$$

where L is a path composed of a ray from $e^{i\alpha}_\infty$ to a point $c < a$ and a ray from c to $e^{i\beta}_\infty$ with $-\pi/2 < \alpha < \beta < \pi/2$.

Proposition 1.7. [33] *The Laplace transformation $\tilde{\mathcal{L}}$ is an isomorphism $B^{\exp}_{[a,\infty]} \to \tilde{\mathcal{L}}B^{\exp}_{[a,\infty]}$, where $\tilde{\mathcal{L}}B^{\exp}_{[a,\infty]}$ is the space of all holomorphic functions $\hat{f}(\xi)$ of exponential type defined on a neighborhood Ω of the semi-circle $S = \{e^{i\gamma}_\infty;\ |\gamma| < \pi/2\}$ in O such that*

$$\overline{\lim_{\rho\to\infty}} \frac{1}{\rho} \log|\hat{f}(\rho e^{i\gamma})| \leqslant -a\cos\gamma, \quad |\gamma| < \pi/2. \tag{1.4}$$

If $\hat{f}(\xi) \in \tilde{\mathcal{L}}B^{\exp}_{[a,\infty]}$, then a defining function $F(z)$ of its inverse image f is given by the integral

$$F(z) = \frac{1}{2\pi i}\int_{s_0}^{\infty} e^{\xi z}\hat{f}(\xi)\, d\xi,$$

where s_0 is a fixed point in Ω and the integral part is a convex curve in Ω.

The restriction mapping $\mathcal{O}^{\exp}\big(\mathbf{O} \smallsetminus [a,\infty]\big) \to \mathcal{O}(\mathbb{C}_{x<b} \smallsetminus [a,b))$ induces a natural mapping $\omega : \mathcal{B}^{\exp}_{[a,\infty]} \to \mathcal{B}_{[a,b)}$ which is surjective, but not injective. It has been proved (cf. [33]) that ω is surjective and

$$\mathcal{B}_{[a,b)} \cong \mathcal{B}^{\exp}_{[a,\infty]}/\mathcal{B}^{\exp}_{[b,\infty]}.$$

Consequently,

$$\mathcal{L}\mathcal{B}_{[a,b)} \cong \mathcal{L}\mathcal{B}^{\exp}_{[a,b]}/\mathcal{L}\mathcal{B}^{\exp}_{[b,\infty]}. \tag{1.5}$$

If $g \in \mathcal{B}_{[a,\infty)}$, then $\tilde{\mathcal{L}}g = [\tilde{\mathcal{L}}\tilde{g}] = \big\{\tilde{\mathcal{L}}\tilde{g} + \tilde{\mathcal{L}}h;\ h \in \mathcal{B}^{\exp}_{[b,\infty]}\big\}$, $\tilde{g} \in \mathcal{B}^{\exp}_{[a,\infty]}$, $\tilde{g} \in \omega^{-1}(g)$.

Let $\mathbf{L}_{\mathrm{loc}}^{\exp}([0,\infty))$ denote the space of locally integrable functions q on $[0,\infty)$ satisfying $|q(x)| \leqslant Ce^{H_x}$, $H \in \mathbb{R}$, $x \geqslant 0$. We write $\overline{\theta q}$ for the element in $\mathcal{B}_{[0,\infty)}$ which corresponds to q. Then the classical Laplace transform of q, $\mathcal{L}q(s) = \int_0^\infty e^{-st}q(t)\,dt$, belongs to $\tilde{\mathcal{L}}\mathcal{B}_{[0,\infty]}^{\exp}$ (by Proposition 2.1) and $\widetilde{\theta q}$ may be regarded as the Laplace hyperfunction for which $\tilde{\mathcal{L}}\widetilde{\theta q}(\xi) = \mathcal{L}q(\xi)$; $\widetilde{\theta q}$ is an extension of $\overline{\theta q}$ on $[0,\infty]$. Here θ stands for the Heaviside function.

The delta distribution δ imbedded in $\mathcal{B}_{[0,\infty]}^{\exp}$ is $\delta = \left[-\dfrac{1}{2\pi i}\,\dfrac{1}{z}\right]$ and with the notation

$$\delta^{(\alpha)}(x) = \begin{cases} D^\alpha\delta(x), & \alpha = 0,1,\dots \\ x_+^{-\alpha-1}/\Gamma(-\alpha), & \alpha \neq 0,1,\dots,\ \alpha \in \mathbb{R}_+ \end{cases}$$

where $x_+^{-\alpha-1}$, $\alpha > 0$ is the distribution with support in $[0,\infty)$ and $x_+^{-\alpha-1} = x^{-\alpha-1}$, $x > 0$, then $\tilde{\mathcal{L}}\delta^{(\alpha)}(\xi) = \xi^\alpha$, $\alpha \in \mathbb{R}_+ \cup \{0\}$ and $\tilde{\mathcal{L}}(\delta(x-x_0))(\xi) = e^{-x_0\xi}$, $x_0 > 0$. If $f, g \in \mathbf{L}_{\mathrm{loc}}([0,\infty))$, then $\overline{\theta f} * \overline{\theta g} = (\widetilde{\theta f}) * (\widetilde{\theta g})\big|_{(-\infty,\infty)} = \overline{\theta(f*g)}$. Here $f*g = \int_0^t f(t-\tau)g(\tau)\,d\tau$ is the convolution.

1.2.3. *Final comments.* Let us remark the following facts which concern the hyperfunctions:

1. It is a very large space containing the most interesting functions and generalized function spaces.

2. The Laplace transform of hyperfunctions is defined by (1.5), for hyperfunctions having an arbitrary growth order. Specially, every locally integrable function on $[0,\infty)$ has the Laplace transform in this sense.

3. The space $\tilde{\mathcal{L}}\mathcal{B}_{[a,\infty]}^{\exp}$ has been characterized by (1.4).

4. The Laplace transform $\tilde{\mathcal{L}}$ is a generalization of the classical one. If $f \in \mathbf{L}_{\mathrm{loc}}([0,\infty))$ and has a classical Laplace transform $\hat{f}(s)$, then it has $\tilde{\mathcal{L}}f(s)$ and $\tilde{\mathcal{L}}f(s) = \hat{f}(s)$.

5. The properties of $\tilde{\mathcal{L}}$ are the same as the properties of the Laplace transform of tempered distributions cited in Section 1.1.5.

At the end we mention that H. Komatsu extended the theory of Laplace hyperfunctions to the hyperfunctions having values in a complex Banach space (cf. [36]) and applied it to find solutions to some partial differential equations using the theory of semigroups.

2. Mathematical models of some elastic and viscoelastic rods

In this Section we shall derive equations corresponding to lateral motion of elastic and viscoelastic rods with different boundary conditions, which will be treated in Section 3 or are treated in some of our papers listed in the References.

2.1. Elastic axially loaded rod. Consider a rod shown in Figure 1. We shall consider in-plane motion of the rod. Let $\bar{x} - B - \bar{y}$ be a rectangular Cartesian coordinate system with the origin fixed at the point B of the rod.

The rod is simply supported at end B and connected to the moving support at end C. At the end C the rod is loaded by an axial *force* $\mathbf{F}$ having fixed direction and

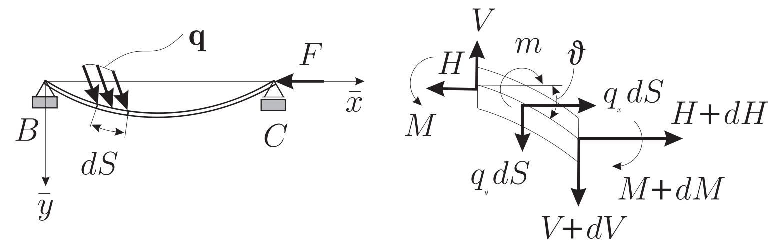

FIGURE 1. Coordinate system and load configuration

of intensity $F(t)$ that may be a function of time. Also the rod is loaded by *distributed forces* of intensity $\mathbf{q}(S,t)$ per unit length of the rod axis in the undeformed state. We further assume that the rod is initial straight and that its length is L. Let S be the arc length of the rod axis. We consider an element of the rod of length dS in the undeformed state. In the deformed state the length of this element is ds, so that

$$\varepsilon = \frac{ds - dS}{dS}, \tag{2.1}$$

is the *strain of the rod axis*. We shall use S as the independent space variable, so that $S \in [0, L]$. In an arbitrary section of the rod the *contact force* $\mathbf{Q}$ and the *contact couple* M represent the influence of the part $[0, S)$ on the part $(S, L]$ of the rod. Let $\mathbf{Q} = H\mathbf{e}_1 + V\mathbf{e}_2$ and $\mathbf{q}(S,t) = q_x\mathbf{e}_1 + q_y\mathbf{e}_2$ where $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors along the $\bar{x}$ and $\bar{y}$ axis, respectively. Then, the *equilibrium equations*, written in the deformed configuration, for and element of the rod of the length dS in the undeformed state read

$$\frac{\partial H}{\partial S} = -q_x, \quad \frac{\partial V}{\partial S} = -q_y, \quad \frac{\partial M}{\partial S} = -V\frac{\partial x}{\partial S} + H\frac{\partial y}{\partial S} - m, \tag{2.2}$$

where m denotes the intensity of the distributed couples along the length of the rod. To equations (2.2) we adjoin the following *geometrical conditions*

$$\frac{\partial x}{\partial S} = (1+\varepsilon)\cos\vartheta, \qquad \frac{\partial y}{\partial S} = (1+\varepsilon)\sin\vartheta,$$

where ϑ is the angle between the tangent to the rod axis at an arbitrary cross section and $\bar{x}$ axis.

Next we formulate the constitutive equations. We neglect the influence of the shear stresses so that the cross section of the rod that is orthogonal to the rod axis in the undeformed state is orthogonal in the deformed state too (for more general rod theories that take into account the influence of the shear stresses see [2], for example). Then, the *strain measures* are $\partial\vartheta/\partial S$ and ε. Note that $\partial\vartheta/\partial S$ is *not* the curvature κ of the rod axis in the deformed state. Indeed, let s be the arc length of the rod axis in the deformed state. Then the curvature is $\kappa = \partial\vartheta/\partial s$, so that

$$\frac{\partial\vartheta}{\partial S} = \frac{\partial\vartheta}{\partial s}(1+\varepsilon) = \kappa(1+\varepsilon),$$

where we used (2.1). We treat materially linear rod so that the contact couple is proportional to $\partial\vartheta/\partial S$ and the strain of the rod axis ε is proportional to the component of the contact force in the direction normal to the cross section in the deformed state (i.e., in the direction of the tangent to the rod axis). Let $\mathbf{t} = \cos\vartheta\mathbf{e}_1 + \sin\vartheta\mathbf{e}_2$ be a unit tangent to the rod axis. Then the component of $\mathbf{Q}$ in the direction of $\mathbf{t}$ is $N = V\cos\vartheta + H\sin\vartheta$. With this, the constitutive equations that we use are

$$M = EI\frac{d\vartheta}{dS}; \quad \varepsilon = \frac{N}{EA} = \frac{V\cos\vartheta + H\sin\vartheta}{EA}. \tag{2.3}$$

In (2.3) E is the *modulus of elasticity* of the material of the rod, A is the cross-sectional area and $I = \int_A \eta^2 dA$ is the *second moment of inertia* of the cross section with respect to the principal axes of the cross section passing through the center of gravity. The constants EI and EA are called bending and extensional rigidity of the rod, respectively. We note that for the case of a rod with variable cross section both EI and EA become functions of the arc length S. The constitutive equations (2.3) were given by Pflüger [45]. Note however that $(2.3)_2$ does not have the important property that $N \to \infty$ as $\varepsilon \to -1$. Thus, $(2.3)_2$ is valid only for $\varepsilon > -1$. There are several generalizations of $(2.3)_2$ that satisfy the property $N \to \infty$ as $\varepsilon \to -1$. For example, in [37] the relation

$$N = EA\frac{\varepsilon^3}{1+\varepsilon},$$

was proposed, with $EA > 0$ being a constant. In [1] more complicated relation

$$N = \frac{EA}{1+\gamma}\left(\varepsilon + 1 - \frac{1}{(\varepsilon+1)^\gamma}\right), \tag{2.4}$$

with $\gamma > 0$ was proposed. For ε small, i.e., $|\varepsilon| \ll 1$, the normal force N obtained from (2.4) is of the form $N = EA\varepsilon + O(\varepsilon^2)$, that is (2.4) approximates the Hooke's law in the limit when $|\varepsilon| \to 0$. For further discussion on $(2.3)_2$ see [2] and [39]. We shall use $(2.3)_2$ but with the restriction

$$\varepsilon > -1. \tag{2.5}$$

Finally we define q_x, q_y and m. By using the *D'Alembert's principle* (active and inertial forces and couples are in equilibrium) we shall add to the active distributed forces and couples the inertial terms and obtain from the system (2.2)–(2.3) equations of motion of the rod. Thus, we assume that

$$q_x = -\rho\frac{\partial^2 x}{\partial t^2} + q_x^{\text{pres.}}, \quad q_x = -\rho\frac{\partial^2 y}{\partial t^2} + q_y^{\text{pres.}}, \quad m = -J\frac{\partial^2 \vartheta}{\partial t^2} + m^{\text{pres.}}, \tag{2.6}$$

where ρ is the mass density of the rod (mass of the rod per unit length of the rod axis in the undeformed state), J is the moment of inertia of the rod cross-section, $q_x^{\text{pres.}}, q_y^{\text{pres.}}$ are prescribed values of the distributed forces along the $\bar{x}$ and $\bar{y}$ axes respectively and $m^{\text{pres.}}$ is the value of the prescribed distributed couples.

With (2.6) we can write the complete system of equations describing in plane motion of an elastic rod with extensible axis

$$\frac{\partial H}{\partial S} = \rho\frac{\partial^2 x}{\partial t^2} - q_x^{\text{pres.}};$$

$$
\begin{aligned}
\frac{\partial V}{\partial S} &= -\rho\frac{\partial^2 y}{\partial t^2} + q_y^{\text{pres.}};\\
\frac{\partial M}{\partial S} &= -V\left(1+\frac{V\cos\vartheta + H\sin\vartheta}{EA}\right)\cos\vartheta\\
&\quad + H\left(1+\frac{V\cos\vartheta + H\sin\vartheta}{EA}\right)\sin\vartheta + J\frac{\partial^2\vartheta}{\partial t^2} - m^{\text{pres.}};\\
\frac{\partial x}{\partial S} &= \left(1+\frac{V\cos\vartheta + H\sin\vartheta}{EA}\right)\cos\vartheta;\\
\frac{\partial y}{\partial S} &= \left(1+\frac{V\cos\vartheta + H\sin\vartheta}{EA}\right)\sin\vartheta;\\
\frac{\partial \vartheta}{\partial S} &= \frac{M}{EI},
\end{aligned}
\tag{2.7}
$$

To the system (2.7) we must add the boundary conditions. For the rod shown in Figure 1 those conditions read

$$
\begin{aligned}
H(L,t) = -F, \quad M(0,t) = 0, \quad M(L,t) = 0,\\
x(0,t) = 0, \quad y(0,t) = 0, \quad y(L,t) = 0.
\end{aligned}
\tag{2.8}
$$

We define as a *trivial* solution the solution of (2.7), (2.8) in which the rod axis remains straight. Suppose that $q_x^{\text{pres.}}(S,t) = q_y^{\text{pres.}}(S,t) = m^{\text{pres.}}(S,t) = 0$. It is easy to see that the trivial solution of (2.7),(2.8) is[1]

$$
\begin{aligned}
H^0(S,t) = -F, \quad V^0(S,t) = 0, \quad M^0(S,t) = 0,\\
x^0(S,t) = \Big(1-\frac{F}{EA}\Big)S, \quad y^0(S,t) = 0, \quad \vartheta^0(S,t) = 0.
\end{aligned}
$$

We study the disturbed motion of the trivial state. Thus, let us denote by $\Delta H(S,t)$, $\dots, \Delta V(S,t), \dots, \Delta\vartheta(S,t)$ the perturbations of the variables $H^0(S,t), \dots, \Delta\vartheta^0(S,t)$ describing the trivial configuration. Then for the disturbed state we have

$$
\begin{aligned}
H(S,t) &= H^0(S,t) + \Delta H(S,t),\\
V(S,t) &= V^0(S,t) + \Delta V(S,t),\\
&= \cdots\cdots\cdots\cdots\cdots .\\
\vartheta(S,t) &= \vartheta^0(S,t) + \Delta\vartheta(S,t).
\end{aligned}
\tag{2.9}
$$

By substituting (2.9) into (2.7) and neglecting the higher order terms in $\Delta H(S,t)$, $\dots, \Delta\vartheta(S,t)$ we obtain

$$
\begin{aligned}
\frac{\partial \Delta H}{\partial S} &= \rho\frac{\partial^2 \Delta x}{\partial t^2},\\
\frac{\partial \Delta V}{\partial S} &= -\rho\frac{\partial^2 \Delta y}{\partial t^2},\\
\frac{\partial \Delta M}{\partial S} &= -\Delta V\Big(1-\frac{F}{EA}\Big) - F\Big(1-\frac{F}{EA}\Big)\Delta\vartheta + J\frac{\partial^2\Delta\vartheta}{\partial t^2},
\end{aligned}
$$

[1]Note that (2.5) requires that $F < EA$

$$\begin{aligned}\frac{\partial \Delta x}{\partial S} &= \Big(1 - \frac{1}{EA}\Big),\\ \frac{\partial \Delta y}{\partial S} &= \Big(1 - \frac{F}{EA}\Big)\Delta\vartheta,\\ \frac{\partial \Delta\vartheta}{\partial S} &= \frac{\Delta M}{EI}.\end{aligned} \tag{2.10}$$

The system (2.10) could be simplified if we assume that we can differentiate the functions involved. Thus, by differentiating $(2.10)_2$ with respect to S and by using $(2.10)_2$ and $(2.10)_{5,6}$ we obtain

$$EI\frac{\partial^4 \Delta y}{\partial S^4} + F\frac{\partial^2 y}{\partial S^2} - J\frac{1}{(1-F/EA)}\frac{\partial^4 \Delta y}{\partial S^2 \partial t^2} + \rho\left(1 - \frac{F}{EA}\right)\frac{\partial^2 \Delta y}{\partial t^2} = 0, \tag{2.11}$$

subject to

$$\Delta y(0,t) = 0, \quad \Delta y(L,t) = 0, \quad \frac{\partial^2 \Delta y}{\partial S^2}(0,t) = 0, \quad \frac{\partial^2 \Delta y}{\partial S^2}(L,t) = 0. \tag{2.12}$$

