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# DIRICHLET'S PRINCIPLE, UNIQUENESS OF HARMONIC MAPS AND EXTREMAL QC MAPPINGS

**CONTENTS**

## Preface

This expository paper consists of the various uniqueness theorems which follow, in general, from the length-area principle of Grötzsch. The structure of this paper is as follows. In Section I we give the main ideas and basic results. In the subsections A, B, C, D and E we discuss connections between the Grötzsch principle, Teichmüller's approach, the Main Inequality and Dirichlet's principle. In the subsections F and G we consider extremal problems for quasiconformal mappings. In particular, we give short review of new results and solve some problems, which originally were subject of investigation of Teichmüller, Reich, Strebel and the other mathematicians.

In Section II we give the outline of proofs of some properties of harmonic maps, using different tools: Dirichlet's principle, minimizing sequences, different versions of Reich–Strebel inequality. We also give a proof of well-known Beurling theorem.

In Section III using Lemma C1 we prove the inequality of Reich and Strebel for Riemann surfaces of finite analytic type and new version of an inequality of Reich and Strebel. We use this result to study the uniqueness properties of harmonic mappings (see section II).

Section IV is an extended version of the lecture given by the author at VIII Romanian–Finish Seminar, Iassy, August '99.

Recently, in [MM1], [BMM] and [BLMM], characterizations of unique extremality and example of unique extremal dilatation of nonconstant modulus have been obtained.

In this section our primary purpose is to give a short exposition of some of the main result of the authors' joint papers, mentioned above, and sketch further progress in the study of a more general concept. We also discuss the Beltrami equation, and we show the necessity of the Hamilton–Krushkal condition.

The most part of the paper consists of the lectures communicated by the author and the other members of the Seminar, at the University of Belgrade during several last years. The author also talked extensively about this subject in a number of places.

# I. Introduction

**A. The problem of Grötzsch.** If $Q$ is a square and $R$ is a rectangle, not a square, there is no conformal mapping of $Q$ on $R$ which maps vertices on vertices. Instead, Grötzsch asked for the most nearly conformal mapping of this kind and took the first step toward the creation of a theory of quasiconformal mappings.

Let $w = f(z)$ be a mapping from one region to another. Recall that we use notation $df = p\,dz + q\,d\bar{z}$, where $p = \partial f$ and $q = \overline{\partial} f$. The complex (Beltrami) dilatation is $\mu_f = \mathrm{Belt}\,[f] = q/p$. The dilatation of $f$ is

$$D_f = \frac{|p| + |q|}{|p| - |q|}.$$

We pass to the Grötzsch problem and give it a precise meaning by saying that $f$ is most nearly conformal if $\sup D_f$ is as small as possible.

Let $R, R'$ be two rectangles with sides $a, b$ and $a', b'$. We may assume that $K = \frac{b'}{a'} : \frac{b}{a} \geqslant 1$. The mapping $f$ is supposed to be $C^1$-homeomorphism from $\overline{R}$ onto $\overline{R'}$, which takes $a$-sides into $a$-sides and $b$-sides into $b$-sides. Next, let $\Gamma_x$ be the vertical segment which is the intersection of the line $\mathrm{Re}\, z = x$ with $\overline{R}$ and $\gamma_x$ the curve which is the image of $\Gamma_x$ under $f$. Using the geometric obvious inequality $b' \leqslant \mathrm{length}(\gamma_x)$ and the Cauchy-Schwarz inequality one gets $K \leqslant \sup D_f$. The minimum is attained for the affine mapping. Note that the restriction to $C^1$-mapping is not essential. The last inequality holds for quasiconformal mapping (see, for example, [Ah2]).

**B. Teichmüller approach.** Teicmüller, following Grötzsch, showed that any homotopic equivalence class of quasiconformal mappings from a compact Riemann surface $M$ to a compact Riemann surface $N$ contains a unique mapping whose maximal dilatation $K$ is minimum. Moreover, this unique mapping can be described geometrically in terms of holomorphic quadratic differentials on $M$. Such a differential gives a way of cutting up the surface $M$ into the euclidean rectangles. These quadratic differentials locally have the form $\Phi = \varphi(z)\,dz^2$, where $\varphi$ is holomorphic and they admit a picture as an orthogonal pair of foliations (horizontal and vertical lines), given locally by lines $\{\mathrm{Re}\,\zeta = \mathrm{const}\}$ and $\{\mathrm{Im}\,\zeta = \mathrm{const}\}$, where $\zeta = \int \sqrt{\varphi(z)}\,dz$, away from zeros of $\Phi$, is natural parameter. The Teichmüller map has the form of a stretching by a factor $K^{1/2}$ along the horizontal lines in this rectangles and a shrinking by a factor $K^{-1/2}$ along vertical lines.

**C. The Main Inequality.** This expository paper consists of the various uniqueness theorems which follow, in general, from the length–area principle of Grötzsch. A powerful version of this principle was given by Marden and Strebel. They called it the minimum norm property for holomorphic quadratic differentials.

Marden and Strebel stated the principle by way of comparison with harmonic quadratic differentials. Gardiner gave two improvements of this principle. In the first version, one takes a minimum over all $L^1$-measurable quadratic differentials. These differentials satisfy an inequality of line integrals taken over arcs which are segments of regular vertical trajectories of a given quadratic differential.

In the second version, the minimum is taken over all conformal quadratic differentials satisfying an inequality of line integrals over all homotopy classes of simple closed curves. These principles lead to the following results: The Main Inequality of Reich and Strebel, the uniqueness part of Teichmüller's theorem, the sufficiency of the Hamilton-Krushkal condition for extremal dilatation.

Let $\Delta$ denote the unit disc and

$$T_\mu\varphi(z) = \frac{\left|1-\mu(z)\frac{\varphi(z)}{|\varphi(z)|}\right|^2}{1-|\mu(z)|^2}.$$

We will refer to the following result as the Reich–Strebel inequality or the Main Inequality.

**Theorem RS** (Reich and Strebel). *Suppose that $f$ is a quasiconformal homeomorphism of $\Delta$ onto itself which is the identity on $\partial\Delta$. Then, with $\mu = \mu_f$*

$$\iint\limits_{\Delta} |\varphi(z)|\,dx\,dy \leqslant \iint\limits_{\Delta} |\varphi(z)| T_\mu\varphi(z)\,dx\,dy,$$

*for every analytic integrable function $\varphi$ on $\Delta$.*

Various forms of this result play a major role in the theory of quasiconformal mappings and have many applications. For applications to extremal and uniquely extremal quasiconformal mappings, we refer the interested reader to the book by Gardiner [G], and for some recent results to [MM1], [BMM], [BLMM] and [Re3].

**D. The energy integral.** Let $M$ and $N$ be two Riemann surfaces with local conformal metrics $\sigma(z)|dz|^2$ and $\rho(z)|dw|^2$ and let $f : M \mapsto N$. It is convenient for us to use the notation in local coordinates $df = (\partial f)\,dz + (\bar\partial f)\,d\bar z$ and $p = \partial f$, $q = \bar\partial f$. The energy integral (Douglas–Dirichlet functional) of $f$ is

$$E(f,\rho) = \int_M e(f)\sigma\,dx\,dy,$$

where $e(f)$ is the energy density defined by

$$e(f)(z) = \left(|p|^2 + |q|^2\right)\frac{\rho\circ f(z)}{\sigma(z)}.$$

If $\rho$ is the euclidean metric ($\rho = 1$ on $N$), then the energy integral of $f$ is Dirichlet integral.

A critical point of the energy functional is called harmonic mapping. The Euler–Lagrange equation for the energy functional is:

$$f_{z\bar z} + (\partial(\log\rho))\circ f\,pq = 0.$$

Thus harmonic maps arise from a geometric variational problem and as far as we know, one can not study solutions of this equation, using classical theory of elliptic equations.

In order to explain our ideas and results it is convenient to suppose that $M$ and $N$ are domains in $\mathbb{C}$. Recall that $\Delta$ denote the unit disc. Now, we will state a simple, but useful, property of harmonic maps (related to natural parameter). Again, we suppose, as at the beginning, that $f$ is a harmonic mapping between Riemann surfaces $M$ and $N$. Then $\varphi(z)\,dz^2$ is a holomorphic quadratic differential on $M$, where $\varphi = \rho\circ f p\,\bar{q}$ in a local coordinate.

Let $P$ be a regular point for $\varphi(z)\,dz^2$ on $M$ and let $\zeta$ be a natural parameter centered at $P$. If we compute $p$ and $q$ with respect to natural parameter, then we have useful formula $\rho\circ f p\bar{q} = 1$.

In Section II we will give an outline of proofs of some properties of harmonic maps, using different tools: Dirichlet's principle, minimizing sequences, different versions of Reich–Strebel inequality, etc. For general properties of harmonic maps we refer the interested reader to Eells and Lemaire ([EL1], [EL2]), Jost [J], Schoen [Sc], Schoen and Yau [SY] and further references there.

**E. The Main Inequality and Dirichlet's principle.** Now we will state a formula for the energy density which explains connection between Dirichlet principle for harmonic maps (in general sense) and via the Main Inequality with Grötzsch principle (an integral version of this formula appears in [ReS2], see also [We] and [M3]). Suppose that $\rho$ is a metric density on $\Delta$, $f$ is $C^1$ function on $\overline{\Delta}$ and let $h$ be a diffeomorphism of $\overline{\Delta}$ onto itself which is the identity on the boundary of $\Delta$. If $\nu = \mathrm{Belt}[h]$, then

$$e(f\circ h^{-1}) = \Big[\frac{1+|\nu|^2}{1-|\nu|^2}e(f) - 4\,\mathrm{Re}\,\frac{\nu}{1-|\nu|^2}\varphi\Big],$$

where $\varphi = \varphi(f) = \rho\circ f\,p\bar{q}$. Hence,

$$e(f\circ h^{-1}) - e(f) = 2(|\varphi| T_\nu\varphi - |\varphi|) + r(h),$$

where

$$r(h) = r(h,f) = \frac{2|\nu|^2}{1-|\nu|^2}(e(f) - 2|\varphi|) \geqslant 0.$$

If $f$ is a harmonic mapping (with respect to $\rho$), then $\varphi = \varphi(f)$ is a holomorphic function on $\Delta$. Hence, using the Main Inequality we obtain a version of Dirichlet principle for harmonic mappings (in general sense):

$$E(f\circ h^{-1}) \geqslant E(f).$$

We expect further applications of the Main Inequality in this direction. In order to illustrate this we will outline a short proof of Dirichlet's principle (Theorem DP) for the euclidean harmonic functions using trajectories of holomorphic quadratic differentials.

**Theorem DP.** (Dirichtet's principle) *Suppose that*

($a$) *$g$ is continuous function on $\overline{\Delta}$.*

($b$) *$g$ has the first partial derivatives which are continuous on $\Delta$*

($c$) *the energy integral of $g$ is finite.*

*If $u$ is continuous on $\overline{\Delta}$, harmonic on $\Delta$ and if $u = g$ on the boundary of $\Delta$, then $D(g) \geqslant D(u)$, where the inequality equals if and only if $u = g$ on $\Delta$.*

*Proof.* Suppose that $g$ and $u$ are real functions and $u$ is harmonic on $\overline{\Delta}$. There exists holomorphic function $f$ on $\overline{\Delta}$ such that $\operatorname{Re} f = u$ on $\overline{\Delta}$. If we consider this mapping $f$ as a natural parameter, we can divide $\Delta$ on a finite number of disjoint quadrilaterals $\Sigma_k$. In each $\Sigma_k$ the mapping $f$ is univalent and maps each $\Sigma_k$ on a horizontally convex domains $D_k$. Using an approach as in Grötzsch principle, one can conclude

$$\iint_{\Sigma_k} |\operatorname{grad} g|^2 dx\, dy \geqslant \operatorname{area}(D_k) = \iint_{\Sigma_k} (u_x^2 + u_y^2) dx\, dy.$$

Summing these inequalities we get the Dirichlet's principle.

In Section II we state a version of Dirichlet's principle for harmonic mappings and generalize the classical area theorem in different directions (see [M1]). Also, in Section II we study uniqueness of harmonic mappings using Dirichlet's principle, minimizing sequences and different versions of the main inequality (see [MM2], [MM3] and [M1]) and we give a proof of Beurling theorem (Theorem B2) using the vertical trajectory of the corresponding holomorphic function. A review of known results in this direction is given in this section too.

**F. Extremal dilatation.** In this section we give a short report concerning extremal mappings. Interested reader can learn more about extremal mappings from Strebel's survey article [S6], Reich's papers [Re8], [Re9] and Earle–Li Zhong's [ELi], all of which we highly recommend. Extremal mappings have been one of the main topics in the theory of quasiconformal mappings, since its earliest days, when Grötzsch solved the extremal problem for two rectangles. In order to discuss them we need to review some familiar definitions. A homeomorphism $f$ from a domain $G$ onto another is called quasiconformal if $f$ is ACL (absolutely continuous on lines) in $G$ and $|f_{\bar z}| \leqslant k|f_z|$ a.e. in $G$, for some real number $k$, with $0 \leqslant k < 1$. It is well known that if $f$ is a quasiconformal mapping defined on the region $G$, then the function $f_z$ is nonzero a.e. in $G$. The function $\mu_f = f_{\bar z}/f_z$ is therefore a well defined bounded measurable function on $G$, called the complex dilatation or Beltrami coefficient of $f$. Let QC denote the space of all quasiconformal mappings from $\Delta$ onto itself. Two elements $f, g \in$QC are equivalent if $f = g$ on $\partial\Delta$. For a given $f \in$QC we denote the equivalence class of $f \in$QC by $Q_f = [f]$ or $[\mu]$, where $\mu = \mu_f$. We also use the notation $k_0([f]) = \inf\{\|\mu_g\|_\infty : g \in Q_f\}$. We let $L^\infty = L^\infty(\Delta)$ be the space of essentially bounded complex-valued measurable functions on $\Delta$, and let $M$ be the open unit ball in $L^\infty$. For any $\mu$ in $M$ there exists a quasiconformal solution $f : \Delta \mapsto \Delta$ of the Beltrami equation

$$\overline{\partial} f = \mu\, \partial f \tag{F1}$$

unique up to a postcomposition by a Möbius transformation. We let $f^\mu$ be the solution $f$ of (F1) normalized by $f(i) = i$, $f(1) = 1$ and $f(-1) = -1$. Two elements $\mu_0$ and $\mu_1$ in $M$ are equivalent if $f^{\mu_0}$ and $f^{\mu_1}$ coincide on $\partial\Delta$. For given $\mu \in M$ the equivalence class $[\mu]$ contains at least one element $\mu_0$ such that $\|\mu_0\|_\infty = \inf\{\|\nu\|_\infty : \nu \in [\mu]\}$. Such a $\mu_0$ is referred to as an extremal complex dilatation and $f_0 = f^{\mu_0}$ as an extremal quasiconformal mapping (abbreviated EQC mapping).

Let $Q$ be the Banach space consisting of holomorphic functions $\varphi$, belonging to $L^1 = L^1(\Delta)$, with norm

$$\|\varphi\| = \iint_\Delta |\varphi(z)|\,dx\,dy \,<\, \infty,\ \varphi \in Q.$$

Instead of $Q$ the notations $A$ and $L^1_a$ are also used.

For $\mu \in L^\infty$ we consider the linear functional $\Lambda_\mu(\varphi) = (\mu,\varphi)$, $\varphi \in Q$, where

$$(\mu,\varphi) = \iint_\Delta \mu(z)\varphi(z)\,dx\,dy,$$

and denote by $\|\mu\|_* = \|\Lambda_\mu\|$ the norm of $\mu$ as an element of the dual space of $Q$. For $\mu \in L^\infty$ we say that it satisfies the Hamilton-Krushkal condition if $\|\mu\|_* = \|\mu\|_\infty$.

We are now ready to state the main result about extremal complex dilatations.

**Theorem HKRS.** (Hamilton–Krushkal and Reich–Strebel) *Let $\mu \in M$. A necessary and sufficient condition that $f^\mu$ is an EQC mapping is that $\|\mu\|_* = \|\mu\|_\infty$.*

**G. Unique extremality.** Ahlfors and Bers showed that $T$ has a complex structure with tangent space at the base point isomorphic to Banach space $Q^*$. Two tangent vectors $\mu$ and $\nu$ in the tangent space to $M$ determine the same tangent vector in $T$ if and only if

$$\int_\Delta \varphi\mu = \int_\Delta \varphi\nu,\ \text{ for all } \varphi \in Q.$$

If $\mu$ and $\nu$ have this property, we write $\mu \sim^* \nu$ and we say that they represent the same Teichmüller infinitesimal equivalence class or, more briefly, that they are infinitesimally equivalent. The space of equivalence classes is denoted by $B$. A given $\mu$ is said to be extremal in its infinitesimal Teichmüller class if $\|\mu\|_\infty \leqslant \|\nu\|_\infty$, for any $\nu$ infinitesimally equivalent to $\mu$.

Recall that Hamilton, Krushkal, Reich and Strebel showed that a Beltrami coefficient $\nu$ in $M$ is extremal in its class in $T$ if and only if $\nu$ is extremal in its class in $B$. It was natural to consider whether the analogous statement holds for the unique extremality. In several articles Reich showed that in many special situations the two notions of unique extremality coincide and he conjectured that the notions may coincide in general. In [BLMM] (see also [MM1] and [BMM]) we have recently proved the answer to this conjecture is affirmative.

**Theorem G1.** (The Equivalence Theorem I) *$\mu$ is uniquely extremal in its Teichmüller class if and only if $\mu$ is extremal in its infinitesimally class.*

The proof of this theorem is based on estimates which allow us to compare two Beltrami coefficients $\mu$ and $\nu$ in the same global equivalence class and two Beltrami differentials in the same infinitesimal equivalence class. These estimates generalize Reich's Delta inequality for Beltrami differentials in the same equivalence class (see [R8]). Unlike Reich's forms of the Delta inequalities, our forms do not require either one of the Beltrami coefficients to have constant absolute value.

The generalized Delta inequality is our first step towards obtaining the criterion for the unique extremality of Beltrami differentials. The next important step is the analysis of the proof of Hahn-Banach theorem and its applications to our setting. In particular, we obtain the following necessary and sufficient criterion for the unique extremality of given Beltrami coefficient $\chi$.

**Theorem G2.** (Characterization Theorem I) *Beltrami coefficient $\chi$ is uniquely extremal if and only if for every admissible variation $\eta$ of $\chi$ there exists a sequence $\varphi_n$ in $Q(\Delta)$ such that*

(a) $\delta(\varphi_n) = \|\varphi_n\| \, \|\eta\|_\infty - \mathrm{Re} \int_\Delta \varphi_n \eta \to 0$

(b) $\lim_{n\to\infty} \inf |\varphi_n(z)| > 0$, *for almost all $z$ in $E(\eta)$.*

Here, an admissible variation $\eta$ of $\chi$ is any Beltrami differential that does not increase the $L^\infty$-norm of $\chi$, and which is allowed to differ from $\chi$ only on the set where $|\chi(z)| \leqslant s < \|\chi\|_\infty$, where $s$ is a constant, and the extremal set $E(\eta)$ is the set where $\eta(z) = \|\eta\|_\infty$. This criterion is analogous to the Hamilton-Krushkal, Reich–Strebel necessary and sufficient criterion for the extremality. Namely, $\chi$ is extremal if and only if there is a sequence $\varphi_n$ of holomorphic quadratic differential of norm 1 such that

$$\|\chi\|_\infty - \mathrm{Re} \int_\Delta \eta\varphi_n \to 0.$$

This criterion is among listed in the theorem in Section 11, in [BLMM], which we called the Characterization Theorem. The Characterization Theorem applies to many interesting situations. For instance, we can say precisely when a Beltrami differential of the form $k|\varphi(z)|/\varphi(z)$, with $\varphi$ a holomorphic quadratic differential with $\|\varphi\| = \infty$, is uniquely extremal.

