Neda Bokan

TORSION FREE CONNECTIONS, TOPOLOGY, GEOMETRY AND DIFFERENTIAL OPERATORS ON SMOOTH MANIFOLDS

Contents

INTRODUCTION

This lecture notes give a survey of basic facts related to geometry of manifolds endowed with a torsion free connection. We pay the attention especially on geometries which come from the existence on some characteristic torsion free connection closely related to some metric, in general case of an arbitrary signature. So we study in this spirit affine differential geometry, Weyl and Codazzi geometries. One can join naturally these two structures: a torsion free connection and a metric, and the corresponding groups of transformations. We present basic facts related to these groups. To study geometry of manifolds endowed with a torsion free connection, a powerful tool is, of course, the corresponding curvatures. Therefore the curvature is appeared in all five sections of this lecture notes. In Section I we give the definitions of curvatures and some of their properties. Section II is devoted to curvatures which are invariant with respect to some groups of transformations. These groups are closely related to classical groups: $GL(m, \mathbb{R}), U(m), SO(m)$ etc. We use well developed representation theory of these groups to enlight curvature from this point of view. It is the content of Section III. To prove the irreducibility of some vector space of curvatures one can use different methods. We pay the attention especially on these ones closely related to the Weyl classical invariance theory. It allows to study also some relations between topology and analysis of these manifolds with their geometry. So we develop the theory of characteristic classes in Section IV and differential operators of Laplace type in Section V. Among characteristic classes we pay the attention on Chern classes and their dependence on some groups of transformations and curvature symmetries. To fulfill our programme related to the influence of Weyl classical invariance theory into the theory of differential operators of Laplace type we study the heat equation method. Several operators of Laplace type are studied more sistematically.

Finally, there are various possibilities to present some material related to this topic. The author of this lecture notes choose this one closely related to her main interest through previous twenty years. Her interest yields in the cooperation with other colleagues a series of results which are presented too.

We omit the proofs as it is far from the framework of these notes. We rather give the advantage to the results to present the riches of this topic to motivate the readers into further investigations. Of course to go into this level we suggest to use the corresponding monographs and papers, mentioned in the convenient moment throughout this notes.

As it is usual the contribution of colleagues friends and institutions to the quality of manuscript is significant. I would like to acknowledge all of them, bur first of all to Prof. B. Stanković, who has initiated and encouraged writing this manuscript.

I. MANIFOLDS WITH A TORSION FREE CONNECTION

I.1. Definitions and basic notions

The straight lines play a very important role in the geometry of a plane. Therefore, it would be useful to have lines on surfaces with the analogous properties to these ones of straight lines. But the definition of such lines on surfaces is not so evident as straight lines have several characteristic properties and hence it is not clear which one should characterize "straight lines" on surfaces, i.e., which one can be generalized, and specially which generalizations give the same and which one give different lines. Among these properties are the followings:

(PL1) The curvature of a straight line (in a plane) vanishes.

(PL2) For any two points there exists the unique straight line which consists both of them.

(PL3) The tangent vectors on a straight line are mutually parallel

All of these properties can be generalized for lines on a surface. To obtain this one we use heavily a linear connection. Studying the same problem on a smooth manifold M^m of the dimension m we need also a linear connection. Hence we give its definition in a full generality. More details one can find in [**61**], [**82**], [**85**], etc.

Definition 1.1. Let $\mathfrak{X}$ be the modul of vector fields over the ring of smooth functions $C^\infty(M)$ on M. *A linear connection* on the manifold M is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, such that for all $x, y, z \in \mathfrak{X}(M)$, $r \in \mathbb{R}$ and $f \in C^\infty(M)$ it yields

(i) $$\nabla_x(y+z) = \nabla_x y + \nabla_x z \quad \text{and} \quad \nabla_x ry = r\nabla_x y,$$

(ii) $$\nabla_{x+y} z = \nabla_x z + \nabla_y z \quad \text{and} \quad \nabla_{fx} y = f\nabla_x y,$$

(iii) $$\nabla_x fy = (xf)y + f\nabla_x y \quad \text{(Leibnitz formula)}.$$

The operator $\nabla_x : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is *the covariant derivative in the direction of a vector field* x. □

If (u, U) is a chart and $\{\partial/\partial u^i\}_p, 1 \le i \le m$ the corresponding coordinate base of tangent space T_pM, for any $p \in U$, then arbitrary vector field x can be given in the following way $\sum X^i \frac{\partial}{\partial u^i}$, $X^i \in C^\infty(M)$. A linear connection ∇ is determined by the vector fields $\sum_{\partial/\partial u^i}(\partial/\partial u^j)$. It allows to introduce the Christoffel symbols of ∇.

Definition 1.2. Let ∇ be a linear connection on the manifold M and (u, U) a chart. *Christoffel symbols of* ∇ with respect to the chart (u, U) are functions

$\Gamma_{ij}^k \in C^\infty(M)$ defined by

$$\nabla_{\partial/\partial u^i}\Big(\frac{\partial}{\partial u^j}\Big) = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial u^k}. \qquad (\Box)$$

Let $\alpha : I \to M$ be a curve. *The tangent vector field* T_α of a curve α is given by $(T_\alpha)_{\alpha(t)} = (\alpha_*)(d/dt), t \in \mathbb{R}$. Usually we use the notation T for T_α if there is no confusion. Locally we have

$$T_{\alpha(t)} = \sum \frac{d\alpha^i}{dt}(t) \left(\frac{\partial}{\partial u^i}\right)_{\alpha(t)}.$$

Let y be a vector field defined along a curve α. We say y is *parallel* along α if $\nabla_{T_\alpha} y = 0$. A curve α on a manifold M is *geodesic* (with respect to the connection ∇) if $\nabla_{T_\alpha} T_\alpha = 0$. Let (M, g) be a Riemannian manifold endowed with a linear connection ∇. The connection ∇ is *metric* if it satisfies

$$xg(y, z) = g(\nabla_x y, z) + g(y, \nabla_x z),$$

for all $x, y, z \in \mathfrak{X}(M)$. A linear connection ∇ is *symmetric* or *torsion free* if we have

$$\nabla_x y - \nabla_y x = [x, y] \qquad (1.1)$$

for all $x, y \in \mathfrak{X}(M)$. A connection ∇ is torsion free if and only if it yields $\Gamma_{ij}^k = \Gamma_{ji}^k$ for all $1 \le i, j, k \le m$ in an arbitrary coordinate chart. There exists a unique metric symmetric connection ∇ on a Riemannian manifold (M, g). This connection ∇ is called *Levi-Civita connection*.

Definition 1.3. *The curvature tensor* of type (1,3) of arbitrary connection ∇ is the map $R : \mathfrak{X}\times\mathfrak{X}\times\mathfrak{X} \to \mathfrak{X}$ defined by relation $R(x, y)z = \nabla_x\nabla_y z - \nabla_y\nabla_x z - \nabla_{[x,y]}z$. The curvature tensor of Levi-Civita connection is called *Riemann curvature tensor*. $\Box$

In a local coordinate system one can find

$$R\Big(\frac{\partial}{\partial u^j}, \frac{\partial}{\partial u^k}\Big)\frac{\partial}{\partial u^i} = \sum R_{jki}{}^l \frac{\partial}{\partial u^l},$$

where the components $R_{jki}{}^l$ are defined by

$$R_{jki}{}^l = \Gamma_{ji,k}^l - \Gamma_{ki,j}^l + \sum_m \Gamma_{ji}^m \Gamma_{mk}^l \quad \text{and} \quad \Gamma_{ji,k}^l = \frac{\partial}{\partial u^k}(\Gamma_{ji}^l).$$

Riemann curvature tensor of type (0,4) is the map $R : \mathfrak{X}\times\mathfrak{X}\times\mathfrak{X}\times\mathfrak{X} \to C^\infty(M)$, given by the relation $R(x, y, z, w) = g(R(x, y)z, w)$.

The curvature tensor of type (1,3) of arbitrary connection ∇ satisfies the following relation

$$R(x,y)z = -R(y,x)z; \tag{1.2}$$

for torsion free connection we have also

$$R(x,y)z + R(z,x)y + R(y,z)x = 0, \tag{1.3}$$

(*the first Bianchi identity*), and

$$(\nabla_v R)(x,y)z + (\nabla_x R)(y,v)z + (\nabla_y R)(v,x)z = 0 \tag{1.4}$$

(*the second Bianchi identity*). Riemann curvature tensor fulfills all these relations (1.2)–(1.4) and

$$R(x,y,z,w) = -R(x,y,w,z), \tag{1.5}$$

$$R(w,z,x,y) = R(x,y,w,z). \tag{1.6}$$

The curvature tensor of a metric connection satisfies symmetry relations (1.2), (1.5) and (1.6).

Let Π be a 2-dimensional subspace of tangent space T_pM. *The sectional curvature* of Π is $K_p(\Pi) = R(x,y,y,x)(p)$, where $\{x,y\}$ is an orthonormal base of Π. If x, y are two arbitrary vectors in Π, then

$$K_p(\Pi) = \frac{R(x,y,y,x)}{\|x\|^2\|y\|^2 - g(x,y)^2},$$

where $\|x\|^2 = g(x,x)$. M is a space of *the constant sectional curvature* if $K_p(\Pi)$ is independent of the choice of Π in T_pM, where p is an arbitrary point of M and depends on $p \in M$. The Riemann curvature tensor of this space is given by

$$R(u,v,z,w) = K_p(g(u,z)g(v,w) - g(u,w)g(v,z)).$$

If $K_p(\Pi)$ is independent of Π in T_pM in all $p \in M$ then K_p is same everywhere on M.

Some information about the geometry of M give Ricci and scalar curvatures. These curvatures are very powerful tool in studying of Einstein spaces and other topics. Let $\Theta_p(x_p, y_p) : T_pM \to T_pM$ be the map defined by the relation

$$\Theta_p(x_p, y_p)v_p = R(v_p, x_p)y_p.$$

Then $\Theta_p(x_p, y_p)$ is linear for all $p \in M$ and $x_p, y_p \in T_pM$. Consequently, there exists the trace of $\Theta_p(x_p, y_p)$.

Definition 1.4. *Ricci curvature tensor* ρ is the correspondence between points $p \in M$ and maps $S_p : T_pM \times T_pM \to R$, given by $\rho_p(x_p, y_p) = \mathrm{trace}(\Theta_p(x_p, y_p))$. *Ricci curvature in a direction* x is $\rho_p(x, x)$. □

Definition 1.5. Let (M, g) be a Riemann manifold with Ricci curvature ρ. *The scalar curvature* τ of M in a point p is defined by $\tau = \sum_{i=1}^{n} \rho_p((x_i)_p, (x_i)_p)$, where $\{x_{1p}, \dots, x_{np}\}$ is an orthonormal base of the tangent space T_pM. □

Einstein space is Riemann space (M, g) such that $\rho_p = \frac{\tau}{m} g_p$.

In general case Ricci curvature tensor is neither symmetric nor skew-symmetric. But, ρ_p corresponding to Levi-Civita connection is symmetric. Manifolds endowed with special type connections will be studied in next sections.

A skew-symmetric Ricci tensor naturally appeared on manifolds which admit absolute parallelizability of directions (see for example [**100**, §§49, 89]). More precisely, it means the following. Let (M, ∇) be a differentiable manifold with a symmetric connection ∇. If a vector field v defined along a curve γ collinear with some parallel vector field w we say *the direction* of v *is parallel*. A manifold M admits *absolute parallelizability of directions* if every direction given in a $p \in M$ can be included in some field with parallel directions along every curve.

A skew-symmetric Ricci tensor is appeared also in the complete decomposition of a curvature tensor for ∇ in the spirit of the representation theory of classical groups (see Section III).

I.2. Affine differential geometry

Torsion free, Ricci symmetric connections arise naturally in affine differential geometry and motivate the discussion of the previous section. We review this geometry briefly and refer to [**10**], [**33**], [**97**], [**111**], [**124**] for further details.

Let $\mathfrak{A}$ be a real affine space which is modeled on a vector space V of dimension $m+1$. Let V^* be the dual space. If $a \in \mathfrak{A}$, we may identify $T_a\mathfrak{A} = V$ and $T_a^*\mathfrak{A} = V^*$.

Let $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$ be the natural pairing between V^* and V. Let x be a smooth hypersurface immersion of M into $\mathfrak{A}$. If $p \in M$, let

$$C(M)_p = \{X \in V^* : \langle X, dx(v) \rangle = 0, \forall v \in T_pM\}$$

be the conormal space at p; we let $C(M)$ be the corresponding conormal line bundle over M. We assume $C(M)$ is trivial and choose a non vanishing conormal field X.

We say the hypersurface $x(M)$ is *regular* if and only if $\mathrm{rank}\,(X, dX) = m + 1$, for every point of M; we impose this condition henceforth. Then X is an immersion $X : M \to V^*$ which is transversal to $X(M)$. Define $y = y(X) : M \to V$ by the conditions $\langle X, y \rangle = 1$ and $\langle dX, y \rangle = 0$.

The triple (x, X, y) is called *a hypersurface with relative normalization*; we remark that y need not be an immersion. The relative structure equations given below contain the fundamental geometric quantities of hypersurface theory: two connections ∇, ∇^*, the relative shape (Weingarten) operator S, and two symmetric forms h and $\hat{S}$. Let $\bar{\nabla}$ be the flat affine connection on $\mathfrak{A}$.

$$\begin{aligned}\bar{\nabla}_v y &= dy(v) = -dx(S(v)),\\ \bar{\nabla}_w dx(v) &= dx(\bar{\nabla}_w v) + h(v,w)y,\\ \bar{\nabla}_w dX(v) &= dX(\nabla^*_w v) - \hat{S}(v,w)X.\end{aligned} \tag{2.1}$$

The first equation is called *the Weingarten equation*, the second two are *the Gauss equations*. Symmetric form h is called *the Blaschke metric*. Generally, it is indefinite. If h is positive definite, this means that the immersed hypersurface $x(M)$ is locally strongly convex.

The relative shape operator S is self-adjoint with respect to h and is related to the auxiliary shape operator $\hat{S}$ by the identity $\hat{S}(v,w) = h(S(v),w) = h(v,S(w))$. It is useful to define a $(1,2)$ *difference tensor* C, *a totally symmetric relative cubic form* $\hat{C}$, and *the Tchebychef form* $\hat{T}$ by:

$$C := \frac{1}{2}(\nabla - \nabla^*), \quad \hat{C}(v,w,z) := h(C(v,w),z), \quad \hat{T}(z) := m^{-1}\operatorname{Tr}_h(C(z,\cdot)).$$

Let $';'$ denotes multiple covariant differentiation with respect to the Levi-Civita connection $\nabla(h)$. $\hat{T}$ has the following useful symmetry property [**123**]: $\hat{T}_{i;j} = \hat{T}_{j;i}$.

We note that both ∇ (*the induced connection*) and ∇^* (*the conormal connection*) are torsion free connections on TM. They are conjugate with respect to the Levi-Civita connection; which implies $\frac{1}{2}(\nabla + \nabla^*) = \nabla(h)$. Consequently, we may express $\nabla = \nabla(h) + C$ and $\nabla^* = \nabla(h) - C$.

The curvature tensors R, R^*, $R(h)$ of ∇, ∇^*, $\nabla(h)$ respectively can be expressed by the Gauss equations

$$\begin{aligned}R(u,v)w &= h(v,w)Su - h(u,w)Sv,\\ R^*(u,v)w &= \hat{S}(v,w)u - \hat{S}(u,w)v,\\ R(h)(w,v)u &= C(C(w,u),v) - C(C(v,u),w)\\ &\quad + \frac{1}{2}\{\hat{S}(v,u)w - \hat{S}(w,u)v + h(v,u)S(w) - h(w,u)S(v)\}.\end{aligned} \tag{2.2}$$

Let R_{ij}, R^*_{ij}, $R(h)_{ij}$ be the components of Ricci tensors Ric, Ric*, Ric(h) for ∇, ∇^*, $\nabla(h)$ respectively relative to a local orthonormal frame. We use the metric to raise and lower indices and identify $\hat{S} = S$. Then:

$$R_{ij} = \delta_{ij}S_{kk} - Sij \quad \text{and} \quad R^*_{ij} = (m-1)S_{ij}.$$

We denote *the normalized mean curvature* by $H := m^{-1}S_{ii}$; the normalized traces are then equal

$$(m-1)^{-1}\operatorname{Tr}_h(\mathrm{Ric}) = (m-1)^{-1}\operatorname{Tr}_h \mathrm{Ric}^* = mH.$$

We construct the extrinsic curvature invariants of relative geometry from S and h. Let $\{\lambda^1, \dots, \lambda^m\}$ be the eigenvalues of S relative to h:

$$\det(S - \lambda h) = 0.$$

These are *the principal curvatures*. Let $\{H_1, \dots, H_m\}$ be the corresponding normed elementary symmetric functions. For example the relative mean curvature is given by $mH_1 = \lambda^1 + \dots + \lambda^m$.

We fix a volume form or determinant on $\mathfrak{A}$. Then there is, up to orientation, a unique equiaffine unimodular normalization which is invariant under the unimodular group. Admitting arbitrary volume forms, all such normalizations differ by a non-zero constant factor. The class of equiaffine normalizations can be characterized within the class of relative normalizations by the vanishing of the Tchebychev form. We call such a hypersurface with equiaffine normalization a *Blaschke hypersurface*. The vanishing of T simplifies the local invariants greatly.

We take $y = -x$ to define *the centroaffine normalization*. This geometry is invariant with respect to the subgroup of regular affine mappings which fix the origin of $\mathfrak{A}$. Let X_e be the equiaffine conormal field and let $\zeta = -\langle x, X_e \rangle$ be the equiaffine support function. We choose the orientation so that $\zeta > 0$. Then

$$\hat{S} = h, \quad H_r = 1 \quad \text{for} \quad r = 1, \dots, m, \quad \text{and} \quad \hat{T} = \frac{2+m}{2m} d \ln(\zeta).$$

Among relative normalizations significant one is Euclidean normalization. Let $x : M \to E$ be a hypersurface, let Y be a conormal and y transversal. The pair $\{Y, y\}$ is called *a Euclidean normalization* with respect to the given Euclidean structure of E if Y and y can be identified by the Riesz theorem and $\langle Y, y \rangle = 1$. We write $Y = y = \mu$.

As a consequence of this definition one can express the regularity of a hypersurface in terms of Euclidean hypersurface geometry. More precisely, let $x : M \to E$ be a hypersurface. Then the following properties are equivalent:
(i) x is non-degenerate.
(ii) The Euclidean Gauss-map is an immersion.
(iii) The Euclidean Weingarten operator is regular.
(iv) The third fundamental form III is positive definite on M.
(v) The second fundamental form II is regular.

We express the relative quantities (in the following on the left) for the Euclidean normalization in terms of quantities of Euclidean hypersurface theory (on the right).
(a) $S(E) = b$, (b) $h(E) = II$, (c) $\nabla(E) = \nabla(I)$,
(d) $\hat{S}(E) = III$, (e) $\nabla^*(E) = \nabla(III)$, (f) $\nabla(h) = \nabla(II)$,
(g) $-2\hat{C}(E) = \nabla(I)II = -\nabla(III)II$, (h) $-2C(E)(v, w) = b^{-1}((\nabla(I)_v b)(w))$,
(i) $C(E) = \frac{1}{2}(\nabla(I) - \nabla(III)) = \nabla(I) - \nabla(II) = \nabla(II) - \nabla(III)$,
(j) $\hat{T}(E) = -\frac{1}{2m} d \lg |H_m(E)|$, where b is the Weingarten operator.

Consider a pair of non-degenerate hypersurfaces $x : M \to V$ and ${}^*x : M \to V^*$ such that $\langle x, {}^*x \rangle = -1$, $\langle d^*x, x \rangle = 0$, and $\langle x, dx \rangle = 0$. Such a pair is called *a polar* *pair*. These relations are satisfied for a non degenerate hypersurface $x : M \to V$ and its centroaffine conormal map $^*x := X : M \to V^*$. This indicates the important role which centroaffine differential geometry has for the investigation of polar pairs. We recall some facts about the controlling geometry of polar pairs and refer to [**OS**, §7.2] for further details

$$\hat{S} = h = {}^*h = {}^*\hat{S},$$
$$\nabla = {}^*\nabla^*, \quad \nabla^* = {}^*\nabla,$$
$$C = -{}^*C, \quad \hat{T} = -{}^*\hat{T}, \quad R = {}^*R\,.$$

Let μ be a unit normal on an Euclidean sphere $S^m(r) \subset E$ of radius r and with center x_0. $S^m(r)$ can be characterized by the relation $r\mu = x - x_0$, or more generally by μ and $(x - x_0)$ being parallel. Studying quadrics we conclude that all quadrics with center x_0 have the property that the equiaffine normal satisfies $y(e) = -H(e)x + x_0$. One can generalizes this notion in relative geometry as follows.

Let $x : M \to \mathfrak{A}$ be a regular hypersurface with relative normalization $\{Y, y\}$. Then $\{x, Y, y\}$ is called *a proper relative sphere* with center x_0 if

$$y = \lambda(x - x_0), \quad \lambda \in C^\infty(M). \tag{2.3}$$

$\{x, Y, y\}$ is called *an improper relative sphere* if $y = \text{const} \neq 0$. A point $p \in M$ is called *a relative umbilic* if the relative principal curvatures coincide

$$k_1(p) = k_2(p) = \cdots = k_m(p).$$

A consequence of the Weingarten equation in (2.1) is that $\lambda = \text{const}$ in (2.3).

Since $y = -x$ per definition in centroaffine geometry it follows any hypersurface with centroaffine normalization is a relative sphere with respect to this normalization.

Usually relative spheres with respect to the equiaffine normalization are called *affine spheres* (instead of equiaffine spheres). Any quadric is an affine sphere. If a regular quadric has a center x_0, it is a proper affine sphere with center x_0 (examples: ellipsoids, hyperboloids are proper affine spheres; paraboloids are improper affine spheres). In the following theorems we give some characterizations of relative spheres and affine ones.