We write next the system (2.11), (2.12) in the dimensionless form. By introducing the following quantities

$$\begin{aligned}&\xi = \frac{S}{L}, \quad u = \frac{\Delta y}{L}, \quad i^2 = \frac{I}{A}, \quad \mu = \frac{L}{i},\\ &\lambda = \frac{FL^2}{EI}, \quad \tau = t\left(\frac{EI}{\rho L^4}\right)^{1/2}, \quad \alpha = \frac{J}{\rho L^2},\end{aligned} \tag{2.13}$$

the system (2.11), (2.12) becomes

$$\begin{aligned}\frac{\partial^4 u}{\partial \xi^4} + \lambda\frac{\partial^2 u}{\partial \xi^2} - \frac{\alpha}{(1-\lambda/\mu^2)}\frac{\partial^4 u}{\partial \xi^2 \partial \tau^2} + \Big(1 - \frac{\lambda}{\mu^2}\Big)\frac{\partial^2 u}{\partial \tau^2} &= 0,\\ \tau > 0, \quad 0 < \xi < 1,&\end{aligned} \tag{2.14}$$

and

$$u(0,\tau) = 0, \quad u(1,\tau) = 0, \quad \frac{\partial^2 u}{\partial \xi^2}(0,\tau) = 0, \quad \frac{\partial^2 u}{\partial \xi^2}(1,\tau) = 0. \tag{2.15}$$

Equation (2.14) reduces to several special cases well known in mathematical physics. For example, suppose that we neglect compressibility of the rod axis. Then $EA \to \infty$ and $i^2 \to 0$ (see $(2.13)_3$) so that in this case the parameter μ, called *slenderness ratio*, tends to infinity i.e., $\mu \to \infty$. By using this, from (2.14) we obtain

$$\frac{\partial^4 u}{\partial \xi^4} + \lambda\frac{\partial^2 u}{\partial \xi^2} - \alpha\frac{\partial^4 u}{\partial \xi^2 \partial \tau^2} + \frac{\partial^2 u}{\partial \tau^2} = 0, \quad \tau > 0, \quad 0 < \xi < 1. \tag{2.16}$$

Equation (2.16) is valid for long and thin rods. Suppose further that the rotary inertia term is small i.e., $J \to 0$. In this case $\alpha \to 0$ and the equation (2.16) becomes

$$\frac{\partial^4 u}{\partial \xi^4} + \lambda\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \tau^2} = 0, \quad \tau > 0, \quad 0 < \xi < 1. \tag{2.17}$$

Note that the parameter λ could be constant or a function of time. The most interesting cases are

$$\lambda = A + B\delta(\tau - \tau_0), \quad \lambda = A + B\sin\Omega\tau, \tag{2.18}$$

where A, B, τ_0 and Ω are constants and $\delta(\tau)$ is Dirac distribution.

Finally, for the case when the axial force is equal to zero, i.e., $\lambda = 0$ equation (2.17) becomes

$$\frac{\partial^4 u}{\partial \xi^4} + \frac{\partial^2 u}{\partial \tau^2} = 0, \quad \tau > 0, \quad 0 < \xi < 1. \tag{2.19}$$

Equation (2.19) is a well known equation of lateral vibrations of an elastic rod, without the axial force.

We mention here a model, similar to (2.16) with $\lambda = 0$ recently proposed in [49] and [50]. It reads (in our notation)

$$\frac{\partial^4 u}{\partial \xi^4} - \alpha\frac{\partial^3 u}{\partial \xi^2 \partial \tau} + \frac{\partial^2 u}{\partial \tau^2} = 0; \quad \tau > 0, \quad 0 < \xi < 1.$$

In physical terms, the model (2.19) has the damping proportional to the rate of change of the curvature of the rod. No derivation (or further physical explanation) of the term $-\alpha\dfrac{\partial^3 u}{\partial \xi^2 \partial \tau}$ is given in [49] and [50]. However it is stated that this new model has good mathematical properties. Some of those properties are examined in [28].

To each of the equations (2.16), (2.17), (2.19) the *boundary conditions* such as (2.15) should be adjoined. For the sake of completeness we list here other, frequently used, boundary conditions:

- Left end clamped, right end free

$$u(0,\tau) = 0, \quad \frac{\partial u}{\partial \xi}(0,\tau) = 0, \quad \frac{\partial^2 u}{\partial \xi^2}(1,\tau) = 0, \quad \frac{\partial^3 u}{\partial \xi^3}(1,\tau) = 0.$$

- Left and right ends simply supported

$$u(0,\tau) = 0, \quad \frac{\partial^2 u}{\partial \xi^2}(0,\tau) = 0, \quad u(1,\tau) = 0, \quad \frac{\partial^2 u}{\partial \xi^2}(1,\tau) = 0.$$

- Left end clamped, right end simply supported

$$u(0,\tau) = 0, \quad \frac{\partial u}{\partial \xi}(0,\tau) = 0, \quad u(1,\tau) = 0, \quad \frac{\partial^2 u}{\partial \xi^2}(1,\tau) = 0.$$

- Left end clamped, right end clamped and free for axial movement

$$u(0,\tau) = 0, \quad \frac{\partial u}{\partial \xi}(0,\tau) = 0, \quad \frac{\partial u}{\partial \xi}(1,\tau) = 0, \quad \frac{\partial u}{\partial \xi}(1,\tau) = 0.$$

- Left end clamped, right end loaded by a follower force

$$u(0,\tau) = 0, \quad \frac{\partial u}{\partial \xi}(0,\tau) = 0, \quad \frac{\partial^2 u}{\partial \xi^2}(1,\tau) = 0, \quad \frac{\partial^3 u}{\partial \xi^3}(1,\tau) = 0.$$

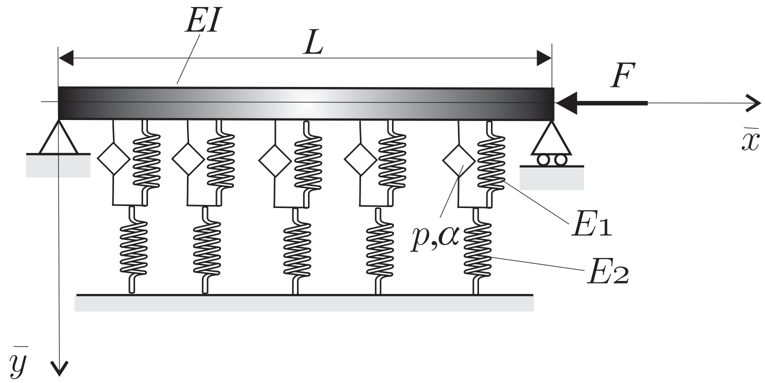

FIGURE 2. Elastic rod on a viscoleastic foundation

- Left end clamped, right end welded to a movable rigid plate (free for a transversal movement)

$$u(0,\tau)=0, \quad \frac{\partial u}{\partial \xi}(0,\tau)=0, \quad \frac{\partial u}{\partial \xi}(1,\tau)=0, \quad \frac{\partial^3 u}{\partial \xi^3}(1,\tau)=0.$$

We note that (2.17) for the rod with $\lambda = \text{const}$ and with different boundary conditions was analyzed in many publications (see [2] for references). Equation (2.16) with the boundary conditions corresponding to a simply supported rod and with $\lambda(t)$ of the form (2.18) is treated, recently, in [63].

2.2. Elastic axially loaded rod on elastic and viscoelastic foundation. We consider an elastic axially compressed rod on a special type of foundation, shown in Figure 2.

The foundation is such that it produces a distributed force f in the vertical direction, along the rod so that $q_y^{\text{pres.}} = f(S,t)$. The function $f(S,t)$ is determined by the constitutive equation of the foundation. For example, if

$$f = -cy, \tag{2.20}$$

then foundation is called *Winkler foundation*. By substituting (2.20) into (2.7) and performing the same steps as before, we obtain instead of (2.14) and (2.15) the following equation

$$\frac{\partial^4 u}{\partial \xi^4} + \lambda \frac{\partial^2 u}{\partial \xi^2} - \frac{\alpha}{(1-\lambda/\mu^2)} \frac{\partial^4 u}{\partial \xi^2 \partial \tau^2} + \left(1 - \frac{\lambda}{\mu^2}\right) \frac{\partial^2 u}{\partial \tau^2} + \beta u = 0, \tag{2.21}$$
$$\tau > 0, \quad 0 < \xi < 1,$$

subject to

$$u(0,\tau)=0, \quad \frac{\partial^2 u}{\partial \xi^2}(0,\tau)=0, \quad u(1,\tau)=0, \quad \frac{\partial^2 u}{\partial \xi^2}(1,\tau)=0. \tag{2.22}$$

In (2.21) the constant β is given as $\beta = cL^3/EI$. In Section 4 we shall analyze the system (2.21), (2.22) for a special case when the rod is thin and long. In this case

$\alpha = J/\rho L^2 \to 0$ (see $(2.13)_7$). Also since the second moment of inertia I and the cross sectional area are connected as $I = cA^m$, where $c > 0$ and $m > 1$ we have (see $(2.13)_2$) that the radius of gyration becomes $i^2 = cA$. Thus for thin and long rods $i^2 \to 0$ and $\mu = \frac{L}{i} \to \infty$ so that (2.21) becomes

$$\frac{\partial^4 u}{\partial \xi^4} + \lambda \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \tau^2} + \beta u = 0; \quad \tau > 0, \quad 0 < \xi < 1. \tag{2.23}$$

Often foundation is made of viscoelastic material. In this case the functional relation between f and y is more complicated than (2.20). For example in rail track problems (see [23]) the following type of viscoelastic foundation is used

$$f + \tau_Q f^{(\alpha)} = E\big(y + \tau_y y^{(\alpha)}\big), \tag{2.24}$$

where E_p, τ_Q, τ_y and $0 < \alpha < 1$ are constants. In (2.24) we used $(\cdot)^{(\alpha)}$ to denote the α-th derivative of a function $(\cdot)$ taken in Riemann–Liouville form as (see [42], [51] and Definition 1.1 in Section 2)

$$\frac{d^\alpha}{dt^\alpha} g(t) = g^{(\alpha)} \equiv \frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{g(\xi)\, d\xi}{(t-\xi)^\alpha} = \frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{g(t-\xi)\, d\xi}{\xi^\alpha}.$$

The dimension of the constants τ_y and τ_Q is $[\text{time}]^\alpha$. The constants E_p, τ_Q and τ_y in (2.24) are called of the pad and the *relaxation times*, respectively. We assume that, as a *consequence of the second law of thermodynamics*, the following inequality, is satisfied (see [11] and [3])[2]

$$E > 0, \quad \tau_Q > 0, \quad \tau_y > \tau_Q. \tag{2.25}$$

Now, by introducing new dimensionless function $F = f/EL$ the system (2.23), (2.24) becomes

$$\frac{\partial^4 u}{\partial \xi^4} + \lambda \frac{\partial^2 u}{\partial \xi^2} - \frac{\alpha}{(1-\lambda/\mu^2)} \frac{\partial^4 u}{\partial \xi^2 \partial \tau^2} + \Big(1 - \frac{\lambda}{\mu^2}\Big) \frac{\partial^2 u}{\partial \tau^2} + F = 0, \quad \tau > 0, \quad 0 < \xi < 1, \tag{2.26}$$

where

$$F + aF^{(\alpha)} = u + bu^{(\alpha)}, \tag{2.27}$$

subject to

$$u(0,t) = 0; \quad u(1,t) = 0; \quad \frac{\partial^2 u}{\partial \xi^2}(0,t) = 0; \quad \frac{\partial^2 u}{\partial \xi^2}(1,t) = 0,$$

and with the restriction $b > a > 0$, following from (2.25). The system (2.26),(2.27) in the special case $\alpha = 0$ was analyzed in [6].

Another important case is the case of an elastic rod on viscoelastic foundation loaded by a concentrated force at the free end (see Figure 3). The follower type concentrated force is a force having (in our case) constant intensity and the direction

[2] If one uses a rheological model shown under the rod in Figure 2, then the constants in (2.24) are given as $E = E_1 E_2/(E_1 + E_2)$, $\tau_Q = p/(E_1 + E_2)$, $\tau_y/E = pE_2/(E_1 + E_2)$ (see [54], [44]). Here E_1, E_2 are spring constants and p is the characteristic of a "springpot" an element whose stress-strin law is given as $\sigma = p\varepsilon^{(\alpha)}$.

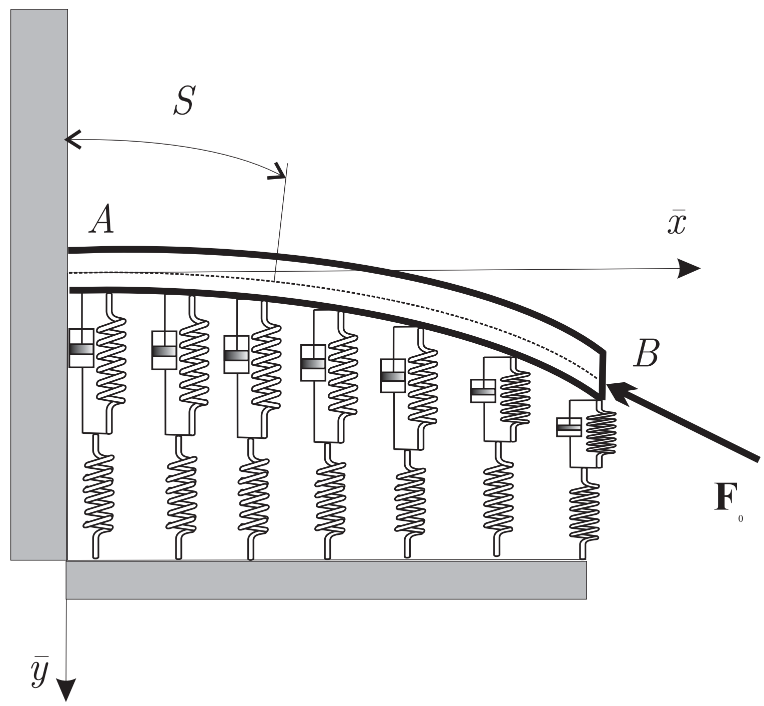

FIGURE 3. Elastic rod on viscoelastic foundation with the follower force

coinciding with the tangent to the rod axis at the point of application of force. For the case of an elastic rod with follower force and without foundation (the so called *Beck's rod*) there exists lot of results, some of them presented in [2] and [17].

The differential equations of the problem, for the rod shown in Figure 3, may be obtained by the same procedure as those used deriving (2.26) and are (see [8])

$$\frac{\partial^4 u}{\partial \xi^4} + \lambda \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \tau^2} + \beta f = 0, \quad \tau > 0; \quad 0 < \xi < 1. \tag{2.28}$$

and

$$f + a f^{(\alpha)} = u + b u^{(\alpha)}, \tag{2.29}$$

with $0 < \alpha < 1$. The boundary conditions are

$$u(0,\tau) = 0, \quad \frac{\partial u}{\partial \xi}(0,\tau) = 0, \quad \frac{\partial^2 u}{\partial \xi^2}(1,\tau) = 0, \quad \frac{\partial^3 u}{\partial \xi^3}(1,\tau) = 0, \quad \tau > 0. \tag{2.30}$$

The problems of existence and stability of the solution to (2.28)–(2.30) were treated in [8]. The conclusion about stability of the system (2.28)–(2.30) i.e., the condition that guarantees that the solution $u(\xi,\tau)$ is bounded when $\tau \to \infty$ is very interesting. Namely, it is shown that the critical value λ_{cr} of the parameter λ (the rod is stable if $\lambda \leqslant \lambda_{cr}$) does not depend on parameter β. Thus, the viscoelastic foundation does not increase the stability bound! This is known to hold for elastic column with follower force on elastic foundation and constitutes the so called Herman–Smith paradox. In [8] it was shown that the same holds when elastic foundation is replaced with the viscoelastic foundation of fractional derivative type described by (2.29).

Finally we mention the problem of determining stability boundary of an elastic rod with rotary inertia positioned on viscoelastic foundation. In this case the problem is described by the system of equations (2.26), (2.27) with $\alpha \neq 0$. The stability analysis and properties of the solution are examined in [9].

2.3. Viscoelastic axially loaded rod. We consider a special type of viscoelastic rod made of material described by fractional derivatives of a strain. Suppose that the rod is made of a material whose stress-strain relation is of the form (2.24). This model is known as the *generalized Zener model* (see [11], [3], [54])

$$\sigma(t) + \tau_\sigma D_t^\beta \sigma(t) = E_0 \left[\varepsilon(t) + \tau_\varepsilon D_t^\alpha \varepsilon(t)\right], \quad t \geqslant 0, \tag{2.31}$$

where $\tau_\sigma, \beta, E_0, \tau_\varepsilon$ and α are real constants. We note that (2.31) is a special case of a stress strain relation treated in [5], [7]. By using the plane cross-section hypothesis [2] we conclude that the strain in an element of the cross-section that is on the distance z from the neutral plane is $\varepsilon_z = z/r = (\partial\vartheta/\partial S)\, z$. Thus, by multiplying (2.31) by z and integrating over the cross-section of the rod A, we obtain

$$M(t) + \tau_\sigma D_t^\beta M(t) = E_0 I \left[\frac{\partial\vartheta}{\partial S} + \tau_\varepsilon D_t^\alpha \frac{\partial\vartheta}{\partial S}\right], \tag{2.32}$$

where I is the second moment of inertia, i.e., $I = \int_A z^2 dA$. For the linearized version of the system (2.32) we can substitute $\partial\vartheta/\partial S$ with $\partial^2 y/\partial S^2$ so that (2.32) becomes

$$M(t) + \tau_\sigma D_t^\beta M(t) = E_0 I \left[\frac{\partial^2 y}{\partial S^2} + \tau_\varepsilon D_t^\alpha \frac{\partial^2 y}{\partial S^2}\right]. \tag{2.33}$$

Equation (2.33) with $\tau_\sigma = 0$ was used in [10] and in its general form (2.33) in [38] and [4].

