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AN INTRODUCTION INTO THE THEORY OF SEQUENCE SPACES AND MEASURES OF NONCOMPACTNESS

CONTENTS

Preface

This paper gives a self–contained, comprehensive treatment of the theories of sequence spaces and measures of noncompactness, as well as a survey of some of the authors' recent research results in these fields. It contains subjects of lectures at the universities of Niš, Novi Sad and Belgrade, Giessen (Germany) and Irbid (Jordan), and talks given by the authors at various international conferences in the Czech Republic, Germany, Hungary, India, Italy, Jordan, Poland and Yugoslavia.

For the first time, methods from the fields of summability, in particular of sequence spaces and matrix transformations on one hand, and of measures of noncompactness on the other are successfully linked on a large scale to obtain necessary and sufficient conditions for matrix maps between certain sequence spaces of a general class to be compact operators. The original idea for research in this field dates back to the classical paper of L. W. Cohen and N. Dunford [10]. In this paper they gave necessary and sufficient conditions for matrix transformations from l_1 to l_p, l_p to c_0 and l_p to l_1, and found the norm of these transformations. Furthermore they established necessary and sufficient conditions for these operators to be compact. Although the concept of measure of noncompactness is not explicitly mentioned in their paper, their studies and techniques are very closely related to our research.

These notes are addressed to both experts and nonexperts with an interest in getting acquainted with sequence spaces and measures of noncompactness. They could also be used as a guideline for research and teaching at graduate and post graduate levels.

Sections 1 and 2 deal with the necessary basic concepts and results of the theory of FK spaces, their duals, matrix transformations and measures of noncompactness. Although most of the results presented are well known and can be found, for instance, in [105, 108, 107, 91] concerning Section 1, and in [1, 7, 86] concerning Section 2, proofs are given in almost all cases to make the paper self–contained.

In Section 3, the authors give their own research results and apply the methods and results of the first two chapters to characterize matrix transformations between sequence spaces closely related to various concepts of summability, such as ordinary and strong summability, spaces of difference sequences of higher order and of strongly convergent and bounded sequences. Finally they apply the Hausdorff measure of noncompactness to give necessary and sufficient conditions for a matrix map between these spaces to be a compact operator.

Although there is a very wide range of problems for further research related to the presented topics, only the most closely related and possibly interesting will be mentioned here. We hope that the results presented here will be a useful introduction to further studies in the following fields. Concerning measures of noncompactness it seems most interesting to study when operators between sequence spaces are strictly singular [45, 104], Fredholm or semi–Fredholm [16, 17, 19, 21]. Results in this direction could also be applied in the perturbation theory of Fredholm and semi–Fredholm operators. The authors' research is also connected with A. Wilansky's results [106], and will be in this direction. Concerning the theory of sequence spaces and matrix transformations it would be interesting to extend the author's results to mixed normed spaces [22, 23, 35, 36, 38]. Furthermore, a generalization should be considered of the spaces l_p, w_0^p to $l(p)$, $w_0(p)$ by replacing the constant p in the exponent by a sequence $p = (p_k)_{k=0}^{\infty}$ of positive reals [95, 43, 44, 49, 50, 52, 53, 55, 56, 48, 90, 61, 57, 58, 60, 62].

Finally, matrix domains of general infinite matrices should be considered instead of matrix domains of triangles.

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1. FK spaces

In this section, we shall give a short introduction into the general theory of *FK spaces* and apply the results to characterize matrix transformations between the classical sequence spaces. For further studies the reader is referred to [108, 91, 37, 114, 54, 111, 112, 113]. Results closely related to our studies can be found in [33]. For a comprehensive survey on results on matrix transformations we recommend [97].

1.1. Linear metric and paranormed spaces. In this subsection, we shall introduce the concepts of *linear metric* and *paranormed spaces*. They will play an important role in our studies of sequence spaces.

The concept of a linear space involves an algebraic structure given by the definition of two operations, namely the sum of any two of its vectors and the product of any scalar with any vector. On the other hand a topological structure of a set may be given by a metric. If a set is both a linear and metric space, then it will be natural to require the algebraic operations to be continuous with respect to the metric.

Let X be a linear space and d a metric on X. Then (X, d), or X for short, is said to be a *linear metric space*, if the algebraic operations on X are continuous functions. A complete linear metric space is said to be a *Fréchet space* (cf. [105, Definition 5.3.2, p. 78]). (Unfortunately this terminology is not universally agreed on. Some authors call a complete linear metric space an F space and a locally convex F space a Fréchet space (see e.g. [92, p. 8], [40, p. 208] and [96, p. 8]), which Wilansky calls an F–space. We shall follow Wilansky's terminology in [105, 107, 108]. The continuity of the algebraic operations of a linear metric space (X, d) means the following: If (x_n) and (y_n) are two sequences in X and (λ_n) is a sequence of scalars with $x_n \to x$, $y_n \to y$ and $\lambda_n \to \lambda$ $(n \to \infty)$, then $x_n + y_n \to x + y$ and $\lambda_n x_n \to \lambda x$ $(n \to \infty)$. This means that $d(x_n, x)$, $d(y_n, y) \to 0$ and $\lambda_n \to \lambda$ $(n \to \infty)$ together imply $d(x_n + y_n, x + y) \to 0$ and $d(\lambda_n x_n, \lambda x) \to 0$ $(n \to \infty)$. By the *completeness of a metric space* we mean that every Cauchy sequence converges.

The concept of *paranorm* is closely related to linear metric spaces. It is a generalization of that of absolute value. The paranorm of a vector x may be thought of as the distance from x to the origin 0.

Definition 1.1. Let X be a linear space. A function $p : X \mapsto \mathbb{R}$ is called *paranorm*, if

(P.1) $p(0) = 0$
(P.2) $p(x) \geq 0$ for all $x \in X$
(P.3) $p(-x) = p(x)$ for all $x \in X$
(P.4) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$ (*triangle inequality*)

(P.5) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ $(n \to \infty)$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ $(n \to \infty)$, then $p(\lambda_n x_n - \lambda x) \to 0 \quad (n \to \infty)$ (*continuity of multiplication by scalars*).

If p is a paranorm on X, then (X, p), or X for short, is called a *paranormed space*. A paranorm p for which $p(x) = 0$ implies $x = 0$ is called *total*. For any two paranorms p and q, p is called *stronger* than q if, whenever (x_n) is a sequence such that $p(x_n) \to 0$ $(n \to \infty)$, then also $q(x_n) \to 0$ $(n \to \infty)$. If p is stronger than q, then q is said to be *weaker* than p. If p is stronger than q and q is stronger than p, then p and q are called *equivalent*. If p is stronger than q, but p and q are not equivalent, then p is said to be *strictly stronger* than q, and q is called *strictly weaker* than p.

It is easy to see that *every totally paranormed space is a linear metric space*. The converse is also true. *The metric of any linear metric space is given by some total paranorm* (cf. [105, Theorem 10.4.2, p. 183]). A sequence of paranorms may be used to define a paranorm.

Theorem 1.2. *Let $(p_k)_{k=1}^{\infty}$ be a sequence of paranorms on a linear space X. We define the so-called Fréchet combination of (p_k) by*

$$p(x) = \sum_{n=0}^{\infty} \frac{1}{2^k} \frac{p_k(x)}{1 + p_k(x)} \quad \text{for all } x \in X. \tag{1.1}$$

Then:

(a) *p is a paranorm on X and satisfies*

$$p(x_n) \to 0 \ (n \to \infty) \text{ if and only if } p_k(x_n) \to 0 \ (n \to \infty) \quad \text{for each } k; \tag{1.2}$$

(b) *p is the weakest paranorm which is stronger than every p_k;*

(c) *p is total if and only if every p_k is total.*

Proof. (a) Conditions (P.1), (P.2) and (P.3) in Definition 1.1 are obvious, since every p_k is a paranorm. To prove condition (P.4), we observe that, for all reals a and b with $0 \le a \le b$, we have $a(1 + b) = a + ab \le b + ab = b(1 + a)$ and so $a/(1 + a) \le b/(1 + b)$. Applying this with $0 \le a = p_k(x + y) \le p_k(a) + p_k(y) = b$, we conclude

$$\frac{p_k(x+y)}{1 + p_k(x+y)} \le \frac{p_k(x) + p_k(y)}{1 + p_k(x) + p_k(y)} \le \frac{p_k(x)}{1 + p_k(x)} + \frac{p_k(y)}{1 + p_k(y)} \quad \text{for all } k,$$

and from this $p(x + y) \le p(x) + p(y)$. To prove the statement in (1.2), we first assume $p_k(x_n) \to 0$ $(n \to \infty)$ for each k. Since

$$0 \le \frac{p_k(x_n)}{1 + p_k(x_n)} \le 1 \quad \text{for all } n, k$$

and $\sum_{k=1}^{\infty} 1/2^k$ converges, the series

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x_n)}{1+p_k(x_n)}$$

converges uniformly in n. Thus $\lim_{n\to\infty} p(x_n) = 0$.

Conversely, we assume $p(x_n) \to 0$, $(n \to \infty)$ and fix k. Then

$$\frac{1}{2^k} \frac{p_k(x_n)}{1+p_k(x_n)} \leq p(x_n)$$

implies $p_k(x_n) \leq 2^k p(x_n) + 2^k p_k(x_n) p(x_n)$. Since $p(x_n) \to 0$ $(n \to \infty)$, it follows that $2^k p(x_n) < 1$ for all sufficiently large n, hence $p_k(x_n)(1 - 2^k p(x_n)) \leq 2^k p(x_n)$ for all sufficiently large n, and consequently

$$p_k(x_n) \leq \frac{2^k p(x_n)}{1 - 2^k p(x_n)} \quad \text{for all sufficiently large } n.$$

This implies $p_k(x_n) \to 0$ $(n \to \infty)$.

To prove condition (P.5), let $\lambda_n \to \lambda$ and $p(x_n - x) \to 0$ $(n \to \infty)$. By the statement in (1.2), $p_k(x_n - x) \to 0$ $(n \to \infty)$ for all k, and, since every p_k is a paranorm, this implies $p_k(\lambda_n x_n - \lambda x) \to 0$ $(n \to \infty)$ for all k. Now it follows from the statement in (1.2) that $p(\lambda_n x_n - \lambda x) \to 0$ $(n \to \infty)$.

(b) Let q be a paranorm which is stronger than every p_k. Then $q(x_n) \to 0$ $(n \to \infty)$ implies $p_k(x_n) \to 0$ $(n \to \infty)$ for all k, and, by the statement in (1.2), $p(x_n) \to 0$ $(n \to \infty)$. Thus q is stronger than p.

(c) Part (c) is trivial. □

Let us recall that a subset S of a linear space X is said to be *absorbing* if for each $x \in X$ there is $\varepsilon > 0$ such that $\lambda x \in S$ for all scalars λ with $|\lambda| \leq \varepsilon$.

Remark 1.3. Let (X, p) be a paranormed space. Then the open neighbourhoods of 0, $N_r(0) = \{x \in X : p(x) < r\}$, are absorbing for all $r > 0$.

Proof. We assume that is $N_r(0)$ is not absorbing for some $r > 0$. Then there are $x \in X$ and a sequence $(\lambda_n)_{n=0}^{\infty}$ of scalars with $\lambda_n \to 0$ $(n \to \infty)$ and $\lambda_n x \notin N_r(0)$ for all $n = 0, 1, \ldots$. But this means $p(\lambda_n x) \geq r$ for all n contradicting condition (P.5) in Definition 1.1. □

Example 1.4. The set $\mathbb{C}$ of complex numbers with the usual algebraic operations and $p = |\cdot|$, the modulus, is a totally paranormed space. If we put $d(z, w) = |z - w|$ for all $z, w \in \mathbb{C}$, then $(\mathbb{C}, d)$ is a Fréchet space.

By ω, we denote the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$ which becomes a linear space with $x + y = (x_k + y_k)_{k=0}^{\infty}$ and $\lambda x = (\lambda x_k)_{k=0}^{\infty}$ or all $x, y \in \omega$ and $\lambda \in \mathbb{C}$. As an immediate consequence of Theorem 1.2 and Example 1.4, we obtain

Theorem 1.5. *The set ω is a Fréchet space with respect to the metric d defined by*

$$d(x,y) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{|x_k - y_k|}{1 + |x_k - y_k|} \quad \text{for all } x, y \in \omega. \tag{1.3}$$

Furthermore convergence in (ω, d) and coordinatewise convergence are equivalent, that is $x^{(n)} \to x$ $(n \to \infty)$ in (ω, d) if and only if $x_k^{(n)} \to x_k$ $(n \to \infty)$ for every k.

Now we introduce the concept of a *Schauder basis*. For further studies on bases we refer the reader to [46, 74].

Definition 1.6. A *Schauder basis* of a linear metric space X is a sequence (b_n) of vectors such that for each vector $x \in P$ there is a unique sequence (λ_n) of scalars with $\sum_{n=1}^{\infty} \lambda_n b_n = x$, that is $\lim_{m\to\infty} \sum_{n=1}^{m} \lambda_n b_n = x$.

For finite dimensional spaces, the concepts of Schauder and algebraic bases coincide. In most cases of interest, however, the concepts differ. Every linear space has an algebraic basis. But there are linear metric spaces without a Schauder basis, as we shall see later in this subsection.

Example 1.7. For each $n = 0, 1, \dots$, let $e^{(n)}$ be the sequence with $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$. Then $(e^{(n)})_{n=0}^{\infty}$ is a Schauder basis of ω. More precisely, every sequence $x = (x_k)_{k=0}^{\infty} \in \omega$ has a unique representation $x = \sum_{k=0}^{\infty} x_k e^{(k)}$ that is $\lim_{m\to\infty} x^{[m]} = x$ for $x^{[m]} = \sum_{k=0}^{m} x_k e^{(k)}$, the *$m$–section of* x.

A metric space (X, d) is called *separable* if it has a *countable dense set.* That means there is a countable set $A \subset X$ such that for all $\varepsilon > 0$ and for all $x \in X$ there is an element $a \in A$ with $d(x, a) < \varepsilon$.

Theorem 1.8. *Every complex linear metric space X with Schauder basis is separable.*

Proof. Let (b_n) be a Schauder basis of X. For each $m \in \mathbb{N}$, we put

$$A_m = \left\{ \sum_{n=1}^{m} \rho_n b_n : \rho_n \in \mathbb{Q} + i\mathbb{Q} \ (n = 1, 2, \dots, m) \right\} \text{ and } A = \bigcup_{m=1}^{\infty} A_m.$$

Then A is a countable set in X and it is easy to see that A is dense in X. □

Example 1.9. The set $l_\infty = \{x \in \omega : \sup_k |x_k| < \infty\}$ of all bounded sequences is a Banach space with $\|x\|_\infty = \sup_k |x_k|$ $(x \in l_\infty)$ which has no Schauder basis.

Proof. The proof that $(l_\infty, \|\cdot\|_\infty)$ is a Banach space is standard and left to the reader. To show that l_∞ has no Schauder basis, we show that l_∞ is not separable and apply Theorem 1.8. We assume that l_∞ is separable. Then there is a countable dense set $A = \{a_n : n = 0, 1, \dots\} \subset l_\infty$. For every n, let $U_n = N_{1/3}(a_n) = \{x \in l_\infty : \|x - a_n\|_\infty < 1/3\}$. Since $A \subset l_\infty$ is dense, $l_\infty \subset \bigcup_{n=0}^{\infty} U_n$. The set

$$B = \{0,1\}^{\mathbb{N}_0} = \big\{x \in \omega : x_k \in \{0,1\} \text{ for all } k = 0, 1, \dots\big\} \subset l_\infty$$

is uncountable. Therefore there must be a set U_m which contains at least two distinct sequences x and x' of B. Then

$$\|x - x'\|_\infty \geq 1 \quad \text{and} \quad \|x - x'\|_\infty \leq \|x - a_m\|_\infty + \|a_m - x'\|_\infty < 2/3,$$

a contradiction. Therefore l_∞ cannot be separable. □

At the end of this subsection we study the so–called *classical sequence spaces*

$$\begin{aligned} l_\infty &= \{x \in \omega : \sup_k |x_k| < \infty\}, \\ c &= \{x \in \omega : \lim_{k\to\infty} x_k = l \text{ for some } l \in \mathbb{C}\}, \\ c_0 &= \{x \in \omega : \lim_{k\to\infty} x_k = 0\} \end{aligned}$$

of all *bounded*, *convergent* and *null sequences*, and

$$l_p = \left\{x \in \omega : \sum_{k=0}^{\infty} |x_k|^p < \infty\right\} \text{ for } 1 \leq p < \infty.$$

The following result gives the algebraic and topological properties of the sets l_∞, c, c_0 and l_p.

Theorem 1.10. (a) *Each of the sets l_∞, c_0 and c is a Banach space with $\|\cdot\|_\infty$ defined by $\|x\|_\infty = \sup_k |x_k|$. Moreover $|x_k| \leq \|x\|_\infty$ for all $k = 0, 1, \dots$.*

(b) *The sets l_p are Banach spaces for $1 \leq p < \infty$ with $\|\cdot\|_p$ defined by $\|x\|_p = \left(\sum_{k=0}^\infty |x_k|^p\right)^{1/p}$. Moreover $|x_k| \leq \|x\|_p$ for all $k = 0, 1, \dots$.*

(c) *The sequence $(e^{(n)})_{n=0}^\infty$ is a Schauder basis for each of the spaces c_0 and l_p for $1 \leq p < \infty$. More precisely, every sequence $x = (x_n)_{n=0}^\infty$ in any of these spaces has a unique representation $x = \sum_{n=0}^\infty x_n e^{(n)}$.*

(d) *Let e be the sequence with $e_k = 1$ for all $k = 0, 1, \dots$. We put $b^{(0)} = e$ and $b^{(n)} = e^{(n-1)}$ for $n = 1, 2, \dots$. Then the sequence $(b^{(n)})_{n=0}^\infty$ is a Schauder basis for c. More precisely, every sequence $x = (x_n)_{n=0}^\infty \in c$ has a unique representation $x = le + \sum_{n=0}^\infty (x_n - l)e^{(n)}$ where $l = l(x) = \lim_{n\to\infty} x_n$.*

(e) *The space l_∞ has no Schauder basis.*

Proof. Part (e) is Example 1.9. Parts (a) to (d) are standard and therefore left to the reader. (The triangle inequality for $\|\cdot\|_p$ follows by Minkowski's inequality (see appendix A.4.2).) □

1.2. Introduction into the theory of FK spaces. In this subsection, we shall give an introduction into the general theory of *FK spaces*. It is the most powerful tool for the solution of problems of various kinds in summability, in particular in the *characterization of matrix transformations between sequence spaces*. Most of the results of this subsection can be found in [108].

We saw in Theorem 1.5 that the set ω is a Fréchet space with the metric d defined in (1.3) and that convergence in ω and coordinatewise convergence are equivalent. Furthermore, by Theorem 1.10, the spaces l_∞, c_0, c and l_p $(1 \leq p < \infty)$ are Banach spaces with the norms $\|\cdot\|_\infty$ and $\|\cdot\|_p$, and convergence in any one of these spaces implies coordinatewise convergence by the inequalities in Theorem 1.10 parts (a) and (b). Thus the metric generated by these norms is stronger than the metric of ω on them.

Definition 1.11. A Fréchet sequence space (X, d_X) is said to be an *FK space* if its metric d_X is stronger than the metric $d|_X$ of ω on X. A *BK space* is an FK space which is a Banach space.

Remark 1.12. (a) Some authors include *local convexity* in the definition of FK spaces. But much of the theory can be developed without local convexity.

(b) By definition, an FK space X is continuously embedded in ω, that is the *inclusion map* $\iota : (X, d_X) \mapsto (\omega, d)$ defined by $\iota(x) = x$ $(x \in X)$ is continuous. An FK space X is a Fréchet sequence space with continuous *coordinates* $P_k : X \mapsto \mathbb{C}$ defined by $P_k(x) = x_k$ $(k = 0, 1, \dots)$ for all $x \in X$.

Example 1.13. The space ω is an FK space with its natural metric d. The spaces l_∞, c, c_0 and l_p $(1 \leq p < \infty)$ are BK spaces with their natural norms.

Theorem 1.14. *Let (X, d_X) be a Fréchet space, (Y, d_Y) an FK space and $f : X \mapsto Y$ a linear map. Then $f : (X, d_X) \mapsto (Y, d|_Y)$ is continuous if and only if $f : (X, d_X) \mapsto (Y, d_Y)$ is continuous.*

Proof. First we assume that $f : (X, d_X) \mapsto (Y, d_Y)$ is continuous. Since Y is an FK space its metric d_Y is stronger than the metric $d|_Y$ of ω on Y. So $f : (X, d_X) \mapsto (Y, d|_Y)$ is continuous.

Conversely we assume that $f : (X, d_X) \mapsto (Y, d|_Y)$ is continuous. Since $(Y, d|_Y)$ is a Hausdorff space and f is continuous, the *graph of* f, $\text{graph}(f) = \{(x, f(x)) : x \in X\}$, is a closed set in $(X, d_X) \times (Y, d|_Y)$ by the *closed graph lemma* (see appendix A.4.4), hence a closed set in $(X, d_X) \times (Y, d_Y)$, since the FK metric d_Y is stronger than $d|_Y$. By the *closed graph theorem* (see appendix A.4.5), the map $f : (X, d_X) \mapsto (Y, d_Y)$ is continuous. □

Corollary 1.15. *Let X be a Fréchet space, Y an FK space, $f : X \mapsto Y$ a linear map and P_n the n-th co-ordinate, that is $P_n(y) = y_n$ $(y \in Y)$ for all $n = 0, 1, \dots$. If each map $P_n \circ f : X \mapsto \mathbb{C}$ is continuous, so is $f : X \mapsto Y$.*

Proof. Since $P_n \circ f : X \mapsto \mathbb{C}$ is continuous for each n, the map $f : X \mapsto \omega$ is continuous by the equivalence of coordinatewise convergence and convergence in ω. By Theorem 1.14, $f : X \mapsto Y$ is continuous. □

By ϕ we denote the set of all *finite sequences* that is of sequences that terminate in zeros.

We shall frequently make use of the following result.

Remark 1.16. Let $X \supset \phi$ be an FK space and $a \in \omega$. If the series $\sum_{k=0}^\infty a_k x_k$ converges for each $x \in X$, then the linear functional $f_a : X \mapsto \mathbb{C}$ defined by

$$f_a(x) = \sum_{k=0}^\infty a_k x_k \quad \text{for all } x \in X$$

is continuous.

Proof. For each $n \in \mathbb{N}_0$, we define the linear functional $f_{a,n} : X \mapsto \mathbb{C}$ by $f_{a,n}(x) = \sum_{k=0}^{n} a_k x_k$ for all $x \in X$. Since X is an FK space, the coordinates $P_k : X \mapsto \mathbb{C}$ are continuous on X for all $k = 0, 1, \dots$, and so are the functionals $f_{a,n} = \sum_{k=0}^{n} a_k P_k$ $(n = 0, 1, \dots)$. For each $x \in X$, $f_a(x) = \lim_{n\to\infty} f_{a,n}(x)$ exists, and so $f_a : X \mapsto \mathbb{C}$ is continuous by the *Banach–Steinhaus theorem* (see appendix A.4.6). □

For the next result we shall need some notations.

Given any two subsets X and Y of ω and any infinite matrix $A = (a_{nk})_{n,k=0}^{\infty}$ of complex numbers, we shall write $A_n = (a_{n,k})_{k=0}^{\infty}$ for the sequence in the n-th row of A,

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \quad (x \in X) \text{ for all } n = 0, 1, \dots$$

(provided the series converge) and

$$A(x) = (A_n(x))_{n=0}^{\infty}.$$

Furthermore let (X, Y) be the class of all matrices A that map X into Y, that is for which the series $A_n(x)$ converge for all $x \in X$ and for all n, and $A(x) \in Y$ for all $x \in X$.

Theorem 1.17. *Any matrix map between FK spaces is continuous.*

Proof. Let X and Y be FK spaces, $A \in (X, Y)$ and the map $f_A : X \mapsto Y$ be defined by $f_A(x) = A(x)$ for all $x \in X$. Since the maps $P_n \circ f_A : X \mapsto \mathbb{C}$ are continuous for all $n \in \mathbb{N}_0$ by Remark 1.16, the linear map f_A is continuous by Corollary 1.15. □

Definition 1.18. An FK space $X \supset \phi$ has *AK* if, for every sequence $x = (x_k)_{k=0}^{\infty} \in X$,

$$x = \sum_{k=0}^{\infty} x_k e^{(k)}, \quad \text{that is} \quad x^{[m]} = \sum_{k=0}^{m} x_k e^{(k)} \to x \ (m \to \infty),$$

and X has *AD* if ϕ is dense in X. If an FK space has AK or AD we also say that it is an *AK* or *AD space*.

Remark 1.19. Every AK space has AD. The converse is not true in general.

Proof. The first part is trivial, and the second part can be found in [108, Example 5.2.5, p. 78]. □

Example 1.20. The spaces ω, c_0 and l_p $(1 \le p < \infty)$ all have AK by Example 1.7 and Theorem 1.10.

The FK metric of an FK space will turn out to be unique.

Theorem 1.21. *Let X and Y be FK spaces and $X \subset Y$. Then the metric d_X on X is stronger than the metric $d_Y|_X$ of Y on X. The metrics are equivalent if and only if X is a closed subspace of Y. In particular, the metric of an FK space is unique, this means there is at most one way to make a linear subspace of ω into an FK space.*

Proof. Let $\iota : (X, d_X) \mapsto (Y, d_Y)$ be the inclusion map. Since X is an FK space, $\iota : (X, d_X) \mapsto (Y, d|_Y)$ is continuous, and so is $\iota : (X, d_X) \mapsto (Y, d_Y)$ by Theorem 1.14. Thus d_X is stronger than $d_Y|_X$. The uniqueness of an FK space is shown in exactly the same way. Let X be closed in Y, then X becomes an FK space with $d_Y|_X$, and the uniqueness of an FK metric implies that d_X and $d_Y|_X$ are equivalent.

Conversely, if d_X and $d_Y|_X$ are equivalent, then X is a complete subspace of Y, hence a closed subspace of Y. □

Example 1.22. The BK spaces c_0 and c are closed subspaces of l_∞. Thus the BK norms on c_0, c and l_∞ must be the same. The BK space l_1 is a subspace of l_∞ which is not closed in l_∞. Thus its BK norm $\|\cdot\|_1$ is strictly stronger than the BK norm $\|\cdot\|_\infty$ on l_∞.

1.3. Matrix transformations into l_∞, c and c_0. In this subsection we shall apply the results of Subsection 1.2 to characterize classes (X, Y) where X is any FK space and Y is any of the spaces l_∞, c and c_0. We shall need some more notations.

If $X \subset \omega$ is a linear metric space with respect to d_X and $a, x_0 \in X$, then we shall write

$$S_\delta[x_0] = S_{X,\delta}[x_0] = \{x \in X : d_X(x, x_0) \le \delta\} \ (\delta > 0)$$

$$\|a\|_D^* = \|a\|_{X,D}^* = \sup\left\{\left|\sum_{k=0}^{\infty} a_k x_k\right| : x \in S_{1/D}[0]\right\} \ (D > 0)$$

provided the expression on the right exists and is finite. By Remark 1.6, this is the case whenever X is an FK space and the series $\sum_{k=0}^{\infty} a_k x_k$ converge for all $x \in X$. If X is a BK space we write

$$\|a\|^* = \|a\|_X^* = \sup\left\{\left|\sum_{k=0}^{\infty} a_k x_k\right| : \|x\| = 1\right\}.$$

Let X and Y be two Fréchet spaces. By $B(X, Y)$ we denote the set of all continuous linear operators $L : X \mapsto Y$, and we write $X' = B(X, \mathbb{C})$ for the set of all continuous linear functionals on X, the set X' is called the *continuous dual of* X. If X and Y are normed spaces and $L \in B(X, Y)$, then we write

$$\|L\| = \sup\{\|L(x)\| : \|x\| = 1\} \qquad \text{for all } L \in B(X, Y). \tag{1.4}$$

for the *operator norm of* L; furthermore we write X^* for X' with the norm in (1.4), that is $\|f\| = \sup\{|f(x)| : \|x\| = 1\}$ for all $f \in X'$.

Let A be an infinite matrix, D a positive real and X an FK space. Then we put

$$M^*_{A,D}(X, l_\infty) = \sup_n \|A_n\|^*_D$$

and, if X is a BK space, then we write

$$M^*_A(X, l_\infty) = \sup_n \|A_n\|^*.$$

Theorem 1.23. *Let X and Y be FK spaces.*

(a) *Then $(X, Y) \subset B(X, Y)$, that is, every $A \in (X, Y)$ defines a linear operator $L_A \in B(X, Y)$ where $L_A(x) = A(x)$ for all $x \in X$.*

(b) *Then $A \in (X, l_\infty)$ if and only if*

$$\|A\|^*_D = M^*_{A,D}(X, l_\infty) < \infty \quad \text{for some } D > 0. \tag{1.5}$$

If X is a BK space and $A \in (X, l_\infty)$, then $\|A\|^* = M^*_A(X, l_\infty) = \|L_A\| < \infty$.

(c) *If $(b^k)_{k=0}^\infty$ is a Schauder basis for X, and Y_1 a closed FK space in Y, then $A \in (X, Y_1)$ if and only if $A \in (X, Y)$ and $A(b^{(k)}) \in Y_1$ for all $k = 0, 1, \ldots$.*

Proof. Part (a) is Theorem 1.17. (b) First we assume that condition (1.5) holds. Then, for all $x \in S_{1/D}[0]$, the series $A_n(x)$ $(n = 0, 1, \ldots)$ converge and $A(x) \in l_\infty$. Since the set $S_{1/D}[0]$ is absorbing by Remark 1.3, we conclude that $A_n(x)$ converges for each $x \in X$ and $A(x) \in l_\infty$ for all $x \in X$, hence $A \in (X, l_\infty)$.

Conversely, we assume $A \in (X, l_\infty)$. Then L_A is continuous by part (a). Hence there exist a neighbourhood N of 0 in X and a real $D > 0$ such that $S_{1/D}[0] \subset N$ and $\|L_A(x)\| < 1$ for all $x \in N$. This implies condition (1.5). If X is a BK space, then $L_A \in B(X, Y)$ implies

$$\|A(x)\|_\infty = \sup_n |A_n(x)| = \|L_A(x)\|_\infty \le \|L_A\| \quad \text{for all } x \in X \text{ with } \|x\| = 1.$$

Thus $|A_n(x)| \le \|L_A$ for all n and for all $x \in X$ with $\|x\| = 1$, and, by the definition of the norm $\|\cdot\|^*$,

$$\|A\|^* = \sup_n \|A_n\|^* \le \|L_A\|. \tag{1.6}$$

Further, given $\varepsilon > 0$, there is $x \in X$ with $\|x\| = 1$ such that $\|A(x)\|_\infty \ge \|L_A\| - \varepsilon/2$, and there is $n(x) \in \mathbb{N}_0$ with $|A_{n(x)}(x)| > \|A(x)\|_\infty - \varepsilon/2$, consequently $|A_{n(x)}(x)| \ge \|L_A\| - \varepsilon$. Therefore $\|A\|^* = \sup_n \|A_n\|^* \ge \|L_A\| - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\|A\|^* \ge \|L_A\|$, and, with (1.6), we have $\|A\|^* = \|L_A\|$.

(c) The necessity of the conditions for $A \in (X, Y_1)$ is trivial.

Conversely, if $A \in (A, Y)$, then $L_A \in B(X, Y)$. Since Y_1 is a closed subspace of Y, the FK metrics of Y_1 and Y are the same by Theorem 1.21. Consequently, if S is any subset in Y_1, then, for its closures $\text{clos}_{Y_1}(S)$ and $\text{clos}_{Y|_{Y_1}}(S)$ with respect to the metrics d_{Y_1} and $d_{Y|_{Y_1}}$, we have

$$\text{clos}_{Y_1}(S) = \text{clos}_{Y|_{Y_1}}(S). \tag{1.7}$$

Let $x \in X$ and $SB = \left\{\sum_{k=0}^{m} \lambda_k b^{(k)} : m \in \mathbb{N}_0,\ \lambda_k \in \mathbb{C}\ (k = 0, 1, \dots)\right\}$ denote the *span of* $\{b^{(k)} : k = 0, 1, \dots\}$. Since $L_A(b^{(k)}) \in Y_1$ for all $k = 0, 1, ..$ and the metrics d_{Y_1} and $d_{Y|_{Y_1}}$ are equivalent, the map $L_A\big|_{SB} : (X, d_X) \mapsto (Y_1, d_{Y_1})$ is continuous. Further, since $(b^k)_{k=0}^{\infty}$ is a basis of X, we have $\overline{SB} = X$. Therefore, by (1.7) and the continuity of $L_A\big|_{SB}$, we have

$$\begin{aligned} L_A(X) = L_A(\overline{SB}) &= \text{clos}_{Y_1}\left(L_A\big|_{SB}(SB)\right) = \text{clos}_{Y|_{Y_1}}\left(L_A\big|_{SB}(SB)\right) \\ &\subset \text{clos}_{Y|_{Y_1}}(Y_1) = Y_1 \end{aligned}$$

Thus $A(x) \in Y$ for all $x \in X$. □

1.4. The α– and β–duals of sets of sequences. In this subsection we shall study the so–called α–, β– and *continuous dual spaces* of sets of sequences. The first two kinds of dual spaces naturally arise in the study of absolute and ordinary convergence of sequences from a subset of ω.

Furthermore the conditions given in Subsection 1.3 for an infinite matrix A to be in the classes (X, l_∞), (X, c) and (X, c_0) for arbitrary FK spaces X involved the norm of the operator L_A defined by $L_A(x) = A(x)$. Since $A \in (X, Y)$ can only hold if $A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k$ converges for all $x \in X$ and for all $n = 0, 1, \dots$, it is essential to know the set of all sequences $a \in \omega$ for which $\sum_{k=0}^{\infty} a_k x_k$ converges for all $x \in X$, the so–called *β–dual of* X. Finally, if X and Y are given FK spaces, then we intend to replace the operator norm in the conditions for $A \in (X, Y)$ by conditions for the entries of the matrix A. In many cases this can be achieved by replacing the operator norm by the natural norm on the β–dual of X.

The α– and β–duals are special cases of the so–called *multiplier spaces*.

Definition 1.24. Let X and Y be subsets of ω.

(a) For all $z \in \omega$, we write $z^{-1} * Y = \left\{x \in \omega : xz = (x_k z_k)_{k=0}^{\infty} \in Y\right\}$. The set $Z = M(X, Y) = \bigcap_{x \in X} x^{-1} * Y = \left\{a \in \omega : ax \in Y \text{ for all } x \in X\right\}$ is called the *multiplier space of* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$.