Theorem 2.1. *(a) Each of the following properties (i)–(vi) characterizes a relative sphere:*

- *(i)* $S = \lambda \cdot \mathrm{id}$ *on* M *(where* $\lambda \in C^\infty(M)$ *and* $\lambda \neq 0$ *for proper relative spheres and* $\lambda = 0$ *for improper relative spheres).*
- *(ii)* $\hat{S} = \lambda \cdot h$ *on* M, $\lambda \in C^\infty(M)$.
- *(iii)* $m\nabla(h)\hat{T} = \div C$ *on* M.
- *(iv)* $\nabla(h)\hat{C}$ *is totally symmetric on* M.
- *(v)* $\nabla\hat{C}$ *is totally symmetric on* M.

(vi) $\mathrm{Ric} = \mathrm{Ric}^*$ *on* M.

(b) $\hat{S} = \lambda h$ *implies* $\lambda = \mathrm{const} = H$.

(c) If x *is a relative sphere, then for each* $p \in M$*:*

(i) p *is an umbilic;*

(ii) $\sum_{i<j}(k_i - k_j)^2 = m\|Hh - \hat{S}\|^2 = m[\|\hat{S}\|^2 - mH^2] = 0.$ □

Theorem 2.2. *Let* x *be regular with relative normalization* $\{Y, y\}$*. Then* x *is a proper relative sphere with center* x_0 *if and only if* $\rho(x_0) = \langle Y, x_0 - x\rangle = \mathrm{const} \neq 0$. □

Theorem 2.3. *A regular hypersurface* x *with relative normalization* $\{Y, y\}$ *is an improper relative sphere if and only if* $S = 0$. □

We refer to [**10**], [**11**], [**32**], [**68**]–[**72**], [**74**]–[**76**], [**95**], [**111**], [**122**], [**134**], for many examples including classifications of subclasses of affine spheres.

The complete classification is yet unknown. One tries to classify subclasses of affine spheres. In the following theorem is given a result related to this topic.

Theorem 2.4. *A locally strongly convex affine hypersphere with constant equiaffine sectional curvature is either a quadric or equiaffinely equivalent to the hypersurface* $x^1x^2\dots x^{m+1} = 1$*, where* $x^i : \mathfrak{A} \to \mathbb{R}$ *is a coordinate function.* □

We refer to [**133**] for the proof.

In case of an indefinite metric there are classifications for $m = 2$ in [**75**], [**120**], and for $m = 3$ in [**76**]. Other classifications results one can find in [**31**], [**117**], [**134**], etc.

There is a serious of results about compact affine spheres where many of the results are related to the spectral geometry of the equiaffine Laplacian (see [**114**], [**115**], [**119**] etc). We refer also Section V of this paper.

In [**122**] was studied existence and uniqueness problem about 2-spheres. Certain types of PDE's play an important role for the local and global classification of affine spheres in the equiaffine theory. One of the first PDE which was used in the theory of affine spheres is an expression for the Laplacian of the Pick invariant (see [**10**, §76]). Simon [**121**] extends this PDE to non-degenerate hypersurface. Monge-Ampère equations are used to investigate improper affine spheres and hyperbolic affine spheres.

A characterization of quadrics and improper affine spheres in terms of symmetry properties of $\nabla\hat{C}$ and $\nabla^2\hat{C}$ is given in [**22**].

I.3. Weyl geometry

As we know the metric h of a semi-Riemannian manifold (M, h) is parallel or covariantly constant with respect to the corresponding Levi-Civita connection. The main purpose of this section is to study a torsion free connection ${}^{\mathfrak{w}}\nabla$ satisfying the recurrence condition for the metric. This connection has been introduced by H. Weyl.

Weyl [**138**] attempted a unification of gravitation and electromagnetism in a model of space-time geometry combining both structures. His particular approach failed for physical reasons but his model is still studied in mathematics (see, for example, [**42**], [**49**]–[**53**], [**106**], [**130**],) and in mathematical physics (see, for example [**54**]).

We begin our discussion by introducing some notational conventions. Let (M, h) be a semi-Riemannian manifold of dimension $m \geq 2$. Fix a torsion free connection ${}^{\mathfrak{w}}\nabla$, called *the Weyl connection*, on the tangent bundle of M. We begin the definition of a Weyl structure by assuming that there exists a one-form $\hat{\theta} = \hat{\theta}_h$ so that

$$ {}^{\mathfrak{w}}\nabla h = 2\hat{\theta}_h \otimes h. \tag{3.1} $$

Let $\mathfrak{C} = \mathfrak{C}\{\mathfrak{W}\}$ be the conformal class defined by h (for more details, see Section II, 4), and let $\mathfrak{T} = \mathfrak{T}\{\mathfrak{W}\}$ be the corresponding collection of one-forms $\hat{\theta}_h$. Here and in the following we identify metrics in $\mathfrak{C}$ which merely differ by a constant positive factor. So there is a bijective correspondence between elements of $\mathfrak{C}$ and of $\mathfrak{T}$. We will call the triple $\mathfrak{W} = ({}^{w}\nabla, \mathfrak{C}, \mathfrak{T})$ *a Weyl structure* on M and we will call $(M, \mathfrak{W})$ *a Weyl manifold*.

The compatibility condition described in equation (3.1) is invariant under so-called gauge transformations

$$ h \to {}_{\beta}h := \beta h \quad \text{and} \quad \hat{\theta} \to {}_{\beta}\hat{\theta} := \hat{\theta} + d\frac{1}{2}(\ln \beta), \tag{3.2} $$

for $\beta \in C^{\infty}_{+}(M)$. We note that $C^{\infty}_{+}(M)$ acts transitively on $\mathfrak{C}$ and on $\mathfrak{T}$.

It is well known that a Weyl structure $\mathfrak{W}$ can be generated from a given pair $\{h, \hat{\theta}\}$ (where h is a semi-Riemannian metric and where $\hat{\theta}$ is a 1-form) in the following way. Let, $u, v, \ldots$ be vector fields on M and let ${}^{h}\nabla = \nabla(h)$ be the Levi-Civita connection of h. Let θ be the vector field dual to the 1-form $\hat{\theta}$, i.e., $h(w, \theta) = \hat{\theta}(w)$. We define $\alpha(u, v, w) := h(({}^{w}\nabla_u - {}^{h}\nabla_u)v, w)$. Since ${}^{w}\nabla$ and ${}^{h}\nabla$ are torsion free, $\alpha(u, v, w) = \alpha(v, u, w)$. Since ${}^{h}\nabla h = 0$ and since ${}^{\mathfrak{w}}\nabla$ satisfies equation (3.1), we have

$$ \begin{aligned} \alpha(u, v, w) + \alpha(u, w, v) + 2\hat{\theta}(u)h(v, w) = 0, \\ \alpha(u, v, w) = -\hat{\theta}(u)h(v, w) - \hat{\theta}(v)h(u, w) + \hat{\theta}(w)h(u, v), \\ {}^{w}\nabla_u v = {}^{h}\nabla_u v - \hat{\theta}(u)v - \hat{\theta}(v)u + h(u, v)\theta. \end{aligned} \tag{3.3} $$

Conversely, if equations (3.3) are satisfied, then ${}^{w}\nabla = {}^{w}\nabla(h, \hat{\theta})$ is a torsion free connection and equation (3.1) is satisfied. One can generates a Weyl structure from an arbitrary semi-Riemannian metric h and from an arbitrary 1-form $\hat{\theta}$ by using equation (3.3) to define ${}^{w}\nabla$ and using the action of $C^{\infty}_{+}(M)$ defined in equation (3.2) to generate the classes $\mathfrak{C}$ and $\mathfrak{T}$; see [**138**] or [**42**] for further details.

We use the sign convention of [**61**] to define the curvature of ${}^{w}\nabla$. Hence

$$ {}^{\mathfrak{w}}R(u, v) := {}^{\mathfrak{w}}\nabla_u\, {}^{\mathfrak{w}}\nabla_v - {}^{\mathfrak{w}}\nabla_v\, {}^{\mathfrak{w}}\nabla_u - {}^{\mathfrak{w}}\nabla_{[u,v]} $$

is the curvature corresponding to Weyl connection ${}^{\mathfrak{w}}\nabla$. For $h \in \mathfrak{C}$, Weyl introduced the 2-form ${}^{\mathfrak{w}}F := d\hat{\theta}_h$ as a gauge invariant of a given Weyl structure. He called it *the length curvature* or *distance curvature* [**138**, p. 124]. We have that F and ${}^{\mathfrak{w}}R$ are related by the equation

$$h(z, {}^{\mathfrak{w}}R(u,v)z) = {}^{\mathfrak{w}}F(u,v)h(z,z). \tag{3.4}$$

Weyl defined *the directional curvature* ${}^{\mathfrak{w}}K$ by

$${}^{\mathfrak{w}}K(u,v)w := {}^{\mathfrak{w}}R(u,v)w - {}^{\mathfrak{w}}F(u,v)w. \tag{3.5}$$

The curvature ${}^{\mathfrak{w}}R$ of ${}^{\mathfrak{w}}\nabla$ and the Weyl directional curvature ${}^{\mathfrak{w}}K$ are also gauge invariants. Relations (3.4) and (3.5) imply the orthogonality relation $h({}^{\mathfrak{w}}K(u,v)w,w) = 0$, for any $h \in \mathfrak{C}$ and for any vector field w. Moreover ${}^{\mathfrak{w}}F$ and ${}^{\mathfrak{w}}K$ satisfy respectively symmetry and skew-symmetry relations

$$h({}^{\mathfrak{w}}F(u,v)w,z) = h({}^{\mathfrak{w}}F(u,v)z,w),$$
$$h({}^{\mathfrak{w}}K(u,v)w,z) = -h(K(u,v)z,w).$$

As a local result the following is known: if the Weyl connection ${}^{\mathfrak{w}}\nabla$ is metric, then the length curvature vanishes identically. Conversely, if $F = d\hat{\theta}_h = 0$, equation (3.2) implies that the cohomology class $[\hat{\theta}_h(\mathfrak{W})] \in H^1(M)$ of the closed form $\hat{\theta}_h(\mathfrak{W})$ does not depend on the choice of a metric in $\mathfrak{W}$. Conversely, if ${}^{\mathfrak{w}}F = d\hat{\theta}_h = 0$ equation (3.2) implies that the cohomology class $[\hat{\theta}_h(\mathfrak{W})] \in H^1(M)$ of the closed form $\hat{\theta}_h(\mathfrak{W})$ is gauge invariant and does not depend on the choice of a metric in $\mathfrak{W}$. The following is well known; see, for example, [**52**], [**106**], [**130**].

Proposition 3.1. *The following assertions are equivalent:*
(i) we have ${}^{\mathfrak{w}}F(\mathfrak{W}) = 0$ *and* $[\hat{\theta}_h(\mathfrak{W})] = 0$ *in* $H^1(M)$;
(ii) there exists $h \in \mathfrak{C}(\mathfrak{W})$ *such that* ${}^{\mathfrak{w}}\nabla h = 0$; *i.e.,* ${}^{\mathfrak{w}}\nabla$ *is the Levi-Civita connection of* h. □

II. SOME TRANSFORMATIONS OF SMOOTH MANIFOLDS

II.1. Projective transformations

The main purpose of this section is to study projective transformations of a smooth manifold (M,∇) endowed with a torsion free connection ∇. More details one can find in [**41**], [**64**], [**109**], [**112**].

A map $f : (\tilde{M},\tilde{\nabla}) \to (M.\nabla)$ of manifolds with torsion free connections is called *projective* if for each geodesic γ of $\tilde{\nabla}$, $f \circ \gamma$ is a reparametrization of a geodesic of ∇, i.e., there exists a strictly increasing C^∞ function h on some open interval such that $f \circ \gamma \circ h$ is a ∇-geodesic. Linear connections $\tilde{\nabla}$ and ∇ on M are *projectively equivalent* if the identity map of M is projective. A *projective transformation* of

(M, ∇) is a diffeomorphism which is projective. The transformation s is projective on M, if the pull back $s^*\nabla$ of the connection is projectively related to ∇, i.e., if there exists a global 1-form $\pi = \pi(s)$ on M such that

$$s^*\nabla_u v = \nabla_u v + \pi(u)v + \pi(v)u, \tag{1.1}$$

for arbitrary smooth vector fields $u, v \in \mathfrak{X}(M)$. Having (1.1) in mind, if s and t are two projective transformations, we find $\pi(st) = \pi(s) + \hat{s} \cdot \pi(t)$, where $\hat{s}$ is the cotangent map, i.e. $[\hat{s} \cdot \pi]_{s(p)} = \hat{s} \cdot [\pi]p$.

If a transformation s of M preserves geodesics and the affine character of the parameter on each geodesic, then s is called *an affine transformation of the connection* ∇ or simply of the manifold M, and we say that s leaves the connection ∇ invariant.

It is well-known that *the Weyl projective curvature tensor* has the form

$$\begin{aligned} P(R)(u,v)w = R(u,v)w &+ \frac{1}{m^2-1}[m\rho(u,w) + \rho(w,u)]v \\ &- \frac{1}{m^2-1}[m\rho(v,w) + \rho(w,v)]u \\ &+ \frac{1}{m+1}[\rho(u,v) - \rho(v,u)]w, \end{aligned} \tag{1.2}$$

for any $m > 2$, and for $m = 2$ we have $P(R)(u,v)w = 0$, where $u, v, w, \dots \in \mathfrak{X}(M)$ (see for example [**110**], [**112**], [**136**]). $P(R)$ is a tensor that is invariant with respect to each projective transformation of M. $P(R)$ characterizes a space of constant sectional curvature in very nice way: $P(R) = 0$ if and only if M^m $(m > 2)$ is space of constant curvature (in that case R is the Riemannian curvature of M^m).

A manifold (M, ∇) is said to be a *projectively flat*, if it can be related to a flat space by a projective map. We know that the curvature tensor of a flat space is equal to zero: $R(u,v) = 0$, and therefore the Ricci tensor ρ is equal to zero also. Due to this fact from (1.2) we have the Weyl projective curvature tensor $P(R)$ of a flat space vanishes. Since the tensor $P(R)$ is invariant with respect to projective transformations, we have immediately $P(R)$ of a projectively flat space vanishes. The inverse theorem is valid also. Namely, if $P(R)$ of a manifold (M, ∇) vanishes then (M, ∇) is a projectively flat space.

One can use (2.2) in Section I to see (M, ∇^*) is a projectively flat space.

Ishihara studied in [**56**] the groups of projective and affine transformations. Among others he investigated the conditions that these groups coincide.

Theorem 1.1. *If (M, ∇) is a compact manifold with a torsion free connection ∇ and the Ricci tensor of ∇ vanishes identically in M, then the group of projective transformations of M coincides with its subgroup of affine transformations.* □

If M is also irreducible then Ishihara has proved that the group of projective transformations of M coincides with its group of isometries.

Projective transformations are closely related with projective structures (see [**60**]). *A projective structure on a m-dimensional manifold* is determined if there exists an atlas on M with transition functions being projective transformations [**65**]. Projective structure can be considered also in terms of subbundles of principal fiber bundles of 2-frames which structure group satisfies certain conditions (see [**60**]).

We refer also [**83**] for the references related this topic.

II.2. Holomorphically projective transformations

Before studying holomorphically projective transformations we need to introduce an almost complex structure.

An almost complex structure J on a smooth manifold M^{2m} is an endomorphism J such that $J^2 = -I$ on TM, where I is the identity. We say ∇ is a *complex symmetric connection* if it satisfies (1.1) of Section I and the following relation

$$\nabla J = 0. \tag{2.1}$$

The curvature R of ∇ satisfies besides of (1.2)–(1.4) of Section I also *the Kähler identity* $R(u,v)\circ J = J \circ R(u,v)$, for $u, v \in \mathfrak{X}(M)$. A manifold M^{2m} endowed with an almost complex structure J is *an almost complex manifold* (M,J). An almost complex manifold (M,J) may be endowed with a complex symmetric connection ∇ if *the Nijenhuis tensor* S of M, given by

$$S(u,v) = [u,v] + J[Ju,v] + J[u,Jv] - [Ju,Jv]$$

vanishes (see [**101**], [**105**]). An almost complex manifold (M,J) such that $S=0$ may be also endowed with *a complex atlas*, i.e., with complex coordinates. This manifold is called *a complex manifold*.

Especially, a complex symmetric connection ∇ is *a holomorphic affine connection* if $R(u,v) = -R(Ju,Jv)$, or *an affine Kähler connection* when one has $R(u,v) = R(Ju,Jv)$.

Holomorphic affine connections naturally appeared in the context of semi-Riemannian manifolds with the metric of signature (n,n) as well as in complex affine and projective differential geometry (see [**38**], [**59**], [**96**], [**98**], [**99**] for more details).

If a semi-Riemannian manifold (M,g) is endowed with an almost complex structure J satisfying (2.1) with respect to the corresponding Levi-Civita connection then (M,g,J) is *a Kähler manifold*.

Let Π_H be a 2-dimensional subspace of tangent space T_pM, spanned by vectors (u,Ju), for any unit vector $u \in T_pM$. *The holomorphic sectional curvature* of Π_H is $KH_p(\Pi_H) = R(u,Ju,Ju,u)(p)$. M is *a space of the constant holomorphic sectional curvature* if $KH_p(\Pi_H)$ is independent of the choice of Π_H in T_pM, where p is an arbitrary point of M and depends on $p \in M$. Its Riemann curvature tensor is given by

$$\begin{aligned} R(u,v,z,w) = KH_p(&g(u,z)g(v,w) \\ &- g(u,w)g(v,z) + g(u,Jz)g(v,Jw) \\ &- g(u,Jw)g(v,Jz) + 2g(u,Jv)g(z,Jw)). \end{aligned}$$

Let (M^{2m}, g, J) be a connected Kähler manifold $(m \geq 2)$. If $KH_p(\Pi_H)$ is invariant by J, depends only on p, then M is *a space of constant holomorphic sectional curvature.*

A Hermitian manifold is a complex manifold endowed with a Riemannian metric g such that

$$g(Ju, Jv) = g(u, v), \tag{2.2}$$

for all $u, v \in \mathfrak{X}(M)$. More details about other types of almost complex manifolds endowed with a metric g satisfying (2.2) one can find in [**46**].

A curve γ is *the holomorphically planar curve* if its tangent vector field T belongs to the plane spanned by the vectors T and JT under parallel displacement with respect to a complex symmetric connection ∇ along the curve γ; i.e., if T satisfies the relation $\nabla_T T = \rho(t)T + \sigma(t)JT$, where $\rho(t)$ and $\sigma(t)$ are some functions of a real parameter t.

A diffeomorphism $f : (\tilde{M}, \tilde{\nabla}) \to (M, \nabla)$ of manifolds with complex symmetric connections is called *holomorphically projective* if the image of any holomorphically planar curve of $\tilde{M}$ is also holomorphically planar curve of M. More details about these diffeomorphisms and curves one can find in [**83**].

Let (M^{2m}, ∇, J) be a complex manifold, where ∇ is the corresponding complex symmetric connection. Tashiro [**129**] has studied some transformations of (M^{2m}, ∇, J). *The transformation* s is *holomorphically projective* on (M^{2m}, ∇, J) if it preserves the system of holomorphically planar curves, i.e., if the pull back $s^*\nabla$ of the complex symmetric connection ∇ is holomorphically projective related to ∇, i.e., if there exists a global 1-form $\pi = \pi(s)$ on M such that

$$s^*\nabla_u v = \nabla_u v + \pi(u)v + \pi(v)u - \pi(Ju)Jv - \pi(Jv)Ju,$$

for arbitrary smooth vector fields u, v. He has proved that *the holomorphically projective curvature tensor*

$$\begin{aligned} HP(R)(u,v)w = R(u,v), w + P(v,w)u - P(u,w)v - P(u,v)w + P(v,u)w \\ - P(v,Jw)Ju + P(u,Jw)Jv + P(u,Jv)Jw - P(v,Ju)Jw, \end{aligned}$$

where

$$P(u,v) = -\frac{1}{2(m+1)}\Big[\rho(u,v) + \frac{1}{2(m-1)}(\rho(u,v) + \rho(v,u) - \rho(Ju,Jv) - \rho(Jv,Ju))\Big],$$

is invariant with respect to each holomorphically projective transformation of ∇. This tensor plays a similar role in studying of a manifold endowed with a complex symmetric connection as the Weyl projective curvature tensor in studying of manifolds with a torsion free connection. So, $HP(R)$ of *a holomorphically projective flat* space vanishes. A complex manifold (M^{2m}, J, ∇) is said to be a holomorphically projective flat, if it can be related to a flat space by a holomorphically projective map. $HP(R)$ characterizes a space of constant holomorphical sectional curvature as follows: $HP(R) = 0$ if and only if M^{2m} is a space of constant holomorphic sectional curvature. Ishihara in [**55**] has found the conditions that the group of holomorphically projective transformations coincides with its subgroup of affine transformations. More precisely, he proved the following theorem.

Theorem 2.1. *If a complex manifold M of complex dimension $m > 1$ is complete with respect to a complex symmetric connection ∇ and the Ricci tensor of M vanishes identically in M, then the group of holomorphically projective transformations of M coincides with its subgroup of affine transformations.* □

Moreover, if M is a compact Kähler manifold then Ishihara has proved that the identity component of its group of holomorphically projective transformations for the Levi-Civita connection coincides with the identity component of its group of isometries.

II.3. C-holomorphically projective transformations

As we have seen in II.2 a C^∞ differentiable manifold M^{2m} is said to have an almost complex structure if there exists on TM a field J of endomorphisms of tangent spaces such that $J^2 = -I$, I being the identity transformation. Every manifold carrying an almost complex structure must have an even dimension.

The notion of almost contact structure generalizes these structures in the case of the odd dimension. A $(2m+1)$-dimensional C^∞ manifold M is said to have *an almost contact structure* (φ, ξ, η) if it admits a field of endomorphisms φ, a vector field ξ and a 1-form η such that $\varphi^2 = -I + \eta \otimes \xi$, $4\eta(\xi) = 1$. The following relations also hold $\varphi(\xi) = 0$, $\eta \circ \varphi = 0$, $\operatorname{rank} \varphi = 2m$. We remark that any odd dimensional orientable compact manifold M has Euler characteristic equal to zero, and there exists at least one non singular vector field ξ on M. On every almost contact manifold M there exists a Riemannian metric g satisfying

$$g(x, \xi) = \eta(x), \quad g(\varphi x, \varphi y) = g(x, y) - \eta(x)\eta(y),$$

g is said to be *compatible with the structure* (φ, ξ, η) and (φ, ξ, η, g) is called *an almost contact metric* structure. We refer to [**7**] for more details.