There are many examples of extremal Beltrami differentials with nonconstant modulus, but all examples of uniquely extremal Beltrami differentials known up to our papers [BLMM] and [BMM] were of the general Teichmüller type. Moreover, many results obtained studying the extremal problems speak in favor of the conjecture that all uniquely extremal Beltrami differentials $\mu$ satisfy $|\mu(z)| = \|\mu\|_\infty$, for almost all $z$. Surprisingly, we disprove this conjecture and show that there are uniquely extremal Beltrami differentials with nonconstant modulus.

## II. Dirichlet's Principle, Uniqueness of Harmonic maps and Related Problems

**A. Introduction and some basic properties.** The main purpose of this section is to give a short review of some results related to harmonic maps, communicated by the author and the other members of the Seminar, at University of Belgrade, during several last years. Also in this section we give a review of known results in this direction.

**A1.** Let $M$ and $N$ be two Riemann surfaces with local conformal metrics $\sigma(z)|dz|^2$ and $\rho(z)|dw|^2$ and let $f : M \mapsto N$. It is convenient for us to use notation in local coordinates $df = (\partial f)dz + (\bar\partial f)d\bar z$ and $p = \partial f$, $q = \bar\partial f$. The energy integral of $f$ is

$$E(f,\rho) = \int_M e(f)\,\sigma\,dx\,dy,$$

where $e(f)$ is the energy density defined by

$$e(f)(z) = (|p|^2 + |q|^2)\frac{\rho \circ f(z)}{\sigma(z)}.$$

A critical point of the energy functional is called harmonic mapping. The Euler-Lagrange equation for the energy functional is:

$$(A1) \qquad f_{z\bar{z}} + \Big(\partial(\log\rho)\Big) \circ f\, pq = 0.$$

Thus harmonic maps arise from a geometric variational problem and as far as we know, one can not study solutions of this equation, using classical theory of elliptic equations.

In this section we will give an outline of the proofs of some properties of harmonic maps, using different tools: Dirichlet's principle, minimizing sequences, different versions of Reich–Strebel inequality, etc. For general properties of harmonic maps we refer the interested reader to Eells and Lemaire ([EL1], [EL2]), Jost [J], Schoen [Sc], Schoen and Yau [SY] and further references there. In order to explain our ideas and results it is convenient to suppose that $M$ and $N$ are the domains in $\mathbb{C}$. Let $\Delta$ denote the unit disc. If $f : M \mapsto N$ is harmonic map, then $\varphi = \rho \circ f\, p\bar{q}$ is a holomorphic function. For the sake of the reader, we will sketch a proof of this result in the case when $M = \Delta$ and $N$ is a domain in $\mathbb{C}$, with the metric $\rho(w)|dw|$.

Let $\lambda$ be a complex valued function of class $C^1$ with compact support in $\Delta$ and let $\Phi_\epsilon(z) = z + \epsilon\lambda(z)$. Then,

$$\nu_\epsilon = \mathrm{Belt}[\Phi_\epsilon] = \frac{\epsilon\lambda_{\bar{z}}}{1 + \epsilon\lambda_z}.$$

If $f$ is a stationary point of the energy integral, using an expression (see [ReS2]) for $E(f \circ \Phi_\epsilon^{-1}, \rho) - E(f, \rho)$, we conclude that

$$\iint_\Delta \bar{\partial}\lambda(z)\varphi(z)\, dx\, dy = 0.$$

Since $\varphi$ is integrable function on $\Delta$, it follows that $\varphi$ is an analytic function on $\Delta$, by Weyl's lemma.

Now, we will state some simple, but useful, properties of harmonic maps.

**A2. Properties of harmonic maps related to natural parameter.** Again, we suppose, as at the beginning, that $f$ is harmonic mapping between Riemann surfaces $M$ and $N$. Then $\varphi(z)dz^2$ is a holomorphic quadratic differential on $M$, where $\varphi = \rho \circ f p\, \bar{q}$ in a local coordinate.

Let $P$ be a regular point for $\varphi(z)dz^2$ on $M$ and let $\zeta$ be a natural parameter centered at $P$. If we compute $p$ and $q$ with respect to natural parameter, then we have important formula

$$(A2) \qquad \rho \circ f p\bar{q} = 1$$

Now, easy computation gives:

$$p\bar{q} = \frac{1}{4}\big(|f_\xi|^2 - |f_\eta|^2 - \, 2\, i\, \mathrm{Re}\, \bar{f}_\xi f_\eta\big)$$

Combining this formula with (A2), we find that $f_\xi$ and $f_\eta$ are orthogonal (if we consider them as vectors). Also, we can show that Jacobian $J = |p|^2 - |q|^2 = 0$ if and only if $f_\eta = 0$.

**A3.** Using Aronszajn's generalization of Carleman's result we can prove the following uniqueness property:

**Theorem S.** *If $f$ is a harmonic mapping of an open connected set $D \subset M$ and $f = 0$ on an open subset of $D$, then $f = 0$ throughout $D$.*

General version of this result, which is concerned with the case when $M$ and $N$ are Riemannian manifolds, is known as Sampson's Unique Continuation Theorem (see [Sa] and [EL2]).

**A4. The symmetry property.**

**Theorem RP.** (The reflection principle) *Suppose $L$ is a segment of the real axis, $\Omega^+$ is a region in $H^+ = \{z : \operatorname{Im} z > 0\}$, and every $t \in L$ is the center of an open disc $B_t$ such that $H^+ \cap B_t$ lies in $\Omega^+$. Let $\Omega^-$ be the reflection of $\Omega^+$: $\Omega^- = \{\overline{z} : z \in \Omega^+\}$. Suppose $u$ is harmonic in $\Omega^+$ and $\lim_{n \mapsto \infty} u(z_n) = 0$ for every sequence $\{z_n\}$ in $\Omega^+$ which converges to a point on $L$. Then there is a function $U$, harmonic in $\Omega = \Omega^+ \cup L \cup \Omega^-$ such that $U = u$ in $\Omega^+$. This function $U$ satisfies the relation $U(z) = -U(\overline{z})$, $z \in \Omega$.*

*Proof.* We extend $u$ to $\Omega$ by defining $U(z) = 0$, for $z \in L$, and $U(z) = -U(\overline{z})$, for $z \in \Omega^-$.

*Example* 1. It is not difficult to verify that function $f(z) = 2x + i \cos y$ is harmonic mapping from $C$ into $C$ with respect to the corresponding metric. This function is periodic with respect to $y$. The next result shows that this periodicity is typical.

**Theorem M1.** *Suppose that $f : \mathbb{C} \mapsto \mathbb{C}$ is a harmonic mapping, given w.r.t. natural parameter and that Jacobian of $f$ equals zero on the real axis. Then $f(z) = f(\bar{z})$.*

The proof of this result is based on the Theorem S.

**B. Dirichlet's principle and related problems. B1.** If the metric density $\rho \equiv 1$ on $N$, then the equation (1) reduces to $f_{z\bar{z}} = 0$. In this case we say that $f$ is a harmonic function and write $D[f]$ instead of $E(f, 1)$ for the energy integral. Recall that $\Delta$ denote the unit disc. Also we will use the notation $D[\phi, \psi] = \iint_\Delta (\phi_x \psi_x + \phi_y \psi_y)\, dx\, dy$.

The following lemma is crucial in the proof of Dirichle's principle.

**Lemma DP.** *Suppose that*

*(a) $u$ and $h$ are continuous on $\overline{\Delta}$ and $h \equiv 0$ on $\partial\Delta$*

*(b) $u$ is harmonic on $\Delta$ and $h$ has the continuous partial derivatives of the first order on $\Delta$*

*(c) $u$ and $h$ have the finite Dirichlet's integral on $\Delta$.*

*Then $D[u, h] = 0$.*

First we will state the Dirichlet's principle for harmonic function.

**Theorem DP.** (Dirichtet's principle) *Suppose that*
($a$) *$g$ is continuous function on $\overline{\Delta}$.*
($b$) *$g$ has the first partial derivatives which are continuous on $\Delta$*
($c$) *the energy integral of $g$ is finite.*
*If $u$ is continuous on $\overline{\Delta}$, harmonic on $\Delta$ and if $u = g$ on the boundary of $\Delta$, then $D(g) \geqslant D(u)$, where the inequality equals if and only if $u = g$ on $\Delta$.*

*Proof.* If $h = g - u$, then Lemma DP shows that,

$$D[g] = D[u] + 2D[u,h] + D[h] = D[u] + D[h] > D[u],$$

unless $D[h] = 0$, i.e., $h$ has the constant value zero.

Now, we are going to discuss some results related to Dirichlet's principle. In [M1] we gave a proof of Theorem M2 (see bellow) based on Dirichlet's principle. Before we state this result we need some definitions and we will state the area theorem and a result of Lehto–Kühnau, which motivated us.

**B2. An area theorem of Lehto–Kühnau type for harmonic maps**. First, we are going to prove the area theorem, which is an important tool in theory of univalent functions.

**Theorem A.** (The area theorem) *Let $w = f(z) = z + \dfrac{a_1}{z} + \cdots + \dfrac{a_n}{z^n} + \cdots$ be an univalent analytic function on $E = \{z : |z| > 1\}$ and let $G = \mathbb{C} \setminus f(E)$ be the omitted set. Then*

$$\pi\left(1 - \sum_{k=1}^{\infty} k|a_k|^2\right) = \operatorname{area}(G).$$

*Proof.* Let $K_\rho$ be the circle $|z| = \rho > 1$, with the positive orientation, and set

$$I_\rho = I_\rho(f) = \frac{i}{2}\int_{K_\rho} f\, d\overline{f}.$$

If $f = u + iv$ and if $\gamma_\rho$ denotes the image curve of $K_\rho$, we have

$$I_\rho = \int_{\gamma_\rho} u\, dv$$

and by elementary calculus this represent the area enclosed by $\gamma_\rho$. Hence $I_\rho > 0$.

Direct calculation gives

$$\begin{aligned} I_\rho &= \frac{i}{2}\int_{K_\rho}\left(z + \sum_{k=1}^{\infty}\frac{a_k}{z^k}\right)\left(1 - \sum_{k=1}^{\infty} k\bar{a}_k \bar{z}^{-k-1}\right)d\bar{z} \\ &= \frac{1}{2}\int_{K_\rho}\left(z + \sum_{k=1}^{\infty}\frac{a_k}{z^k}\right)\left(\bar{z} - \sum_{k=1}^{\infty} k\bar{a}_k \bar{z}^{-k}\right)d\theta \\ &= \pi\left[\rho^2 - \sum_{k=1}^{\infty} k|a_k|^2\rho^{-2k}\right]. \end{aligned}$$

Thus $\sum\limits_{k=1}^{\infty} k|a_k|^2\rho^{-2k} < \rho^2$, and theorem follows for $\rho \mapsto 1$.

Let us consider conformal mapping $h$ which belongs to class $\Sigma$, i.e., $h$ is univalent in $E = \{z : |z| > 1\}$ and has a power series expansion of the form

$$h(z) = z + \sum_{n=1}^{\infty} a_n z^{-n}$$

in $E$. If $h$ has a quasiconformal extension to the plane with complex dilatation $\mu$, satisfying the inequality $\|\mu\|_\infty = k < 1$, we say that $h$ belongs to the subclass $\Sigma_k$ of $\Sigma$. Lehto [L1], [L2] and Kühnau [K] established the area theorem for $\Sigma_k$.

**Theorem LK.** (Lehto–Kühnau) *Let $h \in \Sigma_k$. Then $\sum\limits_{n=1}^{\infty} n|a_n|^2 \leqslant k^2$. The estimate is sharp.*

If we denote by $P$ the area of the omitted set of $h(E)$, then Theorem LK states that $P \geqslant \pi(1-k^2)$.

Before we state the Theorem M2, which is a generalization of Theorem LK to univalent harmonic mappings, we need some definitions. Let $\Sigma'$ be the set of all harmonic, orientation-preserving, univalent mappings

$$h(z) = z + f(z) + \overline{g(z)} + A \log|z|$$

on $E$, where $f(z) = \sum_{n=1}^{\infty} a_n z^{-n}$ and $g(z) = \sum_{n=1}^{\infty} b_n z^{-n}$ are analytic on $E$ and $A \in C$. Let $\Sigma'_k$ denote the set of all homeomorphisms $h$ of $C$ onto itself such that: (a) the restriction of $h$ on $E$ belongs to $\Sigma'$ and (b) the restriction of $h$ on the unit disk $U = \{z : |z| < 1\}$ is a quasiconformal mapping with complex dilatation $\mu$ satisfying $\|\mu\|_\infty \leqslant k < 1$.

The Area theorem can be established for $\Sigma'_k$. Recall, that $P$ denote the area of the omitted set of $h(E)$. Also, it is convenient to use notations $\tau = \sum_{n=1}^{\infty} n|a_n|^2$ and $s = 1 + 2\operatorname{Re} b_1 + l$, where $l = \sum_{n=1}^{\infty} n|b_n|^2$.

**Theorem M2.** *Let $h \in \Sigma'_k$. Then*
*(a)* $P \geqslant \pi(1-k^2)s$; *(b) The equality holds in (a) if and only if*

$$h(z) = z + cz^{-1} + cg(z) + \overline{g(z)} + A\log|z|,$$

*where $g(z) = \sum\limits_{n=1}^{\infty} b_n z^{-n}$ is analytic on $E$; and $|c| = k$, $A \in \mathbb{C}$.*

Since $P = \pi(s - \tau)$ the next result follows immediately from Theorem M1.

**Corollary M1.** *If $h \in \Sigma_k^{'}$, then $\tau \leqslant k^2 s$.*

Finally we state a generalization of the area theorem to analytic functions.

**Theorem A1.** *Let $w = f(z) = \lambda z + \dfrac{a_1}{z} + \dots + \dfrac{a_n}{z^n} + \cdots$ be an analytic function on $E = \{z : |z| > 1\}$ and let $G = \mathbb{C} \setminus f(E)$ be the omitted set. Then*

$$(B1) \qquad \pi\left(|\lambda|^2 - \sum_{k=1}^{\infty} k|a_k|^2\right) \leqslant \operatorname{area}(G).$$

*Equality holds if and only if $f$ is a univalent function on $E$.*

*Proof.* Let $K_\rho$ be the circle $|z| = \rho$ with positive orientation and let $\gamma_\rho$ be the curve defined by the equation $w = f_\rho(e^{it}) = f(\rho e^{it}), 0 \leqslant t \leqslant 2\pi$. For given $w \neq \infty$ let $n(w)$ be the number of roots of $f(z) = w$ in $|z| > \rho$. Assume that $f \neq w$ on $K_\rho$ and $\lambda \neq 0$. Since $f$ has a pole of order 1 at $\infty$, we have $f(z) \neq w$ in $|z| \geqslant r$ for a large $r$ and consequently, by the argument principle,

$$(B2) \qquad n(w) = \frac{1}{2\pi i}\int_{K_r - K_\rho} \frac{f'(z)}{f(z) - w}\, dz = 1 - \chi(\gamma_\rho, w),$$

where $\chi = \chi(\gamma_\rho, w)$ is the winding number (or index) of the curve $\gamma_\rho$ with respect to the point $w$. By the analytic Green's theorem (see, for example [Po]), the area

$$(B3) \qquad I_\rho = \frac{1}{2\pi i}\int_{\gamma_\rho} \bar{w}\, dw = \frac{1}{\pi}\int_{\mathbb{R}^2} \chi(\gamma_\rho, w)\, du\, dv.$$

Let $G_\rho$ be the set omitted by $f$ on $E_\rho = \{|z| > \rho\}$. By (1) $w \in G_\rho$ if and only if $\chi(\gamma_\rho, w) = 1$. Also, it follows from (1) that $\chi(\gamma_\rho, w)$ is an integer less than or equal to zero if $w \notin \bar{G_\rho}$. This together with (B3) gives

$$(B4) \qquad \pi I_\rho \leqslant \text{area}(G_\rho).$$

Direct calculation as in the proof of area theorem gives (B1). For the case of equality see [M].

**B3. Extremal metrics and modulus**. In this item we are going to give a proof of a Beurling result, which is a modification of the proof in [Ah1]. Also, we outline a new proof of the Beurling result, using minimizing sequences. Our approach is influenced by Courant's book (see [C]) and Gehring's work in $\mathbb{R}^3$ space (see [Ge1] and [Ge2]). Some generalizations of Gehring's results are presented in [AMŠ].

In unpublished work Beurling has given the following elegant and useful criterion. Before we state Beurling result we need a few definitions.

Let $\Omega$ be a region in the plane, and let $\Gamma$ be family of curves and let $\rho(z) \geqslant 0$, be Borel measurable function defined in the $z$-plane. We say that $\rho$ is admissible for $\Gamma$, if for every rectifiable $\gamma \in \Gamma$, $\int_\gamma \rho|dz|$ exists and $\infty \geqslant \int_\gamma \rho|dz| \geqslant 1$. In these circumstances every rectifiable arc $\gamma$ has a well defined $\rho$-length

$$L(\gamma, \rho) = \int_\gamma \rho|dz|,$$

which may be infinite, and the open set $\Omega$ has a $\rho$-area $A = A_\rho = A(\rho, \Omega)$. The modulus of $\Gamma$, $M = M_\Omega(\Gamma)$, with respect to $\Omega$, is defined as $\inf A(\Omega, \rho)$ for admissible $\rho$. The extremal length of $\Gamma$ in $\Omega$ is defined as the reciprocal of the modulus. The extremal length is denoted by $\lambda = \lambda_\Omega(\Gamma)$.

**Theorem B1.** (Beurling's theorem) *The metric $\rho_0$ is extremal for $\Gamma$ if $\Gamma$ contains a subfamily $\Gamma_0$ with the following properties:*

(*a*) $\int_\gamma \rho_0|dz| = 1$, *for all* $\gamma \in \Gamma_0$;

(*b*) *for real-valued $h$ in $\Omega$ the conditions $\int_\gamma h|dz| > 0$ for all $\gamma \in \Gamma_0$ imply* $\iint_\Omega h\rho_0\, dx\, dy \geqslant 0$.

Let $\Omega$ be an open set and let $E_1$, $E_2$ be two sets in the closure of $\Omega$. Take $\Gamma$ to be the set of connected arcs in $\Omega$ which join $E_1$ and $E_2$. The extremal length $\lambda(\Gamma)$ is called the extremal distance of $E_1$ and $E_2$ in $\Omega$, and we denote it by $d_\Omega(E_1, E_2)$.

*Example* 1. The extremal distance between vertical sides of a rectangle $R = \{z = x + iy : a < x < b,\ c < y < d\}$ is $\lambda = \frac{b-a}{d-c}$.

*Proof.* Let $\Lambda_y = [a + iy,\ b + iy]$ and $\Gamma_0$ is the family of curves $\{\Lambda_y : c \leqslant y \leqslant d\}$. If we take $\rho_0 = 1$ Beurling's criterion is satisfied, and $\rho_0 = 1$ is extremal metric.