(b) By cs and bs, we denote the set of all *convergent* and *bounded series*, respectively, that is $cs = \{x \in \omega : \sum_{k=0}^{\infty} x_k \text{ converges}\}$ and $bs = \{x \in \omega : (\sum_{k=0}^{n} x_k)_{n=0}^{\infty} \in l_\infty\}$, and we define the norm $\|\cdot\|_{bs}$ on cs and bs by $\|x\|_{bs} = \sup_n |\sum_{k=0}^{n} x_k|$. In the special case where $Y = l_1$ or $Y = cs$, the multiplier spaces $X^\alpha = M(X, l_1)$ and $X^\beta = M(X, cs)$ are called the $\boldsymbol{\alpha}$– or *Köthe–Toeplitz* and *$\boldsymbol{\beta}$–duals of* $\boldsymbol{X}$. If † denotes either of the symbols α or β, then $X \subset \omega$ is said to be **†**–*perfect* if $X^{\dagger\dagger} = (X^\dagger)^\dagger = X$.

Lemma 1.25. *Let $X, Y, Z \subset \omega$ and $\{X_\delta : \delta \in A\}$ be any collection of subsets of ω. Then:*

(*i*) $X \subset M(M(X, Y), Y)$

(*ii*) $X \subset Z$ *implies* $M(Z, Y) \subset M(X, Y)$

(*iii*) $M(X, Y) = M(M(M(X, Y), Y), Y)$

(*iv*) $M\left(\bigcup_{\delta \in A} X_\delta, Y\right) = \bigcap_{\delta \in A} M(X_\delta, Y)$.

Proof. (i) If $x \in X$, then $ax \in Y$ for all $a \in M(X, Y)$, and consequently $x \in M(M(X, Y), Y)$.

(ii) Let $X \subset Z$. If $a \in M(Z,Y)$, then $ax \in Y$ for all $x \in Z$, hence $ax \in Y$ for all $x \in X$, since $X \subset Z$. Thus $a \in M(X,Y)$.

(iii) We apply (i) with X replaced by $M(X,Y)$ to obtain

$$M(X,Y) \subset M(M(M(X,Y),Y),Y).$$

Conversely, by (i), $X \subset M(M(X,Y),Y)$, and so (ii) with $Z = M(M(X,Y),Y)$ yields $M(M(M(X,Y),Y),Y) \subset M(X,Y)$.

(iv) First $X_\delta \subset \bigcup\limits_{\delta\in A} X_\delta$ for all $\delta \in A$ implies $M\Big(\bigcup\limits_{\delta\in A} X_\delta, Y\Big) \subset \bigcap\limits_{\delta\in A} M(X_\delta, Y)$ by part (i).

Conversely, if $a \in \bigcap_\delta M(X_\delta, Y)$, then $a \in M(X_\delta, Y)$ for all $\delta \in A$, and so we have $ax \in Y$ for all $\delta \in A$ and for all $x \in X_\delta$. This implies $ax \in Y$ for all $x \in \bigcup_{\delta\in A} X_\delta$, hence $a \in M(\bigcup_{\delta\in A} X_\delta, Y)$. Thus $\bigcap_{\delta\in A} M(X_\delta, Y) \subset M(\bigcup_{\delta\in A} X_\delta, Y)$. □

As an immediate consequence of Lemma 1.25 we obtain

Corollary 1.26. *Let $X, Y \subset \omega$ and $\{X_\delta : \delta \in A\}$ be a collection of subsets of ω. If $\dagger$ denotes either of the symbols α or β, then*

(i)	$X \subset X^{\dagger\dagger}$	*(ii)*	$X \subset Y$ *implies* $Y^\dagger \subset X^\dagger$
(iii)	$X^\dagger = X^{\dagger\dagger\dagger}$	*(iv)*	$\left(\bigcup_{\delta\in A} X_\delta\right)^\dagger = \bigcap_{\delta\in A} X_\delta^\dagger$.

A subset X of ω is said to be *normal* if $x \in X$ and $|\tilde{x}_k| \le |x_k|$ $(k = 0, 1, \dots)$ together imply $\tilde{x} \in X$.

Remark 1.27. Obviously $X^\alpha \subset X^\beta$ for arbitrary $X \subset \omega$. If X is a normal subset of ω, then $X^\alpha = X^\beta$.

Proof. The first part is obvious. For the second part, we have to show $X^\beta \subset X^\alpha$. Let $a \in X^\beta$ and $x \in X$ be given. We define the sequence y by $y_k = \operatorname{sgn}(x_k)|x_k|$ for $k = 0, 1, \dots$. Then obviously $|y_k| \le |x_k|$ for all k, and consequently $y \in X$, since X is normal, and so $ax \in cs$. Further, by the definition of the sequence y, $ay = (|a_k||x_k|)_{k=0}^\infty = |ax| \in cs$, hence $ax \in l_1$. Since $x \in X$ was arbitrary, $a \in X^\alpha$. This shows $X^\beta \subset X^\alpha$. □

Example 1.28. We have:
(i) $M(c_0, c) = l_\infty$, (ii) $M(c,c) = c$, (iii) $M(l_\infty, c) = c_0$.

Proof. (i) If $a \in l_\infty$, then $ax \in c$ for all $x \in c_0$, and so $l_\infty \subset M(c_0, c)$.

Conversely we assume $a \notin l_\infty$. Then there is a subsequence $(a_{k_j})_{j=0}^\infty$ of the sequence a such that $|a_{k_j}| > j + 1$ for all $j = 0, 1, \dots$. We define the sequence x by

$$x_k = \begin{cases} (-1)^j / a_{k_j} & \text{for } k = k_j \\ 0 & \text{for } k \ne k_j \end{cases} \quad (j = 0, 1, \dots). \tag{1.8}$$

Then $x \in c_0$ and $a_{k_j} x_{k_j} = (-1)^j$ for all $j = 0, 1, \dots$, hence $ax \notin c$. This shows $M(c_0, c) \subset l_\infty$.

(ii) If $a \in c$, then $ax \in c$ for all $x \in c$, and so $c \subset M(c,c)$.

Conversely we assume $a \notin c$. Since $e \in c$ and $ae = a \notin c$, we have $a \notin M(c,c)$. This shows $M(c,c) \subset c$.

(iii) If $a \in c_0$, then $ax \in c$ for all $x \in l_\infty$, and so $c_0 \subset M(l_\infty, c)$.

Conversely we assume $a \notin c_0$. Then there are a real $b > 0$ and a subsequence $(a_{k_j})_{j=0}^\infty$ of the sequence a such that $|a_{k_j}| > b$ for all $j = 0, 1, \dots$. We define the sequence x as in (1.8). Then $x \in l_\infty$ and $a_{k_j} x_{k_j} = (-1)^j$ for $j = 0, 1, \dots$, hence $a \notin M(l_\infty, c)$. This shows $M(l_\infty, c) \subset c_0$. □

Now we shall give the α– and β–duals of the classical sequence spaces.

Theorem 1.29. *Let † denote either of the symbols α or β. Then*
(a) $\omega^\dagger = \phi$ *and* $\phi^\dagger = \omega$.
(b) $l_1^\dagger = l_\infty$, $l_p^\dagger = l_q$ *for* $1 < p < \infty$ *and* $q = p/(p-1)$, *and for all* $a \in l_p^\beta$, $\|a\|_{l_1}^* = \|a\|_\infty$ *and* $\|a\|_{l_p}^* = \|a\|_q$ *for* $1 < p < \infty$.
(c) $c_0^\dagger = c^\dagger = l_\infty^\dagger = l_1$ *and* $\|a\|_{c_0}^* = \|a\|_c^* = \|a\|_{l_\infty}^* = \|a\|_1$ *for all* $a \in l_\infty^\beta$.

The multiplier space of two BK spaces will turn out to be a BK space.

Theorem 1.30. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be BK spaces with $X \supset \phi$ and $Z = M(X,Y)$. Then Z is a BK space with $\|\cdot\|$ defined by*

$$\|z\| = \|z\|_X^* = \sup\{\|xz\|_Y : \|x\|_X = 1\} \qquad \text{for all } z \in Z.$$

Proof. It is well known that $B = B(X,Y)$ is a Banach space. Each $z \in Z$ defines a diagonal matrix map $\hat{z} : X \mapsto Y$ where $\hat{z}(x) = xz$ for all $x \in X$ which is continuous by Theorem 1.17. This embeds Z in B, for if $\hat{z} = 0$, then $\hat{z}(e^{(n)}) = (z_n)_{n=0}^\infty = 0 = z$. To see that the coordinates are continuous, we fix $n \in \mathbb{N}_0$ and put $u = 1/\|e^{(n)}\|_X$ and $v = \|e^{(n)}\|_Y$. Then $\|ue^{(n)}\|_X = 1$ and

$$uv|z_n| = u\|z_n e^{(n)}\|_Y = u\|e^{(n)} z\|_Y = \|(ue^{(n)})z\|_Y \le \|z\|_X^* = \|z\| \qquad \text{for all } n.$$

It remains to show that Z is a closed subspace of B. Let $(\hat{z}^{(m)})_{m=0}^\infty$ be a sequence in B with $\hat{z}^{(m)} \to T \in B$. For each fixed $X \in X$, we obtain $\hat{z}^{(m)}(x) \to T(x) \in Y$ $(m \to \infty)$ and since Y is a BK space, this implies $(\hat{z}^{(m)}(x))_k \to (T(x))_k$, that is $z_k^{(m)} x_k \to (T(x))_k$ $(m \to \infty)$ for each fixed k. We put $x = e^{(k)}$. Then $z_k^{(m)} \to (T(e^{(k)}))_k = t_k$ $(m \to \infty)$, and so $x_k z_k^{(m)} = (T(x))_k$ $(m \to \infty)$ and $x_k z_k^{(m)} \to (T(x))_k$ $(m \to \infty)$. Therefore $T(x) = xt$ for all $x \in X$, and so $T = \hat{t}$. □

Corollary 1.31. *The α– and β– duals of a BK space X are BK spaces with respect to $\|a\|_\alpha = \|a\|_{X,\alpha} = \sup\{\|ax\|_1 = \sum_{k=0}^\infty |a_k x_k| : \|x\| \le 1\}$ and $\|a\|_\beta = \|a\|_{X,\beta} = \sup\{\|ax\|_{bs} = \sup_n |\sum_{k=0}^n a_k x_k| : \|x\| \le 1\}$.*

Example 1.32. Let X be any of the spaces l_∞, c, c_0 and l_p for $1 \le p < \infty$. Then the norms $\|\cdot\|_{X^\beta}$, $\|\cdot\|_X^*$, $\|\cdot\|_{X,\alpha}$ and $\|\cdot\|_{X,\beta}$ are equivalent on X^β.

Proof. The norm $\|\cdot\|_X^*$ and the natural norm $\|\cdot\|_{X^\beta}$ are equal on X^β by Theorem 1.29. Since each set X^β is a BK space with its natural norm, $\|\cdot\|_{X^\beta}$ and

$\|\cdot\|_{X,\beta}$ are equivalent by Corollary 1.31 and Theorem 1.21. Finally, since $X^\alpha = X^\beta$ for each set X, the norms $\|\cdot\|_{X,\alpha}$ and $\|\cdot\|_{X,\beta}$ are equivalent by Corollary 1.31 and Theorem 1.21. □

The analogues of Theorem 1.30 and Corollary 1.31 do not hold for FK spaces in general.

Remark 1.33. The space ω is an FK space and $\omega^\alpha = \omega^\beta = \phi$ and ϕ has no Fréchet metric (cf. [108, 4.0.5, p. 51]).

1.5. The continuous duals of the classical sequence spaces. In this subsection we shall give the continuous duals of the spaces l_p for $1 \leq p < \infty$, c and c_0.

There is a close relation between the β–dual and the continuous dual of an FK space which is very useful in the determination of the continuous duals of the spaces l_p, c and c_0.

Theorem 1.34. *Let X be a BK space and $X \supset \phi$. Then there is a linear one-to-one map $T : X^\beta \mapsto X'$; we denote this by $X^\beta \subset X'$. If X has AK, then T is onto.*

Proof. We define the map T on X^β as follows. For every $a \in X^\beta$, let $Ta : X \mapsto \mathbb{C}$ be defined by $(Ta)(x) = \sum_{k=0}^\infty a_k x_k$ for all $x \in X$. Since $a \in X^\beta$, the series $\sum_{k=0}^\infty a_k x_k$ converge for all $x \in X$, and obviously Ta is linear. Further, since X is an FK space, $Ta \in X'$ for ach $a \in X^\beta$. Therefore $T : X^\beta \mapsto X'$. Further it is easy to see that T is linear.

To show that T is one-to-one, we assume $a, b \in X^\beta$ with $Ta = Tb$. This means $(Ta)(x) = (Tb)(x)$ for all $x \in X$. Since $\phi \subset X$, we may choose $x = e^{(k)}$ for each $k \in \mathbb{N}_0$ and obtain $(Ta)(e^{(k)}) = a_k = b_k = (Tb)(e^{(k)})$ for $k = 0, 1, \ldots$, and so $a = b$.

Now we assume that X has AK and $f \in X'$. We put $a_n = f(e^{(n)})$ for $n = 0, 1, \ldots$. Let $x \in X$ be given. Then $x = \sum_{k=0}^\infty x_k e^{(k)}$, since X has AK, and $f \in X'$ implies $f(x) = \sum_{k=0}^\infty x_k f(e^{(k)}) = \sum_{k=0}^\infty a_k x_k = (Ta)(x)$. As $x \in X$ was arbitrary and the series converge, $a \in X^\beta$ and $f = Ta$. This shows that T is onto X'. □

Now we shall give the continuous duals of the classical sequence spaces.

Two linear spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are called *norm isomorphic* if there is an isomorphism $T : X \mapsto Y$ such that $\|T(x)\|_Y = \|x\|_X$ for all $x \in X$; we shall write $X \simeq Y$.

Theorem 1.35. *We have:*

(a) *$l_p^* \simeq l_\infty$ for $0 < p \leq 1$ and $l_p^* \simeq l_q$ for $1 < p < \infty$ where $q = p/(p-1)$;*

(b) *$c_0^* \simeq l_1$;*

(c) *$f \in c^*$ if and only if $f(x) = l\chi_f + \sum_{k=0}^\infty a_k x_k$ with $a \in l_1$ where $l = \lim_{k\to\infty} x_k$ and $\chi_f = f(e) - \sum_{k=0}^\infty a_k$. Furthermore $\|f\|^* = |\chi_f| + \|a\|_1$.*

It is worth mentioning that the continuous dual of l_∞ is not isomorphic to a sequence space (cf. [40, 31.1, pp. 427, 428] or [105, Example 6.4.8, pp. 93, 94]).

For further studies concerning multiplier spaces, some important special cases, f– and continuous duals, we refer the reader to [108, 91, 22, 23].

1.6. Matrix transformations between some classical sequence spaces. We now apply the results of the previous subsections to characterize certain classes of matrix transformations between some classical sequence spaces by giving necessary and sufficient conditions on the entries of a matrix to belong to the respective class.

Let A be an infinite matrix. We write $q = p/(p-1)$ for $1 < p < \infty$, $q = \infty$ for $p = 1$ and $q = 1$ for $p = \infty$, put

$$M_A(l_p, l_\infty) = \begin{cases} \|A\|_\infty = \sup_{n,k} |a_{nk}| & (p = 1) \\ \|A\|_q = \sup_n \left(\sum_{k=0}^\infty |a_{nk}|^q\right) & (1 < p \le \infty) \end{cases}$$

and consider the conditions

$$\lim_{n\to\infty} a_{nk} = 0 \quad (k = 0, 1, \dots), \tag{1.9}$$

$$\lim_{n\to\infty} \left(\sum_{k=0}^\infty a_{nk}\right) = 0, \tag{1.10}$$

$$\lim_{n\to\infty} a_{nk} = l_k \quad \text{for some } l_k \in \mathbb{C} \ (k = 0, 1, \dots) \tag{1.11}$$

$$\lim_{n\to\infty} \left(\sum_{k=0}^\infty a_{nk}\right) = l \quad \text{for some } l \in \mathbb{C}. \tag{1.12}$$

Theorem 1.36. *We have*
(a) $(c_0, l_\infty) = (c, l_\infty) = (l_\infty, l_\infty)$ *and* $A \in (l_\infty, l_\infty)$ *if and only if*

$$M_A(l_\infty, l_\infty) = \sup_n \left(\sum_{k=0}^\infty |a_{nk}|\right) < \infty, \tag{1.13}$$

(b) $A \in (c_0, c_0)$ *if and only if conditions* (1.13) *and* (1.9) *hold;*
(c) $A \in (c, c_0)$ *if and only if conditions* (1.13), (1.9) *and* (1.10) *hold;*
(d) $A \in (c_0, c)$ *if and only if conditions* (1.13) *and* (1.11) *hold;*
(e) $A \in (c, c)$ *if and only if conditions* (1.13), (1.11) *and* (1.12) *hold.*

Proof. (a) We have $A \in (l_\infty, l_\infty)$ if and only if condition (1.13) holds by Theorems 1.23 and 1.29.

Further, if condition (1.13) holds, then $A \in (l_\infty, l_\infty) \subset (c_0, c)$, since $c_0 \subset l_\infty$.

Conversely, let $A \in (c_0, l_\infty)$. Then $\sup_n \|A_n\|_{c_0}^* < \infty$ by Theorem 1.23 (b). Since the series $A_n(x)$ converge for all x and n, we have $f_{A_n} \in c_0^*$ for all n where $f_{A_n}(x) = \sum_{k=0}^\infty a_{nk}x_k$ for all $x \in c_0$, hence $|f_{A_n}(x)| \le \|f_{A_n}\| = \|A_n\|_{c_0}^*$. We fix $n \in \mathbb{N}_0$. Let $m \in \mathbb{N}_0$ be arbitrary. We define the sequence $x^{[m,n]}$ by $x^{[m,n]} = \sum_{k=0}^m \mathrm{sgn}(a_nk)e^{(k)}$. Then $x^{[m,n]} \in c_0$, $\|x^{[m,n]}\|_\infty \le 1$ and $|f_{A_n}(x^{[m,n]})| = \sum_{k=0}^m |a_{nk}| \le \|A_n\|_{c_0}^*$. Since $m \in \mathbb{N}_0$ was arbitrary, $\|A_n\|_1 = \sum_{k=0}^\infty |a_{nk}| \le \|A_n\|_{c_0}^*$ for all $n = 0, 1, \dots$. Therefore condition (1.13) must hold. Finally $c_0 \subset c \subset l_\infty$ and $(c_0, l_\infty) = (l_\infty, l_\infty)$ together imply $(c, l_\infty) = (l_\infty, l_\infty)$.

Parts (b) to (e) follow from part (a), Theorem 1.23 (c) and Theorem 1.10. □

Similarly, we obtain

Theorem 1.37. *Let* $1 < p < \infty$. *Then:*
(a) $A \in (l_p, l_\infty)$ *if and only if*

$$M(l_p, l_\infty) < \infty; \tag{1.14}$$

(b) $A \in (l_p, c_0)$ *if and only if conditions* (1.14) *and* (1.9) *hold;*
(c) $A \in (l_p, c)$ *if and only if conditions* (1.14) *and* (1.11) *hold.*

2. Measures of concompactness

In Section 1 we developed and applied parts of the FK space theory to give necessary and sufficient conditions for $A \in (X, Y)$ for given sequence spaces. The most important result was that matrix transformations between FK spaces are continuous. It is quite natural to find conditions for a matrix map between FK spaces to define a compact operator. This can be achieved by applying the *Hausdorff measure of noncompactness.* The first *measure of noncompactness*, the function α, was defined and studied by Kuratowski [41] in 1930. It is surprising that later in 1955 Darbo [12] was the first who continued to use the function α. Darbo proved that if T is a continuous self-mapping of a nonempty, bounded, closed and convex subset C of a Banach space X such that $\alpha(T(Q)) \leq k\alpha(Q)$ for all $Q \subset C$, where $k \in (0, 1)$ is a constant, then T has at least one fixed point in the set C. Darbo's fixed point theorem is a very important generalization of Schauder's fixed point theorem and it includes the existence part of Banach's fixed point theorem.

Other measures were introduced by Goldenstein, Gohberg and Markus (the *ball measures of noncompactness, Hausdorff measure of noncompactness*) [19] in 1957 (later studied by Goldenstein and Markus [20] in 1968), Istrăţesku [30] in 1972 and others. Apparently Goldenstein, Gohberg and Markus were not aware of the work of Kuratowski and Darbo. It is surprising that Darbo's theorem was almost never noticed and applied, not till in the seventies mathematicians working in operator theory, functional analysis and differential equations begun to apply Darbo's theorem and developed the theory connected with measures of noncompactness.

The use of these measures is discussed for example in the monographs [1, 6, 7, 24, 25, 28, 31, 42, 86, 99, 100], Ph. D. theses [2, 4, 75, 77, 83, 102] and expository papers [47, 93, 104]. We refer the reader to these works with references given there.

2.1. Introduction.

Let us recall some definitions and results which are probably well known. If M and S are subsets of a metric space (X, d) and $\epsilon > 0$, then the set S is called *ϵ-net* of M if for any $x \in M$ there exists $s \in S$, such that $d(x, s) < \epsilon$. If the set S is finite, then the ϵ-net S of M is called *finite ϵ-net.* The set M is said to be *totally bounded* if it has a finite ϵ-net for every $\epsilon > 0$. It is well known, that a subset M of a metric space X is compact if every sequence (x_n) in M has a convergent subsequence, and in this case the limit of that subsequence is in M. The set M is said to be *relatively compact* if the closure $\overline{M}$ of M is a compact set. If the set M is relatively compact, then M is totally bounded. If the metric space (X, d) is complete, then the set M is relatively compact if and only if it is totally bounded. It is easy to prove that a subset M of a metric space X is relatively compact if and only if every sequence (x_n) in M has a convergent subsequence; in that case the limit of that subsequence need not be in M.

If $x \in X$ and $r > 0$, then the open ball with centre at x and radius r is denoted by $B(x, r)$, $B(x, r) = \{y \in X : d(x, y) < r\}$. If X is a normed space, then we denote by B_X the closed unit ball in X and by S_X the unit sphere in X. Let $\mathcal{M}_X$ (or simply $\mathcal{M}$) be the set of all nonempty and bounded subsets of a metric space (X, d), and let $\mathcal{M}_X^c$ (or simply $\mathcal{M}^c$) be the subfamily of $\mathcal{M}_X$ consisting of all closed sets. Further, let $\mathcal{N}_X$ (or simply $\mathcal{N}$) be the set of all nonempty and relatively compact subsets of (X, d). Let $d_H : \mathcal{M}_X \times \mathcal{M}_X \mapsto \mathbb{R}$ be the function defined by

$$d_H(S, Q) = \max\{\sup_{x\in S} d(x, Q), \sup_{y\in Q} d(y, S)\} \quad (S, Q \in \mathcal{M}_X). \tag{2.1}$$

The function d_H is called *Hausdorff distance*, and $d_H(S, Q)$ $(S,\, Q \in \mathcal{M}_X)$ is the *Hausdorff distance of sets S and Q*.

Let us remark that if $\emptyset \neq F \subset X$, $r > 0$ and

$$B(F, r) = \bigcup_{x\in F} B(x, r) = \{y \in X : d(y, F) < r\}$$

is the *open ball with centre in F and radius r*, then (2.1) is equivalent to

$$d_H(S, Q) = \inf\{\epsilon > 0 : S \subset B(Q, \epsilon) \quad \text{and} \quad Q \subset K(S, \epsilon)\}, \quad (S, Q \in \mathcal{M}_X).$$

It is well known that $(\mathcal{M}_X, d_H)$ is a pseudometric space and that $(\mathcal{M}_X^c, d_H)$ is a metric space.

Let X and Y be infinite-dimensional complex Banach spaces and denote the set of bounded linear operators from X into Y by $B(X, Y)$. We put $B(X) = B(X, X)$. For T in $B(X, Y)$, $N(T)$ and $R(T)$ will denote, respectively, the null space and the range space of T. A linear operator L from X to Y is called *compact* (or *completely continuous*) if $D(L) = X$ for the domain of L, and for every sequence $\{x_n\} \subset X$ such that $\|x_n\| \le C$, the sequence $\{L(x_n)\}$ has a subsequence which converges in Y. A compact operator is bounded. An operator L in $B(X, Y)$ is of *finite rank* if $\dim R(L) < \infty$. An operator of finite rank is clearly compact. Let $F(X, Y)$, $K(X, Y)$ denote the set of all finite rank and compact operators from X to Y, respectively. Set $F(X) = F(X, X)$ and $K(X) = K(X, X)$.

Let X be a vector space over the field $\mathbb{F}$. A subset E of X is said to be *convex* if $\lambda x + (1-\lambda)y \in E$ for all $x, y \in E$ and for all $\lambda \in (0, 1)$.

Clearly the intersection of any family of convex sets is a convex set. If F is a subset of X, then the intersection of all convex sets that contain F is called *convex cover* or *convex hull* of F denoted by $\mathrm{co}(F)$.

The vector subspace $\mathrm{lin}F$ is the set of all linear combinations of elements in F. We shall prove that there is an analogous representation of the set $\mathrm{co}(F)$. Let us mention that a *convex combination of elements* of the set F is an element of the form

$$\lambda_1 x_1 + \cdots + \lambda_n x_n \left(x_i \in F,\, \lambda_i \ge 0\, (i = 1, \ldots, n),\, \sum_{i=1}^{n} \lambda_i = 1\, (n \in \mathbb{N})\right).$$

Let us write $\mathrm{cvx}(F)$ for the set of all convex combinations of elements of the set F.

Theorem 2.1. *If X is a vector space over the field $\mathbb{F}$ and E, $E_1, \dots E_n$ are convex subsets of X and $F \subset X$, then*

$$\operatorname{cvx}(E) \subset E, \tag{2.2}$$

$$\operatorname{co}(F) = \operatorname{cvx}(F), \tag{2.3}$$

$$\operatorname{co}\Big(\bigcup_{i=1}^{n} E_i\Big) = \Big\{\sum_{i=1}^{n} \lambda_i E_i : \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, \, i = 1, \dots, n\Big\}. \tag{2.4}$$

Proof. To prove (2.2), it suffices to show that for any $n \geq 2$

$$x_i \in E, \lambda_i \geq 0 \, (i = 1, \dots n) \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i = 1 \quad \text{together} \\ \text{imply} \quad \lambda_1 x_1 + \cdots + \lambda_n x_n \in E. \tag{2.5}$$

We shall use the method of mathematical induction. For $n = 2$ the statement clearly is true. Suppose that the statement in (2.5) is true for a natural number $n > 2$, and let us prove the statement for $n+1$. If $x_i \in E$, $\lambda_i \geq 0$ $(i = 1, \dots, n+1)$ and $\sum_{i=1}^{n+1} \lambda_i = 1$, then there are two cases: first, if $\sum_{i=1}^{n} \lambda_i = 0$, then $\lambda_i = 0$ $(i = 1, \dots, n)$ and $\lambda_1 x_1 + \cdots + \lambda_{n+1} x_{n+1} = x_{n+1} \in E$; second, if $\lambda \equiv \sum_{i=1}^{n} \lambda_i \neq 0$, then $\lambda_1 x_1 + \cdots + \lambda_{n+1} x_{n+1} = \lambda(\lambda_1 \lambda^{-1} x_1 + \cdots + \lambda_n \lambda^{-1} x_n) + \lambda_{n+1} x_{n+1} \in E$. Thus we have shown inclusion (2.2).

It follows from (2.2) that $\operatorname{cvx}(F) \subset \operatorname{co}(F)$. Hence, since $\operatorname{co}(F)$ is a convex subset of X, it suffices to show that $\operatorname{cvx}(F)$ is convex. Suppose that $\lambda \in (0,1)$, and $x, y \in \operatorname{cvx}(F)$. Then there exist $n, m \in \mathbb{N}$, α_i, x_i $(i = 1, \dots, n)$ with $\sum_{i=1}^{n} \alpha_i = 1$, β_j, y_j $(j = 1, \dots, m)$ with $\sum_{j=1}^{m} \beta_j = 1$ such that $x = \sum_{i=1}^{n} \alpha_i x_i$ and $y = \sum_{j=1}^{m} \beta_j y_j$. Now $\sum_{i=1}^{n} \lambda \alpha_i + \sum_{j=1}^{m} (1-\lambda)\beta_j = \lambda + (1-\lambda) = 1$ implies $\lambda x + (1-\lambda) y \in \operatorname{cvx}(F)$. Hence we have proved (2.3).

We put $S = \{\sum_{i=1}^{n} \lambda_i E_i : \lambda_i \geq 0, \ (i = 1, \dots, n) \sum_{i=1}^{n} \lambda_i = 1\}$. By (2.2) it follows that $S \subset \operatorname{co}(\bigcup_{i=1}^{n} E_i)$. Since $\bigcup_{i=1}^{n} E_i \subset S$, to prove (2.4), it suffices to show that S is convex. Suppose that $\lambda \in (0,1)$ and $x, y \in S$. Now there exist $\alpha_i, x_i \, (i = 1, \dots, n)$ with $\sum_{i=1}^{n} \alpha_i = 1$, $\beta_i, y_i \, (i = 1, \dots, n,)$ with $\sum_{i=1}^{n} \beta_i = 1$ such that $x = \sum_{i=1}^{n} \alpha_i x_i$, $y = \sum_{i=1}^{n} \beta_i y_i$. We put $\gamma_i = \lambda \alpha_i + (1-\lambda)\beta_i \, (i = 1, \dots n)$. Since $E_1, \dots E_n$, are convex, there exist $z_i \in E_i \, (i = 1, \dots n)$ such that

$$\lambda \alpha_i x_i + (1-\lambda)\beta_i y_i = \gamma_i z_i \quad \text{for } i = 1, \dots, n. \tag{2.6}$$

Let us remark

$$\sum_{i=1}^{n} \gamma_i = \lambda \sum_{i=1}^{n} \alpha_i + (1-\lambda) \sum_{i=1}^{n} \beta_i = \lambda + (1-\lambda) = 1. \tag{2.7}$$

By (2.6) and (2.7) we have $\lambda x + (1-\lambda) y = \sum_{i=1}^{n} \gamma_i z_i \in S$. □

We continue with the study of convex sets in normed spaces.

Lemma 2.2. *Let Q be a bounded subset of a normed space X. Then for any $x \in X$*

$$\sup_{y \in \mathrm{co}(Q)} \|x - y\| = \sup_{z \in Q} \|x - z\|. \tag{2.8}$$

Proof. To prove (2.8), it suffices to show the inequality "$\leq$". If $y \in$ co (Q), then there exist $x_i \in Q$, $\lambda_i \geq 0\,(i = 1, \dots n)$ such that $\sum_{i=1}^n \lambda_i = 1$ and $y = \sum_{i=1}^n \lambda_i x_i$. From $x - y = \sum_{i=1}^n \lambda_i x - \sum_{i=1}^n \lambda_i x_i = \sum_{i=1}^n \lambda_i (x - x_i)$, it follows that $\|x - y\| \leq \sum_{i=1}^n \lambda_i \|x - x_i\| \leq \sup_{z \in Q} \|x - z\|$. □

Corollary 2.3. *Let Q be a bounded subset of a normed space X. Then the sets Q and* $\mathrm{co}(Q)$ *have equal diameter, that is* $\mathrm{diam}(Q) = \mathrm{diam}(\mathrm{co}(Q))$.

Proof. This follows by Lemma 2.2. □

Let Q be a nonempty and bounded subset of a normed space X. Then the *convex closure* of Q, is denoted by $\mathrm{Conv}(Q)$, and $\mathrm{Conv}(Q)$ is the smallest convex and closed subset of X that contains Q. It is easy to prove that $\mathrm{Conv}(Q) = \overline{\mathrm{co}(Q)}$.

Corollary 2.4. *Let Q be a bounded subset of a normed space X. Then the sets Q and* $\mathrm{Conv}(Q)$ *have equal diameters, that is* $\mathrm{diam}(Q) = \mathrm{diam}(\mathrm{Conv}(Q))$.

Proof. This follows from Corollary 2.3.

□

2.2. The Kuratowski measure of noncompactness. The notation of measure of noncompactness (α– measure or set-measure), introduced by Kuratowski [41], and the associated notion of an α– contraction, have proved useful in several areas of functional analysis, operator theory and differential equations (see for example, [1, 6, 7]). We start with some results from Kuratowski [41, 42].

Definition 2.5. Let (X, d) be a metric space and Q a bounded subset of X. Then the *Kuratowski measure of noncompactness of* ***Q*** , denoted by $\alpha(Q)$, is the infimum of the set of all numbers $\epsilon > 0$ such that Q can be covered by a finite number of sets with diameters $< \epsilon$, that is

$$\alpha(Q) = \inf\left\{\epsilon > 0 : Q \subset \bigcup_{i=1}^n S_i,\ S_i \subset X,\ \mathrm{diam}(S_i) < \epsilon\ (i = 1, \dots, n;\ n \in \mathbb{N})\right\}. \tag{2.9}$$

The function α is called *Kuratowski's measure of noncompactness.* Clearly

$$\alpha(Q) \leq \mathrm{diam}(Q) \quad \text{for each bounded subset } Q \text{ of } X. \tag{2.10}$$

As an immediate consequence of Definition 2.5, we obtain.

Lemma 2.6. *Let Q, Q_1 and Q_2 be bounded subsets of a complete metric space (X, d). Then:*

$$\alpha(Q) = 0 \quad \text{if and only if} \quad \overline{Q} \text{ is compact,} \tag{2.11}$$

$$\alpha(Q) = \alpha(\overline{Q}), \tag{2.12}$$

$$Q_1 \subset Q_2 \quad \text{implies} \quad \alpha(Q_1) \leq \alpha(Q_2), \tag{2.13}$$

$$\alpha(Q_1 \cup Q_2) = \max\{\alpha(Q_1), \alpha(Q_2)\}, \tag{2.14}$$

$$\alpha(Q_1 \cap Q_2) \leq \min\{\alpha(Q_1), \alpha(Q_2)\}. \tag{2.15}$$

Proof. The statements in (2.11) and (2.13) follow from Definition 2.5.

Clearly $\alpha(Q) \leq \alpha(\overline{Q})$. Let $\epsilon > 0$, S_i be a bounded subset of X with $\operatorname{diam}(S_i) < \epsilon$ for $i = 1, \dots, n$, and $Q \subset \bigcup_{i=1}^{n} S_i$. Then $\overline{Q} \subset \overline{\bigcup_{i=1}^{n} S_i} = \bigcup_{i=1}^{n} \overline{S_i}$. Since $\operatorname{diam}(S_i) = \operatorname{diam}(\overline{S_i})$, we conclude $\alpha(\overline{Q}) \leq \alpha(Q)$. This proves equality (2.12).

From (2.13), we have $\alpha(Q_1) \leq \alpha(Q_1 \cup Q_2)$ and $\alpha(Q_2) \leq \alpha(Q_1 \cup Q_2)$, and so

$$\max\{\alpha(Q_1), \alpha(Q_2)\} \leq \alpha(Q_1 \cup Q_2). \tag{2.16}$$

Let $\max\{\alpha(Q_1), \alpha(Q_2)\} = s$ and $\epsilon > 0$. By Definition 2.5 we know that Q_1 and Q_2 can be covered by a finite number of subsets of diameter smaller than $s + \epsilon$. Obviously, the union of these covers is a finite cover of $Q_1 \cup Q_2$. Hence, we have $\alpha(Q_1 \cup Q_2) \leq s + \epsilon$, and now we obtain (2.14) from (2.16). From $Q_1 \cap Q_2 \subset Q_1$ and $Q_1 \cap Q_2 \subset Q_2$ we obtain $\alpha(Q_1 \cap Q_2) \leq \alpha(Q_1)$ and $\alpha(Q_1 \cap Q_2) \leq \alpha(Q_2)$. Hence $\alpha(Q_1 \cap Q_2) \leq \min\{\alpha(Q_1), \alpha(Q_2)\}$. This proves inequality (2.15). □

The next theorem is a generalization of the well–known Cantor intersection theorem.