Example 3.1. Let M^{2m+1} be a C^∞ orientable hypersurface of an almost Hermitian manifold $\bar{M}^{2m+2}$ with almost complex structure J and Hermitian metric G.

Then there exists a vector field C along M^{2m+1} transverse to M^{2m+1} such that JC is tangent to M^{2m+1} (otherwise an almost complex structure on M^{2m+1} would exist, which is impossible). Thus, we can find a vector field ξ on M^{2m+1} such that $C = J\xi$ is transverse to M^{2m+1}. The relation $Ju = \varphi u + \eta(u)C$ defines the tensor field φ of type (1,1) and the 1-form η on M^{2m+1} satisfying $\varphi^2 = -I + \eta \otimes \xi$ and $\eta \circ \varphi = 0$. Since $\varphi\xi = 0$ and $\eta(\xi) = 1$ also hold, (φ, ξ, η) defines an almost contact structure on M^{2m+1}. Moreover, the metric g induced by G is the metric compatible with the almost contact structure (φ, ξ, η). □

Example 3.2. Let M^{2m} be an almost complex manifold with almost complex structure J. We consider the manifold $M^{2m+1} = M^{2m} \times \mathbb{R}$, though a similar construction can be made for the product $M^{2m} \times S^1$. Denote a vector field on M^{2m+1} by $(u, f\frac{d}{dt})$ where u is tangent to M^{2m}, t is the coordinate of $\mathbb{R}$ and f is a C^∞ function on M^{2m+1}. Then $\eta = dt$, $\xi = (0, \frac{d}{dt})$ and $\varphi(u, f\frac{d}{dt}) = (Ju, 0)$ define an almost contact structure (φ, ξ, η) on M^{2m+1}. □

An odd-dimensional parallelizable manifold, especially any odd-dimensional Lie group, carries an almost contact structure.

As is well known, if $(M^{2m+1}, \varphi, \xi, \eta)$ is an almost contact manifold, the linear map J defined on the product $M^{2m+1} \times \mathbb{R}$ by the relation

$$J\Big(u, f\frac{d}{dt}\Big) = \Big(\varphi u - f\xi, \eta(u)\frac{d}{dt}\Big),$$

where f is a C^∞ real-valued function on $M^{2m+1} \times \mathbb{R}$, is an almost complex structure on $M^{2m+1} \times \mathbb{R}$; thus we have $J^2 = -I$. In particular, if J is integrable, the almost contact structure (φ, ξ, η) *is normal.*

The almost contact structure (φ, ξ, η) is said to be *normal* if and only if the tensors $N, N^{(1)}, N^{(2)}, N^{(3)}$ vanish on M^{2m+1}, where

$$\begin{aligned} N(u,v) &= [\varphi,\varphi](u,v) + d\eta(u,v)\xi, \quad N^{(2)}(u,v) = (\mathfrak{L}_v\varphi)(u), \\ N^{(1)}(u,v) &= (\mathfrak{L}_{\varphi u}\eta)(v) - (\mathfrak{L}_{\varphi v}\eta)(u), \qquad N^{(3)}(u) = (\mathfrak{L}_\xi\eta)(u), \end{aligned} \tag{3.6}$$

$\mathfrak{L}$ denotes the *Lie differentiation* and $[\varphi, \varphi]$ is *the Nijenhuis torsion tensor* of φ.

The normal almost contact structure generalizes, in the odd dimension, the complex structure.

If an almost complex structure J is integrable then $[J, J] = 0$. As a consequence there exists a torsion free adopted connection $\bar{\nabla}$, i.e., satisfying $\bar{\nabla} J = 0$. Thus it appears of interest to construct some connection ∇ on the almost contact manifold $(M^{2m+1}, \varphi, \xi, \eta)$ which gives rise to an adapted connection $\bar{\nabla}$ on $M^{2m+1} \times \mathbb{R}$.

Definition 3.3. [**79**] A linear connection ∇ on an almost contact manifold $(M^{2m+1}, \varphi, \xi, \eta)$ is called *an adopted connection* if it satisfies the following system

$$\begin{aligned} (\nabla_u\varphi)v &= \eta(v)hu + \frac{1}{4}(d\eta(\varphi u, hv) - d\eta(u, \varphi v))\xi, \\ (\nabla_u\eta)(v) &= \frac{1}{4}(d\eta(u,v) + d\eta(\varphi u, \varphi v)), \\ \nabla_u\xi &= \varphi u - \frac{1}{4}d\eta(u,\xi)\xi, \end{aligned} \tag{3.1}$$

where $h = I - \xi \otimes \eta$. □

We refer to [**80**] for more details related to the results which follow. Notice that the system (3.1) is not the only solution to our initial problem $\nabla J = 0$. One can check that the general family of the adopted connections ∇ on the almost contact manifold $(M^{2m+1}, \varphi, \xi, \eta)$ is given by the equation

$$\nabla_u v = \breve{\nabla}_u v + P(u, v),$$

where $\breve{\nabla}$ is an arbitrary initial connection and P is given by

$$P(u,v) = \frac{1}{2}(\breve{\nabla}_u\varphi)\varphi v - (\breve{\nabla}_u\xi)\eta(v) + \frac{1}{2}(\breve{\nabla}_u\varphi)(v)\xi + \frac{1}{2}\eta(\breve{\nabla}_u\xi)\eta(v)\xi$$
$$+ \eta(v)\varphi u - \frac{1}{4}\{d\eta(u,v) + d\eta(\varphi u, \varphi v)\}\xi + (\Phi - \Theta)Q(u,v).$$

Here Q denotes an arbitrary tensor field of type (1.2) and $\Phi = \frac{1}{2}(I\otimes I - \varphi\otimes\varphi)$, $\Theta = \frac{1}{2}(I\otimes I - h\otimes h)$.

We remark the curvature tensor as well as the Ricci tensor of an adopted connection on a normal almost contact manifold $(M^{2m+1}, \varphi, \xi, \eta)$ have some interesting properties which allow us to consider some transformations, in the spirit of the sections II.2.

Definition 3.4. Let ∇ be a torsion free connection adopted to the normal almost contact structure (φ, ξ, η) on M^{2m+1}. A curve γ is *C-flat* (*almost-contact flat*) with respect to ∇ if $\nabla_T T = \rho(t)T + \sigma(t)\varphi T$, where T denotes the vector tangent to γ and ρ, σ are smooth real valued functions along γ. □

We remark that in this case the subspace spanned by T and φT is not transported by parallelism along γ. Namely, $\nabla_T(\varphi T)$ does not belong to the space spanned by T and φT. However, one can show that the dimension of this subspace is constant along γ and this dimension can be 2,1 or 0.

Remark. We introduced in [**25**] the concept of C-flat paths, obtaining a C-projective tensor in normal almost contact manifolds, endowed, with a torsion free connection whose fundamental tensors φ, ξ and η are parallel.

The torsion free linear connections $\nabla, \tilde{\nabla}$ adapted to the normal almost contact structure (φ, ξ, η) are *C-projectively related* if they have the same C-flat curve. One can show that two torsion free connections $\nabla, \tilde{\nabla}$ adopted to the normal almost contact structure (φ, ξ, η) are C-projectively related if and only if

$$\tilde{\nabla}_u v = \nabla_u v + P(u,v),$$

where $P(u,v) = \alpha(u)hv + \alpha(v)hu - \beta(u)\varphi v - \beta(v)\varphi u$, with α an arbitrary 1-form satisfying $\alpha(\xi) = 0$, $\beta(u) = \alpha(\varphi u)$. Consequently, since $\nabla, \tilde{\nabla}$ fulfill the same conditions (3.1), their difference tensor P satisfies the same relations as in the case where φ, ξ, η are parallel.

Matzeu has proved in [**80**] that the tensor field $W(R)$ given by

$$W(R)(u,v)z = hR(u,v)z + \{L(u,v) - L(v,u)\}hz + \{L(u,v) + \eta(u)\eta(z)\}hv$$
$$- \{L(v,z) + \eta(v)\eta(z)\}hu - \{L(u,\varphi v) - L(v,\varphi u) + d\eta(u,v)\}\varphi z$$
$$- \Big\{L(u,\varphi z) + \frac{1}{2}d\eta(u,z)\Big\}\varphi v + \Big\{L(v,\varphi z) + \frac{1}{2}d\eta(v,z)\Big\}\varphi u, \tag{3.2}$$

with

$$L(u,v) = \frac{1}{2(m+1)}\Big\{\rho(R)(u,hv) + \frac{1}{2(m-1)}[\rho(R)(hu,v) + \rho(R)(hv,u) - \rho(R)(\varphi u,\varphi v) - \rho(R)(\varphi v,\varphi u)]\Big\} + kd\eta(u,\varphi v), \quad k = \text{const}, \tag{3.3}$$

is C-projectively invariant. Moreover, if k in (3.3) is given by $k = \dfrac{1}{2m+2}$, all traces of $W(R)$

$$\begin{array}{ll} \mathrm{trace}(W(R)(u,v)), & \mathrm{trace}(u \to W(R)(u,v)z) \\ \mathrm{trace}(\varphi W(R)(u,v)), & \mathrm{trace}(u \to \varphi W(R)(u,v)z) \end{array}$$

vanish.

We say torsion free connection adopted to the normal almost contact structure (φ, ξ, η) is *C-projectively flat* if its *C-projective curvature tensor* $W(R)$ vanishes.

We refer [**80**] for the proof of the following theorem.

Theorem 3.5. *For $m > 2$ the torsion free adopted connection ∇ is C-projectively flat if and only if it can be transformed locally by a C-projective transformation into a torsion free adopted connection $\tilde{\nabla}$ whose curvature tensor $\tilde{R}$ satisfies the condition*

$$h\tilde{R}(u,v)z = \{-\eta(u)hv + \eta(v)hu\}\eta(z) + d\eta(u,v)\varphi z + \frac{1}{2}d\eta(u,z)\varphi v - \frac{1}{2}d\eta(v,z)\varphi u. \square$$

The case $m = 1$ has been studied by Oproiu in [**103**]. It is also interesting that there does not exist a flat adopted connection from a C-projective transformation. But, in the framework of ∇ with parallelizable (φ, ξ, η) it exists.

A special class of normal almost contact metric spaces $(M^{2m+1}, \varphi, \xi, \eta, g)$ is Sasakian one satisfying the condition $\eta \wedge d\eta^m \neq 0$ ($d\eta^m$ is m-th exterior power). The Levi-Civita connection ∇ of g for Sasakian manifold is an adopted one. A Sasakian manifold is C-projectively flat if and only if it has constant φ-sectional curvature.

II.4. Conformal transformations

Let (M, g) be an m-dimensional Riemannian manifold. Locally the metric is given by $ds^2 = g_{ij}dx^i dx^j$, where the g_{ij} are the components of g with respect to the natural frames of a local coordinate system (x^i). A metric g^* on M is said to be conformally related to g if it is proportional to g, that is, if there is a function $\beta > 0$ on M such that $g^* = \beta^2 g$. We denote by $\mathfrak{C}$ a conformal class of Riemannian metrics on a smooth manifold M, of dimension $m \geq 2$. By *a conformal transformation* of M is meant a differentiable homeomorphism f of M onto itself with the property that $f^*(ds^2) = \beta^2 ds^2$, where f^* is the induced map in the bundle of frames and β is a positive function on M. The set of conformal transformations of M forms a group. Moreover, it can be shown that it is a Lie transformation group. A diffeomorphism f of M onto itself is called an isometry if it preserves the metric tensor.

Under a conformal transformation of metric, the curvature tensor $R(u,v)w$ will be transformed into

$$R^*(u,v)w = R(u,v)w - \sigma(w,u)v + \sigma(w,v)u - g(w,u)v + g(w,v)u,$$

where σ is the tensor of type (0,2) with components $\sigma_{jk} = \beta_{j;k} - \beta_j\beta_k + \frac{1}{2}g^{bc}\beta_b\beta_c g_{jk}$, and μ the corresponding type (1,1) tensor with components

$$\mu^i_l = \sigma_{kl}g^{ki}, \quad \Big(\beta_j = \frac{\partial \log \beta}{\partial u^j}\Big).$$

Let $m > 2$. The tensor

$$C(u,v)w = R(u,v)w - \frac{1}{m-2}(\rho(w,u)v - \rho(w,v)u + g(w,u)S(v) - g(w,v)S(u)) + \frac{\tau}{(m-1)(m-2)}(g(w,u)v - g(w,v)u),$$

where S is the Ricci endomorphism $g(Su,v) = \rho(u,v)$, is invariant under a conformal transformation of a metric, i.e. $C^*(u,v)w = C(u,v)w$. This tensor is called *the Weyl conformal curvature tensor*. The case $m = 3$ is interesting. Indeed, by choosing an orthogonal coordinate system ($g_{ij} = 0$, $i \neq j$) at a point, it is readily shown that the Weyl conformal curvature tensor vanishes.

Consider a Riemannian manifold (M,g) and let g^* be a conformally related locally flat metric. Under these circumstances M is said to be locally conformally flat. Let

$$C(u,v,w) = \frac{1}{m-2}((\nabla_w\rho)(u,v) - (\nabla_v\rho)(w,v)) - \frac{1}{2(m-1)(m-2)}(g(u,v)\nabla_w\tau - g(u,w)\nabla_v\tau).$$

One can prove the following theorem

Theorem 4.1. *A necessary and sufficient condition that a Riemannian manifold of dimension $m > 3$ be a conformally flat is that its Weyl conformal curvature tensor vanish. For $m = 3$, it is necessary and sufficient that the tensor $C(u,v,w)$ vanishes, i.e. $C(u,v,w) = 0$.* □

There exist numerous examples of conformally flat spaces. For example, a Riemannian manifold of constant curvature is conformally flat, provided $m \geq 3$.

Any two 2-dimensional Riemannian manifolds are conformally related, as the quadratic form ds^2 for $m = 2$ is reducible to the form $\lambda[(du^1)^2 + (du^2)^2]$ (in infinitely many ways).

For more details one can use [**45**], [**64**], [**140**] etc.

II.5. Codazzi geometry

Codazzi structure is constructed from a conformal and a projective structure using the Codazzi equations. A torsion free connection ${}^*\nabla$ and a semi-Riemannian metric h are said to *satisfy the Codazzi equation* or to be *Codazzi compatible* if

$$({}^*\nabla_u h)(v,w) = ({}^*\nabla_v h)(u,w). \tag{5.1}$$

A projective class $\mathfrak{P}$ of torsion free connections and a conformal class $\mathfrak{C}$ of semi-Riemannian metrics are said to be *Codazzi compatible* if there exists ${}^*\nabla \in \mathfrak{P}$ and $h \in \mathfrak{C}$ which are Codazzi compatible. We extend the action of the gauge group to define ${}^*\nabla \to {}^*_\beta\nabla$ where ${}^*_\beta\nabla$ is defined by taking $\pi = d\ln\beta$ in (1.1):

$$
{}^*_\beta\nabla_u v = {}^*\nabla_u\, v + d\ln\beta(u)v + d\ln\beta(v)u. \tag{5.2}
$$

One can check easily the Codazzi equations are preserved by gauge equivalence. *A Codazzi structure* $\mathfrak{K}$ on M is a pair $(\mathfrak{C}, \mathfrak{P})$ where the conformal class of semi-Riemannian metrics $\mathfrak{C}$ and the projective class $\mathfrak{P}$ are Codazzi compatible. *A Codazzi manifold* $(M, \mathfrak{K})$ is a manifold endowed with the Codazzi structure.

Suppose now that $(h, {}^*\nabla)$ are Codazzi compatible. Let $C := {}^*\nabla - \nabla(h)$ be a (1,2) tensor and let $\check{C}$ be the associated cubic form. Since ${}^*\nabla$ and $\nabla(h)$ are torsion free, C is a symmetric (1,2) tensor and $\check{C}(u,v,w) = \check{C}(v,u,w)$. The relation (5.1) and this symmetry implies $\check{C}(u,v,w) = \check{C}(w,v,u)$ and consequently $\check{C}$ is totally symmetric.

Assuming that h is a semi-Riemannian metric and $\check{C}$ is a totally symmetric cubic form one can construct *a conjugate triple* $({}^*\nabla, h, \nabla)$, i.e. a triple $({}^*\nabla, h, \nabla)$ satisfying

$$
uh(v,w) = h(\nabla_u v, w) + h(v, {}^*\nabla_u w). \tag{5.3}
$$

Therefore, let ${}^*\nabla := \nabla(h) + C$, where C is the associated symmetric (1,2) tensor field. Since $\check{C}$ is symmetric, ${}^*\nabla$ is torsion free and the Codazzi equation (5.1) is satisfied. If we put $\nabla := \nabla(h) - C$ one can check the triple $({}^*\nabla, h, \nabla)$ satisfies (5.3), i.e., it is a conjugate triple.

If $\mathfrak{W}$ is a Weyl structure one can define an associated Codazzi structure $\mathfrak{K}(\mathfrak{W})$. We may recover also the Weyl structure from the associated Codazzi structure. We refer to [**20**] for more details.

III. DECOMPOSITIONS OF CURVATURE TENSORS UNDER THE ACTION OF SOME CLASSICAL GROUPS AND THEIR APPLICATIONS

The main purpose of this section is to consider a curvature for a torsion free connection from the algebraic point of view and to see why it does provide insight in some problems of differential geometry, topology etc. It is possible also to study the various curvatures which appear in differential geometry in different context (see [**73**]). Let us mention that it is possible in this spirit to study some classification of almost Hermitian manifolds [**48**], Riemannian homogeneous structure [**132**] etc. Of course all these decompositions are, in principle, consequences of general theorems of groups representations (see [**135**]).

More precisely, the proofs of theorems are based on the following facts. Let $\mathfrak{G}$ be a Lie group, V a real vector space and V^* its dual space. When ξ is a $\mathfrak{G}$-concomitant between two spaces, $\mathfrak{G}$ acting on these spaces then the image for ξ

of an invariant subspace is also invariant. Further, the image is irreducible when the first space is irreducible. Also an invariant subspace of $\otimes^r V^*$ is irreducible for the action of some group if and only if the space of its quadratic invariants is 1-dimensional.

III.1. Some historical remarks

The development of the theory of the decomposition was initiated by Singer and Thorpe [**125**]. Let (V, g) be an m-dimensional real vector space with positive definite inner product and denote by $\mathcal{R}_b(V)$ the vector space of all symmetric linear transformations of the space of 2-vectors of V. All tensors having the same symmetries as the Riemannian curvature tensor including the first Bianchi identity belong also to $\mathcal{R}_b(V)$. Singer and Thorpe gave a geometrically useful description of the splitting of $\mathcal{R}_b(V)$ under the action of $O(n)$ into four components. One of the projections gives the Weyl conformal tensor. Their considerations are as follows.

Let a tensor R of type (1,3) over V be a bilinear mapping

$$R : V \times V \to \mathrm{Hom}(V, V) : (x, y) \mapsto R(x, y).$$

We use the notation $R(x, y, z, w) = g(R(x, y)z, w)$. Let $\mathcal{R}_b(V)$ and $\mathcal{R}(V)$ be the subspaces of $\otimes^4 V^*$ consisting of all tensors having the same symmetries as the curvature tensor, the first for metric connections, the second for Levi-Civita connections. It means, $R \in \mathcal{R}_b(V)$ if it yields

(a) $R(x, y) = -R(y, x)$

(b) $R(x, y)$ is a skew-symmetric endomorphism of V, i.e.,

$$R(x, y, z, w) + R(x, y, w, z) = 0$$

and $R \in \mathcal{R}(V)$ if R satisfies (a), (b) and the first Bianchi identity

(c) $\sigma R(x, y)z = 0$, where σ denotes the cyclic sum over x, y and z.

The Ricci tensor $\rho(R)$ of type (0,2) associated with R is symmetric bilinear function on $V \times V$ defined by $\rho(R)(x, y) = \mathrm{trace}\,(z \in V \mapsto R(z, x)y \in V)$. Then the Ricci tensor $Q = Q(R)$ of type (1,1) is given by $\rho(R)(x, y) = g(Qx, y)$ and the trace of Q is called the scalar curvature $\tau = \tau(R)$ of R.

Further, let α be the standard representation of the orthogonal group $O(n)$ in V. Then there is a natural *induced representation* $\tilde{\alpha}$ of $O(n)$ in $\mathcal{R}_b(V)$ given by

$$\tilde{\alpha}(a)(R)(x, y, z, w) = R(\alpha(a^{-1})x, \alpha(a^{-1})y, \alpha(a^{-1})z, \alpha(a^{-1})w).$$

for all $x, y, z, w \in V$, $R \in \mathcal{R}_b(V)$ and $a \in O(n)$.

Theorem 1.1. $\mathcal{R}_b(V) = \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \mathcal{R}_3 \oplus \mathcal{R}_4$, $\mathcal{R}(V) = \mathcal{R}_2 \oplus \mathcal{R}_3 \oplus \mathcal{R}_4$.

(i) $\mathcal{R} \in \mathcal{R}_1$ *iff the sectional curvature is zero.*

(ii) $R \in \mathcal{R}_1 \oplus \mathcal{R}_2$ *iff the sectional curvature of* R *is constant.*

(iii) $R \in \mathcal{R}_1 \oplus \mathcal{R}_3$ *iff the Ricci tensor of* R *is zero.*

(iv) $R \in \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \mathcal{R}_3$ *iff the Ricci tensor of* R *is a scalar multiple of the identity.*

(v) $R \in \mathcal{R}_1 \oplus \mathcal{R}_3 \oplus \mathcal{R}_4$ *iff the scalar curvature of* R *is zero.* □

Furthermore, the action of $O(n)$ in $\mathcal{R}_b(V)$ is irreducible on each $\mathcal{R}_i$, $i = 1,2,3,4$. Since a curvature tensor R, corresponding to the Levi Civita connection ∇ of a Riemannian manifold M satisfies the first Bianchi identity, we have $R \in \mathcal{R}_1^{\perp} = \mathcal{R}_2 \oplus \mathcal{R}_3 \oplus \mathcal{R}_4$. Statement (iv) of Theorem 1.1 implies a very nice characterization of an Einstein space as follows: a Riemannian manifold M has curvature tensor in $\mathcal{E} = \mathcal{R}_2 \oplus \mathcal{R}_3$ at each point if and only if M is an Einstein space. The $\mathcal{R}_3$-component of a curvature tensor of M is its Weyl conformal curvature tensor.