*Example* 2. Let $A$ be the ring $A = A(r_1, r_2) = \{z : r_1 < |z| < r_2\}$. If $\Gamma$ is the family of arcs in $\overline{A}$, which join circles $K_{r_1} = \{z : |z| = r_1\}$ and $K_{r_2} = \{z : |z| = r_2\}$, then

$$L(\Gamma) = \frac{1}{2\pi} \ln \frac{r_2}{r_1}. \tag{B5}$$

*Proof.* Let $A' = A \setminus (r_1, r_2)$ and $R = \{w : \ln r_1 < u < \ln r_2,\ 0 < v < 2\pi\}$. Since *exp* maps conformally $R$ onto $A'$, using the Example 1 we get (B5).

Now, we state a result of Beurling, which express the Dirichlet's integral by means of extremal distance (see [Ah1]).

**Theorem B2.** (Beurling's theorem) *Let $\Omega$ be a region in the complex plane bounded by a finite number of analytic Jordan curves, let $E_0$ and $E_1$ be disjoint and consist of finite number of closed arcs or curves in the boundary of $\Omega$. Then the extremal distance $d_\Omega(E_0, E_1)$ is the reciprocal of the Dirichlet integral*

$$D(u) = \iint_\Omega (u_x^2 + u_y^2)\, dx\, dy,$$

*where $u$ satisfies:*

*($i$) $u$ is bounded and harmonic in $\Omega$*

*($ii$) $u$ has a continuous extension to $\Omega \cup E_0^o \cup E_1^o$, and $u = 0$ on $E_0$ and $u = 1$ on $E_1$*

*($iii$) the normal derivative $\partial u/\partial n$ exists and vanishes on $C_0$ ($C$ denote the full boundary of $\Omega$, $C_0 = C - (E_0 \cup E_1)$, and $E_0^o$ and $E_1^o$ denote relative interiors of $E_0$ and $E_1$ as a subset of $C$).*

The proof of this theorem in [Ah1] is based on two important ingredients:

1) the existence of solution of a mixed Dirichlet-Neuman problem (we denote it by $u$)

2) decomposition of a domain on rings and quadrilateral subdomains using, in fact, the orthogonal and vertical trajectories of quadratic differential defined by $u$.

For the theory of trajectories of holomorphic quadratic differentials see [Ga] and [S2].

*Proof of Theorem* B2. Let $A$ be the set of the end points of the $E_1$ and $E_2$ as subsets of $C$. The reflection principle shows that $u$ has a harmonic extension across $\partial\Omega \setminus A$.

Let $z_0 \in A$, for example, $z_0 \in E_1$. We can chose a local conjugate $v$ in $\Omega$ near $z_0$ such that, on the boundary, $u = 0$ on one side of $z_0$ and $v = 0$ on the other side of $z_0$. Then, by the reflection principle, there exists neighborhood $V$ of $z_0$ and an analytic function $\varphi$ in $V \setminus \{z_0\}$ such that $\varphi = (u + iv)^2$ in $\Omega \cap V$. Hence, $\varphi$ is an analytic function on $V$ and has a simple zero at $z_0$. Therefore, $u_x - iu_y$ must tend to $\infty$, and the number of critical points in $\overline{\Omega} \setminus A$ is finite.

Locally, for every $z_0 \in \partial\Omega \setminus A$ there exists a neighborhood $V$ of $z_0$ and an analytic function $f$ on $V$ such that $\operatorname{Re} f = u$ on $V$. Hence, we can define horizontal trajectories with respect to $w = f(z)$.

The part of noncritical horizontal trajectory $\gamma$ which is in $\overline{\Omega}$ can be parameterized with parameter interval $I = [0, 1]$ such that:

1. $\gamma$ join $E_1$ and $E_2$ in $\Omega$ (more precisely $\gamma(0, 1) \subset \Omega$, $\gamma(0) \in E_1$ and $\gamma(1) \in E_2$).
2. $\operatorname{Re} \gamma$ is strictly increasing function on $I$ and $\operatorname{Re} \gamma(0) = 0$, $\operatorname{Re} \gamma(1) = 1$.

Hence, we conclude that up to a set of Lebesgue 2-dimensional measure zero there exists finite number of disjoint quadrilateral $\Sigma_k$, $k = 1, 2, ..., n$, such that:

1. $\Omega = \bigcup_{k=1}^{n} \Sigma_k$
2. Each $\Sigma_k$ is swept out with noncritical horizontal trajectories
3. There exists rectangles $R_k$ of width 1 and height $m_k$ and conformal (univalent) mapping $\Phi = \Phi_k$ of $\Sigma_k$ onto $R_k$ such that $\operatorname{Re} \Phi_k = u$ on $\Sigma_k$. Hence,

$$m_k = \iint_{\Sigma_k} |\Phi'|^2 dx\, dy \quad \text{and} \quad m = \sum_{k=1}^{n} m_k = D(u).$$

Together rectangles $R_k$ fill out a rectangle with sides 1 and $D(u)$. After appropriate identification we obtain a model of $\Omega$ with $E_1$ and $E_2$ as vertical sides.

From this model and Beurling theorem (Theorem B1) it is immediately clear that the euclidean metric is extremal, and we conclude that $d_\Omega(E_1, E_2) = 1/D(u)$.

Our first purpose was to give more elementary proof of this result (that is, with no use of these two subjects), using a minimizing sequence (see, for example Courant's book [C]), and to derive some equalities not contained in the proof of Beurling's theorem. During our work on this problem we become aware of Gehring's works (see [Ge1] and [Ge2]), which strongly influenced our research.

In [Ge1] and [Ge2] Gehring proved that essentially Väisälä's definition of extremal distance between $E_0$ and $E_1$ in $\Omega$ is equivalent to the Dirichlet's integral definition due to Loewner (see [Lo]) if $\Omega$ is a ring domain in $\mathbb{R}^3$, and $E_0$ and $E_1$ are boundary components of $\Omega$. Gehring used this result to study quasiconformal mappings in space. We generalize this result to the setting of smooth domains in $\mathbb{R}^n$. An application of this result gives a short proof of Beurling's Theorem. As we understand, there are additional technical difficulties if we work with general domains instead of ring domains. Before we state the result we need a few definitions.

**Definition B1.** Let $\Omega$ be an open set in $\mathbb{R}^n$ and $\Gamma$ a set whose elements $\gamma$ are rectifiable arcs in $\Omega$. Let $\rho$ be a nonnegative Borel measurable function in $\Omega$ (such $\rho$ we will call metric). We can define the $\rho$-length of $\gamma$ by

$$L(\gamma, \rho) = \int_\gamma \rho\, |dx|$$

the $\rho$-volume of $\Omega$ as

$$V(\Omega,\rho)=\int_\Omega \rho^n\,dV(x)$$

where $dV$ is the $n$-dimensional Lebesgue measure in $\mathbb{R}^n$, and the minimum length of $\Gamma$ by $L(\Gamma,\rho)=\inf_{\gamma\in\Gamma}L(\gamma,\rho)$. The modulus of $\Gamma$ in $\Omega$ is defined by

$$\operatorname{mod}_\Omega(\Gamma)=\inf_\rho\frac{V(\Omega,\rho)}{L(\Gamma,\rho)^n},$$

where $\rho$ is subject to the condition $0<V(\Omega,\rho)<\infty$. The extremal length of $\Gamma$ in $\Omega$ is defined as $\Lambda_\Omega(\Gamma)=\operatorname{mod}_\Omega(\Gamma)^{1/1-n}$.

**Definition B2.** Let $\Omega$ be an open set in $\mathbb{R}^n$, and let $E_0,E_1$ be two sets in the closure of $\Omega$. Take $\Gamma$ to be the set of connected arcs in $\Omega$ which join $E_0$ and $E_1$, i.e. each $\gamma\in\Gamma$ has one endpoint in $E_0$ and the other in $E_1$ and all other points of $\gamma$ are in $\Omega$. The extremal length $\Lambda(\Gamma)$ is called the extremal distance of $E_0$ and $E_1$ in $\Omega$, and we denote it by $d_\Omega(E_0,E_1)$.

Now, let $\Omega$ be a bounded region whose boundary consists of a finite number of $C^1$ hypersurfaces. If $E_0$ and $E_1$ are disjoint, and each is a finite union of closed hypersurfaces contained in the boundary of $\Omega$, then we define the conformal $n$-capacity of $\Omega$ as

$$C[\Omega,E_0,E_1]=\inf_u\int_\Omega|\nabla u|^n\,dV(x),$$

where infimum is taken over all functions $u:\Omega\to\mathbb{R}$ which are differentiable in $\Omega$, continuous in $\overline{\Omega}$ and have boundary values 0 on $E_0$ and 1 on $E_1$.

The proof of the following theorem is given in [AMŠ].

**Theorem AMŠ.** *If $\Omega$ is a bounded domain, whose boundary consists of a finite number of $C^1$ hypersurfaces, and if $E_0$ and $E_1$ are disjoint sets of the boundary of $\Omega$ consisting of finite number of closed hypersurfaces, then we have*

$$\operatorname{mod}_\Omega(\Gamma)=\inf_f\frac{V(\Omega,f)}{L(\Gamma,f)^n}=C[\Omega,E_0,E_1],$$

*where $f$ is any metric in $\Omega$ and $\Gamma$ is the family of all Jordan arcs joining $E_0$ and $E_1$ inside $\Omega$.*

The case $n=2$ of previous theorem enables us to give a short proof of Theorem B. In fact, the proof immediately follows from Theorem 1.3 (see [C]), which gives a solution of a mixed Dirichlet-Neuman problem.

The proof of Theorem 1.3 in Courant's book [C], is based on using minimizing sequences. We believe that we can use minimizing sequences as Gehring in [Ge1] to show existence of the extremal admissible function $u\in E(\Omega,E_0,E_1)$ such that

$$C[\Omega,E_0,E_1]=\int_\Omega|\nabla u|^n dV.$$

**B4. Dirichlet's principle for harmonic mappings**. Let $N$ be complete Riemannian manifold of dimension $n$ and let its metric in local coordinates be given by $(g_{ik})$, with Christoffel symbols $\Gamma^i_{kl}$. For $f \in H^{1,2}(M,N)$ we define the energy density

$$e(f)(z) = \frac{1}{\sigma^2} \sum g_{ik}(f)\,(f^i_x f^k_x + f^i_y f^k_y),$$

and the energy as

$$E(f) = \frac{1}{2}\int_M e(f)\sigma^2 dx\,dy = \frac{1}{2}\int_M \sum g_{ik}(f),(f^i_x f^k_x + f^i_y f^k_y)\,dx\,dy,$$

where we write $f = (f^1, f^2, \dots, f^n)$ in local coordinates. A solution of the corresponding Euler–Lagrange equation $\Delta f^i + \Gamma^i_{kl}(f^k_x f^l_x + f^k_y f^l_y) = 0$, $i = 1, 2, \dots, n$, is called harmonic map.

**Theorem M3.** (Dirichlet's principle for harmonic mappings) *Let $N$ be Riemannian $n$-dimensional manifold and $f : \overline{\Delta} \mapsto N$ be a harmonic mapping. If $\Phi$ is diffeomorphism of $\overline{\Delta}$ onto itself, which is identity on $\partial\Delta$, then $E(f \circ \Phi) \geqslant E(f)$.*

**C. Uniqueness of harmonic maps.** Our further discussion is concerned mainly with the case when $M$ and $N$ are domains in complex plane $\mathbb{C}$. Recall, that the following result enables us to use theory of trajectory of holomorphic quadratic differentials.

**C1.** If $f$ is a harmonic mapping between Riemann surfaces $M$ and $N$ with local conformal metrics $\sigma(z)|dz|^2$ and $\rho(w)|dw|^2$, respectively, then $\varphi = \rho p \bar{q}\, dz^2$ is a holomorphic quadratic differential. For example if $M$ and $N$ are subset of the complex plane $C$, this simply means that the function $\rho\, p\, \bar{q}$ is a holomorphic function. This enables us to use the techniques and results from the theory of holomorphic functions.

**C2.** Marković and the author, using a version of Reich–Strebel inequality, proved the following uniqueness property.

**Theorem MM.** *Suppose that*
(*a*) *$f$ and $g$ are harmonic diffeomorphisms of $\Delta$ onto itself*
(*b*) *$f$ and $g$ are continuous on $\overline{\Delta}$*
(*c*) *$f = g$ on $\partial\Delta$.*
*If, in addition, we suppose that the energy integrals of $f$ and $g$ are finite, they are identical.*

This result was communicated on our Seminar at Belgrade University in 1996. and at Nevannlina Colloquium, Switzerland 1997. The proof is based on the next lemma if $f$ and $g$ are diffeomorphisms of $\overline{\Delta}$ onto itself and on a new version of Reich–Strebel inequality in general case.

**Lemma MM.** *Suppose that $f$ and $g$ are diffeomorphisms of $\overline{\Delta}$ onto $\bar{\Delta}$ and that $f$ is harmonic with respect to conformal metric $ds = \rho(w)|dw|$ on $\bar{\Delta}$. If we suppose in addition, that $E(f) < +\infty$ and that $f = g$ on $\partial\Delta$, then*

$$\int_\Delta \tilde{\rho}(\zeta)\,d\xi\,d\eta \leqslant \int_\Delta \tilde{\rho}(\zeta)\frac{1 - |\tilde{\mu}(\zeta)|}{1 + |\tilde{\mu}(\zeta)|}\frac{\left|1 + \frac{\tilde{\chi}(\zeta)}{\tilde{\mu}(\zeta)}|\tilde{\mu}(\zeta)|\right|^2}{1 - |\tilde{\chi}(\zeta)|^2}\,d\xi\,d\eta,$$

*where* $\tilde{\mu} = \mathrm{Belt}(f^{-1})$, $\tilde{\chi} = \mathrm{Belt}(g^{-1})$ *and* $\tilde{\rho}(\zeta) = \rho(\zeta)\dfrac{|\tilde{\mu}(\zeta)|}{1-|\tilde{\mu}(\zeta)|^2}$.

We will outline a proof of Theorem MM in the case that $f = Id$ on $\partial\Delta$ and that $f$ is diffeomorphism of $\overline{\Delta}$ onto itself. For the proof it is useful to observe that if $f$ is harmonic, then Beltrami dilatation $\mu$ of $f$ has the form

$$\mu(z) = s(z)\frac{|\varphi(z)|}{\varphi(z)},$$

where $s$ is non-negative measurable function and $\varphi = \rho\circ f p\bar{q}$ is an analytic function. Thus we have that expression $\mu\varphi/|\varphi|$, which appears in Reich–Strebel inequality equals $|\mu|$ and we get

$$\int_\Delta |\varphi|\,dx\,dy \leqslant \int_\Delta |\varphi|\frac{1-|\mu|}{1+|\mu|}\,dx\,dy.$$

If $\varphi$ is not identically zero we get $\mu = 0$ a.e. Hence we conclude that $f$ is conformal mapping. Since $f = \mathrm{Id}$ on $\partial\Delta$, we get that $f = \mathrm{Id}$ on $\Delta$. In general, we need a version of main inequality which holds for the mapping whose maximal dilatation can be 1.

**C3.** Marković and the author have proved that $f = g$ under weaker conditions, then in Theorem MM. The following two results will appear in [MM3].

**Theorem MM1.** *Suppose that*
($a$) *$f$ is homeomorphism of $\overline{\Delta}$ onto itself*
($b$) *$f$ has the first generalized derivatives on $\Delta$*
($c$) *$f$ is identity on $\partial\Delta$*
($d$) *$f$ is harmonic w.r.t. some metric density $\rho$ on $\Delta$*
($e$) *Hopf differential of $f$ is integrable on $\Delta$.*
*Then $f$ is the identity on $\Delta$.*

**Theorem MM2.** (The uniqueness property). *Suppose that*
($a$) *$f$ and $g$ are homeomorphisms of $\overline{\Delta}$ onto itself and $f = g$ on $\partial\Delta$*
($b$) *$f$ and $g$ are loc. q.c. on $\Delta$*
($c$) *$f$ and $g$ are harmonic w.r.t. some metric density $\rho$ on $\Delta$*
($d$) *Hopf differentials of $f$ and $g$ are integrable on $\Delta$.*
*Then $f$ and $g$ are identical.*

Also, we might add that we have a generalization of this result if instead of the unit disk, we consider Riemann surfaces. Recall, if the metric $\rho \equiv 1$ on $N$, which is open subset of complex plane $\mathbb{C}$ (euclidean case), we will say harmonic function instead of harmonic mapping. Thus in euclidean case this result says that the solution of classical Dirichlet problem is unique.

The proof of Theorem MM2 is based on a new version of Riech-Strebel inequality. Note that if $f$ and $g$ are harmonic property ($A$) says that function $\varphi = \rho\circ f p\bar{q}$ and $\psi = \rho\circ g A\bar{B}$ are holomorphic functions on the unit disk, where we use notation $A = \partial g$, $B = \bar{\partial} g$. The idea of the proof is to apply a new version of Reich–Strebel inequality to functions $\varphi$ and $\psi$.

In the next item we are going to give a short discussion of known result related to uniqueness of harmonic maps.

**C4.** We refer the interested reader to [J] for the global uniqueness theorem of Al'ber and Hartman, for the result of Jäger and Kaul and for further references.

**Theorem AH** (Al'ber and Hartman). *Let $u : M \mapsto N$ be a harmonic map between compact Riemannian manifolds (without boundary). Suppose $N$ has negative sectional curvature. Then $u$ is unique harmonic map in its homotopy class unless $u(M)$ is a point or a closed geodesic.*

*If the sectional curvature of $N$ is non-positive, then for any two homotopic harmonic $u_0, u_1 : M \mapsto N$, there exist a family $u_t : M \mapsto N$ of harmonic maps, with the property that the curves $u_t(x)$, for fixed $x \in M$, $t \in [0,1]$ varying, constitute a family of parallel geodesics, parameterized proportionally to arc length. In particular, all maps $u_t$ have the same energy.*

**Theorem JK** (Jäger and Kaul). *Suppose that $u_i : \overline{\Omega} \mapsto N$ $(i = 1,2)$ are harmonic maps of class $C^0(\overline{\Omega}, N) \cap C^2(\Omega, N)$, $\Omega$ is a bounded domain in some Riemannian manifold, and $u_i(\overline{\Omega}) \subset B(p,\rho)$, where $B(p,\rho)$ is a geodesic ball in $N$, disjoint to the cut locus of $p$ and with radius $\rho < \pi/2\kappa$, where $\kappa^2$ is an upper bound for the sectional curvature of $B(p,\rho)$. If $u_1 = u_2$ on $\partial\Omega$, then $u_1 \equiv u_2$.*

We refer the interested reader to the Schoen-Yau book [SY] for uniqueness theorems concerning harmonic maps into non-positive curved metric spaces and further references.

After writing the previous version E. Reich pointed out to us that H. Wei [We] studied uniqueness property of harmonic mappings. Also, we became aware of the Coron–Helein paper [CH]. H. Wei using the formula for the energy of variation of a mapping (see [ReS2]) and Reich–Strebel inequality, proved a weaker version of Theorem 2 concerning q.c. mapping. Namely, H. Wei proved Theorem MM2 under additional hypotheses that

(c) $f$ and $g$ are q.c. mappings on the unit disk $\Delta$ onto itself

(d) the metric density $\rho$ is an integrable function on $\Delta$.