Theorem 2.7. (Kuratowski [41]) *Let (X, d) be a complete metric space. If (F_n) is a decreasing sequence of nonempty, closed and bounded subsets of X such that $\lim_{n\to\infty} \alpha(F_n) = 0$, then the intersection $F_\infty = \bigcap_{n=1}^{\infty} F_n$ is a nonempty and compact subset of X.*

Proof. The set F_∞ is a closed subset of X. Since $F_\infty \subset F_n$ for all $n = 1, 2, \dots$, we obtain from (2.11) and (2.1.3) that F_∞ is a compact set. Now we show $F_\infty \neq \emptyset$. Let $x_n \in F_n$ $(n = 1, 2, \dots)$ and $X_n = \{x_i : i \geq n\}$ for $n = 1, 2, \dots$. Since $X_n \subset F_n$, we obtain from (2.11), (2.13) and (2.14)

$$\alpha(X_1) = \alpha(X_n) \leq \alpha(F_n) \quad \text{for each } n. \tag{2.17}$$

The assumption of the theorem and (2.17) together imply $\alpha(X_1) = 0$, hence X_1 is a relatively compact set. Thus the sequence (x_n) has a convergent subsequence with limit $x \in X$, say. Since F_n is closed in X, we get $x \in F_n$ for all $n = 1, 2, \dots$, that is $x \in F_\infty$. □

If X is a normed space, then the function α has some additional properties connected with the vector (linear) structures of a normed space [12].

Theorem 2.8. (Darbo [12]) *Let Q, Q_1 and Q_2 be bounded subsets of a normed space X. Then:*

$$\alpha(Q_1 + Q_2) \leq \alpha(Q_1) + \alpha(Q_2), \tag{2.18}$$

$$\alpha(Q + x) = \alpha(Q) \quad \textit{for each } x \in X, \tag{2.19}$$

$$\alpha(\lambda Q) = |\lambda|\alpha(Q) \quad \textit{for each } \lambda \in \mathbb{F}, \tag{2.20}$$

$$\alpha(Q) = \alpha(\mathrm{co}(Q)). \tag{2.21}$$

Proof. Let S_i be a bounded subset of X with $\mathrm{diam}(S_i) < d$ for each $i = 1, \dots, n$ and $Q_1 \subset \bigcup_{i=1}^n S_i$. Furthermore, let G_j be a bounded subset of X with $\mathrm{diam}(G_j) < p$ for each $j = 1, \dots, m$ and $Q_2 \subset \bigcup_{j=1}^m G_j$. Then

$$Q_1 + Q_2 \subset \bigcup_{i=1}^n \bigcup_{j=1}^m (S_i + G_j) \quad \text{and} \quad \mathrm{diam}(S_i + G_j) < d + p. \tag{2.22}$$

It follows from (2.22) that $\alpha(Q_1 + Q_2) < d + p$. This shows inequality (2.18). Let $x \in X$. By (2.18) it follows that

$$\alpha(Q + x) \leq \alpha(Q) + \alpha(\{x\}) = \alpha(Q), \tag{2.23}$$

and by the same argument we have

$$\alpha(Q) = \alpha((Q + x) + (-x)) \leq \alpha(Q + x) + \alpha(\{-x\}) = \alpha(Q + x). \tag{2.24}$$

Now we obtain (2.19) from (2.23) and (2.24).

For $\lambda = 0$, equality (2.20) is obvious. Let S_i be a bounded subset of X with $\mathrm{diam}(S_i) < d$ for $i = 1, \dots, n$ and $Q_1 \subset \bigcup_{i=1}^n S_i$. Then for any $\lambda \in \mathbb{F}$, $\lambda Q \subset \bigcup_{i=1}^n \lambda S_i$ and $\mathrm{diam}(\lambda S_i) = |\lambda| \,\mathrm{diam}\, S_i$. Hence it follows that $\alpha(\lambda Q) \leq |\lambda| \,\alpha(Q)$. If $\lambda \neq 0$, analogously we have $\alpha(Q) = \alpha(\lambda^{-1}(\lambda Q)) \leq |\lambda^{-1}| \,\alpha(\lambda Q)$, that is $|\lambda|\alpha(Q) \leq \alpha(\lambda Q)$. This proves (2.20).

Now we prove (2.21). Clearly $\alpha(Q) \leq \alpha(\mathrm{co}\ Q)$, and it suffices to show $\alpha(\mathrm{co}\ Q) \leq \alpha(Q)$. Let S_i be a bounded subset of X with $\mathrm{diam}(S_i) < d$ for each $i = 1, \dots, n$ and $Q = \bigcup_{i=1}^n S_i$. By Theorem 2.1 it follows that

$$\mathrm{co}(Q) = \left\{ \sum_{i=1}^n \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1,\, x_i \in \mathrm{co}(S_i)\, (i = 1, \dots, n) \right\}. \tag{2.25}$$

Let $\epsilon > 0$ and $\mathbf{S}=\{(\lambda_1, \dots, \lambda_n) : \sum_{i=1}^n \lambda_i = 1, \lambda_i \geq 0\, (i = 1, \dots, n)\}$. Then $\mathbf{S}$ is a compact subset of $(\mathbb{R}^n, \|\cdot\|_\infty)$, where $\|(\lambda_1, \dots, \lambda_n)\|_\infty = \sup_{1 \leq i \leq n} |\lambda_i|$. We put $M = \sup\{\|x\| : x \in \bigcup_{i=1}^n \mathrm{co}(S_i)\}$. Let $\mathbf{T} = \{(t_{j,1}, \dots, t_{j,n}) : j = 1, \dots, m\} \subset \mathbf{S}$ be a finite $\epsilon/(Mn)$ -net for $\mathbf{S}$, with respect to the $\|\cdot\|_\infty$ -norm. Hence, if $\sum_{i=1}^n \lambda_i x_i$ is a convex combination of elements of Q, where we suppose that $x_i \in \mathrm{co}(S_i)$ for $i = 1, \dots, n$, then there exists $(t_{j,1}, \dots, t_{j,n}) \in \mathbf{T}$ such that

$$\|(\lambda_1, \dots, \lambda_n) - (t_{j,1}, \dots, t_{j,n})\|_\infty < \frac{\epsilon}{M} n. \tag{2.26}$$

Since

$$\sum_{i=1}^{n}\lambda_i x_i = \sum_{i=1}^{n} t_{j,i}x_i + \sum_{i=1}^{n}(\lambda_i - t_{j,i})x_i, \tag{2.27}$$

it follows from (2.25), (2.26) and (2.27) that

$$\operatorname{co}(Q) \subset \bigcup_{j=1}^{m}\left\{\sum_{i=1}^{n} t_{j,i}\ \operatorname{co}(S_i)\right\} + \frac{\epsilon}{Mn}\sum_{i=1}^{n} B_i, \tag{2.28}$$

where $B_i = \{x \in X : \|x\| \leq M\}$ for $i = 1, 2, \dots, n$. Now, by (2.4), (2.5), (2.18), (2.20), Corollary 2.3 and (2.28), we have

$$\begin{aligned}
\alpha(\operatorname{co}(Q)) &\leq \alpha\left(\bigcup_{j=1}^{m}\left\{\sum_{i=1}^{n} t_{j,i}\operatorname{co}(S_i)\right\}\right) + \alpha\left(\frac{\epsilon}{Mn}\sum_{i=1}^{n} B_i\right)\\
&\leq \max_{1\leq j\leq m}\alpha\left(\sum_{i=1}^{n} t_{j,i}\operatorname{co}(S_i)\right) + \frac{\epsilon}{Mn}\sum_{i=1}^{n}\alpha(B_i)\\
&< \max_{1\leq j\leq m}\sum_{i=1}^{n} t_{j,i}\alpha(\operatorname{co}(S_i)) + \frac{\epsilon}{Mn}2nM\\
&< d\max_{1\leq j\leq m}\sum_{i=1}^{n} t_{j,i} + 2\epsilon < d + 2\epsilon.
\end{aligned}$$

□

Let us remark that Darbo [12] proved (2.21) and, then applied it in the proof of his famous fixed point theorem [12, 1, 7, 86, 100]. His fixed point theorem is a very important generalization of the Schauder fixed point theorem, and is the first important result with applications of Kuratowski's measure of noncompactness.

Let X be an infinite–dimensional normed space and B_X the closed unit ball in X. Then, clearly $\alpha(B_X) \leq 2$, but Furi and Vignoli [18] and Nussbaum [79] have shown more precisely:

Theorem 2.9. (Furi-Vignoli [18], Nussbaum [79]) *Let X be an infinite-dimensional normed space. Then $\alpha(B_X) = 2$.*

Proof. Clearly $\alpha(B_X) \leq 2$. If $\alpha(B_X) < 2$, then there exist bounded and closed subsets Q_i of X with $\operatorname{diam}(Q_i) < 2$ for $i = 1, \dots, n$ such that $B_X \subset \bigcup_{i=1}^{n} Q_i$. Let $\{x_1, \dots\ x_n\}$ be a linearly independent subset of X and E_n be the set of all linear combinations of elements of the set $\{x_1, \dots, x_n\}$ with real coefficients. Clearly, E_n is a real n-dimensional normed space (the norm on E_n, of course, being the restriction of the norm on X). By $S_n = \{x \in E_n : \|x\| = 1\}$, we denote the unit sphere of E_n. Let us mention that $S_n \subset \bigcup_{i=1}^{n} S_n \cap Q_i$, $\operatorname{diam}(S_n \cap Q_i) < 2$ and $S_n \cap Q_i$ is a closed subset of E_n for each $i = 1, \dots\ n$. This is a contradiction to the well–known Ljusternik-Šnirelman-Borsuk theorem (see the proof in [86] or in [15, pp. 303–307]: If S_n is the unit sphere of an n-dimensional real normed space E_n, F_i a closed subset of E_n for each $i = 1, \dots, n$ and $S_n \subset \bigcup_{i=1}^n F_i$, then there exists $i_0 \in \{1, \dots, n\}$ such that the set $S_n \cap F_{i_0}$ contains a pair of antipodial points, that is, there exists $x_0 \in S_n \cap F_{i_0}$, such that $\{x_0, -x_0\} \subset S_n \cap F_{i_0}$. □

2.3. The Hausdorff measure of noncompactness. Usually it is complicated to find the exact value of $\alpha(Q)$. Another measure of noncompactness, which is more applicable in many cases, were introduced and studied by Goldenstein, Gohberg and Markus (the *ball measures of noncompactness, Hausdorff measure of noncompactness*) [19] in 1957 (later studied by Goldenstein and Markus [20] in 1968), is given in the next definition.

Definition 2.10. Let (X, d) be a metric space and Q a bounded subset of X. Then the *Hausdorff measure of noncompactness of the set* $\boldsymbol{Q}$, denoted by $\chi(Q)$ is defined to be the infimum of the set of all reals $\epsilon > 0$ such that Q can be covered by a finite number of balls of radii $< \epsilon$, that is

$$\chi(Q) = \inf\{\epsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i, r_i),\ x_i \in X,\ r_i < \epsilon\, (i = 1, \dots\ n)\, n \in \mathbb{N}\}. \tag{2.29}$$

The function χ is called *Hausdorff measure of noncompactness.*

Let us remark that in the definition of the Hausdorff measure of noncompactness of the set Q it is not supposed that centres of the balls which cover Q belong to Q. Hence, (2.29) can equivalently be stated as follows:

$$\chi(Q) = \inf\{\epsilon > 0 : Q \text{ has a finite } \epsilon\text{–net in } X\}. \tag{2.30}$$

The Hausdorff measure of noncompactness is often called *ball measure of noncompactness.* The next lemma and theorem could be proved analogously as in the case of the Kuratowski measure of noncompactness.

Lemma 2.11. *Let Q, Q_1 and Q_2 be bounded subsets of the metric space (X, d). Then*

$$\begin{aligned}
&\chi(Q) = 0 \quad \textit{if and only if} \quad Q \textit{ is totally bounded},\\
&\chi(Q) = \chi(\overline{Q}),\\
&Q_1 \subset Q_2 \quad \textit{implies} \quad \chi(Q_1) \le \chi(Q_2),\\
&\chi(Q_1 \cup Q_2) = \max\{\chi(Q_1), \chi(Q_2)\},\\
&\chi(Q_1 \cap Q_2) \le \min\{\chi(Q_1), \chi(Q_2)\}.
\end{aligned}$$

Proof. The proof is left as an exercise for the reader. □

Theorem 2.12. *Let Q, Q_1 and Q_2 be bounded subsets of the normed space X. Then*

$$\chi(Q_1 + Q_2) \le \chi(Q_1) + \chi(Q_2),$$
$$\chi(Q + x) = \chi(Q) \quad \text{for each } x \in X,$$
$$\chi(\lambda Q) = |\lambda|\chi(Q) \quad \text{for each } \lambda \in \mathbb{F}$$
$$\chi(Q) = \chi(\mathrm{co}(Q)). \tag{2.31}$$

Proof. The proof is left as an exercise for the reader. □

The next theorem shows that the functions α and χ are in some sense equivalent.

Theorem 2.13. *Let (X, d) be a metric space and Q be a bounded subset of X. Then*

$$\chi(Q) \le \alpha(Q) \le 2\chi(Q). \tag{2.32}$$

Proof. Let $\epsilon > 0$. If $\{x_1, \dots, x_n\}$ is an ϵ-net of Q, then $\{Q \cap B(x_i, \epsilon)\}_{i=1}^n$ is a cover of Q with sets of diameter $< 2\epsilon$. This shows $\alpha(Q) \le 2\chi(Q)$. To prove the left side inequality in (2.32), let us suppose that $\{S_i\}_{i=1}^k$ is a cover of Q with sets of diameter $< \epsilon$ and $y_i \in S_i$ for $i = 1, \dots k$. Now $\{y_1, \dots, y_k\}$ is an ϵ-net of Q. This proves $\chi(Q) \le \alpha(Q)$. □

Let us remark that the inequalities (2.32) are best possible in general, as an example shows. These measures are closely related to geometric properties of the space and it is possible to improve the inequality $\chi(Q) \le \alpha(Q)$ in certain spaces (see e.g. Dominguez Benavides and Ayerbe [14], Webb and Weiyu Zhao [103]). For example (see [1], [7]) in Hilbert space, $\sqrt{2}\chi(Q) \le \alpha(Q) \le 2\chi(Q)$, and in l^p for $1 \le p < \infty$, $\sqrt[p]{2}\chi(Q) \le \alpha(Q) \le 2\chi(Q)$.

Theorem 2.14. *Let X be an infinite-dimensional normed space and B_X be the closed unit ball of X. Then $\chi(B_X) = 1$.*

Proof. Obviously $\chi(B_X) \le 1$. If $\chi(B_X) = q < 1$, then we choose $\epsilon > 0$ such that $q + \epsilon < 1$. Now there exists a $(q + \epsilon)$-net of B_X, say $\{x_1, \dots, x_k\}$. Hence

$$B_X \subset \bigcup_{i=1}^{k} \{x_i + (q + \epsilon)B_X\}. \tag{2.33}$$

Now it follows from Lemmas 2.11 and 2.12 that

$$q = \chi(B_X) \le \max_{1 \le i \le k} \chi(\{x_i + (q + \epsilon)B_X\}) = (q + \epsilon)q. \tag{2.34}$$

Since $q + \epsilon < 1$, by (2.33) we have $q = 0$, that is B_X is a totally bounded set. But this is impossible since X is an infinite-dimensional space. Hence $\chi(B_X) = 1$. □

Let us remark that Theorem 2.14 follows from Theorems 2.9 and 2.13. (But we offer another proof.)

Now we shall show how to compute the Hausdorff measure of noncompactness in the spaces ℓ_p for $1 \le p < \infty$ and c_0.

Theorem 2.15. *Let Q be a bounded subset of the normed space X, where X is ℓ_p for $1 \leq p < \infty$ or c_0. If $P_n : X \mapsto X$ is the operator defined by $P_n(x_1, x_2, \dots) = (x_1, x_2, \dots, x_n, 0, 0, \dots)$ for $(x_1, x_2, \dots) \in X$; then*

$$\chi(Q) = \lim_{n\to\infty} \sup_{x\in Q} \|(I - P_n)x\|. \tag{2.35}$$

Proof. Clearly

$$Q \subset P_nQ + (I - P_n)Q. \tag{2.36}$$

It follows from Lemma 2.11, Theorem 2.12 and (2.36) that

$$\chi(Q) \leq \chi(P_nQ) + \chi((I - P_n)Q) = \chi((I - P_n)Q) \leq \sup_{x\in Q} \|(I - P_n)x\|. \tag{2.37}$$

Since the limit in (2.35) clearly exists, we have by (2.37)

$$\chi(Q) \leq \lim_{n\to\infty} \sup_{x\in Q} \|(I - P_n)x\|. \tag{2.38}$$

Now we prove the converse inequality in (2.38). Let $\epsilon > 0$ and $\{z_1, \dots, z_k\}$ be a $[\chi(Q) + \epsilon]$-net of Q. It is easy to prove that

$$Q \subset \{z_1, \dots, z_k\} + [\chi(Q) + \epsilon]B_X. \tag{2.39}$$

It follows from (2.39) that for any $x \in Q$ there exist $z \in \{z_1, \dots, z_k\}$ and $s \in B_X$ such that $x = z + [\chi(Q) + \epsilon]s$. Hence

$$\sup_{x\in Q} \|(I - P_n)x\| \leq \sup_{1\leq i\leq k} \|(I - P_n)z_i\| + [\chi(Q) + \epsilon]. \tag{2.40}$$

Finally, (2.40) implies $\lim_{n\to\infty} \sup_{x\in Q} \|(I - P_n)x\| \leq \chi(Q) + \epsilon$. □

Concerning the space $\ell_\infty(\mathbb{R})$, to the best of our knowledge is the following theorem [13, Proposition 3.5].

Theorem 2.16. (Dominguez Benavides [13]) *Let ℓ_∞ be the real normed space of bounded sequences with* sup*-norm and Q be a bounded subset of ℓ_∞. Then $\alpha(Q) = 2\chi(Q)$.*

Proof. We know that $\alpha(Q) \leq 2\chi(Q)$. Let $\epsilon > 0$ and $Q_1, \dots, Q_n$ be subsets of $\ell_\infty(\mathbb{R})$ such that $Q \subset \bigcup_{i=1}^n Q_i$ and $\operatorname{diam} Q_i < \alpha(Q) + \epsilon$. For any $k \in \mathbb{N}$ we put $\alpha_{k,i} = \inf\{x_k : (x_j) \in Q_i\}$, $\beta_{k,i} = \sup\{x_k : (x_j) \in Q_i\}$, $c_{k,i} = (\alpha_{k,i} + \beta_{k,i})/2$, $B_i = B((c_{k,i})_{k=1}^\infty, (\alpha(Q) + \epsilon)/2)$ for $i = 1, \dots, n$. It is easy to prove that $Q_i \subset B_i$. Hence $\chi(Q) \leq (\alpha(Q) + \epsilon)/2$, that is $2\chi(Q) \leq \alpha(Q)$. □

We shall prove that the Hausdorff measure of noncompactness is connected with the Hausdorff distance.

Theorem 2.17. *Let (X,d) be a metric space. Then $(\mathcal{M}_X^c, d_H)$ is a metric space.*

Proof. Clearly $d_H(S,Q) = 0$ if and only if $S = Q$, and $d_H(S,Q) = d_H(Q,S)$ for all $S, Q \in \mathcal{M}_X^c$.

To show the triangle inequality, suppose $S, Q, F \in \mathcal{M}_X^c$, $x \in S$, $y \in Q$ and $z \in F$. It is easy to prove $d(x,F) \leq d(x,y) + d(y,F) \leq d(x,y) + d_H(Q,F)$, and this implies

$$\begin{aligned} d(x,F) &\leq \inf_{y \in Q} d(x,y) + d_H(Q,F) = d(x,Q) + d_H(Q,F) \\ &\leq d_H(S,Q) + d_H(Q,F). \end{aligned} \tag{2.41}$$

Replacing x and F by z and S in (2.41), respectively, we obtain

$$d(z,S) \leq d_H(F,Q) + d_H(Q,S) \tag{2.42}$$

Finally, (2.41) and (2.42) together imply $d_H(S,F) \leq d_H(S,Q) + d_H(Q,F)$. □

Theorem 2.18. *Let (X,d) be a metric space, $Q, Q_1, Q_2 \in \mathcal{M}_X$, and $\mathcal{N}_X^c$ be the set of all nonempty and compact subsets of (X,d). Then*

$$|\chi(Q_1) - \chi(Q_2)| \leq d_H(Q_1, Q_2), \tag{2.43}$$

$$\chi(Q) = d_H(Q, \mathcal{N}_X^c). \tag{2.44}$$

Proof. Let $\epsilon > 0$ and $d = d_H(Q_1, Q_2)$. Then it follows from (2.29) and (2.1) that there exists a finite set $S \subset X$, such that

$$Q_1 \subset B(Q_2, d + \epsilon) \quad \text{and} \quad Q_2 \subset B(S, \chi(Q_2) + \epsilon). \tag{2.45}$$

Furthermore, (2.45) implies

$$Q_1 \subset B(S, d + \chi(Q_2) + 2\epsilon), \tag{2.46}$$

and so we conclude

$$\chi(Q_1) \leq \chi(Q_2) + d + 2\epsilon. \tag{2.47}$$

Now (2.43) clearly follows from (2.47).

To prove (2.44), let us remark that the inequality $\leq$ in (2.44) follows from (2.43). Therefore it suffices to show the inequality $\geq$. If $\epsilon > 0$, then there exists a finite set $F \subset X$, such that

$$Q \subset B(F, \chi(Q) + \epsilon) \quad \text{and} \quad F \subset B(Q, \chi(Q) + \epsilon). \tag{2.48}$$

Now (2.48) and (2.1) together imply $d_H(Q, \mathcal{N}_X^c) \leq d_H(Q,F) \leq \chi(Q) + \epsilon$. □

Corollary 2.19. *Let $\mathcal{N}_X^c$ be the set of all nonempty and compact subsets of a complete metric space (X, d). Then $\mathcal{N}_X^c$ is a closed subset of $(\mathcal{M}_X^c, d_H)$.*

Proof. This is an immediate consequence of (2.44). □

If the centres of the balls in Definition 2.10 are in Q we have

Definition 2.20. Let (X, d) be a metric space and Q a bounded subset of X. Then the *inner Hausdorff measure of noncompactness* of the set Q, denoted by $\chi_i(Q)$ is defined to be the infimum of the set of all reals $\epsilon > 0$ such that Q can be covered by a finite number of balls of radii $< \epsilon$ and centers in Q, that is

$$\chi_i(Q) = \inf\left\{\epsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i, r_i),\ x_i \in Q,\ r_i < \epsilon\,(i = 1, \dots\ n)\, n \in \mathbb{N}\right\}.$$

The function χ_i is called *inner Hausdorff measure of noncompactness.* Hence the formula in Definition 2.20 can equivalently be stated as follows:

$$\chi_i(Q) = \inf\{\epsilon > 0 : Q \text{ has a finite } \epsilon\text{–net in } Q\}.$$

If Q, Q_1 and Q_2 are bounded subsets of the metric space (X, d), then

$$\chi_i(Q) = 0 \quad \text{if and only if} \quad Q \quad \text{is totally bounded,}$$
$$\chi_i(Q) = \chi_i(\overline{Q}),$$

but in general

$$Q_1 \subset Q_2 \quad \text{does not imply} \quad \chi_i(Q_1) \le \chi_i(Q_2),$$

and

$$\chi_i(Q_1 \cup Q_2) \ne \max\{\chi_i(Q_1), \chi_i(Q_2)\}.$$

Let Q, Q_1 and Q_2 be bounded subset of the normed space X. Then

$$\chi_i(Q_1 + Q_2) \le \chi_i(Q_1) + \chi_i(Q_2),$$
$$\chi_i(Q + x) = \chi_i(Q) \quad \text{for each } x \in X,$$
$$\chi_i(\lambda Q) = |\lambda|\chi_i(Q) \quad \text{for each } \lambda \in \mathbb{F},$$

but in general

$$\chi_i(Q) \ne \chi_i(\text{co}(Q)).$$

In the fixed point theory in normed space (or more generally in locally convex spaces) the relation $\alpha(Q) = \alpha(\text{co}(Q))$ is of great importance. Let us remark that O. Hadžić [26], among other things, studied the inner Hausdorff measure of noncompactness in paranormed spaces. She proved under some additional conditions the inequality $\chi_i(\text{co}(Q)) \le \varphi[\chi_i(Q)]$, where $\varphi : [0, \infty) \mapsto [0, \infty)$, and, then got some fixed point theorems for multivalued mappings in general topological vector spaces.

Istrăţescu's measure of noncompactness is closely related to the Hausdorff and Kuratowski measures of noncompactness. Before we give its definition, we need to recall that a bounded subset Q of a complete metric space (X,d) is to be said ϵ–*discrete* if $d(x,y) \geq \epsilon$ for all $x, y \in Q$ with $x \neq y$. Obviously, the set Q is relatively compact if and only if every ϵ–discrete set is finite for all $\epsilon > 0$.

Definition 2.21. (Istrăţescu, [30]) Let (X,d) be a complete metric space and Q a bounded subset of X. Then the *Istrăţescu measure of noncompactness (β–measure, I –measure)* of Q, is denoted by $\beta(Q)$, and defined by

$$\beta(Q) = \inf\{\epsilon > 0 : Q \text{ has no infinite } \epsilon\text{–discrete subsets}\}.$$

The function β is called *Istrăţesku's measure of noncompactness.* Let us remark [11] that β can be defined also by

$$\beta(Q) = \sup\{\epsilon > 0 : Q \text{ contains an infinite } \epsilon\text{–discrete set}\},$$

and the above mentioned properties of α are also valid for β (see e.g. [1, 7, 11]).

Theorem 2.22. (Daneš, [11]) *Let (X,d) be a metric space and Q be a bounded subset of X. Then*

$$\chi(Q) \leq \chi_i(Q) \leq \beta(Q) \leq \alpha(Q) \leq 2\chi(Q).$$

Hence, in particular, $\frac{1}{2}\alpha(Q) \leq \beta(Q) \leq \alpha(Q)$ *and* $\chi(Q) \leq \beta(Q) \leq 2\chi(Q)$.

Now we shall point out the well–known result of Goldenštein, Gohberg and Markus [19, Theorem 1] (see also [7, Theorem 6.1.1] or [1, 1.8.1]) concerning the Hausdorff measure of noncompactness in Banach spaces with Schauder basis. Let X be a Banach space with a Schauder basis $\{e_1, e_2, \dots\}$. Then each element $x \in X$ has a unique representation $x = \sum_{i=1}^{\infty} \phi_i(x)e_i$ where the functions ϕ_i are the basis functionals. Let $P_n : X \mapsto X$ be the projector onto the linear span of $\{e_1, e_2, \dots, e_n\}$, that is $P_n(x) = \sum_{i=1}^{n} \phi_i(x)e_i$. Then, in view of the Banach-Steinhaus theorem, all operators P_n and $I - P_n$ are equibounded. Now we shall prove

Theorem 2.23. (Goldenštein, Gohberg and Markus [19]) *Let X be a Banach space with a Schauder basis $\{e_1, e_2, \dots\}$, Q be a bounded subset of X, and $P_n : X \mapsto X$ the projector onto the linear span of $\{e_1, e_2, \dots, e_n\}$. Then*

$$\begin{aligned} &\frac{1}{a}\limsup_{n\to\infty}\Big(\sup_{x\in Q}\|(I-P_n)(x)\|\Big) \leq \chi(Q) \leq \\ &\leq \inf_n \sup_{x\in Q}\|(I-P_n)(x)\| \leq \limsup_{n\to\infty}\Big(\sup_{x\in Q}\|(I-P_n)(x)\|\Big), \end{aligned} \tag{2.49}$$

where $a = \limsup_{n\to\infty}\|I - P_n\|$.

Proof. Clearly, for any natural number n we have

$$Q \subset P_nQ + (I - P_n)Q. \tag{2.50}$$

It follows from Lemma 2.11, Theorem 2.12 and (2.50) that

$$\chi(Q) \leq \chi(P_nQ) + \chi((I-P_n)Q) = \chi((I-P_n)Q) \leq \sup_{x\in Q} \|(I-P_n)(x)\|. \tag{2.51}$$

Now we obtain

$$\chi(Q) \leq \inf_n \sup_{x\in Q} \|(I-P_n)(x)\| \leq \limsup_{n\to\infty} \left(\sup_{x\in Q} \|(I-P_n)(x)\| \right), \tag{2.52}$$

Hence it suffices to show the first inequality in (2.49). Let $\epsilon > 0$ and $\{z_1, \dots, z_k\}$ be a $[\chi(Q)+\epsilon]$-net of Q. It is easy to show that $Q \subset \{z_1, \dots, z_k\} + [\chi(Q)+\epsilon]B_X$. This implies that for any $x \in Q$ there exist $z \in \{z_1, \dots, z_k\}$ and $s \in B_X$ such that $x = z + [\chi(Q)+\epsilon]s$, and so

$$\sup_{x\in Q} \|(I-P_n)(x)\| \leq \sup_{1\leq i\leq k} \|(I-P_n)(z_i)\| + [\chi(Q)+\epsilon]\, \|(I-P_n)\|.$$

This implies

$$\limsup_{n\to\infty} \Big(\sup_{x\in Q} \|(I-P_n)(x)\| \Big) \leq (\chi(Q)+\epsilon)\, \limsup_{n\to\infty} \|I-P_n\|.$$

□

Let us mention that concerning the number a in Theorem 2.23, if $X = c_0$, then $a = 1$, but if $X = c$, then $a = 2$ (see e.g. [7, p. 22]).

2.4. Operators. So far we "measured" the noncompactness of a bounded subset of a metric space. Now we "measure" the noncompactness of an operator.

Definition 2.24. Let κ_1 and κ_2 any of the measures of noncompactness defined above on the Banach spaces X and Y, respectively. An operator $L : X \mapsto Y$ is said to be $(\boldsymbol{\kappa_1}, \boldsymbol{\kappa_2})$*–bounded* if

$$L(Q) \in \mathcal{M}_Y \quad \text{for each } Q \in \mathcal{M}_X \tag{2.53}$$

and there exists a real k with $0 \leq k < \infty$ such that

$$\kappa_2(L(Q)) \leq k\kappa_1(Q) \quad \text{for each } Q \in \mathcal{M}_X. \tag{2.54}$$

If an operator L is (κ_1, κ_2)–*bounded* then the number $\|K\|_{\kappa_1,\kappa_2}$ defined by

$$\|L\|_{\kappa_1,\kappa_2} = \inf\{k \geq 0 : \kappa_2(L(Q)) \leq k\kappa_1(Q) \quad \text{for each } Q \in \mathcal{M}_X\} \tag{2.55}$$

is called $(\boldsymbol{\kappa_1}, \boldsymbol{\kappa_2})$*–operator norm of* $\boldsymbol{L}$, or $(\boldsymbol{\kappa_1}, \boldsymbol{\kappa_2})$*–measure of noncompactness of* $\boldsymbol{L}$, or simply *measures of noncompactness of* $\boldsymbol{L}$.

If $\kappa_1 = \kappa_2 = \kappa$, then we write $\|L\|_\kappa$ instead of $\|L\|_{\kappa,\kappa}$.

The next theorem is related to the Hausdorff measure of noncompactness.

Theorem 2.25. *Let X and Y be Banach spaces and $L \in B(X,Y)$. Then $\|L\|_\chi = \chi(L(S_X)) = \chi(L(B_X))$.*

Proof. We write $B = B_X$ and $S = S_X$. Since $\text{co}(S) = B_X$ and $L(\text{co}(S)) = \text{co}(L(S))$, it follows from (2.31) that

$$\chi(L(B)) = \chi(L(\text{co}(S)) = \chi(\text{co}L(S)) = \chi(L(S)), \tag{2.56}$$

hence we have by (2.55) and Theorem 2.14 $\chi(L(B)) \le \|L\|_\chi$. Now we show $\|L\|_\chi \le \chi(L(B))$. Let $Q \in \mathcal{M}$ and $\{x_i\}_{i=1}^n$ be a finite r-net of Q. Then $Q \subset \bigcup_{i=1}^n B(x_i, r)$ and obviously

$$L(Q) \subset \bigcup_{i=1}^n L(B(x_i, r)). \tag{2.57}$$

It follows from (2.57), Lemma 2.11 and Theorem 2.12 that

$$\chi(L(Q)) \le \chi\Bigg(\bigcup_{i=1}^n L(B(x_i, r))) = \chi(L(B(0,r)\Bigg) = r\chi(L(B)),$$

and we have $\chi(L(Q)) \le \chi(Q)\chi(L(B))$ □

Corollary 2.26. *Let X, Y and Z be Banach spaces, $L \in B(X,Y)$, $\tilde{L} \in B(Y,Z)$ and $\|\cdot\|_K$ the quotient norm on the Banach space $B(X,Y)/K(X,Y)$. Then $\|\cdot\|_\chi$ is a seminorm on $B(X,Y)$ and*

$$\|L\|_\chi = 0 \quad \textit{if and only if} \quad L \in K(X,Y), \tag{2.58}$$

$$\|L\|_\chi \le \|L\|, \tag{2.59}$$

$$\|L + K\|_\chi = \|L\|_\chi, \ \textit{for each } K \in K(X,Y), \tag{2.60}$$

$$\|\tilde{L} \circ L\|_\chi \le \|\tilde{L}\|_\chi \, \|L\|_\chi. \tag{2.61}$$

$$\|L\|_\chi \le \|L\|_K. \tag{2.62}$$

Proof. The proof is left as an exercise to the reader. □

The following results will give a technique for the evaluation of the Hausdorff measure of noncompactness of an operator on the space l_1.

Theorem 2.27. *We have $L \in B(l_1, l_1)$ if and only if there exists an infinite matrix $A = (a_{nk})_{n,k=0}^\infty$ of complex numbers such that*

$$\|A\| = \sup_k \sum_{n=0}^\infty |a_{nk}| < \infty \tag{2.63}$$

$$L(x) = A(x) \quad \textit{for all } x \in l_1. \tag{2.64}$$

In this case

$$\|L\| = \|A\|, \tag{2.65}$$

and the operator L *uniquely determines the matrix* $A = (a_{nk})_{n,k}^{\infty}$. *The operator* L *is said to be given (defined) by the matrix* A.