By substituting (2.33) in (2.10) we obtain

$$\begin{gathered} \frac{\partial^2 m}{\partial \xi^2} + \lambda \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \tau^2} = 0, \\ \frac{\partial^2 u}{\partial \xi^2} + \mu_1 D_\tau^\alpha \frac{\partial^2 u}{\partial \xi^2} - m - \mu D_\tau^\alpha m = 0, \quad \tau > 0, \quad 0 < \xi < 1, \end{gathered} \tag{2.34}$$

subject to

$$m(0,\tau) = 0, \quad m(1,\tau) = 0, \quad u(0,\tau) = 0, \quad u(1,\tau) = 0.$$

In (2.34) we used the following dimensionless quantities

$$u = \frac{\Delta y}{L}, \quad m = \frac{\Delta M L}{E_0 I}, \quad \tau = t\sqrt{\frac{E_0 I}{\rho L^4}}, \quad \lambda = \frac{F L^2}{E_0 I},$$

$$\xi = \frac{S}{L}, \quad \mu = \tau_\sigma \Big(\frac{IE_0}{\rho L^4}\Big)^{\alpha/2}, \quad \mu_1 = \tau_\varepsilon \Big(\frac{IE_0}{\rho L^4}\Big)^{\alpha/2}. \tag{2.35}$$

The second law of thermodynamics requires that $\mu_1 > \mu$.

An important generalization of (2.31) represents the so called *five-parameter model of viscoelastic body* studied in [46] and [48]. Suppose we use constitutive equation connecting the stress σ and strain ε in the form

$$\sigma_z(t) + \tau_\sigma D_t^\alpha \sigma_z(t) = E_0 \left[\varepsilon + \tau_\varepsilon D_t^\alpha \varepsilon + \tau_\gamma D^\gamma \varepsilon\right].$$

The plane cross-section hypothesis, together with the linearization of the expression for curvature, leads to

$$M(t) + \tau_\sigma D_t^\alpha M(t) = E_0 I \Big[\frac{\partial^2 y}{\partial S^2} + \tau_\varepsilon D_t^\alpha \frac{\partial^2 y}{\partial S^2} + (\tau_\varepsilon)^{\gamma/\alpha} D_t^\gamma \frac{\partial^2 y}{\partial S^2}\Big]. \tag{2.36}$$

The second law of thermodynamics in the case (2.36) requires that (see [11], [3], [46] and [7])

$$\gamma > \alpha; \quad \tau_\varepsilon > \tau_\sigma > 0. \tag{2.37}$$

Introducing a dimensionless quantities (2.35) and

$$\mu_2 = (\mu_1)^{\gamma/\alpha} = (\tau_\varepsilon)^{\gamma/\alpha} \Big(\frac{IE_0}{\rho L^4}\Big)^{\gamma/2},$$

we obtain, instead of the system (2.34), the following system of partial differential equations of integer and fractional order

$$\frac{\partial^2 m}{\partial \xi^2} + \lambda \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \tau^2} = 0;$$
$$\frac{\partial^2 u}{\partial \xi^2} + \mu_1 D_\tau^\alpha \frac{\partial^2 u}{\partial \xi^2} + \mu_2 D_\tau^\gamma \frac{\partial^2 u}{\partial \xi^2} - m - \mu D_\tau^\alpha m = 0; \quad \tau > 0, \quad 0 < \xi < 1,$$

with the boundary conditions $m(0,\tau) = 0$, $m(1,\tau) = 0$, $u(0,\tau) = 0$, $u(1,\tau) = 0$. The thermodynamic restrictions (2.37) become

$$\mu_1 > \mu; \quad \gamma \geqslant \alpha.$$

We note that in all cases formulated up to now, the dimensionless axial force λ can have both constant and time dependent part. For the case when an axial load is constant equal to B and additional load D is applied suddenly, at the time instant τ_0, we have $\lambda = B + C\theta(\tau - \tau_0)$. Also if we have constant axial force and at the time instant τ_0 an *impulsive force* is applied the axial force λ in this case is given as $\lambda = B + D\delta(\tau - \tau_0)$, where D is a constant.

Finally we present one more generalization of (2.31) and the corresponding constitutive equation for moments. Suppose that the stress strain relation is given in the form of so called *distributed derivative model* (see [5])

$$\int_0^1 \phi_\sigma(\gamma) \sigma^{(\gamma)} d\gamma = \int_0^1 \phi_\varepsilon(\gamma) \varepsilon^{(\gamma)} d\gamma,$$

where $\phi_\sigma(\gamma)$ and $\phi_\varepsilon(\gamma)$ are known functions that are determined from experiments (constitutive functions). Then, the plane cross-section hypothesis and the procedure used in deriving (2.32) and (2.36) leads to the constitutive equation for the bending moment in the form

$$\int_0^1 \phi_\sigma(\gamma) M^{(\gamma)}(t)\, d\gamma = I \int_0^1 \phi_\varepsilon(\gamma) \Big(\frac{\partial^2 y}{\partial \xi^2}(t)\Big)^{(\gamma)} d\gamma,$$

where I is, again, the second the moment of inertia of the rod cross-section A. The restrictions that the functions $\phi_\sigma(\gamma)$ and $\phi_\varepsilon(\gamma)$ must satisfy in order that the second law of thermodynamics is not violated, are derived in [7].

3. Generalized solutions to some partial differential equations

3.1. Equation in a space of generalized functions which corresponds to a partial differential equation. We denote by Ω an open set belonging to $\mathbb{R}^2$. Let

$$P(\partial)u(x,t) = f(x,t), \quad (x,t) \in \Omega, \quad f \in \mathcal{C}(\Omega), \tag{3.1}$$

be a linear partial differential equation with coefficients belonging to $\mathcal{C}^\infty(\Omega)$. To equation (3.1), by the property 4 of the derivative in $\mathcal{D}'(\Omega)$ (cf. Section 1, Subsection 1.1.2), it corresponds in $\mathcal{D}'(\Omega)$ the equation

$$P(D)[u(x,t)] = [f(x,t)]. \tag{3.2}$$

If there exists a solution $u(\xi,t)$ to (3.1) such that $u \in \mathcal{C}^p(\Omega)$, where $p = (p_1, p_2)$ is the degree of the equation (3.1), this solution is called the *classical solution.* A classical solution defines a distribution (regular) which is a solution to (3.2). If the solution to (3.2) is not defined by a function from $\mathcal{C}^p(\Omega)$ it is called *generalized solution* to (3.1). Conversely, if a solution w to (3.2) is the regular distribution $[u(x,t)]$, where $u(x,t) \in \mathcal{C}^p(\Omega)$, then $u(x,t)$ is a solution to (3.1). In this paper we use the so defined notations of a generalized and classical solution to (3.1).

Which generalized solution can be used depends on every concrete case. We are here interested in those mathematical models which are coming from mechanics. We are also going to point at the possibility to use the classical results in construction of a generalized solution.

We will not give a general theory, but illustrate it by some special cases in which generalized functions can improve the classical results or methods. However, there is a general procedure which will be conducted in solving equations to obtain classical and generalized solutions. It is the following:

First we find the equation (3.2) in $\mathcal{D}'(\Omega)$ which corresponds to the given equation (3.1). Then we apply certain methods to solve such equation (3.2). Usually these methods in $\mathcal{D}'(\Omega)$ are less restrictive than the methods in spaces of numerical functions.

If we find a solution u to (3.2), then it can happen that it is defined by a function $u(x,t)$, $u = [u(x,t)]$. This function $u(x,t)$ can belong to $\mathcal{C}^p(\Omega)$ and consequently be a classical solution to (3.1). Also it can belong to $\mathcal{C}^q(\Omega)$, $0 \leqslant q < p$, or to $\mathbf{L}_{\rm loc}(\Omega)$ and then u represents a generalized solution to (3.1). But in this case we can see why $u(x,t)$ can not be a classical solution to (3.1). Sometimes having generalized solutions to (3.1) we can construct the classical ones as well. For example, let $P(D)$ in (3.2) be with constant coefficients, $\Omega = \mathbb{R}^2$ and $f = 0$. Suppose that u is a solution to such homogeneous equation (3.2). Then u is a limit in $\mathcal{D}'(\mathbb{R}^2)$ of a sequence $\{u_j\}_{j\in\mathbb{N}}$ of classical solutions to (3.1) with $f = 0$. Let $\{\delta_j\}$ be a δ-sequence, ($\delta_j \in \mathcal{D}(\mathbb{R}^2)$ and δ_j converges to δ in $\mathcal{D}'(\mathbb{R}^2)$). Now, the sequence $\{u_j\}_{j\in\mathbb{N}}$ can be $\{u * \delta_j\}_{j\in\mathbb{N}}$ (cf. [64, p. 243]).

3.2. Construction of solutions by using fundamental solutions. We have seen in Section 3.1 that to a linear partial differential equation

$$P(\partial)u(x,t) = f(x,t), \ \ (x,t) \subset \mathbb{R}^2, \quad f \in \mathcal{C}(\mathbb{R}^2), \tag{3.3}$$

with coefficients belonging to $\mathcal{C}^\infty(\mathbb{R}^2)$, it corresponds in $\mathcal{D}'(\mathbb{R}^2)$ the equation

$$P(D)[u(x,t)] = \big[f(x,t)\big].$$

A distribution $E \in \mathcal{D}'(\mathbb{R}^2)$ is called a *fundamental solution* of the operator $P(\partial)$, by definition, if it satisfies the equation $P(\partial)E = \delta$. If f in (3.2) is such that the convolution $E * [f(x,t)]$ exists and the operator P has constant coefficients, then $W = E * [f(x,t)]$ is a solution to (3.2). In that case W is a generalized solution to (3.3) belonging to $\mathcal{D}'(\mathbb{R}^2)$. In the mathematical literature one can find fundamental solutions for different differential operators. (cf. for example [43]).

As an illustration we consider the equation

$$\frac{\partial^4}{\partial\xi^4}u(t,\xi) + \lambda\frac{\partial^2}{\partial\xi^2}u(t,\xi) + \frac{\partial^2}{\partial t^2}u(t,\xi) = 0, \quad t > 0, \quad \xi \in \mathbb{R}, \tag{3.4}$$

which appears in mathematical models for many different phenomena subject to different boundary and initial conditions (cf. Section 2 (2.1.18)).

It is well known that a solution to (3.4) is $u(t,\xi) = Y(\xi)T(t)$, where Y and T have the analytical from:

$$Y(\xi) = C_1 \cosh r_1\xi + C_2 \sinh r_1\xi + C_3 \cos r_2\xi + C_4 \sin r_2\xi \tag{3.5}$$

$$T(t) = C_5 \cos\omega t + C_6 \sin\omega t, \qquad \omega^2 \in \mathbb{R}_+, \tag{3.6}$$

where

$$r_1 = \sqrt{\frac{\sqrt{\lambda^2 + 4\omega^2} - \lambda}{2}}, \quad r_2 = \sqrt{\frac{\sqrt{\lambda^2 + 4\omega^2} + \lambda}{2}},$$

(cf. [2]). For ω any complex number (cf. [2], [55]).

To find generalized solutions to (3.4) belonging to $\mathcal{D}'(\mathbb{R}^2)$ we have first to find the equation in $\mathcal{D}'(\mathbb{R}^2)$ which corresponds to (3.4). In fact we seek for the corresponding equation in $\mathcal{D}'(\overline{\mathbb{R}}_+ \times \mathbb{R})$, because this space is more suitable to find a fundamental solution.

Suppose that there exists $u(t,\xi) \in \mathcal{C}_t^{(2)}(\mathbb{R}_+,\mathbb{R})$ such that:

1. $u(t,\xi)$ is a solution to (3.4);
2. There exist $\lim\limits_{t\to 0^+} u(t,\xi) = u_1(\xi) \in \mathcal{C}(\mathbb{R})$, $\quad \lim\limits_{t\to 0^+} u_t^{(1)}(t,\xi) = u_2(\xi) \in \mathcal{C}(\mathbb{R})$.

Let $[\theta u]$ denote the regular distribution defined by the function $\theta(t)u(t,\xi)$, where θ is the *Heaviside* function ($\theta(t) = 0,\ t < 0;\ \theta(t) = 1,\ t \geqslant 0$). By the property of derivatives in $\mathcal{D}'$ (cf. 8 in Section 1.1.2), to (3.4) there corresponds in $\mathcal{D}'(\overline{\mathbb{R}}_+ \times \mathbb{R}) \subset \mathcal{D}'(\mathbb{R}^2)$ the following equation:

$$(D_\xi^4 + \lambda D_\xi^2 + D_t^2)\tilde{u} = [u_1(\xi)] \otimes \delta^{(1)}(t) + [u_2(\xi)] \otimes \delta(t),$$

or

$$(D_t^2 + P(D_\xi))\tilde{u} = f, \tag{3.7}$$

where $P(D_\xi) = D_\xi^4 + \lambda D_\xi^2$, $f = [u_1(\xi)] \otimes \delta^{(1)}(t) + [u_2(\xi)] \otimes \delta(t)$ and $\tilde{u} \in \mathcal{D}'(\mathbb{R}^2)$. We seek for solutions to (3.7) with the property $\operatorname{supp} \tilde{u} \subset \overline{\mathbb{R}}_+ \times \mathbb{R}$.

By the lemma in [43, p. 30], the operator $D_t^2 + P(D_\xi)$ is *quasihyperbolic* with respect to t if and only if the following condition is satisfied:

$$\exists c > 0, d \in \mathbb{R}, \forall \xi \in \mathbb{R} : \operatorname{Re} P(i\xi) - c(\operatorname{Im} P(i\xi))^2 \geqslant d.$$

In our case $P(i\xi) = \xi^4 - \lambda\xi^2$. For every $\xi \in \mathbb{R}$, $\xi^4 - \lambda\xi^2 \geqslant -\lambda^2/4$. Consequently the operator $D_t^2 + P(D_\xi)$ is quasihyperbolic.

By Proposition 5 in [43, p. 32] the unique fundamental solution E of $D_t^2 + P(D_\xi)$ with support in $\overline{\mathbb{R}}_+ \times \mathbb{R}$ and $E \in e^{\alpha t}\mathcal{S}'$ for an $\alpha \in \mathbb{R}$ is given by

$$E(t, \xi) = H(t)\mathcal{F}_x^{-1}\bigg(\frac{\sin\big(t\sqrt{P(2\pi i x)}\big)}{\sqrt{P(2\pi i x)}}\bigg)(t, \xi),$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform.

Using Bochner's formula (cf. [56, VII, 7, 22], or [43, p. 19])

$$E(t, |\xi|) = H(t)2\pi|\xi|^{1/2}\int_0^\infty \frac{\sin\big(t\sqrt{P(2\pi i x)}\big)}{\sqrt{P(2\pi i x)}} x^{1/2} J_{-1/2}(2\pi|\xi|x)\, dx,$$

where J_v is the Bessel function. Since $J_{-1/2}(2\pi|\xi|x) = \dfrac{1}{\pi}\dfrac{\cos 2\pi|\xi|x}{\sqrt{|\xi|x}}$, we have

$$E(t, \xi) = 2H(t)\int_0^\infty \frac{\sin\big(t\sqrt{P(2\pi i x)}\big)}{\sqrt{P(2\pi i x)}}\frac{\cos(2\pi|\xi|x)}{\sqrt{x}} dx. \tag{3.8}$$

Suppose now that $u_1(\xi)$ and $u_2(\xi)$ in (3.7) have the properties that:

$$\big([u_2(\xi)] \otimes \delta(t)\big) * [E(t, \xi)], \quad \big([u_1(\xi)] \otimes \delta^{(1)}(t)\big) * [E(t, \xi)] \tag{3.9}$$

exist, then there is a solution $\tilde{u}$ to (3.7) in $\mathcal{D}'(\mathbb{R}^2)$ with support in $\bar{\mathbb{R}}_+ \times \mathbb{R}$

$$\begin{aligned}\tilde{u} &= \Big(\big([u_1(\xi)] \otimes \delta^{(1)}(t)\big)\Big) + \big([u_2(\xi)] \otimes \delta(t)\big)\Big) * [E(t, \xi)] \\ &= [u_2(\xi)] * [E(t, \xi)] + [u_1(\xi)] * D_t[E(t, \xi)].\end{aligned}$$

This solution is unique in the vector space $\mathcal{G} \subset \mathcal{D}'(\mathbb{R}^2)$. $\mathcal{G}$ consists of all $q \in \mathcal{D}'(\mathbb{R}^2)$ for which there exists $E * q$ (cf. [67, Chapter III, §11.3]). We proved the following:

Theorem 3.1. [61] *Let E be given by* (3.8) *and let $\mathcal{G}$ be the vector space belonging to $\mathcal{D}'(\mathbb{R}^2)$ such that for every $g \in \mathcal{G}$ there exists $[E] * g$. Suppose that $u_1(\xi)$ and $u_2(\xi)$ are in $\mathcal{C}(\mathbb{R})$ such that the convolutions* (3.9) *exist. Then*

$$\tilde{u} = [u_2(\xi)] * [E(t,\xi)] + [u_1(\xi)] * D_t[E(t,\xi)]$$

is a solution to $(D_\xi^4 + \lambda D_\xi^2 + D_t^2)\tilde{u} = 0$ in $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{R})$. But it is also the unique solution in the space $\mathcal{G} \subset D'(\mathbb{R}^2)$ satisfying the initial condition in t in the sense that

$$(D_\xi^4 + \lambda D_\xi^2 + D_t^2)\tilde{u} = [u_2(\xi)] \otimes \delta(t) + [u_1(\xi)] \otimes \delta^{(1)}(t).$$

Remarks. 1. If $u_1(\xi)$ and $u_2(\xi)$ also belong to $\mathcal{C}^4(\mathbb{R})$, then by the property of convolution (cf. Section 1 Subsection 1.2, property 9)

$$D_\xi^i \tilde{u} = [u_2^{(i)}(\xi)] * [E(t,\xi)] + [u_1^{(i)}(\xi)] * D_t[E(t,\xi)], \quad i = 1, \dots, 4.$$

2. If we have two solutions $u_1(t,\xi)$ and $u_2(t,\xi)$ to (3.4) with some initial condition

$$u_1(0,\xi) = u_2(0,\xi) \text{ and } \frac{d}{dt}u_1(t,\xi)|_{t=0} = \frac{d}{dt}u_2(t,\xi)|_{t=0}, \quad \xi \in \mathbb{R},$$

then $[u_2(t,\xi)] = [u_1(t,\xi)] + h$, where $h = 0$ or $h \notin \mathcal{G}$. Let us prove it. The function $U(t,\xi) = u_2(t,\xi) - u_1(t,\xi)$ satisfies (3.4) with initial condition $U_t^{(i)}(t,\xi)|_{t=0} = 0$, $i = 0, 1$, $\xi \in \mathbb{R}$, consequently the regular distribution $[U(t,\xi)] \in \mathcal{D}'(\mathbb{R}^2)$ satisfies (3.7) with $f = 0$. Then $[U(t,\xi)] = h$, where $h = 0$ or $h \notin \mathcal{G}$. Hence $[U(t,\xi)] = [u_2(t,\xi)] - [u_1(t,\xi)] = h$.