To study the action of $SO(n)$, especially for $\dim V = 4$, Singer and Thorpe have used the star operator $*$. They studied also in [**125**] the problem of a *normal form* for the curvature tensor of a 4-dimensional oriented Einstein manifold by analyzing the critical point behavior of the sectional curvature function σ. In this case, the function σ on each 2-plane is equal to its value on the orthogonal complement. Using this characterization, they have shown that the curvature function σ is completely determined by its critical point behaviour and they have shown what the locus of critical points looks like.

The relationship between the Euler-Poincaré characteristic, the arithmetic genus $\alpha(M)$ and the decomposition of the space of curvature operators at a point of 4-dimensional compact Riemannian manifold has been studied by Gray [**47**]. Applications of the decomposition of $\mathcal{R}(V)$ involving orthogonal Radon transformations were given by Strichartz [**127**]. An algebraic interpretation of the Weyl conformal curvature tensor due to Singer and Thorpe makes possible the development of the theory of submanifolds in conformal differential geometry more up to date (see [**67**]).

The complete decomposition of $\mathcal{R}_k(V) \subset \mathcal{R}(V)$, $\dim V = 2m$, satisfying the Kähler identity, under the action of $U(V)$ was treated by Sitaramaya [**126**] (see also [**57**], [**88**]). Tricerri and Vanhecke [**131**] have found the complete decomposition of $\mathcal{R}(V)$ under the action of $U(V)$. They have obtained new conformal invariants among components of the complete decomposition of $\mathcal{R}(V)$ on almost Hermitian manifolds.

We refer to [**24**] for more details.

III.2. The action of general linear group

The main purpose of this section is to interpret the Weyl projective curvature tensor as one of the projection operators in the decomposition of tensors having all the symmetries of curvature tensors for torsion free connections under the action of the general linear group $GL(V)$. We refer to [**127**] for some details.

In this section V denotes m-dimensional ($m \geq 2$) real vector space, V^* its dual space, and $\mathcal{V}_3^1$ the space of (1,3) tensors $T(u,v,z,w)$ with $u,v,z \in V$ and $w \in V^*$. The group $GL(V)$ acts naturally on $\mathcal{V}_3^1$ by

$$\pi(g)T(u,v,z,w) = T(g^{-1}u, g^{-1}v, g^{-1}z, (g^{-1})^T w).$$

Let $\pi(m)$ be representations of $GL(V)$, where $m = (m_1, \ldots, m_m)$ is the highest weight of the representation, with $m_1 \geq m_2 \geq \cdots \geq m_m$, all m_i integers, and for simplicity of notation we delete strings of zeroes (so that $\pi(2,-1)$ stands for $\pi(2,0,\ldots,0,-1)$. Let $\mathcal{R}(V)$ be a subspace of tensors with symmetries as the curvature of a torsion free connection. So for $R \in \mathcal{R}(V)$ we have

$$R(u,v,z,w) = -R(v,u,z,w), \tag{2.1}$$

$$R(u,v,z,w) + R(v,z,u,w) + R(z,u,v,w) = 0, \tag{2.2}$$

where $R(u,v,z,w) = \langle R(u,v)z, w\rangle$. We denote the Ricci contraction by $\rho(R) = \mathrm{con}(1,4)R$. It maps $\mathcal{R}(V)$ onto $\mathcal{V}_2$. The space $\mathcal{V}_2$ splits as $\pi(2) \oplus \pi(1,1)$, the symmetric and skew-symmetric tensors. Consequently, we have

$$\mathcal{R}(V) = \pi(2) \oplus \pi(1,1) \oplus \ker(\rho).$$

One can check easily that $ker(\rho)$ is also irreducible. We introduce two special products $\odot_1$ and $\odot_2$ to describe the corresponding projection operators. For $Q \in \mathcal{V}_1^1$ and $S \in \mathcal{V}_2$ we have

$$\begin{aligned} Q \odot S(u,v,z,w) &= Q(v,w)(S(u,z) + S(z,u) - Q(u,w)(S(v,z) + S(z,v)) \\ Q \odot_2 S(u,v,z,w) &= Q(v,w)(S(u,z) - S(z,u)) - Q(u,w)(S(v,z) - S(z,v)) \\ &\quad + 2Q(z,w)(S(u,v) - S(v,u)). \end{aligned}$$

By direct computation one can check $Q \odot_1 S$, $Q \odot_2 S \in \mathcal{R}(V)$ with $\rho(Q \odot_1 S)$ symmetric and ρ $(Q \odot_2 S)$ skew-symmetric. Henceforth we have

Theorem 2.1. *Under the action of $GL(V)$, the space $\mathcal{R}(V)$ decomposes as*

$$\pi(2) \oplus \pi(1,1) \oplus \pi(2,1,-1),$$

(when $m = 2$ the third component is deleted) with corresponding projections

$$\begin{aligned} P_{(2)}R &= \frac{1}{2(m-1)}\delta \odot_1 \rho(R), \quad P_{(1,1)}R = \frac{1}{2(m+1)}\delta \odot_2 \rho(R), \\ P_{(2,1,-1)}R &= R - \frac{1}{2(m-1)}\delta \odot_1 \rho(R) - \frac{1}{2(m+1)}\delta \odot_2 \rho(R), \end{aligned}$$

where δ is Kronecker symbol.

The $\pi(2,1,-1)$ component is the kernel of ρ, while

$$\begin{aligned} \rho(P_{(z)}R)(u,v) &= \frac{1}{2}(\rho(R)(u,v) + \rho(R)(v,u)) \\ \rho(P_{(1,1)}R)(u,v) &= \frac{1}{2}(\rho(R)(v,u) - \rho(T)(u,v)). \end{aligned}$$

The corresponding dimensions and highest weight vectors are as follows:

Components	Dimension	Highest Weight Vector
$\pi(2)$	$\frac{1}{2}m(m+1)$	$\sum_k (e_1 \wedge e_k) \otimes e_1 \otimes e_k^*$
$\pi(1,1)$	$\frac{1}{2}m(m-1)$	$\sum_k (2(e_1 \wedge e_2) \otimes e_k \otimes e_k^* + (e_1 \wedge e_k) \otimes e_2 \otimes e_k^* + (e_k \wedge e_2) \otimes e_1 \otimes e_k^*)$
$\pi(2,1,-1)$	$\frac{1}{3}m^2(m^2-4)$	$(e_1 \wedge e_2) \otimes e_1 \otimes e_m^*$
$\mathcal{R}(V)$	$\frac{1}{2}m^2(m^2-1)$	

If $V = T_pM$ then one can compare (1.2) in Section II with $P_{(2,,1,-1)}R$ to see that they coincide and consequently the Weyl projective curvature tensor is really an irreducible component in the proceeding decomposition. One can see easily that the Weyl projective curvature tensor fulfills the algebraic conditions (2.2), (2.3) and $\rho(P(R)) = 0$.

Strichartz in [**127**] has studied the complete decomposition of the vector space of the first covariant derivative of curvature tensors for torsion free connections under the action of $GL(V)$. Using these decompositions he has proved that a projectively flat affine manifold with skew-symmetric Ricci curvature must be locally affine symmetric.

III.3. The action of the group $SO(m)$

Studying projective transformations of a Riemannian manifold (M, g) we naturally combine two structures: the positive definite metric g and torsion free connections ∇, which can be, for example, projectively equivalent to the Levi-Civita connection. Therefore we are interested now in the complete decomposition of $\mathcal{R}(V)$ from the section III.2 under the action of $SO(m)$. We refer to [**23**] for more details.

Let V be an m-dimensional real vector space endowed with positive definite inner product $\langle \cdot, \cdot \rangle$. A tensor R of type (1,3) over V is a bilinear mapping

$$R : V \times V \to \mathrm{Hom}(V, V) : (x, y) \mapsto R(x, y).$$

R is called *a curvature tensor over* V if it has the following properties for all $x, y, z, w \in V$:

(i) $R(x, y) = -R(y, x)$,

(ii) *the first Bianchi identity*, i.e. $\sigma R(x, y)z = 0$, where σ denotes the cyclic sum with respect to x, y and z.

We also use the notation $R(x, y, z, w) = g(R(x, y)z, w)$. We denote by $\mathcal{R}(V)$ the vector space of all curvature tensors over V. In addition to *the Ricci tensor*

$\rho(R)$ for a curvature tensor $R \in \mathcal{R}(V)$ it makes sense to define *the second trace* $\hat{\rho}(R)$ by

$$\hat{\rho}(R)(x,y) = \sum_{i=1}^{m} R(e_i, x, e_i, y), \quad x, y \in V,$$

where $\{e_i\}$ is an arbitrary orthonormal basis of V. The traces $\rho(R)$ and $\hat{\rho}(R)$ are orthogonal. Moreover, they are neither symmetric nor skew-symmetric in general case. The scalar curvature $\tau = \tau(R)$ of R is defined as the trace of $Q = Q(R)$, given by $\rho(R)(x,y) = \langle Qx, y\rangle$. Now one can define all the components of the decomposition of $\mathcal{R}(V)$. We put

$$\begin{aligned}
\mathcal{R}^a(V) &= \{R \in \mathcal{R}(V) \mid \rho(R) \text{ and } \hat{\rho}(R) \text{ are skew-symmetric}\},\\
\mathcal{R}^s(V) &= \{R \in \mathcal{R}(V) \mid \rho(R) \text{ and } \hat{\rho}(R) \text{ are symmetric}\},\\
\mathcal{R}_p(V) &= \{R \in \mathcal{R}(V) \mid \rho(r) \text{ is zero}\},\\
\mathcal{R}_0(V) &= \mathcal{R}^a(V) \cap \mathcal{R}^s(V) = \{R \in \mathcal{R}(V) \mid \rho(R) \text{ and } \hat{\rho}(R) \text{ are zero}\},\\
W_4 &= \text{orthogonal complement of } \mathcal{R}_0(V) \text{ in } \mathcal{R}_p(V) \cap \mathcal{R}^a(V),\\
W_5 &= \text{orthogonal complement of } \mathcal{R}_0(V) \text{ in } \mathcal{R}_p(V) \cap \mathcal{R}^s(V),\\
W_3 &= \text{orthogonal complement of } \mathcal{R}_p(V) \cap \mathcal{R}^a(v) \text{ in } \mathcal{R}^a(V),\\
W_1 \oplus W_2 &= \text{orthogonal complement of } \mathcal{R}_p(V) \cap \mathcal{R}^s(V) \text{ in } \mathcal{R}^s(V),\\
W_2 &= \{R \in W_1 \oplus W_2 \mid \tau(R) \text{ is zero}\},\\
W_1 &= \text{orthogonal complement of } W_2 \text{ in } W_1 \oplus W_2,\\
W_6 &= \{R \in \mathcal{R}_0(V) \mid R(x,y,z,w) = -R(x,y,w,z);\ x,y,z,w \in V\},\\
W_7 &= \{R \in \mathcal{R}_0(V) \mid R(x,y,z,w) = R(x,y,w,z);\ x,y,z,w \in V\},\\
W_8 &= \text{orthogonal complement of } W_6 \oplus W_7 \text{ in } \mathcal{R}_0(V).
\end{aligned}$$

So we can state the decomposition theorem for $\mathcal{R}(V)$.

Theorem 3.1. *We have*

$$\mathcal{R}(V) = W_1 \oplus \cdots \oplus W_8, \tag{3.1}$$

where W_i are orthogonal invariant subspaces under the action of $SO(V)$ $(m \geq 2)$. Moreover,

(i) The decomposition (3.1) is irreducible for $m \geq 4$.

(ii) For $m = 4$ we have $W_6 = W^+ \oplus W^-$ where $W^\pm = \{R \in W_6 \mid R* = \pm R\}$, and the other factors are irreducible.

(iii) For $m = 3$ we have $W_6 = W_8 = \{0\}$ and the other factors are irreducible.

(iv) For $m = 2$ we have $W_4 = W_5 = W_6 = W_7 = W_8 = \{0\}$ and the other factors are irreducible. □

We compare the decompositions in subsections III.2 and III.3 to see

$$\pi(2) = W_1 \oplus W_2, \quad \pi(1,1) = W_3, \quad \pi(2,1,-1) = \mathcal{R}_p(V).$$

If we suppose $x, y, z, \ldots \in \mathfrak{X}(M)$, the algebra of C^∞ vector fields on (M, g) and R and ρ the curvature and the Ricci tensor respectively then the projective curvature tensor $P(R)$ associated with R is the orthogonal projection of R on $\mathcal{R}_p(\mathfrak{X}(M))$. We recall that the Weyl conformal curvature tensor belongs to the $\mathcal{R}_3$-component from Singer-Thorpe decomposition, which is irreducible under the action of the group $O(V)$. The projective component $\mathcal{R}_p(V)$ is not irreducible under the action of $O(V)$ or $SO(V)$.

The complete decomposition of $\mathcal{R}(V)$ given by Theorem 3.1. is very useful in the study of the group of projective transformations on some manifold M^m and its subgroups. We recall that an equiaffine transformation is an affine volume preserving transformation of a manifold (M, ∇). On a manifold (M^m, ∇) there exists an equiaffine transformation if and only if the Ricci tensor $\rho(R)$, corresponding to any symmetric connection ∇, is symmetric. Let us mention some of these results.

Theorem 3.2. *Let (M, ∇, g) be a Riemannian manifold endowed with a symmetric connection ∇ such that $W_3 = 0$. Then the group of affine transformations coincides with its subgroup of equiaffine transformations.* □

Theorem 3.3. *If a Riemannian manifold (M^m, g) is compact and $W_1 = W_2 = W_3 = 0$ then the group $\mathcal{P}(M)$ of all projective transformations coincides with its subgroup $\mathcal{A}(M)$ of all affine transformations.* □

Some of components in (3.1) have other interesting geometric properties. So, Nikčević has proved in [**91**] that $\mathcal{R}_0(\mathfrak{X}(M))$ is conformally invariant.

Let us point out that for some torsion free connections the corresponding curvature tensor has some of its projections on W_i $(i = 1, \ldots, 8)$ equal to zero. Namely, if ∇ is the Levi-Civita connection then we have

$$\begin{aligned} &P(R) \in W_5 \oplus W_6, && \text{for } m > 4,\\ &P(R) \in W_5 \oplus W^+ \oplus W^-, && \text{for } m = 4,\\ &P(R) \in W_5, && \text{for } m = 3. \end{aligned}$$

We refer to [**28**] for more details.

The decomposition (3.1) is not unique. The second one is given in [**23**] which is closely related with the decomposition of curvature tensors for Weyl connections under the action of the conformal group $CO(m)$ [**53**].

III.4. The action of the group $U(m)$

The main purpose of this section is to study algebraic properties of a holomorphically projective curvature tensor on Hermitian manifold. Therefore we start with some considerations in a vector space endowed with some structures. We refer to [**81**] for some details.

Let V be a $2m$-dimensional real vector space endowed with the complex structure J, compatible with the positive definite inner product g, i.e.

$$J^2 = -I, \quad g(Jx, Jy) = g(x, y),$$

for all $x, y \in V$ and where I denotes the identity transformation of V. A tensor R of type (1,3) over V is bilinear mapping

$$R : V \times V \to \mathrm{Hom}(V, V) : (x, y) \mapsto R(x, y).$$

R is called *a curvature tensor over* V if it has the following properties for all $x, y, z, w \in V$:

(i) $R(x, y) = -R(y, x)$,
(ii) $\sigma R(x, y)z = 0$ (the first Bianchi identity)
(iii) $JR(x, y) = R(x, y)J$ (the Kähler identity).

We use also the notation $R(x, y, z, w) = g(R(x, y)z, w)$.

Let $\mathcal{R}(V)$ denotes the vector space of all curvature tensors over V. This space has a natural inner product defined with that on V:

$$\langle R, \tilde{R} \rangle = \sum_{i,j,k=1}^{2m} g(R(e_i, e_j)e_k, \tilde{R}(e_i.e_j)e_k),$$

where $R, \tilde{R} \in \mathcal{R}(V)$ and $\{e_i\}$ is an orthonormal basis of V. *A natural induced representation of* $U(m)$ *in* $\mathcal{R}(V)$ is the same as of $O(m)$ in the previous sections.

To describe a complete decomposition of $\mathcal{R}(V)$ under the action of $U(m)$ we need some basic notations. There are independent traces as follows:

$$\rho(R)(x, y) = \sum_{i=1}^{2m} R(e_i, x, y, e_i), \qquad \tau(R) = \sum_{i,j=1}^{2m} R(e_i, e_j, e_j, e_i),$$

$$\tilde{\rho}(R)(x, y) = \sum_{i=1}^{2m} R(e_i, x, e_i, y), \qquad \tau^*(R) = \sum_{i,j=1}^{2m} R(e_i, e_j, Je_j, e_i),$$

where $\{e_1, \ldots, e_m, Je_1, \ldots, Je_m\}$ is an arbitrary basis of V. The trace $\rho = \rho(R)$, as we have seen in the Section I, is called *the Ricci tensor*, and $\tau = \tau(R)$ is *the scalar curvature* of R.

In general, the traces ρ and $\hat{\rho}$ are neither symmetric nor skew-symmetric and always we have $\hat{\rho}(Jx, Jy) = \hat{\rho}(x, y)$; ρ and $\hat{\rho}$ belong to $\mathcal{V}^2 = V^* \otimes V^*$, where V^* is the dual space of V. Let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathcal{V}^2(V)$ given by:

$$\langle \alpha, \beta \rangle = \sum_{i,j=1}^{2n} \alpha(e_i, e_j)\beta(e_i, e_j), \quad \text{for} \alpha, \beta \in \mathcal{V}^2(V).$$

Now we introduce some tensors, and operators that we need to define components in the complete decomposition of $\mathcal{R}(V)$.

$$\pi_1(x, y)z := g(x, z)y - g(y, z)x + g(Jx, z)Jy - g(Jy, z)Jx + 2g(Jx, y)Jz,$$
$$\pi_2(x, y)z := g(Jx, z)y - g(Jy, z)x + 2g(Jx, y)z - g(x, z)Jy + g(y, z)Jx.$$

Let ϕ be the operator defined by

$$\phi(R)(x,y,z,w) := R(Jx,Jy,z,w),\ \ R \in \mathcal{R}(V),$$

and $\mathcal{R}^+$ and $\mathcal{R}^-$ be the vector subspaces of $\mathcal{R}(V)$ given by

$$\mathcal{R}^+ : \{R \in \mathcal{R}(V) \mid \phi(R) = R\}, \quad \mathcal{R}^- := \{R \in \mathcal{R}(V) \mid \phi(R) = -R\}.$$

The vector space $\mathcal{R}(V)$ consists some subspaces which one can define in terms of traces symmetry properties. So we have

$$\begin{aligned}
W_0 &: = \{R \in \mathcal{R}(V) \mid \tau(R) = \tau^*(R) = 0\},\\
R_H &: = \{R \in \mathcal{R}(V) \mid \rho(R) = 0\},\\
\mathcal{R}_0 &: = \{R \in \mathcal{R}(V) \mid \rho(R) = \hat{\rho}(R) = 0\},\\
\mathcal{R}^s_\rho &: = \{R \in \mathcal{R}(V) \mid \rho(R) \neq 0 \text{ and } \rho(R)(x,y) = \rho(R)(y,x)\},\\
\mathcal{R}^a_\rho &: = \{R \in \mathcal{R}(V) \mid \rho(R) \neq 0 \text{ and } \rho(R)(x,y) = -\rho(R)(y,x)\},\\
\mathcal{R}^s_{\hat{\rho}} &: = \{R \in \mathcal{R}(V) \mid \hat{\rho}(R)(x,y) = \hat{\rho}(R)(y,x)\},\\
\mathcal{R}^a_{\hat{\rho}} &: = \{R \in \mathcal{R}(V) \mid \hat{\rho}(R)(x,y) = -\hat{\rho}(y,x)\}.
\end{aligned}$$

Now we can define all components in the compete decomposition of $\mathcal{R}(V)$.

Definition 4.1. We put

$$\begin{aligned}
W_9 &: = \{R \in \mathcal{R}^+ \cap \mathcal{R}_0 \mid R(x,y,z,w) = -R(x,y,w,z)\},\\
W_{10} &: = \{R \in \mathcal{R}^+ \cap \mathcal{R}_0 \mid R(x,y,z,w) = R(x,y,w,z)\},\\
W_{11} &: = \text{orthogonal complement of } W_9 \oplus W_{10} \text{ in } \mathcal{R}^+ \cap \mathcal{R}_0,\\
W_1 &: = \mathcal{R}^+ \cap \mathcal{R}^s_\rho, \qquad W_5 := \mathcal{L}(\pi_1);\\
W_3 &: = \mathcal{R}^+ \cap \mathcal{R}^a_\rho, \qquad W_6 := \mathcal{L}(\pi_2) :\\
W_7 &: = \text{orthogonal complement of } \mathcal{R}_0 \text{ in } \mathcal{R}_H \cap \mathcal{R}^s_{\hat{\rho}},\\
W_8 &: = \text{orthogonal complement of } \mathcal{R}_0 \text{ in } \mathcal{R}_H \cap \mathcal{R}^a_{\hat{\rho}},\\
W_{12} &: = \mathcal{R}^- \cap \mathcal{R}_H,\\
W_2 &: = \mathcal{R}^- \cap \mathcal{R}^s_\rho = \text{ orthogonal complement of } W_1 \text{ in } \mathcal{R}^s_\rho,\\
W_4 &: = \mathcal{R}^- \cap \mathcal{R}^a_\rho = \text{orthogonal complement of } W_3 \text{ in } \mathcal{R}^a_\rho.
\end{aligned}$$

Thus we obtain

Theorem 4.2. *If* $\dim V = 2m$, $m \geq 3$, *then* $\mathcal{R}(V) = W_1 \oplus \cdots \oplus W_{12}$; *if* $m = 2, W_{11} = W_{12} = \{0\}$ *and* $\mathcal{R}(V) = W_1 \oplus \cdots \oplus W_{10}$. *These subspaces are mutually orthogonal and invariant under the action of* $U(m)$. □

Recalling that an invariant subspace is irreducible if it does not contain a nontrivial invariant subspace, we have also

Theorem 4.3. *The decomposition of $\mathcal{R}(V)$ given above is irreducible under the action of $U(m)$.* □

The projections of $R \in \mathcal{R}(V)$ on W_i $(i = 1, 2, \dots, 12)$ and the dimensions of W_i have been done also in [**81**].