Proof. First we assume $L \in B(X,Y)$. We write $L_n = P_n \circ L$ for all n where P_n denotes the n–th coordinate, and put $a_{nk} = L_n(e^{(k)})$ for all $n, k = 0, 1, \dots$. Since l_1 is a BK space, we have $L_n \in l_1^*$ for each n and so $L_n(x) = A_n(x)$ for each n by Theorem 1.35. This yields the representation in (2.64). If we choose $x = e^{(k)}$, then

$$\|L(e^{(k)})\|_1 = \sum_{n=0}^{\infty} |L_n(e^{(k)})| = \sum_{n=0}^{\infty} |a_{nk}| \le \|L\| \|e^{(k)}\|_1 = \|L\| \quad \text{for all },$$

that is

$$\|A\| = \sup_k \sum_{n=0}^{\infty} |a_{nk}| \le \|L\| < \infty \tag{2.66}$$

and (2.63) holds. Further

$$\|L(x)\|_1 = \sum_{k=0}^{\infty} |A_n(x)| \le \sum_{k=0}^{\infty} |x_k| \sum_{n=0}^{\infty} |a_{nk}| \le \|A\| \|x\|_1 \quad \text{for all } x \in l_1, \tag{2.67}$$

and so $\|L\| \le \|A\|$. This and (2.66) together yield (2.65).

Conversely let condition (2.63) hold. Then obviously $\sup_k |a_{nk}| < \infty$ for all n, that is $A_n \in X^{\beta}$ for all n. Let $x \in l_1$. As in (2.67), we obtain $A(x) \in l_1$ by (2.63), whence $A \in (l_1, l_1)$. We define the linear operator $L : l_1 \mapsto l_1$ by (2.64). Then $L \in B(l_1, l_1)$ by Theorem 1.23 (a). □

Theorem 2.28. (Goldenštein, Gohberg and Markus [19]) *Let* $L \in B(l_1, l_1)$ *be given by an infinite matrix* $A = (a_{nk})_{n,k=0}^{\infty}$. *Then*

$$\|L\|_{\chi} = \lim_{m\to\infty} \sup_k \sum_{n=m}^{\infty} |a_{nk}|. \tag{2.68}$$

Proof. We write $S = S_{l_1}$. It follows from Theorems 2.15 and 2.27 that

$$\|L\|_{\chi} = \chi(L(S)) = \lim_{m\to\infty} \sup_{x \in S} \sum_{n=m}^{\infty} \left| \sum_{k=0}^{\infty} a_{nk} x_k \right|. \tag{2.69}$$

The limit in (2.68) obviously exists. From

$$\sup_{x\in S}\sum_{n=m}^{\infty}\left|\sum_{k=0}^{\infty}a_{nk}x_k\right| \le \sup_{x\in S}\sum_{n=m}^{\infty}\sum_{k=0}^{\infty}|a_{nk}x_k| = \sup_{x\in S}\sum_{k=0}^{\infty}\sum_{n=m}^{\infty}|a_{nk}||x_k|$$
$$\le \sup_k \sum_{n=m}^{\infty}|a_{nk}|$$

and (2.69) we obtain

$$\|L\|_\chi \le \lim_{m\to\infty}\sup_k\sum_{n=m}^{\infty}|a_{nk}|. \tag{2.70}$$

To prove the converse inequality, we choose $x = e^{(k)} \in l_1$. Since $L(e^{(k)}) = A^k = (a_{nk})_{n=0}^{\infty}$, Theorem 2.15 implies

$$\chi(\{L(e^{(k)}) : k = 0, 1, \dots\}) = \lim_{m\to\infty}\sup_k\sum_{n=m}^{\infty}|a_{nk}| \le \chi(L(S)).$$

This and inequality (2.70) together yield (2.68). □

As an immediate consequence of Theorem 2.28, we have

Corollary 2.29. *Let $L \in B(l_1, l_1)$ be given by the infinite matrix $A = (a_{nk})_{n,k=0}^{\infty}$. Then L is compact if and only if*

$$\lim_{m\to\infty}\sup_k\sum_{n=m}^{\infty}|a_{nk}| = 0.$$

Let us mention that measures of noncompactness are of special interest in *spectral theory*, the theory of *Fredholm* and *semi–Fredholm operators* (see e.g. [8, 9, 17, 19, 21, 45, 78, 87, 88, 89, 93, 94, 101, 102, 110, 115, 116].

3. Matrix domains

In this section, we shall deal with sequence spaces related to the concepts of ordinary and strong summability, spaces of sequences of differences and sequences that are strongly convergent and bounded. We shall characterize matrix transformations between these spaces and apply the Hausdorff measure of noncompactness to give necessary and sufficient conditions for these matrix maps to be compact operators. This section contains some of our recent research results which can be found in [34, 64, 65, 68, 69, 70, 71, 72, 73] and in the survey articles [32, 66, 67].

Let A be an infinite matrix and $x = (x_k)_{k=0}^{\infty}$ be a sequence. The sequence x is said to be *A–summable to $l \in \mathbb{C}$* if

$$A_n(x) = \sum_{k=0}^{\infty}a_{nk}x_k \to l \quad (n\to\infty); \quad \text{we shall write } x \to l(A).$$

This means $A_n \in x^\beta = x^{-1} * cs$ for all n and $A(x) \in c$.

The sequence x is said to be *strongly summable* A *to* $l \in \mathbb{C}$ if

$$\sum_{k=0}^{\infty} a_{nk}|x_k - l| \to 0 \quad (n \to \infty); \qquad \text{we shall write } x \to l[A].$$

The sequence x is said to be *absolutely summable* A if

$$\sum_{n=0}^{\infty} |A_n(x)| < \infty.$$

We shall mainly be interested in the first two concepts.

3.1. Ordinary and strong matrix domains. In this subsection, we define *ordinary* and *strong matrix domains* and study their topological properties.

Definition 3.1. Let X be a set of sequences and A an infinite matrix. The sets

$$X_A = \{x \in \omega : A(x) \in X\}$$

and

$$X_{[A]} = \{x \in \omega : A(|x|) = \Big(\sum_{k=0}^{\infty} a_{nk}|x_k|\Big)_{n=0}^{\infty} \in X\}$$

are called the *(ordinary) matrix domain* and *strong matrix domain of* $\boldsymbol{A}$. In the special case where $X = c$, the sets c_A and $c_{[A]}$ are called *convergence domain* and *strong convergence domain of* $\boldsymbol{A}$.

The sets c_A and $c_{[A]}$ are closely related to the concepts of ordinary and strong summability. Obviously $x \to l(A)$ if and only if $x \in c_A$ and $x \to l[A]$ if and only if $x - le \in (c_0)_{[A]}$.

It is known that the ordinary matrix domain of an FK space again is an FK space [108, Theorem 4.3.12, p. 63] or [91, Proposition 4.2.1, p. 101]. Since we shall here confine our studies to BK spaces and matrix domains of triangles, we shall only prove the special result. We need the following

Lemma 3.2. *Let X be a linear space, $(Y, \|\cdot\|)$ a normed space and $T : X \mapsto Y$ a linear one-to-one map. Then X becomes a normed space with $\|x\|_X = \|T(x)\|$. If, in addition, Y is a Banach space and T is onto Y, then $(X, \|\cdot\|_X)$ is a Banach space.*

Proof. The proof is elementary and left to the reader. □

Theorem 3.3. *Let T be a triangle and $(X, \|\cdot\|)$ be a BK space. Then X_T is a BK space with $\|x\|_T = \|T(x)\|$.*

Proof. We define the map $L_T : X_T \mapsto X$ by $L_T(x) = T(x)$ for all $x \in X_T$. Then L_T is linear, one–to–one, since T is a triangle, and onto X, since $X_T = L_T^{-1}(X)$ and L_T is one–to–one. By Lemma 3.2, X_T is a Banach space.

To show that the coordinates are continuous in X_T, let $x^{(n)} \to x$ in X_T. Then $y_k^{(n)} = T_k(x^{(n)}) \to y_k = T_k(x)$, since X is a BK space. Let S be the inverse of T, also a triangle. Then $x_k^{(n)} = \sum_{j=0}^{k} s_{kj} y_j^{(n)} \to \sum_{j=0}^{k} s_{kj} y_j = x_k$. This shows that the coordinates are continuous on X_T. □

As a special case of Theorem 3.3, we obtain

Corollary 3.4. [108, Theorem 4.3.13, p. 64] *Let T be a triangle. Then c_T is a BK space with $\|x\|_{T,\infty} = \|T(x)\|_\infty$.*

Theorem 3.5. [108, Theorem 4.3.14, p. 64] *If X is a closed subspace of Y, then X_A is a closed subspace of Y_A.*

Proof. Define the map $f : Y_A \mapsto Y$ by $f(y) = A(y)$, a continuous map. Then f_A is continuous by Theorem 1.17, and so $X_A = f^{-1}(X)$ is closed. □

A result similar to Theorem 3.3 holds for the strong matrix domains of triangles. We call a norm $\|\cdot\|$ a sequence space X *monotone*, if $|\tilde{x}_k| \le |x_k|$ $(k = 0, 1, \dots)$ implies $\|\tilde{x}\| \le \|x\|$.

Theorem 3.6. [34, Theorem 1] *Let X be a normal BK space with monotone norm $\|\cdot\|$, T a triangle and B a positive triangle. Then $X_{[B]}$ is a BK space with $\|x\|_{X_{[B]}} = \|B(|x|)\|$ for all $x \in X_{[B]}$.*

Proof. We write $\|\cdot\|' = \|\cdot\|_{X_{[B]}}$ for short. Obviously, $\|\cdot\|'$ is a norm on $X_{[B]}$. Further, since X is a BK space,

$$\|x^{(m)} - x\|' = \|B(|x^{(m)} - x|)\| \to 0 \quad (m \to \infty)$$

implies $B_n(|x^{(m)} - x|) = \sum_{k=0}^{n} b_{nk}|x_k^{(m)} - x_k| \to 0$ $(m \to \infty)$ for all n. Thus

$$|x_n^{(m)} - x_n \le \frac{1}{b_{nn}} B_n(|x^{(m)} - x|) \to 0 \quad (m \to \infty) \quad \text{for all } n.$$

Hence the norm $\|\cdot\|'$ is stronger than the metric of ω on $X_{[B]}$. Let $(x^{(m)})_{m=0}^\infty$ be a Cauchy sequence in $X_{[B]}$, hence in ω by what we have just shown. Then there is $y \in \omega$ such that

$$x^{(m)} \to y \quad \text{in } \omega. \tag{3.1}$$

Further, by the completeness of X, there is $z \in X$ such that

$$B(|x^{(m)}|) \to z \quad \text{in } X. \tag{3.2}$$

From (3.1), we conclude $x_k^{(m)} \to y_k$ $(m \to \infty)$ for each fixed k, hence $B_n(|x^{(m)}|) \to B_n(|y|)$ $(m \to \infty)$ for all n, and consequently

$$B(|x^{(m)}|) \to B(|y|) \quad \text{in } \omega. \tag{3.3}$$

Finally (3.2) and (3.3) together imply $z = B(|y|) \in X$, that is $y \in X_{[B]}$. □

There is no general method to find the Schauder basis of a matrix domain X_A or $X_{[A]}$ from that of X not even when A is a positive triangle. We give a special result which will be applied later.

Theorem 3.7. [34, Theorem 2] (a) *Let X be a BK space with basis $(b^k)_{k=0}^{\infty}$, $\mathcal{U} = \{u \in \omega : u_k \neq 0$ for all $k\}$ $u \in \mathcal{U}$ and $c^{(k)} = (1/u) \cdot b^{(k)}$ $(k = 0, 1, \dots)$ where $1/u = (1/u_k)_{k=0}^{\infty}$. Then $(c^{(k)})_{k=0}^{\infty}$ is a basis for $Y = u^{-1} * X$.*

(b) *Let $u \in \mathcal{U}$ be a sequence such that $|u_0| \leq |u_1| \leq \dots$ and $|u_n| \to \infty$ for $n \to \infty$, and T a triangle with $t_{nk} = 1/u_n$ $(0 \leq k \leq n)$ and $t_{nk} = 0$ $(k > n)$ for all $n = 0, 1, \dots$. Then $(c_0)_T$ has AK.*

Proof. (a) Let $\|\cdot\|$ be the BK norm on X. Then Y is a BK space with $\|y\|_u = \|u \cdot y\|$ $(y \in Y)$ by Theorem 3.3. Further $u \cdot c^{(k)} = b^{(k)} \in X$ $(k = 0, 1, \dots)$ implies $c^{(k)} \in Y$ $(k = 0, 1, \dots)$. Finally let $y \in Y$ be given. Then $u \cdot y = x \in X$ and $x^{\langle m \rangle} = \sum_{k=0}^{m} \lambda_k b^{(k)} \to x$ $(m \to \infty)$ in X. We put $y^{\langle m \rangle} = (1/u) \cdot x^{\langle m \rangle}$. Then $u \cdot y^{\langle m \rangle} = x^{\langle m \rangle} \to x = u \cdot y$ in$X$, hence $y^{\langle m \rangle} \to y$ in Y, that is $y = \sum_{k=0}^{\infty} \lambda_k c^{(k)}$. Obviously, this representation is unique.

(b) $(c_0)_T$ is a BK space with respect to $\|x\|_{(c_0)_T} = \sup_n \left| \dfrac{1}{u_n} \sum\limits_{k=0}^{n} x_k \right|$, by Theorem 3.3. Further $|u_n| \to \infty$ $(n \to \infty)$ implies $\phi \subset (c_0)_T$. Let $\varepsilon > 0$ and $x \in (c_0)_T$ be given. Then there is a nonnegative integer n_0 such that $|T_n(x)| < \varepsilon/2$ for all $n \geq n_0$. Let $m > n_0$. Then

$$\|x - x^{[m]}\|_{(c_0)_T} = \sup_{n \geq m+1} \left| \frac{1}{u_n} \sum_{k=m+1}^{n} x_k \right| \leq \sup_{n \geq m+1} |T_n(x) < +|T_m(x)| < \varepsilon.$$

Obviously, the representation is unique. □

3.2. Matrix transformations into matrix domains. In this subsection, we shall show that, for triangles T, the characterizations of the classes (X, Y) and $(X, Y_{[T]})$ can be reduced to that of (X, Y).

Theorem 3.8. [65, Theorem 1], [71, Proposition 3.4] *Let T be a triangle.*

(a) *Then, for arbitrary subsets X and Y of ω, $A \in (X, Y_T)$ if and only if $B = TA \in (X, Y)$.*

(b) *Further, if X and Y are BK spaces and $A \in (X, Y_T)$, then*

$$\|L_A\| = \|L_B\|. \tag{3.4}$$

Proof. (a) The proof of part (a) is straightforward and can be found in [65, Theorem 1].

(b) Let $A \in (X, Y_T)$. Since Y is a BK space and T a triangle, Y_T is a BK space with

$$\|y\|_{Y_T} = \|T(y)\|_Y \quad (y \in Y_T) \tag{3.5}$$

by Theorem 3.3. Thus A is continuous by Theorem 1.17 and consequently

$$\|L_A\| = \sup\{\|L_A(x)\|_{Y_T} : \|x\| = 1\} = \sup\{\|A(x)\|_{Y_T} : \|x\| = 1\} < \infty. \tag{3.6}$$

Further, since B is continuous,

$$\|L_B\| = \sup\{\|L_B(x)\|_Y : \|x\| = 1\} = \sup\{\|B(x)\|_Y : \|x\| = 1\} < \infty. \tag{3.7}$$

Let $x \in X$. Since $A_n \in X^\beta$ for all $n = 0, 1, \dots$, we have $x \in \omega_A$. Further $T_n \in \phi$ $(n = 0, 1, \dots)$, since T is a triangle. Thus $B(x) = (TA)(x) = T(A(x))$ (cf. [108, Theorem 1.4.4, p. 8]), and (3.4) follows from (3.5), (3.6) and (3.7). □

For the characterization of the class $(X, Y_{[T]})$, we need the following lemma.

Lemma 3.9. [81] *Let* $a_0, a_1, \dots, a_n \in \mathbb{C}$. *Then*

$$\sum_{k=0}^{n} |a_k| \le 4 \cdot \max_{N \in \{0,\dots,n\}} \left|\sum_{k \in N} a_k\right|.$$

Proof. First we consider the case where $a_0, a_1, \dots, a_n \in \mathbb{R}$. We put $N^+ = \{k \in \{0, \dots, n\} : a_k \ge 0\}$ and $N^- = \{k \in \{0, \dots, n\} : a_k < 0\}$. Then

$$\sum_{k=0}^{n} |a_k| = \left|\sum_{k \in N^+} a_k\right| + \left|\sum_{k \in N^-} a_k\right| \le 2 \cdot \max_{N \in \{0,\dots,n\}} \left|\sum_{k \in N} a_k\right|.$$

Now let $a_0, a_1, \dots, a_n \in \mathbb{C}$. We write $a_k = \alpha_k + i\beta_k$ $(k = 0, 1, \dots, n)$. For any subset N of $\{0, \dots, n\}$, we write

$$x_N = \sum_{k \in N} \alpha_k,\ y_N = \sum_{k \in N} \beta_k \quad \text{and} \quad z_N = x_N + iy_N = \sum_{k \in N} a_k.$$

Now we choose subsets N_r, N_i and N_* of $\{0, \dots, n\}$ such that

$$|x_{N_r}| = \max_{N \subset \{0,\dots,n\}} |x_N|,\ |y_{N_i}| = \max_{N \subset \{0,\dots,n\}} |y_N| \quad \text{and} \quad |z_{N_*}| = \max_{N \subset \{0,\dots,n\}} |z_N|.$$

Then, for all $N \subset \{0, \dots, n\}$, we have $|x_N|, |y_N| \le |z_{N_*}|$ and $|x_{N_r}| + |y_{N_i}| \le 2 \cdot |z_{N_*}|$. Thus, by the first part of the proof,

$$\sum_{k=0}^{n} |a_k| \le \sum_{k=0}^{n} |\alpha_k| + \sum_{k=0}^{n} |\beta_k| \le 2(|x_{N_r}| + |y_{N_i}|) \le$$
$$\le 4|z_{N_*}| = 4 \cdot \max_{N \subset \{0,\dots,n\}} \left|\sum_{k \in N} z_k\right|.$$

□

Theorem 3.10. [70, Theorem 2] *Let A be an infinite matrix, B a positive triangle. For each $m \in \mathbb{N}_0$, let N_m be a subset of the set $\{0,1,\dots,m\}$, $N = (N_m)_{m=0}^{\infty}$ the sequence of the subsets N_m and $\mathcal{N}$ the set of all such sequences N. Furthermore, for each $N \in \mathcal{N}$, we define the matrix $S^N = S^N(A)$ by*

$$s_{mk}^N = \sum_{n \in N_m} b_{mn} a_{nk} \ (m,k = 0,1,\dots).$$

Then, for arbitrary subsets X of ω and any normal set Y of sequences, $A \in (X, Y_{[B]})$ if and only if $S^N(A) \in (X,Y)$ for all sequences N in $\mathcal{N}$.

Proof. First we assume $A \in (X, Y_{[B]})$. Then $A_n \in X^\beta$ $(n = 0,1,\dots)$ implies $S_m^N \in X^\beta$ for all m and all $N \in \mathcal{N}$. For each $x \in X$, we put $y = B(|(A(x)|)$. Then $A(x) \in Y_{[B]}$, that is $y \in Y$, and

$$|S_m^N(x)| = \left|\sum_{k=0}^{\infty} s_{mk}^N x_k\right| = \left|\sum_{n \in N_m} b_{mn} \sum_{k=0}^{\infty} a_{nk} x_k\right| \le |y_m| \ (m = 0,1,\dots)$$

for all $N \in \mathcal{N}$ together imply $S^N(x) \in Y$ for all $N \in \mathcal{N}$, since Y is normal. Thus $S^N \in (X,Y)$ for all $N \in \mathcal{N}$.

Conversely we assume $S^N \in (X,Y)$ for all $N \in \mathcal{N}$. Then $S_m^N \in X^\beta$ for all m and for all $N \in \mathcal{N}$, in particular, for $N = (\{m\})_{m=0}^{\infty}$, $S_m^N = b_{mm} A_m \in X^\beta$, hence $A_m \in X^\beta$, since $b_{mm} \neq 0$. Further, let $x \in X$ be given. For every $m = 0,1,\dots$, we choose the set $N_m^{(0)} \subset \{0,\dots,\}$ such that

$$\left|\sum_{n \in N_m^{(0)}} b_{mn} A_n(x)\right| = \left|\max_{N_m \subset \{0,\dots,m\}} b_{mn} A_n(x)\right|.$$

Then, by Lemma 3.9,

$$|y_m| \le 4 \cdot \left|\sum_{n \in N_m^{(0)}} b_{mn} A_n(x)\right| = 4 \cdot \left|S^{N^{(0)}}(x)\right|.$$

By hypothesis, $S^{N^{(0)}}(x) \in Y$, and the normality of Y implies $y = B(|A(x)|) \in Y$, that is $A \in (X, Y_{[B]})$. □

3.3. Bounded and convergent difference sequences of order m. Now we apply the results of the previous subsections to *sets of of bounded and convergent sequences of order m* which may be considered as ordinary matrix domains of a certain triangle. We shall give their Schauder bases and their α– and β–duals. The results may be found in [69] and [39, 63] in the special case $m = 1$.

Let m denote a positive integer throughout and the operators $\Delta^{(m)}$, $\sum^{(m)} : \omega \mapsto \omega$ be defined by

$$(\Delta^{(1)} x)_k = \Delta^{(1)} x_k = x_k - x_{k-1}, \qquad \left(\sum^{(1)} x\right)_k = \sum_{j=0}^{k} x_j \qquad (k = 0,1,\dots),$$

$$\Delta^{(m)} = \Delta^{(1)} \circ \Delta^{(m-1)}, \qquad \sum^{(m)} = \sum^{(1)} \circ \sum^{m-1} \qquad (m \ge 2).$$

We shall write $\Delta = \Delta^{(1)}$ for short and use the convention that any term with a negative subscript is equal to naught. For any subset X of ω let

$$X(\Delta^{(m)}) = \{x \in \omega : \Delta^{(m)} x \in X\}.$$

We shall be interested in the cases where $X = c_0$, $X = c$ or $X = l_\infty$. The following results are well known and can be found in [27]:

$$(\Delta^{(m)} x)_k = \sum_{j=0}^{m} (-1)^j \binom{m}{j} x_{k-j} = \sum_{j=\max\{0,k-m\}}^{m} (-1)^{k-j} \binom{m}{k-j} x_j \quad (k = 0, 1, \dots), \tag{3.8}$$

$$\left(\sum^{(m)} x\right)_k = \sum_{j=0}^{k} \binom{m+k-j-1}{k-j} x_j \qquad (k = 0, 1, \dots), \tag{3.9}$$

$$\sum^{(m)} \circ \Delta^{(m)} = \Delta^{(m)} \circ \sum^{(m)} = \text{id}, \quad \text{the identity on } \omega, \tag{3.10}$$

$$\sum_{j=0}^{k} \binom{m+j-1}{j} = \binom{m+k}{k} \quad (k = 0, 1, \dots) \tag{3.11}$$

$$\begin{cases} \text{there are positive constants } M_1,\ M_2 \text{ such that} \\ M_1 k^m \le \dbinom{m+k}{k} \le M_2 k^m \text{ for all } k = 1, 2 \dots \end{cases} \tag{3.12}$$

As an immediate consequence of Example 1.13 and Theorems 3.3 and 3.5, we obtain

Corollary 3.11. [69, Proposition 1] *Let m be a positive integer. Then the sets $l_\infty(\Delta^{(m)})$, $c(\Delta^{(m)})$ and $c_0(\Delta^{(m)})$ are BK spaces with $\|\cdot\|$ defined by*

$$\|x\| = \sup_k \left| \left(\Delta^{(m)} x\right)_k \right| = \sup_k \left| \sum_{j=0}^{m} (-1)^j \binom{m}{j} x_{k-j} \right|$$

and $c_0(\Delta^{(m)})$ and $c(\Delta^{(m)})$ are closed subspaces of $l_\infty(\Delta^{(m)})$.

Now we shall give Schauder bases for the spaces $c_0(\Delta^{(m)})$ and $c(\Delta^{(m)})$.

Theorem 3.12. [69, Theorem 1] *Let m be a positive integer. We define the sequences $b^k(m)$ by*

$$b_n^{(-1)}(m) = \binom{m+n}{n} \quad (n = 0, 1, \dots),$$
$$b_n^{(k)}(m) = \begin{cases} 0 & (n \le k-1) \\ \binom{m+n-k-1}{n-k} & (n \ge k) \end{cases} \quad (k = 0, 1, \dots).$$

(a) *Then $(b^{(k)}(m))_{k=0}^{\infty}$ is a basis of $c_0(\Delta^{(m)})$. More precisely, every sequence $x = (x_k)_{k=0}^{\infty} \in c_0(\Delta^{(m)})$ has a unique representation*

$$x = \sum_{k=0}^{\infty} \lambda_k(m) b^{(k)}(m) \ \ \textit{where } \lambda_k(m) = (\Delta^{(m)} x)_k \quad (k = 0, 1, \dots). \tag{3.13}$$

(b) *Then $(b^{(k)}(m))_{k=-1}^{\infty}$ is a basis of $c(\Delta^{(m)})$. More precisely, every sequence $x = (x_k)_{k=0}^{\infty} \in c(\Delta^{(m)})$ has a unique representation*

$$x = l b^{(-1)}(m) + \sum_{k=0}^{\infty} (\lambda_k(m) - l) b^{(k)}(m) \ \ \textit{where } l = \lim_{k \to \infty} (\Delta^{(m)} x)_k. \tag{3.14}$$

Proof. (a) For $k = 0, 1, \dots,$ we put

$$b^{(k)} = e - \sum_{p=0}^{k-1} e^{(p)}, \text{ that is } b_j^{(k)} = \begin{cases} 0 & (j \le k-1) \\ 1 & (j \ge k). \end{cases}$$

Then by (3.9) and (3.11),

$$\left(\sum^{(m-1)} b^{(k)} \right)_n = \sum_{j=0}^{n} \binom{m-1+n-j-1}{n-j} b_j^{(k)}$$
$$= \begin{cases} \sum_{j=k}^{n} \binom{m-1+n-j-1}{n-j} & (n \ge k) \\ 0 & (n \le k-1), \end{cases}$$
$$\sum_{j=k}^{n} \binom{m-1+n-j-1}{n-j} = \sum_{l=0}^{n-k} \binom{m-1+l-1}{l} = \binom{m-1+n-k}{n-k} \quad (n \ge k),$$

hence $b^{(k)}(m) = \sum^{(m-1)} b^{(k)} \quad (k = 0, 1, \dots)$, and by (3.10),

$$\Delta^{(m)} b^{(k)}(m) = \Delta b^{(k)} = e^{(k)} \in c_0 \quad (k = 0, 1, \dots) \tag{3.15}$$

Thus

$$b^{(k)}(m) \in c_0(\Delta^{(m)}) \quad (k = 0, 1, \dots). \tag{3.16}$$

Let $x = (x_k)_{k=0}^{\infty} \in c_0(\Delta^{(m)})$ be given. For every nonnegative integer p, we put $x^{\langle p\rangle} = \sum_{k=0}^{p} \lambda_k(m) b^{(k)}(m)$. Then by the linearity of $\Delta^{(m)}$ and by (3.15)

$$\Delta^{(m)} x^{\langle p\rangle} = \sum_{k=0}^{p} \lambda_k(m) \Delta^{(m)} b^{(k)}(m) = \sum_{k=0}^{p} (\Delta^{(m)} x)_k e^{(k)}$$

$$\Delta^{(m)}(x - x^{\langle p\rangle})_n = \begin{cases} 0 & (n \le p) \\ (\Delta^{(m)} x)_n & (n \ge p+1). \end{cases}$$

Given $\varepsilon > 0$ there is an integer p_0 such that $|(\Delta^{(m)} x)_p| < \varepsilon/2$ for all $p \ge p_0$, hence

$$\|x - x^{\langle p\rangle}\| = \sup_{n \ge p} |(\Delta^{(m)} x)_n| \le \sup_{n \ge p_0} |(\Delta^{(m)})_n| \le \varepsilon/2 < \varepsilon$$

for all $p \ge p_0$. This proves the representation in (3.13). To show the uniqueness of this representation we assume $x = \sum_{k=0}^{\infty} \mu_k b^{(k)}$. Since $\Delta^{(m)} : c_0(\Delta^{(m)}) \to c_0$ obviously is a continuous linear operator, we have by (3.15)

$$(\Delta^{(m)} x)_n = \sum_{k=0}^{\infty} \mu_k (\Delta^{(m)} b^{(k)})_n = \sum_{k=0}^{\infty} \mu_k e_n^{(k)} = \mu_n \quad (n = 0, 1, \dots).$$

(b) First $b^{(-1)}(m) = \sum^{(m)} e$ implies $\Delta^{(m)} b^{(-1)}(m) = e \in c$, that is $b^{(-1)}(m) \in c(\Delta^{(m)})$. In view of (3.16) and the fact that $c_0(\Delta^{(m)}) \subset c(\Delta^{(m)})$, we have $b^{(k)}(m) \in c(\Delta^{(m)})$ for all $k = -1, 0, 1, \dots$. Let $x = (x_k)_{k=0}^{\infty} \in c(\Delta^{(m)})$ be given. Then there is a unique number l such that (3.14) holds. We put $y = x - l \cdot b^{(-1)}(m)$. Then

$$\Delta^{(m)} y = \Delta^{(m)}(x - l b^{(-1)}(m)) = \Delta^{(m)} x - le, \text{ that is } y \in c_0(\Delta^{(m)}),$$

and it follows from part (a) that x has a unique representation (3.14). □

Now we shall give the α–duals of the sets $c_0(\Delta^{(m)})$, $c(\Delta^{(m)})$ and $l_\infty(\Delta^{(m)})$. If $u \in \mathcal{U}$, then obviously

(3.17) $\quad (u^{-1} * X)^\dagger = (1/u)^{-1} * X^\dagger \quad (\dagger \in \{\alpha, \beta\})$ for every subset X of ω.

Theorem 3.13. [69, Theorem 2] *Let m be a positive integer.*
(a) *We put $M^\alpha(m) = \{a \in \omega : \sum_{k=0}^{\infty} |a_k| k^m < \infty\}$. Then*

(3.18) $$\left(c_0(\Delta^{(m)})\right)^\alpha = \left(c(\Delta^{(m)})\right)^\alpha = \left(l_\infty(\Delta^{(m)})\right)^\alpha = M^\alpha(m).$$

(b) *We put $M^{\alpha\alpha}(m) = \{a \in \omega : \sup_{k \ge 1} |a_k| k^{-m} < \infty\}$. Then*

(3.19) $$\left(c_0(\Delta^{(m)})\right)^{\alpha\alpha} = \left(c(\Delta^{(m)})\right)^{\alpha\alpha} = \left(l_\infty(\Delta^{(m)})\right)^{\alpha\alpha} = M^{\alpha\alpha}(m).$$

Proof. (a) First we assume $a \in M^\alpha(m)$. Then

$$\sum_{k=0}^{\infty} |a_k| k^m < \infty. \tag{3.20}$$

Let $x \in l_\infty(\Delta^{(m)})$. Then there is a positive constant M such that $|(\Delta^{(m)}x)_k| \le M$ $(k = 0, 1, \dots)$, and by (3.10), (3.9), (3.11), (3.12) and (3.20)

$$\begin{aligned}\sum_{k=0}^{\infty} |a_k x_k| &= \sum_{k=0}^{\infty} |a_k| \left| \left(\sum^{(m)} \left(\Delta^{(m)} x \right) \right)_k \right| \\ &\le \sum_{k=0}^{\infty} |a_k| \sum_{j=0}^{k} \binom{m+k-j-1}{k-j} \left| \left(\Delta^{(m)} x \right)_j \right| \\ &\le M \sum_{k=0}^{\infty} |a_k| \binom{m+k}{k} \le M \cdot M_2 \sum_{k=0}^{\infty} |a_k| k^m < \infty.\end{aligned}$$

Thus we have shown

$$M^\alpha(m) \subset \left(l_\infty(\Delta^{(m)}) \right)^\alpha. \tag{3.21}$$

Conversely let $a \notin M^\alpha(m)$. By (3.12), there is a sequence $(k(s))_{s=0}^\infty$ of integers $0 = k(0) < k(1) < \dots$ such that

$$\sum_{k=k(s)}^{k(s+1)-1} |a_k| \binom{m+k}{k} \ge s+1 \quad (s = 0, 1, \dots). \tag{3.22}$$

We define the sequence x by

$$x_k = \sum_{l=0}^{s-1} \frac{1}{l+1} \sum_{j=k(l)}^{k(l+1)-1} \binom{m+k-j-1}{k-j} + \frac{1}{s+1} \sum_{j=k(s)}^{k} \binom{m+k-j-1}{k-j}$$
$$(k(s) \le k \le k(s+1)-1;\, s = 0, 1, \dots).$$

If we define the sequence $y \in c_0$ by $y_k = 1/(s+1)$ for $k(s) \le k \le k(s+1) - 1$ $(s = 0, 1, \dots)$, then it easily follows from (3.9) that $x = \sum^{(m)} y$. Thus $\Delta^{(m)} x = y \in c_0$ and $x \in c_0(\Delta^{(m)})$. On the other hand by (3.11) and (3.22)

$$\begin{aligned}\sum_{k=k(s)}^{k(s+1)-1} |a_k x_k| &\ge \sum_{k=k(s)}^{k(s+1)-1} |a_k| \frac{1}{s+1} \sum_{j=0}^{k} \binom{m+j-1}{j} \\ &= \frac{1}{s+1} \sum_{k=k(s)}^{k(s+1)-1} |a_k| \binom{m+k}{k} \ge 1 \quad (s = 0, 1, \dots).\end{aligned}$$

Thus $a \notin c_0(\Delta^{(m)})$, and we have shown

$$\left(c_0(\Delta^{(m)})\right)^{\alpha} \subset M^{\alpha}(m). \tag{3.23}$$

Since $c_0(\Delta^{(m)}) \subset c(\Delta^{(m)}) \subset l_\infty(\Delta^{(m)})$, (3.18) follows from (3.21) and (3.23).

(b) Since $y = (x_{k+1})_{k=0}^\infty \in M^\alpha(m)$ if and only if $y \in ((k+1)^m)_{k=0}^\infty)^{-1} * l_1$, and since $l_1^\alpha = l_\infty$, identity (3.19) follows from (3.17) and part (a). □

To determine the β–duals of the sets $c_0(\Delta^{(m)})$, $c(\Delta^{(m)})$ and $l_\infty(\Delta^{(m)})$, we need a few results.