3. The well-known solution to (3.4) $u(t,\xi) = Y(\xi)T(t)$, where Y and T are given by (3.5) and (3.6), has not the convolution with $E(t,\xi)$ in the sense of distributions, i.e., $[u(t,\xi)] * [E(t,\xi)]$ does not exist. If it were true that $[u(t,\xi)] * [E(t,\xi)]$ exists, then by 3.4 and the property of convolution:

$$\begin{aligned}[u(t,\xi)] &= [u(t,\xi)] * \delta(t,\xi) = [u(t,\xi)] * \big(D_t^2 + P(D_\xi)\big)[E(t,\xi)] \\ &= \big((D_t^2 + P(D_\xi))[u(t,\xi)]\big) * [E(t,\xi)] \\ &= \Big[\Big(\frac{\partial^2}{\partial t^2} + \frac{\partial^4}{\partial \xi^4} + \frac{\partial^2}{\partial \xi^2}\Big)u(t,\xi)\Big] * [E(t,\xi)] = 0.\end{aligned}$$

Thus $u(t,\xi) = 0$, $t > 0$, $\xi \in \mathbb{R}$.

4. If equation (3.7) with $f = 0$ has a solution belonging to $\mathcal{D}'(\mathbb{R}^2)$, it does not belong to $\mathcal{G}$.

Proof. A solution to (3.4) in $\mathcal{D}'(\mathbb{R}^2)$ is $u(t,\xi) \equiv 0$, $(t,\xi) \in \mathbb{R}^2$. By 2 if there is a solution to (3.4) belonging to $\mathcal{D}'(\mathbb{R}^2)$ which is not identical zero, then it does not belong to $\mathcal{G}$ and the proof is complete. □

The solution $u(t,\xi) = Y(\xi)T(t)$, where Y and T have been given by (3.5) and (3.6) respectively, is in fact a solution to

$$(Y^{(4)}(\xi) + \lambda Y^{(2)}(\xi) + \omega^2 Y(\xi))T(t) + (T^{(2)}(t) - \omega^2 T(t))Y(\xi) = 0, \;\; t > 0, \; \xi \in \mathbb{R},$$

for $\omega^2 \in \mathbb{R} \setminus \{0\}$. This equation can be written in the form

$$\Big(P\Big(\frac{d}{d\xi}\Big) + \frac{d^2}{dt^2} - \omega^2\Big)Y(\xi)T(t) = 0,$$

where $P\Big(\dfrac{d}{d\xi}\Big) = \dfrac{d^4}{d\xi^4} + \lambda\dfrac{d^2}{d\xi^2} + \omega^2$. Let us suppose that $\omega^2 > 0$. Since

$$P(i\xi) = \xi^4 - \lambda\xi^2 + \omega^2 > 0, \quad \xi \in \mathbb{R}, \quad \omega^2 - \lambda^2/4 > 0,$$

by Proposition 6 in [43] there is the unique fundamental solution $E_\omega(t,\xi)$ of

$$P\Big(\frac{d}{d\xi}\Big) + \frac{d^2}{dt^2} - \omega^2$$

with support in $\overline{\mathbb{R}}_+ \times \mathbb{R}$ and belonging to $e^{\alpha t}\mathcal{S}'$ for an $\alpha \in \mathbb{R}$. It has the following representation

$$E_\omega(t,\xi) = E(t,\xi) - \omega H(t)\int_0^t \frac{\tau}{\sqrt{t^2-\tau^2}} J_1\big(\omega\sqrt{t^2-\tau^2}\big)E(\tau,\xi)\,d\tau, \tag{3.10}$$

where $E(t,\xi)$ is given by (3.8).

Theorem 3.2. *If in the Theorem* 1.1 *instead of* $E(t,\xi)$ *we take* $E_\omega(t,\xi)$*, given by* (3.10)*, then we obtain an other form of solutions to*

$$\Big(P\Big(\frac{d}{d\xi}\Big) + \frac{d^2}{dt^2} - \omega^2\Big)[u(t,\xi)] = 0$$

with $P\Big(\dfrac{d}{d\xi}\Big) = \dfrac{d^4}{d\xi^4} + \lambda\dfrac{d^2}{d\xi^2} + \omega^2$*, where* $\omega^2 - \lambda^2/4 > 0$*,* $\omega^2 > 0$*.*

3.3. Weak solutions to partial differential equation with boundary conditions.

We consider, as an illustration, the partial differential equations for the vibration rod and for lateral vibrating of an elastic rod on Winkler foundation (cf. Section 2, Subsection 2.2). To find weak (generalized) solutions we use the classical well-known results. That is the reason to consider them as a preliminary.

In this part we use some facts from the theory of linear differential operators and from Fredholm theory of integral equations. We repeat them. Let L denote a *linear differential operator* defined by the *differential expression*

$$l(u) = a_0u^{(n)}(x) + \cdots + a_{n-1}u^{(1)}(x) + a_nu(x), \quad x_1 < x < x_2,$$

and by the homogeneous boundary condition $U_\nu(u) = 0$, $\nu = 1,\dots,n$, so to say a *differential problem* is defined. *Eigenvalues* and *eigenfunctions* of the operator L have been given by $l(u) = 0$, $U_\nu(u) = 0$, $\nu = 1,\dots,n$. *Green's function* of the operator L is the function $G(x,\xi)$ with the following properties:

(1) $G(x,\xi)$ with its $(n-2)$ derivatives in x is continuous for $x,\xi \in (x_1,x_2)$ and satisfies the prescribed boundary conditions $U_\nu(u) = 0$, $\nu = 1,\dots,n$.

(2) Except at the point $x = \xi$ the $(n-1)$-th and the n-th derivative in x are continuous for $x,\xi \in (x_1,x_2)$. At the point $x = \xi$ the $(n-1)$-th derivative in x has a jump discontinuity given by

$$\frac{\partial^{n-1}}{\partial x^{n-1}}G(\xi+0,\xi) - \frac{\partial^{n-1}}{\partial x^{n-1}}G(\xi-0,\xi) = -\frac{1}{a_0(\xi)}, \quad \xi \in (x_1,x_2).$$

(3) $G(x,\xi)$ considered as a function of x satisfies the differential equation $l(u) = 0$, $x, \xi \in (x_1, x_2)$, $x \neq \xi$.

Proposition 3.1. *If the differential problem*

$$l(u) = 0,\ U_\nu(u) = 0,\ \ \nu = 1, \dots, n$$

has only the trivial solution $u = 0$, then L has one and only one Green's function $G(x,\xi)$. This function $G(x,\xi)$ is the kernel of the integral equation

$$u(x) = \lambda \int_0^\pi G(x,\xi)u(\xi)\, d\xi + \int_0^\pi G(x,\xi) f(\xi)\, d\xi \tag{3.11}$$

which is equivalent to the differential problem

$$l(u) + \lambda u = -f,\ U_\nu(u) = 0,\ \ \nu = 1, \dots, n.$$

(cf. [19, I, p. 353]).

If a kernel $K(x,\xi)$ of the integral equation (3.11) has the property that

$$J(\varphi,\varphi) = \iint K(s,\xi)\varphi(s)\varphi(\xi)\, ds\, d\xi$$

can assume only positive or only negative values (unless φ vanishes identically) it is said to be *positive definite* or *negative definite* in both cases it is *definite*. φ is any function which is continuous or piecewise continuous in the basic domain.

Proposition 3.2. *If $K(x,\xi)$ is a continuous symmetric kernel of the integral equation* (3.11), *then every function g of the form*

$$g(x) = \int_0^\pi K(x,\xi)h(\xi)\, d\xi,$$

where h is a piecewise continuous function on $[0,\pi]$, can be expanded in a series in the orthonormal eigenfunctions of $K(x,\xi)$

$$g(x) = \sum_{i=1}^{\infty} g_i v_i(x), \quad g_i(g, v_i) = \frac{\langle h, v_i\rangle}{\lambda_i},$$

where $\langle g, v_i\rangle \equiv \int_0^\pi g(\xi)v_i(\xi)\, d\xi$. This series converges uniformly and absolutely (cf. [19, I, p. 136]).

From the proof of this Proposition we will use the following:
For every $\varepsilon > 0$ there exists $\mathbb{N}_0(\varepsilon)$ such that:

$$\sum_{i=m}^{n} |g_i|\, |v_i(x)| < \varepsilon, \quad n, m \geqslant \mathbb{N}_0(\varepsilon),\ \ x \in [0,\pi]. \tag{3.12}$$

3.3.1. *The classical theory of a vibrating rod.* The mathematical model of the vibrating rod is (cf. Section 2, Subsection 2.1)

$$\frac{\partial^4}{\partial x^4}u(x,t)+\frac{\partial^2}{\partial t^2}u(x,t)=0,\ \ 0<x<\pi,\ \ t>0. \tag{3.13}$$

Since for the construction of generalized solutions to (3.13) we use the classical results, we quote some of them (cf. [19, I]).

If we suppose that the solution to (3.13) has the form $u(x,t)=v(x)g(t)$, then equation (3.13) decomposes to two differential equations

$$v^{(4)}(x)-\lambda v(x)=0,\ \ 0<x<\pi;\ \ g^{(2)}(t)+\lambda g(t)=0,\ \ t>0. \tag{3.14}$$

In [19] five various types of boundary conditions have been analyzed (see also Section 2, Subsection 2.1):

(3.15)

1. $v^{(2)}(x)=v^{(3)}(x)=0,$ for $x=0$ and $x=\pi$, i.e., free ends
2. $v(x)=v^{(2)}(x)=0,$ for $x=0$ and $x=\pi$, i.e., simply supported ends
3. $v(x)=v^{(1)}(x)=0,$ for $x=0$ and $x=\pi$, i.e., clamped ends
4. $v^{(1)}(x)=v^{(3)}(x)=0,$ for $x=0$ and $x=\pi$, i.e., moving clamped ends
5. $v(0)=v(\pi),\ v^{(1)}(0)=v^{(1)}(\pi),\ v^{(2)}(0)=v^{(2)}(\pi),\ v^{(3)}(0)=v^{(3)}(\pi),$ periodicity conditions.

In all these cases eigenvalues and eigenfunctions can be given explicitly. The next Proposition gives the properties of these eigenvalues and eigenfunctions.

Proposition 3.3. *For the differential problem* $(3.14)_1$ *and one of boundary conditions* (3.15)*, there exists a denumerable infinite system of eigenvalues* $\lambda_i\geqslant 0$*,* $i\in N$ *and associated eigenfunctions,* v_i*,* $i\in N$*. Note that* $\{\lambda_i\}_{i\in\mathbb{N}}$ *is not a bounded set;* $\{v_i\}_{i\in\mathbb{N}}$ *is a complete system and arbitrary functions possessing continuous first and second and piecewise continuous third and fourth derivatives may be expanded in terms of these eigenfunctions.*

By the solutions to equations (3.14) we can construct a family of solutions to (3.15)

$$u_i(x,t)=v_i(x)(a_i\cos\nu_i t+b_i\sin\nu_i t),\ \ i\in\mathbb{N}, \tag{3.16}$$

where a_i,b_i are arbitrary constants and $\nu_i=\sqrt{\lambda_i}$ ($\sqrt{\lambda_i}$ is the principal branch), $i\in\mathbb{N}$. This form of solutions contains also the initial condition in t:

$$u_i(x,0)=a_iv_i(x);\ \frac{\partial}{\partial t}u_i(x,t)\Big|_{t=0}=b_i\nu_iv_i(x).$$

It is easily seen that every finite sum $\sum u_i(x,t)$ is a solution to (3.13), as well.

Let us go back to equation $(3.14)_1$ with the boundary condition $U_\nu(v)=0$, $\nu=1,\dots,4$, which is one of the type (3.15). In this case we have that a linear homogeneous operator L is given by $l(v)=v^{(4)}(x)=0$ and $U_\nu(v)=0$, $\nu=1,\dots,4$. From $v^{(4)}(x)=0$, it follows that $v(x)=C_1+C_2x+C_3x^2+C_4x^3$, where C_i, $i=1,\dots,4$ are arbitrary constants. For the boundary condition $U_\nu(v)=0$, $\nu=1,\dots,4$ we take for example $(3.15)_3$. Then we have to find C_i, $i=1,2,3,4$ in such a way that the chosen condition $U_\nu(v) = 0$, $\nu = 1, \dots, 4$ is satisfied. It is easily seen that all the $C_i = 0$, $i = 1, \dots, 4$. Consequently $v = 0$.

By Proposition 3.1, there exists one and only one Green's function $G(x, \xi)$ for L. This Green's function in our case is definite (cf. [19, p. 363]).

3.3.2. *Construction of generalized solutions to* (3.13), (3.14). Now, the equation (3.13) can be drowned in $\mathcal{D}'((0,\pi)\times(0,\infty))$ by the property 4 in Section 1, Subsection 1.2 of the distributional derivative. To (3.13) in $\mathcal{D}'(0,\pi)\times(0,\infty))$ it corresponds

$$D_x^4[u(x,t)] + D_t^2[u(x,t)] = 0. \tag{3.17}$$

Every solution to (3.13) defines a regular distribution, which is a solution to (3.17).

To $u(x,t) = v(x)g(t)$ corresponds in $\mathcal{D}'((0,\pi)\times(0,\infty))$ the distribution $[u(x,t)] = [v(x)] \times [g(t)]$ (*tensor product*). We know that (cf. [64, p. 120])

$$D_x^4[v(x)g(t)] = D_x^4[v(x)] \times [g(t)],$$
$$D_t^2[v(x)g(t)] = [v(x)] \times D_t^2[g(t)].$$

We proceed to find $[v(x)]$ and $[g(t)]$ in such a way that $[v(x)g(t)]$ satisfies (3.17). This equation (3.17) can be written in the form:

$$D_x^4[v(x)] \times [g(t)] - \lambda[v(x)] \times [g(t)] + [v(x)] \times D_t^2[g(t)] + \lambda[v(x)] \times [g(t)] = 0.$$

Let us find λ, $[v(x)]$ and $[g(t)]$ so that

$$D_x^4[v(x)] - \lambda[v(x)] = 0, \quad D_t^2[g(t)] + \lambda[g(t)] = 0. \tag{3.18}$$

It is well known (cf. Property 7 in Section 1, Subsection 1.2 of the distributional derivative) that these two equations (3.18) have only solutions defined by the solutions to equations (3.14). Then solutions to (3.17) have been defined by functions of the form (3.16) or by finite sums of them. Consequently we have nothing new for equation (3.17).

To find generalized solutions to (3.13), which are interesting for our differential problem (3.13), (3.15) we shall start from the classical results for the equation (3.13), we cited in Proposition 3.1.

The Green function $G(x, \xi)$ for the operator L defined on the end of the Section 3.3.1 has all the properties we need so that Proposition 3.2 can be applied.

Let $w_1(x)$ and $w_2(x)$ be continuous functions and $h_i(x)$, $i = 1, 2$, piecewise continuous functions such that

$$\omega_i(x) = \int_0^\pi G(x,\xi) h_i(\xi)\, d\xi, \quad x \in [0,\pi], \ i = 1, 2. \tag{3.19}$$

Then by Proposition 3.2 we have

$$\omega_i(x) = \sum_{j=1}^\infty \omega_{ij} v_j(x), \quad i = 1, 2, \tag{3.20}$$

where $\{v_j\}_{j\in\mathbb{N}}$ is the sequence of eigenfunctions of $G(x, \xi)$.

From (3.19) and properties of Green's function it follows by (3.20) that the functions $\omega_i(x)$, $i = 1, 2$ are not only continuous, but they have also continuous first and second order derivatives. They satisfy the boundary condition, as well. Because of the properties of eigenfunctions $v_i(x)$, $i \in \mathbb{N}$, to be continuous, to have continuous first derivative and that $v_i(0) = 0$, for every $i \in \mathbb{N}$, there exists $x_i \in (0, \pi)$, such that

$$\max_{0 \leqslant x \leqslant \pi} |v_i^{(1)}(x)| = |v_i^{(1)}(x_i)| \equiv M_i \neq 0, \ \ i \in \mathbb{N}, \tag{3.21}$$

and there exists $x_i' \in (0, \pi)$, such that

$$|v_i(x_i')|/M_i < 1, \quad i \in \mathbb{N}. \tag{3.22}$$

We will also use the property of the set $\{\lambda_i\}_{i\in\mathbb{N}}$ of eigenvalues, not to be bounded. Consequently there exists $i_0 \in \mathbb{N}$ such that

$$\lambda_i^{-1} < 1, \quad i \geqslant i_0. \tag{3.23}$$

We can now construct the function $W(x,t)$

$$W(x,t) = \sum_{j=1}^{\infty} v_j(x)(a_j \cos \nu_j t + b_j \sin \nu_j t), \ \ 0 \leqslant x \leqslant \pi, \ t \geqslant 0. \tag{3.24}$$

We consider two cases for constant $a_j, b_j \in \mathbb{N}$:

$$\text{(i)} \quad a_j = \frac{\omega_{1j} v_j(x_j')}{M_j}, \quad b_j = \frac{\omega_{2j} v_j(x_j')}{M_j};$$

$$\text{(ii)} \quad a_j = \frac{\omega_{1j} v_j(x_j')}{M_j \nu_j}, \quad b_j = \frac{\omega_{2j} v_j(x_j')}{M_j \nu_j},$$

where $\nu_j = \sqrt{\lambda_j}$, $\lambda_j \geqslant 0$, $j \in \mathbb{N}$.

The function $W(x,t)$ has the following properties:

1) In case (i) it is a continuous function with a continuous first derivative in x on $[0, \pi) \times [0, \infty)$. In case (ii) it has also a continuous derivative in t.

First we prove the continuity proving that the two series which constitute the function $W(x,t)$ are uniformly convergent on $[0, \pi] \times [0, \infty)$.

Case (i): By (3.12) and (3.22) we have for the first series

$$\Big| \sum_{j=m}^{n} \frac{\omega_{1j} v_j(x_j')}{M_j} v_j(x) \cos v_j t \Big|^2 \leqslant \Big(\sum_{j=m}^{n} |\omega_{1j}||v_j(x)| \Big)^2 < \varepsilon$$

$n, m \geqslant \mathbb{N}_0$, $(x,t) \in [0, \pi] \times [0, \infty)$.
The proof for the second series is just the same.

Case (ii): We use now (3.23) in the proof of the continuity.