Let M be a $2m$-dimensional C^∞ manifold with an almost complex structure J and a Hermitian inner product g. Then, for all $u, v \in \mathfrak{X}(M)$, the Lie algebra of C^∞ vector fields on M, we have $J^2u = -u$, $g(Ju, Jv) = g(u, v)$. It is known (see [**90**], [**139**]) that the existence on M of an arbitrary torsion free connection ∇ s.t. $\nabla J = 0$ is equivalent, to the vanishing of the Nijenhuis tensor defined by

$$N_J(u, v) = [Ju, Jv] - J[Ju, v] - J[u, Jv] - [u, v], \quad u; v \in \mathfrak{X}(M).$$

For every $p \in M$, the tangent space T_pM has a Hermitian structure given by $(J_{|p}, g_{|p})$. Now let $\mathcal{R}(V)$ be the vector bundle on M with fibre $\mathcal{R}(T_pM)$; the decomposition of $\mathcal{R}(T_pM)$ gives rise to a decomposition of $\mathcal{R}(M)$ into orthogonal subbundles with respect to the fibre metric introduced by g on $\mathcal{R}(M)$. We shall still denote the components of this decomposition by W_i, $i = 1, 2, \dots, 12$. If ∇ is an arbitrary linear torsion free connection, the corresponding curvature tensor is a section of the vector bundle $\mathcal{R}(M)$ and it is not difficult to check that its $HP(R)$-component in each point $p \in M$ gives the well-known holomorphical projective curvature tensor associated with ∇ (see [**27**], [**139**]); as a consequence, every subspace W_i, $i = 7, 8, \dots, 12$ of the decomposition is holomorphically projective invariant.

If some of the W_i vanish then the corresponding manifold has special groups of transformations and we have the following theorems (we refer [**92**]) for more details).

Theorem 4.4. *Let (M, g) be a Hermitian manifold with a torsion free connection. If the homogeneous holonomy group of M has no invariant hyperplane, or if the restricted homogeneous holonomy group has no invariant covariant vector and $W_1 = \cdots = W_6 = 0$ then $\mathcal{HP}(M) = \mathcal{A}(M)$.* □

We denote here by $\mathcal{HP}(M)$ the group of all holomorphically projective transformations of M and denote by $A(M)$ the group of all affine transformations of M.

Theorem 4.5. *If a Hermitian manifold (M, g) endowed with a torsion free connection ∇ is complete with respect to ∇ and $W_1 = \cdots = W_6 = 0$ then $\mathcal{HP}(M) = \mathcal{A}(M)$.* □

III.5. The action of the group $U(m) \times 1$

In Section II.3 we have introduced C-projective transformations on a normal almost contact manifold and have found C-projective curvature tensor - invariant with respect to these transformations. The key point was the existence of a torsion free adopted connection ∇. The main purpose of this section is to study the curvature tensor R of ∇, especially its C-projective curvature tensor $W(R)$ from algebraic point of view.

We use (3.2) of Section II to check

$$W(R)(u,v)\varphi = \varphi W(R)(u,v), \quad W(R)(u,v)\xi = 0. \tag{5.1}$$

Now starting from (5.1) we shall define certain special curvature tensor fields on M^{2m+1}, which will become useful for our discussion.

Definition 5.1. Let $(M^{2m+1}, \varphi, \xi, \eta)$ be a normal almost contact manifold. We define the *difference curvature tensor field of the torsion free adopted connection* ∇ as $K(R) = hR - h\tilde{R}$, where R is the curvature tensor field of ∇ and $h\tilde{R}$ is given by

(5.2)

$$h\tilde{R}(u,v)z = \{-\eta(u)hv + \eta(v)hu\}\eta(z) + d\eta(u,v)\varphi z + \frac{1}{2}d\eta(u,z)\varphi v - \frac{1}{2}d\eta(v,z)\varphi u. \square$$

Notice that $h\tilde{R}$ can be considered as the component on the vector subbundle $H = Ker\eta$ of TM^{2m+1} of the curvature tensor field $\tilde{R}$ of a torsion free adopted connection $\tilde{\nabla}$ on M^{2m+1}. Taking into account the properties of R and $\rho(R)$ we find

$$K(R)(u,v)z = -K(R)(v,u)z, \tag{5.3}$$

$$\underset{uvz}{\sigma} K(R)(u,v)z = 0 \quad \text{(the first Bianchi identity)}, \tag{5.4}$$

$$K(R)(u,v)\varphi z = \varphi K(R)(u,v)z, \quad K(R)(u,v)\xi = 0, \tag{5.5}$$

$$\begin{aligned} \rho(K(R))(u,v) = \mathrm{tr}(z \to K(R)(z,u)v) = \rho(R)(u,v) - \rho(\tilde{R})(u,v), \\ \rho(K(R))(u,\xi) = 0. \end{aligned} \tag{5.6}$$

III.5.1. The vector space $\mathcal{K}(V)$. Let V be an $(2m+1)$-dimensional real vector space endowed with an almost contact structure (φ, ξ, η) and a compatible inner product g and let V^* be the dual of V. Then, the (1,1) tensor φ, the vector $\xi \in V$ and the one-form $\eta \in V^*$ satisfy the relations:

$$\begin{gathered} \varphi^2 = -I_V + \xi \otimes \eta, \quad \eta(\xi) = 1, \\ \varphi\xi = 0, \quad \eta \circ \varphi = 0 \\ g(\varphi x, \varphi y) = g(x,y) - \eta(x)\eta(y), \quad x, y \in V. \end{gathered}$$

A tensor R of type (1,3) *over* V is a bilinear mapping $R : V \times V \to \mathrm{Hom}(V,V)$, $(x,y) \mapsto R(x,y)$. We say that R is *a curvature tensor over V* if

$$R(x,y) = -R(y,x), \text{ and } \underset{x,y,z}{\sigma} R(x,y)z = 0$$

We denote by $\mathcal{R}(V)$ the vector space of all curvature tensors over V. One can consider the following inner product, induced by g:

$$\langle R, \bar{R} \rangle = \sum_{1}^{2m+1} g(R(e_i,e_j)e_k, \bar{R}(e_i,e_j)e_k), \qquad R, \bar{R} \in \mathcal{R}(V),$$

where $\{e_i\}$, $i = 1, \dots, 2m+1$ is an arbitrary orthonormal basis of V. Furthermore, the representation α of $U(m) \times 1$ in V induces a representation $\tilde{\alpha}$ of $U(m) \times 1$ in $\mathcal{R}(V)$ in the following way

$$\tilde{\alpha} : U(m) \times 1 \to \mathfrak{gl}(\mathcal{R}(V)), \quad r \mapsto \tilde{\alpha}(r),\ r \in U(m) \times 1,$$

where $\tilde{\alpha}(r)(R)(x, y, z, w) = R(\alpha(r^{-1})x,\ \alpha(r^{-1})y,\ \alpha(r^{-1})z,\ \alpha(r^{-1})w)$, for all x, y, z, $w \in V$. It follows that the mapping $R \mapsto \tilde{\alpha}(r)R$ is an isometry for $\mathcal{R}(V)$; therefore $\langle \tilde{\alpha}(r)R,\ \tilde{\alpha}(r)\bar{R}\rangle = \langle R, \bar{R}\rangle$, which implies that the orthogonal complement of an invariant subspace of $\mathcal{R}(V)$ is also invariant and the representation $\tilde{\alpha}$ is completely reducible.

Taking into account the properties (5.3)–(5.6) of a "difference curvature tensor field" we shall denote by K the curvature tensors over V such that

$$K(x, y)\varphi z = \varphi K(x, y)z \text{ and } K(x, y)\xi = 0, \tag{5.7}$$

for all $x, y, z \in V$, or equivalently, if $K(x, y, z, w) = g(K(x, y)z, w)$, we have

$$\begin{aligned} &K(x, y, z, w) = K(x, y, \varphi z, \varphi w), \\ &K(x, y, \xi, w) = 0, \quad K(x, y, z, \xi) = \eta(K(x, y)z) = 0. \end{aligned} \tag{5.8}$$

Let $\mathcal{K}(V)$ be the vector subspace of $\mathcal{R}(V)$, whose elements are all K, satisfying (5.7). This subspace of $\mathcal{R}(V)$ is invariant for $\tilde{\alpha}$.

$\mathcal{K}(V)$ may be splited into direct sum of two subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$, defined as follows

$$\begin{aligned} &\mathcal{K}_1 = \{K \in \mathcal{K}(V) \mid K(x, \xi, z, w) = 0\}, \\ &\mathcal{K}_2 = \{K \in \mathcal{K}(V) \mid K(x, y, z, w) = \eta(x)K(\xi, y, z, w) + \eta(y)K(x, \xi, z, w)\}. \end{aligned}$$

It means

$$\mathcal{K}(V) = \mathcal{K}_1 \oplus \mathcal{K}_2, \tag{5.9}$$

and moreover $\mathcal{K}_1$ and $\mathcal{K}_2$ are mutually orthogonal and invariant with respect to the action of $U(m) \times 1$.

Now let $H = \operatorname{Ker} \eta$; H is a $2m$-dimensional Hermitian vector space with $(\varphi|_H, g|_H)$ as Hermitian structure and $U(m) \times 1|_H \simeq U(m)$; further, the vector space $\mathcal{K}_1$ is naturally isomorphic to the vector space $\mathcal{K}(H)$ given by the curvature tensors over H which satisfy the Kähler identity. This isomorphism allows us to use the results of Section III.4. concerning the decomposition of $\mathcal{K}(H)$ with respect to the action of $U(m)$ to obtain the decomposition of $\mathcal{K}_1$.

To simplify our notation, in the following, we shall denote for every $x \in V$ the component on H by $\dot{x}$; that is, $\dot{x} = hx$, where $h = I_V - \eta \otimes \xi$ is the projection on

H. First, we notice that for any $K \in \mathcal{K}(V)$ there are only two possible independent traces associated with K, analogously to these ones in III.4. i.e.

$$\rho(K)(y,z) = \sum_{i=1}^{2m+1} K(e_i, y, z, e_i),$$
$$\hat{\rho}(K)(y,z) = \sum_{i} K(e_i, y, e_i, z), \quad y, z \in V,$$

where $\{e_i\}$, $i = 1, 2, \ldots, 2m+1$ is an arbitrary orthonormal basis of V. Further, we have two scalar curvatures

$$\tau(K) = \sum_{i,j} K(e_i, e_j, e_j, e_i),$$
$$\bar{\tau}(K) = \sum_{i,j} K(e_i, e_j, \varphi e_j, e_i).$$

One can check easily

$$\rho(K)(y,z) = \rho(K)(\dot{y}, \dot{z}) + \eta(y)\rho(K)(\xi, \dot{z}),$$

$$\hat{\rho}(K)(\varphi y, \varphi z) = \hat{\rho}(K)(y,z).$$

In general, $\rho(K)$ and $\hat{\rho}(K)$ are neither symmetric nor antisymmetric; moreover $\rho(K)(\xi, z) = 0$ for every $K \in \mathcal{K}_1$, while for $K \in \mathcal{K}_2$, $\rho(K)$ reduces to $\eta(y)\rho(K)(\xi, \dot{z})$ and $\hat{\rho}(K) = 0$.

We omit all details related to the decomposition of $\mathcal{K}_1$, because of the previous comments, and pay the attention only on the decomposition of $\mathcal{K}_2 \subset \mathcal{K}$. First of all, we note that $K(\xi, y, z, w) = K(\xi, z, y, w)$, for every $K \in \mathcal{K}_2$ Next, we introduce the endomorphism δ on $\mathcal{K}_2$ defined by

$$\begin{aligned}\delta(K)(x,y,z,w) = -\frac{1}{2m+2}\{&\eta(x)[g(\varphi y, \varphi w)\rho(K)(\xi, \dot{z}) + g(\varphi z, \varphi w)\rho(K)(\xi, \dot{y})\\ &- g(\varphi y, w)\rho(K)(\xi, \varphi z) - g(\varphi z, w)\rho(K)(\xi, \varphi y)]\\ &- \eta(y)[g(\varphi x, \varphi w)\rho(K)(\xi, \dot{z}) + g(\varphi z, \varphi w)\rho(K)(\xi, \dot{x})\\ &- g(\varphi x, w)\rho(K)(\xi, \varphi z) - g(\varphi z, w)\rho(K)(\xi, \varphi x)]\},\end{aligned}$$

for any $x, y, z, w \in V$. If we take into account $\rho(\delta(K)) = \rho(K)$ we can check easily $\delta(K) \in \mathcal{K}_2$, $\delta^2 = \delta$ and δ commutes with the action of $U(m) \times 1$.

Now we define the following subspaces of $\mathcal{K}_2$

$$W_{13} = \operatorname{Ker}\delta = \{K \in \mathcal{K}_2 \mid \rho(K) = 0\}, \quad W_{14} = \operatorname{Im}\delta,$$

and state the following theorem.

Theorem 5.2. *If* $\dim V = 2m+1$, $m \geq 2$, *then* $\mathcal{K}_2 = W_{13} \oplus W_{14}$. *The subspaces* W_{13} *and* W_{14} *are mutually orthogonal and invariant under the action of* $U(m) \times 1$. *In particular, for* $m = 1$, $W_{13} = \{0\}$ *and* $\mathcal{K}_2 = W_{14}$. □

We use now (5.8), the isomorphism of $\mathcal{K}_1$ and $\mathcal{R}(H)$, Theorem 4.2. and Theorem 5.2. to obtain the following decomposition theorem for $\mathcal{K}(V)$:

Theorem 5.3. *If* $\dim V = 2m+1$, $m \geq 3$, *then*

$$\mathcal{K}(V) = W_1 \oplus \cdots \oplus W_{14} \tag{5.10}$$

and the subspaces W_i *are* $U(m) \times 1$ *- invariant and mutually orthogonal. For* $m = 2$, $W_{11} = W_{12} = \{0\}$ *and when* $m = 1$, *the decomposition reduces to* $\mathcal{K}(V) = W_5 \oplus W_6 \oplus W_{14}$. □

III.5.2. Some geometric results. Let $(M^{2m+1}, \varphi, \xi, \eta, g)$ be a normal almost contact metric manifold. For every $p \in M^{2m+1}$, the vector space T_pM^{2m+1} has an induced almost contact structure $(\varphi_p, \eta_p, \xi_p)$ with compatible inner product g_p. If we denote by $\mathcal{K}(M^{2m+1})$ the vector bundle on M^{2m+1} with fibre $\mathcal{K}(T_pM^{2m+1})$, the decomposition (5.10) gives rise to a decomposition of $\mathcal{K}(M^{2m+1})$ into orthogonal subbundles with respect to the fibre metric induced by g on $\mathcal{K}(M^{2m+1})$. We use the same notation W_i, $i = 1, \ldots, 14$ for the components of this decomposition.

Let ∇ be a torsion free adapted connection on M^{2m+1} with curvature tensor R. Then, the difference tensor field $K(R)$ is a section of $\mathcal{K}(M^{2m+1})$. Let Q_i be the projections of K on the subspaces W_i $(i = 1, \ldots, 14)$. Recalling that $K(R) = hR - h\tilde{R}$, where $h\tilde{R}$ is given by (5.2) with $W(\tilde{R}) = 0$, we can state

Proposition 5.4. *Let* $(M^{2m+1}, \varphi, \xi, \eta, g)$ *be a normal almost contact metric manifold. If* ∇ *is an adapted torsion free connection on* M^{2m+1} *with curvature tensor* R, *we have*

$$W(R) = \sum_{i=7}^{13} Q_i(K(R)), \quad K(R) = hR - h\tilde{R}$$

and the spaces W_i, $i = 7, 8, \ldots, 13$ *of the decomposition (5.9) are* C*-projectively invariant.* □

If $(M^{2m+1}, \varphi, \xi, \eta, g)$ is a Sasakian manifold, then $d\eta(u,v) = 2g(\varphi u, v)$, where $u, v, z, \cdots \in \mathfrak{X}(M^{2m+1})$.

As we know, the Levi-Civita connection ∇ on M^{2m+1} is one of adapted connections, and the system (3.1) in Section II is reduced to the simpler one

$$(\nabla_u \varphi)v = \eta(v)u - g(u,v)\xi, \quad (\nabla_u x\eta)(v) = g(\varphi u, v),$$
$$\nabla_u \xi = \varphi u, \quad \nabla_u g = 0$$
$$(\nabla_u d\eta)(v,z) = 2\eta(v)g(u,z) - 2\eta(z)g(u,v) = 2\eta(R(v,z)u).$$

Among Sasakian manifolds one can characterizes these ones of constant φ-sectional curvature using the previous results. More precisely, we have

Proposition 5.5. *Let* $(M^{2m+1}, \varphi, \xi, \eta, g)$, $m \geq 3$ *be a Sasakian manifold. Then* $K(R) = Q_5(K(R)) \in W_5$ *if and only if it has constant* φ*-sectional curvature* $c \neq -3 \quad (M^{2m+1} \neq R^{2m+1}(-3))$. □

Corollary 5.6. *Let* $(M^{2m+1}, \varphi, \xi, \eta, g)$, $m \geq 3$ *be a Sasakian manifold* $\neq R^{2m+1}(-3)$. *Then* $K(R) = Q_5(K(R)) \in W_5$, *if and only if it is* C*-projectively flat.* □

We refer to [**78**] for characterization of other classes of Sasakian manifolds using the decomposition of curvature tensors and the corresponding examples (see also [**8**], [**9**], [**66**] etc.).

A normal almost contact metric manifold $(M^{2m+1}, \varphi, \xi, \eta, g)$ has *a cosymplectic structure* if the fundamental 2-form Ω defined by $\Omega(u, v) = 2g(\varphi u, v)$ and the 1-form η are closed on M^{2m+1}. Examples of cosymplectic manifolds are provided by the products $\bar{M} \times S^1$, where $\bar{M}$ is any Kähler manifold. For a cosymplectic manifold Matzeu [**78**] has proved

(i) $K(R) = R = Q_5(K(R)) \in W_5$ if and only if it has constant φ-sectional curvature $c = \frac{\tau(K)}{m(m+1)}$,

(ii) $K(R) = R = Q_5(K(R)) + Q_9(K(R))$ if and only if it is η-Einstein, i.e. $\rho(K) = a(g - \eta \otimes \eta)$, where $a = \frac{\tau(K)}{2m}$ is constant.

We refer also to [**78**] for the studying of real hypersurfaces on complex space forms in this spirit.

IV. THE CHARACTERISTIC CLASSES

IV.1. Some basis notions and definitions

Let $GL(m, \mathbb{R})$ be the full general linear group and $\mathrm{gl}(m, \mathbb{R})$ be the Lie algebra of $GL(m, \mathbb{R})$; this is the Lie algebra of real $m \times m$ matrices. A map $Q : \mathrm{gl}(m, \mathbb{R}) \to \mathbb{C}$ is invariant if $Q(gAg^{-1}) = Q(A)$ for all $A \in \mathrm{gl}(m, \mathbb{R})$ and for all $g \in GL(m, \mathbb{R})$. Let $\mathcal{Q}$ be the ring of invariant polynomials. One can decompose $\mathcal{Q} = \oplus \mathcal{Q}_\nu$ as a graded ring, where $\mathcal{Q}_\nu$ is the subspace of invariant polynomials which are homogeneous of degree ν. Let

$$\mathrm{Ch}(A) := \sum_\nu \mathrm{Ch}_\nu \quad \text{for } \mathrm{Ch}_\nu(A) := \mathrm{Tr}\Big\{\Big(\frac{\sqrt{-1}}{2\pi} A\Big)^\nu\Big\},$$

$$C(A) := \det\Big(I + \frac{\sqrt{-1}}{2\pi} A\Big) = 1 + C_1(A) + \cdots + C_m(A)$$

define *the Chern character* and *total Chern polynomial*; $\mathrm{Ch}_\nu \in \mathcal{Q}_\nu$ and $C_\nu \in \mathcal{Q}_\nu$. The Chern characters and the Chern polynomials generate the *characteristic ring*: $\mathcal{Q} = \mathbb{C}[C_1, \dots, C_m]$ and $\mathcal{Q} = \mathbb{C}[\mathrm{Ch}_1, \dots, \mathrm{Ch}_m]$. If $Q \in \mathcal{Q}_\nu$ we polarize Q to define a multilinear form $Q(A_1, \dots, A_\nu)$ so that $Q(A) = Q(A, \dots, A)$ and $Q(A_1, \dots, A_\nu) = Q(gA_1g^{-1}, \dots, gA_\nu g^{-1})$.

We shall restrict our attention to the tangent bundle TM henceforth; let ∇ be an arbitrary connection on TM and R the corresponding curvature tensor. If $\{e_i\}$ is a local frame for TM, then $R(e_i, e_j)e_k = R_{ijk}{}^l e_l$. We shall let

$$\mathcal{R} = \mathcal{R}_k^l := \frac{1}{2} R_{ijk}{}^l e^i \wedge e^j$$

be the associated 2-form valued endomorphism. As we are not assuming that a metric is given, we do not restrict to orthonormal frames. Thus the structure group is the full general linear group $GL(m, \mathbb{R})$ and not the orthogonal group $O(m)$.

If $Q \in \mathcal{Q}_\nu$, we define

$$Q(\nabla) := Q(\mathcal{R}, \dots, \mathcal{R}) \in C^\infty(\Lambda^{2\nu} M)$$

by substitution; the value is independent of the frame chosen and associates a closed differential form of degree 2ν to any connection ∇ on TM. The corresponding cohomology class $[Q(\nabla)] \in H^{2\nu}(M; \mathbb{C})$ is independent of the connection ∇ chosen; as we shall see later. These are the characteristic forms and classes. For more details one can use also [**35**], [**40**], [**61**].