Note that the hypotheses (c) and (d) provide that the energy integral of $f$ and $g$ are finite.

In [CH], Coron–Helein used completely different approach then H. Wei in [We] to study minimizing harmonic mappings. Their approach was based on decomposition of given metric $g$ on $\Delta$ as the sum of two metrics $c$ and $h$ such that $c$ is conformal metric of the euclidean metric $e$, $h$ has non-positive Gaussian curvature and $Id$ is harmonic map between $(\Delta, e)$ and $(\Delta, h)$.

**Theorem CH** (Coron–Helein). *Let $(M,h)$ and $(N,g)$ be two Riemannian compact surfaces of class $C^\infty$ possibly with boundary. Then any smooth harmonic diffeomorphism between $(M,h)$ and $(N,g)$ is minimizing in its homotopy class. Moreover, if $\partial M$ is nonempty or if the genus of $M$ is strictly larger then one, then such a diffeomorphism is the unique minimizing map in its homotopy class.*

**D. Related results.** First, we will give an application of Theorem MM2 in the case when the energy integral is infinite.

**D1**. Suppose that

(a) $f$ and $g$ are harmonic diffeomorphisms from the $\Delta$ onto itself w.r.t. Poincaré metric.

(b) Hopf differentials $\varphi = \text{Hopf}(f)$ and $\psi = \text{Hopf}(g)$ are integrable on $\Delta$.
Since $\varphi$ and $\psi$ belong to Bers space (see, for example, [Ah2], [W] and [AMM] for definition and properties of Bers space) a result of Wan [W] shows that $f$ and $g$ are q.c. mappings of $\Delta$ onto itself. If, in addition, we suppose that $f = g$ on the boundary of the unit disk, an application of Theorem MM2 shows that $f$ and $g$ are identical.

Note that every harmonic diffeomorphism of $\Delta$ onto itself w.r.t. Poincaré metric has infinite energy integral.

The following example shows that without assumption that Hopf differentials are integrable Theorem MM1 is not valid.

**D2**. Let $\varphi$ be the conformal mapping of the unit disk $\Delta$ onto upper half-plane $H$ and let $\rho(w) = |\varphi'(w)|$. Next, let $g = \psi \circ h \circ \varphi$, where $\psi$ is the inverse function of $\varphi$ and $h$ is given by $h(z) = x + iky$, $k > 0$. We leave to the reader to verify that $g$ is q.c. harmonic mapping (w.r.t. $\rho$) of the unit disk $\Delta$ onto itself and that $g = \text{Id}$ on the boundary of $\Delta$.

Although, the metric defined by the density $\rho$ is flat on the complex plane $C$ except at one point, Theorem MM1 is not valid.

**D3**. In connection with the parts (D1) and (D2) of this section, we will give a short discussion (we follow Schoen [Sc]).

There is an interesting conjecture which is due to Schoen (see also [Sc]).

*Conjecture*. The q.c. harmonic homeomorphisms from the unit disk $\Delta$ onto itself, w.r.t. Poincaré metric, are parameterized by the boundary values of q.c. maps of the disk.

This is a question which involves proving both an existence and a uniqueness theorem. The existence result for this ideal boundary value problem has been shown by Li and Tam [LT1] under the additional hypothesis that boundary map be sufficiently differentiable. They have also obtained counterexamples to uniqueness without the quasi-conformal hypothesis (but with continuity) and then proved the uniqueness part of Schoen's conjecture (see [LT2]).

A result of Wan [W] gives a parameterisation of the q.c. harmonic homeomorphisms of $\Delta$ in terms of bounded holomorphic quadratic differentials on $\Delta$. Wan has shown that if $f$ is q.c. mapping, then Hopf differential of $f$ is bounded w.r.t. the Poincaré metric on $\Delta$. Conversely, he has shown that for any bounded holomorphic quadratic differential $\Phi$ on $\Delta$ there is a unique q.c. harmonic homeomorphism $f : \Delta \mapsto \Delta$ such that $\text{Hopf}(f) = \Phi$.

**D4**. Theorem MM2 remains valid if the condition (b) (in the hypotheses of Theorem MM2 is replaced by the following.

(e) $f$, $g$ and their inverse mapping have $L^2$-derivatives.

The idea of the proof is as follows. If the condition (e) holds then one can get that $f \circ g^{-1}$ and $g \circ f^{-1}$ have $L^1$-derivatives and its partial derivatives satisfy the chain rule (for a details see Lemma 6.4 of [LV, p. 151]).

It is well-known that the condition (b) implies the condition (e) (see, for example, [LV]).

For a development of theory of harmonic mappings by means of Sobolev spaces, we refer to Schoen-Yau book [SY].

**D5. Harmonic maps and extremal QC mapping**. Before we state the results, we need some notations. Suppose that $f$ is quasi-conformal mapping of the unit disk $\Delta$ onto itself. Let $k[f] = \operatorname{ess\,sup}\{|\mu_f(z)| : z \in \Delta\}$ and let $Q(f)$ denote the collection of all q.c. mappings of $\Delta$ whose pointwise boundary values on $\partial\Delta$ agree with those of $f$. We call $f$ extremal (in its Teichmüller class) if $k[f] \leqslant k[g]$ for every $g \in Q(f)$. An extremal q.c. mapping $f$ is uniquely extremal (in its Teichmüller class) if $k[f] < k[g]$ for every other $g$ in $Q(f)$.

**Theorem M4.** (The first removable singularity theorem). *Suppose that*

($a$) *$f$ is q.c. mapping from $\Delta$ onto $\Delta$*

($b$) *$f$ is a harmonic function with respect to the metric density $\rho$ on $\Delta \setminus K$, where $K$ is compact subset of $\Delta$*

($c$) *$f$ is extremal in its Teichmüller class*

($d$) *there are two positive constant $m$ and $M$ such that $m \leqslant |\varphi(z)| \leqslant M$ for each $z \in \Delta \setminus K$, where $\varphi$ is Hopf differential of $f$.*

*Then $\varphi$ has an analytic extension $\tilde\varphi$ from $\Delta \setminus K$ to $\Delta$ and $\mu(z) = k\,|\tilde\varphi(z)|/\tilde\varphi(z)$ a.e. in $\Delta$, where $k$ is a constant.*

**Theorem M5.** (The second removable singularity theorem). *Suppose that*

($a$) *$f$ is uniquely extremal q.c. mapping, in its class, from $\Delta$ onto $\Delta$*

($b$) *$f$ is a harmonic function with respect to the metric density $\rho$ on $\Delta \setminus K$, where $K$ is compact subset of $\Delta$.*

*Then we have the same conclusion as in the previous theorem.*

During our work with Božin on the problems related to uniquely extremal q.c. mapping [BMM], we also obtained some results of this type.

## III. New version of the Main Inequality

Analyzing the proof of the Grötzsch principle we discovered the following lemma. Let $D$ be a vertically convex domain of finite area in the complex plane $\mathbb{C}$ and let $F$ be a mapping from the domain $D$ onto the domain $G$. Suppose that we have metric $ds = \rho(w)|dw|$ on $G$. Let $\Gamma_x$ be the interval which is the intersection of $D$ by the straight line $\operatorname{Re} z = x$ and let $\gamma_x$ be the curve which is the image of $\Gamma_x$ under $F$. Let $p(x, y) = x$ be the projection and let $(\alpha, \beta) = p(D)$.

**Lemma 1.** *With the notation and hypothesis just stated, suppose (in addition) that the mapping $F$ is homeomorphism which has the first generalized derivatives and that*

$$\operatorname{length}(\Gamma_x) \leqslant \int_{\gamma_x} \rho(w)|dw| \quad \text{a.e. in } (\alpha, \beta).$$

*Then*

$$\operatorname{area}(D) \leqslant \left[\iint_G \rho^2(w)\,du\,dv\right]^{1/2}\left[\iint_D T_\nu\,d\xi\,d\eta\right]^{1/2},$$

*where* $\nu = \operatorname{Belt}[F]$.

The proof of this lemma will be given in the section C of this section. Note that this lemma enables us to get a new version of the Main Inequality (Theorem 1, section D) which is applicable to mappings which are not quasiconformal. But first, we are going to give a proof of the Reich–Strebel inequality in the case of Riemann surfaces of finite analytic type using this lemma.

## A. The inequality of Reich and Strebel.

It is convenient to use notation

$$T_\mu\varphi(z) = \frac{\left|1-\mu(z)\frac{\varphi(z)}{|\varphi(z)|}\right|^2}{1-|\mu(z)|^2}.$$

**Theorem RS.** (Reich and Strebel) *Let $R$ be a Riemann surface of finite analytic type and let $\varphi$ be integrable holomorphic quadratic differential on $R$. Let $f$ be a quasiconformal self mapping of $R$ which is homotopic to the identity map and let $\mu$ be the Beltrami coefficient of $f$. Then*

$$(A1)\qquad \iint\limits_R |\varphi(z)|\,dx\,dy \leqslant \iint\limits_R |\varphi(z)| T_\mu\varphi(z)\,dx\,dy.$$

*Proof.* Suppose that all noncritical trajectory of $\varphi$ are closed. Up to a set of Lebesgue 2-dimensional measure zero $R=\bigcup\Sigma_k$, where $\Sigma_k$ are disjoint ring domains. Each $\Sigma_k$ is swept out by a family of vertical trajectories of the holomorphic quadratic differential $\varphi(z)dz^2$, and in each $\Sigma_k$ there exists a single valued univalent branch $\zeta=\Phi_k(z)$ of $\int\sqrt{\varphi(z)}\,dz$.

Each region $\Phi_k(\Sigma_k)$ is a rectangle $R_k=\{\zeta : 0<\xi<a_k,\ 0<\eta<b_k\}$. Let $\Psi_k=\Phi_k^{-1}$, $F_k=f\circ\Psi_k$, $\theta=\theta_\xi=\Psi_k(\Gamma_\xi)$, where $\Gamma_\xi$ is vertical interval which is the intersection of $R_k$ with the straight line $\operatorname{Re}\zeta=\xi$, and let $\gamma=\gamma_\xi=f(\theta_\xi)$. Since the closed trajectories are shortest in their homotopy class (see Theorem 17.1 of [S2]), we obtain

$$b_k=\int_\theta \sqrt{|\varphi(z)|}|dz| \leqslant \int_\gamma \sqrt{|\varphi(w)|}|dw|.$$

If $G_k=f(\Sigma_k)$, $\nu=\nu_k=\operatorname{Belt}[F_k]$,

$$A_k=\left[\iint\limits_{G_k}|\varphi(w)|\,du\,dv\right]^{1/2} \quad\text{and}\quad B_k=\left[\iint\limits_{D_k} T_\nu\,d\xi\,d\eta\right]^{1/2},$$

then by Lemma 1 $a_k\,b_k\leqslant A_k\,B_k$. Using the change of variables $z=\Phi_k^{-1}(\zeta)$, we get

$$B_k{}^2=\iint\limits_{\Sigma_k}|\varphi(z)|\;T_\mu\varphi(z)\,dx\,dy.$$

Now, an application of Cauchy-Schwarz inequality, as in the proof of the new version of the Main Inequality (see bellow), gives (A1).

Using the fact that quadratic differentials with closed trajectories are dense in $Q$ (see Theorem 25.2 of [S2]), we can get the Main Inequality in general.

**B. The Problem of Grötzsch. B1**. In order to motivate the statement and proof of our version of the Main Inequality we will emphasize the main points in the proof of Grötzsch's principle. We will follow [MM2], where we announced the results of this section.

If $Q$ is a square and $R$ is a rectangle, not a square, there is no conformal mapping of $Q$ on $R$ which maps vertices on vertices. Instead, Grötzsch asked for the most nearly conformal mapping of this kind and took the first step toward the creation of a theory of quasiconformal mappings.

Let $w = f(z)$ be a mapping from one region to another. Recall that we use notation $df = p\,dz + q\,d\overline{z}$, where $p = \partial f$ and $q = \overline{\partial} f$. The complex (Beltrami) dilatataion is $\mu_f = \mathrm{Belt}[f] = q/p$. The dilatation of $f$ is:

$$D_f = \frac{|p| + |q|}{|p| - |q|}.$$

We pass to the Grötzsch problem and give it a precise meaning by saying that $f$ is most nearly conformal if $\sup D_f$ is as small as possible. Let $R, R'$ be two rectangles with sides $a, b$ and $a', b'$. We may assume that $K = \frac{b'}{a'} : \frac{b}{a} \geqslant 1$. The mapping $f$ is supposed to be $C^1$-homeomorphism from $\overline{R}$ onto $\overline{R'}$, which takes $a$-sides into $a$-sides and $b$-sides into $b$-sides. Next, let $\Gamma_x$ be the vertical segment which is the intersection of the line $\mathrm{Re}\, z = x$ with $\overline{R}$ and $\gamma_x$ the curve which is image of $\Gamma_x$ under $f$.

The starting point of Grötzsch's approach is the geometric obvious inequality

$$(B1) \qquad b' \leqslant \mathrm{length}(\gamma_x) = \int_0^b |p - q|\, dy.$$

Using:

$$(B2) \qquad \iint_R J_f\, dx\, dy = a'b',$$

where $J_f$ denotes the Jacobian of $f$, and the Cauchy-Schwarz inequality one gets

$$(B3) \qquad K \leqslant \sup D_f.$$

The minimum is attained for the affine mapping. We note the following connections to with Grötzsch's problem. The restriction to $C^1$-mapping is not essential. The inequality (B3) holds for quasiconformal mapping (see, for example, [Ah2]).

In order to give a version of Grötzsch principle concerning mappings with $L^1$-derivatives, we need the following definition.

**B2. Definition of $L^p$-derivatives.** Let $D$ be a domain in $C$. We say that a function $f : D \mapsto C$ has $L^p$-derivatives, $p \geqslant 1$, if it satisfies the following two conditions:

(a) $f$ is absolutely continuous on lines in $D$

(b) The partial derivatives $f_x$ and $f_y$ belong to $L^p$ in every compact subset of $D$.

When we say that $f$ has first generalized derivatives in $D$ this means that $f$ has $L^1$-derivatives in $D$. For various characterizations of functions with $L^p$-derivatives and their important role in the theory of quasiconformal mappings we refer to Chapters III to VI of the book by Lehto Virtanen [LV].

**B3. A version of Grötzsch's principle.** Before, we give further extension of Grötzsch's principle, it is useful to consider the following example when (B1) and (B3) does not hold.

*Example* 1. Let $\alpha : I \mapsto I$, where $I = [0, 1]$, be Cantor function and let $f(z) = x + i(y + \alpha(y))$.

Note that this function does not satisfy ACL property and that the known formula for the length of curve by means of first partial derivatives does not hold.

Suppose that

(a) $f$ is a homeomorphism of closed rectangle $\overline{R}$ onto the closed rectangle $\overline{R'}$ which maps $a$-sides onto $a'$-sides and $b$-sides onto $b'$-sides.

(b) $f$ has the first generalized derivatives on $R$.

In order to get some conclusion we can follow the outline of the proof of Grötzsch principle from the subsection B1 of this section. We need the following definition. At the point $z$ where $\mu(z)$ is defined and $|\mu(z)| \neq 1$ we define $T_\mu(z)$ by

$$T_\mu(z) = \frac{|1 - \mu(z)|^2}{1 - |\mu(z)|^2}$$

Also, at point $z$ where $|p(z)| = |q(z)|$ we define $T_\mu(z)$ to be zero if $p(z) = q(z)$ and $+\infty$ if $p(z) \neq q(z)$.

Now, we can give the precise meaning of $T_\mu \varphi$ by means of $T_\chi$, where $\chi = \mu\varphi/|\varphi|$.

Since $f$ satisfies the ACL-property, inequality (B1) holds for a.a. $x \in [0, a]$.

In order to prove inequality (B4) (see below), we can suppose that $T_\mu$ is defined and finite a.e. on $R$, because otherwise the right-hand side of (B4) is infinite.

Next, we can integrate w.r.t. $dx$ over $[0, a]$ and use the fact that the Jacobian

$$J_f = |p|^2(1 - |\mu|^2) \quad \text{a.e. on } R.$$

Instead of (B2), we have

$$\int_R J_f \, dx \, dy \leqslant \text{area}(R') = a'b'.$$

Now, an application of Cauchy-Schwarz inequality gives

$$(B4)\qquad \operatorname{area}(R)\frac{b'}{b} \leqslant [\operatorname{area}(R')]^{1/2}\left[\iint\limits_R T_\mu\right]^{1/2}.$$

Further development of the ideas outlined above leads us to Lemma 1 (see below), which will be used in the proof of the new version of the Main Inequality.

## C. Proofs of Lemma 1 and Lemma 2.

For the sake of the reader recall the statement of the Lemma 1.

Let $D$ be a vertically convex domain of finite area in the complex plane $\mathbb{C}$ and let $F$ be a mapping from the domain $D$ onto the domain $G$. Suppose that we have metric $ds = \rho(w)|dw|$ on $G$. Let $\Gamma_x$ be the interval which is the intersection of $D$ by the straight line $\operatorname{Re} z = x$ and let $\gamma_x$ be the curve which is the image of $\Gamma_x$ under $F$. Let $p(x, y) = x$ be the projection and let $(\alpha, \beta) = p(D)$.

**Lemma 1.** *With the notation and hypothesis just stated, suppose (in addition) that the mapping $F$ is homeomorphism which has first generalized derivatives and that*

$$(C1)\qquad \operatorname{length}(\Gamma_x) \leqslant \int\limits_{\gamma_x} \rho(w)|dw| \quad a.e.\ in\ (\alpha, \beta).$$

*Then*

$$(C2)\qquad \operatorname{area}(D) \leqslant \left[\iint\limits_G \rho^2(w)\, du\, dv\right]^{1/2}\left[\iint\limits_D T_\nu\, d\xi\, d\eta\right]^{1/2},$$

*where* $\nu = \operatorname{Belt}[F]$.

*Proof.* We will use the notation $dF = P\, d\zeta + Q\, d\overline{\zeta}$, where $P = \partial F$ and $Q = \overline{\partial} F$. We can suppose that $T_\nu$ is defined and finite a.e. on $D$, because otherwise the right-hand side of (C2) is infinite. With definition of $T_\nu$ in mind, this means that $P = Q$ a.e. on $A$, where $A$ is the set on which Jacobian $J_F$ equals zero. Since $F$ is absolutely continuous on $\Gamma_x$ for a.a. $x \in (\alpha, \beta)$, we find

$$\rho - \operatorname{length}(\gamma_x) = \int\limits_{\Gamma_x} (\rho \circ F)(\zeta)|P||1 - \nu|\, d\eta.$$

By Fubini's theorem and assumption (C1),

$$(\operatorname{area})(D) \leqslant \iint\limits_D (\rho \circ F)(\zeta)|P||1 - \nu|\, d\xi\, d\eta.$$

Since $J_F = |P|^2(1 - |\nu|^2)$ a.e. on $D$, the term on the right can be written in the form

$$\tau = \iint\limits_D (\rho \circ F)(\zeta) J_F^{1/2} T_\nu^{1/2}\, d\xi\, d\eta.$$

Next, using the Cauchy-Schwarz inequality we conclude that $\tau \leqslant A^{1/2} \cdot B^{1/2}$, where

$$A = \iint\limits_D (\rho^2 \circ F)(z)\, J_F(z)\, dx\, dy \quad \text{and} \quad B = \left[\iint\limits_D T_\nu \, d\xi \, d\eta\right].$$

Let $C = \iint\limits_G \rho^2(w)\, dudv$. We need the following lemma to finish the proof.