Lemma 3.14. [69, Lemma 1] *Let m be a positive integer. Then for arbitrary sequences a*

$$\left(a_k\binom{m+k}{k}\right)_{k=1}^\infty \in cs \quad \textit{if and only if} \quad \left(a_k k^m\right)_{k=1}^\infty \in cs.$$

Proof. We define the sequences b and c by

$$b_k = \frac{\binom{m+k}{k}}{k^m} \quad (k = 1, 2, \dots) \quad \text{and} \quad c = 1/b.$$

Since $cs^\beta = bv$ [108, Theorem 7.3.5(v), p. 110], it suffices to show that $b, c \in bv$. It is well known that $\lim_{k\to\infty} b_k = 1/m!$ [27, p. 97]. Therefore we have to show that b is monotone. We define the function f on $[0, 1/2]$ by $f(x) = (1+mx)(1-x)^m$. Then $f'(x) = -m(1-x)^{m-1}(m+1)x \leq 0$ for all $x \in [0, 1/2]$, whence $f(x) \leq f(0) = 1$ for all $x \in [0, 1/2]$. Thus

$$\frac{b_{k+1}}{b_k} = \frac{k+1+m}{k+1}\,\frac{k^m}{(k+1)^m} = \left(1 + \frac{m}{k+1}\right)\left(1 - \frac{1}{k+1}\right)^m = f\left(\frac{1}{k+1}\right) \leq 1$$

for all $k \geq 1$. □

Lemma 3.15. [63, Lemma 1] *Let (P_n) be a sequence of non decreasing positive reals. Then $y \in cs$ implies*

$$\lim_{n\to\infty}\left(P_n \sum_{k=1}^\infty \frac{y_{n+k-1}}{P_{n+k}}\right) = 0.$$

Proof. The proof can be found in [39, Lemma 3] and [63, Lemma 1]. □

Corollary 3.16. [63, Corollary 1] *Let $(P_n)_{n=1}^\infty$ be a sequence of nondecreasing positive reals. Then $a \in (P_n)^{-1} * cs$ implies $R \in (P_n)^{-1} * c_o$ where $R_n = \sum_{k=n+1}^\infty a_k$ $(n = 1, 2, \dots)$.*

Proof. Put $y_k = P_{k+1}a_{k+1}$ $(k = 1, 2, \dots)$ in Lemma 3.15. □

We shall frequently apply the following two versions of Abel's summation by parts: Let $b, c \in \omega$. We put

$$s = s(c) = \sum^{(1)} c \quad \text{and, if } c \in cs,\ R_k = R_k(c) = \sum_{j=k}^{\infty} c_k \ (k = 0, 1, \dots).$$

Then

$$\sum_{k=0}^{n} b_k c_k = -\sum_{k=0}^{n} s_k \Delta b_{k+1} + s_n b_{n+1} \quad (n = 0, 1, \dots) \tag{3.24}$$

$$\sum_{k=0}^{n} b_k c_k = \sum_{k=0}^{n} R_k \Delta b_k - b_n R_{n+1} \quad (n = 0, 1, \dots). \tag{3.25}$$

Theorem 3.17. [69, Theorem 3] *Let m be a positive integer.*

(a) *We put $R_k^1 = R_k = \sum_{j=k}^{\infty} a_j$, $R_k^{(m)} = \sum_{j=k}^{\infty} R_j^{(m-1)}$ $(k = 0, 1, \dots)$ for $m \geq 2$ and*

$$M_{\infty}^{\beta}(m) = \left\{ a \in \omega : \sum_{k=0}^{\infty} a_k k^m \text{ converges and } \sum_{k=0}^{\infty} |R_k^{(m)}| < \infty \right\}.$$

Then

$$\left(c(\Delta^{(m)})\right)^{\beta} = \left(l_{\infty}(\Delta^{(m)})\right)^{\beta} = M_{\infty}^{\beta}(m). \tag{3.26}$$

(b) *Further, let c_0^+ denote the set of all positive sequences in c_0. We put*

$$M_0^{\beta}(m) = \left\{ a \in \omega : \sum_{k=0}^{\infty} a_k \sum_{j=0}^{k} \binom{m+k-j-1}{k-j} v_j \text{ converges for all } v \in c_0^+ \right\}$$
$$\bigcap \left\{ a \in \omega : \sum_{k=0}^{\infty} |R_k^{(m)}| < \infty \right\}.$$

Then

$$\left(c_0(\Delta^{(m)})\right)^{\beta} = M_0^{\beta}(m). \tag{3.27}$$

Proof. (a) For all positive integers p let $s^{(p)} = \sum^{(p)} e$; we write $s = s^{(1)}$. First we assume $m = 1$ and write $M_{\infty}^{\beta} = M_{\infty}^{\beta}(1)$. Let $a \in M_{\infty}^{\beta}$. Then

$$R \in l_1 \tag{3.28}$$

$$as \in cs \tag{3.29}$$

Now condition (3.29) and Corollary 3.16 together imply

$$\big(R_{n+1}s_n\big)_{n=0}^{\infty} \in c_0. \tag{3.30}$$

Let $x \in l_\infty(\Delta)$. From (3.25) with $b = x$ and $c = a$, we have

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} R_k(\Delta x)_k - R_{n+1}x_n \ (n = 0, 1, \dots). \tag{3.31}$$

Since $\Delta x \in l_\infty$, condition (3.28) implies

$$R\Delta x \in cs. \tag{3.32}$$

Further there is a constant $M > 0$ such that $|(\Delta x)_k| \leq M$ for all k and so $|x_n| \leq |(\sum^{(1)}(\Delta^{(1)}x))_k| \leq M\dot{(}n + 1) = M\dot{s}_n$ for all n. Now condition (3.30) implies

$$\big(R_{n+1}x_n\big)_{n=0}^{\infty} \in c_0. \tag{3.33}$$

Finally (3.31), (3.32) and (3.33) together imply $ax \in cs$ for all $x \in l_\infty$, that is $a \in (l_\infty(\Delta))^\beta$.

Conversely, let $a \in (c(\Delta))^\beta$. Then $ax \in cs$ for all $x \in c(\Delta)$. First $e \in c(\Delta)$ implies $a = ae \in cs$, hence the sequence R is defined. Further, for $x = s$, we have $\Delta x = e \in c$, that is $x \in c(\Delta)$, and condition (3.29) holds. By Corollary 3.16, we have (3.30), and again this yields (3.33) for all $x \in c(\Delta)$. From (3.31) we conclude $R\Delta x \in cs$ for all $x \in c(\Delta)$, and so $R \in c^\beta = l_1$. Now we assume that identity (3.26) holds for some integer $m \geq 1$. Let $a \in M_\infty^\beta(m + 1)$. Then by

$$R^{(m+1)} = R^m(R) \in l_1. \tag{3.34}$$

and, by Lemma 3.14,

$$as^{(m+1)} \in cs \tag{3.35}$$

Applying identity (3.24) with $b = R$ and $c = s^{(m)}$ we obtain

$$\sum_{k=0}^{n} s_k^{(m)} R_k = \sum_{k=0}^{n} a_k s_k^{(m+1)} + R_{n+1}s_n^{(m+1)} \quad (n = 0, 1, \dots). \tag{3.36}$$

By Corollary 3.16, condition (3.35) implies

$$\big(R_{n+1}s_n^{(m+1)}\big)_{n=0}^{\infty} \in c_0 \tag{3.37}$$

and consequently by (3.36)

$$s^{(m)}R \in cs \tag{3.38}$$

Now, by assumption, (3.34) and (3.38) together imply

$$R \in \left(l_\infty(\Delta^{(m)})\right)^\beta. \tag{3.39}$$

Let $x \in l_\infty(\Delta^{(m+1)})$ be given. Since $x \in l_\infty(\Delta^{(m+1)})$ if and only if $y = \Delta x \in l_\infty(\Delta^{(m)})$, condition (3.39) implies

$$R\Delta x \in cs \quad \text{for all } x \in l_\infty(\Delta^{(m+1)}). \tag{3.40}$$

Further there is a positive constant M such that $|(\Delta^{(m+1)}x)_j| \leq M$ $(j = 0, 1, \dots)$ and thus

$$\begin{aligned} |x_k| &= \left| \left(\sum^{(m+1)} (\Delta^{(m+1)}(x)) \right)_k \right| \leq \sum_{j=0}^{k} \binom{m+k-j}{k-j} \left| \left(\Delta^{(m+1)} x \right)_j \right| \\ &\leq M \cdot \sum_{j=0}^{k} \binom{m+k-j}{k-j} = M \left(\sum^{(m+1)} e \right)_k = M \cdot s_k^{(m+1)} \end{aligned}$$

for $k = 0, 1, \dots$, and condition (3.37) implies

$$\left(R_{n+1}x_n\right)_{n=0}^\infty \in c_0. \tag{3.41}$$

Finally (3.31), (3.40) and (3.41) together imply $ax \in cs$ for all $x \in l_\infty(\Delta^{(m+1)})$, consequently $a \in \left(l_\infty(\Delta^{(m+1)})\right)^\beta$.

Conversely let $a \in \left(c(\Delta^{(m+1)})\right)^\beta$. Then $ax \in cs$ for all $x \in c(\Delta^{(m+1)})$. First, $e \in c(\Delta^{(m+1)})$ implies $a = ae \in cs$, hence the sequence R is defined. Further for $x = s^{(m+1)}$ we have $\Delta^{(m+1)}x = \Delta^{(m+1)}(\sum^{(m+1)} e) = e \in c$, that is $x \in c(\Delta^{(m+1)})$, and condition (3.35) is satisfied. By Corollary 3.16, we have (3.37) and again this yields (3.41) for all $x \in c(\Delta^{(m+1)})$. From (3.31) we conclude $R\Delta x \in cs$ for all $x \in c(\Delta^{(m+1)})$ and consequently $R \in (c(\Delta^{(m)}))^\beta$. This implies $R^{(m+1)} = R^{(m)}(R) \in l_1$ by assumption.

(b) For all positive integers p and all sequences $v \in c_0$, let $t^{(p)}(v) = \sum^{(p)}(v)$; we write $t(v) = t^{(1)}(v)$. The proof of part (b) is exactly the same as that of part (a) with s, $s^{(m)}$ and $s^{(m+1)}$ replaced by $t(v)$, $t^{(m)}(v)$ and $t^{(m+1)}(v)$. □

Remark 3.18. By [63, Theorem 2 (c)] it is obvious that $(c_0(\Delta^{(m)}))^\beta \neq (l_\infty(\Delta^{(m)}))^\beta$.

3.4. Matrix transformations in the spaces $c_0(\Delta^{(m)})$, $c(\Delta^{(m)})$ and $l_\infty(\Delta^{(m)})$ and their measures of noncompactness. In this subsection we shall characterize matrix transformations between the spaces of bounded and convergent m–th order difference sequences and apply the Hausdorff measure of noncompactness to give necessary and sufficient conditions for these matrix maps to be compact operators.

Lemma 3.19. [69, Lemma 4] *Let m be a positive integer Then*

$$\|a\|^* = \|R^{(m)}\|_1 = \sum_{k=0}^{\infty} |R_k^{(m)}| \tag{3.42}$$

on any of the spaces $(c_0(\Delta^{(m)}))^\beta$, $(c(\Delta^{(m)}))^\beta$ and $(l_\infty(\Delta^{(m)}))^\beta$.

Proof. Let X be any of the sequences c_0, c or l_∞. If $m = 1$ and $a \in (X(\Delta^{(1)}))^\beta$, then

$$\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} R_k^{(k)} \Delta^{(1)} x_k \quad \text{for all } x \in X(\Delta^{(1)}) \tag{3.43}$$

by the proof of Theorem 3.17. Since $x \in X(\Delta^{(1)})$ if and only if $\Delta^{(1)}x \in X$, this implies $R^{(1)} \in X^\beta = l_1$. It is well known that $\| \cdot \|^* = \| \cdot \|_1$ on X^β, and (3.42) follows from the definition of the norm on $X(\Delta^{(1)})$.

Now we assume that (3.42) holds for some integer $m \geq 1$. Let $a \in X(\Delta^{(m+1)})$. Again, by the proof of Theorem 3.17, (3.43) holds for all $x \in X(\Delta^{(m+1)})$. Since $x \in X(\Delta^{(m+1)})$ if and only if $\Delta^{(1)} \in X(\Delta^{(m)})$, this implies $R^{(1)} \in (X(\Delta^{(m)}))^\beta$, and by assumption $\|a\|^* = \|R^{(m)}(R^{(1)})\|_1 = \|R^{(m+1)}\|_1$. □

Theorem 3.20. [69, Theorem 4] *Let m be a positive integer and A be an infinite matrix. For each n, we put $R_{nk}^{(1)} = R_{nk} = \sum_{j=k}^{\infty} a_{nj}$ and $R_n^{(m)} = \sum_{j=k}^{\infty} R_{nj}^{(m-1)}$ for $m \geq 2$.*

(a) *Then $A \in (l_\infty(\Delta^{(m)}), l_\infty)$ if and only if*

$$\sum_{k=0}^{\infty} k^m a_{nk} \quad \textit{converges for all } n = 0, 1, \ldots \tag{3.44}$$

and

$$\sup_n \sum_{k=0}^{\infty} |R_{nk}^{(m)}| < \infty. \tag{3.45}$$

Further $(l_\infty(\Delta^{(m)}), l_\infty) = (c(\Delta^{(m)}), l_\infty)$.

(b) *Then $A \in (c_0(\Delta^{(m)}), l_\infty)$ if and only if condition (3.45) holds and*

$$\sum_{k=0}^{\infty} a_{nk} \sum_{j=0}^{k} \binom{m+k-j-1}{k-j} v_j \quad \textit{converges for all } v \in c_0^+ \ \textit{and for all } n = 0, 1, \ldots. \tag{3.46}$$

(c) *Then $A \in (c_0(\Delta^{(m)}), c_0)$ if and only if conditions (3.45) and (3.46) hold and*

$$\lim_{n \to \infty} \left(\sum_{j=k}^{\infty} \binom{m-1+j-k}{j-k} a_{nj} \right) = 0 \quad (k = 0, 1, \ldots). \tag{3.47}$$

(d) *Then* $A \in (c_0(\Delta^{(m)}), c)$ *if and only if conditions (3.45) and (3.46) hold and*

$$\lim_{n\to\infty}\left(\sum_{j=k}^{\infty}\binom{m-1+j-k}{j-k}a_{nj}\right) = l_k \quad (k=0,1,\dots). \tag{3.48}$$

(e) *Then* $A \in (c(\Delta^{(m)}), c_0)$ *if and only if conditions (3.45), (3.46), (3.47) hold and*

$$\lim_{n\to\infty}\left(\sum_{j=0}^{\infty}\binom{m+j}{j}a_{nj}\right) = 0. \tag{3.48}$$

(f) *Then* $A \in (c(\Delta^{(m)}), c)$ *if and only if conditions (3.45), (3.46), (3.48) hold and*

$$\lim_{n\to\infty}\left(\sum_{j=0}^{\infty}\binom{m+j}{j}a_{nj}\right) = l_{-1}. \tag{3.50}$$

Proof. (a) Let $A \in (l_\infty(\Delta^{(m)}), l_\infty)$. Then $A_n \in (l_\infty(\Delta^{(m)}))^\beta$ for $n = 0,1,\dots$, and, by Theorem 3.17 (a), condition (3.44) holds for all n and $\sum_{n=0}^\infty |R_{nk}^{(m)}| < \infty$ $(n = 0,1,\dots)$. Further $\|A\|^* = \sup_n(\sum_{k=0}^\infty |R_{nk}^{(m)}|) < \infty$ by Theorem 1.23 (b) and Lemma 3.19. Conversely let conditions (3.44) and (3.45) hold. By Theorem 3.17 (a), this implies $A_n \in (l_\infty(\Delta^{(m)}))^\beta$ for all n, and again Theorem 1.23 (b) and Lemma 3.19 together imply $A \in (l_\infty(\Delta^{(m)}), l_\infty)$. Clearly $(l_\infty(\Delta^{(m)}), l_\infty) \subset (c(\Delta^{(m)}), l_\infty)$. If $A \in (c(\Delta^{(m)}), l_\infty)$, then condition (3.45) follows from Theorems 1.23 (b) and 3.17 (a) and Lemma 3.19. Further $A_n \in (c(\Delta^{(m)}))^\beta = (l_\infty(\Delta^{(m)}))^\beta$ and condition (3.44) holds (see Theorem 3.17 (a)). Therefore $A \in (l_\infty(\Delta^{(m)}), l_\infty)$ by what we have shown above.

(b) The proof of part (b) is exactly the same as that of the first part of part (a) with condition (4.3) and Theorem 3.17 (a) replaced by condition (3.46) and Theorem 3.17 (b).

Parts (c) to (f) follow from Theorem 1.23 (c) and parts (a) or (b), since $c_0(\Delta^{(m)})$ and $c(\Delta^{(m)})$ are closed subspaces of $l_\infty(\Delta^{(m)})$ by Corollary 3.11. □

As a corollary of Theorems 1.23 and 3.8, we have

Corollary 3.21. ([71, Corollary 3.5] *Let* X *be a BK space.*

(a) *Then* $A \in (X, l_\infty(\Delta^{(m)}))$ *if and only if*

$$M(X, l_\infty(\Delta^{(m)})) = \sup_n \left\| \sum_{l=\max\{0,n-m\}}^{n} (-1)^{n-l}\binom{m}{n-l}A_l \right\|^* < \infty. \tag{3.51}$$

(b) *Further, if* $(b^k)_{k=0}^\infty$ *is a basis of* X*, then* $A \in (X, c_0(\Delta^{(m)}))$ *if and only if condition (3.51) holds and*

$$\lim_{n\to\infty}\left(\sum_{l=\max\{0,n-m\}}^{n} (-1)^{n-l}\binom{m}{n-l}A_l(b^{(k)})\right) = 0 \quad \textit{for each } k; \tag{3.52}$$

$A \in (X, c(\Delta^{(m)}))$ *if and only if condition (3.51) holds and*

$$(3.53)\quad \lim_{n\to\infty}\left(\sum_{l=\max\{0,n-m\}}^{n}(-1)^{n-l}\binom{m}{n-l}A_l(b^{(k)})\right) = \alpha_k \quad \text{for each } k = 0, 1, \dots;$$

Remark 3.22. (a) If $X = l_p$ $(1 \le p < \infty)$ and Y is any of the spaces $l_\infty(\Delta^{(m)})$, $c(\Delta^{(m)})$ and $c_0(\Delta^{(m)})$, then the conditions for $A \in (X, Y)$ follow from the respective ones in Corollary 3.21 by replacing the norm $\|\cdot\|^*$ in condition (3.51) by the natural norm on the β–dual of l_p, that is on l_q $(q = p/(p-1), 1 < p < \infty; q = \infty, p = 1)$ which is norm isomorphic to l_p^*. Hence we have

$$M(l_p, l_\infty(\Delta^{(m)})) = \begin{cases} \sup_n\left(\sum_{k=0}^{\infty}\left|\sum_{l=\max\{0,n-m\}}^{n}(-1)^{n-l}\binom{m}{n-l}a_{lk}\right|^q\right) & (1 < p < \infty) \\ \sup_{n,k}\left|\sum_{l=\max\{0,n-m\}}^{n}(-1)^{n-l}\binom{m}{n-l}a_{lk}\right| & (p = 1). \end{cases}$$

(b) Let s be a nonnegative integer. If X is any of the spaces $l_\infty(\Delta^{(s)})$, $c(\Delta^{(s)})$ and $c_0(\Delta^{(s)})$, and Y is any of the spaces $l_\infty(\Delta^{(m)})$, $c(\Delta^{(m)})$ and $c_0(\Delta^{(m)})$, then the conditions for $A \in (X, Y)$ are obtained from the respective ones in Theorem 3.20 by replacing the entries of the matrix A by those of the matrix $B = TA$, for instance

$$\sup_n \|B_n\|^* = \sup_n \|R^{(s)}(B_n)\|_1 < \infty$$

where

$$B_n = \sum_{l=\max\{0,n-m\}}(-1)^{n-l}\binom{m}{n-l}A_l.$$

Theorem 3.23. [71, Theorem 4] *Let A be as in Theorem 3.20, and for any integers m, n, r, $n > r$, set*

$$(3.54)\qquad \|A\|^{(r)} = \sup_{n>r}\|R^{(m)}(A_n)\|_1$$

Let X be either $c_0(\Delta^{(m)})$ or $X = c(\Delta^{(m)})$, and let $A \in (X, c_0)$. Then we have

$$(3.55)\qquad \|L_A\|_\chi = \lim_{r\to\infty}\|A\|^{(r)}.$$

Let X be either $c_0(\Delta^{(m)})$ or $X = c(\Delta^{(m)})$, and let $A \in (X, c)$. Then we have

$$(3.56)\qquad \frac{1}{2}\cdot\lim_{r\to\infty}\|A\|^{(r)} \le \|L_A\|_\chi \le \lim_{r\to\infty}\|A\|^{(r)}.$$

Let X be either $l_\infty(\Delta^{(m)})$, $c_0(\Delta^{(m)})$ or $X = c(\Delta^{(m)})$, and let $A \in (X, l_\infty)$. Then we have

$$(3.57)\qquad 0 \le \|L_A\|_\chi \le \lim_{r\to\infty}\|A\|^{(r)}.$$

Proof. Let us remark that the limits in (3.55), (3.56) and (3.57) exist. We put $B = \{x \in X : \|x\| \leq 1\}$. In the case $A \in (X, c_0)$ for $X = c_0(\Delta(^{(m)})$ or $X = c(\Delta(^{(m)})$, we have by Theorem 2.23

$$\|L_A\|_\chi = \chi(A(B)) = \lim_{r\to\infty}\left[\sup_{x\in B}\|(I-P_r)(A(x))\|\right], \tag{3.58}$$

where $P_r : c_0 \mapsto c_0$ for $r = 0, 1, \ldots$ is the projector on the first $r+1$ coordinates, that is $P_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots)$ for $x = (x_k) \in c_0$; (let us remark that $\|I - P_r\| = 1$ for $r = 0, 1, \ldots$). Further we have by Theorem 3.20

$$\|A\|^{(r)} = \sup_{x\in B}\|(I-P_r)(A(x))\|, \tag{3.59}$$

and by (3.58) we get (3.55).

To prove (3.56) let us remark that every sequence $x = (x_k)_{k=0}^{\infty} \in c$ has a unique representation $x = le + \sum_{k=0}^{\infty}(x_k - l)e^{(k)}$ where $l \in \mathbb{C}$ is such that $x - le \in c_0$. Let us define $P_r : c \mapsto c$ by $P_r(x) = le + \sum_{k=0}^{r}(x_k - l)e^{(k)}$ for $r = 0, 1, \ldots$. It is easy to prove that $\|I - P_r\| = 2$ for $r = 0, 1, \ldots$. Now the proof of (3.56) is similar as in the case (3.55), and we omit it.

To prove (3.57), we define $P_r : l_\infty \mapsto l_\infty$ by $P_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots)$ for $x = (x_k) \in l_\infty$ and $r = 0, 1, \ldots$. It is clear that $A(B) \subset P_r(A(B)) + (I - P_r)(A(B))$. Now, by the elementary properties of function the χ we have

$$\begin{aligned}\chi(A(B)) &\leq \chi(P_r(A(B))) + \chi((I-P_r)(A(B))) = \chi((I-P_r)(A(B)) \\ &\leq \sup_{x\in B}\|(I-P_r)(A(x))\|.\end{aligned} \tag{3.60}$$

Finally we get (3.57) by Theorem 3.20. □

As a corollary of the theorem above, we have

Corollary 3.24. [71, Corollary 4.3] *Let A be as in Theorem 3.23. Then if $A \in (X, c_0)$ for $X = c_0(\Delta^{(m)})$ or $X = c(\Delta^{(m)})$, or if $A \in (X, c)$ for $X = c_0(\Delta^{(m)})$ or $X = c(\Delta^{(m)})$, then in all cases we have*

$$L_A \quad \textit{is compact if and only if} \quad \lim_{r\to\infty}\|A\|^{(r)} = 0. \tag{3.61}$$

Further, if $A \in (X, l_\infty)$ for $X = l_\infty(\Delta^{(m)})$, $X = c_0(\Delta^{(m)})$ or $X = c(\Delta^{(m)})$, then we have

$$L_A \quad \textit{is compact if} \quad \lim_{r\to\infty}\|A\|^{(r)} = 0. \tag{3.62}$$

The following example will show that it is possible for L_A in (3.62) to be compact in the case $\lim_{r\to\infty}\|A\|^{(r)} > 0$, and hence in general we have just “if” in (3.62).

Example 3.25. Let the matrix A be defined by $A_n = e^{(0)}$ $(n = 0, 1, \dots)$. Then obviously $R^{(m)}(A_n) = e^{(0)}$ for all n, and $A \in (l_\infty(\Delta^{(m)}), l_\infty)$. Further,

$$\|A\|^{(r)} = \sup_{n>r} \|R^{(m)}(A_n)\|_1 = \sup_{n>r} \|e^{(0)}\|_1 = 1 > 0 \quad \text{for all } r,$$

whence $\lim_{r\to\infty} \|A\|^{(r)} > 0$. Since $A(x) = x_0 e$ for all $x \in l_\infty(\Delta^{(m)})$, A is a compact operator.

Concerning Corollary 3.2.1 and the measures of noncompactness we have

Theorem 3.26. [71, Theorem 4.5] *Let X be a BK space and let A be as in Corollary 3.21 Then for all integers m, n, r, $n > r$, we put*

$$\|A\|_\Delta^{(r)} = \sup_{n>r} \left\| \sum_{j=\max\{0,n-m\}}^{n} (-1)^{n-j} \binom{m}{n-j} A_j \right\|^* . \tag{3.63}$$

Further, if X has a Schauder basis, and $A \in (X, c_0(\Delta^{(m)}))$, then we have

$$\|L_A\|_\chi = \lim_{r\to\infty} \|A\|_\Delta^{(r)}. \tag{3.64}$$

If X has a Schauder basis, and $A \in (X, c(\Delta^{(m)}))$, then we have

$$\frac{1}{2} \cdot \lim_{r\to\infty} \|A\|_\Delta^{(r)} \le \|L_A\|_\chi \le \lim_{r\to\infty} \|A\|_\Delta^{(r)}. \tag{3.65}$$

Finally, if $A \in (X, l_\infty(\Delta^{(m)}))$, then we have

$$0 \le \|L_A\|_\chi \le \lim_{r\to\infty} \|A\|_\Delta^{(r)}. \tag{3.66}$$

Proof. Let us remark that the limits in (3.64), (3.65) and (3.66) exist. We put $B = \{x \in X : \|x\| \le 1\}$. To prove (3.64), we have by Theorems 3.12 and 2.23

$$\|L_A\|_\chi = \chi(A(B)) = \lim_{r\to\infty} \left[\sup_{x\in B} \|(I - P_r)(A(x))\| \right], \tag{3.67}$$

where $P_r : c_0(\Delta^{(m)}) \mapsto c_0(\Delta^{(m)})$ $(r = 0, 1, \dots)$ is the projector defined by

$$P_r(x) = \sum_{k=0}^{r} \lambda_k(m) b^{(k)}(m), \tag{3.68}$$

for $x = \sum_{k=0}^\infty \lambda_k(m) b^{(k)}(m) \in c_0(\Delta^{(m)})$ and the Schauder basis $(b^{(k)}(m))_{k=0}^\infty$ of $c_0(\Delta^{(m)})$ (see Theorem 3.12). Let us remark that $\|I - P_r\| = 1$ for $(r = 0, 1, \dots)$. Further we have by Theorem 3.8

$$\|A\|_\Delta^{(r)} = \sup_{x\in B} \|(I - P_r)(A(x))\|, \tag{3.69}$$

To prove (3.65), let us remark (see Theorem 3.12) that $c(\Delta^{(m)})$ has the Schauder basis $b^{(k)}(m)$ $k=-1,0,1,\dots$, and every $x \in c(\Delta^{(m)})$ has a unique representation

$$x = lb^{(-1)}(m) + \sum_{k=0}^{\infty}(\lambda_k(m) - l)b^{(k)}(m) \quad \text{where} \quad l = \lim_{k\to\infty}(\Delta^{(m)}x)_k. \tag{3.70}$$

Now let us define $P_r : c(\Delta^{(m)}) \mapsto c(\Delta^{(m)})$ $(r = 0,1,\dots)$ by

$$P_r(x) = lb^{(-1)}(m) + \sum_{k=0}^{r}(\lambda_k(m) - l)b^{(k)}(m). \tag{3.71}$$

It is easy to show that $\|I - P_r\| = 2$ for $r = 0,1,\dots$. Now the proof of (3.65) is similar as in the case (3.64), and we omit it.

Finally in order to prove (3.66), we define $P_r : l_\infty(\Delta^{(m)}) \mapsto l_\infty(\Delta^{(m)})$, by $P_r(x) = (x_0, x_1, \dots, x_r, 0, 0, \dots)$ for $x = (x_k) \in l_\infty(\Delta^{(m)})$ and $r = 0,1,\dots$. It is clear that $A(B) \subset P_r(A(B)) + (I - P_r)(A(B))$. Now, by the elementary properties of the function χ, we again have (3.60) and, then (3.66) by Theorem 3.8 and Corollary 3.21. □

As a corollary of the theorem above, we have

Corollary 3.27. [71, Corollary 4.6] *Let X be a BK space and let A and $\|A\|_\Delta^{(r)}$ be as in Theorem 3.26. If X has a Schauder basis, and either $A \in (X, c_0(\Delta^{(m)}))$ or $A \in (X, c(\Delta^{(m)}))$, then L_A is compact if and only if* $\lim_{r\to\infty}\|A\|_\Delta^{(r)} = 0$. *Further, if $A \in (X, l_\infty(\Delta^{(m)}))$, then L_A is compact if* $\lim_{r\to\infty}\|A\|_\Delta^{(r)} = 0$.

Finally we obtain several corollaries concerning Remark 3.22.

Corollary 3.28. [71, Corollary 4.7] *If either $A \in (l^p, c_0(\Delta^{(m)}))$ or $A \in (l^p, c(\Delta^{(m)}))$ $(1 < p < \infty)$, then*

L_A is compact if and only if

$$\lim_{r\to\infty}\sup_{n>r}\left(\sum_{k=0}^{\infty}\left|\sum_{j=\max\{0,n-m\}}^{n}(-1)^{n-j}\binom{m}{n-j}a_{jk}\right|^q\right) = 0, \quad q = p/(p-1).$$

Further, if either $A \in (l_\infty, c_0(\Delta^{(m)}))$ or $A \in (l_\infty, c(\Delta^{(m)}))$, then

L_A is compact if and only if

$$\lim_{r\to\infty}\sup_{n>r,k}\left|\sum_{j=\max\{0,n-m\}}^{n}(-1)^{n-j}\binom{m}{n-j}a_{jk}\right| = 0.$$

If $A \in (l^p, l_\infty(\Delta^{(m)}))$ for $1 < p < \infty$, then

L_A is compact if

$$\lim_{r\to\infty}\sup_{n>r}\left(\sum_{k=0}^{\infty}\left|\sum_{j=\max\{0,n-m\}}^{n}(-1)^{n-j}\binom{m}{n-j}a_{jk}\right|^q\right) = 0, \quad q = p/(p-1).$$

Finally, if $A \in (l_\infty, l_\infty(\Delta^{(m)}))$, *then*

$$L_A \text{ is compact if } \lim_{r\to\infty} \sup_{n>r,k} \left| \sum_{j=\max\{0,n-m\}}^{n} (-1)^{n-j} \binom{m}{n-j} a_{jk} \right| = 0.$$

From Corollary 3.24, Theorem 3.20 and Remark 3.22, we have

Corollary 3.29. [71, Corollary 4.8] *Let* s *and* m *be non negative integers. If* $A \in (X, c_0(\Delta^{(m)})$ *for* $X = c_0(\Delta^{(s)})$ *or* $X = c(\Delta^{(s)})$, *or if* $A \in (X, c(\Delta^{(m)})$ *for* $X = c_0(\Delta^{(s)})$ *or* $X = c(\Delta^{(s)})$, *then in all cases we have*

L_A is compact if and only if

$$\lim_{r\to\infty} \sup_{n>r} \left\| R^{(s)} \left(\sum_{j=\max\{0,n-m\}}^{n} (-1)^{n-j} \binom{m}{n-j} A_j \right) \right\|_1 = 0.$$

Further, if $A \in (X, l_\infty(\Delta^{(m)})$ *for* $X = l_\infty(\Delta^{(s)})$, $X = c_0(\Delta^{(s)})$ *or* $X = c(\Delta^{(s)})$, *then we have*

$$L_A \text{ is compact if } \lim_{r\to\infty} \sup_{n>r} \left\| R^{(s)} \left(\sum_{j=\max\{0,n-m\}}^{n} (-1)^{n-j} \binom{m}{n-j} A_j \right) \right\|_1 = 0.$$

3.5. Sequences of weighted means. In this subsection, we shall study sets of *weighted means sequences* , give their bases and determine their β– and continuous duals. The results can be found in [34].

Let $(q_k)_{k=0}^\infty$ be a positive sequence, Q be the sequence with $Q_n = \sum_{k=0}^n q_k$ $(n = 0, 1, \dots)$ and the matrix $\bar{N}_q$ be defined by

$$(\bar{N}_q)_{n,k} = \begin{cases} q_k/Q_n & (0 \le k \le n) \\ 0 & (k > n) \end{cases} \quad (n = 0, 1, \dots).$$

Then we define the sets $(\bar{N}, q)_0 = (c_0)_{\bar{N}_q}$, $(\bar{N}, q) = c_{\bar{N}_q}$ and $(\bar{N}, q)_\infty = (l_\infty)_{\bar{N}_q}$ of sequences that are $(\bar{N}, q)$ summable to naught, summable and bounded, respectively.

For any $x \in X$, we write $\tau = \tau(x)$ for the sequence defined by

$$\tau_n = (\bar{N}_q)_n(x) = \frac{1}{Q_n} \sum_{k=0}^{n} q_k x_k \ (n = 0, 1, \dots),$$

and τ is called the sequence of the $\bar{N}_q$ or *weighted means of* x. As an immediate consequence of Example 1.13 and Theorems 3.3 and 3.5, we obtain

Proposition 3.30. (cf. [34, Corollary 1]) *Each of the sets* $(\bar{N},q)_0$, $(\bar{N},q)$ *and* $(\bar{N},q)_\infty$ *is a BK space with*

$$\|x\|_{\bar{N}_q} = \sup_n \left| \frac{1}{Q_n} \sum_{k=0}^n q_k x_k \right|.$$

Further, if $Q_n \to \infty$ $(n \to \infty)$, *then* $(\bar{N},q)_0$ *has AK, and every sequence* $x = (x_k)_{k=0}^\infty \in (\bar{N},q)$ *has a unique representation* $x = le + \sum_{k=0}^\infty (x_k - l)e^{(k)}$ *where* $l \in \mathbb{C}$ *is such that* $x - le \in (\bar{N},q)_0$.

We define the operator $\Delta^+ : \omega \mapsto \omega$ by $\Delta^+ x = ((\Delta^+ x)_k)_{k=0}^\infty = (x_k - x_{k+1})_{k=0}^\infty$.

Theorem 3.31. [34, Theorem 6] *Let* $q = (q_k)_{k=0}^\infty$ *be a positive sequence and* Q *the sequence with* $Q_n = \sum_{k=0}^n q_k$ $(n = 0, 1, \dots)$. *We write* $1/q$ *for the sequence* $(1/q_n)_{n=0}^\infty$, *and put* $M_1 = \{a \in \omega : Q(\Delta^+ a) \in l_1\}$, $\mathcal{N}_0 = (1/q)^{-1} * (M_1 \cap (Q^{-1} * l_\infty))$, $\mathcal{N} = (1/q)^{-1} * (M_1 \cap (Q^{-1} * c))$ *and* $\mathcal{N}_\infty = (1/q)^{-1} * (M_1 \cap (Q^{-1} * c_0))$. *Then* $(\bar{N},q)_0^\beta = \mathcal{N}_0$, $(\bar{N},q)^\beta = \mathcal{N}$ *and* $(\bar{N},q)_\infty^\beta = \mathcal{N}_\infty$.