From now on we deal with complex manifolds.

We express now C_1, C_1^2 and C_2 by using a suitable chosen frame of TM. Let E_1, $JE_1, \dots, E_m$, JE_m, be a real base for tangent space T_pM and $w^1, \bar{w}^1, \dots, w^m$, $\bar{w}^m$ the corresponding dual base for T_p^*M. Then we will write $E_{m+s} = JE_s = E_{\bar{s}}$ and similarly $w^{m+r} = \bar{w}^r$, $1 \le s$, $r \le m$. We suppose summation for every pair of repeated indexes. We use also the following ranges for indexes $i, j, s, r = 1, 2, \dots, m$, and $I, J, S, R = 1, 2, \dots, 2m$. We denote $JE_S = E_{\bar{S}}$ and $R(u,v)E_S = R_{uvS}{}^R E_R$. For $u = E_I$; $v = E_J$ we simplify notation and write $R_{E_I E_J S}{}^R = R_{IJS}{}^R$.

It will be useful for our consideration of Chern classes to introduce the following traces:

$$\mu(u,v) = \frac{1}{2} \operatorname{tr}\{w \mapsto R(u,v)w\} = R_{uvi}{}^i, \tag{1.1}$$

$$\bar{\mu}(u,v) = \frac{1}{2} \{w \mapsto J \circ R(u, Jv)w\} = R_{uJv\bar{i}}{}^i, \tag{1.2}$$

for $u, v \in T_pM \otimes \mathbb{C}$ and $w \in T_pM$. After some computations one can express these traces in terms of the Ricci tensor as follows

$$2\bar{\mu}(u,v) = \rho(u,v) + \rho(Jv, Ju),$$
$$2\mu(u,v) = \rho(v,u) - \rho(u,v).$$

We put $\mathcal{R}_I^J(u,v) = R_{uvI}{}^J$, i.e., $\mathcal{R}_I{}^J = R_{RSI}{}^J \omega^R \wedge \omega^S$ and

$$\Theta_i^j(u,v) = -(\mathcal{R}_i^j(u,v) - \sqrt{-1}\mathcal{R}_{\bar{i}}^j(u,v)),$$

for $u, v \in T_pM \otimes \mathbb{C}$. Then (Θ_i^j) is a matrix of complex 2-forms and

$$\det\Big(\delta_i^j - \frac{1}{2\pi\sqrt{-1}}\Theta_i^j\Big) = 1 + C_1 + \dots + C_m$$

is a globally defined closed form which represents *the total Chern class* of M via de Rham's theorem (see [**61**, vol. II, p. 307]). Chern classes determined by C_1, C_2 are denoted by c_1, c_2 respectively. In particular, the Chern forms C_1, C_2, and C_1^2 are given by

$$
\begin{aligned}
C_1 &= \frac{\sqrt{-1}}{2\pi}\sum \Theta_i^i = \frac{\sqrt{-1}}{2\pi}(\mathcal{R}_i^i - \sqrt{-1}\mathcal{R}_{\bar i}^i),\\
C_1^2 &= -\frac{1}{4\pi^2}\sum_{1\le i<j\le m} \Theta_i^i \wedge \Theta_j^j\\
&= -\frac{1}{4\pi^2}\sum_{1\le i<j\le m}\{(\mathcal{R}_i^i\wedge\mathcal{R}_j^j - \mathcal{R}_{\bar i}^i\wedge\mathcal{R}_{\bar j}^j) - \sqrt{-1}(\mathcal{R}_i^i\wedge\mathcal{R}_{\bar j}^j + \mathcal{R}_i{}^{\bar i}\wedge\mathcal{R}_j^j)\},\\
C_2 &= -\frac{1}{4\pi^2}\sum_{1\le i<j\le m}\{\Theta_i^i\wedge\Theta_j^j - \Theta_i^j\wedge\Theta_j^i\}\\
&= -\frac{1}{4\pi^2}\sum_{1\le i<j\le m}\{(\mathcal{R}_i^i\wedge\mathcal{R}_j^j - \mathcal{R}_{\bar i}^i\wedge\mathcal{R}_{\bar j}^j - \mathcal{R}_i^j\wedge\mathcal{R}_j^i + \mathcal{R}_{\bar i}^j\wedge\mathcal{R}_{\bar j}^i)\\
&\qquad\qquad - \sqrt{-1}(\mathcal{R}_i^i\wedge\mathcal{R}_{\bar j}^j + \mathcal{R}_{\bar i}^i\wedge\mathcal{R}_j^j - \mathcal{R}_i^j\wedge\mathcal{R}_{\bar j}^i - \mathcal{R}_{\bar i}^i\wedge\mathcal{R}_j^i)\}.
\end{aligned}
$$

We consider Chern numbers $\gamma_2(M) = \int_M C_2$ and $\gamma_1^2(M) = \int_M C_1^2$ for a compact complex surface M and similarly $\gamma_1^m = \int_M C_1^m$ for an arbitrary complex compact m-dimensional manifold.

Let $A \in \sigma(m)$ be a skew-symmetric matrix. Then $C_{2\nu+1}(A) = 0$ and we define $P_\nu(A) = (-1)^\nu C_{2\nu}(A)$; $P = \sum_\nu P_\nu(A)$ is *the total Pontrjagin polynomial*. The $\{P_\nu\}$ for $2\nu \le m$ generate *the characteristic ring of the orthogonal group* $O(m)$. We can always choose a Riemannian metric g for M and use the associated Levi-Civita connection $\nabla(g)$ to compute the characteristic classes of the tangent bundle. This reduces the structure group to $O(m)$ and shows that only the Pontrjagin classes are relevant in the study of the primary characteristic classes of TM. From the point of view of cohomology, the connection plays an unessential role; however, in many geometrical applications one must work with differential forms not cohomology classes. We illustrate it by the following facts. Let dx be the volume element of compact 4-dimensional orientable M, where M is without boundary. The Chern-Gauss-Bonnet formula [**36**] and the Atiyah-Patodi-Singer formula [**1**] yields formulas for *the Euler-Poincaré characteristic* $\chi(M)$ and *the signature* $\mathrm{Sign}(M)$:

$$
\chi(M) = \int_M E_4(\nabla(g))dx, \quad \mathrm{Sign}(M) = \frac{1}{3}\int_M P_1(\nabla(g)),
$$

where

$$
\begin{aligned}
E_4(\nabla(g)) &= \frac{1}{32\pi^2}(R_{ijji}R_{kllk} - 4R_{ijjk}R_{illk} + R_{ijkl}R_{ijkl}),\\
P_1(\nabla(g)) &= -\frac{1}{32\pi^2}R_{ijk_1k_2}R_{jik_3k_4}e^{k_1}\wedge e^{k_2}\wedge e^{k_3}\wedge e^{k_4}.
\end{aligned}
$$

The interior integrands E_4 and P_1 are primary characteristic forms, not characteristic classes. But to express $\chi(M)$ and $\mathrm{Sign}(M)$ of compact 4-dimensional orientable manifold M with smooth boundary $\partial M \neq \phi$ we need also secondary characteristic forms.

We introduce firstly relative secondary characteristic forms and later absolute ones.

The space of all connections is an affine space; the space of torsion free connections is an affine subspace. If ∇_i are connections on TM, let $\nabla_t := t\nabla_1 + (1-t)\nabla_0$. Let $\Psi = \nabla_1 - \nabla_0$; Ψ is an invariantly defined 1-form valued endomorphism. Let $R(t)$ be the associated curvature. Let $Q \in \mathcal{Q}_\nu$. Let

$$\begin{aligned} TQ(\nabla_1, \nabla_0) &: = \nu \int_0^1 Q(\Psi, R(t), \ldots, R(t))dt; \\ dTQ(\nabla_1, \nabla_0) &= Q(\nabla_1) - Q(\nabla_0). \end{aligned} \tag{1.3}$$

This shows that $[Q(\nabla_1)] = [Q(\nabla_0)]$ in de Rham cohomology. Note that we have:

$$TQ(\nabla_0, \nabla_1) + TQ(\nabla_1, \nabla_2) = TQ(\nabla_0, \nabla_2) + \text{exact form}.$$

Suppose now that M is a 4-dimensional Riemannian manifold with smooth non-empty boundary ∂M. Let g be a Riemannian metric on M. Let indices i, j, k and l range from 1 to 4 and index a local orthonormal frame $\{e_i\}$ for the tangent bundle. At a point of the boundary of M, we assume e_4 is the inward unit normal and let indices a, b, c range from 1 to 3. Let $L_{ab} := (\nabla(g)_{e_2} e_b, e_4)$ be the second fundamental form on ∂M. We choose $x = (y, t)$ to be local coordinates for M near ∂M so the curves $t \mapsto (y, t)$ are unit speed geodesics perpendicular to ∂M. This identifies a neighborhood of ∂M in M with a collared neighborhood $\mathcal{K} = \partial M \times (0, \epsilon)$ for some $\epsilon > 0$. Let h_0 be the associated product metric. We denote by ∇_1, ∇_0 the Levi-Civita connections of h, h_0 respectively. The $TP_1(\nabla_1, \nabla_0)$ is given by

$$TP_1(\nabla_1, \nabla_0) = TP_1(L, \nabla_t) := -\frac{1}{16\pi^2} L_{ab} R_{4acd} e^b \wedge e^c \wedge e^d,$$

and consequently

$$\text{Sign}(M) = \frac{1}{3}\int_M P_1(\nabla(h)) - \frac{1}{3}\int_{\partial M} TP_1(L, \nabla(h)) - \eta(\partial M),$$

where the invariant $\eta(\partial M)$ is intrinsic to ∂M and we will not be concerned with this invariant here; see [**40**] for details.

To define absolute secondary characteristic forms we need the principal frame bundle $\pi : \mathcal{P} \to M$ for TM. A local section e to $\mathcal{P}$ is a frame $e = \{e_i\}$ for TM. Let g be the natural inclusion of $GL(m, \mathbb{R})$ in the Lie algebra $\mathfrak{gl}(m, \mathbb{R})$ of $m \times m$ real matrices. The Maurer-Cartan form dgg^{-1} on $GL(m, \mathbb{R})$ is a $\mathfrak{gl}(m, \mathbb{R})$ valued 1-form on $GL(m, \mathbb{R})$ which is invariant under right multiplication. Let ∇ be a connection on TM. Fix a local frame field e for TM; this is often called a choice of gauge. We denote by w the associated connection 1-form, $\nabla e_i = w_i^j e_j$. Let

$$\begin{aligned} \Theta &: = \Theta(\nabla) := dgg^{-1} + g\omega g^{-1}, \\ \Omega &: = \Omega(\nabla) := g(d\omega - \omega \wedge \omega)g^{-1} = g(\pi^* R)g^{-1}. \end{aligned}$$

These are Lie algebra valued forms on the principal bundle $\mathcal{P}$ which do not depend on the local frame field chosen. If $Q \in \mathcal{Q}_\nu$, then we have $Q(\Omega) = \pi^* Q(\nabla)$. We set $\Omega(t) = td\Theta - t^2\Theta \wedge \Theta = t\Omega + (t - t^2)\Theta \wedge \Theta$ and define

$$\mathcal{T}Q(\nabla) := \nu \int_0^1 Q(\Theta, \Omega(t), \dots, \Omega(t))dt. \tag{1.4}$$

We refer to Chern and Simons [**37**, Propositions 3.2, 3.7 and 3.8] for the proof of:

Theorem 1.1. *Let $Q \in \mathcal{Q}_\ni$ and $\tilde{Q} \in \mathcal{Q}_\mu$. (1) We have $d\mathcal{T}Q(\nabla) = \pi^* Q(\nabla)$.*
(2) We have $\mathcal{T}(Q\tilde{Q})(\nabla) = \mathcal{T}Q(\nabla) \wedge \pi^*\tilde{Q}(\nabla) +$ exact $= \pi^* Q(\nabla) \wedge \mathcal{T}\tilde{Q}(\nabla) +$ exact.
(3) Let ∇_ρ be a smooth 1 parameter family of connections. Let $A := \partial_\rho \nabla_\rho|_{\rho=0}$. Then $\partial_\rho \mathcal{T}Q(\nabla_\rho)|_{\rho=0} = \nu Q(A, \Omega_0, \dots, \Omega_0) +$ exact. □

Suppose M is parallelizable. Let e be a global frame for the principal frame bundle $\mathcal{P}$. Let ${}^e\nabla e = 0$ define the connection ${}^e\nabla$. We use equations (1.3) and (1.4) to see that

$$e^*\mathcal{T}Q(\nabla) = \int_0^1 Q(w_e, \mathcal{R}_t, \dots, \mathcal{R}_t) = TQ[\nabla, {}^e\nabla),$$

where $w_e = \nabla e$ and $\mathcal{R}_t = tdw_e - t^2 w_e \wedge w_e = t\mathcal{R} + (t - t^2)w_e \wedge w_e$.

We note that $\mathcal{R}_t$ is the curvature of the connection $t^e\nabla + (1-t)\nabla$. Fix $g \in GL(m, \mathbb{R})$. Since Q is GL invariant, we have $e^*\mathcal{T}Q(\nabla) = (ge)^*\mathcal{T}Q(\nabla)$.

Let $Q \in \mathcal{Q}_\nu$. Suppose that $Q(\nabla) = 0$. Then $e^*\mathcal{T}Q(\nabla)$ is a closed form on M of degree $2\nu - 1$ and $[e^*\mathcal{T}Q(\nabla)]$ in $H^{2\nu-1}(M; C)$ is independent of the homotopy class of e. We say that Q is integral if Q is the image of an integral class in the classifying space; see [**37**, §3] for details; the Pontrjagin polynomials are integral.

Theorem 1.2. *Let $Q \in \mathcal{Q}_\nu$. Assume that M is parallelizable and $Q(\nabla) = 0$.*
(1) If Q is integral, then $[e^*\mathcal{T}Q(\nabla)]$ is independent of e in $H^{2\nu-1}(M; \mathbb{C}/\mathbb{Z})$.
(2) If ν is odd, then $[e^*\mathcal{T}Q(\nabla)]$ is independent of e in $H^{2\nu-1}(M; \mathbb{C})$. □

IV.2. Characteristic classes and symmetries of a curvature tensor

The main purpose of this section is to study topology of a manifold endowed with a torsion free connection which curvature tensor has symmetries, invariant under the action of some classical groups in the spirit of Section II.

The relations between topology and the existence of some flat connection have been studied by Milnor [**87**], Auslander [**2**], Benzécri [**5**] etc.

The topological obstruction of the existence of a complex torsion free connection with skew-symmetric Ricci tensor has been studied in [**12**]. More precisely, we proved if ∇ is a complex torsion free connection on a Riemann surface M and the Ricci tensor ρ for ∇ is skew-symmetric then $\gamma_1(M) = \int_M C_1 = 0$.

Let us remark if M is a sphere S^2 we have $\gamma_1(M) \neq 0$. Therefore there is no a complex torsion free connection $\tilde{\nabla}$ with the skew-symmetric Ricci tensor globally defined on S^2. The local existence of this connection is proved by its construction. Namely, (S^2, g) is the standard sphere with the standard embedding into the Euclidean space $\mathbb{R}^3$ determined by

$$x = \cos\alpha \sin\beta, \quad y = \sin\alpha \sin\beta, \quad z = \cos\beta, \quad 0 < \alpha < 2\pi,\ 0 < \beta < \pi.$$

The Christoffel symbols for our complex connection are given by the following formulas

$$\begin{aligned} &\tilde{\Gamma}^2_{22} = \frac{1-\cos\beta}{\sin\beta}, \quad \tilde{\Gamma}^1_{22} = \frac{-1+\cos\beta}{\sin^2\beta}, \\ &\tilde{\Gamma}^1_{12} = \tilde{\Gamma}^1_{21} = \frac{1}{\sin\beta}, \quad \tilde{\Gamma}^2_{11} = -\sin\beta, \\ &\tilde{\Gamma}^2_{12} = \tilde{\Gamma}^2_{21} = \tilde{\Gamma}^1_{11} = 1 - \cos\beta; \end{aligned}$$

(see [**12**] for more details).

Having in mind the previously mentioned facts, torus T^2 is a good candidate to permite a globally defined torsion free connection $\tilde{\nabla}$ with the skew-symmetric Ricci tensor. Really, let $x_1 = \cos\alpha$, $x_2 = \sin\alpha$, $x_3 = \cos\beta$, $x_4 = \sin\beta$, $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$, be the standard embedding of the torus into the Euclidean space $\mathbb{R}^4$. Let $\tilde{\Gamma}^k_{ij}\,(i,j,k = 1,2)$ be the Christoffel symbols for a complex torsion free connection $\tilde{\nabla}$. Then

$$\begin{aligned} &\tilde{\Gamma}^1_{11} = \tilde{\Gamma}^2_{12} = \tilde{\Gamma}^2_{21} = -\tilde{\Gamma}^1_{22} = -\cos\alpha\sin\beta, \\ &\tilde{\Gamma}^1_{12} = \tilde{\Gamma}^1_{21} = \tilde{\Gamma}^2_{22} = -\tilde{\Gamma}^2_{11} = \sin\alpha\cos\beta. \end{aligned}$$

We point out these connections belong to the class of affine conformal invariants, studied by Simon in [**123**].

The Chern characteristic classes of complex surfaces endowed with a holomorphic affine connection ∇ have been studied in [**14**]. The following theorem considers complex surfaces with all vanishing characteristic classes.

Theorem 2.1. *Let M be a complex surface* ($\dim M = 2$) *endowed with a holomorphic affine connection ∇. Then its Chern characteristic classes C_2 and C_1^2 vanish. Moreover, if M is a complex equiaffine surface (∇ permits a parallel complex 2-form ${}^c w$) then C_1 also vanishes.* □

One can find in [**14**] the examples of nonflat holomorphic affine connections on the torus T^4 and the Euclidean space $\mathbb{R}^4$.

Let us assume for our complex torsion free connection ∇ to have a symmetric curvature operator, i.e. $R(x,y)$ satisfies the relation $g(R(x,y)z,v) = g(R(x,y)v,z)$. Then R satisfies also the following relations

$$R(Jx,Jy) = R(x,y), \quad \rho(x,y) = -\rho(y,x), \quad \rho(Jx,Jy) = \rho(x,y),$$

(see [**93**] for the proof). Now one can use the results from the section IV.1 to study the Chern characteristic classes of a Hermite surface M endowed with a complex torsion free connection ∇ with the symmetric curvature operator and conclude

$$[C_1(M)] = 0, \quad [C_1^2(M)] = 0, \quad [C_2(M)] = [\gamma\tilde{\delta}_2] = [\hat{\delta}_2],$$

where

$$\tilde{\delta}_2 = \frac{1}{16\pi^2}(2\|\rho\|^2 - \|R\|^2)\Phi^2, \qquad \hat{\delta}_2 = \frac{1}{16\pi^2}(\tau^{*2} - \|R\|^2)\Phi^2,$$

$\|R\|$, $\|\rho\|$ are the norms of the curvature and the Ricci tensor, i.e.

$$\|R\|^2 = \sum R_{PQIK}R_{PQIK}, \qquad \|\rho\|^2 = \sum \rho_{PQ}\rho_{PQ}$$

and $\Phi = \sum w^i \wedge \bar{w}^i$ is the fundamental 2-form; assuming that $(w^i, \bar{w}^i)$ is the corresponding dual base for (E_i, JE_i). Moreover, we have also some geometrical consequences. More precisely we have

Theorem 2.2. *Suppose that a torsion free complex connection ∇ exists on a compact Hermite surface M with $\tau^* = 0$. Then $\gamma_2(M) \leq 0$. The equality holds if and only if ∇ is a flat connection.* □

Corollary 2.3. *Let (M, J) be a compact Hermite surface which admits a Kähler-Einstein metric. Then every complex torsion free connection with $\tau^* = 0$ on M is flat.* □

We refer to [**16**] for more details related to the symmetric curvature operators and topology. One can find also some examples of complex torsion free connections on reducible Hermite surface M with the generic $R \in \mathcal{R}(T_pM)$ or with R belonging to some vector subspaces of $\mathcal{R}(T_pM)$, which are invariant or, irreducible under the action of the unitary group $U(m)$. Some of examples show that the compactness of M is an essential assumption in Theorem 2.2 and Corollary 2.3.

IV.3. The relations between characteristic classes and projective geometry

If we are interested in relations between topology and geometry of a smooth manifold we must work with differential forms. The main purpose of this section is to study invariance of characteristic forms with respect to some group of transformations. We are interested in also does the topology of a manifold M determine the relations between the group of holomorphically projective transformations, the group of projective transformations and the group of affine transformations on M.

First, we are interested in the invariance of characteristic forms. Conformally equivalent metrics and projectively equivalent torsion free connections have the same characteristic forms. More precisely, it yields

Theorem 3.1. *Let $Q \in \mathcal{Q}_\nu$ and let $\beta \in C^\infty_+(M)$.*
(1) Let $\nabla(h)$ be the Levi-Civita connection of a semi-Riemannian metric. Then $Q(\nabla(h)) = Q(\nabla\beta(h))$, where $\beta(h) = \beta h$.
(2) Let ∇ and $\tilde{\nabla}$ be two projectively equivalent torsion free connections. Then $Q(\nabla) = Q(\tilde{\nabla})$. □

We refer [**3**], [**15**] for the proof of this theorem.

Matzeu [**77**] has studied the Chern algebra of the complex vector subbundle H of TM defined as $H = Ker\eta$, where M is normal almost contact manifold.

The corresponding conditions to have invariant Chern forms under C-projective transformations have been found. We refer to Section III.5. for basic notations. Suppose that ∇ is an adapted symmetric connection with symmetric Ricci tensor field and D is its restriction to the vector subbundle H of TM defined as $H = Ker\eta$. The D is the complex connection with the Ricci tensor $\rho(K)$ symmetric too. We refer to [**77**] for the proofs of the following theorems.