**Lemma 2.** *We have* $A \leqslant C$.

*Proof*. Let the measure $\mu$ be defined by

$$\mu(E) = \int\limits_E J_F(z)\, dx\, dy \quad \text{and} \quad \mu_F(E) = m(F(E)),$$

for every Lebesgue measurable set $E$. Since $F$ is a homeomorphism which possesses finite partial derivatives a.e. in $D$, by Lemma 3.3 of [LV, p. 131], we have $\mu(E) \leqslant \mu_F(E)$, and therefore we have the desired result.

**D. Proof of new version of the Main Inequality.** There are several papers of Reich and Strebel which concern various forms of the Main Inequality. Our proof is based on their ideas.

Here, we will give a complete proof of the Theorem 1, because we need to be careful when we work with mappings whose dilatation is not bounded. For convenience of the reader let us recall the statement of Theorem 1.

**Theorem 1.** *Suppose that*
($a$) *$f$ is a homeomorphism of $\overline{\Delta}$ onto itself*
($b$) *$f$ has first generalized derivatives on $\Delta$*
($c$) *$f$ is the identity on $\partial\Delta$.*
*Then the inequality*

$$\iint\limits_\Delta |\varphi(z)|\, dx\, dy \leqslant \iint\limits_\Delta |\varphi(z)| T_\mu \varphi(z)\, dx\, dy$$

*holds for every analytic integrable function $\varphi$ on $\Delta$.*

**D1**. First, we observe that Theorem 1 can be reduced to the case when $\varphi$ is also analytic on $\partial\Delta$. Namely, let $\varphi_r$, $0 < r < 1$, be the function defined by $\varphi_r(z) = \varphi(rz)$, $z \in \Delta$. If Theorem 1 holds for every $\varphi_r$, $0 < r < 1$, then, when $r$ approaches 1 we conclude that the theorem holds for $\varphi$, by Lebesgue dominated convergence theorem.

Suppose that $\varphi$ is an analytic function in $\overline{\Delta}$. The following decomposition is possible (see [S1] and [S5]). Up to a set of Lebesgue 2-dimensional measure zero, $\Delta = \bigcup_{k=1}^n \Sigma_k$, where $\{\Sigma_k\}$ are disjoint simple connected "strip" domains. Each $\Sigma_k$ is swept out by a family of vertical trajectories of the holomorphic quadratic differential $\varphi(z)\, dz^2$ and in each $\Sigma_k$ there exists a single valued schlicht branch $\zeta = \Phi_k(z)$ of $\int \sqrt{\varphi(z)}\, dz$. Each region $D_k = \Phi_k(\Sigma_k)$ is vertically convex.

In [S] it is merely assumed that $\varphi$ is analytic on $\Delta$, instead of $\partial\Delta$, so that countably many, instead of merely finitely many $\Sigma_k$, can occur. Actually, in our use of the strip domains, the advantage of limiting ourselves to finitely many is purely didactic.

For the local and global behavior of the trajectories of holomorphic quadratic differentials we refer reader to Strebel's book ([S2]).

The following fact is important in the proof of Theorem 1.

**D2**. The vertical trajectories of holomorphic quadratic differential are globally geodesics in Teichmüller's metric $ds^2 = |\varphi(z)|\,|dz|^2$. Note that $\zeta = \Phi_k(z)$ is a single-valued branch of $\int \sqrt{\varphi}\,dz$ in $\Sigma_k$ and that $D_k = \Phi_k(\Sigma_k)$ and $\Gamma_x = \Gamma_x^k$ is the interval which is intersection of $D_k$ by straight line $\operatorname{Re} z = x$. Let $\theta_x = \Phi_k^{-1}(\Gamma_x)$ and $G_k = f(\Sigma_k)$. Thus $\theta_x$ is trajectory of holomorphic quadratic differential $\varphi(z)\,dz^2$. Let $\gamma_x = f(\theta_x)$. Since $\theta_x$ is a global geodesic in Teichmüller metric, we have

$$ds^2 = |\varphi(z)|\,|dz|^2,$$

$$\operatorname{length}(\Gamma_x) = \int_{\theta_x} |\varphi(z)|^{1/2}|dz| \leqslant \int_{\gamma_x} |\varphi(w)|^{1/2}\ |dw|.$$

Thus we can apply Lemma 1 to the function $F_k = f \circ \Phi_k^{-1}$ and $\rho(w) = |\varphi(w)|^{1/2}$. Hence, by Lemma 1,

$$(D1) \qquad \iint_{\Sigma_k} |\varphi(z)|\,dx\,dy = \operatorname{area}(D_k) \leqslant A_k B_k,$$

where

$$A_k = \Bigg[\iint_{G_k} |\varphi(w)|\,du\,dv\Bigg]^{1/2} \quad \text{and} \quad B_k = \Bigg[\iint_{D_k} T_\nu\,d\xi\,d\eta\Bigg]^{1/2}, \quad \nu = \nu_k = \operatorname{Belt}(F_k).$$

Using the change of variables $z = \Phi_k^{-1}(\zeta)$, we get

$$B_k{}^2 = \iint_{\Sigma_k} |\varphi(z)|\ T_\mu\varphi(z)\,dx\,dy.$$

Further application of the Cauchy-Schwarz lemma and (D1) gives

$$\sum_{k=1}^{n} \big(\varphi - \operatorname{area}(\Sigma_k)\big) \leqslant \sum_{k=1}^{n} A_k B_k \leqslant A \cdot B,$$

where

$$A = \Bigg(\sum_{k=1}^{n} A_k^2\Bigg)^{1/2} \quad \text{and} \quad B = \Bigg(\sum_{k=1}^{n} B_k^2\Bigg)^{1/2}$$

Now, Theorem 1 follows from the fact that

$$A = \Bigg[\iint_{\Delta} |\varphi(z)|\,dx\,dy\Bigg]^{1/2} \quad \text{and} \quad B = \Bigg[\int_{\Delta} |\varphi| T_\mu\varphi\,dx\,dy\Bigg]^{1/2},$$

where $\mu = \operatorname{Belt}[f]$.

## IV. Extremal QC

**A. Introduction.** This subsections an expanded version of the lecture given by the author at The VIII Romanian-Finish Seminar, Iassy, August '99 (see [**M4**]). Recently, in [MM1], [BMM] and [BLMM], characterizations of unique extremality and example of unique extremal dilatation of nonconstant modulus have been obtained. Our primary purpose is to give a short exposition of some of the main result of the authors' joint papers, mentioned above, and sketch further progress in the study of a more general concept. In order to simplify exposition it is convenient to restrict mainly considerations in this review to the disc $\Delta$.

In order to introduce and discuss a more general concept of unique extremality we need a few definitions. Let QC denote the space of all quasiconformal mappings from $\Delta$ onto itself. Two elements $f, g \in$QC are equivalent if $f = g$ on $\partial\Delta$. For a given $f \in$QC we denote the equivalence class of $f$ by $Q_f = [f]$ or $[\mu]$, where $\mu = \mu_f$. Recall some definitions from the introduction. We also use the notation,

$$k_0([f]) = \inf\{||\mu_g||_\infty : g \in Q_f\}.$$

We let $L^\infty = L^\infty(\Delta)$ be the space of essentially bounded complex-valued measurable functions on $\Delta$, and let $M$ be the open unit ball in $L^\infty$. For any $\mu$ in $M$ there exists a quasiconformal solution $f : \Delta \mapsto \Delta$ of the Beltrami equation

$$(A1) \qquad \overline{\partial} f = \mu\, \partial f$$

unique up to a postcomposition by a Möbius transformation.

We let $f^\mu$ be the solution $f$ of (A1) normalized by $f(i) = i$, $f(1) = 1$ and $f(-1) = -1$.

We say two elements $\mu_0$ and $\mu_1$ in $M$ are equivalent if $f^{\mu_0}$ and $f^{\mu_1}$ coincide on $\partial\Delta$ and write $\mu_0 \sim \mu_1$ .

The universal Teichmüller space $T = T(\Delta)$ is the space of equivalence classes of Beltrami coefficient $\mu$ in the unit ball $M$ of the space $L^\infty = L^\infty(\Delta)$ of all essentially bounded functions on $\Delta$. The equivalence class of the zero dilatation is the base point in $T$. For dilatation $\mu$ the extremal set $E = E(\mu)$ is the set where $|\mu(z)| = \|\mu\|_\infty$. We say that $\chi$ is uniquely extremal on its extremal set $E$ if the hypothesis that $\mu$ is equivalent to $\chi$, in its Teichmüller class together with the condition $\|\mu\|_\infty \leqslant \|\chi\|_\infty$, imply that $\mu = \chi$ a.e. on $E$. One verifies easily that if $\chi$ is uniquely extremal on its extremal set $E$ and if the measure of $E$ is positive, then $\chi$ is an extremal dilatation.

In order to prove that $f \in$QC is extremal (or uniquely extremal) we need estimates which allow us to compare Beltrami dilatation $\mu_f$ of $f$, with $\mu_g$, for the other $g \in Q_f$. It appears that the main inequality of Reich and Strebel is a major tool in theory of extremal quasiconformal mappings. Using the main inequality in [BLMM] we have derived an inequality, which we called Delta inequality, and we have shown that the Delta inequality is suitable for studying unique extremality.

In order to give a characterization of a given dilatation $\chi$, which is uniquely extremal on its extremal set $E = E(\chi)$, we make a variation $\chi_r$ of $\chi$ on a compact set $K \subset E$. It turns out that $[\chi_r]$ is a Strebel point. Using Strebel Frame Mapping Criterion and the main inequality, we show that for $r \in (0, r_0)$, where $r_0$ is a positive number small enough, there is a unit vector $\varphi = \varphi_r$ such that

$$\delta(\varphi) = \delta_\chi(\varphi) = \|\chi\|_\infty - \int_\Delta \chi\varphi\, dx\, dy \leqslant 2r \int_K |\varphi|.$$

Here $Q$ is the subspace of $L^1 = L^1(\Delta)$ consisting of holomorphic function in $\Delta$.

Now, if $\lambda = \lambda_\chi$ is the linear functional defined by $\lambda(\varphi) = \iint_\Delta \chi\varphi\, dx\, dy$, we can elementary show that

(1) $\lambda_\chi$ has a unique norm-preserving extension from $Q$ to $Q_{\tilde{\chi}}$

Here $\tilde{\chi}$ is defined by $\tilde{\chi} = \overline{\chi}$ on $E$, $\tilde{\chi} = 0$ on $\Delta \setminus E$ and $Q_{\tilde{\chi}}$ is the smallest subspace of $L^1$ which contains $Q \cup \{\tilde{\chi}\}$.

Proof that (1) implies that $\chi$ is uniquely extremal on $E$ is based on the Delta inequality. Analysis of the proof of Hahn-Banach Theorem shows that (1) is equivalent to

(2) There exists sequence $\{u_n\}$ in $Q$ such that $\lambda(u_n) = \lambda(\tilde{\chi}) + \|\chi\|_\infty \|\tilde{\chi} - u_n\| + o(1)$, where $o(1) \to 0$, when $n \to \infty$.

Hence, using the classical result that $L^1$-convergence of a sequence of functions $\{f_n\}$ implies that there exists subsequence $\{f_{n_k}\}$, which converges a.e. on the corresponding set, and the Delta inequality, we can prove the following result.

**Theorem A.** *$\chi$ is uniquely extremal on its extremal set $E$ if and only if*
*emph(3) $\chi$ satisfies Reich condition on $E$,*
*that is, there exists sequence $\{\varphi_n\}$ in $Q$, such that*
(a) $\delta(\varphi_n) = \|\varphi_n\| \|\chi\|_\infty - \operatorname{Re} \int_\Delta \varphi_n \chi \to 0$
(b) $\lim_{n\to\infty} \inf |\varphi_n(z)| > 0$ *for almost all $z$ in $E(\chi)$.*

This criterion has many applications concerning precise characterizations (Theorem G2 and Corollary G3) and removable properties (Theorem G3 and Theorem G4) of unique extremal dilatation of Teichmüller type. Also, this criterion can be used to construct unique extremal dilatation of nonconstant modulus.

**B. Extremal dilatation.** In this section we give a short report concerning extremal mappings. The interested reader can learn more about extremal mappings from Strebel's survey article [S6], Reich's papers [Re8], [Re9] and Earle-Li Zhong's [ELi] , all of which we highly recommend. Extremal mappings have been one of the main topics in the theory of quasiconformal mappings, since its earliest days, when Grötzsch solved the extremal problem for two rectangles. In order to discuss them we need to review some familiar definitions.

A homeomorphism $f$ from a domain $G$ onto another is called quasiconformal if $f$ is ACL (absolutely continuous on lines) in $G$ and $|f_{\bar{z}}| \leqslant k|f_z|$ a.e. in $G$, for some real number $k$, with $0 \leqslant k < 1$.

It is well known that if $f$ is a quasiconformal mapping defined on the region $G$, then the function $f_z$ is nonzero a.e. in $G$. The function $\mu_f = f_{\bar{z}}/f_z$ is therefore a well defined bounded measurable function on $G$, called the complex dilatation or Beltrami coefficient of $f$.

The positive number

$$K(f) = \frac{1 + \|\mu_f\|_\infty}{1 - \|\mu_f\|_\infty}$$

is called the maximal dilatation of $f$. We say that $f$ is $K$-quasiconformal if $f$ is a quasiconformal mapping and $K(f) \leqslant K$. Let QC denote the space of all quasiconformal mappings from $\Delta$ onto itself. Two elements $f, g \in$QC are equivalent if $f = g$ on $\partial\Delta$. For a given $f \in$QC we denote the equivalence class of $f \in$QC by $Q_f = [f]$ or $[\mu]$, where $\mu = \mu_f$. Also we use notation

$$k_0([f]) = \inf\{\|\mu_g\|_\infty : g \in Q_f\} \quad \text{and} \quad K_0([f]) = \frac{1+k_0}{1-k_0}.$$

We let $L^\infty = L^\infty(\Delta)$ be the space of essentially bounded complex-valued measurable functions on $\Delta$, and let $M$ be the open unit ball in $L^\infty$. For any $\mu$ in $M$ there exists a quasiconformal solution $f : \Delta \mapsto \Delta$ of the Beltrami equation

$$\overline{\partial} f = \mu\, \partial f \tag{1}$$

unique up to a postcomposition by a Möbius transformation.

We let $f^\mu$ be the solution $f$ of (1) normalized by $f(i) = i$, $f(1) = 1$ and $f(-1) = -1$. Two elements $\mu_0$ and $\mu_1$ in $M$ are equivalent if $f^{\mu_0}$ and $f^{\mu_1}$ coincide on $\partial\Delta$. For given $\mu \in M$ the equivalence class $[\mu]$ contains at least one element $\mu_0$ such that $\|\mu_0\|_\infty = \inf\{\|\nu\|_\infty : \nu \in [\mu]\}$. Such a $\mu_0$ is referred to as an extremal complex dilatation and $f_0 = f^{\mu_0}$ as an extremal quasiconformal mapping (abbreviated EQC mapping).

As we mentioned, we restrict mainly considerations in this review to the disc and only consider a few examples concerning the other Jordan domains.

For discussion concerning subregions of the plane, which are not necessarily simply-connected, we refer the interested reader to the paper [ELi], which we follow in this section.

Let $z_i$, $1 \leqslant i \leqslant 4$, be four distinct points on the unit circle $S^1$, and let $w_i$, $1 \leqslant i \leqslant 4$, be their images under some sense-preserving homeomorphism $h$ of the closed unit disc $\Delta$ onto itself. Let $S$ be the set of all quasiconformal mappings $g$ of the open unit disc $\Delta$ onto itself which maps $z_i$ to $w_i$, $1 \leqslant i \leqslant 4$.

Let $\phi$ and $\psi$ be conformal mappings of the two discs onto horizontal rectangles, $R$ and $R'$, which map the distinguished points (vertices) on vertices of rectangles. Let $A_K$ be the horizontal stretching of $R$ onto $R'$ defined by $A_K(\zeta) = K\xi + i\eta$, where $\zeta = \xi + i\eta$. Then $f = \psi^{-1} \circ A_K \circ \phi$ is the unique extremal mapping.

It is easy to compute the Beltrami dilatation of $f$, that we have just described. One finds that

$$\mu_f = k\frac{\overline{\phi'}}{\phi'} = k\frac{|\phi'|^2}{(\phi')^2}. \tag{2}$$

Note that the holomorphic function $(\phi')^2$ belongs to $Q(\Delta)$ since its $L^1$ norm equals the area of rectangle $R$.

Teichmüller discovered that many extremal quasiconformal mappings have Beltrami dilatation whose form resembles (2).

In recognition of the importance of that discovery, a Beltrami coefficient $\mu$ in a plane region $G$ is called a Teichmüller dilatation if there are number $k \in [0, 1)$ and a holomorphic function $\varphi \in Q(G)$, not identically zero, such that $\mu = k|\varphi|/\varphi$, a.e. in $G$.

If we do not require that $\varphi$ is integrable we will say that $\mu$ is Teichmüller dilatation in general sense (or $\mu$ has Teichmüller type). Here, for a given domain $G \subset \mathbb{C}$, by $Q(G)$ we denote the space of all holomorphic function $\varphi$ in $L^1(G)$. A quasiconformal mapping $f$ whose Beltrami coefficient is a Teichmüller dilatation is called a Teichmüller mapping.

Let $Q$ be the Banach space consisting of holomorphic functions $\varphi$, belonging to $L^1 = L^1(\Delta)$, with norm

$$\|\varphi\| = \iint_{\Delta} |\varphi(z)|\, dx\, dy \;<\; \infty,\;\; \varphi \in Q.$$

For $\mu \in L^\infty$ we consider the linear functional $\Lambda_\mu(\varphi) = (\mu, \varphi)$, $\varphi \in Q$, where

$$(\mu, \varphi) = \iint_{\Delta} \mu(z)\varphi(z)\, dx\, dy,$$

and denote by $\|\mu\|_* = \|\Lambda_\mu\|$ the norm of $\mu$ as an element of the dual space of $Q$. For $\mu \in L^\infty$ we say that it satisfies the Hamilton-Krushkal condition if $\|\mu\|_* = \|\mu\|_\infty$.

We are now ready to state the main result about extremal complex dilatations.

**Theorem HKRS.** (Hamilton–Krushkal and Reich–Strebel) *Let $\mu \in M$. A necessary and sufficient condition that $f^\mu$ is an EQC mapping is that*

$$\|\mu\|_* = \|\mu\|_\infty. \tag{3}$$

We are going to prove the necessity of Hamilton-Krushkal condition for a quasiconformal mapping to be extremal in its Teichmüller class (see Theorem HK bellow). For the proof we need two lemmas.

Let $N$ denote the subspace of $L^\infty(\Delta)$ which is orthogonal to $Q$. Differentials which belong to $N$ are called infinitesimally trivial.

For the proof of the next Lemma see Lehto [L2].