Proof. We put $X_1 = (1/q)^{-1} * M_1$ and observe that

$$x_k = \frac{1}{q_k}(Q_k \tau_k - Q_{k-1}\tau_{k-1}) \ (k = 0, 1, \dots) \quad \text{for all } x \in \omega \tag{3.72}$$

and for all $n = 0, 1, \dots$

$$\sum_{k=0}^n a_k x_k = \sum_{k=0}^n \frac{a_k}{q_k} \Delta(Q_k \tau_k) = \sum_{k=0}^{n-1} \left(Q_k \tau_k \Delta^+ \left(\frac{a_k}{q_k} \right) \right) + \frac{a_n Q_n}{q_n} \tau_n. \tag{3.73}$$

Let $a \in X_1$, that is $Q\Delta^+(a/q) \in l_1 = l_\infty^\beta$. Thus $\tau \cdot (Q\Delta^+(a/q)) \in cs$ for all $\tau \in l_\infty$, hence for all $\tau \in c$ and $\tau \in c_0$. Further $a \in (Q/q)^{-1} * c_0$ implies $(aQ/q)\tau \in c_0$ for all $\tau \in l_\infty$. Since $\tau = \tau(x) \in l_\infty$ if and only if $x \in (\bar{N},q)_\infty$, $ax \in cs$ for all $x \in (\bar{N},q)_\infty$ by (3.73), that is $a \in (\bar{N},q)_\infty^\beta$. Similarly $a \in (Q/q)^{-1} * c$ or $a \in (Q/q) * l_\infty$ imply $a \in (\bar{N},q)^\beta$ or $a \in (\bar{N},q)_0^\beta$. Thus we have proved $\mathcal{N}_0 \subset (\bar{N},q)_0^\beta$, $\mathcal{N} \subset (\bar{N},q)$ and $\mathcal{N}_\infty \subset (\bar{N},q)_\infty^\beta$.

To prove the converse inclusions we first assume $ax \in cs$ for all $x \in (\bar{N},q)_0$. Then $ax \in c_0$ for all $x \in (\bar{N},q)_0$, hence $(a/q)\Delta(Q\tau) \in c_0$ for all $\tau = \tau(x) \in c_0$, whence

$$\frac{a_k}{q_k}\Delta(Q_k(-1)^k|\tau_k|) = (-1)^k \frac{a_k}{q_k}(Q_k|\tau_k| + Q_{k-1}|\tau_{k-1}|) \to 0 \quad \text{for all } \tau \in c_0.$$

This implies $(aQ/q)\tau \in c_0$ for all $\tau \in c_0$, and thus $aQ/q \in l_\infty$ by Example 1.28. From (3.73), we conclude $Q\Delta^+(a/q)\tau \in cs$ for all $\tau \in c_0$, that is $Q\Delta^+(a/q) \in c_0^\beta = l_1$. Thus $a \in X_1$, and we have proved $(\bar{N},q)_0^\beta \subset \mathcal{N}_0$.

Now let $ax \in cs$ for all $x \in (\bar{N},q)$. Then $ax \in cs$ for all $x \in (\bar{N},q)_0$, and consequently $a \in (\bar{N},q)_0^\beta \subset X_1$. Thus by (3.73), $(aQ/q)\tau \in c$ for all $\tau \in c$, hence $aQ/q \in c$ by Example 1.28. This proves $(\bar{N},q)_\infty \subset \mathcal{N}$.

Finally let $ax \in cs$ for all $x \in (\bar{N},q)_\infty$. Then again $a \in X_1$, and by (3.73), $(aQ/q)\tau \in c$ for all $\tau \in l_\infty$, hence $aQ/q \in c_0$ by Example 1.28. This proves $(\bar{N},q)^\beta \subset \mathcal{N}_\infty$. □

3.6. Matrix transformations in the spaces $(\bar{N},q)_0$, $(\bar{N},q)$ and $(\bar{N},q)_\infty$ and their measures of noncompactness. We need the following proposition

Proposition 3.32. [72, Proposition 3.1] *On any of the spaces $(\bar{N},q)_0^\beta$, $(\bar{N},q)^\beta$ and $(\bar{N},q)_\infty^\beta$, we have*

$$\|a\|^* = \sup_n \left(\sum_{k=0}^{n-1} Q_k \left| \frac{a_k}{q_k} - \frac{a_{k+1}}{q_{k+1}} \right| + \left| \frac{a_n Q_n}{q_n} \right| \right).$$

Proof. Given any sequence x we shall write $\tau^{[n]} = \tau(x^{[n]})$ $(n = 0,1,\dots)$ where $x^{[n]}$ is the n–section of x. Let $a \in \mathcal{N}_0$ and n be a nonnegative integer. We define the sequence $b^{[n]}$ by

$$b_k^{[n]} = \begin{cases} Q_k \Delta^+(a/q)_k & (0 \le k \le n) \\ a_n Q_n / q_n & (k = n) \\ 0 & (k > n) \end{cases}$$

and put $\|a\|_{\mathcal{N}} = \sup_n \|b^{[n]}\|_1 = \sup_n (\sum_{k=0}^\infty |b_k^{[n]}|)$. Then

$$\begin{aligned} \left|\sum_{k=0}^\infty a_k x_k^{[n]}\right| &= \left|\sum_{k=0}^n \frac{a_k}{q_k} \Delta(Q\tau^{[n]})_k\right| \le \sum_{k=0}^{n-1} \left| Q_k \tau_k^{[n]} \Delta^+(a/q)_k \right| + \left|\frac{a_n Q_n}{q_n}\right| |\tau_n^{[n]}| \\ &\le \sup_k |\tau_k^{[n]}| \cdot \left(\sum_{k=0}^{n-1} \left| Q_k \Delta^+(a/q)_k \right| + \left| \frac{a_n Q_n}{q_n} \right| \right) \\ &= \|x^{[n]}\|_{\bar{N}_q} \|b^{[n]}\|_1 = \|a\|_{\mathcal{N}} \|x^{[n]}\|_{\bar{N}_q}. \end{aligned}$$

Thus

$$\|a\|^* \le \|a\|_{\mathcal{N}}. \tag{3.74}$$

To prove the converse inequality let n be an arbitrary integer. We define the sequence $x^{(n)}$ by $\tau_k(x^{(n)}) = \operatorname{sign}(b_k^{[n]})$ for $k = 0,1,\dots$. Then $\tau_k(x^{(n)}) =$ for $k > n$, that is $x^{(n)} \in (\bar{N},q)_0$, $\|x^{(n)}\|_{\bar{N}_n} = \|\tau(x^{(n)})\|_\infty \le 1$ and

$$\left|\sum_{k=0}^\infty a_k x_k^{(n)}\right| = \left|\sum_{k=0}^n b_k^{[n]} x_k^{(n)}\right| = \sum_{k=0}^n |b_k^{[n]}| \le \|a\|^*.$$

Since n was arbitrary, we have $\|a\|_{\mathcal{N}} \le \|a\|^*$. This and (3.74) together yield the conclusion. □

As a corollary of Theorems 1.23 and 3.31 and Proposition 3.32 we obtain

Corollary 3.33. [72, Corollary 3.4] *Let* $q = (q_k)_{k=0}^\infty$ *be a positive sequence and* $Q_n = \sum_{k=0}^n q_k \to \infty$ $(n \to \infty)$.

(a) *Then* $A \in ((\bar{N}, q)_\infty, l_\infty)$ *if and only if*

$$M((\bar{N}, q)_\infty, l_\infty) = \sup_{n,m}\left(\sum_{k=0}^{m-1} Q_k \left|\frac{a_{nk}}{q_k} - \frac{a_{n,k+1}}{q_{k+1}}\right| + Q_m \frac{|a_{nm}|}{q_m}\right) < \infty \tag{3.75}$$

and

$$A_n Q/q \in c_0 \quad \text{for all } n = 0, 1, \ldots. \tag{3.76}$$

(b) *Then* $A \in ((\bar{N}, q), l_\infty)$ *if and only if condition (3.75) holds and*

$$A_n Q/q \in c \quad \text{for all } n = 0, 1, \ldots. \tag{3.77}$$

(c) *Then* $A \in ((\bar{N}, q)_0, l_\infty)$ *if and only condition (3.75) holds.*

(d) *Then* $A \in ((\bar{N}, q)_0, c_0)$ *if and only condition (3.75) holds and*

$$\lim_{n\to\infty} a_{nk} = 0 \quad \text{for all } k = 0, 1, \ldots. \tag{3.78}$$

(e) *Then* $A \in ((\bar{N}, q)_0, c)$ *if and only if condition (3.75) holds and*

$$\lim_{n\to\infty} a_{nk} = l_k \quad \text{for all } k = 0, 1, \ldots. \tag{3.79}$$

(f) *Then* $A \in ((\bar{N}, q), c_0)$ *if and only if conditions (3.75), (3.77) and (3.78) hold and*

$$\lim_{n\to\infty} \sum_{k=0}^\infty a_{nk} = 0. \tag{3.80}$$

(g) *Then* $A \in ((\bar{N}, q), c)$ *if and only if conditions (3.75), (3.77) and (3.79) hold and*

$$\lim_{n\to\infty} \sum_{k=0}^\infty a_{nk} = l. \tag{3.81}$$

As a corollary of Theorem 3.8 and Corollary 3.33, we obtain

Corollary 3.34. [72, Corollary 3.5] *Let* X *be a BK space,* $(p_k)_{k=0}^\infty$ *a positive sequence and* $P_n = \sum_{k=0}^n p_k$ $(n = 0, 1, \ldots)$. *Then* $A \in (X, (\bar{N}, p)_\infty)$ *if and only if*

$$M(X, (\bar{N}, p)_\infty) = \sup_m \left\|\frac{1}{P_m}\sum_{n=0}^m p_n A_n\right\|^* < \infty. \tag{3.82}$$

Further, if $(b^k)_{k=0}^{\infty}$ *is a basis of* X*, then* $A \in (X, (\bar{N}, p)_0)$ *if and only if condition (3.82) holds and*

$$\lim_{m\to\infty}\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n A_n(b^{(k)})\right) = 0 \quad \text{for all } k = 0, 1, \dots, \tag{3.83}$$

and $A \in (X, (\bar{N}, p))$ *if and only if condition (3.83) holds and*

$$\lim_{m\to\infty}\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n A_n(b^{(k)})\right) = l_k \quad \text{for all} \quad k = 0, 1, \dots. \tag{3.84}$$

Remark 3.35. (a) If $X = l_r$ $(1 \le r < \infty)$ and Y is any of the spaces $(\bar{N}, p)_\infty$, $(\bar{N}, p)$ and $(\bar{N}, p)_0$, then the conditions for $A \in (X, Y)$ follow from the respective ones in Corollary 3.34 by replacing the norm $\|\cdot\|^*$ in condition (3.82) by the natural norm on l_s where $s = \infty$ for $r = 1$ and $s = r/(r-1)$ for $1 < r < \infty$, that is

$$M(l_r, (\bar{N}, p)_\infty) = \begin{cases} \sup_{m,k}\left|\frac{1}{P_m}\sum_{n=0}^{m} p_n a_{nk}\right| & (r = 1) \\ \sup_m \sum_{k=0}^{\infty}\left|\frac{1}{P_m}\sum_{n=0}^{m} p_n a_{nk}\right|^s & (1 < r < \infty), \end{cases}$$

and by replacing the terms $A_n(b^{(k)})$ in conditions (3.83) and (3.84) by the terms a_{nk}.

(b) We consider the conditions

$$\begin{aligned} &M((\bar{N}, q)_\infty, (\bar{N}, p)_\infty) \\ &\quad = \sup_{m,n}\left(\sum_{k=0}^{n-1} Q_k\left|\frac{1}{P_m}\sum_{l=0}^{m} p_l\left(\Delta^+ A_l/q\right)_k\right| + \left|\frac{Q_n}{q_n P_m}\sum_{l=0}^{m} p_l a_{ln}\right|\right) < \infty, \end{aligned} \tag{3.85}$$

$$\left(\frac{a_{nk}Q_k}{q_k}\right)_{k=0}^{\infty} \in c_0 \qquad (n = 0, 1, \dots), \tag{3.86}$$

$$\left(\frac{a_{nk}Q_k}{q_k}\right)_{k=0}^{\infty} \in c \qquad (n = 0, 1, \dots), \tag{3.87}$$

$$\lim_{m\to\infty}\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n a_{nk}\right) = 0 \qquad (k = 0, 1, \dots), \tag{3.88}$$

$$\lim_{m\to\infty}\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n a_{nk}\right) = l_k \qquad (k = 0, 1, \dots), \tag{3.89}$$

$$\lim_{m\to\infty}\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n\left(\sum_{k=0}^{\infty} a_{nk}\right)\right) = 0, \tag{3.90}$$

$$\lim_{m\to\infty}\left(\frac{1}{P_m}\sum_{n=0}^{m} p_n\left(\sum_{k=0}^{\infty} a_{nk}\right)\right) = l. \tag{3.91}$$

Then

$$\begin{array}{lll} A \in ((\bar{N},q)_\infty, (\bar{N},p)_\infty) & \text{and only if} & (3.85) \text{ and } (3.86); \\ A \in ((\bar{N},q), (\bar{N},p)_\infty) & \text{and only if} & (3.85) \text{ and } (3.87); \\ A \in ((\bar{N},q)_0, (\bar{N},q)_\infty) & \text{and only if} & (3.85); \\ A \in ((\bar{N},q)_0, (\bar{N},p)_0) & \text{and only if} & (3.85) \text{ and } (3.88); \\ A \in ((\bar{N},q)_0, (\bar{N},p)) & \text{and only if} & (3.85) \text{ and } (3.89); \\ A \in ((\bar{N},q), (\bar{N},p)_0) & \text{and only if} & (3.85), (3.87), (3.88) \text{ and } (3.90); \\ A \in ((\bar{N},q), (\bar{N},p)) & \text{and only if} & (3.85), (3.87), (3.89) \text{ and } (3.91). \end{array}$$

Theorem 3.36. [72, Theorem 4.2] *Let A be as in Corollary 3.33, and for all integers n, r, $n > r$, set*

$$\|A\|^{(r)} = \sup_{\substack{m \\ n>r}} \left(\sum_{k=0}^{m-1} Q_k \left| \frac{a_{nk}}{q_k} - \frac{a_{n,k+1}}{q_{k+1}} \right| + Q_m \frac{|a_{nm}|}{q_m} \right) \tag{3.92}$$

Let X be either $(\bar{N},q)_0$ or $X = (\bar{N},q)$, and let $A \in (X, c_0)$. Then we have

$$\|L_A\|_\chi = \lim_{r\to\infty} \|A\|^{(r)}.$$

Let X be either $(\bar{N},q)_0$ or $X = (\bar{N},q)$, and let $A \in (X, c)$. Then we have

$$\frac{1}{2} \cdot \lim_{r\to\infty} \|A\|^{(r)} \le \|L_A\|_\chi \le \lim_{r\to\infty} \|A\|^{(r)}.$$

Let X be either $(\bar{N},q)_0$, $(\bar{N},q)$ or $X = (\bar{N},q)_\infty$, and let $A \in (X, l_\infty)$. Then we have $0 \le \|L_A\|_\chi \le \lim_{r\to\infty} \|A\|^{(r)}$.

Proof. The proof follows exactly the same lines as that of Theorem 3.26. □

As a corollary of the theorem above, we have

Corollary 3.37. [72, Corollary 4.3] *Let A be as in Theorem 3.36. Then if $A \in (X, c_0)$ for $X = (\bar{N},q)_0$ or $X = (\bar{N},q)$, or if $A \in (X, c)$ for $X = (\bar{N},q)_0$ or $X = (\bar{N},q)$, then in all cases we have L_A is compact if and only if $\lim_{r\to\infty} \|A\|^{(r)} = 0$. Further, if $A \in (X, l_\infty)$ for $X = (\bar{N},q)_0$, $X = (\bar{N},q)$ or $X = (\bar{N},q)_\infty$, then we have*

$$L_A \quad \text{is compact if} \quad \lim_{r\to\infty} \|A\|^{(r)} = 0. \tag{3.93}$$

The following example will show that it is possible for L_A in (3.93) to be compact in the case $\lim_{r\to\infty} \|A\|^{(r)} > 0$, and hence in general we have just " if" in (3.93).

Example 3.38. Let the matrix A be defined by $A_n = e^{(0)}$ $(n = 0, 1, \dots)$ and $q_n = 2^n$ for $n = 0, 1, 2, \dots$. Then $M((\bar{N},q)_\infty, l_\infty) = \sup_n [1 + (2 - 2^{-n})] < 3$, and by Corollary 3.33 we know that $A \in ((\bar{N},q)_\infty, l_\infty)$. Further

$$\|A\|^{(r)} = \sup_{n>r} [1 + (2 - \frac{1}{2^n})] = 3 - \frac{1}{2^{r+1}} \quad \text{for all} \quad r,$$

whence $\lim_{r\to\infty} \|A\|^{(r)} = 3 > 0$. Since $A(x) = x_0 e_0$ for all $x \in (\bar{N},q)_\infty$, L_A is a compact operator.

Now we continue with the following auxiliary result.

Lemma 3.39. [72, Lemma 4.5] *Let $q_k > 0$ $(k = 0, 1, \dots)$ and $Q_n = \sum_{k=0}^{n} q_k \to \infty$ $(n \to \infty)$. Let $r \geq 0$ and the operators $B_0^{(r)} : (\bar{N}, q)_0 \mapsto (\bar{N}, q)_0$ and $B^{(r)} : (\bar{N}, q) \mapsto (\bar{N}, q)$ be defined by*

$$B_0^{(r)} x = \sum_{k=r+1}^{\infty} x_k e^{(k)}, \quad (x \in (\bar{N}, q)_0)$$

$$B^{(r)} x = \sum_{k=r+1}^{\infty} (x_k - l) e^{(k)}, \quad (x \in (\bar{N}, q))$$

where $l = \lim_{n\to\infty} \tau_n(x)$. Then

$$\|B_0^{(r)}\| = 1 + \frac{Q_r}{Q_{r+1}} \tag{3.94}$$

$$\|B^{(r)}\| = 2. \tag{3.95}$$

Proof. First we show identity (3.94). Let $x \in (\bar{N}, q)_0$. Since $\tau_n(B_0^{(r)}(x)) = 0$ for $0 \leq n \leq r$, and, for $n \geq r + 1$,

$$\begin{aligned} |\tau_n(B_0^{(r)}(x))| &= \left| \frac{1}{Q_n} \sum_{k=r+1}^{n} q_k x_k \right| = \left| \tau_n(x) - \frac{Q_r}{Q_n} \tau_r(x) \right| \\ &\leq \left(1 + \frac{Q_r}{Q_{r+1}} \right) \|x\|_{(\bar{N},q)_\infty}, \end{aligned}$$

it follows that

$$\left\|B_0^{(r)}(x)\right\|_{(\bar{N},q)_\infty} \leq \left(1 + \frac{Q_r}{Q_{r+1}} \right) \|x\|_{(\bar{N},q)_\infty},$$

and consequently

$$\left\|B_0^{(r)}\right\| \leq 1 + \frac{Q_r}{Q_{r+1}}. \tag{3.96}$$

Defining the sequence x by

$$x_k = \begin{cases} -1 & (0 \leq k \leq r) \\ \dfrac{Q_r + Q_{r+1}}{q_{r+1}} & (k = r + 1) \\ -\dfrac{Q_r + Q_{r+1}}{q_{r+2}} & (k = r + 2) \\ 0 & (k \geq r + 3), \end{cases}$$

we conclude $\tau_n(x) = -1$ for $0 \leq n \leq r$

$$\begin{aligned} \tau_{r+1}(x) &= -\frac{Q_r}{Q_{r+1}} + \frac{Q_r}{Q_{r+1}} + 1 = 1 \\ \tau_n(x) &= \frac{1}{Q_n} \Big(-Q_r + Q_r + Q_{r+1} - (Q_r + Q_{r+1}) \Big) = -\frac{Q_r}{Q_n} \quad (n \geq r + 2). \end{aligned}$$

Since $Q_n \to \infty$ $(n \to \infty)$, we have $x \in (\bar{N}, q)_0$ and $\|x\|_{(\bar{N},q)_\infty} = 1$. Further

$$\tau_{r+1}(B_0^{(r)}(x)) = \frac{1}{Q_{r+1}}\big(Q_r + Q_{r+1}\big) = 1 + \frac{Q_r}{Q_{r+1}}$$

and $\tau_n(B_0^{(r)}(x)) = 0$ for $n \neq r+1$. Therefore

$$\big\|B_0^{(r)}(x)\big\|_{(\bar{N},q)_\infty} = 1 + \frac{Q_r}{Q_{r+1}} = \left(1 + \frac{Q_r}{Q_{r+1}}\right)\|x\|_{(\bar{N},q)_\infty}$$

and $\|B_0^{(r)}\| \geq 1 + Q_r/Q_{r+1}$. Now this and (3.96) together yield identity (3.94).

Now we prove identity (3.95). Let $x \in (\bar{N}, q)$. Since $\tau_n(B^{(r)}(x)) = 0$ for $0 \leq n \leq r$ and, for $n \geq r+1$,

$$\begin{aligned}\big|\tau_n(B^{(r)}(x))\big| &= \left|\frac{1}{Q_n}\sum_{k=r+1}^{n} q_k(x_k - l)\right| = \left|\tau_n(x) - \frac{Q_r}{Q_n}\tau_r(x) - l + \frac{Q_r}{Q_n}l\right| \\ &\leq \left|1 + \frac{Q_r}{Q_n}\right|\|x\|_{(\bar{N},q)_\infty} + \left|1 - \frac{Q_r}{Q_n}\right||l|,\end{aligned}$$

since $|l| = \lim_{n\to\infty}|\tau_n(x)| \leq \|x\|_{(\bar{N},q)_\infty}$, it follows that $|\tau_n(B^{(r)}(x))| \leq 2\|x\|_{(\bar{N},q)_\infty}$ for $n \geq r+1$ and consequently

$$\|B^{(r)}\| \leq 2. \tag{3.97}$$

Defining the sequence x by

$$x_k = \begin{cases} -1 & (0 \leq k \leq r) \\ 2Q_{r+1}/q_{r+1} - 1 & (k = r+1) \\ -1 & (k \geq r+2), \end{cases}$$

we conclude $\tau_n(x) = -1$ for $0 \leq n \leq r$,

$$\begin{aligned}\tau_{r+1}(x) &= \frac{1}{Q_{r+1}}\big(-Q_r + 2Q_r - q_{r+1}\big) = 1 \\ \tau_n(x) &= \frac{1}{Q_n}\left(-Q_r + 2Q_{r+1} - \sum_{k=r+1}^{n} q_k\right) = \frac{1}{Q_n}\big(-Q_n + 2Q_{r+1}\big) \\ &= -1 + 2\frac{Q_{r+1}}{Q_n} \leq 1 \quad (n \geq r+2).\end{aligned}$$

Hence $\|x\|_{(\bar{N},q)_\infty} = 1$ and $\lim_{n\to\infty}\tau_n(x) = -1$, that is $x \in (\bar{N}, q)$. Finally $\tau_n(B^{(r)}(x)) = 0$ $(0 \leq n \leq r)$,

$$\begin{aligned}\tau_{r+1}(B^{(r)}(x)) &= \frac{q_{r+1}}{Q_{r+1}}(x_{r+1} + 1) = 2 \\ \tau_n(B^{(r)}(x)) &= 2\frac{Q_{r+1}}{Q_n} \leq 2 \quad (n \geq r+2).\end{aligned}$$

This implies $\|B^{(r)}\| \geq 2$, and together with (3.97) we obtain (3.95). □

Concerning Corollary 3.34 and the measures of noncompactness we have

Theorem 3.40. [72, Theorem 4.6] *Let X be a BK space, let A be as in Corollary 3.34, and let $P_m \to \infty \,(m \to \infty)$. Then for all integers m, r, $m > r$, we put*

$$\|A\|^{(r)}_{(\bar{N},p)_\infty} = \sup_{m>r} \left\| \frac{1}{P_m} \sum_{n=0}^{m} p_n A_n \right\|^*.$$

Further, if X has a Schauder basis, and $A \in (X, (\bar{N},p)_0)$, then we have

$$\frac{1}{b} \cdot \lim_{r\to\infty} \|A\|^{(r)}_{(\bar{N},p)_\infty} \le \|L_A\|_\chi \le \lim_{r\to\infty} \|A\|^{(r)}_{(\bar{N},p)_\infty}, \tag{3.98}$$

where $b = \limsup_{n\to\infty}(2 - p_n/P_n)$.

If X has a Schauder basis, and $A \in (X, (\bar{N},p))$, then we have

$$\frac{1}{2} \cdot \lim_{r\to\infty} \|A\|^{(r)}_{(\bar{N},p)_\infty} \le \|L_A\|_\chi \le \lim_{r\to\infty} \|A\|^{(r)}_{(\bar{N},p)_\infty}. \tag{3.99}$$

Finally, if $A \in (X, (\bar{N},p)_\infty)$, then we have

$$0 \le \|L_A\|_\chi \le \lim_{r\to\infty} \|A\|^{(r)}_{(\bar{N},p)_\infty}. \tag{3.100}$$

Proof. Let us remark that the limits in (3.98), (3.99) and (3.100) exist. We put $B = \{x \in X : \|x\| \le 1\}$. Suppose that $A \in (X, (\bar{N},p)_0)$. Let $B_0^{(r)} : (\bar{N},p)_0 \mapsto (\bar{N},p)_0$ be the projector defined in Lemma 3.39. Then by (3.94) we have that $\|B_0^{(r)}\| = 2 - p_r/P_r$. Now, to prove (3.98), we have by Theorem 2.23 and Proposition 3.30

$$\frac{1}{b} \limsup_{r\to\infty} \left(\sup_{x\in B} \|B_0^{(r)}(A(x))\| \right) \le \chi(A(B)) \le \limsup_{r\to\infty} \left(\sup_{x\in B} \|B_0^{(r)}(A(x))\| \right),$$

where $b = \limsup_{r\to\infty} \|B_0^{(r)}\|$. This proves (3.98), since

$$\sup_{x\in B} \|B_0^{(r)}(A(x))\| = \|A\|^{(r)}_{(\bar{N},p)_\infty}.$$

To prove (3.99) let us remark (see Proposition 3.30) that $(\bar{N},p)$ has the Schauder basis $e, e^{(k)}, k = 0, 1, \dots,$ and every $(x_k)_{k=0}^\infty \in (\bar{N},q)$ has a unique representation $x = le + \sum_{k=0}^\infty (x_k - l)e^{(k)}$, where $l \in \mathbb{C}$ is such that $x - le \in (\bar{N},p)_0$. Let $B^{(r)} : (\bar{N},p)_0 \mapsto (\bar{N},p)_0$ be the projector defined by (see Lemma 3.39) $B^{(r)}(x) = \sum_{k=r+1}^\infty (x_k - l)e^{(k)}$. Then we have $\|B^{(r)}\| = 2$ by (3.95). Now the proof of (3.99) is similar as in the case (3.98), and we omit it. Let us prove (3.100). Now define $\mathcal{P}_r : (\bar{N},p)_\infty \mapsto (\bar{N},p)_\infty$, by $\mathcal{P}_r(x) = (x_0, x_1, \dots, x_r, 0, 0, \dots)$ for all $x = (x_k) \in (\bar{N},p)_\infty$ and $r = 1, 2, \dots$. It is clear that $A(B) \subset \mathcal{P}_r(A(B)) + (I - \mathcal{P}_r)(A(B))$. By Remark 3.22 (b) it follows that $\mathcal{P}_r$ is a bounded operator, and since it is obviously a finite-rank, it is a compact operator. Now, by the elementary properties of function χ we have

$$\begin{aligned}\chi(A(B)) &\le \chi(\mathcal{P}_r(A(B))) + \chi((I - \mathcal{P}_r)(A(B))) = \chi((I - \mathcal{P}_r)(A(B)) \\ &\le \sup_{x\in B} \|(I - \mathcal{P}_r)(A(x))\| = \|A\|^{(r)}_{(\bar{N},p)_\infty}.\end{aligned}$$

□

As a corollary of the theorem above we have

Corollary 3.41. [72, Corollary 4.7] *Let X be a BK space and let A and $\|A\|^{(r)}_{(\bar{N},p)}$ be as in Theorem 4.6. If X has a Schauder basis, and either $A \in (X,(\bar{N},p)_0)$ or $A \in (X,(\bar{N},p))$, then L_A is compact if and only if $\lim_{r\to\infty} \|A\|^{(r)}_{(\bar{N},p)} = 0$. Further, if $A \in (X,(\bar{N},p)_\infty)$, then L_A is compact if $\lim_{r\to\infty} \|A\|^{(r)}_{(\bar{N},p)} = 0$.*

Now we get several corollaries concerning Remark 3.22.

Corollary 3.42. [72, Corollary 4.8] *If either $A \in (l^u,(\bar{N},p)_0)$ or $A \in (l^u,(\bar{N},p))$ for $1 < u < \infty$, then*

$$L_A \quad \text{is compact if and only if}$$

$$\lim_{r\to\infty}\left[\sup_{m>r}\left(\sum_{k=0}^{\infty}\left|\frac{1}{P_m}\sum_{n=0}^{m} p_n a_{nk}\right|^v\right)^{1/v}\right] = 0 \quad \text{where } v = u/(u-1).$$

Further, if either $A \in (l^1,(\bar{N},p)_0))$ or $A \in (l^1,(\bar{N},p)))$, then

$$L_A \quad \text{is compact if and only if} \quad \lim_{r\to\infty}\left(\sup_{n>r,k}\left|\frac{1}{P_m}\sum_{n=0}^{m} p_n a_{nk}\right|\right) = 0.$$

If $A \in (l^u,(\bar{N},p))$ for $1 < u < \infty$, then

$$L_A \quad \text{is compact if}$$

$$\lim_{r\to\infty}\left[\sup_{m>r}\left(\sum_{k=0}^{\infty}\left|\frac{1}{P_m}\sum_{n=0}^{m} p_n a_{nk}\right|^v\right)^{1/v}\right] = 0 \quad \text{where } v = u/(u-1).$$

Finally, if $A \in (l^1,(\bar{N},p))$, then

$$L_A \quad \text{is compact if} \ \lim_{r\to\infty}\left(\sup_{n>r,k}\left|\frac{1}{P_m}\sum_{n=0}^{m} p_n a_{nk}\right|\right) = 0.$$

From Corollary 3.41, Theorem 3.8 and Remark 3.22 (b), we have

Corollary 3.43. [72, Corollary 4.9] *If $A \in (X,(\bar{N},p)_0)$ for $X = (\bar{N},q)_0$ or $X = (\bar{N},q)$, or if $A \in (X,(\bar{N},p))$ for $X = (\bar{N},q)_0$ or $X = (\bar{N},q)$, then we have in all cases*

$$L_A \quad \text{is compact if and only if}$$

$$\lim_{r\to\infty}\left[\sup_{m>r,n}\left(\sum_{k=0}^{n-1} Q_k\left|\frac{1}{P_m}\sum_{l=0}^{m} p_l(\Delta^+ A_l/q)_k\right| + \left|\frac{Q_n}{q_n P_m}\sum_{l=0}^{m} p_l a_{ln}\right|\right)\right] = 0.$$

Further, if $A \in (X,(\bar{N},p)_\infty)$ for $X = (\bar{N},q)_\infty$, $X = (\bar{N},q)_0$ or $X = (\bar{N},q)$, then we have

$$L_A \quad \text{is compact if}$$

$$\lim_{r\to\infty}\left[\sup_{m>r,n}\left(\sum_{k=0}^{n-1} Q_k\left|\frac{1}{P_m}\sum_{l=0}^{m} p_l(\Delta^+ A_l/q)_k\right| + \left|\frac{Q_n}{q_n P_m}\sum_{l=0}^{m} p_l a_{ln}\right|\right)\right] = 0.$$

3.7. Spaces of strongly summable and convergent sequences. In this subsection, we shall study spaces of *strongly summable* and *strongly convergent* sequences and give their dual spaces. The results of this subsection can be found in [59, 64, 65, 70].

For $X \subset \omega$ and any real $p > 0$, we write

$$X_{[A]^p} = \{x \in \omega : A(|x|^p) \in X\}.$$

If $p = 1$, then we omit the index p and write $X_{[B]} = X_{[B]^1}$ for short.

Let C_1 be the Cesáro matrix of order 1, that is $(C_1)_{nk} = 1/n$ for $1 \le k \le n$ and 0 for $k > n$ $(n = 0, 1, \dots)$. For $0 < p < \infty$, Maddox [51, 54] defined the sets

$$\begin{aligned} w_0^p &= (c_0)_{[C_1]^p} = \left\{x \in \omega : \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|^p\right) = 0\right\}, \\ w_p &= \{x \in \omega : x - le \in w_0^p \text{ for some complex number } l\}, \\ w_\infty^p &= (l_\infty)_{[C_1]^p}. \end{aligned}$$

These sets are special cases of the so-called mixed normed spaces (see e.g. [38, 35, 36, 59, 22, 23]). Here we shall only deal with the cases $1 \le p < \infty$. The following result is well known.

Proposition 3.44. [51, 59] *Each of the set w_0, w^p and w_∞^p is a BK space for $1 \le p < \infty$ with*

$$\|x\| = \sup_{\nu \ge 0}\left(\frac{1}{2^\nu}\sum_{k=2^\nu}^{2^{\nu+1}-1}|x_k|^p\right)^{1/p}; \tag{3.101}$$

w_0^p has AK; every sequence $x = (x_k)_{k=1}^\infty \in w^p$ has a unique representation $x = le + \sum_{k=1}^\infty (x_k - l)e^{(k)}$ where $l \in \mathbb{C}$ is such that $x - le \in w_0^p$.

Let $\mu = (\mu_n)_{n=0}^\infty$ be a nondecreasing sequence of positive reals tending to infinity. If $(n(\nu))_{\nu=0}^\infty$ is a sequence such that $0 = n(0) < n(1) < n(2) < \dots$, then we denote the set of all integers k with $n(\nu) \le k \le n(\nu+1) - 1$ by $K^{\langle\nu\rangle}$, and we write $\sum_\nu$ and $\max_\nu$ for the sum and maximum taken over all k in $K^{\langle\nu\rangle}$. We define the matrices $B = (b_{nk})_{n,k=0}^\infty$ and $\tilde{B} = (\tilde{b}_{\nu k})_{\nu,k=0}^\infty$ by

$$b_{nk} = \begin{cases} 1/\lambda_n & (0 \le k \le n) \\ 0 & (k < n) \end{cases} \quad \text{and} \quad \tilde{b}_{\nu k} = \begin{cases} 1/\lambda_{n(\nu+1)} & (k \in K^{<\nu>}) \\ 0 & (k \notin K^{<\nu>}). \end{cases}$$

Further, let $\Delta(\mu)$ be the matrix with

$$\Delta_{nk}(\mu) = \begin{cases} -\mu_{n-1} & (k = n-1) \\ \mu_n & (k = n) \\ 0 & (\text{otherwise}) \end{cases} \quad (n = 0, 1, \dots) \quad \text{where } \mu_{-1} = 0.$$

We define the sets [76, 64]

$$c_0(\mu) = ((c_0)_{[B]})_{\Delta(\mu)}, \qquad \tilde{c_0}(\mu) = ((c_0)_{[\tilde{B}]})_{\Delta(\mu)},$$
$$c(\mu) = \{x \in \omega : x - le \in c_0(\mu)\}, \qquad \tilde{c}(\mu) = \{x \in \omega : x - le \in \tilde{c}_0(\mu)\},$$
$$c_\infty(\mu) = ((l_\infty)_{[B]})_{\Delta(\mu)}, \qquad \tilde{c_\infty}(\mu) = ((l_\infty)_{[\tilde{B}]})_{\Delta(\mu)}.$$

The following result is well known.