Theorem 3.2. *If the connection D with symmetric Ricci tensor field has the first Chern form proportional to $d\eta$, then all its Chern forms are C-projectively invariant.* □

Theorem 3.3. *All the Chern forms of a C-projectively flat adopted connection are C-projectively invariant.* □

Theorem 3.4. *The Chern classes of a C-projectively flat manifold are trivial.*□

One can use the traces $\mu, \bar{\mu}$ given by (1.1), (1.2) and Chern numbers to prove the following theorems related to the influence of topology of M in the group of projective transformations and its subgroup of affine transformations. We refer to [**13**] for details.

Theorem 3.5. *Let M, $\dim_{\mathbb{C}} M = m$, be a compact complex manifold with a complex symmetric connection ∇. If*
(i) $\rho(u,v) = \rho(v,u)$, *and* $\rho(Ju,Jv) = \rho(u,v)$,
(ii) $\bar{\mu}$ *is a semi-definite bilinear form, of rank* 0 *or* m, *nonnegative if* m *is odd,*
(iii) $\gamma_1^m(M) \leq 0$ *then the group of all projective diffeomorphisms of the connection* ∇ *coincides with the group of all affine diffeomorphisms of the same connection.* □

Theorem 3.6. *Let M be a surface of general type with a complex symmetric connection ∇. If*
(i) $\rho(v,u) = \rho(u,v)$, *and* $\rho(Ju,Jv) = \rho(u,v)$,
(ii) $\bar{\mu}$ *is a semi-definite bilinear form of rank* 0 *or* m, *nonnegative if* m *is odd,*
(iii) $\gamma_2(M) \leq 0$, *then the group of all projective diffeomorphisms of the connection* ∇ *coincides with the group of all affine diffeomorphisms of the same connection.* □

Under the assumptions of Theorems 3.5. or 3.6. one can prove the group of holomorphically projective transformations coincides with the group of affine transformations of ∇.

In the general case the group of projective diffeomorphisms, the group of affine diffeomorphisms and the isometry group do not coincide for a Riemannian manifold (M, g). Nagano [**89**] has proved that if M is a complete Riemannian manifold with parallel Ricci tensor then the largest connected group of projective transformations of M coincides with the largest connected group of affine transformations of M unless M is a space of positive constant sectional curvature. For Kähler manifolds problems of this type have been studied in [**13**]. So we have

Theorem 3.7. *Let M be a compact Kähler manifold of complex dimension $m > 1$. If: (i) $\tau =$ constant, (ii) $c_1 = 0$, then* *(a) the group of all projective diffeomorphisms of the Levi-Civita connection coincides with the group of all affine diffeomorphisms of the same connection;*

(b) the identity component of the group of all holomorphically projective diffeomorphisms of the Levi-Civita connection coincides with the identity component of its group of isometries. □

Kobayashi and Ochiai have studied in [**62**] holomorphic normal projective connections on complex manifolds and classified all compact complex analytic surfaces which admit flat holomorphic projective connections. They have proved in [**63**] a complex analytic surface of general type, which admits a holomorphic (normal) projective connection, is covered by a unit ball $B^2 \subset \mathbb{C}^2$ without ramification.

IV.4. Characteristic classes and affine differential geometry

Let us recall, if x is a nondegenerate embedding of a manifold M as a hypersurface in affine space, we let (x, X, y) be a relative normalization. This defines a triple (∇, h, ∇^*) on M, where h is a semi-Riemannian metric, and where ∇ and ∇^* are torsion free connections on the tangent bundle TM. If Q is an invariant polynomial, then $Q(\nabla) = 0$, $Q(\nabla(h)) = 0$ and $Q(\nabla^*) = 0$. Moreover, the secondary characteristic forms of the connections $\nabla, \nabla^*, \nabla(h)$ vanish. To be more precise we introduce a decomposable invariant polynomial Q by the relation $Q = \sum_i Q_{i,1}Q_{i,2}$, where $0 \neq Q_{i,j} \in \mathcal{Q}_{\nu(i,j)}$ and $\nu(i,j) > 0$. For the proofs of following lemma and theorems we refer [**15**].

Lemma 4.1. *Let (∇, h, ∇^*) be the conjugate triple defined by a relative normalization (x, X, y) of an affine embedding of an orientable manifold M. Let $Q \in \mathcal{Q}_\nu$.*

(1) If Q is decomposable, then $[\mathcal{T}_x Q(\nabla)] = 0$, $[\mathcal{T}_x Q(\nabla(h))] = 0$, and $[\mathcal{T}_x Q(\nabla^*)] = 0$ in $H^{2\nu-1}(M, \mathbb{C})$.

(2) The classes $[\mathcal{T} Q(\nabla)]$, $[\mathcal{T}_x Q(\nabla^*)]$, and $[\mathcal{T} Q(\nabla(h))]$ in $H^{2\nu-1}(M, \mathbb{C})$ are affine invariants; these cohomology classes are independent of the relative normalization chosen. □

Theorem 4.2. *Let (∇, h, ∇^*) be the conjugate triple defined by a relative normalization (x, X, y) of an affine embedding of an orientable manifold M. Let $Q \in \mathcal{Q}_\nu$.*

(1) We have $[\mathcal{T}_x Q(\nabla)] = 0$ in $H^{2\nu-1}(M; \mathbb{C})$.

(2) If Q is integral and if ν is even, then $[\mathcal{T} Q(\nabla^*)] = 0$ in $H^{2\nu-1}(M; \mathbb{C}/\mathbb{Z})$.

(3) If ν is odd, then $[\mathcal{T}_x Q(\nabla^*)] = 0$ in $H^{2\nu-1}(M; \mathbb{C})$.

(4) If ν is even, then $[\mathcal{T}_x Q(\nabla(h))] = 0$ in $H^{2\nu-1}(M; \mathbb{C})$.

(5) If ν is odd and if h is definite, then $[\mathcal{T}_x Q(\nabla(h))] = 0$ in $H^{2\nu-1}(M, \mathbb{C})$. □

One can apply these results to 3-dimensional affine differential geometry to construct obstructions to realizing the conformal class of a Riemannian metric as the second fundamental form of an embedding; this generalizes work of Chern and Simons [**37**].

To state the corresponding theorem we have in mind that if M is a compact orientable 3-dimensional manifold, then M is parallelizable. Hence we can choose a global frame f for TM. If $Q \in \mathcal{Q}_2$, then $Q = cP_1 +$ decomposable, so we need only to study $[\mathcal{T}_x P_1]$, where P_1 is the first Pontrjagin form. Since P_1 is a real integral differential form, we define

$$\Phi(\nabla) = \int_M f^* \mathcal{T} P_1(\nabla) \in \mathbb{R}/\mathbb{Z}.$$

One can prove that $\Phi(\nabla)$ is independent of the particular parallelization f which is chosen. Consequently, Theorem 4.2 implies

Theorem 4.3. *Let (M, g_o) be a 3-dimensional Riemannian manifold.*

(1) If there exists an immersion $x : M \to \mathbb{R}^4$ so that g_0 is conformally equivalent to the first fundamental form of x, then $\Phi(\nabla(g)) = 0$ in $\mathbb{R}/\mathbb{Z}$.

(2) If there exists an immersion $x : M \to \mathbb{R}^4$ so that g_0 is conformally equivalent to the second fundamental form of x, then $\Phi(\nabla(g)) = 0$ in $\mathbb{R}/\mathbb{Z}$. □

We refer to [**15**] for details of the proof of this theorem and also for other references related to other applications of the secondary characteristic forms in 3-dimensional geometry and in mathematical physics.

V. DIFFERENTIAL OPERATORS OF LAPLACE TYPE

The main purpose of this Section is to study the second order differential operators of Laplace type which are naturally appeared in differential geometry. Of course, the most interesting for us are these operators which depend on a torsion free connection, and relations between the spectrum of operators from one side and geometry and topology of a manifold from other side. To study these problems we explore the heat equation method. We refer to monographs [**6**], [**34**], [**44**], and expository papers [**30**], [**39**], [**58**] [**84**] etc. for more details.

V.1. Definitions and basic notations

Let M be a compact Riemannian manifold of dimension m. *The Laplace-Beltrami operator* (shorter *the Laplacian*) is an operator

$$\Delta(f) = - \div (\operatorname{grad} f), \tag{1.1}$$

where $f \in C^\infty(M)$, i.e. in coordinates $\Delta = -\sum_{i,j} g^{-1}\partial_i(gg^{ij}\partial_j)$, for $g = \sqrt{\det(g_{ij})}$. For example if the metric is given by $ds^2 = h(dx^2 + dy^2)$, then $\Delta = g^{-1}(\partial_x^2 + \partial_y^2)$.

Let $f_t(x)$ denote the temperature in a time t and a point $x \in M$. If we assume the heat trasfers into the coolest direction, then $f_t(x)$ satisfies the equation

$$\frac{\partial}{\partial t} f + \Delta f = 0. \tag{1.2}$$

We say $f_0(x)$ is *an eigenfunction* of Δ with *the eigenvalue* $\lambda \in \mathbb{R}$ if it yields $\Delta(f_0) = \lambda \cdot f_0$. One can check then $f_t(x) = e^{-\lambda t} f_0(x)$ satisfies the heat equation (1.2). Therefore, f_t may be interpreted as "a heat wave" with "the frequency" $e^{-\lambda t}$.

The theory of partial differential operators implies that there exist countable set of eigenvalues λ_i and for every λ_i the finite-dimensional family of eigenfunctions f_i, such that we have $\Delta(f_i) = \lambda_i f_i$. Furthermore, λ_i are positive, and $\lambda_i \to \infty$ when $i \to \infty$. The collection $\{\lambda_i\}$, together with the multiplicities of each λ_i, is *the spectrum of a manifold* M.

If one struck M with a mallet than λ_i may be interpreted as the sounds emitted by M, assuming that sound satisfies a similar equation to that of heat.

Weyl [**137**] has proved that the spectrum of M determine one of significant geometrical invariant - volume of M. This was a reason to believe that the spectrum determines completely the geometry of Kac [**58**] formulated this problem in a lovely question: "Can one hear a shape of a drum". The example of two 16th dimensional non-isometric torus [**86**] with the same spectrum have shown that expectations were excessively strong. Many examples have been constructed later on (see [**30**], [**84**] etc.) using different methods to show the same things.

We say manifolds M_1 and M_2 are *isospectral* if they have the same spectrum.

A function $H_t(x, y)$ is *a fundamental solution* of the heat equation (or *a heat kernel*) if

$$\left(\frac{\partial}{\partial t} + \Delta_x\right) H = 0, \tag{1.3}$$

$$\lim_{t \to 0} \int_M H_t(x, y) f(y) dy = f(x), \tag{1.4}$$

for any $f \in C^\infty(M)$. One can use (1.3) and (1.4) to check that the general solution $f_t(x)$ of the heat equation with initial equation $f_0(x) = f$ is given by the formula

$$f_t(x) = \int_M H_t(x, y) f(y) dy.$$

We look for H_t to fulfill the following conditions

(i) $H_t(x, y)$ is uniquely determined by (1.3) and (1.4).

(ii) If M is a compact manifold and $\{f_i\}$ is an orthonormal base of eigenfunctions with corresponding eigenvalues $\{\lambda_i\}$, then

$$H_t(x, y) = \sum_i e^{-\lambda_i t} f_i(x) f_i(y).$$

Now we use (ii) to eliminate $f_i^{,}s$ and describe $\lambda_i^{,}s$. Therefore, we put

$$\operatorname{tr}(H_t) = \int_M H_t(x, x) dx.$$

Consequently

$$\begin{aligned}\operatorname{tr}(H_t) = \operatorname{tr}_{L^2} e^{-t\Delta} &= \int_M \sum e^{-\lambda_i t} f_i^2(x)dx \\ &= \sum e^{-\lambda_i t} \int_M f_i^2 dx = \sum e^{-\lambda_i t}.\end{aligned}$$

If $t \to 0^+$ then there is a power serious expansion, asymptoticaly equivalent to $\sum e^{-\lambda_i t}$, i.e.

$$\sum_i e^{-t\lambda_i} \sim \sum_{n=0}^{\infty} a_n(\Delta) t^{(n-m)/2},$$

where $a_n(\Delta)$ are spectral invariants determined by local geometry of M. If M is a manifold with boundary, i.e. $\partial M \neq \phi$, then $a_{2k+1}(\Delta) \neq 0$ and they depend on the boundary conditions

$$\begin{aligned}&\mathcal{B}_D f = f|_{\partial M} = 0 \qquad \text{(Dirichlet boundary condition)} \qquad \text{or} \\ &\mathcal{B}_N^S f = (\partial_\nu + S)|_{\partial M} \quad \text{(modified Neumann boundary condition).}\end{aligned}$$

These results may be generalized for a partial differential operator D of order $d > 0$ on a smooth vector bundle. We assume the leading symbol of D is self-adjoint and positive definite. If the boundary of M is non-empty, we impose boundary conditions $\mathcal{B}$ and let Domain $(D_{\mathcal{B}}) = \{w \in C^\infty(V) : \mathcal{B}w = 0\}$. We assume the boundary conditions $\mathcal{B}$ are strongly elliptic; see Gilkey [**44**, §1.11].

Let $f \in C^\infty(M)$ be an auxiliary test function. Then there is an asymptotic series at $t \downarrow 0^+$ of the form

$$\operatorname{tr}_{L^2}(f e^{-tD_{\mathcal{B}}}) \sim \sum_{n=0}^{\infty} a_n(f, D, \mathcal{B}) t^{(n-m)/d};$$

see Gilkey [**44**, Theorem 1.11.4] for details. The global invariants $a_n(f, D, \mathcal{B})$ are locally computable. Let $\partial_m^\nu f$ be the ν^{th} normal covariant derivative of f. Then there exists local measure valued invariants $A_n(x, D)$ defined for $x \in M$ and $\mathcal{A}_{n,\nu}^{bd}(y, D, \mathcal{B})$ defined for $y \in \partial M$ such that

$$a_n(f, D, \mathcal{B}) = \int_M f \mathcal{A}_n(x, D) + \sum_{0 \le \nu \le n-1} \int_{\partial M} (\partial_m^\nu f) \mathcal{A}_{n,\nu}^{bd}(y, D, \mathcal{B}). \tag{1.5}$$

From now on we study the local geometry of operators of Laplace type. Let $D = -(g^{\nu\mu}\partial_\nu\partial_\mu + A^\sigma\partial_\sigma + B)$ be an operator of Laplace type on $C^\infty(M)$, for $A^\sigma \in \operatorname{End}(M)$ and $B \in \operatorname{End}(M)$. One can note that Dirichlet and modified Neumann boundary conditions are strongly elliptic for second order operators of Laplace type. We refer to [**43**] for the proof of

Lemma 1.1. *There exists a unique connection ∇_D on $C^\infty(M)$ and a unique function $E_D \in C^\infty(M)$ so that $D = -(\mathrm{tr}(\nabla_D^2) + E_D)$. If w_D is the connection 1-form of ∇_D, then*

$$w_{D,\delta} = \frac{1}{2} g_{\nu\delta}(A^\nu + g^{\mu\sigma}\Gamma_{g,\mu\sigma}{}^\nu), \quad \text{and}$$
$$E_D = B - g^{\nu\mu}(\partial_\mu w_{D,\nu} + w_{D,\nu} w_{D,\mu} - w_{D,\sigma}\Gamma_{g,\nu\mu}{}^\sigma). \qquad \square$$

We set $f = 1$ in (1.5) to recover the invariants $a_n(D, \mathcal{B})$ for an operator of Laplace type. We use now these invariants to express $\mathcal{A}_n(x, D)$. Let $\Omega_{D,ij}$ be the curvature of the connection ∇_D on $C^\infty(M)$ and let ';' be multiple covariant differentiation with respect to the Levi-Civita connection. We refer to Gilkey [**43**], [**44**] for the proof of the following theorem:

Theorem 1.2. *Let $D = D(\nabla_D, E_D)$ on $C^\infty(M)$.*
(a) $\mathcal{A}_0(x, D) = (4\pi)^{-m/2}$.
(b) $\mathcal{A}_2(x, D) = 6^{-1}(4\pi)^{-m/2}(\tau_g + 6E_D)$.
(c) $\mathcal{A}_4(x, D) = 360^{-1}(4\pi)^{-m/2}\{60(E_D);kk + 60\tau_g E_D + 180(E_D)^2 +$
$30\Omega_{D,ij}\Omega_{D,ij} + 12(\tau_g)_{;kk} + 5(\tau_g)^2 - 2|\rho_g|^2 + 2|Rg|^2\}$.

We suppose given some auxiliary geometric structure $\mathcal{J}$ on which $C_+^\infty(M)$ also acts. For $g \in \mathfrak{C}$ and $s \in \mathcal{J}$ we assume given a natural operator $D = D\{g, s\}$ on M which is of Laplace type. Let $D \mapsto {}_\beta D := D({}_\beta g, {}_\beta s)$, where $g \mapsto {}_\beta g = \beta g$, $\beta \in C_+^\infty(M), g \in \mathfrak{C}$, and let $\mathcal{M}(\beta)$ be function multiplication. An operator D is said to transform conformally if ${}_\beta D = \mathcal{M}(\beta^a) \circ D \circ \mathcal{M}(\beta^b)$ for $a + b = -1$. If D transforms conformally, the conformal index theorem of Branson and Orsted [**29**] and Parker and Rosenberg [**104**] shows that $a_m(D) = a_m({}_\beta D)$.

V.2. Asymptotics of Laplacians defined by torsion free connections

In this section we present the heat equation asymptotics of the Laplacians defined by torsion free connections.

V.2.1. Laplacians on the tangent bundle of a manifold without boundary. We assume that (M^m, g) is a compact Riemannian manifold without boundary of dimension $m > 1$. We choose a local coordinates to have ∂_i and dx^i as local coordinate frames for the tangent TM^m and cotangent T^*M^m bundles respectively. If ∇ is a torsion free connection on TM^m we denote by $w_i \in \mathrm{End}(TM^m)$ the connection 1-form of ∇

$$\nabla\partial_j = dx^i \otimes w_i(\partial_j) = w_{ij}{}^k dx^i \otimes \partial_k.$$

Since ∇ is torsion free it follows $w_{ij}{}^k = w_{ji}{}^k$. Let ${}^\nabla R$ be the curvature of ∇. Let ${}^g\nabla = \nabla(g)$ be the Levi-Civita connection corresponding to the metric g. Then

${}^g\nabla(\partial_j) = \Gamma_{ij}{}^k dx^i \otimes \partial_k$ and $\theta = \nabla - {}^g\nabla$ is tensorial and $\theta_{ij}{}^k = w_{ij}{}^k - \Gamma_{ij}{}^k$. We introduce also the tensor $\mathcal{F} \in TM^m$ by contracting the first two indices of θ:

$$\mathcal{F}^i = g^{jk}\theta_{jk}{}^i = g^{jk}(w_{jk}{}^i - \Gamma_{jk}{}^i).$$

The dual connection ∇^* on T^*M^m is defined by $d(u,\alpha) = (\nabla u, \alpha) + (u, \nabla^*\alpha)$, for any smooth vector field u and smooth covector field α. Consequently

$$\nabla^*(dx^j) = -dx^i \otimes w_i^*(\partial_j) = -w_{ik}{}^j dx^i \otimes dx^k.$$

If $\nabla^* \otimes 1 + 1 \otimes \nabla$ is the tensor product connection on $T^*M^m \otimes TM^m$, then

$$\mathcal{P} = \mathcal{P}(\nabla) = -g^{ij}\{\nabla^* \otimes 1 + 1 \otimes \nabla\}_i \nabla_j \quad \text{on} \quad C^\infty(TM^m) \tag{2.1}$$

is a second order PDO of Laplace type.

We shall use Roman indices for a coordinate frame and Greek indices for an orthonormal frame. We refer to [**26**] for the proof of:

Theorem 2.1. *Let $\mathcal{P} = \mathcal{P}(\nabla)$ be a PDO of Laplace type given by (2.1). Then*

(a) $a_0(\mathcal{P}) = m \cdot \mathrm{vol}(M)$,

(b) $a_2(\mathcal{P}) = \frac{m}{12}\int_M (2\tau - 3\mathcal{F}_\nu\mathcal{F}_\nu)$,

(c) $a_4(\mathcal{P}) = \frac{1}{360}\int_M \{m\{5\tau^2 - 2\rho^2 - 2\rho^2 + 2R^2 + 15\tau(2\mathcal{F}_{\nu;\nu} - \mathcal{F}_\nu\mathcal{F}_\nu)$
$+ \frac{45}{4}(2\mathcal{F}_{\nu;\nu} - \mathcal{F}_\nu\mathcal{F}_\nu)^2 + \frac{15}{2}(\mathcal{F}_{\mu;\nu} - \mathcal{F}_{\nu;\mu})^2\} + \mathrm{Tr}(30\Omega^2)\}$. □

We define the Hessian H_∇ for a torsion free connection ∇ on TM by

$$(H_\nabla f)(u,v) = u(v(f)) - \nabla_u v(f).$$

One can check easily that $(H_\nabla f)(u,v) = (H_\nabla f)(v,u)$ and H_∇ is tensorial in X and Y. The normalized Hessian $\mathcal{H}(f) := H_\nabla(f) + (m-1)^{-1} f\rho_\nabla$ arises naturally in the study of Codazzi equations; see [**108**] for details. We contract the normalized Hessian for a torsion free connection with symmetric Ricci tensor ρ_∇ to define an operator of Laplace type

$$Df = D(g,\nabla)f := -\,\mathrm{tr}_g\{H_\nabla(f) + (m-1)^{-1} f\rho_\nabla\}. \tag{2.2}$$

In general case, D need not be self-adjoint.

If $\tilde\nabla$ and ∇ are projectively equivalent, as in (1.1) Section II, then we may choose a local primitive ϕ so $d\phi = \pi$. Then

$$\mathcal{H}_{\tilde\nabla} = e^\phi \mathcal{H}_\nabla e^{-\phi}; \tag{2.3}$$

i.e. the operators $\mathcal{H}_{\tilde\nabla}$ and $\mathcal{H}_\nabla$ are locally conjugate. Furthermore, if $\tilde g = e^{2\psi}g$, then (2.3) implies

$$D(\tilde g, \tilde\nabla) = e^{-2\psi+\phi} D(g,\nabla)e^{-\phi}. \tag{2.4}$$

If $[\pi] = 0$ in the first cohomology group $H^1(M)$, then ϕ is globally defined and $D(g, \tilde{\nabla})$ and $D(g, \nabla)$ are conjugate and hence isospectral. We refer to [**17**]–[**19**], [**21**] for more details related to this operator.