**Lemma B1.** *Let $\nu \in N$ and $\|\nu\|_\infty < 2$. Then, for $0 \leqslant t \leqslant 1/4$, there is a $\sigma_t \in [t\nu]$ such that $\|\sigma_t\|_\infty \leqslant 12\, t^2$.*

**Lemma KRS.** *Let $f^\mu$ be extremal. If $\mu$ and $\chi$ represent the same infinitesimal equivalence class, then $\|\mu\|_\infty \leqslant \|\chi\|_\infty$.*

*Proof.* Set $k_0 = \|\mu\|_\infty$, $k = \|\chi\|_\infty$. We assume that $k < k_0$ and prove that $f^\mu$ cannot be extremal. Writing $\nu = \mu - \chi$, we first prove that for $f^\lambda = f \circ (f^{t\nu})^{-1}$ has a smaller maximal dilatation than $f^\mu$. Since

$$\lambda(\zeta) = \frac{\mu - t\nu}{1 - t\mu\overline{\nu}}\frac{1}{\theta}, \quad \text{where } \theta = \overline{p}/p,\;\; p = \partial f^{t\nu},$$

direct calculation gives $|\lambda(\zeta)| = |\mu(z)| - At + O(t^2)$, where

$$\zeta = f^{t\nu}(z),\;\; A = A_{\mu,\chi}(z) = \frac{1 - |\mu(z)|^2}{|\mu(z)|}\,\mathrm{Re}\,\mu(z)\overline{\nu(z)}$$

and the remainder term $O(t^2)$ is uniformly bounded in $z$.

Write $E_1 = \{z \in \Delta : |\mu(z)| < (k + k_0)/2\}$, $E_2 = \Delta \setminus E_1$. In $E_2$,

$$\operatorname{Re} \frac{\mu\overline{\nu}}{|\mu|} = |\mu| - \operatorname{Re} \frac{\mu}{|\mu|}\overline{\nu} \geqslant \frac{k + k_0}{2} - k = \frac{k_0 - k}{2}$$

and therefore

$$A \geqslant \frac{1}{2}(1 - k_0^2)(k_0 - k).$$

Hence, $|\lambda(\zeta)| < k_0 - m_0 t$. But, if $\sigma_t$ is as in Lemma B1, then $f^\tau = f^\lambda \circ f^{\sigma_t}$ and $f^\mu$ belong to the same Teichmüller class. We have

$$\|\tau\|_\infty \leqslant \frac{\|\lambda\|_\infty + \|\sigma_t\|_\infty}{1 - \|\sigma_t\|_\infty} < \frac{k_0 - m_0 t + 12t^2}{1 - 12t^2}.$$

**Theorem HK.** *If $f^\mu$ is extremal in its equivalence class, then*

$$\|\mu\|_* = \|\mu\|_\infty. \tag{4}$$

*Proof.* By the Hahn-Banach theorem and Riesz representation theorem there exists $\chi \in M$ such that $\mu$ and $\chi$ represent the same infinitesimal class and $\|\chi\|_\infty = \|\Lambda_\mu\| = \|\mu\|_*$. Since $\|\chi\|_\infty = \|\mu\|_* \leqslant \|\mu\|_\infty$, the equality (4) follows from Lemma KRS.

The necessary condition (4) for $\mu$ to be extremal is also sufficient. Reich and Strebel proved this using their "Main Inequality". We find it is convenient to formulate Hamilton-Krushkal's condition in terms of Hamilton sequences.

**Definition B1.** Let $\mu_f$ be the Beltrami coefficient of some quasiconformal mapping $f$ of the unit disc $\Delta$ onto itself. A Hamilton sequence for $\mu_f$, is a sequence in $Q$, such that $\|\varphi_n\| = 1$, for all $n$, and $\lim\limits_{n\to\infty}(\mu, \varphi_n) = \|\mu\|_\infty$.

Now we can state theorem of Hamilton-Krushkal and Reich–Strebel in the form

**Theorem B1.** *Let $f$ be a quasiconformal mapping of the unit disc $\Delta$ onto itself, and let $\mu_f$ be its Beltrami coefficient. Then $f$ is extremal in its class $[f]$ if and only if $\mu_f$ has a Hamilton sequence.*

**Corollary B1.** *Every Teichmüller mapping is extremal in its equivalence class.*

*Proof.* If $f$ is a Teichmüller mapping, then its Beltrami coefficient $\mu_f$ can be written in the form $\mu_f = k|\varphi|/\varphi$, with $0 < k < 1$, $\varphi \in Q$ and $\|\varphi\| = 1$. The sequence $\{\varphi_n\}$, with $\varphi_n = \varphi$, for all $n$, is obviously a Hamilton sequence for $\mu_f$.

Not all extremal quasiconformal mappings are Teichmüller mappings. The first counterexample occurs in the famous paper [BA].

**Example B1.** Let $H$ be the upper half plane and $K > 1$. In the section 5 of [BA] it is shown that the quasiconformal mapping $f(z) = z\,|z|^{K-1}$ of $H$ onto itself is extremal in its class. A simple calculation shows that

$$\mu_f(z) = k\frac{|\psi(z)|}{\psi(z)},\ z \in H,$$

where $k = (K-1)/(K+1)$ and $\psi(z) = z^{-2}$. This has the same form as Teichmüller mapping, but $\psi$ is not Teichmüller mapping, because $\psi$ is not an integrable function on the upper half plane.

The question whether $f$ is uniquely extremal mapping in its class was not considered in [BA]. One method for studying extremal mappings that are not Teichmüller mappings is to use degenerating Hamilton sequences.

Let $\mu_f$ be the Beltrami coefficient of the quasiconformal mapping $f$. The Hamilton sequence $\{\varphi_n\}$ for $\mu_\varphi$ is degenerating if $\varphi_n$ converges zero uniformly on compact subsets on $\Delta$ as $n \to \infty$. The connection between degenerating Hamilton sequences and Teichmüller mappings is given by the following.

**Lemma B2.** (Reich and Strebel) *If $\mu_f$ has a Hamilton sequence that does not degenerate, then $f$ is a Teichmüller mapping.*

For a proof of this Lemma see for example Earle–Li Zhong [EL] (see also Lehto [L2]). Lemma B2 shows that it is desirable to find geometric condition on an extremal mapping that will prevent its Beltrami coefficient from having a degenerating Hamilton sequences. Strebel's Frame Mapping Criterion provides such conditions in terms of the boundary dilatation, which we shall now define.

The boundary dilatation $H([\mu])$ of the Teichmüller class of $\mu$ is the infimum over all elements $\nu$ in the equivalence class of $\mu$ in $T$ of the quantity

$$\frac{1+h^*(\nu)}{1-h^*(\nu)}.$$

Here $h^*(\nu)$ is the infimum over all compact subsets $K$ contained in $\Delta$ of the essential supremum of Beltrami coefficient $\nu(z)$ as $z$ varies over $\Delta \setminus K$. As usual, we let

$$H^*(\mu) = \frac{1+h^*(\mu)}{1-h^*(\mu)}.$$

For $f \in$QC, also, we define $H^*(f) = H^*(\mu_f)$ and $H([f]) = H([\mu_f])$.

**Theorem B2.** (Strebel Frame Mapping Criterion) *Let $f \in QC$ and let $f$ be extremal in its class $Q_f$. If $H([f]) < K(f)$, then*
$(a)$ *$\mu_f$ has no degenerating Hamilton sequences*
$(b)$ *$f$ is a Teichmüller mapping.*

For a proof of this theorem see for example Gardiner [Ga].

**Example B2.** (Strebel's chimney) In [S1], Strebel made another breakthrough by constructing the first example of a nonuniquely extremal Beltrami coefficient. Strebel considered the plane region

$$V = \{z = x + iy : |x| < 1\} \cup \{z = x + iy : y < 0\},$$

now known as Strebel's chimney. For every real number $K > 1$, the quasiconformal homeomorphism $f_K(z) = x + iKy$ of $V$ is extremal in its class $\tau_K \in T(V)$. On the other hand, $\tau_K$ contains infinitely many distinct extremal mappings. For instance, $h_L(z) = f_K(z)$, for $y \geqslant 0$, and $h_L(z) = f_L(z)$, for $y < 0$, is extremal in $\tau_K$ for every $L \in [1/K, K]$.

**C. Unique extremality.** In this section we shortly discuss results of authors' joint paper [BLMM] (see also [MM1] and [BMM]). For studying unique extremality it is convenient to use the following result.

**Proposition C1.** *Let $\varphi$ be conformal mapping of the domain $D$ onto $V$. Then $\nu$ is uniquely extremal on $V$ if and only if $\mu(z) = \nu(\varphi(z))\overline{\varphi'(z)}/\varphi'(z)$ is uniquely extremal on $D$.*

Strebel has proved in [S3] that the horizontal stretching $A(w) = A_K(w) = Ku + iv$, $K > 1$, in $V = \{w : |v| < \pi/4\}$ is the unique extremal in its equivalence class. The proof of Strebel's result has also been given in [ELi], using Reich's method (see also [Re9]).

Using conformal mapping $w = \varphi(z) = \frac{1}{2}\ln z - \frac{\pi}{4}$ of $H$ onto $V$ one can show that the Beurling-Ahlfors mapping $f$ is the uniquely extremal mapping in its class.

Ahlfors and Bers showed that $T$ has a complex structure with tangent space at the base point isomorphic to Banach space $Q^*$. Two tangent vectors $\mu$ and $\nu$ in the tangent space to $M$ determine the same tangent vector in $T$ if and only if

$$\int_\Delta \varphi\mu = \int_\Delta \varphi\nu, \text{ for all } \varphi \in Q.$$

If $\mu$ and $\nu$ have this property, we say that they represent the same Teichmüller infinitesimal equivalence class or, more briefly, that they are infinitesimally equivalent. The space of equivalence classes is denoted by $B$. A given $\mu$ is said to be extremal in its infinitesimal Teichmüller class if $\|\mu\|_\infty \leqslant \|\nu\|_\infty$, for any $\nu$ infinitesimally equivalent to $\mu$.

Recall that Hamilton, Krushkal, Reich and Strebel showed that a Beltrami coefficient $\nu$ in $M$ is extremal in its class in $T$ if and only if $\nu$ is extremal in its class in $B$. It was natural to consider whether the analogous statement holds for the unique extremality. In several articles Reich showed that in many special situations the two notions of unique extremality coincide and conjectured that the notions may coincide in general. In [BLMM] (see also [MM1] and [BMM]) we have recently proved the answer to this conjecture is affirmative.

**Theorem C1.** (The Equivalence Theorem I) *$\mu$ is uniquely extremal in its Teichmüller class if and only if $\mu$ is uniquely extremal in its infinitesimally class.*

Proof of this theorem is based on estimates which allow us to compare two Beltrami coefficients $\mu$ and $\nu$ in the same global equivalence class and two Beltrami differentials in the same infinitesimal equivalence class. These estimates generalize Reich's Delta inequality for Beltrami differentials in the same equivalence class (see [R8]). Unlike Reich's forms of the Delta inequalities, our forms do not require either one of the Beltrami coefficients to have constant absolute value.

The generalized Delta inequality is our first step towards obtaining the criterion for the unique extremality of Beltrami differentials. The next important step is the analysis of the proof of Hahn-Banach theorem and its applications to our setting. In particular, we obtain the following necessary and sufficient criterion for the unique extremality of given Beltrami coefficient $\chi$.

**Theorem C2.** (Characterization Theorem I) *Beltrami coefficient $\chi$ is uniquely extremal if and only if for every admissible variation $\eta$ of $\chi$ there exist a sequence $\varphi_n$ in $A(R)$ such that*

(a) $\delta(\varphi_n) = \|\varphi_n\| \|\eta\|_\infty - \operatorname{Re} \int_R \varphi_n \eta \to 0$

(b) $\lim\limits_{n\to\infty} \inf |\varphi_n(z)| > 0$, *for almost all $z$ in $E(\eta)$.*

Here, an admissible variation $\eta$ of $\chi$ is any Beltrami differential that does not increase the $L^\infty$-norm of $\chi$, and which is allowed to differ from $\chi$ only on the set where $|\chi(z)| \leqslant s < \|\chi\|_\infty$, where $s$ is a constant, and the extremal set $E(\eta)$ is the set where $\eta(z) = \|\eta\|_\infty$. This criterion is analogous to the Hamilton-Krushkal, Reich–Strebel necessary and sufficient criterion for the extremality. Namely, $\chi$ is extremal if and only if there is a sequence $\varphi_n$ of holomorphic quadratic differential of norm 1 such that

$$\|\chi\|_\infty - \operatorname{Re} \int_R \eta\varphi_n \to 0.$$

This criterion is among listed in the theorem in Section 11, in [BLMM], which we called the Characterization Theorem. The Characterization Theorem applies to many interesting situations. For instance, we can say precisely when a Beltrami differential of the form $k|\varphi(z)|/\varphi(z)$, with $\varphi$ a holomorphic quadratic differential with $\|\varphi\| = \infty$, is uniquely extremal.

There are many examples of extremal Beltrami differentials with nonconstant modulus, but all examples of uniquely extremal Beltrami differentials known up to our papers [BLMM] and [BMM] were of the general Teichmüller type. Moreover, many results obtained studying the extremal problems speak in favour of the conjecture that all uniquely extremal Beltrami differentials $\mu$ satisfy $|\mu(z)| = \|\mu\|_\infty$, for almost all $z$. Surprisingly, we disprove this conjecture and show that there are uniquely extremal Beltrami differentials with nonconstant modulus.

**D. The main inequalities.** Let $\Delta$ denote the unit disc,

$$S_\mu\varphi = \Big|1 - \mu(z)\frac{\varphi(z)}{|\varphi(z)|}\Big|^2 \quad \text{and} \quad T_\mu\varphi(z) = \frac{S_\mu\varphi}{1-|\mu(z)|^2} = \frac{\Big|1 - \mu(z)\frac{\varphi(z)}{|\varphi(z)|}\Big|^2}{1-|\mu(z)|^2}.$$

We will refer to the following result as the Reich–Strebel inequality or the Main Inequality.

**Theorem RS.** (Reich and Strebel). *Suppose that $f$ is a quasiconformal homeomorphism of $\Delta$ onto itself which is the identity on $\partial\Delta$. Then, with $\mu = \mu_f$*

$$\iint\limits_\Delta |\varphi(z)|\,dx\,dy \leqslant \iint\limits_\Delta |\varphi(z)| T_\mu\varphi(z)\,dx\,dy,$$

*for every analytic integrable function $\varphi$ on $\Delta$.*

Various forms of this result play a major role in the theory of quasiconformal mappings and have many applications.

For applications to extremal and uniquely extremal quasiconformal mappings, we refer the interested reader to the book by Gardiner ([G]), and for some recent results to [MM1], [BMM], [BLMM], [Re3] and [Re9] .

Let $f$ and $g$ be two equivalent quasiconformal mappings on $\Delta$ and let

$$\mu = \mu_f = \mathrm{Belt}[f],\ \alpha = \mu_{f^{-1}} \circ f,\ \beta = \mu_{g^{-1}} \circ f \text{ and } \tau = \mu \frac{\beta}{\alpha}.$$

Then $g^{-1} \circ f$ is the identity on $\partial\Delta$, and, if we apply the Reich–Strebel inequality to $F = g^{-1} \circ f$, we get

$$1 \leqslant \iint\limits_{\Delta} |\varphi(z)| T_\mu \varphi\, T_{-\tau\theta} \varphi \, dx\, dy, \tag{1}$$

where $\theta = (1 - \overline{\mu\varphi}/|\varphi|)(1 - \mu\varphi/|\varphi|)^{-1}$ and $\varphi \in Q$, $\|\varphi\| = 1$. Note that $\alpha = -\mu p/\overline{p}$, $\tau = -\overline{p}/p\beta$ and

$$T_{-\tau\theta}\varphi = \frac{S_{-\tau\theta}\varphi}{1 - |\beta|^2}.$$

Now, we are going to state two consequences of the Main Inequality, known as the fundamental Reich–Strebel inequalities (inequalities (2) and (3) below). If $K_0 = K_0([\mu])$ and $g \in Q_f$ is an extremal quasiconformal mapping, then inequality (1) yields

$$\frac{1}{K_0} \leqslant \iint\limits_{\Delta} |\varphi| T_\mu \varphi \, dx\, dy. \tag{2}$$

Suppose now that $\mu$ is a Teichmüller differential, i.e., $\mu = k_0|\varphi_0|/\varphi_0$ for some $\varphi_0 \in Q$, with $\|\varphi_0\| = 1$ and $0 < k_0 < 1$. Then $\theta = 1$ and $T_\mu \varphi_0 = K_0^{-1}$, where $K_0 = \frac{1+k_0}{1-k_0}$. Suppose that $\nu$ equivalent to $k_0|\varphi_0|/\varphi_0$, where $\|\varphi_0\| = 1$.

It means that there exists $g = f^\nu$, which is is equivalent to $f = f^\mu$. Therefore, the inequality (1) becomes

$$K_0 \leqslant \iint\limits_{\Delta} |\varphi_0| T_{-\tau} \varphi_0 \, dx\, dy \tag{3}$$

Since $\|\tau\|_\infty = \|\beta\|_\infty = \|\nu\|_\infty$, it follows from (3) that $K_0 \leqslant \frac{1+\|\nu\|_\infty}{1-\|\nu\|_\infty}$, which implies $k_0 \leqslant \|\nu\|_\infty$ and, therefore, $k_0|\varphi_0|/\varphi_0$ has minimal norm among all equivalent Beltrami dilatations $\nu$. Moreover, if $k_0 = \|\nu\|_\infty$, then the inequality (3) yields

$$K_0 \leqslant \iint\limits_{\Delta} |\varphi_0| T_{-\nu} \varphi_0 \, dx\, dy \leqslant K_0, \tag{4}$$

and so (4) is an equality and this obviously implies first that $\alpha = \beta$, i.e., $f^{-1} = g^{-1}$. Hence $f = g$ and therefore $\nu = k_0|\varphi_0|/\varphi_0$ almost everywhere.

We have proved the following theorem.

**Theorem T.** (Teichmüller Uniqueness Theorem) *Suppose that $\mu = k|\varphi|/\varphi$, where $0 < k < 1$ and $\varphi$ is an element of norm 1 in $Q$. Then $\mu = k|\varphi|/\varphi$ is uniquely extremal in its Teichmüller class on $\Delta$.*

Note that Teichmüller Uniqueness Theorem also follows directly from the Delta inequality, which will be considered in the next subsection.

**E. Delta inequality.** One verifies easily that the main inequality of Reich and Strebel can be stated in the following form.

**Theorem E1.** (Reich and Strebel) textitSuppose that $f$ is a quasiconfomal homeomorphism of $\Delta$ onto itself which is the identity on $\partial\Delta$.Then, with $\mu = \mu_f$

$$\operatorname{Re} \iint_{\Delta} \frac{\mu}{1-|\mu|^2}\varphi\, dx\, dy \leqslant \iint_{\Delta} \frac{|\mu|^2}{1-|\mu|^2}|\varphi|\, dx\, dy, \text{ for all } \varphi \in Q.$$

A simple calculation shows that this inequality is equivalent to the inequality (1) (see below), which is a starting point in the proof of the Delta inequality.