Let us consider the series

$$\sum_{j=1}^{\infty} v_j^{(1)}(x) \Big(\frac{\omega_{1j} v_j(x_j')}{M_j} \cos \nu_j + \frac{\omega_{2j} v_j(x_j')}{M_j} \sin \nu_j t \Big). \tag{3.25}$$

By using again (3.12) and (3.21), we have

$$\Big| \sum_{j=m}^{n} v_j^{(1)}(x) \frac{\omega_{1j} v_j(x_j')}{M_j} \cos \nu_j t \Big|^2 \leqslant \Big(\sum_{j=m}^{n} |\omega_{1j}||v_j(x_j')| \Big)^2 < \varepsilon,$$

$n, m \geqslant \mathbb{N}(\varepsilon)$. The treatment of the second series in (3.25) is the same.

Now we can conclude that in case (i) the function $W(x,t)$ given by (3.24) has a continuous derivative in x. This derivative can be obtained by taking the derivative of every member of the series in (3.24).

The proceeding of the proof that in case (ii) we have also the derivative in t does not differ of the proof of the derivative in x.

2) In case (i) and (ii)

$$W(x,0) = \sum_{j=1}^{\infty} v_j(x) a_j,$$

and this is a continuous function with continuous derivative on $[0,\pi]$.

In case (ii) we have

$$\frac{\partial}{\partial t} W(x,t)\Big|_{t=0} = \sum_{j=1}^{\infty} v_j(x) \frac{\omega_{2j} v_j(x'_j)}{M_j},$$

as well. The given series defines also a continuous function on $[0,\pi]$.

3) $W(x,t)$ satisfies the boundary condition we chose $(3.15)_3$.

4) $W(x,t)$ given by (3.24) is the limit of the sequence

$$W_n(x,t) = \sum_{j=1}^{n} v_j(x)(a_j \cos \nu_j t + b_j \sin \nu_j t), \quad n \in \mathbb{N}, \tag{3.26}$$

in $\mathcal{C}\big([0,\pi] \times [0,\infty)\big)$. The elements of the sequence (3.26) are solutions to (3.13) (cf. (3.16)).

It is easy now to prove

Theorem 3.3. *Let us denote by:* 1) $\{\lambda_i\}_{i\in\mathbb{N}}$ *and* $\{v_i\}_{i\in\mathbb{N}}$ *the eigenvalues and eigenfunctions respectively of the differential problem*

$$v^{(4)}(x) - \lambda v(x) = 0,$$
$$v(x) = v^{(1)}(x) = 0, \text{ for } x = 0 \text{ and } x = \pi.$$

2) $\{\nu_i\}_{i\in\mathbb{N}}$ *the sequence defined by* $\nu_i = \sqrt{\lambda_i}$, $\lambda_i \geqslant 0$, *where* $\sqrt{\lambda_i}$ *means the principal branch,* $i \in N$.

3) $\{a_j\}_{j\in\mathbb{N}}$ *and* $\{b_j\}_{j\in\mathbb{N}}$ *the sequences*

$$(i) \qquad a_j = \frac{\omega_{1j} v_j(x'_j)}{M_j}, \quad b_j = \frac{\omega_{2j} v_j(x'_j)}{M_j}, \qquad or$$
$$(ii) \qquad a_j = \frac{\omega_{1j} v_j(x'_j)}{M_j \nu_j}, \quad b_j = \frac{\omega_{2j} v_j(x'_j)}{M_j \nu_j},$$

where

$$M_j = \max_{0\leqslant x\leqslant\pi} \left|v_j^{(1)}(x)\right| \text{ and } x'_j \in (0,\pi), \ \left|v_j(x'_j)\right|/M_j < 1, \ j \in \mathbb{N}. \tag{3.27}$$

Then the function $W(x,t) = \sum_{j=1}^{\infty} v_j(x)(a_j \cos \nu_j t + b_j \sin \nu_j t)$, $0 \leqslant x \leqslant \pi$, $t \geqslant 0$ *defines a regular distribution* $[W(x,t)] \in D'((0,\pi)\times(0,\infty))$. *This distribution is a solution to* (3.17) *and a generalized solution to* (3.13), $(3.15)_3$.

The properties of the function $W(x,t)$ are:

a) In case (i) and (ii) it is a continuous function with continuous first order partial derivative in x on $[0,\pi]\times[0,\infty)$.

b) In case (ii) it has also a continuous first order partial derivative in t on $[0,\pi]\times[0,\infty)$.

c) In case (i) and (ii) we have $W(x,0)=\sum_{j=1}^{\infty} v_j(x)a_j$, $x\in[0,\pi]$, and this is a continuous function with a continuous first order derivative on $[0,\pi]$.

d) In case (ii) we have $\frac{\partial}{\partial t}W(x,t)\Big|_{t=0}=\sum_{j=1}^{\infty} v_j(x)\nu_j b_j$, $x\in[0,\pi]$. The given series defines a continuous function on $[0,\pi]$, as well.

e) $W(x,t)$ satisfies the boundary conditions $W(x,0)=\frac{\partial}{\partial x}W(x,t)=0$, for $x=0$ and $x=\pi$, and $t\geqslant 0$.

f) In case (i) and (ii) $D_x[W(x,t)]=\left[\frac{\partial}{\partial x}W(x,t)\right]$ and in case (ii) $D_t[W(x,t)]=\left[\frac{\partial}{\partial t}W(x,t)\right]$

g) In case (i) and (ii) $W(x,t)$ and in case (ii) $\frac{\partial}{\partial t}W(x,t)$ are bounded on $[0,\pi]\times[0,\infty)$.

Proof. The function $W(x,t)$ given by (3.24) defines a distribution because of its property 1), we proved. □

If the sequence (3.26) consists of solutions to (3.13), $(3.15)_3$, then the sequence $([W_n(x,t)])_{n\in\mathbb{N}}\subset\mathcal{D}'((0,\pi)\times(0,\infty))$ is the sequence of solutions to (3.17). Since the sequence (3.26) converges in $\mathcal{C}\big([0,\pi]\times[0,\infty)\big)$, the sequence $([W_n(x,t)])_{n\in\mathbb{N}}$ converges in $\mathcal{D}'((0,\pi)\times(0,\infty))$ (cf. Section 1, Subsection 1.2). Consequently, $[W(x,t)]$ as the limit of the sequence of solutions to (3.17) is also a solution to (3.17).

The other cited properties of the function $W(x,t)$ one can easily prove.

Remarks. 1) By (3.27) we have a family of functions because the sequence $\{x'_j\}_{j\in\mathbb{N}}\subset(0,\pi)$ has only to satisfy the inequality $|v_j(x'_j)|/M_j<1$, $j\in\mathbb{N}$.

2) If the solution to (3.13), $(3.15)_3$ is of the form $u(x,t)=v(x)g(t)$ we have

$$u(x,0)=g(0)v(x)\ \text{and}\ \frac{\partial}{\partial t}u(x,t)\Big|_{t=0}=g'(0)v(x).$$

But in our case $W(x,t)$ given by (3.27) which defines a generalized solution to (3.13), $(3.15)_3$ satisfies a more general initial condition: in case (i) and (ii) $W(x,0)=\sum_{j=1}^{\infty}a_jv_j(x)$ and in case (ii) we have moreover

$$\frac{\partial}{\partial t}W(x,t)\Big|_{t=0}=\sum_{j=1}^{\infty}b_j\nu_jv_j(x).$$

3.3.3. *Construction of generalized solutions to equation of the lateral vibration of an elastic rod on Winkler foundation.* We consider the equation

$$\frac{\partial^4}{\partial x^4}u(x,t)+\frac{\partial^2}{\partial t^2}u(x,t)+\lambda q(x)u(x,t)=0,\ \ 0<x<\pi,\ t>0, \tag{3.28}$$

where $q(x)\geqslant 0$, $x\in[0,\pi]$ with boundary condition:

$$u(0,t)=\frac{\partial}{\partial x}u(x,t)\Big|_{x=0}=0;\ u(\pi,t)=\frac{\partial}{\partial x}u(x,t)\Big|_{x=\pi}=0,\ t\geqslant 0. \tag{3.29}$$

As in Section 3.3.2, we suppose that a solution of (3.28) is of the form $u(x,t) = v(x)g(t)$; then equation (3.28) becomes

$$\frac{\partial^4}{\partial x^4}v(x)g(t) + \lambda q(x)v(x)g(t) - \omega v(x)g(t) + \frac{\partial^2}{\partial x^2}v(x)g(t) + \omega v(x)g(t) = 0, \quad 0 < x < \pi,\ t > 0.$$

To find v and g we use two equations

$$v^{(4)}(x) + \lambda q(x)v(x) - \omega v(x) = 0, \quad 0 < x < \pi,$$
$$g^{(2)}(t) + \omega g(t) = 0, \quad t > 0,$$

and the boundary condition

$$v(0) = v^{(1)}(0) = 0, \quad v(\pi) = v^{(1)}(\pi) = 0. \tag{3.30}$$

Let L denote the differential expression $L(v) = v^{(4)}(x) + \lambda q(x)v(x)$. Note that L is self adjoint. To prove that L has Green's function we have to show (Proposition 3.1) that from $L(v) = 0$ and (3.30) it follows that $v = 0$. We will do it in two steps. First we consider the differential expression $l(v) = v^{(4)}(x)$ with (3.30). It is easily seen that $v^{(4)}(x) = 0$ with (3.30) gives $v = 0$. Then l has Green's function $G_l(x,\xi)$. We know that $G_l(x,\xi)$ is symmetric and definite (cf. [19, p. 363]).

Now, in the second step, we use the fact that

$$L(v) = v^{(4)}(x) + \lambda q(x)v(x) = 0, \text{ with (2.31)} \tag{3.31}$$

is equivalent to (cf. Proposition 3.1)

$$v(x) = \lambda \int_0^\pi G_l(x,\xi)q(\xi)v(\xi)\,d\xi,$$

or

$$\sqrt{q(x)}\,v(x) = \lambda \int_0^\pi G_l(x,\xi)\sqrt{q(x)q(\xi)}\,\sqrt{q(\xi)}\,v(\xi)\,d\xi.$$

The kernel $K(x,\xi) = G_l(x,\xi)\sqrt{q(x)q(\xi)}$ is also symmetric and definite. Let us denote by $y(x) = \sqrt{q(x)}\,v(x)$. Then (3.31) is equivalent to

$$y(x) = \lambda \int_0^\pi K(x,\xi)y(\xi)\,d\xi. \tag{3.32}$$

Since $K(x,\xi)$ is a continuous and symmetric kernel it possesses eigenvalues and eigenfunctions. Their number is denumerably infinite (cf. [19, p. 22]). Let λ_0 be a real number (positive) which is not an eigenvalue for the kernel $K(x,\xi)$. Then equation (3.32) and consequently equation (3.31) have only $v = 0$ as the solution. Hence we know that Green's function $G_L(x,\xi)$ exists for L with (3.30). Since L is self adjoint, $G_L(x,\xi)$ is symmetric and L has eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$ and eigenfunctions $\{v_i(x)\}_{i\in\mathbb{N}}$. Consequently we can apply Proposition 3.2. The consequence is that we can construct generalized solutions to equation (3.28) (which depends on the chosen number λ_0) with boundary condition (3.29) processing just in the same way as in Section 3.3.2 for equation (3.13) with the same boundary condition.

We have to remark that in this case we do not know that the Green function G_L is positive definite; the eigenvalues have not to be positive. Consequently we can not assert that the function $W(x,t)$ which defines the distributional solution is bounded on $[0,\pi]\times[0,\infty)$. The stability of the solution has to be considered separately.

3.4. The Laplace transform applied to a partial differential equation. The Laplace transform is very useful in solving partial differential equations. But we have always to take into account that as a first condition for applicability of the Laplace transform on a generalized function is to have its support bounded on the left. In such a way when we have a partial differential equations with numerical functions and look for the corresponding equation in a space of generalized functions we have to use the Property 8 in Section 1, Subsection 1.2 of the derivative of a generalized function.

Working with the Laplace transform, when we find a function $F(s)$, $\operatorname{Re} s>\omega>0$ and seek for a generalized function f, such that $\tilde{\mathcal{L}}f(s)=F(s)$, we have first to check if such f exists. For this purpose Propositions 1.4 and Proposition 2.1 in Section 1 can help. Secondly, we have to find such f. In many cases f is a numerical function. Thus, $\tilde{\mathcal{L}}^{-1}(f)$ is the regular distribution $[f]$ defined by the function f. The solution still has not to be a classical one, because the derivatives in, general, exist only in the distributional sense. An illustration how it reflects in solving a partial differential equation one can find in [61]. We consider in 3.4.1 the case when we apply the Laplace transform in one variable and in 3.4.2 in two variables to a partial differential equation.

3.4.1. *$\mathcal{M}$-valued functions as solutions to a partial differential equation.* Let $\mathcal{M}$ denote one of the following spaces: the space of L-functions (cf. [21]), $\mathcal{D}'_\omega(\overline{\mathbb{R}}_+)$ or $\mathcal{B}^{\exp}_{[0,\infty]}$. We use the Laplace transform which is defined for elements of these three spaces, consequently for elements of $\mathcal{M}$.

The partial differential equation we analyze is:

$$\frac{\partial^4}{\partial x^4}u(x,t)+\frac{\partial^2}{\partial t^2}u(x,t)=0,\ \ 0<x<1,\ t>0, \tag{3.33}$$

with the initial conditions

$$u(x,0)=B_0(x),\ \ \frac{\partial}{\partial t}u(x,t)\Big|_{t=0}=B_1(x),\ \ 0<x<1. \tag{3.34}$$

It is well-known that equation (3.33) has a solution of the form $u(x,t)=v(x)g(t)$ (cf. [2], [19]). In this case $B_0(x)=v(x)g(0)$ and $B_1(x)=v(x)g^{(1)}(0)$.

Let $\{[u(x,t)]\}_{0<x<1}$ denote a family of $\mathcal{M}$-valued functions of class $\mathcal{C}^4$ (cf. Section 1, Subsection 1.4). For any fixed x, $[u(x,t)]\in\mathcal{M}$.

By the property 8.1 in Section 1, Subsection 1.2, to equation (3.33) it corresponds in $\mathcal{M}$ the equation

$$\frac{\partial}{\partial x^4}[u(x,t)]+D_t^2[u(x,t)]=B_1(x)\delta(t)+B_0(x)\delta^{(1)}(t),\ \ 0<x<1. \tag{3.35}$$

Now, the solutions to (3.35) are the generalized solutions to (3.33), (3.34). Our aim is to find all the solutions to (3.35), i.e., all the generalized solutions to (3.33) with $B_0(x) = v(x)g(0)$ and $B_1(x) = v(x)g^{(1)}(0)$ which are functions with values in $\mathcal{M}$.

Suppose that we have two such solutions to (3.35) with values in $\mathcal{M}$, $w_1(x)$ and $w_2(x)$. Then $w_0(x) = w_1(x) - w_2(x)$ satisfies the homogenous equation (compare to (3.35))

$$\frac{\partial^4}{\partial x^4} w_0(x) + D_t^2 w_0(x) = 0, \quad 0 < x < 1. \tag{3.36}$$

The Laplace transform in t transforms (3.36) in

$$\frac{\partial^4}{\partial x^4} W_0(x, \widehat{s}) + s^2 W_0(x, \widehat{s}) = 0, \quad 0 < x < 1, \tag{3.37}$$

where $W_0(x, \widehat{s}) = \tilde{\mathcal{L}}_t(w_0(x))(x, \widehat{s})$. The equation (3.37) is a classical differential equation in which s, $\operatorname{Re} s > \omega > 0$, is only a parameter.

The general solution to (3.37) is of the form

$$W_0(x, \widehat{s}) = C_1(s) e^{r_1 x} + C_2(s) e^{r_2 x} + C_3(s) e^{r_3 x} + C_4(s) e^{r_4 x}, \qquad 0 < x < 1, \quad \operatorname{Re} s > \omega, \tag{3.38}$$

where C_i, $i = 1, \ldots, 4$ are functions of s and r_i, $i = 1, \ldots, 4$ are solutions to equation $r^4 + s^2 = 0$.

The Propositions 1.4 and 2.1 in Section 1, give the conditions which $C_i(s)$, $i = 1, \ldots, 4$, have to satisfy that $w_0(x)$ exists such that $\tilde{\mathcal{L}}(w_0(x))(x, \widehat{s}) = W_0(x, \widehat{s})$, $0 < x < 1$.

We known that $[v(x)g(t)]$ is solution to (3.35) with $B_0(x) = v(x)g(0)$ and $B_1(x) = v(x)g^{(1)}(0)$. Then all the solutions to (3.35) with cited values for B_0 and B_1 which are functions with values in $\mathcal{M}$ are $[v(x)g(t)] + w_0(x)$.

In such a way we proved the following theorem:

Theorem 3.4. *Let $u_1(x,t) = v(x)g(t)$ be the well known classical solution to* (3.33) *and let $\mathcal{M}$ denote one of the spaces: The space of L-functions* (cf. [19]), $\mathcal{D}'_\omega\left(\overline{\mathbb{R}}_+\right)$ *or* $\mathcal{B}^{\exp}_{[0,\infty]}$.

All the solutions to (3.35), *i.e., all the generalized solutions to* (3.33) *with initial condition*

$$u(x,0) = v(x)g(0) \text{ and } \frac{\partial}{\partial t} u(x,t)\Big|_{t=0} = v(x)g^{(1)}(0)$$

which are functions in x with values in M are $w(x) = [v(x)g(t)] + w_0(x)$, where $\tilde{\mathcal{L}}(w_0(x))(\widehat{s}) = W_0(x, \widehat{s})$ and $W_0(x, \widehat{s})$ is given by (3.38).

Applying the Laplace transform to (3.35) with any B_0 and B_1 we obtain a nonhomogeneous differential equation. The same procedure as for (3.37) gives us the generalized solutions to (3.33), (3.34) for any B_0, B_1.

3.4.2. *Solution of partial differential equation* (3.33) *by the Laplace transform.* We consider the equation

$$\frac{\partial^4}{\partial x^4}u(x,t)+\frac{\partial^2}{\partial t^2}u(x,t)=0,\ \ (x,t)\in\mathbb{R}^2_+, \tag{3.39}$$

with initial conditions:

$$\begin{aligned} u(0,t)=\frac{\partial}{\partial x}\,u(0,t)=0,\ \ t\geqslant 0,\\ \frac{\partial^k}{\partial x^k}u(0,t)=A_k(t),\ \ k=2,3,\ \ t\geqslant 0,\\ u(x,0)=B_0(x),\quad \frac{\partial}{\partial t}u(x,0)=B_1(x),\ \ x\geqslant 0, \end{aligned} \tag{3.40}$$

where $[\theta(t)A_k(t)]\in e^{pt}\mathcal{S}'(\overline{\mathbb{R}}_+)$, $k=2,3$, and $[\theta(x)B_i(x)]\in e^{pt}\mathcal{S}'(\overline{\mathbb{R}}_+)$, $i=0,1$, $p>0$. To find an equation in $\mathcal{D}'(\overline{\mathbb{R}}^2_+)$ which corresponds to (3.40) for $x>0$, $t>0$, we need the relations between derivatives in the sense of distributions and the classical ones.