Proposition 3.45. [64, Theorem 2 (c)] *Let $\mu = (\mu_n)_{n=0}^\infty$ be a nondecreasing sequence of positive reals tending to infinity. Then each of the spaces $c_0(\mu)$, $c(\mu)$ and $c_\infty(\mu)$ is a BK space*

$$\|x\|' = \|B(|\Delta(\mu)x|))\|_\infty = \sup_{n\geq 0}\left(\frac{1}{\mu_n}\sum_{k=0}^{n}|\mu_k x_k - \mu_{k-1}x_{k-1}|\right);$$

$c_0(\mu)$ has AK; every sequence $x = (x_k)_{k=0}^\infty \in c_\mu$ has a unique representation $x = le + \sum_{k=1}^\infty (x_k - l)e^{(k)}$ where $l \in \mathbb{C}$ is such that $x - le \in c_0(\mu)$.

A sequence $\Lambda = (\lambda_n)_{n=0}^\infty$ of positive reals is called *exponentially bounded* if there is an integer $m \geq 2$ such that for all integers ν there is at least one λ_n in the interval $[m^\nu, m^{\nu+1})$. It is known (cf. [64, Lemma 1]) that a nondecreasing sequence $\Lambda = (\lambda_n)_{n=0}^\infty$ of positive reals is exponentially bounded if and only if there are reals $s \leq t$ such that $0 < s \leq \lambda_{n(\nu)}/\lambda_{n(\nu+1)} \leq t < 1$ for some subsequence $(\lambda_{n(\nu+1)})_{\nu=0}^\infty$ for all $\nu = 0, 1, \dots$; such a subsequence is called an *associated subsequence*.

The following result is well known.

Proposition 3.46. [64, Theorem 2] *Let $\Lambda = (\lambda_n)_{n=0}^\infty$ be a nondecreasing exponentially bounded sequence of positive reals and $(\lambda_{n(\nu+1)})_{\nu=0}^\infty$ an associated subsequence. Then $c_0(\Lambda) = \tilde{c}_0(\Lambda)$, $c(\Lambda) = \tilde{c}(\Lambda)$ and $c_\infty(\Lambda) = \tilde{c}_\infty(\Lambda)$. The norms $\|x\|'$ and*

$$\|x\| = l\|\tilde{B}(|\Delta(\Lambda)x|)\|_\infty = \sup_{\nu\geq 0}\left(\frac{1}{\lambda_{n(\nu+1)}}\sum_{\nu}|\lambda_k x_k - \lambda_{k-1}x_{k-1}|\right)$$

are equivalent on $c_0(\Lambda)$, $c(\Lambda)$ and $c_\infty(\Lambda)$. Thus each of the spaces $c_0(\Lambda)$, $c(\Lambda)$ and $c_\infty(\Lambda)$ is a BK space with $\|\cdot\|$.

Proposition 3.47. [51] and [59, Theorems 4 and 6] *Let $K^{\langle\nu\rangle} = [2^\nu, 2^{\nu+1} - 1]$ $(\nu = 0, 1, \dots)$. We put*

$$\mathcal{M}^p = \begin{cases} \left\{a \in \omega : \sum_{\nu=0}^\infty 2^{\nu/p}\max_\nu |a_k| < \infty\right\} & (p = 1) \\ \left\{a \in \omega : \sum_{\nu=0}^\infty 2^{\nu/p}\left(\sum_\nu |a_k|^q\right)^{1/q} < \infty\right\} & \left(1 < p < \infty;\ q = \dfrac{p}{p-1}\right) \end{cases}$$

$$\|a\|_{\mathcal{M}^p} = \begin{cases} \sum_{\nu=0}^\infty 2^{\nu/p}\max_\nu |a_k| & (p = 1) \\ \sum_{\nu=0}^\infty 2^{\nu/p}\left(\sum_\nu |a_k|^q\right)^{1/q} & (1 < p < \infty) \end{cases} \quad \text{for all } a \in \mathcal{M}^p.$$

Then $(w_0^p)^\beta = (w^p)^\beta = (w_\infty^p)^\beta = \mathcal{M}^p$ and $\|a\|^* = \|a\|_{\mathcal{M}^p}$ on $\mathcal{M}^p$.

Proposition 3.48. [65, Lemma 2] *Let $\Lambda = (\lambda_n)_{n=0}^\infty$ be a nondecreasing exponentially bounded sequence of positive reals and $(\lambda_{n(\nu+1)})_{\nu=0}^\infty$ an associated subsequence. We put*

$$\mathcal{C}(\Lambda) = \left\{ a \in \omega : \sum_{\nu=0}^\infty \lambda_{n(\nu+1)} \max_\nu \left| \sum_{k=n}^\infty \frac{a_k}{\lambda_k} \right| < \infty \right\}$$
$$\|a\|_{\mathcal{C}(\Lambda)} = \sum_{\nu=0}^\infty \lambda_{n(\nu+1)} \max_\nu \left| \sum_{k=n}^\infty \frac{a_k}{\lambda_k} \right| \quad \text{for all } a \in \mathcal{C}(\Lambda).$$

Then $(c_0(\Lambda))^\beta = (c(\Lambda))^\beta = (c_\infty(\Lambda))^\beta = \mathcal{C}(\Lambda)$ and $\|a\|^* = \|a\|_{\mathcal{C}(\Lambda)}$ on $\mathcal{C}(\Lambda)$.

As an immediate consequence of Theorems 3.8 and 3.10 we obtain

Corollary 3.49. [70, Corollary 1] *Let X be an arbitrary FK space. Further, let $\mu = (\mu_n)_{n=0}^\infty$ be a nondecreasing sequence of positive reals tending to infinity. We write $w_0 = w_0^1$ etc., for short and put*

$$M(X, w_\infty) = \sup_{m \geq 1} \left(\max_{N_m \subset \{1,\dots,m\}} \left\| \frac{1}{m} \sum_{n \in N_m} A_n \right\|_D^* \right)$$
$$M(X, c_\infty(\mu)) = \sup_{m \geq 0} \left(\max_{N_m \subset \{0,\dots,m\}} \left\| \frac{1}{\mu_m} \sum_{n \in N_m} (\mu_n A_n - \mu_{n-1} A_{n-1} \right\|_D^* \right).$$

(a) *Then $A \in (X, w_\infty)$ if and only if*

$$M(X, w_\infty) < \infty \quad \text{for some } D > 0. \tag{3.102}$$

Furthermore, if $(b^k)_{k=0}^\infty$ is a basis of X, then $A \in (X, w_0)$ if and only if condition (3.102) holds and

$$\lim_{m \to \infty} \left(\frac{1}{m} \sum_{n=1}^m |A_n(b^{(k)})| \right) = 0 \quad \text{for all } k = 0, 1, \dots; \tag{3.103}$$

$A \in (X, w)$ if and only if condition (3.102) holds and there are complex numbers l_k $(k = 0, 1, \dots)$ such that

$$\lim_{m \to \infty} \left(\frac{1}{m} \sum_{n=1}^m |A_n(b^{(k)}) - l_k)| \right) = 0 \quad \text{for all } k = 0, 1, \dots. \tag{3.104}$$

Finally, if X is a normed space and $A \in (X, Y)$ for $Y = w_0, w$ or w_∞, then, for

$$\|A\|_{w_\infty}^* = \sup_{m \geq 1} \left(\max_{N_m \subset \{1,\dots,m\}} \left\| \frac{1}{m} \sum_{n \in N_m} A_n \right\|^* \right),$$

we have

$$\|A\|^*_{w_\infty} \leq \|L_A\| \leq 4 \cdot \|A\|^*_{w_\infty}. \tag{3.105}$$

(b) *Then* $A \in (X, c_\infty(\mu))$ *if and only if*

$$M(X, c_\infty(\mu)) < \infty \quad \text{for some } D > 0. \tag{3.106}$$

Further, if $(b^k)_{k=0}^\infty$ *is a basis of* X*, then* $A \in (X, c_0(\mu))$ *if and only if condition (3.106) holds and*

$$\lim_{m\to\infty}\left(\frac{1}{\mu_m}\sum_{n=0}^{m}|\mu_n A_n(b^{(k)}) - \mu_{n-1}A_{n-1}(b^{(k)})|\right) = 0 \tag{3.107}$$

for all $k = 0, 1, \ldots$; $A \in (X, c(\mu))$ *if and only if condition (3.107) holds and there are complex numbers* l_k *such that*

$$\lim_{m\to\infty}\left(\frac{1}{\mu_m}\sum_{n=0}^{m}|\mu_n(A_n(b^{(k)}) - l_k) - \mu_{n-1}(A_{n-1}(b^{(k)}) - l_k)|\right) = 0 \tag{3.108}$$

for all $k = 0, 1, \ldots$.

Finally, if X *is normed and* $A \in (X, Y)$ *for* $Y = c_0(\mu)$, $c(\mu)$ *or* $c_\infty(\mu)$*, then, for*

$$\|A\|^*_{c_\infty} = \sup_{m\geq 0}\left(\max_{N_m\subset\{0,\ldots,m\}}\left\|\frac{1}{\mu_m}\sum_{n\in N_m}(\mu_n A_n - \mu_{n-1}A_{n-1}\right\|^*\right),$$

we have

$$\|A\|^*_{c_\infty(\mu)} \leq \|L_A\| \leq 4 \cdot \|A\|^*_{c_\infty(\mu)}. \tag{3.109}$$

Proof. All we have to show are inequalities (3.105) and (3.109). Let $A \in (X, Y)$ where $Y = w_0$, w or w_∞. Then

$$\left|\frac{1}{m}\sum_{n\in N_m}A_n(x)\right| \leq \frac{1}{m}\sum_{n=1}^{m}|A_n(x)| \leq \|L_A\|$$

for all $m = 1, 2, \ldots$, all $N \subset N_m$ and all $\|x\| = 1$. This implies

$$\|A\|^*_{w_\infty} \leq \|L_A\|. \tag{3.110}$$

Further, given $\varepsilon > 0$ there is $x \in X$ with $\|x\| = 1$ such that

$$\|A(x)\| = \sup_{m\geq 1}\left(\frac{1}{m}\sum_{n=1}^{m}|A_n(x)|\right) \geq \|L_A\| - \varepsilon/2,$$

and there is an integer $m(x)$ such that

$$\frac{1}{m(x)}\sum_{n=1}^{m(x)}|A_n(x)| \geq \|A(x)\| - \varepsilon/2,$$

consequently

$$\frac{1}{m(x)}\sum_{n=1}^{m(x)}|A_n(x)| \geq \|L_A\| - \varepsilon.$$

By Lemma 3.9

$$4\cdot \max_{N_{m(x)}\subset\{1,\dots,m(x)\}}\left(\frac{1}{m(x)}\left|\sum_{n\in N_{m(x)}} A_n(x)\right|\right) \geq \frac{1}{m(x)}\sum_{n=1}^{m(x)}|A_n(x)| \geq \|L_A\| - \varepsilon$$

and so $4\cdot\|A\|^*_{w_\infty} \geq \|L_A\| - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $4\cdot\|A\|^*_{w_\infty} \geq \|L_A\|$. Together with inequality (3.110) this yields (3.105). The inequalities in (3.109) are proved similarly. □

Remark 3.50. If X is a given BK space, and Y is any of the spaces w_0, w, w_∞, $c_0(\mu)$, $c(\mu)$ or $c_\infty(\mu)$, then the conditions for $A \in (X,Y)$ follow from the respective ones in Corollary 3.49 by replacing the norms $\|\cdot\|^*_D$ in conditions (3.102) and (3.106) by the natural norms on the β–duals of X. We shall write

$$\max_{N_m} \quad \text{for} \quad \begin{cases} \max\limits_{N_m\subset\{0,\dots,m\}} & \text{if } Y = w_0, w, w_\infty \\ \max\limits_{N_m\subset\{1,\dots,m\}} & \text{if } Y = c_0(\mu), c(\mu), c_\infty(\mu), \end{cases}$$

$q = p/(p-1)$ for $1 < p < \infty$, $\Delta_n(\mu_n a_{nk}) = \mu_n a_{nk} - \mu_{n-1}a_{n-1,k}$

$$\max_\nu \quad \text{for} \quad \max_{2^\nu \leq k \leq 2^{\nu+1}-1}, \quad \sum_\nu \quad \text{for} \quad \sum_{2^\nu\leq k\leq 2^{\nu+1}-1}.$$

(a) For $X = l_p$, we have

$$M(l_p, w_\infty) = \begin{cases} \sup\limits_m\left(\max\limits_{N_m}\left(\sup\limits_k\left|\frac{1}{m}\sum\limits_{n\in N_m} a_{nk}\right|\right)\right) & (p=1) \\ \sup\limits_m\left(\max\limits_{N_m}\left(\sum\limits_{k=1}^\infty\left|\frac{1}{m}\sum\limits_{n\in N_m} a_{nk}\right|^q\right)\right) & (1 < p \leq \infty), \end{cases}$$

$$M(l_p, c_\infty(\mu)) = \begin{cases} \sup\limits_m\left(\max\limits_{N_m}\left(\sup\limits_k\left|\frac{1}{\mu_m}\sum\limits_{n\in N_m} \Delta_n(\mu_n a_{nk})\right|\right)\right) & (p=1) \\ \sup\limits_m\left(\max\limits_{N_m}\left(\sum\limits_{k=1}^\infty\left|\frac{1}{\mu_m}\sum\limits_{n\in N_m} \Delta_n(\mu_n a_{nk})\right|^q\right)\right) & (1 < p < \infty). \end{cases}$$

(b) For $X = w_\infty^p$, we have

$$M(w_\infty^p, w_\infty) = \begin{cases} \sup_m \left(\max_{N_m} \left(\sum_{\nu=0}^\infty 2^{\nu/p} \max_\nu \left| \frac{1}{m} \sum_{n \in N_m} a_{nk} \right| \right) \right) & \text{for } p = 1 \\ \sup_m \left(\max_{N_m} \left(\sum_{\nu=0}^\infty 2^{\nu/p} \left(\sum_\nu \left| \frac{1}{m} \sum_{n \in N_m} a_{nk} \right|^q \right)^{1/q} \right) \right) & \text{for } 1 < p < \infty, \end{cases}$$

and

$$M(w_\infty^p, c_\infty(\mu)) = \begin{cases} \sup_m \left(\max_{N_m} \left(\sum_{\nu=0}^\infty 2^{\nu/p} \max_\nu \left| \frac{1}{\mu_m} \sum_{n \in N_m} \Delta_n(\mu_n a_{nk}) \right| \right) \right) & \text{for } p = 1 \\ \sup_m \left(\max_{N_m} \left(\sum_{\nu=0}^\infty 2^{\nu/p} \left(\sum_\nu \left| \frac{1}{\mu_m} \sum_{n \in N_m} \Delta_n(\mu_n a_{nk}) \right|^q \right)^{1/q} \right) \right) & \text{for } 1 < p < \infty. \end{cases}$$

(c) Let $\Lambda = (\lambda_k)_{k=0}^\infty$ be an exponentially bounded sequence of positive reals and $(\lambda_{k(\nu)})_{\nu=0}^\infty$ be an associated subsequence. We write $\max_\nu$ and $\sum_\nu$ for the maximum and the sum taken over all integers k such that $k(\nu) \le k \le k(\nu+1) - 1$. Then for $X = c_\infty(\Lambda)$ we have

$$M(c_\infty(\Lambda), w_\infty) = \sup_m \left(\max_{N_m} \left(\sum_{\nu=0}^\infty \lambda_{k(\nu+1)} \max_\nu \left| \sum_{j=k}^\infty \frac{1}{\lambda_j} \left(\frac{1}{m} \sum_{n \in N_m} a_{nj} \right) \right| \right) \right),$$

and

$$M(c_\infty(\Lambda), c_\infty(\mu)) = \sup_m \left(\max_{N_m} \left(\sum_{\nu=0}^\infty \lambda_{k(\nu+1)} \max_\nu \left| \sum_{j=k}^\infty \frac{1}{\lambda_j} \left(\frac{1}{\mu_m} \sum_{n \in N_m} \Delta_n(\mu_n a_{nk}) \right) \right| \right) \right).$$

The main result of this subsection is the following theorem (see Corollary 3.49).

Theorem 3.51. [70, Theorem 3] *Let A, X and Y be as in Corollary 3.49*
(a) *If X is a normed space and $A \in (X, Y)$ for $Y = w_0, w$ and w_∞, then, for*

$$\|A\|_{w_\infty}^{(m)} = \sup_{k > m} \left(\max_{N_{m,k} \subset \{m+1, \dots, k\}} \left\| \frac{1}{k} \sum_{i \in N_{m,k}} A_i \right\|^* \right)$$

we have

$$\lim_{m \to \infty} \|A\|_{w_\infty}^{(m)} \le \|L_A\|_\chi \le 4 \cdot \lim_{m \to \infty} \|A\|_{w_\infty}^{(m)} \quad \textit{if } Y = w_0, \tag{3.111}$$

$$\frac{1}{2} \cdot \lim_{m \to \infty} \|A\|_{w_\infty}^{(m)} \le \|L_A\|_\chi \le 4 \cdot \lim_{m \to \infty} \|A\|_{w_\infty}^{(m)} \quad \textit{if } Y = w, \tag{3.112}$$

$$(3.113) \qquad 0 \leq \|L_A\|_\chi \leq 4 \cdot \lim_{m\to\infty} \|A\|_{w_\infty}^{(m)} \qquad \textit{if } Y = w_\infty.$$

(b) *If X is a normed space and $A \in (X,Y)$ for $Y = c_0(\mu), c(\mu)$ and $c_\infty(\mu)$, then, for*

$$(3.114) \qquad \|A\|_{c_\infty}^{(m)} = \sup_{k>m}\left(\max_{N_{m,k}\subset\{m+1,\dots,k\}}\left\|\frac{1}{\mu_k}\sum_{i\in N_{m,k}}\mu_i A_i - \mu_{i-1}A_{i-1}\right\|^*\right)$$

we have

$$(3.115) \qquad \lim_{m\to\infty}\|A\|_{c_\infty}^{(m)} \leq \|L_A\|_\chi \leq 4\cdot \lim_{m\to\infty}\|A\|_{c_\infty}^{(m)} \quad \textit{if } Y = c_0(\mu),$$

$$(3.116) \qquad \frac{1}{2}\cdot \lim_{m\to\infty}\|A\|_{c_\infty}^{(m)} \leq \|L_A\|_\chi \leq 4\cdot \lim_{m\to\infty}\|A\|_{c_\infty}^{(m)} \quad \textit{if } Y = c(\mu),$$

$$(3.117) \qquad 0 \leq \|L_A\|_\chi \leq 4\cdot \lim_{m\to\infty}\|A\|_{c_\infty}^{(m)} \quad \textit{if } Y = c_\infty(\mu).$$

Proof. Let us remark that the limits in (3.111) and (3.115) exist. We put $B = \{x \in X : \|x\| \leq 1\}$. In the case $Y = w_0$ we have by Theorem 2.23

$$(3.118) \qquad \|L_A\|_\chi = \chi(A(B)) = \lim_{m\to\infty}\left[\sup_{x\in B}\|(I-P_m)(A(x))\|\right],$$

where $P_m : w_0 \mapsto w_0$, $m = 1, 2, \dots$, is the projector on the first m coordinates, that is $P_m(x) = (x_1, x_2, \dots, x_m, 0, 0, \dots)$ for $x = (x_i) \in w_0$; (let us remark that $\|I - P_m\| = 1$ for $m = 1, 2, \dots$). For given $\epsilon > 0$ there is $x \in B$ such that

$$(3.119) \qquad \|(I-P_m)(A(x))\| > \|(I-P_m)(A)\| - \epsilon/2.$$

Now there is an integer $k(x) > m$ such that

$$(3.120) \qquad \frac{1}{k(x)}\sum_{i=m+1}^{k(x)}|A_i(x)| > \|(I-P_m)(A(x))\| - \frac{\epsilon}{2}.$$

Further by Lemma 3.9

$$4\cdot\left(\max_{N_{m,k(x)}\subset\{m+1,\dots,k(x)\}}\frac{1}{k(x)}\left|\sum_{i\in N_{m,k(x)}}A_i(x)\right|\right) \geq \frac{1}{k(x)}\sum_{i=m+1}^{k(x)}|A_i(x)|.$$

Now, by (3.119) and (3.120) we get

$$(3.121) \qquad 4\cdot\left(\max_{N_{m,k(x)}\subset\{m+1,\dots,k(x)\}}\frac{1}{k(x)}\left|\sum_{i\in N_{m,k(x)}}A_i(x)\right|\right) \geq \|(I-P_m)(A)\| - \epsilon.$$

Since $\epsilon > 0$ was arbitrary and $x \in B$, from (3.121) we have for each m

$$\|(I - P_m)(A)\| \leq 4 \cdot \left[\sup_{k>m}\left(\max_{N_{m,k}\subset\{m+1,\dots,k\}}\left\|\frac{1}{k}\sum_{i\in N_{m,k}} A_i\right\|\right)\right]. \tag{3.122}$$

Hence, by (3.118) and (3.122) we get the right inequality in (3.111). To prove the left inequality in (3.111), suppose that m is an integer, $k > m$, $N_{m,k} \subset \{m+1, \dots, k\}$ and $x \in B$. Then

$$\left|\frac{1}{k}\sum_{i\in N_{m,k}} A_i(x)\right| \leq \frac{1}{k}\sum_{i\in N_{m,k}} |A_i(x)| \leq \frac{1}{k}\sum_{i=m+1}^{k} |A_i(x)| \leq \|(I - P_m)(A(x))\|.$$

Thus for all m and $k > m$ we have

$$\left\|\frac{1}{k}\sum_{i\in N_{m,k}} A_i\right\| \leq \|(I - P_m)(L_A)\|,$$

and by (3.118) we get the left inequality in (3.111).

To prove (3.112) we recall that every sequence $x = (x_k)_{k=1}^\infty \in w$ has a unique representation $x = le + \sum_{k=1}^\infty (x_k - l)e^{(k)}$ where $l \in \mathbb{C}$ is such that $x - le \in w$. Let us define $P_m : w \mapsto w$ by $P_m(x) = le + \sum_{k=1}^m (x_k - l)e^{(k)}$ for $m = 1, 2, \dots$. It is easy to prove that $\|I - P_m\| = 2$ for $m = 1, 2, \dots$. Now the proof of(3.112) is similar as in the case (3.111), and we omit it.

Let us prove (3.113). Now define $P_m : w_\infty \mapsto w_\infty$ by $P_m(x) = (x_1, x_2, \dots, x_m, 0, \dots)$ for all $x = (x_i) \in w_\infty$ and $m = 1, 2, \dots$. It is clear that $A(B) \subset P_m(A(B)) + (I - P_m)(A(B))$. Now, by the elementary properties of the function χ we have

$$\begin{aligned} \chi(A(B)) &\leq \chi(P_m(A(B))) + \chi((I - P_m)(A(B))) = \chi((I - P_m)(A(B)) \\ &\leq \sup_{x\in B} \|(I - P_m)(A(x))\|. \end{aligned} \tag{3.123}$$

Since the limit in (3.113) obviously exists, by (3.123) and from the proof of the right inequality in (3.111) we get (3.113).

Let us mention that inequalities (3.115), (3.116) and (3.117) are proved similarly as the inequalities (3.111), (3.112) and (3.113). □

Now as a corollary of the theorem above we have

Corollary 3.52. [70, Corollary 2] *Let A, X and Y be as in Theorem 3.51. Then for $A \in (X, Y)$ we have*

$$A \text{ is compact if and only if } \|A\|_{w_\infty} < \infty \text{ and}$$
$$\lim_{m\to\infty} \|A\|_{w_\infty}^{(m)} = 0, \qquad \text{if } Y = w_0 \text{ and } w,$$

$$A \text{ is compact if } \|A\|_{w_\infty} < \infty \text{ and}$$
$$\lim_{m\to\infty} \|A\|^{(m)}_{w_\infty} = 0, \qquad \text{if } Y = w_\infty,$$

$$A \text{ is compact if and only if } \|A\|_{c_\infty(\mu)} < \infty \text{ and}$$
$$\lim_{m\to\infty} \|A\|^{(m)}_{c_\infty(\mu)} = 0, \qquad \text{if } Y = c_0(\mu) \text{ and } c(\mu),$$

$$A \text{ is compact if } \|A\|_{c_\infty(\mu)} < \infty \text{ and}$$
$$\lim_{m\to\infty} \|A\|^{(m)}_{c_\infty(\mu)} = 0, \qquad \text{if } Y = c_\infty(\mu).$$

Now, concerning Remark 3.50, we get several corollaries.

Corollary 3.53. [70, Corollary 3] *Let A, X and Y be as in Theorem 3.51 and in Remark 3.50* (a). *We shall write $\max_{N_{m,k}}$ for $\max_{N_{m,k}\subset\{m+1,\dots,k\}}$. For $A \in (X,Y)$ and $X = l_p$, we set for each m*

$$M(l_p, w_\infty)^{(m)} = \begin{cases} \sup_{k>m}\left(\max_{N_{m,k}}\left(\sup_j \left|\frac{1}{k}\sum_{i\in N_{m,k}} a_{ij}\right|\right)\right) & \text{for } p = 1 \\ \sup_{k>m}\left(\max_{N_{m,k}}\left(\sum_{j=1}^\infty \left|\frac{1}{k}\sum_{i\in N_{m,k}} a_{ij}\right|^q\right)^{1/q}\right) & \text{for } 1 < p < \infty, \end{cases}$$

and

$$M(l_p, c_\infty(\mu))^{(m)} = \begin{cases} \sup_{k>m}\left(\max_{N_{m,k}}\left(\sup_j \left|\frac{1}{\mu_k}\sum_{i\in N_{m,k}} \Delta_i(\mu_i a_{ij})\right|\right)\right) & \text{for } p = 1 \\ \sup_{k>m}\left(\max_{N_{m,k}}\left(\sum_{j=1}^\infty \left|\frac{1}{\mu_k}\sum_{i\in N_{m,k}} \Delta_i(\mu_i a_{ij})\right|^q\right)^{1/q}\right) & \text{for } 1 < p < \infty. \end{cases}$$

Now we have

$$A \text{ is compact if and only if } M(l_p, w_\infty) < \infty \text{ and}$$
$$\lim_{m\to\infty} M(l_p, w_\infty)^{(m)} = 0, \qquad \text{if } Y = w_0 \text{ and } w,$$

$$A \text{ is compact if } M(l_p, w_\infty) < \infty \text{ and}$$
$$\lim_{m\to\infty} M(l_p, w_\infty)^{(m)} = 0, \qquad \text{if } Y = w_\infty,$$

$$A \text{ is compact if and only if } M(l_p, c_\infty(\mu)) < \infty \text{ and}$$
$$\lim_{m\to\infty} M(l_p, c_\infty(\mu))^{(m)} = 0, \qquad \text{if } Y = c_0(\mu) \text{ and } c(\mu),$$

$$A \text{ is compact if } M(l_p, c_\infty(\mu)) < \infty \text{ and}$$
$$\lim_{m\to\infty} M(l_p, c_\infty(\mu))^{(m)} = 0, \qquad \text{if } Y = c_\infty(\mu).$$

Corollary 3.54. [70, Corollary 4] *Let* A, X *and* Y *be as in Theorem 3.51 and in Remark 3.50 (b). We shall write* $\max_{N_{m,k}}$ *for* $\max_{N_{m,k}\subset\{m+1,\dots,k\}}$. *For* $A \in (X, Y)$ *and* $X = w_\infty^p$, *we set for each* m

$$M(w_\infty^p, w_\infty)^{(m)} = \begin{cases} \sup_{k>m}\left(\max_{N_{m,k}}\left(\sum_{\nu=0}^{\infty} 2^{\nu/p}\ \max_{2^\nu\le j\le 2^{\nu+1}-1}\left|\frac{1}{k}\sum_{i\in N_{m,k}} a_{ij}\right|\right)\right) \\ \text{for}\quad p=1 \\ \sup_{k>m}\left(\max_{N_{m,k}}\left(\sum_{\nu=0}^{\infty} 2^{\nu/p}\left(\sum_{2^\nu\le j\le 2^{\nu+1}-1}\left|\frac{1}{k}\sum_{i\in N_{m,k}} a_{ij}\right|^q\right)^{1/q}\right)\right) \\ \text{for}\quad 1<p<\infty, \end{cases}$$

and

$$M(w_\infty^p, c_\infty(\mu))^{(m)} = \begin{cases} \sup_{k>m}\left(\max_{N_{m,k}}\left(\sum_{\nu=1}^{\infty} 2^{\nu/p}\ \max_{2^\nu\le j\le 2^{\nu+1}-1}\left|\frac{1}{\mu_k}\sum_{i\in N_{m,k}} \Delta_i(\mu_i a_{ij})\right|\right)\right) \\ \text{for } p=1 \\ \sup_{k>m}\left(\max_{N_{m,k}}\left(\sum_{\nu=1}^{\infty} 2^{\nu/p}\left(\sum_{2^\nu\le j\le 2^{\nu+1}-1}\left|\frac{1}{\mu_k}\sum_{i\in N_{m,k}} \Delta_i(\mu_i a_{ij})\right|^q\right)^{1/q}\right)\right) \\ \text{for } 1<p<\infty. \end{cases}$$

Now we have

$$A \text{ is compact if and only if } M(w_\infty^p, w_\infty) < \infty \text{ and}$$
$$\lim_{m\to\infty} M(w_\infty^p, w_\infty)^{(m)} = 0, \qquad \text{if } Y = w_0 \text{ and } w,$$

$$A \text{ is compact if } M(w_\infty^p, w_\infty) < \infty \text{ and}$$
$$\lim_{m\to\infty} M(w_\infty^p, w_\infty)^{(m)}, \qquad \text{if } Y = w_\infty,$$

$$A \text{ is compact if and only if } M(w_\infty^p, c_\infty(\mu)) < \infty \text{ and}$$
$$\lim_{m\to\infty} M(w_\infty^p, c_\infty(\mu))^{(m)} = 0, \qquad \text{if } Y = c_0(\mu) \text{ and } c(\mu),$$

$$A \text{ is compact if } M(w_\infty^p, c_\infty(\mu)) < \infty \text{ and}$$
$$\lim_{m\to\infty} M(w_\infty^p, c_\infty(\mu))^{(m)} = 0, \qquad \text{if } Y = c_\infty(\mu).$$

Corollary 3.55. [70, Corollary 5] *Let A, X and Y be as in Theorem 3.51 and in Remark 3.50 (c). We shall write* $\max_{N_{m,k}}$ *for* $\max_{N_{m,k}\subset\{m+1,\dots,k\}}$. *For $A \in (X,Y)$, if $X = c_0(\Lambda)$, $c(\Lambda)$ or $c_\infty(\Lambda)$, we set for each m*

$$M(c_\infty(\Lambda), w_\infty)^{(m)} = \sup_{k>m}\left(\max_{N_{m,k}}\left(\sum_{\nu=0}^{\infty}\lambda_{r(\nu+1)}\max_{r(\nu)\le r\le r(\nu+1)-1}\left|\sum_{j=r}^{\infty}\frac{1}{\lambda_j}\left(\frac{1}{k}\sum_{i\in N_{m,k}}a_{ij}\right)\right|\right)\right)$$

and

$$M(c_\infty(\Lambda), c_\infty(\mu))^{(m)} = \sup_{k>m}\left(\max_{N_{m,k}}\left(\sum_{\nu=0}^{\infty}\lambda_{r(\nu+1)}\max_{r(\nu)\le r\le r(\nu+1)-1}\left|\sum_{j=r}^{\infty}\frac{1}{\lambda_j}\left(\frac{1}{\mu_k}\sum_{i\in N_{m,k}}\Delta_i(\mu_i a_{ij})\right)\right|\right)\right)$$

Now we have

A is compact if and only if $M(c_\infty(\Lambda), w_\infty) < \infty$ *and*
$$\lim_{m\to\infty} M(c_\infty(\Lambda), w_\infty)^{(m)} = 0, \qquad \text{if } Y = w_0 \text{ and } w,$$

A is compact if $M(c_\infty(\Lambda), w_\infty) < \infty$ *and*
$$\lim_{m\to\infty} M(c_\infty(\Lambda), w_\infty)^{(m)} = 0, \qquad \text{if } Y = w_\infty,$$

A is compact if and only if $M(c_\infty(\Lambda), c_\infty(\mu)) < \infty$ *and*
$$\lim_{m\to\infty} M(c_\infty(\Lambda), c_\infty(\mu))^{(m)} = 0, \qquad \text{if } Y = c_0(\mu) \text{ and } c(\mu),$$

A is compact if $M(c_\infty(\Lambda), c_\infty(\mu)) < \infty$ *and*
$$\lim_{m\to\infty} M(c_\infty(\Lambda), c_\infty(\mu))^{(m)} = 0, \qquad \text{if } Y = c_\infty(\mu).$$

3.8. Further results. In this subsection, we shall give the characterizations of the classes (X,Y) where $X = l_1$ and $Y = w_\infty^p$, w^p, w_0^p $(1 \le p < \infty)$, or $X = w_0$, w, w_∞ and $Y = l_p$ $(1 \le p < \infty)$, or $X = w_0$, w, w_∞ and $Y = w_0^p$, w^p and w_∞^p $(1 \le p < \infty)$. Furthermore we shall apply the Hausdorff measure of compactness to give necessary and sufficient conditions for a linear operator between these spaces to be compact. The results can be found in [73].

Let $a \in \omega$. Then we write

$$\|a\|^{**} = \|a\|^*_{\mathcal{M}^p} = \sup\left\{\left|\sum_{k=1}^{\infty}a_k x_k\right| : \|x\|_{\mathcal{M}p} = 1\right\}.$$

Lemma 3.56. [73, Lemma 1] *Let* $1 \le p < \infty$.

(a) *Then* $(w_0^p)^\beta = (w^p)^\beta = (w_\infty^p)^\beta = \mathcal{M}^p$ *and* $\|a\|^* = \|a\|_{\mathcal{M}^p}$ *on* $\mathcal{M}^p$ *(cf. [49] or [65, Lemma 2]). Further the sets* $\mathcal{M}^p$ *are BK spaces with the norms* $\|\cdot\|_{\mathcal{M}^p}$ *(cf. [59, Theorem 2 (a)]), and it is easy to see that the spaces* $\mathcal{M}^p$ *have AK.*

(b) *Then* w_∞^p *is* β*–perfect, that is* $(w_\infty^p)^{\beta\beta} = w_\infty^p$ *and* $(w_0^p)^{\beta\beta} = (w^p)^{\beta\beta} = w_\infty^p$ *(cf. [59, Theorem 4 (b) and (c)]) and* $\|a\|^{**} = \|a\|_{w_\infty^p}$ *on* $(\mathcal{M}^p)^\beta = w_\infty^p$ *(cf. [63, Theorem 6 (b)]).*

If A is an infinite matrix, then we write A^T for its transpose.