**Theorem E2.** (The Delta inequality in $T$) *Let $\mu$ and $\nu$ belong to the same class in $T$, $f = f^\mu$, $g = f^\nu$, $\alpha = \operatorname{Belt}[f^{-1}] \circ f$, $\beta = \operatorname{Belt}[g^{-1}] \circ f$, $\rho = |\alpha(z) - \beta(z)|^2$ and $I = I(\varphi) = \int_\Delta \rho|\varphi|$. If $\|\nu\|_\infty \leqslant k = \|\mu\|_\infty$, then $I(\varphi) \leqslant C\delta_\mu(\varphi)$, $\varphi \in Q$, where $C$ is a constant which depends only on $k = \|\mu\|_\infty$.*

*Proof.* We will prove this result under additional hypothesis that $|\mu|$ is bounded from below by a positive constant $s$, for almost every $z$ in $\Delta$. For a complete proof we refer the interested reader to [BLMM]. Let $P = (\alpha - \beta)(1 - \alpha\overline{\beta})\alpha^{-1}$ and $Q = (1 - |\alpha|^2)(1 - |\beta|^2)$. Then for any $\varphi \in Q$ we have as an easy consequence of the Main Inequality

$$\operatorname{Re} \int_\Delta \frac{P}{Q}\mu\varphi \leqslant \int_\Delta \frac{\rho}{Q}|\varphi|. \tag{1}$$

In order to get an estimate involving $\delta\{\varphi\}$ add to both sides $l = \int_\Delta |\alpha| \frac{\operatorname{Re} P}{Q}|\varphi|$. We get

$$\int_\Delta \frac{|\alpha| \operatorname{Re} P - \rho}{Q}|\varphi| \leqslant \operatorname{Re} \int_\Delta \frac{P}{Q}(|\alpha||\varphi| - \mu\varphi).$$

Furthermore, $|\alpha| \operatorname{Re} P - \rho = A\rho + B(|\alpha| - |\beta|)$, where $A = 2^{-1}|\alpha|^{-1}(1 - |\alpha|)^2$ and $B = 2^{-1}|\alpha|^{-1}(|\alpha| + |\beta|)(1 - |\alpha|^2)$. Hence, the inequality (1) can be rewritten in the form

$$\int_\Delta \frac{A\rho}{Q}|\varphi| \leqslant J(\varphi) + s(\varphi), \tag{2}$$

where

$$J(\varphi) = \operatorname{Re} \int_\Delta \frac{P}{Q}(|\mu||\varphi| - \mu\varphi) \quad \text{and} \quad s(\varphi) = \int_\Delta \frac{B}{Q}(|\beta| - |\alpha|)|\varphi|.$$

Since $\mu$ is bounded from below by a positive constant $s > 0$ in $\Delta$, $A$ and $B$ are bounded, using (2) and the estimate $|\beta| - |\alpha| \leqslant k - |\mu|$, one can show that there is a constant $c$, which depends only on $k = \|\mu\|_\infty$ and $s$, such that $I \leqslant c(I_1 + \tau\{\varphi\})$, where

$$I_1 = \int_\Delta \sqrt{\rho}||\mu\varphi| - \mu\varphi| \quad \text{and} \quad \tau = \tau\{\varphi\} = \int_\Delta (k - |\mu|)|\varphi|.$$

Using Cauchy-Schwarz inequality and the identity $||w|-w|^2 = 2|w|(|w|-\operatorname{Re} w)$, we obtain $I_1 \leqslant cI^{1/2}\delta^{1/2}$. Using this and the inequality $\tau \leqslant \delta$, we get $I \leqslant c(I^{1/2}\delta^{1/2} + \delta)$, and therefore $I \leqslant C(k,s)\delta$.

**F. More general concept of unique extremality.** We say that $\chi \in M$ is uniquely extremal on its extremal set $E$ if $|E| > 0$ and the hypothesis that $\mu$ is equivalent to $\chi$, in its Teichmüller class together with the condition $\|\mu\|_\infty \leqslant \|\chi\|_\infty$, imply that $\mu = \chi$ a.e. on $E$.

We say that $\chi \in L^\infty$ satisfies unique extension property on its extremal set $E$ (or we say that $\chi$ is unique extremal on $E$ in its infinitesimal Teichmüller class $B$), if $|E| > 0$ and the hypothesis that $\mu$ is equivalent to $\chi$, in its infinitesimal Teichmüller class $B$ together with the condition $\|\mu\|_\infty \leqslant \|\chi\|_\infty$, imply that $\mu = \chi$ a.e. on $E$.

**Theorem F1.** (The Equivalence Theorem II) *Let $\chi \in M$ and $E$ be its extremal set. Then the following conditions are equivalent*

(a) *$\chi$ is uniquely extremal on $E$*

(b) *$\chi$ satisfies unique extension property on $E$.*

Note that an immediate consequence of this result is the Equivalence Theorem when dilatation has constant absolute value. First, we give a few definitions and lemmas, on which the proof is based.

Now, we will introduce the variation property and prove Lemma $VT$, Lemma $V$ and two other lemmas (we develop them in $T$ and $B$) which utilize variational property.

**Definition F1.** Let $\mu \in L^\infty$, $k = \|\mu\|_\infty < 1$ and $E$ be its extremal set. We say that $\mu$ satisfies variational property on $E$ in $T$ if for each compact subset $K \subset E$ and for each $r > 0$

$$(1+r)k_0([\mu_r]) > k_0([\mu]),$$

where $\mu_r = \dfrac{\mu}{1+r}$ in $K^c$ and $\mu_r = \mu$ in $K$.

In a parallel manner, we define the variational property in $B$.

**Definition F2.** Let $\mu \in L^\infty$, $k = \|\mu\|_\infty$ and $E$ be its extremal set. We say that $\mu$ satisfies variational property on $E$ in $B$ if for each compact subset $K \subset E$ and for each $r > 0$

$$\|\mu_r\|_* > \|\mu\|_*,$$

where $\mu_r = \mu$ in $K^c$ and $\mu_r = (1+r)\mu$ in $K$.

**Lemma VT1.** *If $\mu$ is uniquely extremal on its extremal set $E$, then $\mu$ satisfies variational property on $E$ in $T$.*

*Proof.* Let $K \subset E$ be a set of positive measure and let $\mu_r$ be the variation of $\mu$ to $K$. Assume that $H(\mu_r) = K_0(\mu_r)$. Then, there exists $\nu$ in Teichmüller class of $\mu_r$ such that $\|\nu\|_\infty \leqslant s_0$.

Let $g = f^\nu \circ g_r$, where $g_r = (f^{\mu_r})^{-1} \circ f^\mu$. As in [BLMM] the reader can verify that $K(g) \leqslant K(f^\mu)$. Using that $g_r$ converges uniformly to the identity on $K$, when $r \to 0+$, one can show that the set $A = K \cap g_r(K)$ has positive measure if $r \in (0, r_0)$

for some positive $r_0$. This means that $f$ and $g$ are distinct on the set $B = g_r^{-1}(A)$. Since $B \subset E$ has positive measure and $f$ is uniquely extremal on $E$ this yields a contradiction.

**Lemma VT2.** *Let $\mu$ satisfies variational property on $E$ in $T$ and let $K \subset E$ be compact set of positive measure on which $|\mu| = \|\mu\|_\infty < 1$. Then for each $r > 0$ there is a unit vector $\varphi \in Q$ such that*

$$\delta_\mu(\varphi) \leqslant 2r \int_K |\varphi|. \tag{1}$$

*Proof.* Let $k_0 = k_0([\mu])$, $s_0 = k_0/(1+r)$, $\mu_r = \frac{\mu}{1+r}$ in $K^c$ and $\mu_r = \mu$ in $K$. Since $H([\mu_r]) \leqslant H^*(\mu_r) \leqslant \frac{1+s_0}{1-s_0}$ and by Lemma VT, $K_0([\mu_r]) > \frac{1+s_0}{1-s_0}$, we conclude that the extremal dilatation $K_0$ is strictly greater then the boundary dilatation $H$. Thus $[\mu_r]$ is a Strebel point in $T$ and by Strebel's frame mapping theorem there exists $s_r = k_0([\mu_r]) > k_0$ and a unit vector $\varphi \in Q$ such that $\mu_r$ and $s_r \frac{|\varphi|}{\varphi}$ are equivalent in $T$.

Therefore, by Reich–Strebel's second fundamental inequality (see the inequality (3) in Section C and also [Ga]),

$$\frac{1+s_0}{1-s_0} \leqslant \frac{1+s_r}{1-s_r} \leqslant \int_\Delta |\varphi| \frac{|1+\mu_r \frac{\varphi}{|\varphi|}|^2}{1-|\mu_r|^2}.$$

A simple calculation as in [BLMM] gives (1).

Now, we are going to prove that condition (a) implies the condition (b) in Theorem F1.

*Proof.* Let $\mu$ be uniquely extremal on its extremal set $E$ in its Teichmüller class and $k_0 = k_0([\mu])$. Suppose that $\mu$ does not satisfy the condition (b). Hence, there exists $\nu$, distinct from $\mu$ on $E$, such that $\mu$ and $\nu$ belong to the same class in $B$ and $\|\nu\|_\infty \leqslant k_0$. Therefore, there exist $\epsilon \in (0, k_0)$ and a compact set $K \subset E$ of positive measure such that $|\mu(z) - \nu(z)| \geqslant 2\epsilon$, a.e. on $K$. Since

$$\delta_\mu(\varphi) = k_0\|\varphi\| - \operatorname{Re}\int_\Delta \frac{\mu+\nu}{2}\varphi,\ \varphi \in Q \quad \text{and} \quad \Big|\frac{\mu+\nu}{2}\Big| \leqslant d,\ \text{a.e. on } K,$$

where $d = \sqrt{k_0^2 - \epsilon^2}$, we conclude that $(k_0 - d)\|\varphi\|_K \leqslant \delta_\mu(\varphi)$. Here we use the notation $\|\varphi\|_K = \iint_K |\varphi|\, dx\, dy$. Hence, using Lemma VT1 and Lemma VT2 one can get a contradiction.

**Lemma V1.** *If $\mu$ satisfies unique extension property on its extremal set $E$, then $\mu$ satisfies variational property on $E$ in $B$.*

*Proof.* Contrary, suppose that $\|\mu_r\|_* \leqslant k = \|\mu\|_\infty$ for some $r > 0$ and some compact set $K \subset E$, where $\mu_r = \mu$ in $K^c$ and $\mu_r = (1+r)\mu$ in $K$. Then there is a non-zero annihilator $\eta \in N$ such that $\|\mu_r + \eta\|_\infty \leqslant k$.

Let $\mu_1 = \mu + \epsilon\eta$, where $\epsilon = (1+r)^{-1}$. By using similarity of the triangles, one can check that $\|\mu_1\| \leqslant k$. Since $\mu_1 \in [\mu]$, we conclude that $\mu_1 = \mu$, i.e., $\eta = 0$. Thus we have a contradiction.

**Lemma V2.** *Let $\mu$ satisfy the variational property on $E$ in $B$ and let $K \subset E$ be a compact set of positive measure on which $|\mu| = \|\mu\|_\infty = k$. Then for each $r > 0$ there is a unit vector $\varphi \in Q$ such that*

$$\delta_\mu(\varphi) \leqslant kr \int_K |\varphi|.$$

*Proof.* Let $\mu_r = \mu$ in $K^c$, $\mu_r = (1+r)\mu$ in $K$ and let $\lambda_r$ be a linear functional defined by $\lambda_r(\varphi) = \mathrm{Re}(\mu_r, \varphi)$, $\varphi \in Q$. Since $\mu$ satisfies variational property in $B$, there exists a unit vector $\varphi \in Q$ such that $\lambda_r(\varphi) \geqslant k$. Therefore,

$$\delta_\mu(\varphi) = k - \lambda(\varphi) \leqslant \lambda_r(\varphi) - \lambda(\varphi).$$

Since

$$\lambda_r(\varphi) - \lambda(\varphi) = \mathrm{Re}(\mu_r - \mu, \varphi) = \mathrm{Re} \int_K r\mu,$$

one can find that

$$\delta_\mu(\varphi) \leqslant |\lambda_r(\varphi) - \lambda(\varphi)| \leqslant rk \int_K |\varphi|.$$

Now, we can complete the proof of Theorem F1.

One can easily verify, by using Lemma V1, Lemma V2 and the Delta inequality, that condition (b) implies condition (a).

**G. Uniquely extremal differentials of Teichmüller type.** In this section we will give some applications of the Characterization Theorem using Reich sequences. Using new tools available in infinitesimal cotangent space $Q$ (such as compactness of certain families of holomorphic functions and mean value theorem) we can prove some properties of uniquely extremal dilatation of Teichmüller type.

Let $X$ be a linear space with norm and let $M$ be a subspace of $X$ and let $\lambda$ be a real bounded linear functional on $M$. We define the functional $\delta = \delta_\lambda$ on $M$ by $\delta(\varphi) = \|\lambda\| \, \|\varphi\| - \lambda(\varphi)$, $\varphi \in M$. We say that a sequence $\{\varphi_n\}$ from $M$ is weak Hamilton sequence for $\lambda$ if $\delta_n = \delta(\varphi_n)$ converges to zero.

Suppose now that $X = L^1(\Delta)$ and $M = Q$. Recall that we denote by $Q = Q(\Delta)$ the space of $L^1$-integrable analytic functions on $\Delta$, and that $\chi \in L^\infty(\Delta)$ is uniquely extremal in infinitesimal class (abbreviated by $\chi \in HBU$) if the linear function $\Lambda_\chi \in Q^*$ induced by $\chi$, $\Lambda_\chi(\varphi) = (\chi, \varphi)$, has a unique norm-preserving extension from $Q$ to a bounded linear functional on $L^1(\Delta)$.

We say that $\chi \in L^\infty(\Delta)$ satisfies Reich condition on a set $S \subset \Delta$ (or, we say that $\varphi_n$ is Reich sequence for $\chi$ on $S$) if

(1) there is a weak Hamilton sequence $\varphi_n$ for $\lambda_\chi$
(2) $\liminf |\varphi_n(z)| > 0$ a.e. in $S$.

If $\chi$ satisfies Reich condition on $\Delta$ we will simply say that $\chi$ satisfies Reich condition and, also, that $\varphi_n$ is Reich sequence. Now, we can state an immediate corollary of Characterization Theorem.

**Corollary G1.** *If $\chi$ is uniquely extremal, then $\chi$ satisfies Reich condition on its extremal set.*

**Corollary G2.** *If $|\chi|$ is constant, then $\chi$ is uniquely extremal if and only if $\chi$ satisfies Reich condition.*

**Example G1.** If $f(z) = Kx + iy$, $K > 1$, is the affine stretch, defined on a plane domain $D$, then the Beltrami coefficient $\mu$ of $f$ has the form

$$\mu(z) = k\frac{|\varphi_0|}{\varphi_0}, \quad \text{with } \varphi_0 = 1 \text{ and } k = \frac{K-1}{K+1}.$$

In [Re5], Reich has shown that if there exist a sequence $\varphi_n$ in $Q(D)$ such that:
(a) $\varphi_n(z) \to \varphi_0(z)$ for all $z \in D$ and (b) $\delta(\varphi_n) = \|\varphi_n\| - \operatorname{Re}\int_D \varphi_n \to 0$,
then $f(z)$ is uniquely extremal.

Also, in [Re5], Reich showed that a sequence $\varphi(z) = e^{-z/n}$ in $Q(D)$ satisfies conditions (a) and (b) for $D = D_\alpha = \{z : y > |x|^\alpha\}$, with $\alpha > 3$, and asked interesting question.

**Question G.** Whether the conditions (a) and (b) are not only sufficient but also necessary for unique extremality.

Theorem G2 and Corollary G3 (see below) provide an affirmative answer to this question in more general situation.

Concerning the Reich sequences, the next example is interesting.

**Example G2.** Let $\psi(z) = (1-z)^{-2}$, $\chi = k\dfrac{|\psi|}{\psi}$, $0 < k < 1$. Then $\chi$ is uniquely extremal on $\Delta$.

It is interesting to note that if $t_n \to 1-$ and $\psi_n(z) = (1-t_n z)^{-2}$, then $\delta(\psi_n) \to \pi(1-\ln 2)$, when $n \to \infty$. Thus, $\psi_n$ is not Reich sequence.

Recall that we say that $\mu$ is Teichmüller differential in general sense on $\Delta$ if $\mu = k|\psi|/\psi$, where $\psi$ is an analytic function on $\Delta$, which is not identically zero. We say that Reich sequence $\psi_n$ for $\mu = k|\psi|/\psi$ is normalized at a point $z_0 \in G$ if $\psi_n(z_0) \to \psi(z_0) \neq 0$.

The outline of the proof of the next lemma shows how one can use the presence of analytic function in definition of Teichmüller differential to show that each normalized Reich sequence forms a normal family.

**Lemma G1.** *Let $\chi$ be a Teichmüller differential in general sense defined by $\chi = k|\psi|/\psi$, where $k$ is number in $[0, 1)$ and $\psi$ is an analytic function on $\Delta$ and let $\psi_n$ be normalized Reich sequence for $\chi$ at a point $z_0$ and let $D = D(z_0, r)$ be the disc such that $\psi$ has no zeros on $\overline{D}$. Then $\psi_n$ converges uniformly to $\psi$ on $\overline{D}$.*

*Proof.* There exist disc $D_1$ with center at $z_0$ and of radius $r_1 > r$, such that $\overline{D} \subset D_1$ and that $\psi$ has no zeros on $D_1$, and positive numbers $m$ and $M$ such that $m \leqslant |\psi(z)| \leqslant M$ for each $z \in D_1$. Let $\varphi_n = \psi_n/\psi$ and

$$\delta_n = \delta_n(D_1) = k\int_{D_1} |\psi_n| - \operatorname{Re}\int_{D_1} k\frac{\psi_n|\psi|}{\psi}. \tag{1}$$

Hence

$$\delta_n \geqslant km\int_{D_1} (|\varphi_n| - \operatorname{Re}\varphi_n).$$

This inequality, with the mean value theorem, shows that $\varphi_n$, and therefore $\psi_n$, form a normal family on $D_1$. Therefore, there is a subsequence $\psi_{n_k}$ which converges uniformly to $\psi_0$ on $\overline{D}$. By letting $n$ to infinity in (1) (by $D$ instead of $D_1$) and using normalization that $\psi_0 = \psi$ at $z_0$, we conclude that $\psi_0 = \psi$ on $D$. This actually shows that $\psi_n$ converges uniformly to $\psi$ on $D$.

The proof of the next result is based on Lemma G1.

**Theorem G2.** *Let $\chi$ be uniquey extremal on $\Delta$ and let $\chi$ be Teichmüller differential (in general sense) defined by an analytic function $\varphi$. Then every normalized Reich sequence $\varphi_n$ converges uniformly on compact subsets of $\Delta$ to $\varphi$.*

**Corollary G3.** [BLMM] *Let $\chi$ be Teichmüller dilatation in general sense defined by some analytic function $\varphi$ in $\Delta$. Then $\chi$ is uniquely extremal if and only if there exists Reich sequence $\varphi_n$ in $Q$, which uniformly converges on compact subsets of $\Delta$.*

Further developments of the ideas outlined in the proof of Lemma G1 leads to the following results.

**Theorem G3.** (The first removable singularity Theorem) *Let $K$ be a compact subset of $\Delta$, $G = \Delta \setminus K$ and $\varphi$ an analytic function on $G$. Suppose that*

(a) *$\mu$ is an extremal dilatation on $\Delta$*

(b) *$\mu = s|\varphi|/\varphi$ on $G$,*

*where $s$ is non-negative measurable function on $G$. If there exist two positive constants $m$ and $M$, such that $m \leqslant |\varphi(z) \leqslant M$, for all $z \in G$, then*

(a) *$\varphi$ has an analytic extension $\tilde{\varphi}$ from $G$ to $\Delta$*

(b) *$\mu = k|\tilde{\varphi}|/\tilde{\varphi}$ a.e. in $\Delta$.*

**Theorem G4.** (The second removable singularity Theorem) *Let $\chi$ be uniquely extremal on $\Delta$ and let $\chi$ be multiple of nonnegative measurable function and Teichmüller differential defined by analytic function $\varphi$ on the complement of a compact set $K \subset \Delta$. Then*

(a) *$\varphi$ has an analytic extension $\tilde{\varphi}$ from $G$ to $\Delta$*

(b) *$\mu = k|\tilde{\varphi}|/\tilde{\varphi}$ a.e. in $\Delta$.*

During author's work with Božin, Lakić and Marković, on the subject concerning unique extremality, we wrote several drafts, in which the proofs of some versions of Theorem G3 and Theorem G4 have been given.