Let $\theta^2(x_1,x_2)=\theta(x_1)\theta(x_2)$, where θ is the Heaviside function. For a function f with continuous partial derivatives on $\mathbb{R}^2$, $[\theta^2 f]$ is the distribution, defined by $\theta^2 f$, belonging to $\mathcal{D}'(\mathbb{R}^2)$ and to $\mathcal{D}'(\overline{\mathbb{R}}^2_+)$, as well. Let $\big(\partial^p f/\partial x_i^p\big)_0$ denote the function equal to $\partial^p f/\partial x_i^p$ on the $\mathbb{R}^2_+$ and equal zero on $\mathbb{R}^2\setminus\overline{\mathbb{R}}^2_+$, but is not defined for $(x_1,x_2)\in\{(0,x_2)\cup(x_1,0);\ x_1\geqslant 0,\ x_2\geqslant 0\}$.

With the notation as above we have (cf. 8.2 in Section 1, Subsection 1.2).

$$\begin{aligned} \frac{\partial^4}{\partial x^4}[u(x,t)]+\frac{\partial^2}{\partial t^2}[u(x,t)]=[\theta(t)A_2(t)]\times\delta^{(1)}(x)+[\theta(t)A_3(t)]\times\delta(x)\\ +[\theta(x)B_1(x)]\times\delta(t)+[\theta(x)B_0(x)]\times\delta^{(1)}(t). \end{aligned} \tag{3.41}$$

Applying the LT we have

$$(z^4+s^2)\mathcal{L}(u)(z,s)=\mathcal{L}(A_2)(s)z+\mathcal{L}(A_3)(s)+\mathcal{L}(B_1)(z)+\mathcal{L}(B_0)(z)s,$$

or

$$\mathcal{L}(u)(z,s)=\frac{Q(z,s)}{z^4+s^2},$$

with $Q(z,s)=\mathcal{L}(A_2)(s)z+\mathcal{L}(A_3)(s)+\mathcal{L}(B_1)(z)+\mathcal{L}(B_0)(z)s$. Since

$$\frac{1}{z^4+s^2}=\frac{1}{2is}\Big(\frac{1}{z^2-is}-\frac{1}{z^2+is}\Big),$$

we have

$$\frac{Q(z,s)}{z^4+s^2}=\frac{Q(z,s)}{2is}\Big(\frac{1}{z^2-is}-\frac{1}{z^2+is}\Big). \tag{3.42}$$

By Proposition 1.4 in Section 1, and the property of the space $\mathcal{H}_+$, $\frac{Q(z,s)}{z^4+s^2}$ has to be holomorphic in $\{(z,s)\in\mathbb{C}^2;\ \operatorname{Re} z>w_1>0,\ \operatorname{Re} s>w_2>0\}$. Since $z^4+s^2=(z-z_1)(z+z_1)(z-z_2)(z+z_2)$, where $z_1=e^{i\pi/4}\sqrt{s}$, $z_2=e^{3i\pi/4}\sqrt{s}$, it is necessary to have

$$Q(e^{i\pi/4}\sqrt{s},s)=0 \text{ and } Q(-e^{3i\pi/4}\sqrt{s},s)=0$$

or equivalently

$$Q(e^{i\pi/4}\sqrt{s}, s) = 0 \text{ and } Q(e^{-i\pi/4}\sqrt{s}, s) = 0. \tag{3.43}$$

Let us consider the first addend in (3.42). Then $(3.43)_1$ has to be satisfied which gives

$$\mathcal{L}(A_2)(s)e^{i\pi/4}\sqrt{s} + \mathcal{L}(A_3)(s) + \mathcal{L}(B_1)(e^{i\pi/4}\sqrt{s}) + s\mathcal{L}(B_0)(e^{i\pi/4}\sqrt{s}) = 0.$$

Now we can express $\mathcal{L}(A_3)(s)$,

$$\mathcal{L}(A_3)(s) = -\mathcal{L}(A_2)(s)e^{i\pi/4}\sqrt{s} - \mathcal{L}(B_1)(e^{i\pi/4}\sqrt{s}) - s\mathcal{L}(B_0)(e^{i\pi/4}\sqrt{s}).$$

With such expressed $\mathcal{L}(A_3)(s)$ the first addend in (3.42) is:

$$\begin{aligned}
\frac{Q(z,s)}{2is(z^2-is)} &= \frac{\mathcal{L}(A_2)(s)(z-e^{i\pi/4}\sqrt{s})}{2is(z^2-is)} \\
&\quad + \frac{\mathcal{L}(B_1)(z) - \mathcal{L}(B_1)(e^{i\pi/4}\sqrt{s}) + s(\mathcal{L}(B_0)(z) - \mathcal{L}(B_0(e^{i\pi/4}\sqrt{s}))}{2is(z^2-is)} \\
&= \frac{\mathcal{L}(A_2)(s)}{2is(z+e^{i\pi/4}\sqrt{s})} + \left(\frac{\mathcal{L}(B_1)(z)-\mathcal{L}(B_1)(e^{i\pi/4}\sqrt{s})}{4ise^{i\pi/4}\sqrt{s}} + \frac{\mathcal{L}(B_0)(z)-\mathcal{L}(B_0)(e^{i\pi/4}\sqrt{s})}{4ie^{i\pi/4}\sqrt{s}}\right) \\
&\quad \times \left(\frac{1}{z-e^{i\pi/4}\sqrt{s})} - \frac{1}{z+e^{i\pi/4}\sqrt{s})}\right).
\end{aligned} \tag{3.44}$$

By using the following formulas for the Laplace transform

$$\begin{aligned}
\mathcal{L}_z^{-1}\Big(\frac{1}{z+a\sqrt{s}}\Big) &= \theta(x)e^{-ax\sqrt{s}} \\
\mathcal{L}_s^{-1}\Big(\frac{1}{\sqrt{s}}e^{-ax\sqrt{s}}\Big) &= \frac{\theta(t)}{\sqrt{\pi t}}e^{-(ax)^2/(4t)},\ x>0,\ \operatorname{Re} a > 0 \\
&= \theta(t)\chi(ax,t).
\end{aligned}$$

We can find the Laplace transforms in (3.44). Let us do it

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{\mathcal{L}(A_2)(s)}{2is(z+e^{i\pi/4}\sqrt{s})}\right) &= \mathcal{L}_s^{-1}\circ\left(\mathcal{L}_z^{-1}\left(\frac{1}{z+e^{i\pi/4}\sqrt{s}}\right)\frac{\mathcal{L}(A_2)(s)}{2is}\right) \\
&= \frac{1}{2i}\mathcal{L}_s^{-1}\left(\frac{1}{\sqrt{s}}\,e^{-e^{i\pi/4}\sqrt{s}x}\right)\frac{1}{\sqrt{s}}\mathcal{L}(A_2)(s) \\
&= \frac{\theta(x)\theta(t)}{2i\Gamma(1/2)}\,\chi(e^{i\pi/4}x,t) \overset{t}{*} \int_0^t (t-\tau)^{-1/2}A_2(\tau)\,d\tau.
\end{aligned}$$

The second addend in (3.44) is:

$$\frac{\mathcal{L}(B_1)(z) - \mathcal{L}(B_1)(e^{i\pi/4}\sqrt{s})}{4ise^{i\pi/4}\sqrt{s}}\left(\frac{1}{z-e^{i\pi/4}\sqrt{s}} - \frac{1}{z+e^{i\pi/4}\sqrt{s}}\right). \tag{3.45}$$

We shall start with:

$$\mathcal{L}^{-1}\left(\frac{\mathcal{L}(B_1)(z) - \mathcal{L}(B_1)(e^{i\pi/4}\sqrt{s})}{4ise^{i\pi/4}\sqrt{s}(z+e^{i\pi/4}\sqrt{s})}\right) \tag{3.46}$$

$$= \mathcal{L}_z^{-1} \circ \mathcal{L}_s^{-1}\left(\frac{\mathcal{L}(B_1)(z)}{4ise^{i\pi/4}\sqrt{s}(z+e^{i\pi/4}\sqrt{s})}\right) - \mathcal{L}_s^{-1} \circ \mathcal{L}_z^{-1}\left(\frac{\mathcal{L}(B_1)(e^{i\pi/4}\sqrt{s})}{4ise^{i\pi/4}\sqrt{s}(z+e^{i\pi/4}\sqrt{s})}\right).$$

The first addend in (3.46) is

$$\begin{aligned} &\mathcal{L}_z^{-1}\left(B_1(z)\mathcal{L}_s^{-1}\left(\frac{1}{4(e^{i\pi/4}\sqrt{s})^3(e^{i\pi/4}\sqrt{s}+z)}\right)\right) \\ &= \mathcal{L}_z^{-1}\left(B_1(z)\mathcal{L}_s^{-1}\frac{1}{4e^{3i\pi/4}s} \overset{t}{*} \mathcal{L}_s^{-1}\frac{1}{(z+e^{i\pi/4}\sqrt{s})\sqrt{s}}\right) \\ &= \frac{1}{4e^{3i\pi/4}}\int_0^t \chi(e^{i\pi/4}x,\tau)\,d\tau \overset{x}{*} B_1(x). \end{aligned} \tag{3.47}$$

For the second addend in (3.46) we have

$$\begin{aligned} &-\mathcal{L}_s^{-1} \circ \mathcal{L}_z^{-1}\left(\frac{\mathcal{L}(B_1)(e^{i\pi/4}\sqrt{s})}{4ise^{i\pi/4}\sqrt{s}(z+e^{i\pi/4}\sqrt{s})}\right) \\ &= -\mathcal{L}_s^{-1}\left(\mathcal{L}_s(B_1)(e^{i\pi/4}\sqrt{s}) \cdot \frac{1}{4e^{3i\pi/4}s} \cdot \frac{1}{\sqrt{s}}\mathcal{L}_z^{-1}\left(\frac{1}{z+e^{i\pi/4}\sqrt{s}}\right)\right) \\ &= -\mathcal{L}_s^{-1}\left(\frac{1}{4e^{3i\pi/4}s}\frac{\theta(x)}{\sqrt{s}}\, e^{-e^{i\pi/4}x\sqrt{s}}\int_0^\infty e^{-e^{i\pi/4}\sqrt{s}\tau}B_1(\tau)\,d\tau\right) \\ &= d - \frac{1}{4e^{3i\pi/4}} \overset{t}{*} \mathcal{L}_s^{-1}\left(\frac{1}{\sqrt{s}}\int_0^\infty e^{-e^{i\pi/4}\sqrt{s}(x+\tau)}B_1(\tau)\,d\tau\right) \\ &= -\frac{1}{4e^{3i\pi/4}} \overset{t}{*} \int_0^\infty e^{-\frac{1}{4}i(x+\tau)^2/t}\frac{1}{\sqrt{\pi t}}B_1(\tau)\,d\tau \\ &= -\frac{1}{4ie^{i\pi/4}}\int_0^t du\int_0^\infty \chi(e^{i\pi/4}(x+\tau),u)B_1(\tau)\,d\tau. \end{aligned}$$

Applying the inverse Laplace transformation the first fraction in (3.45) becomes

$$\begin{aligned} &\mathcal{L}^{-1}\left(\frac{\mathcal{L}(B_1)(z)-\mathcal{L}(B_1)(e^{i\pi/4}\sqrt{s})}{4ise^{i\pi/4}s\sqrt{s}(z-e^{i\pi/4}\sqrt{s})}\right) \\ &= \mathcal{L}^{-1}\frac{\mathcal{L}(B_1)(z)}{4ie^{i\pi/4}s\sqrt{s}(z-e^{i\pi/4}\sqrt{s})} - \mathcal{L}^{-1}\frac{\mathcal{L}(B_1)(e^{i\pi/4}\sqrt{s})}{4ie^{i\pi/4}s\sqrt{s}(z-e^{i\pi/4}\sqrt{s})} \\ &= \frac{1}{4ie^{i\pi/4}}\left(\mathcal{L}_s^{-1}\frac{1}{s\sqrt{s}}\,e^{e^{i\pi/4}x\sqrt{s}} \overset{x}{*} B_1(x) - \mathcal{L}_s^{-1}\frac{1}{s\sqrt{s}}\,e^{e^{i\pi/4}sx}\int_0^\infty e^{-e^{i\pi/4}\sqrt{s}u}B_1(u)\,du\right) \\ &= \frac{1}{4ie^{i\pi/4}}\left(\mathcal{L}_s^{-1}\frac{1}{s\sqrt{s}}\int_0^x e^{i\frac{\pi}{4}(x-u)\sqrt{s}}B_1(u)\,du - \mathcal{L}_s^{-1}\frac{1}{s\sqrt{s}}\int_0^\infty e^{-e^{i\frac{\pi}{4}(u-x)\sqrt{s}}}B_1(u)\,du\right. \end{aligned}$$

$$= \frac{-1}{4ie^{i\pi/4}} \mathcal{L}_s^{-1}\left(\frac{1}{s\sqrt{s}} \int_x^\infty e^{-e^{i\frac{\pi}{4}(u-x)\sqrt{s}}} B_1(u)\, du \right)$$

(3.48)

$$= \frac{-1}{4ie^{i\pi/4}} \int_0^t \int_x^\infty \chi(e^{i\pi/4}(u-x),\tau) B_1(u)\, du\, d\tau.$$

If we collect all the results obtained in (3.47)–(3.48), then the inverse LT of (3.45) is a function denoted by $F(B_1,x,t,\pi/4)$,

$$\begin{aligned} F\Big(B_1,x,t,\frac{\pi}{4}\Big) &= -\frac{1}{4ie^{i\pi/4}} \int_0^t \int_x^\infty \chi(e^{i\pi/4}(u-x),\tau) B_1(u)\, du\, d\tau \\ &\quad - \frac{1}{4ie^{i\pi/4}} \int_0^t \chi(e^{i\pi/4}x,\tau)\, d\tau \overset{x}{*} B_1(x) \\ &\quad + \frac{1}{4ie^{i\pi/4}} \int_0^t du \int_0^\infty \chi(e^{i\pi/4}(x+\tau),u) B_1(\tau)\, d\tau. \end{aligned}$$

To find the inverse LT of (3.44), it is yet to be find the inverse LT of

$$\text{(3.49)} \qquad \frac{s(\mathcal{L}(B_0)(z) - \mathcal{L}(B_0)(e^{i\pi/4}\sqrt{s}))}{4e^{3i\pi/4}s\sqrt{s}} \left(\frac{1}{z - e^{i\pi/4}\sqrt{s}} - \frac{1}{z + e^{i\pi/4}\sqrt{s}} \right).$$

If we compare (3.49) with (3.45), we can observe that in the structure of (3.49) we have additionally only a product by s. Since $F(B_0,x,0,\frac{\pi}{4}) = 0$, the inverse LT of (3.49) is $\partial F(B_0,x,t,\pi/4)/\partial t$.

The procedure of finding the inverse Laplace transform of the second addend in (3.42) is just the same as for the first one. The details, the complete solution and the comments one can find in [61].

Remark. If in equation (3.39), $(x,t) \in (0,1) \times \mathbb{R}_+$, then we can consider the equation (3.41) in $\mathcal{D}'_\omega((0,1)\times\mathbb{R}_+)$ (cf. Section 1, Subsection 1.5).

3.5. The case in which a generalized function appears just in the model. We shall study the existence and properties of the solutions to the following system of coupled partial differential equations (cf. Section 2, (2.3.4)):

$$\text{(3.50)} \qquad \begin{aligned} &\frac{\partial^2 m}{\partial \xi^2} + \lambda \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial t^2} = 0, \\ &\frac{\partial^2 u}{\partial \xi^2} + \mu_1 D_t^\alpha \frac{\partial^2 u}{\partial \xi^2} - m - \mu D_t^\alpha m = 0, \quad t > 0, \quad 0 < \xi < 1, \end{aligned}$$

with boundary conditions

$$\text{(3.51)} \qquad m(0,t) = 0,\ m(1,t) = 0,\ \ u(0,t) = 0,\ \ u(1,t) = 0,\ \ t \geqslant 0.$$

We assume solutions to (3.50), (3.51) in the form $m(\xi,t) = M(\xi)V(t)$, $u(\xi,t) = U(\xi)T(t)$. Then for every $k = \pm 1, \pm 2, \dots$ the system (3.50), (3.51) reduces to

$$M_k(\xi) = C_k \sin k\pi\xi, \quad U_k = C_k \sin k\pi\xi, \tag{3.52}$$

and

$$\begin{gathered} -(k\pi)^2 V_k(t) - \lambda (k\pi)^2 T_k(t) + T_k^{(2)}(t) = 0, \\ V_k(t) + \mu V_k^{(\alpha)}(t) + (k\pi)^2 T_k(t) + \mu_1 (k\pi)^2 T_k^{(\alpha)}(t) = 0, \quad k \in \pm\mathbb{N}, \end{gathered} \tag{3.53}$$

where C_k are arbitrary constants.

Throughout this example we shall assume that, firstly $\mu \neq 0$, $\mu_1 \neq 0$ and secondly

$$\lambda \equiv \frac{FL^2}{E_0 I} = B + A\delta(t - t_0), \quad t_0 > 0.$$

The second assumption means that the axial force is subject to an impulsive change. Consequently, in equation $(4.4)_1$ we have the product $\delta(t-t_0)T_k(t)$. Since δ can be treated as a measure, this product has a meaning for any $t_0 > 0$ if $T_k \in \mathcal{C}([0,\infty))$. Then $\delta(t - t_0)T_k(t) = T_k(t_0)\delta(t - t_0)$ (cf. [56]). This fact one has to take into account when we construct the generalized solutions. Such solution can be only a regular generalized function defined by a continuous function $T_k(t)$.

To solve the system (3.53) we will use the Laplace transform (cf. Section 1, Subsections 1.5 and 2.2) applied on functions or generalized functions with support in $\overline{\mathbb{R}}_+$. A function and its derivatives with the support in $\overline{\mathbb{R}}_+$ can have discontinuities at zero. For this reason, when we construct the system in $\mathcal{D}'(\mathbb{R})$ which corresponds to the system (3.53), we have to take care of the property 8.1 of a derivative given in Section 1, Subsection 1.2. Let us take for short in (3.53) that $k = 1$.