Theorem 3.57. [73, Theorem 1] *Let* $1 \le p < \infty$. *Then*

(a) $A \in (l_1, w_\infty^p)$ *if and only if*

$$M(l_1, w_\infty^p) = \sup_{m,k}\left(\frac{1}{m}\sum_{n=1}^{m}|a_{nk}|^p\right) < \infty; \tag{3.124}$$

(b) $A \in (l_1, w_0^p)$ *if and only if condition (3.124) holds and*

$$\lim_{m\to\infty}\left(\frac{1}{m}\sum_{n=1}^{m}|a_{nk}|^p\right) = 0 \quad \text{for all } k; \tag{3.125}$$

(c) $A \in (l_1, w^p)$ *if and only if condition (3.124) holds and there is a sequence* $(\lambda_k)_{k=1}^\infty \in \omega$ *such that*

$$\lim_{m\to\infty}\left(\frac{1}{m}\sum_{n=1}^{m}|a_{nk} - \lambda_k|^p\right) = 0 \quad \text{for all } k. \tag{3.126}$$

Proof. (a) condition (3.124) follows from [108, Example 8.4.1, p. 126] with $Y = w_\infty^p$.

(b) Parts (b) and (c) follow from part (a) and [70, Theorem 1 (c)]. □

By T we denote the set of all strictly increasing sequences $(t_\nu)_{\nu=0}^\infty$ of integers such that for each ν there is one and only one t_ν with $2^\nu \le t_\nu \le 2^{\nu+1} - 1$. We put

$$M(w_0, l_p) = \begin{cases} \sup\limits_{\substack{N \subset \mathbb{N} \\ N \text{ finite}}} \left(\sum\limits_{\nu=0}^{\infty} 2^\nu \max\limits_{\nu}\left|\sum\limits_{n\in N} a_{nk}\right|\right) & (p = 1) \\ \sup\limits_{N \subset \mathbb{N}_0}\left(\sup\limits_{t\in T}\left(\sum\limits_{n=1}^{\infty}\left|\sum\limits_{\nu\in N} 2^\nu a_{n,t_\nu}\right|^p\right)\right) & (1 < p < \infty) \\ \sup\limits_{N \subset \mathbb{N}_0}\left(\sup\limits_{t\in T}\left(\sup\limits_{n}\left|\sum\limits_{\nu\in N} 2^\nu a_{n,t_\nu}\right|\right)\right) & (p = \infty). \end{cases}$$

Theorem 3.58. [73, Theorem 2] *Let* $1 \le p \le \infty$. *Then*

$$(w_0, l_p) = (w, l_p) = (w_\infty, l_p); \tag{3.127}$$

further $A \in (w_0, l_p)$ *if and only if*

$$M(w_0, l_p) < \infty. \tag{3.128}$$

Proof. The case $p = 1$ follows from Lemma 3.56 (a) and from [63, Theorem 1] with $X = w_0, w, w_\infty$.

For $1 < p \le \infty$, we apply [108, Theorem 8.3.9, p.124] with $X = w_0$ and $Z = l_q$ where $q = 1$ for $p = \infty$ and $q = p/(p-1)$ for $1 < p < \infty$. Since X and Z are BK spaces with AK, we obtain $(w_0, l_p) = (w_0^{\beta\beta}, l_p) = (w_\infty, l_p)$. (The second equality holds in view of Lemma 3.56 (b).) Since $w_0 \subset w \subset w_\infty$, we have established the identities in (3.127). Further, by [108, Theorem 8.3.9], $A \in (w_0, l_p)$ if and only if $A^T \in (l_q, w_0^\beta) = (l_q, \mathcal{M}^1)$, by Lemma 3.56 (a). Finally, by [59, Theorem 7], $A^T \in (l_q, \mathcal{M}^1)$ if and only if $M(w_0, l_p) < \infty$. □

Let us remark that an application of [70, Theorem 1 (b)] and Lemma 3.56 (a) yields $A \in (w_0, l_\infty)$ if and only if $\sup_n \sum_{\nu=0}^\infty 2^\nu \max_\nu |a_{nk}| < \infty$, a condition equivalent to condition (3.128) in Theorem 3.58 for $p = \infty$.

We write $N^{\langle\mu\rangle}$ for the set of all integers n with $2^\mu \le n \le 2^{\mu+1} - 1$, and we put

$$M(w_0, w_\infty^p) = \begin{cases} \sup_{\mu \in \mathbb{N}_0} \left(\max_{N_\mu \subset N^{\langle\mu\rangle}} \left(\sum_{\nu=0}^\infty 2^\nu \max_\nu \left| \frac{1}{2^\mu} \sum_{n \in N_\mu} a_{nk} \right| \right) \right) & (p = 1) \\ \sup_{N \subset \mathbb{N}_0} \left(\sup_{t \in T} \left(\sup_m \left(\frac{1}{m} \sum_{n=1}^m \left| \sum_{\nu \in N} \frac{1}{2^\nu} a_{n,t_\nu} \right|^p \right) \right) \right) & (1 < p < \infty). \end{cases}$$

Theorem 3.59. [73, Theorem 3] *Let* $1 \le p < \infty$.
(a) *Then*

$$(w_0, w_\infty^p) = (w, w_\infty^p) = (w_\infty, w_\infty^p); \tag{3.129}$$

further $A \in (w_0, w_\infty^p)$ *if and only if*

$$M(w_0, w_\infty^p) < \infty; \tag{3.130}$$

(b) $A \in (w_0, w_0^p)$ *if and only if conditions (3.130) and (3.125) hold;* $A \in (w_0, w^p)$ *if and only if conditions (3.130) and (3.126) hold;* $A \in (w, w_0^p)$ *if and only if conditions (3.130) and (3.125) hold and*

$$\lim_{m \to \infty} \left(\frac{1}{m} \sum_{n=1}^m \left| \sum_{k=1}^\infty a_{nk} \right|^p \right) = 0; \tag{3.131}$$

$A \in (w, w^p)$ if and only if conditions (3.130) and (3.126) hold and

$$\lim_{m\to\infty}\left(\frac{1}{m}\sum_{n=1}^{m}\left|\sum_{k=1}^{\infty}a_{nk}-\lambda\right|^p\right)=0 \quad \textit{for some complex number } \lambda. \tag{3.132}$$

Proof. (a) For $p = 1$, part (a) is an immediate consequence of [63, Korollar 2] and Lemma 3.56 (a).

For $1 < p < \infty$, the identities in (3.129) follow by an argument similar to that in the proof of Theorem 3.58. We apply [108, Theorem 8.3.9] with $X = w_0$ and $Z = \mathcal{M}^p$ to conclude $A \in (w_0, w_\infty^p)$ if and only if $A^T \in (\mathcal{M}^p, w_0^p) = (\mathcal{M}^p, \mathcal{M}^1)$. Finally, by [59, Theorem 7], $A^T \in (\mathcal{M}^p, \mathcal{M}^1)$ if and only if $M(w_0, w_\infty^p) < \infty$.

(b) Part (b) follows from [70, Theorem 1 (c)], the fact that w_0 has AK and the representation for sequences in w given in Proposition 3.45. □

Now we shall give estimates for the operator norm $\|L_A\|$. We put

$$M_A^*(l_1, \tilde{w}_\infty^p) = \sup_{m,k}\left(\frac{1}{m}\sum_{n=1}^{m}|a_{nk}|^p\right)^{1/p} \quad (1 \le p < \infty),$$

and for any BK space X

$$\begin{aligned}
M_A^*(X, l_1) &= \sup_{\substack{N\subset\mathbb{N}\\ N \text{ finite}}}\left\|\sum_{n\in N}A_n\right\|^*,\\
M_A^*(X, l_\infty) &= \sup_n \|A_n\|^*,\\
M_A^*(X, w_\infty) &= \sup_\mu\left(\max_{N_\mu\subset N^{\langle\mu\rangle}}\left\|\frac{1}{2^\mu}\sum_{n\in N_\mu}A_n\right\|^*\right)\\
M_A^*(X, \mathcal{M}^1) &= \sup_{N\subset\mathbb{N}_0}\left(\left\|\sum_{\mu\in N}2^\mu A_{t_\mu}\right\|^*\right).
\end{aligned}$$

Theorem 3.60. [73, Theorem 4] (a) *Let $1 \le p < \infty$, $\|\cdot\|_{\tilde{w}_\infty^p}$ the norm on w_∞^p defined by*

$$\|x\|_{\tilde{w}_\infty^p} = \sup_n\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|^p\right)^{1/p} \tag{3.133}$$

and $A \in (l_1, w_\infty^p)$. Then

$$\|L_A\| = M_A^*(l_1, \tilde{w}_\infty^p). \tag{3.134}$$

(b) *Let X be an arbitrary BK space. If $A \in (X, l_1)$, then*

$$M_A^*(X, l_1) \le \|L_A\| \le 4 \cdot M_A^*(X, l_1). \tag{3.135}$$

If $A \in (X, l_\infty)$, then

$$\|L_A\| = M_A^*(X, l_\infty). \tag{3.136}$$

If $\|\cdot\|_{w_\infty}$ is the norm on w_∞ defined in (3.101) and $A \in (X, w_\infty)$, then

$$M_A^*(X, w_\infty) \le \|L_A\| \le 4 \cdot M_A^*(X, w_\infty). \tag{3.137}$$

(c) Let X be an arbitrary BK space and $A \in (X, \mathcal{M}^1)$, then

$$M_A^*(X, \mathcal{M}^1) \le \|L_A\| \le 4 \cdot M_A^*(X, \mathcal{M}^1). \tag{3.138}$$

Proof. (a) Let $A \in (l_1, w_\infty^p)$, $x \in l_1$ with $\|x\|_1 = \sum_{k=1}^\infty |x_k| = 1$ and $m \in \mathbb{N}$ be given. Then we have by Minkowski's inequality

$$\begin{aligned}\left(\frac{1}{m}\sum_{n=1}^m |A_n(x)|^p\right)^{1/p} &= \left(\frac{1}{m}\sum_{n=1}^m \left|\sum_{k=1}^\infty a_{nk}x_k\right|^p\right)^{1/p} \le \sum_{k=1}^\infty |x_k| \left(\frac{1}{m}\sum_{n=1}^m |a_{nk}|^p\right)^{1/p} \\ &\le M_A^*(l_1, \tilde{w}_\infty^p),\end{aligned}$$

hence, since m was arbitrary, $\|A(x)\|_{\tilde{w}_\infty^p} \le M_A^*(l_1, \tilde{w}_\infty^p)$ and consequently

$$\|L_A\| = \sup\{\|A(x)\|_{\tilde{w}_\infty^p} : \|x\|_1 = 1\} \le M_A(l_1, \tilde{w}_\infty^p). \tag{3.139}$$

Now let $x = e^{(k)}$ $(k = 1, 2, \dots)$. Then $x \in l_1$, $\|x\|_1 = 1$ and

$$\|A(x)\|_{w_\infty^p} = \sup\left(\frac{1}{m}\sum_{n=1}^m |a_{nk}|^p\right)^{1/p} \le \|L_A\|$$

together imply

$$M_A^*(l_1, \tilde{w}_\infty^p) \le \|L_A\|. \tag{3.140}$$

Finally, from (3.139) and (3.140), we conclude (3.134).

(b) First we show (3.135). Let $A \in (X, l_1)$, $x \in X$ with $\|x\| = 1$ and $m \in \mathbb{N}$ be given. Then

$$\sum_{n=1}^m |A_n(x)| \le 4 \cdot \max_{N \subset \{1,\dots,m\}} \left|\sum_{k=1}^\infty \left(\sum_{n \in N} a_{nk}\right) x_k\right| \le 4 \cdot M_A^*(X, l_1).$$

Since m was arbitrary, we conclude $\|A(x)\|_1 \le 4 \cdot M_A^*(X, l_1)$ and consequently

$$\|L_A\| \le 4 \cdot M_A^*(X, l_1). \tag{3.141}$$

Conversely, let $N \subset \mathbb{N}$ be an arbitrary finite set. Then given $\varepsilon > 0$ there is a sequence $x = x(N, \varepsilon) \in X$ such that $\|x\| = 1$ and $\left\|\sum_{n \in N} A_n\right\|^* \le \left|\sum_{n \in N} A_n(x)\right| + \varepsilon$.

Therefore

$$\left\|\sum_{n\in N} A_n\right\|^* \leq \sum_{n=1}^{\infty} |A_n(x)| + \varepsilon \leq \|A(x)\|_1 + \varepsilon \leq \|L_A\| + \varepsilon.$$

Since $N \subset \mathbb{N}$ and $\varepsilon > 0$ were arbitrary,

$$M_A^*(X, l_1) \leq \|L_A\|. \tag{3.142}$$

Finally, from (3.141) and (3.142), we conclude (3.135) Equality (3.136) is Theorem 1.23 (b). The inequalities in (3.137) are shown in exactly the same way as those in [70, Corollary 1 (a), (2.8)] with m, $N_m \subset \{1, \dots, m\}$ and $1/m$ replaced by μ, $N_\mu \subset N^{\langle\mu\rangle}$ and $1/2^\mu$.

(c) Let $A \in (X, \mathcal{M}^1)$, $x \in X$ with $\|x\| = 1$ and $\mu_0 \in \mathbb{N}_0$ be given. We choose $n_\mu \in N^{\langle\mu\rangle}$ $(\mu = 0, 1, \dots)$ such that $|A_{n_\mu}(x)| = \max_{n\in N^{\langle\mu\rangle}} |A_n(x)|$. Then we have

$$\begin{aligned}\sum_{\mu=0}^{\mu_0} 2^\mu |A_{n_\mu}(x)| &\leq 4 \cdot \max_{N\subset\{0,\dots,\mu_0\}} \left|\sum_{\mu\in N} 2^\mu A_{n_\mu}(x)\right| \\ &= 4 \cdot \max_{N\subset\{0,\dots,\mu_0\}} \left|\sum_{k=1}^{\infty}\left(\sum_{\mu\in N} 2^\mu a_{n_\mu,k}\right) x_k\right| \\ &\leq 4 \cdot \sup_{N\in\mathbb{N}_0}\left(\sup_{t\in T}\left\|\sum_{\mu\in N} 2^\mu A_{t_\mu}\right\|^*\right) = 4 \cdot M_A^*(X, \mathcal{M}^1).\end{aligned}$$

Since this holds for all $\mu_0 \in \mathbb{N}_0$, we conclude $\|A(x)\|_{\mathcal{M}^1} \leq 4 \cdot M_A^*(X, \mathcal{M}^1)$ and consequently

$$\|L_A\| \leq 4 \cdot M_A^*(X, \mathcal{M}^1) \tag{3.143}$$

Conversely, let $N \in \mathbb{N}_0$, $t \in T$ and $\varepsilon > 0$ be given. Then there is a sequence $x = x(N, t, \varepsilon) \in X$ such that $\|x\| = 1$ and $\|\sum_{\mu\in N} 2^\mu A_{t_\mu}\|^* \leq |\sum_{\mu\in N} 2^\mu A_{t_\mu}(x)| + \varepsilon$. Therefore

$$\left\|\sum_{\mu\in N} 2^\mu A_{t_\mu}\right\|^* \leq \sum_{\mu=0}^{\infty} 2^\mu \max_{n\in N^{\langle\mu\rangle}} |A_n(x)| + \varepsilon = \|A(x)\|_{\mathcal{M}^1} + \varepsilon \leq \|L_A\| + \varepsilon.$$

Since $N \in \mathbb{N}_0$, $t \in T$ and $\varepsilon > 0$ were arbitrary, we have $M_A^*(X, \mathcal{M}^1) \leq \|L_A\|$. Finally, from this and (3.143), we conclude (3.138). □

Now we apply the previous results to estimate the operator norms of the matrix transformations characterized in Theorems 3.57, 3.58 and 3.59.

Let X be any of the spaces w_0, w and w_∞. We put

$$M_A^*(X,l_1) = \sup_{\substack{N\subset\mathbb{N}\\ N \text{ finite}}}\left(\sum_{\nu=0}^{\infty} 2^\nu \max_\nu \left|\sum_{n\in N} a_{nk}\right|\right),$$

$$M_A^*(X,l_\infty) = \sup_n \left(\sum_{\nu=0}^{\infty} 2^\nu \max_\nu |a_{nk}|\right),$$

$$M_A^*(X,w_\infty) = \sup_\mu \left(\max_{N_\mu\subset N^{\langle\mu\rangle}} \left(\sum_{\nu=0}^{\infty} 2^\nu \max_\nu \left|\frac{1}{2^\mu}\sum_{n\in N_\mu} a_{nk}\right|\right)\right),$$

and, for $1 < p < \infty$ and $q = p/(p-1)$,

$$M_{A^T}^*(l_q,\mathcal{M}^1) = \sup_{N\subset\mathbb{N}_0}\left(\sup_{t\in T}\left(\sum_{n=1}^{\infty}\left|\sum_{\nu\in N} 2^\nu a_{n,t_\nu}\right|^p\right)^{1/p}\right),$$

$$M_{A^T}^*(\mathcal{M}^p,\mathcal{M}^1) = \sup_{N\subset\mathbb{N}_0}\left(\sup_{t\in T}\left(\sup_\mu\left(\frac{1}{2^\mu}\sum_{n\in N^{\langle\mu\rangle}}\left|\sum_{\nu\in N} 2^\nu a_{n,t_\nu}\right|^p\right)^{1/p}\right)\right).$$

Corollary 3.61. [73, Corollary 1] *Let X be any of the spaces w_0, w and w_∞ and $\|\cdot\|_{w_\infty^p}$ the norm defined in (3.101).*
If $A \in (X,l_1)$, then $M_A^*(X,l_1) \le \|L_A\| \le 4\cdot M_A^*(X,l_1)$.
If $A \in (X,l_\infty)$, then $\|L_A\| = M_A^*(X,l_\infty)$.
If $A \in (X,l_p)$ $(1 < p < \infty, q = \frac{p}{p-1})$, then $M_{A^T}^*(l_q,\mathcal{M}^1) \le \|L_A\| \le 4\cdot M_{A^T}^*(l_q,\mathcal{M}^1)$.
If $A \in (X,w_\infty)$, then $M_A^*(X,w_\infty) \le \|L_A\| \le 4\cdot M_A^*(X,w_\infty)$.
If $A \in (X,w_\infty^p)$ $(1 < p < \infty)$, then $M_{A^T}^*(\mathcal{M}^p,\mathcal{M}^1) \le \|L_A\| \le 4\cdot M_{A^T}^*(\mathcal{M}^p,\mathcal{M}^1)$.

We need with the following auxiliary lemma.

Lemma 3.62. [73, Lemma 2] (a) *Let $P_m : w_0^p \mapsto w_0^p$ for $1 \le p < \infty$ and $m = 1,2,\dots$ be the projector on the first m coordinates, that is $P_m(x) = (x_1,x_2,\dots,x_m, 0,0,\dots)$ for $x = (x_i) \in w_0^p$. Then $\|I-P_m\| = 1, m = 1,2,\dots$.*

(b) *For $x \in w^p$, we use the representation in Proposition 3.44 and define $P_m : w^p \mapsto w^p$ by $P_m(x) = le + \sum_{k=1}^m (x_k - l)e^{(k)}$ for $m = 1,2,\dots$. Then $\|I-P_m\| = 2$ for $m = 1,2,\dots$.*

Proof. (a) It is clear that $\|I-P_m\| \le 1$. Since $I - P_m \ne O$ is a bounded linear operator and projector, we have $\|I-P_m\| \ge 1$. This proves (a).

(b) Let $x = (x_k)_{k=1}^\infty \in w^p$. Then x has the representation in Proposition 3.44, and we obtain

$$\|(I-P_m)(x)\| = \|\underbrace{(0,\dots,0}_{m}, x_{m+1}-l, x_{m+2}-l,\dots\| \le \|x\| + |l| \le 2\|x\|.$$

Hence $\|I-P_m\| \le 2$, $m = 1,2,\dots$. To prove that $\|I-P_m\| \ge 2$, let $\epsilon > 0$. Then, since

$$2\left(\frac{k}{m+k}\right)^{1/p} \to 2 \quad (k\to\infty),$$

there exists $k_0 \in \mathbb{N}$ such that

$$2\left(\frac{k_0}{m+k_0}\right)^{1/p} > 2-\epsilon.$$

Let $u_0 \in w^p$ be defined by

$$u_0 = (\underbrace{1,\dots,1}_{m}, \underbrace{-1,\dots,-1}_{k_0}, 1,1,1,\dots).$$

Then $\|u_0\| = 1$, $l = 1$ and

$$\|(I-P_m)(u_0)\| \geq \left(\frac{1}{m+k_0}\cdot 2^p k_0\right)^{1/p} = 2\left(\frac{k_0}{m+k_0}\right)^{1/p} > 2-\epsilon.$$

Hence $\|I-P_m\| > 2-\epsilon$, that is $\|I-P_m\| \geq 2$. □

Theorem 3.63. [73, Theorem 5] *Let $1 \leq p < \infty$, $\|\cdot\|_{\tilde{w}^p_\infty}$ the norm on w_0^p, w^p and w^p_∞ defined in (3.133). We put*

$$M_A^*(l_1, \tilde{w}^p_\infty)_{(m)} = \sup_{\substack{k\\ u>m}} \left(\frac{1}{u}\sum_{n=m+1}^{u} |a_{nk}|^p\right)^{1/p}.$$

(a) *If $A \in (l_1, w_0^p)$, then*

$$\|L_A\|_\chi = \lim_{m\to\infty} M_A^*(l_1, \tilde{w}^p_\infty)_{(m)}. \tag{3.144}$$

(b) *If $A \in (l_1, w^p)$, then*

$$\frac{1}{2}\cdot \lim_{m\to\infty} M_A^*(l_1, \tilde{w}^p_\infty)_{(m)} \leq \|L_A\|_\chi \leq \lim_{m\to\infty} M_A^*(l_1, \tilde{w}^p_\infty)_{(m)}. \tag{3.145}$$

(c) *If $A \in (l_1, w^p_\infty)$, then*

$$0 \leq \|L_A\|_\chi \leq \lim_{m\to\infty} M_A^*(l_1, \tilde{w}^p_\infty)_{(m)}. \tag{3.146}$$

Proof. Let us remark that the limits in (3.144), (3.145) and (3.146) exist. We put $B = \{x \in l_1 : \|x\| \leq 1\}$. In the case (a) we have by the inequality in Theorem 2.23

$$\|L_A\|_\chi = \chi(A(B)) = \lim_{m\to\infty}\left[\sup_{x\in B} \|(I-P_m)(A(x))\|\right], \tag{3.147}$$

where $P_m : w_0^p \mapsto w_0^p$ for $m = 1,2,\dots$ is the projector on the first m coordinates, that is $P_m(x) = (x_1, x_2, \dots, x_m, 0, 0, \dots)$ for $x = (x_k) \in w_0^p$. Let us recall that by

Lemma 3.62 (c) we have $\|I - P_m\| = 1, m = 1, 2, \dots$. Let $A_{(m)} = (\tilde{a}_{nk})$ be the infinite matrix defined by $\tilde{a}_{nk} = 0$ if $1 \le n \le m$ and $\tilde{a}_{nk} = a_{nk}$ if $m < n$. Now, by (3.134) we have

$$\sup_{x \in B} \|(I - P_m)(A(x))\| = \|L_{A_{(m)}}\| = M^*_{A_{(m)}}(l_1, \tilde{w}^p_\infty)_{(m)} = M^*_A(l_1, \tilde{w}^p_\infty)_{(m)}$$

Part (a) now follows from this and (3.147).

(b) Let $x = (x_k)_{k=1}^\infty \in w^p_0$. Then x has the representation in Proposition 3.44, and we define $P_m : w^p \mapsto w^p$ by $P_m(x) = le + \sum_{k=1}^m (x_k - l)e^{(k)}$ for $m = 1, 2, \dots$. By Lemma 3.62 (b) we know that $\|I - P_m\| = 2$ for $m = 1, 2, \dots$. Now the proof of (b) is similar as in the case (a), and we omit it.

Let us prove (3.146). Now define $P_m : w^p_\infty \mapsto w^p_\infty$ by $P_m(x) = (x_1, x_2, \dots, x_m, 0, \dots)$ for $x = (x_i) \in w^p_\infty$ and $m = 1, 2, \dots$. It is clear that $A(B) \subset P_m(A(B)) + (I - P_m)(A(B))$. Now, by the elementary properties of the function χ we have

$$\begin{aligned} \chi(A(B)) &\le \chi(P_m(A(B))) + \chi((I - P_m)(A(B))) = \chi((I - P_m)(A(B)) \\ &\le \sup_{x \in B} \|(I - P_m)(A(x))\| = \|L_{A_{(m)}}\|. \end{aligned} \tag{3.148}$$

Since the limit in (3.146) obviously exists, by (3.148) and (3.135) we get (3.146). □

Now as a corollary of the above theorem we have

Corollary 3.64. [73, Corollary 2] *If either $A \in (l_1, w^p_0)$ or $A \in (l_1, w^p_0)$, then*

$$L_A \quad \text{is compact if and only if} \lim_{m \to \infty} M^*_A(l_1, \tilde{w}^p_\infty)_{(m)} = 0.$$

If $A \in (l_1, w^p_\infty)$, then

$$L_A \quad \text{is compact if} \lim_{m \to \infty} M^*_A(l_1, \tilde{w}^p_\infty)_{(m)} = 0. \tag{3.149}$$

The following example will show that it is possible for L_A in (3.149) to be compact in the case $\lim_{m\to\infty} M^*_A(l_1, \tilde{w}^p_\infty)_{(m)} > 0$, and hence in general we have just "if" in (3.149).

Example 3.65. [73, Example 1] Let the matrix A be defined by $a_{nk} = 1$ if $n = 1$ and $a_{nk} = 0$ if $n \ne 1$. Then $M^*_A(l_1, \tilde{w}^p_\infty) = 1$ and $A \in (l_1, \tilde{w}^p_\infty)$. Further

$$M^*_A(l_1, \tilde{w}^p_\infty)_{(m)} = \sup_{k \ge 1, u > m} \left(\frac{1}{u} \sum_{n=m+1}^u |a_{nk}|^p \right)^{1/p} = \sup_{k \ge 1, u > m} \left(\frac{u - m}{u} \right)^{1/p} = 1.$$

Whence $\lim_{m\to\infty} M^*_A(l_1, \tilde{w}^p_\infty)_{(m)} = 1 > 0$. Since $A(x) = x_1 e$ for all $x \in l_1$, L_A is a compact operator.

Now, concerning Corollary 3.61 we continue to study the measures of noncompactness of operators when the final spaces are the spaces l_p and w_∞^p. Let X be any of the spaces w_0, w and w_∞. For $m \in \mathbb{N}$ we put

$$M_A^*(X, l_1)_{(m)} = \sup_{\substack{N \subset \mathbb{N} \setminus \{1,2,\dots,m\} \\ N \text{ finite}}} \left(\sum_{\nu=0}^{\infty} 2^\nu \max_\nu \left| \sum_{n \in N} a_{nk} \right| \right),$$

$$M_A^*(X, l_\infty)_{(m)} = \sup_{n > m} \left(\sum_{\nu=0}^{\infty} 2^\nu \max_\nu |a_{nk}| \right),$$

$$M_A^*(X, w_\infty)_{(m)} = \sup_{\mu > m} \left(\max_{N_\mu \subset N^{\langle \mu \rangle}} \left(\sum_{\nu=0}^{\infty} 2^\nu \max_\nu \left| \frac{1}{2^\mu} \sum_{n \in N_\mu} a_{nk} \right| \right) \right),$$

and, for $1 < p < \infty$ and $q = p/(p-1)$,

$$M_{A^T}^*(l_q, \mathcal{M}^1)_{(m)} = \sup_{N \subset \mathbb{N}_0} \left(\sup_{t \in T} \left(\sum_{n=m+1}^{\infty} \left| \sum_{\nu \in N} 2^\nu a_{n,t_\nu} \right|^p \right)^{1/p} \right),$$

$$M_{A^T}^*(\mathcal{M}^p, \mathcal{M}^1)_{(m)} = \sup_{N \subset \mathbb{N}_0 \setminus \{1,2,\dots,m\}} \left(\sup_{t \in T} \left(\sup_\mu \left(\frac{1}{2^\mu} \sum_{n \in N^{\langle \mu \rangle}} \left| \sum_{\nu \in N} 2^\nu a_{n,t_\nu} \right|^p \right)^{1/p} \right) \right).$$

Theorem 3.66. [73, Theorem 6] *Let X be any of the spaces w_0, w and w_∞ and $\|\cdot\|_{w_\infty^p}$ the norm defined in (3.101). If $A \in (X, l_1)$, then*

$$\lim_{m\to\infty} M_A^*(X, l_1)_{(m)} \le \|L_A\|_\chi \le 4 \cdot \lim_{m\to\infty} M_A^*(X, l_1)_{(m)}. \tag{3.150}$$

If $A \in (X, l_\infty)$, then

$$\|L_A\|_\chi \le \lim_{m\to\infty} M_A^*(X, l_\infty)_{(m)}. \tag{3.151}$$

If $A \in (X, l_p)$ $(1 < p < \infty, q = p/(p-1))$, then

$$\lim_{m\to\infty} M_{A^T}^*(l_q, \mathcal{M}^1)_{(m)} \le \|L_A\|_\chi \le 4 \cdot \lim_{m\to\infty} M_{A^T}^*(l_q, \mathcal{M}^1)_{(m)}. \tag{3.152}$$

If $A \in (X, w_\infty)$, then

$$\|L_A\|_\chi \le 4 \cdot \lim_{m\to\infty} M_A^*(X, w_\infty)_{(m)}. \tag{3.153}$$

If $A \in (X, w_\infty^p)$ $(1 < p < \infty)$, then

$$\|L_A\|_\chi \le 4 \cdot \lim_{m\to\infty} M_{A^T}^*(\mathcal{M}^p, \mathcal{M}^1)_{(m)}. \tag{3.154}$$

Proof. Let us remark that the limits in (3.150) to (3.154) exist. Let $P_m : l^p \mapsto l^p$ for $m = 1, 2, \dots$ and $1 \le p < \infty$ be the projector on the first m coordinates, that is $P_m(x) = (x_1, x_2, \dots, x_m, 0, 0, \dots)$ for $x = (x_i) \in l^p$. It is easy to check that $\|I - P_m\| = 1, m = 1, 2, \dots$. Now the proof of (3.150) and (3.152) (when final spaces have a basis) can be given by the method of proof of Theorem 3.63 (a), while in the proof of (3.151), (3.153), and (3.154) (when final spaces have no basis) we can use the method of the proof of Theorem 3.63 (c). □

Now as a corollary of the theorem above we have

Corollary 3.67. [73, Corollary 3] *Let X be any of the spaces w_0, w and w_∞ and $\|\cdot\|_{w_\infty^p}$ the norm defined in (3.101). If $A \in (X, l_1)$, then*

$$L_A \quad \text{is compact if and only if} \quad \lim_{m\to\infty} M_A^*(X, l_1)_{(m)} = 0. \tag{3.155}$$

If $A \in (X, l_\infty)$, then

$$L_A \quad \text{is compact if} \quad \lim_{m\to\infty} M_A^*(X, l_\infty)_{(m)} = 0. \tag{3.156}$$

If $A \in (X, l_p)$ $(1 < p < \infty, q = p/(p-1))$, then

$$L_A \quad \text{is compact if and only if} \quad \lim_{m\to\infty} M_{A^T}^*(l_q, \mathcal{M}^1)_{(m)} = 0. \tag{3.157}$$

If $A \in (X, w_\infty)$, then

$$L_A \quad \text{is compact if} \quad \lim_{m\to\infty} M_A^*(X, w_\infty)_{(m)} = 0. \tag{3.158}$$

If $A \in (X, w_\infty^p)$ $(1 < p < \infty)$, then

$$L_A \quad \text{is compact if} \quad \lim_{m\to\infty} M_{A^T}^*(\mathcal{M}^p, \mathcal{M}^1)_{(m)} = 0. \tag{3.159}$$

Let us remark that it is possible for L_A in (3.156), (3.158) and (3.159) to be compact in the cases $\lim_{m\to\infty} M_A^*(X, l_\infty)_{(m)} > 0$, $\lim_{m\to\infty} M_A^*(X, w_\infty)_{(m)} > 0$ and $\lim_{m\to\infty} M_{A^T}^*(\mathcal{M}^p, \mathcal{M}^1)_{(m)} > 0$, respectively. This can be proved by Example 3.65.

4 Appendix

In this appendix, we collect the results from Functional Analysis needed in the previous sections.

4.1. Inequalities.

Theorem A.4.1. (Hölder's inequality) *Let $1 < p < \infty$, $q = p/(p-1)$ and $x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_n \in \mathbb{C}$. Then*

$$\sum_{k=0}^{n} |x_k y_k| \le \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p} \left(\sum_{k=0}^{\infty} |y_k|^q\right)^{1/q}.$$

If $x \in l_p$ and $y \in l_q$, then $xy = (x_k y_k)_{k=0}^\infty \in l_1$ and $\|xy\|_1 \le \|x\|_p \|y\|_q$.

Theorem A.4.2. (Minkowski's inequality) *Let $1 \le p < \infty$ and $x_0, x_1, \dots, x_n$, $y_0, y_1, \dots, y_n \in \mathbb{C}$. Then*

$$\left(\sum_{k=0}^{n} |x_k + y_k|^p\right)^{1/p} \le \left(\sum_{k=0}^{n} |x_k|^p\right)^{1/p} + \left(\sum_{k=0}^{n} |y_k|^p\right)^{1/p}$$

If $x, y \in l_p$, then $x + y \in l_p$ and $\|x + y\|_p \le \|x\|_p + \|y\|_p$.

Theorem A.4.3. (Jensen's inequality) *Let* $p > 0$ *and* $x_0, x_1, \dots, x_n \in \mathbb{C}$. *Then*

$$\left(\sum_{k=0}^{n} |x_k|^p\right)^{1/p} \quad \text{is a decreasing function in } p,$$

that is, if $r > s > 0$, *then*

$$\left(\sum_{k=0}^{n} |x_k|^r\right)^{1/r} \leq \left(\sum_{k=0}^{n} |x_k|^s\right)^{1/s}.$$

If $p > p'$, *then* $l_{p'} \subset l_p$.

4.2. The closed graph theorem and the Banach–Steinhaus theorem.

Theorem A.4.4. (Closed graph lemma) *Any continuous map into a Hausdorff space has closed graph* [105, Theorem 11.1.1, p. 195].

Theorem A.4.5. (Closed graph theorem) *If* X *and* Y *are Fréchet spaces and* $f : X \mapsto Y$ *is a linear map with closed graph, then* f *is continuous* [105, Theorem 11.2.2, p. 200].

Theorem A.4.6. (Banach–Steinhaus theorem) *Let* $(f_n)_{n=0}^{\infty}$ *be a pointwise convergent sequence of continuous linear functionals on a Fréchet space* X. *Then* f *defined by*

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \text{for all } x \in X,$$

is continuous [105, Corollary 11.2.4, p. 200].

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