**H. Unique extremality and approximation sequences.** In this section we briefly discuss some results from [BLMM] and only announce new results (see below Lemma H3, Proposition H3–H4 and Theorem H1–H2).

Let $M$ be a subspace of a normed linear space $X$ and let $\Lambda$ be a linear functional on $M$ and $\lambda$ be its real part. Put

$$\overline{\lambda}(x_0) = \inf\{\lambda(y) + \|\lambda\|\|y - x_0\| : y \in M\}$$
$$\underline{\lambda}(x_0) = \sup\{\lambda(x) - \|\lambda\|\|x - x_0\| : x \in M\}.$$

Analysis of the proof of the Hahn-Banach Theorem leads to the following.

**Proposition H1.** *Linear functional $\Lambda$ has a unique Hahn-Banach norm-preserving extension from $M$ to $X$ if and only if $\underline{\lambda}(x_0) = \overline{\lambda}(x_0)$, for each $x_0 \in X \setminus M$.*

The details of the proof are left to the reader.

**Lemma H1.** *Let $\lambda$ have unique Hahn-Banach norm-preserving extension from $M$ to $X$. Then for each $x_0 \in X \setminus M$ there exist sequences $u_n$, $v_n \in M$ such that*
(1) $\lambda(u_n) = \lambda(x_0) + \|\lambda\|\|u_n - x_0\| + o(1)$
(2) $\lambda(v_n) = \lambda(x_0) - \|\lambda\|\|v_n - x_0\| + o(1)$
(3) $\lambda(w_n) = \|\lambda\|(\|x_0 - u_n\| + \|x_0 - v_n\|) + o(1)$, *where* $w_n = u_n - v_n$
(4) $w_n$ *is weak Hamilton sequence for* $\lambda$.

We say that $\lambda$ satisfies unique approximation property at $x_0 \in X \setminus M$ if there exist sequences $\{u_n\}$ and $\{v_n\}$, in $M$, such that the condition (3) is satisfied.
The following result follows from Lemma 1 and Proposition 1.

**Proposition H2.** *Let $\Lambda$ be a bounded linear functional on $M$ and $\lambda$ be its real part. Then $\Lambda$ has unique Hahn-Banach norm-preserving extension from $M$ to $X$ if and only if $\lambda$ satisfies unique approximation property at each $x_0 \in X \setminus M$.*

We say that $\psi_0 \in L^1(\Delta)$ is an extremal vector for linear function $\lambda = \lambda_\chi$, if $\lambda(\psi_0) = k\|\psi_0\|$, where $k = \|\chi\|_\infty$.
For a given $\chi \in L^\infty(\Delta)$, it is convenient to mark the extremal vector $\tilde{\chi}$ defined by $\tilde{\chi} = \overline{\chi}$, on $E$ and $\tilde{\chi} = 0$, on $\Delta \setminus E$.
The further discussion will show that $\tilde{\chi}$ has an important role in characterizations of uniquely extremal dilatation $\chi$.

**Lemma H2.** *If $\chi \in HBU$, then $\chi$ satisfies Reich condition on its extremal set $E$.*

*Proof.* Applying the part (1) of Lemma to the function $\psi$, defined by $\psi = \tilde{\chi}$, we find that there exist a sequence $u_n \in A$ such that

$$\lambda(\psi) + \|\psi - u_n\| + o(1) = \lambda(u_n), \text{ where } \lambda \text{ denotes } \lambda_\chi. \tag{5}$$

Since $\lambda(\psi) = \|\psi\|$ and $\lambda(u_n) \leqslant \|u_n\| \leqslant \|\psi\| + \|u_n - \psi\|$, we obtain, using (5), that

$$\|\psi\| + \|u_n - \psi\| + o(1) = \|u_n\|. \tag{6}$$

Hence, $\lambda(u_n) = \|u_n\| + o(1)$. Thus $u_n$ is weak Hamilton sequence for $\lambda$.
Using (6) we conclude that $\|\alpha_n\| \to 0$, $n \to \infty$, where $\alpha_n = |\psi| + |\psi - u_n| - |u_n|$. Hence, there exist subsequence $\alpha_{n_k}$ which converges to zero a.e. on $\Delta$. Since, $|u_n| \geqslant |\psi| - \alpha_n$, it follows that $\lim_{k\to+\infty} \inf |u_{n_k}(z)| \geqslant |\psi(z)|$ a.e. on $\Delta$.

Recall that $E = \{z \in \Delta : |\chi(z)| = k\}$.

**Lemma H3.** *Let $\psi_0$ be an extremal vector for $\lambda_\chi$ and $A = \{z \in \Delta : \psi_0(z) \neq 0\}$. Then*

$$\chi(z) = k\frac{|\psi_0(z)|}{\psi_0(z)}, \; z \in A \cap E.$$

**Proposition H3.** *If $\lambda(\tilde{\chi}) = \underline{\lambda}(\tilde{\chi})$, then $\chi$ is uniquely extremal on its extremal set $E$.*

Let $Q_{\tilde{\chi}}$ be the smallest subset of $L^1$ which contains $Q \cup \{\tilde{\chi}\}$. For a given $\mu \in L^\infty$, it is convenient to consider $\lambda = \lambda_\mu$ as a linear functional on $L^1$, defined by $\lambda_\mu(f) = \mathrm{Re}(\mu, f),\ f \in L^1$.

**Proposition H4.** *If $\mu$ is a extremal dilatation on $\Delta$ and $\psi_0 \in L^1$ an extremal vector, then $\overline{\lambda}(\psi_0) = \lambda(\psi_0)$, where $\lambda = \lambda_\mu$.*

**Theorem H1.** (The Characterization Theorem II) *Let $\chi \in L^\infty$. The following conditions are equivalent*

(7) *$\chi$ is uniquely extremal on its extremal set*

(8) *$\lambda_\chi$ has a unique norm-preserving extension from $Q$ to $Q_{\tilde{\chi}}$*

(9) *There exists sequence $\{\varphi_n\}$, with $\varphi_n \in Q$ for all $n$, such that*

$$\lambda(\varphi_n) = \lambda(\tilde{\chi}) + \|\chi\|_* \|\varphi_n - \tilde{\chi}\| + o(1)$$

(10) *$\chi$ satisfies Reich condition on its extremal set $E$.*

**Theorem H2.** *Let $\chi \in L^\infty(\Delta)$ and $k = \|\chi\|_\infty$. If there exist Reich sequence $\varphi_n$ for $\chi$, on its extremal set $E$, then*

(11) *$k|\varphi_n|/\varphi_n$ converges to $\chi$ a.e. on its extremal set $E$*

(12) *If two-dimensional Lebesgue measure of $E$ is positive, then*

$$\lim_{n\to\infty} \int_E \chi \frac{|\varphi_n|}{\varphi_n} = k.$$

**I. Construction.** Before we state next results we introduce a class of sets which is important in our investigation. Let $K$ be a compact subset of $\mathbb{C}$ whose complement is connected in $\mathbb{C}$ and interior is empty. We say that $K$ is a special Mergelyan's set (a special $M$-set). Motivation for this definition is famous Mergelyan's Theorem.

**Mergelyan's Theorem.** *If $K$ is a compact set in the plane whose complement is connected, if $f$ is a continuous complex function on $K$ which is holomorphic in the interior of $K$, and if $\epsilon > 0$, then there exist a polynomial $P$ such that $|f(z) - P(z)| < \epsilon$ for all $z \in K$.*

Note that $K$ need not be connected.

Recall that we denote by $Q = Q(\Delta)$ the space of $L^1$-integrable analytic functions on $\Delta$, and that $\chi \in L^\infty(\Delta)$ is uniquely extremal in infinitesimal class (abbreviated by $\chi \in HBU$) if the linear function $\Lambda_\chi \in Q^*$ induced by $\chi$, $\Lambda_\chi(\varphi) = (\chi, \varphi)$, has a unique norm-preserving extension from $Q$ to a bounded linear functional on $L^1(\Delta)$.

If $\chi \in HBU$ and $k = ||\chi||_\infty$, then $\operatorname{ess\,sup} |\chi(z)| = k$ over each open set $G \subset \Delta$, as has been observed by Reich [Re7]. Therefore, the following question (see [Re7] and [S6]) is natural.

**Question I.** Does $\chi \in HBU$ actually imply that $|\chi(z)| = k$ a.e.?

The next theorem shows that the answer to corresponding question, concerning the more general concept of unique extremal dilation, is negative.

**Theorem I1.** *Let $K \subset \Delta$ be a compact set of positive measure, whose complement is connected and let $\Omega = \Delta \setminus K$. Then there exist $\chi \in L^\infty(\Delta)$ such that $\chi$ is zero on $K$, $\chi$ is uniquely extremal on $\Omega$ and $|\chi(z)| = k > 0$ a.e. on $\Omega$.*

*Outline of proof.* Inductively, we can find a sequence of polynomials $\{P_n\}$ and increasing sequence of compact special $M$-sets $K_n$ such that

(1) $K_n$ and $K$ are disjunct
(2) complement of $F_n = K \cup K_n$ is connected
(3) $|\bigcup_{n=1}^{\infty} K_n| = |\Omega|$
(4) $|P_n(z)| < 1/2$ for each $z \in K$ and $|P_n(z)| > 2$ for each $z \in K_n$
(5) $\alpha_n |1 - |P_{n+1}(z)|\,/P_{n+1}(z)| \leqslant 2^{-n}$, for each $z \in K_{n+1}$, where $\varphi_n = P_1 P_2 \dots P_n$, $\alpha_n = \max\{|\varphi_n(z)| : z \in K_{n+1}\}$
(6) $\int_\Delta |\varphi_n| - \int_{F_{n+1}} |\varphi_n| \leqslant 1/n$, where $F_n = K \cup K_n$.

Let $\chi_n = k|\varphi_n|/\varphi_n$, $0 < k < 1$ and define $\chi$ to be zero on $K$. We leave to the reader to show that $\chi_n(z)$ is a Cauchy sequence a.e. on $\Omega$, that is $\chi(z) = \lim \chi_n(z)$ exists a.e. on $\Omega$; and that $\varphi_n$ is Reich sequence on $\Omega$ for $\chi$.

It is interesting that the following surprisingly simple lemma plays a role in construction unique extremal dilatation with nonconstant modulus.

**Lemma Re.** *If $K$ is a special Mergelyan's set and $\nu$ annihilator of $Q$ in $L^\infty$ such that* $\operatorname{supp}\nu \subset K$, *then $\nu = 0$.*

This lemma was proved by Reich in [Re7] (see also [MM1] and [BLMM]). The following result is an immediate corollary of Theorem I1 and Lemma Re.

**Theorem I2** [BLMM]. *Let $K \subset \Delta$ be a special Mergelyan's set of positive measure. Then there exist $\chi \in HBU$ such that $\chi(z) = 0$ in $K$ and $|\chi(z)| = k > 0$ a.e. in $K^c = \Delta \setminus K$.*

We refer to the proof of this result as the construction of uniquely extremal dilatation with nonconstant modulus (shortly the construction).

After writing the final version of this paper, Reich [Re9] has modified the proof of Theorem I2, given in [BLMM], using Runge theorem instead of Mergelyan's theorem.

Further simplifications of the construction has been given by author during his lectures at Scoala Normala Superioara Buchurest (SNSB), 2003–2004 (to appear in [M10]).

**Outline of new construction**. Recall, if $K \subset \mathbb{C}$ is a compact set and do not separate the plane, we say that $K$ is $M$-set ("Mergelyan set"); we call $K$ a special $M$-set if in addition $K$ has empty interior and positive 2-dimensional measure.

Using Runge, we can prove the following result.

**Lemma Ma.** *Let $K$ be a $M$-set and $G$ be a Jordan domain such that $\overline{G} \cap K = \emptyset$. For given positive numbers $p$, $q$ and $\varepsilon$, there exists polynomial $Q$ such that*

$(a_1)$ $|Q| < p$ *on* $K$; $(a_2)$ $q < |Q|$ *on* $\overline{G}$; $(a_3)$ $|1 - |Q|/Q| < \varepsilon$ *on* $F = K \cup \overline{G}$

*Remark*: There is an entire function which satisfies the above conditions and in addition has no zeros in $\mathbb{C}$. For suitable $Q$ close to constant functions on $K$ and $\overline{G}$ the entire function $\phi = e^{Q-1}$ satisfies the above conditions.

Let $K$ be a $M$-set and $D$ be a Jordan domain such that $K \subset D$. Then there exists sequence of Jordan-domains $J_n$ such that

$$(1) \qquad J_n \subset \operatorname{Int} J_{n+1}, \quad \bigcup_1^\infty \overline{J_k} = D \setminus K.$$

We say that sequence of Jordan-domains $J_n$ exhaust $D \setminus K$. Inductively, we will find a sequence of polynomials $P_n$ and an increasing sequence of Jordan-domains $G_n$ which exhaust $D \setminus K$ such that:

($b_1$) $|P_n| < 1/2$ on $K$ and $|P_n| > 2$ on $\overline{G_n}$

($b_2$) $\int_D |\varphi_n| - \int_{F_{n+1}} |\varphi_n| < 1/n$, where $\varphi_n = P_1 \cdots P_n$ and $F_n = K \cup \overline{G_n}$.

($b_3$) $\alpha_n |1 - |P_{n+1}|/P_{n+1}| < 2^{-n-1}$, where $\alpha_n = \max\{|\varphi_n(z)| : z \in F_{n+1}\}$.

We call $L_n = D \setminus F_n$ canal. Roughly speaking, by ($b_1$) and ($b_2$), we control polynomial $\varphi_n$ respectively on $F_n$ and $L_{n+1}$, but we do not control on canal $L_n$ (more precisely on $L_n \setminus L_{n+1}$). Thus, we do not have any estimates of growth of $\alpha_n$ from above. At this point, it seems that it is difficult to overcome this problem. However, we can overcome this problem using ($b_3$), which has a crucial role. More precisely, applications of Lemma Ma ($a_3$) shows that there exists polynomial $P_{n+1}$ such that estimate in ($b_3$) holds. Now, we can modify $\varphi_n$ on $F_{n+1}$ by means of polynomial $P_{n+1}$; i.e., we construct the function $\varphi_{n+1} = P_{n+1}\varphi_n$.

The function $|\varphi_n|/\varphi_n$ is defined except on the set $Z_n$ of zeros of polynomial $\varphi_n$. Let $Z = \bigcup_1^\infty Z_n$. If we define $\mu_n$ to be 1 on $Z$ for every $n \geqslant 1$ and $\mu_n = |\varphi_n|/\varphi_n$ on $\mathbb{C} \setminus Z$. Since $\varphi_{n+1} = P_{n+1}\varphi_n$, by $b_3$) we have $\alpha_n|\mu_{n+1}(z) - \mu_n(z)| < 2^{-n-1}$ for $z \in F_{n+1}$.

Since $\alpha_n$ is obviously increasing sequence, then a standard argument shows that $\mu_n(z)$ is a Cauchy sequence on $D$; that is $\mu(z) = \lim \mu_n(z)$ exists on $D$ and that $\alpha_n|\mu(z) - \mu_n(z)| < 2^{-n}$ for $z \in F_{n+1}$. Hence, $\|\varphi_n\| = \Lambda_\mu[\varphi_n] + 0(1)$ on $D$. Since $|\varphi_n(z)| \to \infty$ on $D \setminus (K \cup Z)$, $\varphi_n$ satisfies Re-condition on $D \setminus K$ and therefore $\mu$ is uniquely extremal on $D \setminus K$. Hence, we get

**Proposition.** *If $K$ is a special $M$-set and $\chi$ measurable function, which is equal $\mu$ on $D \setminus K$ and $\|\chi\|_\infty \leqslant 1$, then $\chi$ is uniquely extremal on $D$.*

Note that Theorem I2 can be considered as a corollary of this result.

**J. Beltrami equation.** Suppose that $f$ has $L^1$ derivatives in the complex plane $\mathbb{C}$ and that $f(z) \to 0$ as $z \to \infty$. With the notation

$$T\omega = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\omega(\zeta)}{\zeta - z} d\xi \, d\eta$$

we then obtain from Green's formula

$$(1) \qquad f = T\overline{\partial} f.$$

For smooth $\omega$ with compact support we define the Hilbert transform $H\omega$ of $\omega$ by $H\omega = \partial(T\omega)$. By differentiation we obtain an expression for $H$ as principle value

$$(H\omega)(z) = \lim_{\epsilon \to 0_+} -\frac{1}{\pi} \iint_{A_\epsilon} \frac{\omega(\zeta)}{(\zeta - z)^2} d\xi \, d\eta,$$

where $A_\epsilon = \{\zeta : \epsilon < |\zeta| < 1/\epsilon\}$.

Fix $0 < k < 1$, and let $L^\infty(k,R)$ denote the measurable functions on $\mathbb{C}$, bounded by $k$, and supported in the disc $B_R$. We let $QC^1(k,R)$ denote the continuous differentiable homeomorphisms $f$ of $\mathbb{C}$ such that $\overline{\partial} f = \mu \partial f$, for some $\mu \in L^\infty(k,R)$, normalized so that $f(z) = z + O(1/z)$, as $z \mapsto \infty$.

Let $f \in QC^1(k,R)$. Then by (1),

$$f(z) - z = T(\overline{\partial} f)(z).$$

Thus, if we set $g = \partial f - 1$ and use $\overline{\partial} f = \mu \partial f$, we obtain

$$g = H(\overline{\partial} f) = H(\mu \partial f) = H(\mu g) + H(\mu).$$

In terms of the operator $H_\mu(g) = H(\mu g)$, $g \in L^p$, we obtain the equation

$$(I - H_\mu)g = H(\mu). \tag{2}$$

If we fix $p = p(k) > 2$ so that $\|H_\mu\| < 1$, then $I - H_\mu$ is invertible. Thus, we can solve the equation (2) for $g$ to obtain

$$g = (I - H_\mu)^{-1} H(\mu) \in L^p. \tag{3}$$

**Theorem 1.** *Fix $0 < k < 1$, $R > 0$ and $p = p(k) > 2$ as above. For $\mu \in L^\infty(k,R)$, there is a function $f$ on $\mathbb{C}$, normalized so that $f(z) = z + O(\frac{1}{z})$ at $\infty$, with distribution derivatives satisfying the Beltrami equation $\overline{\partial} f = \mu \partial f$.*

*Outline of the proof*: Define $g$ by (3) and define $f(z) = z + T(\mu g + \mu)$. Since $T$ is the convolution operator with kernel $1/z$ locally in $L^1$, $f$ is continuous. Moreover, $f$ is normalized at $\infty$, and

$$\overline{\partial} f = \mu g + \mu, \quad \partial f = 1 + H(\mu g + \mu) = 1 + g$$

in the sense of distributions, so $f$ satisfies the Beltrami equation.

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