So if T is bounded in $[0, \varepsilon)$, for an $\varepsilon > 0$ (an assumption which is supposed to be satisfied in this case), then

$$\begin{aligned} D_t^\alpha[\theta(t)T(t)] &= [\theta(t) D_t^\alpha T(t)], \quad 0 < \alpha < 1, \\ D_t^{(2)}[\theta(t)T(t)] &= [\theta(t) T^{(2)}(t)] + T^{(1)}(0)\delta(t) + T(0)\delta^{(1)}(t). \end{aligned}$$

Consequently, to (3.53) it corresponds in $\mathcal{D}'(\overline{\mathbb{R}}_+)$

$$\begin{gathered} D^2[\theta T] - B\pi^2[\theta T] - \pi^2[\theta V] = T(0)\delta^{(1)}(t) + T^{(1)}(0)\delta(t) + \pi^2 A T(t_0)\delta(t - t_0), \\ \mu D^\alpha[\theta V] + \mu_1 \pi^2 D^\alpha[\theta T] + [\theta V] + \pi^2[\theta T] = 0. \end{gathered}$$

Applying the generalized Laplace transform (cf. 1.1.5) with the following notation: $\mathcal{L}[(\theta T)](s) = \widehat{T}(s)$, $\mathcal{L}[\theta V](s) = \widehat{V}(s)$, $T(0) = T_0$, and $T^{(1)}(0) = T_0^1$, we have

$$\begin{gathered} -\pi^2 \widehat{V}(s) - (B\pi^2 - s^2)\widehat{T}(s) = T_0 s + T_0^1 + \pi^2 A T(t_0) e^{-t_0 s}, \\ (1 + \mu s^\alpha)\widehat{V}(s) + \pi^2(1 + \mu_1 s^\alpha)\widehat{T}(s) = 0. \end{gathered} \tag{3.54}$$

The solution to system (3.54) is

$$\widehat{T}(s) = \frac{s^\alpha + 1/\mu}{\Delta(s)} \left(T_0^1 \mu + T_0 \mu s + \pi^2 A T(t_0) \mu e^{-t_0 s} \right),$$

(3.55) $$-\widehat{V}(s) = \frac{s^\alpha + 1/\mu_1}{\Delta(s)} \big(T_0^1\mu_1 + T_0\mu_1 s + \pi^2 AT(t_0)\mu_1 e^{-t_0 s}\big),$$

where

$$\Delta(s) = \mu s^{\alpha+2} + s^2 + (\mu_1\pi^2 - B\mu)\pi^2 s^\alpha + (\pi^2 - B)\pi^2 = \mu s^{\alpha+2} + s^2 + as^\alpha + d,$$

and

$$a = \pi^2(\mu_1\pi^2 - B\mu); \quad d = \pi^2(\pi^2 - B).$$

The next step is to find the distribution which corresponds to (3.55). The main part to the solution (3.55) is the function

(3.56) $$\hat{f}(s) = \frac{s^\alpha + 1/\mu}{\Delta(s)}.$$

To the function $\hat{f}(s)$ we can apply Theorem 3 in [19, Vol 1, p. 263], as well. In fact, there exists $f \in L_{\mathrm{loc}}[0,\infty)$ and $x_1 > 0$, such that

(3.57) $$f(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{ts}\widehat{f}(s)\,ds, \quad x > x_1, \ \ t \geqslant 0,$$

$(\mathcal{L}f)(s) = \hat{f}(s)$. Here $(\mathcal{L}f)(s)$ denotes the classical Laplace transform of f defined as $(\mathcal{L}f)(s) = \int_0^\infty e^{-st} f(t)\,dt$.

Since the integral in (3.57) converges uniformly for $0 \leqslant t_0 \leqslant t \leqslant t_1 < \infty$, $f(t)$ is a continuous function in $[0,\infty)$. Consequently, $f(t)$ is bounded in the interval $[0,\varepsilon]$, $0 < \varepsilon < \infty$. How such an integral can be calculated, see for example [25]. But we will find an analytic form for f which is, in our opinion, more suitable then integral (3.57) (cf. [59]).

Let us analyze the function f defined by (3.57). Put $c = \frac{1}{\mu}\big(d - a/\mu\big)$. Then

$$\begin{aligned}\frac{1}{\Delta(s)} &= \frac{1/\mu}{(s^2 + a/\mu)(s^\alpha + 1/\mu) + c} \\ &= \frac{1/\mu}{(s^2 + a/\mu)(s^\alpha + 1/\mu)} \times \left(1 + \sum_{\nu=1}^{\infty}(-c)^\nu \Big(\frac{1}{s^2 + a/\mu}\Big)^\nu \Big(\frac{1}{s^\alpha + 1/\mu}\Big)^\nu\right).\end{aligned}$$

First we find the function $\phi_\alpha(t)$, $t \geqslant 0$, such that

(3.58) $$(\mathcal{L}\phi_a)(s) = \sum_{\nu=1}^{\infty}(-c)^\nu \Big(\frac{1}{s^2 + a/\mu}\Big)^\nu \Big(\frac{1}{s^\alpha + 1/\mu}\Big)^\nu.$$

Then,

$$\frac{1}{\Delta(s)} = \frac{1}{\mu}\frac{1}{(s^2 + a/\mu)(s^\alpha + 1/\mu)}\big(1 + (\mathcal{L}\phi_a)(s)\big).$$

We will denote by $\omega(t)$ the function

(3.59) $$\omega(t) = \alpha t^{\alpha-1} E_\alpha^{(1)}(z),$$

where $z = -t^\alpha/\mu$, $t \geqslant 0$ and $E_\alpha(z)$ is Mittag-Leffler's function (see [22] and [26]). We know that $(\mathcal{L}\omega)(s) = (s^\alpha + 1/\mu)^{-1}$ (cf. [25]). In our analysis of the terms of the series (3.58), we have to distinguish three cases: $a > 0$, $a = 0$ and $a < 0$. Thus,

$$\Big(\frac{1}{s^2 + a/\mu}\Big)^\nu \Big(\frac{1}{s^\alpha + 1/\mu}\Big) = \begin{cases} \Big(\sqrt{\frac{\mu}{a}}\Big)^\nu \mathcal{L}\Big(\Big(\sin\sqrt{\frac{a}{\mu}}t * \omega(t)\Big)^{*\nu}\Big)(s), & a > 0\\ \mathcal{L}((t*\omega(t))^{*\nu})(s), & a = 0\\ \Big(\sqrt{\frac{\mu}{-a}}\Big)^\nu \mathcal{L}\Big(\Big(\sinh\sqrt{\frac{-a}{\mu}}t * \omega(t)\Big)^{*\nu}\Big)(s)(s), & a < 0, \end{cases}$$

where $f^{*\nu}$ means ν-fold convolution of f. We have to evaluate the obtained convolutions. First for the function ω given by (3.59) we need some properties of the Mittag-Leffler function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

Namely, $E_\alpha(z)$ is an entire function with the properties:

$$E_\alpha(z) = \frac{-1}{\Gamma(1-\alpha)}\,\frac{1}{z} + O(|z|^{-2}), \quad |\arg(-z)| < (1-\alpha/2)\pi, \quad z \to \infty,$$

$$E_\alpha^{(1)}(z) = \sum_{k=1}^{\infty} \frac{kz^{k-1}}{\Gamma(\alpha k + 1)}.$$

By [13, p. 36],

$$E_\alpha^{(1)}(z) = \frac{1}{\Gamma(1-\alpha)}\,\frac{1}{z^2} + O(|z|^{-3}), \quad |\arg(-z)| < (1-3\alpha/4)\pi, \quad z \to \infty.$$

Consequently,

$$\omega(t) \sim \frac{\alpha}{\Gamma(1+\alpha)}\, t^{\alpha-1} = \frac{1}{\Gamma(\alpha)}\, t^{\alpha-1}, \quad t \to 0;$$

$$\omega(t) \sim \frac{\alpha}{\Gamma(1-\alpha)}\, t^{\alpha-1}\mu^2 \frac{1}{t^{2\alpha}} = -\frac{\mu^2}{\Gamma(-\alpha)}\, t^{-(1+\alpha)}, \quad t \to \infty,$$

and

$$\omega(t) \sim O(t^{\alpha-1}), \quad t \to 0;$$

$$\omega(t) \sim O(t^{\alpha-1}), \quad t \to \infty.$$

Then, there exists a constant C_1 such that $|\omega(t)| \leqslant C_1 t^{\alpha-1}$, $0 < t < \infty$. Now, we can estimate the terms in the series

$$\phi_a(t) = \sum_{\nu=1}^{\infty} (-c)^\nu \big(\sqrt{\mu/a}\big)^\nu \big(\sin\sqrt{a/\mu}\,\tau * \omega(\tau)\big)^{*\nu}(t), \quad t \geqslant 0, \tag{3.60}$$

in our three cases: $a > 0$, $a = 0$, and $a < 0$. Let us start with $a > 0$. If $\nu = 1$, $a > 0$:

$$\big|\big(\sin\sqrt{a/\mu}\,\tau * \omega(\tau)\big)(t)\big| \leqslant C_1 \int_0^t \tau^{\alpha-1} d\tau = \frac{C_1}{\alpha} t^\alpha = C_2 \frac{t^\alpha}{\Gamma(\alpha+1)}, \quad t \geqslant 0.$$

For any $\nu \in \mathbb{N}$ and $t \geqslant 0$,

$$\begin{aligned}\left|\left(\sin\sqrt{a/\mu}\,\tau * \omega(\tau)\right)^{*\nu}(t)\right| &\leqslant C_2^\nu\left(\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right)^{*\nu}(t)\\ &\leqslant C_2^\nu \mathcal{L}^{-1}\frac{1}{s^{(\alpha+1)\nu}} \leqslant C_2^\nu\frac{t^{(\alpha+1)\nu-1}}{\Gamma(\nu(\alpha+1))}.\end{aligned}$$

Let us set $F_\nu(t) = (-c)^\nu\left(\sqrt{\mu/a}\right)^\nu\left(\sin\sqrt{a/\mu}\,\tau * \omega(\tau)\right)^{*\nu}$, $\nu = 1, 2, \ldots$. Then for $t \geqslant 0$,

$$\sum_{\nu=1}^\infty |F_\nu(t)| \leqslant t^{-1}\sum_{\nu=1}^\infty\left(|C|\sqrt{\mu/a}\,t^{\alpha+1}\right)^\nu\frac{1}{\Gamma(\nu\alpha+1)} \leqslant t^{-1}\left(E_\alpha\left(|C|\sqrt{\mu/a}\,t^{\alpha+1}\right)-1\right).$$

Hence, the series (3.60) is absolute convergent for $t \geqslant 0$ and $\phi_\alpha(t)$ is bounded on every compact set $K \subset [0, \infty)$.

Now we have the following properties of F_ν, $\nu \in \mathbb{N}$:

(1) $\mathcal{L}(F_\nu)(s) = (-c)^\nu\left(\frac{1}{s^2+a/\mu}\right)^\nu\left(\frac{1}{s^\alpha+1/\mu}\right)^\nu$;

(2) $\int\limits_0^\infty e^{-x_0t}|F_\nu(t)|dt \leqslant |c|^\nu\left(\sqrt{\mu/a}\right)^\nu C_2^\nu\frac{1}{x_0^{\nu(\alpha+1)}}$, $\nu = 1, 2, \ldots$

(3) The series $\sum\limits_{\nu=1}^\infty\int\limits_0^\infty e^{-x_0t}|F_\nu(t)|dt \leqslant \sum\limits_{\nu=1}^\infty\left(\frac{|c|\sqrt{\mu/a}\,C_2}{x_0^{\alpha+1}}\right)^\nu$ converges, if $x_0^{\alpha+1} > |c|\sqrt{\frac{\mu}{a}}C_2$.

By Theorem 2, in [19, Vol. 1, p. 305], $\mathcal{L}(\phi_a)(x) = \widehat{\phi}_a(s)$, $a > 0$, with $\widehat{\phi}_a(s)$ given by (3.58).

In the other two cases the procedure is just the same. We have only to use the following evaluations, for $\nu \in \mathbb{N}$ and $t \geqslant 0$:

$$\begin{aligned}|(\tau * \omega(\tau))^{*\nu}(t)| &\leqslant (\tau * (C_1\tau^{\alpha-1}))^{*\nu}(t) = C_2^\nu\left(\tau * \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}\right)^{*\nu}(t)\\ &\leqslant C_2^\nu\left(\frac{\tau^{\alpha+1}}{\Gamma(\alpha+2)}\right)^{*\nu}(t) = C_2^\nu\frac{t^{(\alpha+2)\nu-1}}{\Gamma(\nu(\alpha+2))},\end{aligned}$$

and

$$\left|\left(\sin h\sqrt{-a/\mu}\,\tau\right)^{*\nu}(t)\right| \leqslant \frac{t^{\nu-1}}{\Gamma(\nu)}\,e^{\sqrt{-a/\mu}\,t},\quad t \geqslant 0.$$

Now, the function $\hat{f}(s)$ in (3.56) is:

$$\hat{f}(s) = \frac{s^\alpha + 1/\mu}{\Delta(s)} = \frac{1}{\mu}\,\frac{1}{s^2+a/\mu}(1+\widehat{\phi}_a(s)),$$

where $\widehat{\phi}_a(s)$ is given by (3.58). Consequently for $t \geqslant 0$,

$$f(t) = (\mathcal{L}^{-1}\hat{f})(t) = \frac{1}{\mu}\left[\left(\mathcal{L}^{-1}\frac{1}{s^2+a/\mu}\right)(t) + \left(\left(\mathcal{L}^{-1}\frac{1}{s^2+a/\mu}\right) * \phi_a\right)(t)\right], \tag{3.61}$$

and

$$f^{(1)}(t) = \left(\mathcal{L}^{-1}s\hat{f}(s)\right)(t) = \frac{1}{\mu}\left[\left(\mathcal{L}^{-1}\frac{s}{s^2+a/\mu}\right)(t) + \left(\left(\mathcal{L}^{-1}\frac{s}{s^2+a/\mu}\right) * \phi_a\right)(t)\right], \tag{3.62}$$

where

$$\phi_a(t) = \sum_{\nu=1}^{\infty} (-c)^{\nu} \Big(\Big(\mathcal{L}^{-1} \frac{1}{s^2 + a/\mu} \Big)^{*\nu} * \omega^{*\nu} \Big)(t).$$

Note that in all three case: $a > 0$, $a = 0$ and $a < 0$, we have $f(0) = 0$. Hence, $s\hat{f}(s) = (\mathcal{L} f^{(1)})(s)$. Also

$$\frac{1}{\Delta(s)} = \frac{1}{s^{\alpha} + 1/\mu} \hat{f}(s) = \mathcal{L}(\omega * f)(s), \quad \frac{s}{\Delta(s)} = \mathcal{L}(\omega * f^{(1)})(s).$$

Now we can fix the form of the solution to (3.53), for $t \geqslant 0$,

$$\begin{aligned} T(t) &= T_0^1 \mu f(t) = T_0 \mu f^{(1)}(t) + \pi^2 A T(t_0) \mu \theta(t - t_0) f(t - t_0); \\ -V(t) &= T_0^1 \mu_1 \Big(f(t) + \Big(\frac{1}{\mu_1} - \frac{1}{\mu} \Big)(\omega * f)(t) \Big) \\ &\quad + T_0 \mu_1 \Big(f^{(1)}(t) + \Big(\frac{1}{\mu_1} - \frac{1}{\mu} \Big)(\omega * f^{(1)})(t) \Big) \\ &\quad + \pi^2 A T(t_0) \mu_1 \theta(t - t_0) \Big(f(t - t_0) + \Big(\frac{1}{\mu_1} - \frac{1}{\mu} \Big)(\omega * f)(t - t_0) \Big) \end{aligned} \tag{3.63}$$

where $f(t)$ and $f^{(1)}(t)$ are given by (3.61) and (3.62). To analyze the character of the solution (3.63) we will find the first and second derivatives of f. By (3.62) $f^{(1)}(t)$, $f^{(2)}(t)$ and $f^{(3)}(t)$ belong to $\mathcal{C}_{[0,\infty)}$.

From the properties of the generalized Laplace transform it follows that (3.63) is the unique solution in $L_{\rm loc}([0,\infty))$ such that T and V are bounded in $[0,\varepsilon]$ for $\varepsilon > 0$.

We state now the main results of this section:

Theorem 3.5. *A solution to* (3.53) *is given by* (3.63). *This solution is continuous on* $[0,\infty)$. *If* $A = 0$, *then the solution belongs to* $\mathcal{C}^2_{[0,\infty)}$ *and is a classical one; it can be obtained by the classical Laplace transform. In the general case the functions* $T(t)$ *and* $V(t)$ *define regular distributions* $[T(t)]$ *and* $[V(t)]$ *which are solutions to* (**??**) *and generalized solution to* (3.53).

Remark. The continuity of T and V follows from the fact that $f(0) = 0$.

Theorem 3.6. *A family of solutions to* (3.50) *and* (3.51) *is*

$$m_k(\xi, t) = M_k(\xi) V_k(t), \ \ u_k(\xi, t) = U_k(\xi) T_k(t), \ \ k \in \pm\mathbb{N},$$

where M_k *and* U_k *are given by* (3.52), $k \in \mathbb{N}$, *and* V_k *and* T_k *are given by* (3.63) *when instead of* π *we take* πk.

3.6. Localization of the solution. The mathematical model of lateral vibration of a viscoelastic axially loaded rod (cf. Section 2, (2.3.12)) is

$$\begin{aligned} &\frac{\partial^2 m}{\partial \xi^2} + \lambda \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial t^2} = 0; \\ &\frac{\partial^2 u}{\partial \xi^2} + \mu_1 D_t^{\alpha} \frac{\partial^2 u}{\partial \xi^2} + \mu_2 D_t^{\beta} \frac{\partial^2 u}{\partial \xi^2} = m + \mu D_t^{\alpha} m, \ \ 0 < t, \ 0 < \xi < 1, \end{aligned} \tag{3.64}$$

with the boundary conditions:

$$m(0,t)=0;\ \ m(1,t)=0;\ \ u(0,t)=0,\ \ u(1,t)=0,\ \ t\geqslant 0. \tag{3.65}$$

We consider the vibrations of the rod when it is loaded by a compressive axial force F such that the intensity λ of the force F is $\lambda = B + A\theta(t-t_0)$, $t_0>0$, where θ is Heaviside's function and A, B are constants.

To stress possibilities of the Laplace transform of generalized functions (cf. Section 1, Subsections 1.5 and 1.2) we consider more general system which can appear as a model of an other situation, as well, namely:

$$\begin{gathered}\frac{\partial^2 m}{\partial\xi^2}+\lambda\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial t^2}=g(t)\sin\,k\,\pi\,\xi,\ \ k\in\mathbb{N},\\ \frac{\partial^2 u}{\partial\xi^2}+\mu_1\,D_t^{\alpha}\frac{\partial^2 u}{\partial\xi^2}+\mu_2 D_t^{\beta}\,\frac{\partial^2 u}{\partial\xi^2}=m+\mu\,D_t^{\alpha}m,\end{gathered} \tag{3.66}$$

$0<t$, $0<\xi<1$, with the same boundary conditions (3.65), where $g\in\mathcal{C}([0,\infty))$ and without any growth condition. In case $g=0$ system (3.66) becomes (3.64).

Let us remark that in system (3.66) we have a coefficient which is a discontinuous function with a discontinuity in $t=t_0>0$. Since the product of a discontinuous function and a generalized function, e.g., of a distribution and a hyperfunction, is not defined, we can not to expect such a generalized solution to (3.66). So we have to localize the procedure of the construction of the solutions to (3.66). Therefore, we construct a solution for the domain $D_1=\{(\xi,t);\ 0<\xi<1,\ 0<t<t_0\}$ with boundary conditions (3.65) and initial conditions in $t=0$ and then for the domain $D_2=\{(\xi,t);\ 0<\xi<1,\ t_0<t\}$ using the Laplace transform presented in Section 1, Subsection 1.1.5. At the end we try to find a "global" solution to (3.66).

We start with the separation of variables.

Let us suppose that the solutions of the system (3.66), (3.65) have the from

$$m(\xi,t)=M(\xi)V(t),\ u(\xi,t)=U(\xi)T(t).$$

It is easily seen that for M and U, which satisfy the boundary conditions from (5.2), we have a family of solutions:

$$M_k(\xi)=C_k\sin\,k\,\pi\,\xi;\ \ U_k(\xi)=C_k\sin\,k\,\pi\,\xi,\ \ k\in\mathbb{N}.$$

In order to find the corresponding values T_k and V_k we have to solve the system:

$$\begin{gathered}T_k^{(2)}(t)-\lambda(k\pi)^2T_k(t)-(k\pi)^2V_k(t)=g(t);\\ V_k(t)+\mu V_k^{(\alpha)}(t)+(k\pi)^2T_k(t)+\mu_1(k\pi)^2T_k^{(\alpha)}+\mu_2(k\pi)^2T_k^{(\beta)}(t)=0,\ \ 0<t.\end{gathered} \tag{3.67}$$

We start with the domain D_1. Then we analyze system (3.67) in the interval $(0,t_0)$ with initial condition in $t=0$ and with $\lambda=B$. In this case to (3.67) it corresponds in $\mathcal{D}'_\omega([0,t_0))$ (cf. 8.1 Section 1, Subsection 1.1.2):

$$\begin{gathered}D^2[H_0T_k]-B(k\pi)^2[H_0T_k]-(k\pi)^2[H_0V_k]=[H_0g]+T_{k0}\delta^{(1)}(t)+T^1_{k0}\delta(t),\\ [H_0V_k]+\mu D^{\alpha}[H_0V_k]+(k\pi)^2[H_0T_k]+\mu_1(k\pi)^2D^{\alpha}[H_0T_k]+\mu_2(k\pi)^2D^{\beta}[H_0T_k]=0,\end{gathered} \tag{3.68}$$

where $T_{k0} = T_k(0),\ T^1_{k0} = T^{(1)}_k(0)$. Applying the LT to (3.68) we get

$$\begin{aligned}(s^2 - B(k\pi)^2)\widehat{\widehat{T}}_k(s) - (k\pi)^2\widehat{\widehat{V}}_k(s) &= \widehat{g}(s) + T_{k0}s + T^1_{k0} + \widehat{r}_1(s);\\ (1+\mu s^\alpha)\widehat{\widehat{V}}_k(s) + (k\pi)^2(1+\mu_1 s^\alpha + \mu_2 s^\beta)\widehat{\widehat{T}}_k(s) &= \widehat{r}_2(s),\end{aligned} \tag{3.69}$$

where $r_1, r_2 \in \mathcal{A}$. For simplicity we solve system (3.68) for $k = 1$. Let

$$\begin{aligned}\Delta_{10}(s) &= \begin{vmatrix} s^2 - B\pi^2 & -\pi^2 \\ \pi^2(1+\mu_1 s^\alpha + \mu_2 s^\beta) & (1+\mu s^\alpha)\end{vmatrix}\\ &= \mu s^{2+\alpha} + s^2 + \pi^2(\mu_1\pi^2 - B\mu)s^\alpha + \pi^4\mu_2 s^\beta + \pi^2(\pi^2 - B)\\ &= \mu s^{2+\alpha} + s^2 + a s^\alpha + b s^\beta + d,\end{aligned}$$

where $a = \pi^2(\mu_1\pi^2 - B\mu)$, $b = \pi^4\mu_2$, $d = \pi^2(\pi^2 - B)$,

$$\begin{aligned}\Delta_{11}(s) &= \begin{vmatrix} T_{10}s + T^1_{10} + \widehat{g}(s) + \widehat{r}_1(s) & -\pi^2 \\ \widehat{r}_2(s) & (1+\mu s^\alpha)\end{vmatrix}\\ &= \mu(s^\alpha + 1/\mu)(T_{10}s + T^1_{10} + \widehat{g}(s) + \widehat{r}_1(s)) + \pi^2\widehat{r}_2(s),\end{aligned}$$

$$\begin{aligned}\Delta_{12}(s) &= \begin{vmatrix} s^2 - B\pi^2 & T_{10}s + T^1_{10} + \widehat{g}(s) + \widehat{r}_1(s) \\ \pi^2(1+\mu_1 s^\alpha + \mu_2 s^\beta) & \widehat{r}_2 s\end{vmatrix}\\ &= -\pi^2(T_{10}\mu_2 s^{1+\beta} + T_{10}\mu_1 s^{1+\alpha} + T_{10}s + T^1_{10}\mu_1 s^\alpha + T^1_{10}\mu_2 s^\beta + T^1_{10})\\ &\quad - \pi^4(1+\mu_1 s^\alpha + \mu_2 s^\beta)\widehat{g}(s) - \pi^2(1+\mu_1 s^\alpha + \mu_2 s^\beta)\widehat{r}_1(s) + (s^2 - B\pi^2)\widehat{r}_2(s).\end{aligned}$$

If in Δ_{10}, Δ_{11} and Δ_{12} we replace π with $k\pi$, then we have Δ_{k0}, Δ_{k1} and Δ_{k2} respectively. The solutions $\widehat{\widehat{T}}_k(s)$, $\widehat{\widehat{V}}_k(s)$, $k \in \mathbb{N}$ to system (3.69) are

$$\widehat{\widehat{T}}_k(s) = \frac{\Delta_{k1}(s)}{\Delta_{k0}(s)};\quad \widehat{\widehat{V}}_k(s) = \frac{\Delta_{k2}(s)}{\Delta_{k0}(s)}.$$

Suppose that $\Delta_{k0}(s) \neq 0$, $\operatorname{Re} s > x^0_k > 0$, $k \in \mathbb{N}$. Let us introduce the new variable $\zeta_k = s - x^0_k$ in $\Delta_{ki}(s)/\Delta_{k0}(s)$, $i = 1, 2$,

$$\frac{\Delta_{ki}(s)}{\Delta_{k0}(s)} = \frac{\Delta_{ki}(\zeta_k + x^0_k)}{\Delta_{k0}(\zeta_k + x^0_k)} \equiv Q_{ki}(\zeta_k),\ i = 1, 2,\quad k \in \mathbb{N}.$$

Now the functions $Q_{ki}(\zeta_k)$ are holomorphic on $\mathbb{R}_+ + i\mathbb{R}$ and belong to the space $\mathcal{H}(\mathbb{R}_+)$. Hence, there exist $q_{ki} \in \mathcal{S}'(\overline{\mathbb{R}}_+)$ such that

$$\langle q_{ki}(t), e^{-\zeta_k t}\rangle = Q_{ki}(\zeta_k),\ i = 1, 2,\quad k \in \mathbb{N},$$

or

$$\langle q_{ki}(t),\ e^{-(s - x^0_k)t}\rangle = \frac{\Delta_{ki}(s)}{\Delta_{k0}(s)},\ \ \operatorname{Re} s > x^0_k,\ i = 1, 2.\ \ k \in \mathbb{N}.$$

Hence, a solution to the system (3.68) for a fixed $k \in \mathbb{N}$ is:

$$\begin{aligned}T^0_k(t) &= e^{x^0_k t} q_{k1}(t)|_{[0,b)};\\ V^0_k(t) &= e^{x^0_k t} q_{k2}(t)|_{[0,b)}.\end{aligned}$$

Note that T_k^0 and V_k^0 belong to $\mathcal{D}'([0,b))$ for every b, $0 < b < \infty$ (cf. Section 1, Subsection 1.1.5).

By the similar method as we applied in [59] we can prove that T_k^0 and V_k^0 are regular distributions defined by T_k and V_k which have the following properties for $k \in \mathbb{N}$:

(1) $T_k \in \mathcal{C}^2((0,t_0]) \cap \mathcal{C}^1([0,t_0])$, $T_k^{(2)} \in \mathbf{L}^1([0,t_0]) \cap \mathcal{C}((0,t_0])$; $T_k^{(2)}(t)$ is not bounded in $t = 0$, $\lim\limits_{t\to 0^+} T_k(t) = T_{k0}$, $k \in \mathbb{N}$.

(2) $V_k \in \mathbf{L}^1([0,t_0]) \cap \mathcal{C}((0,t_0])$ and $V_k(t)$ is not bounded at $t = 0$, $V_k(t) = O(t^{-(\beta-\alpha)})$, $k \in \mathbb{N}$ but it satisfies Proposition 1.1 in Section 1.

If additionally $T_k(0) = 0$, then $T_k \in \mathcal{C}^2([0,t_0])$ and $V_k \in \mathcal{C}([0,t_0])$, $V_k^{(1)} \in \mathbf{L}^1([0,t_0]) \cap \mathcal{C}((0,t_0))$.

Consequently, by our definition of the classical solution and generalized solution, in D_1 we have a classical solution to (3.66). The functions T_k and V_k, $k \in \mathbb{N}$, which satisfy (3.67) in $(0,t_0)$ are

$$\begin{aligned} T_k(t) &= T_k(0)\big(\mu F_{\alpha+1}(t) + F_1(t)\big) \\ &\quad + T_k^{(1)}(0)\big(\mu F_\alpha(t) + F_0(t)\big) + \big((\mu F_\alpha + F_0) * g\big)(t); \end{aligned} \tag{3.70}$$

and

$$\begin{aligned} V_k(t) = -(k\pi)^2\big\{&T_k(0)[\mu_2 F_{1+\beta}(t) + \mu_1 F_{1+\alpha}(t) + F_1(t)] \\ &+ T_k^{(1)}(0)[\mu_1 F_\alpha(t) + \mu_2 F_\beta(t) + F_0(t)] \\ &+ \big((\mu_1 F_\alpha + \mu_2 F_\beta + F_0) * g\big)(t)\big\}, \quad 0 < t < t_0, \end{aligned} \tag{3.71}$$

where $F_p(t) = \mathcal{L}^{-1}(s^p/\Delta_{k0}(s))(t)$, $0 \leqslant t < t_0$ (cf. [62]).

With regard to domain D_2 we have to find a solution to system (3.67) but in the interval (t_0,b) for any $b > t_0$, and $\lambda = B+A$. We proceed in the following way: First we have to localize the supposed solution to (3.67) on the interval (t_0,b). Then we suppose that there exists a solution T_k, V_k to (3.67) such that $H_{t_0}T_k \in \mathcal{C}^1([t_0,b))$, $(H_{t_0}T_k)^{(2)} \in \mathbf{L}^1([t_0,b))$; $V_k \in \mathcal{C}((t_0,b)) \cap \mathbf{L}^1([t_0,b))$.

By (1.1) and Proposition 1.1 in Section 1, to (3.67), on the interval $[t_0,b)$ it corresponds in $\mathcal{D}'_\omega([t_0,b))$

$$\begin{aligned} D^2[H_{t_0}T_k] - (A+B)(k\pi)^2[H_{t_0}T_k] - (k\pi)^2[H_{t_0}V_k] \\ = T_k(t_0)D^1\delta(t-t_0) + T_k^{(1)}(t_0)\delta(t-t_0) + [H_{t_0}g]; \end{aligned} \tag{3.72}$$

and

$$[H_{t_0}V_k] + \mu D^\alpha[H_{t_0}V_k] + (k\pi)^2[H_{t_0}T_k] + \mu_1(k\pi)^2 D^\alpha[H_{t_0}T_k] + \mu_2(k\pi)^2 D^\beta[H_{t_0}T_k] = 0.$$

Let

$$\begin{aligned} \overline{T}_k &\in e^{\omega t}\mathcal{S}'\big(\overline{\mathbb{R}}_+ + [t_0,b)\big) \text{ such that } \overline{T}_k|_{(-\infty,b)} = H_{t_0}T_k, \\ \overline{V}_k &\in e^{\omega t}\mathcal{S}'\big(\overline{\mathbb{R}}_+ + [t_0,b)\big) \text{ such that } \overline{V}_k|_{(-\infty,b)} = H_{t_0}V_k \\ \overline{g} &\in e^{\omega t}\mathcal{S}'\big(\overline{\mathbb{R}}_+ + [t_0,b)\big) \text{ such that } \overline{g}|_{(-\infty,b)} = H_{t_0}g. \end{aligned}$$

Applying to (3.72) the defined LT, we get

$$\big(s^2-(A+B)(k\pi)^2\big)\widehat{\overline{T}}_k(s)-(k\pi)^2\widehat{\overline{V}}_k(s)=T_k(t_0)se^{-t_0s}+T_k^{(1)}(t_0)e^{-t_0s}+\widehat{\overline{g}}(s)+\widehat{r}_1(s);$$
$$(1+\mu s^\alpha)\widehat{\overline{V}}_k(s)+(k\pi)^2(1+\mu_1 s^\alpha+\mu_2 s^\beta)\widehat{\overline{T}}_k(s)=\widehat{r}_2(s),$$

where r_1 and $r_2\in\mathcal{A}$. By $\widehat{\overline{T}}_k$ is denoted the LT of $\overline{T}_k$.

When we solve this system in $\widehat{\overline{T}}_k$, $\widehat{\overline{V}}_k$ and use the inverse LT, we get

$$\begin{aligned}(H_{t_0}T_k)(t)&=T_k(t_0)\theta(t-t_0)\big(\mu G_{\alpha+1}(t-t_0)+G_1(t-t_0)\big)\\&\quad+T_k^{(1)}(t_0)\theta(t-t_0)\big(\mu G_\alpha(t-t_0)+G_0(t-t_0)\big)\\&\quad+\big((\mu G_\alpha+G_0)*H_{t_0}g\big)(t);\end{aligned}\tag{3.73}$$

$$\begin{aligned}&(H_{t_0}V_k)(t)\\&=-(k\pi)^2\big\{T_k(t_0)\theta(t-t_0)[\mu_2G_{1+\beta}(t-t_0)+\mu_1G_{\alpha+1}(t-t_0)+G_1(t-t_0)]\\&\quad+T_k^{(1)}(t_0)\theta(t-t_0)[\mu_1G_\alpha(t-t_0)+\mu_2G_\beta(t-t_0)+G_0(t-t_0)]\\&\quad+[(\mu_1G_\alpha+\mu_2G_\beta+G_0)*(H_{t_0}g)](t)\big\},\quad t_0<t<b,\end{aligned}\tag{3.74}$$

where $G_p(t)=\mathcal{L}^{-1}(s^p/\Delta'_{k0})$, Δ'_{k0} equals Δ_{k0} in which instead of B we have $A+B$. Therefore, we can use the properties of solution (3.70), (3.71) to system (3.67) taking into account that we have $A+B$ instead of B.

We have now a solution for the domain D_1, given by (3.70), (3.71) and a solution for the domain D_2 given by (3.73), (3.74). The properties of $H_{t_0}T_k$ and $H_{t_0}V_k$ in $t=t_0$ follow by the properties of T_k and V_k in $t=0$.

Theorem 3.7. *If in the system* (3.66) *with the boundary condition* (3.65)*,* $\lambda=B$ *and* $g\in\mathcal{C}([0,b))$*, for any* $b>0$*, then we have the classical solutions in* $(0,1)\times(0,b)$ *for every* $b>0$*. These solutions are*

$$m_k(\xi,t)=C_k\sin k\pi\xi V_k(t),\quad u_k(\xi,t)=C_k\sin k\pi\xi V_k(t),\quad k\in\mathbb{N},\tag{3.75}$$

where T_k*,* V_k *are of the form* (3.70), (3.71) *for* $0<t<b$*. In case* $\lambda=B+A\theta(t-t_0)$*,* $0<t_0$*,* $A\neq 0$*, there exist the regular distributions* $R_k,Q_k\in D'((0,1)\times(0,b))$ *defined by the functions* $r_k(\xi,t)$ *and* $q_k(\xi,t)$ *respectively,* $k\in N$*, such that:*

1. *r_k and q_k belong to* $\mathcal{C}^\infty([0,1])\times L^1([0,b])$*,* $0<t_0<b<\infty$*.*
2. *The restriction of* $r_k(\xi,t)$ *and* $q_k(\xi,t)$ *to* D_1 *are* $m_k(\xi,t)$ *and*$u_k(\xi,t)$ *given by* (3.75)*, where* V_k *and* T_k *have been given by* (3.70) *and* (3.71)*;*
3. *The restriction of* $r_k(\xi,t)$ *and* $q_k(\xi,t)$ *to* D_2 *are the same functions* m_k *and* u_k *given by* (3.75) *in which instead of* T_k *and* V_k *we have* $H_{t_0}T_k$ *and* $H_{t_0}V_k$ *given by* (3.73), (3.74)*.*

Proof. We have only to prove that two regular distributions defined on $D_1\cup D_2$ by $m_k(\xi,t)$ and $u_k(\xi,t)$ for a fixed $k\in\mathbb{N}$ can be extended to $(0,1)\times(0,b)$ for any b, $0<t_0<b<\infty$. By the properties of V_k and T_k, we cited it is easily seen that the condition of Proposition 1.6 in Section 1 is satisfied. Consequently such extension exists. □

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