Let $\mathcal{K}$ be a Codazzi structure, let $(g,^* \nabla) \in \mathcal{K}$ and ${}^w\nabla$ be the Weyl connection defined by $\mathcal{K}$. Then we use (2.2) and define

(i) Let ${}^*D := D\{g,^* \nabla\}$ be the trace of the normalized Hessian of ${}^*\nabla$.
(ii) Let ${}^wD := D\{g,^w \nabla\}$ be the trace of the normalized Hessian of ${}^w\nabla$.
(iii) Let ${}^w\Delta := -\operatorname{tr}_g {}^w\nabla d$ be the scalar Laplacian of ${}^w\nabla$.
(iv) Let ${}^g\Box := -\operatorname{tr}_g \delta_g d + (m-2)\tau(g)/4(m-1)$ be the conformal Laplacian.

V.3. Geometry reflected by the spectrum

As we already know torsion free connections arise naturally in affine differential geometry and Weyl geometry. The main purpose of this section is to study geometry of a manifold in the framework previously mentioned, reflected by the spectrum of an differential operator of Laplace type. One can find more details in [**17**]–[**21**] and [**26**].

V.3.1. Affine differential geometry reflected by the spectrum. In this subsection we deal with smooth hypersurfaces M^m immersed into an affine space $\mathcal{A}^{m+1}$. Because of the convinience reason througout this subsection we denote by ${}^1\nabla$, ${}^2\nabla$ the induced connection and the conormal connection. We suppose the Blaschke metric G positive definite henceforth; this means that the immersed hypersurface $x(M^m)$ is locally strongly convex. Let $\mathcal{P} = \mathcal{P}({}^1\nabla, G) = \mathcal{P}(x, X, y)$ on $C^\infty(TM^m)$ be defined by (2.1). The spectral geometry of $\mathcal{P}$ should play an important role in affine geometry. Since the Blaschke metric G and first affine connection ${}^1\nabla$ are defined by expressions which are invariant under the group of affine transformations, the operator $\mathcal{P}$ and its spectrum are affinely invariant.

One can compute the heat equation invariants.

Lemma 3.1. *(a)* $C_{ij}{}^k = \theta_{ij}{}^k$, *(b)* $w_{ij}{}^k = {}^1\Gamma_{ij}{}^k$. *(c)* $\mathcal{F}_\nu = m\tilde{T}_\nu$,
(d) $\operatorname{tr}(\Omega^2) = {}^1R_{ijk}{}^l\, {}^1R_{ijl}{}^k = -2\{m(m-1)H^2 - \frac{1}{m}\sum_{i<j}(\lambda^i - \lambda^j)^2\}$. □

We combine now Lemma 3.1 and Theorem 2.1 with results from Section I.2. to prove the following theorems.

Theorem 3.2. *Let x and $\bar{x} : M^m \to \mathcal{A}^{m+1}$ define hyperovaloids with the same regular relative spherical indicatrix $y = \bar{y}$ which are $\mathcal{P}$ isospectral. Then x and $\bar{x}$ are translation equivalent.* □

Theorem 3.3. *Let x and $\bar{x} : M^2 \to \mathcal{A}^3$ be ovaloids with centroaffine normalization which are $\mathcal{P}$ isospectral. If $x(M^2)$ is an ellipsoid, then $\bar{x}(M^2)$ is an ellipsoid.* □

Theorem 3.4. *Let M_i be ovaloids with equiaffine normalization and M_1 an ellipsoid. If for $M_1 := M$, $M_2 := \bar{M}$*

(i) $\int_M H = \int_{\bar{M}} \bar{H}$, $\int_M \operatorname{tr}(1) = \int_{\bar{M}} \operatorname{tr}(1)$ *or*

(ii) $\int_M H^2 = \int_{\bar{M}} \bar{H}^2$, $\int_M \mathrm{tr}(1) = \int_{\bar{M}} \mathrm{tr}(1)$, $H^2 > 0$
(H^2 is the second elementary curvature function)
then $\bar{M} = M_2$ is an ellipsoid. □

We study now affine geometry reflected by the spectrum of the operator D given by (2.2). Since the ${}^i\nabla$ are torsion free, Ricci symmetric connections, we can apply the results of Section V.2. to this setting. Let

$$ {}^1D := D(G, {}^1\nabla), \quad {}^2D := D(G, {}^2\nabla) \quad \text{and} \quad {}^GD := D(G, {}^G\nabla) $$

be the associated operators of Laplace type on $C^\infty(M)$; these operators and their spectra are affine invariants of (x, X, y).

We now compute the expressions of Lemma 1.1, which we need to obtain the coefficients in the corresponding heat equation asymptotics.

Lemma 3.5. *Let $D = {}^1D$ and let $\epsilon = 1$ or let $D = {}^2D$ and let $\epsilon = -1$. Then:*

$$ {}^D\theta = \epsilon C, \quad w_D = -\frac{1}{2}\epsilon m\tilde{T}, \quad \Omega_D = 0, $$
$$ E_D = mH - \frac{1}{4}m^2|\tilde{T}|^2 + \frac{1}{2}\epsilon m\tilde{T}i; i. $$
□

Theorem 3.6. *Let $x : M \to \mathcal{A}$ be a hyperovaloid. Let $D = {}^1D$ and let $\epsilon = 1$ or let $D = {}^2D$ and let $\epsilon = -1$. Then:*
(a) $\mathcal{A}_0(x, D) = (4\pi)^{-m/2}$. $a_0(D) = (4\pi)^{-m/2} \int_M 1$.
(b) $\mathcal{A}_2(x, D) = (4\pi)^{-m/2}\{\frac{1}{6}\tau_g + mH - \frac{1}{4}m^2|\tilde{T}|^2 + \frac{1}{2}\epsilon m\tilde{T}_{i;i}\}$
$a_2(D) = (4\pi)^{-m/2} \int_M \{\frac{1}{6}\tau_g + mH - \frac{1}{4}m^2|\tilde{T}|^2\}$.
(c) $\mathcal{A}_4(x, D) = (4\pi)^{-m/2} 360^{-1}\{60\tau_G(mH - \frac{1}{4}m^2|\tilde{T}|^2 + \frac{1}{2}\epsilon m\tilde{T}_{i;i}) +$
$180(mH - \frac{1}{4}m^2|\tilde{T}|^2 + \frac{1}{2}\epsilon mT_{i;i})^2 + 60(mH - \frac{1}{4}m^2|\tilde{T}|^2 + \frac{1}{2}\epsilon m\hat{T}_{i;i})_{;jj} +$
$12\tau_{G;kk} + 5\tau_G^2 - 2|\rho_G|^2 + 2|R_G|^2\}$. □

One can combine Lemma 3.5 and Theorem 3.6 with results from Section I.2. to prove the following theorems, which consider affine geometry reflected by the spectrums of ${}^iD$.

Theorem 3.7. *Let $D = {}^1D$ or $D = {}^2D$.*
(a) Let (x, X, y) be a relative normalization. Then

$$ (4\pi)^{m/2}\Big\{a_2(D) - \frac{m-1}{m+5}a_2({}^GD)\Big\} \le m\int_M H. $$

Equality holds if and only if the normalization is equiaffine.
(b) If the normalization is equiaffine, then

$$ m(m+5)\int_M H \le 6(4\pi)^{m/2}a_2(D) \le m(m+5)\int_m (H + J). $$

Equality holds on the left or on the right hand side if and only if the hyperovaloid is an ellipsoid. □

Theorem 3.8. *Let $D = {}^1D$ or $D = {}^2D$. With the centroaffine normalization:*
(a) $a_0(D) = (4\pi)^{-m/2} \int_M 1.$ $\quad a_2(D) = (4\pi)^{-m/2} \int_M (\frac{1}{6}\tau_G + m - \frac{1}{4}m^2|\tilde{T}|^2).$
(b) $6(4\pi)^{m/2}\{a_2(D) - ma_0(D)\} \leq \int_m \tau_G$ *with equality if and only if the hyperovaloid is an ellipsoid.*
(c) Let $x(M^2)$ and $\tilde{x}(M^2)$ be ovaloids in 3 space with centroaffine normalization which are D isospectral. Then x is an ellipsoid if and only if $\tilde{x}$ is an ellipsoid. □

As we have mentioned already the operators ${}^iD$ are not self-adjoint in general case. The following theorem deals with the conditions to have self-adjoint operators ${}^iD$.

Theorem 3.9. *(1) Let $\{x, X, y\}$ be the Euclidean normalization. Then the following assertions are equivalent*
1-a) The Gauss-Kronecker curvature $K = K_n$ is constant.
1-b) We have ${}^1D = {}^2D$.
1-c) The operator ${}^1D$ or the operator ${}^2D$ is self-adjoint.
(2) Let x be a compact centroaffine hypersurface with non-empty boundary. The following assertions are equivalent:
2-a) We have ${}^1D = {}^2D$.
2-b) We have that ${}^1D$ or ${}^2D$ is self-adjoint.
2-c) We have that x is a proper affine sphere.
(3) Let x be a compact centroaffine hypersurface without boundary. Then the following assertions are equivalent:
3-a) We have ${}^1D ={}^2 D$.
3-b) We have that ${}^1D$ or ${}^2D$ is self-adjoint.
3-c) We have that x is a hyperovalloid. □

Dirichlet boundary conditions on affine hypersurface with boundary were considered by Schwenk [**113**], [**128**] and Simon [**116**]. Their methods are different from this one.

V.3.2. Projective geometry reflected by the spectrum. In general, constructing projective invariants is quite difficult. One such example is the projective curvature tensor of H. Weyl (see Section II.1., especialy the formula (1.2)). This subsection deals with spectral invariants which are also projectively invariant. More precisely we have

Theorem 3.10. *Let $\nabla, \tilde{\nabla}$ be torsion free projectively equivalent connections on a Riemannian manifold (M, g). Let $D = D(g, \nabla)$ and $\tilde{D} = D(g, \tilde{\nabla})$.*
(a) $\mathcal{A}_n(x, \tilde{D}) = \mathcal{A}_n(x, D)$ *and* $\mathcal{A}_n^{bd}(y, \tilde{D}, \mathcal{B}) = \mathcal{A}_n^{bd}(y, D, \mathcal{B})$.
(b) $a_n(\tilde{D}, \mathcal{B}) = a_n(D, \mathcal{B})$. □

If m is odd, and if the boundary of M is empty, there is a global spectral invariant called *the functional determinant* which can be defined in this context.

For $\mathrm{Re}(s) \gg 0$, let $\zeta(s, D) := \mathrm{tr}_{L^2}(D^{-s})$, where we project on the complement of the kernel of D to avoid the 0-spectrum. This has a meromorphic extension to

$\mathbb{C}$ with isolated simple poles on the real axis. The origin is a regular value and $\zeta'(0) := -\log(\det(D))$ is a global invariant of D.

Theorem 3.11. *Let $\nabla, \tilde{\nabla}$ be torsion free projectively equivalent connections on a Riemannian manifold (M, g) and $\tilde{g} \in \mathfrak{C}(g)$. If the boundary of M is empty and if $m = \dim M$ is odd, then $\zeta'(0, D) = \zeta'(0, \tilde{D})$.* □

V.3.3. Invariants of Codazzi and Weyl structures.

The relation (2.4) informs us that the operator D, given by (2.2) transforms conformally. This implies one can apply the conformal index theorem of Branson and Orsted [**29**] and Parker and Rosenberg [**104**] to prove the following lemma.

Lema 3.12. *(i) Let D be an operator of Laplace type given by (2.2). Then $a_m(D(\tilde{g}, \tilde{\nabla})) = a_m(D(g, \nabla))$.*

(ii) We have that $a_m({}^*D), a_m({}^wD), a_m({}^w\Delta)$, and $a_m({}^g\Box)$ are gauge invariants of a Codazzi structure $\mathcal{K}$. □

One can compute the endomorphism E and the curvature Ω for four natural operators defined in Section V.2.

Lemma 3.13. We have

(i) $E\{{}^*D\} = \{(m+2)\tau(g, {}^w\nabla) - (m-2)\tau(g)\}/4(m-1)$.

(ii) $\Omega\{{}^*D\} = -(m+2){}^wF/2$.

(iii) $E\{{}^wD\} = -(m-2)\delta_g\hat{\theta}/2 - (m-2)\|\hat{\theta}\|_g^2/4 + (m-1)^{-1}\tau(g, {}^w\nabla)$.

(iv) $\Omega\{{}^wD\} = -(m-2){}^wF/2$

(v) $E\{{}^w\Delta\} = -(m-2)\delta_g\hat{\theta}/2 - (m-2)^2\|\hat{\theta}\|_g^2/4$.

(vi) $\Omega({}^w\Delta) = -(m-2){}^wF/2$.

(vii) $E\{{}^g\Box\} = -(m-2)\tau(g)/4(m-1)$ and $\Omega\{{}^g\Box\} = 0$. □

We refer to [**20**] for the proof of this Lemma and more details related to the operators ${}^*D$, ${}^wD$, ${}^w\nabla$ and ${}^g\Box$.

We use now Lemma 3.13 and Theorem 1.2 in dimensions $m = 2$ and $m = 4$. Let $\chi(M)$ be the Euler-Poincare characteristic of M. The Chern Gauss Bonnet theorem yields

$$\chi(M^2) = (4\pi)^{-1} \int_M \tau(g)(x) d\nu_g(x),$$

$$\chi(M^4) = (32\pi^2)^{-1} \int_M \{\|{}^gR\|_g^2 - 4\|{}^g\rho\|_g^2 + \tau(g)^2\}(x) d\nu_g(x).$$

Theorem 3.14. *Let $\dim(M) = 2$. Then*

(i) $a_2({}^*D) = \chi(M)/6 + (4\pi)^{-1} \int_M \tau(g, {}^w\nabla) d\nu_g(x)$.

(ii) $a_2({}^wD) = \chi(M)/6 + (4\pi)^{-1} \int_M \tau(g, {}^w\nabla)(x) d\nu_g(x)$.

(iii) $a_2({}^w\Delta) = \chi(M)/6$.

(iv) $a_2({}^g\Box) = \chi(M)/6$. □

Theorem 3.15. *Let $\dim(M) = 4$. Let ${}^gW$ be the Weyl conformal curvature. Then*

(i) $a_4({}^*D) = -\frac{\chi(M)}{180} + \frac{1}{(4\pi)^2 360}\int_M \{3\|{}^gW\|_g^2 + 270\|{}^wF\|_g^2 + 45\tau(g,{}^w\nabla)^2\}d\nu_g(x)$.

(ii) $a_4({}^wD) = -\frac{\chi(M)}{180} + \frac{1}{(4\pi)^2 360}\int_M \{3\|{}^gW\|_g^2 + 30\|{}^wF\|_g^2 + 45\tau(g,{}^w\nabla)^2\}d\nu_g(x)$.

(iii) $a_4({}^w\Delta) = -\frac{\chi(M)}{180} + \frac{1}{(4\pi)^2 360}\int_M \{3\|{}^gW\|_g^2 + 30\|{}^wF\|_g^2 + 5\tau(h,{}^w\nabla^2\}d\nu_g(x)$.

(iv) $a_4({}^g\Box) = -\frac{\chi(M)}{180} + \frac{1}{(4\pi)^2 360}\int_M \{3\|{}^gW\|_g^2\}d\nu_g(x)$. □

If f is a scalar invariant, let $f[M] := \int_M f(x)d\nu_g(x)$. The Euler characteristic is a topological invariant of M which does not depend on the Codazzi structure. Then we use Theorem 3.15 to prove the following Corollary:

Corollary 3.16. *(i) The invariants $\tau(g,{}^w\nabla)^2[M]$, $\|{}^wF\|_g^2[M]$ and $\|{}^gW\|_g^2[M]$ of a Weyl structure on M are determined by $\chi(M)$ and by the spectrum of the operators ${}^*D$, ${}^wD$, and ${}^w\Delta$.*

(ii) We have $32\pi^2\chi(M^4) \geq 45\tau(g,{}^w\nabla)^2[M] + 270\|{}^wF\|_g^2[M] - (4\pi)^2 360 a_4({}^*D)$ with equality if, and only if, the class $\mathfrak{C}$ is conformally flat.

(iii) We have $32\pi^2\chi(M^4) \geq 45\tau(g,{}^w\nabla)^2[M] + 3\|{}^gW\|_g^2[M] - (4\pi)^2 360 a_4({}^*D)$ with equality if, and only if, the length curvature ${}^wF = 0$. □

Bibliography

[1] M.F. Atiyah, V.K. Patodi, and I.M. Singer, *Spectral asymmetry, and Riemannian geometry I, II, III*, Math. Proc. Cambridge Phil. Soc. **77** (1975), 43–69, **78** (1975), 405–432, **79** (1976), 71–99.

[2] L. Auslander, *On the Euler characteristic of compact locally affine spaces*, Comment. Math. Helv. **35** (1961), 25–27.

[3] A. Avez, *Remarques sur les formes de Pontrjagin*, C.R. Acad. Sci. Paris **270** (1970), 1248.

[4] J.P. Benzécri, *Sur les variété s localement affines et localement projectives*, Bull. Soc. Math. France **88** (1960), 229–332.

[5] J.P. Benzécri, *Sur la classe d'Euler de fibré s affins plats*, C.R. Acad. Sci. Paris **260** (1965), 5442–5444.

[6] M. Berger, P. Gauduchon and E. Mazet, *Le Spectre d'une Variété Riemannienne*, Lecture Notes in Math. **194**, Springer-Verlag, 1971.

[7] D.E. Blair, *Contact manifolds in Riemannian Geometry*, Lecture Notes in Math. **509**, Springer-Verlag, Berlin, New York, 1976.

[8] D.E. Blair, L. Vanhecke, *New characterization of φ-symmetric spaces*, Kodai Math. J. **10** (1987), 102–107.

[9] D.E. Blair, L. Vanhecke, *Symmetries and φ-symmetric spaces*, Tohoku Math. J. **39** (1987), 373–382.

[10] W. Blaschke, *Vorlesungen über Differentialgeometrie, II. Affine Differential geometrie*, Springer-Verlag, Berlin, 1923.

[11] W. Blaschke, *Gesammelte Werke, Vol. 4. Affine Differentialgeometrie, Differentialgeometrie der Kreis- und Kugelgruppen*, Thales Verlag, Essen, 1985.

[12] N. Blažić and N. Bokan, *Compact Riemann surfaces with the skew-symmetric Ricci tensor*, Izvestiya VUZ Matematika and Soviet Math. **9(376)** (1993), 8–12.

[13] N. Blažić and N. Bokan, *Projective and affine transformations of a complex symmetric connection*, Bull. Austral. Math. Soc. **50** (1994), 337–347.

[14] N. Blažić and N. Bokan, *Invariance theory and affine differential geometry*, in: J. Janyska (ed.) et al., *Differential Geometry and Applications. Proc. 6th Internat. Conf., Brno*, Masaryk University, 1996, pp. 249–260.

[15] N. Blažić, N. Bokan and P.B. Gilkey, *Pontrjagin forms, Chern Simons classes, Codazzi transformations and affine hypersurfaces*, J. Geom. Phys. **27** (1998), 333–349.

[16] N. Blažić and N. Bokan, *Some topological obstructions to symmetric curvature operators*, Publ. Math. Debrecen **42** (1993), 357–368.

[17] N. Bokan, P. Gilkey and U. Simon, *Applications of spectral geometry to affine and projective geometry*, Beiträge Algebra Geom. **35** (1994), 283–314.

[18] N. Bokan, P. Gilkey and U. Simon, *Second order local affine invariants*, Izvestiya VUZ Matematika and Soviet Math. **5** (1995), 11–15.

[19] N. Bokan, P. Gilkey and U. Simon, *Relating the second and fourth order terms in the heat equation*, J. Geom. **59** (1997), 61–65.

[20] N. Bokan, P. Gilkey and U. Simon, *Geometry of differential operators on Weyl manifolds*, Proc. Roy. Soc. London Ser. A **453** (1996), 2527–2536.

[21] N. Bokan, P. Gilkey and U. Simon, *Spectral invariants of affine hypersurfaces*, Publ. Inst. Math. (Beograd) **64(78)** (1998), 133–145.

[22] N. Bokan, K. Nomizu, U. Simon, *Affine hypersurfaces with parallel cubic forms*, Tohoku Math. J. **42** (1990), 101–108.

[23] N. Bokan, *On the complete decomposition of curvature tensors of Riemannian manifolds with symmetric connection*, Rend. Circ. Mat. Palermo (3) **39** (1990), 331–380.

[24] N. Bokan, *The decomposition theory and its application*, in: G. M. Rassias (ed.), *The Mathematical Heritage of C. F. Gauss*, World Scientific, Singapore, 1991, pp. 66–99.

[25] N. Bokan, *C-holomorphically projective transformations in a normal almost contact manifold*, Tensor N.S. **33** (1979), 189–199.

[26] N. Bokan and P. Gilkey, *Asymptotics of Laplacians defined by Symmetric Connections*, Results in Math. **20** (1991), 589–599.

[27] N. Bokan, *Curvature tensors on Hermitian manifolds*, Colloquia Mathematica Societatis Janos Bolyai 46, Topics in Diff. Geom. Debrecen (Hungary (1984), 213–239.

[28] N. Bokan and S. Nikčević, *A characterization of projective and holomorphic projective structures*, Arch. Math. **62** (1994), 368-377.

[29] T. Branson and B. Orsted, *Conformal indices of Riemannian manifolds*, Comp. Math. **60** (1986), 261–293.

[30] R. Brooks, *Constructing Isospectral Problems*, Amer. Math. Monthly Nov. (1988), 823–839.

[31] W. Burau, U. Simon, *Blaschkes Beiträge zur affinen Differentialgeometrie*, In [11], 11–34.

[32] E. Calabi, *Complete affine hyperspheres I*, Symposia Math. **10** (1972), 19–38.

[33] E. Calabi, *Hypersurfaces with maximal affinely invariant area*, Amer. J. Math. **104** (1982), 91–126.

[34] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, 1984.

[35] S.S. Chern, *Geometry of characteristic classes*, Canadian Math. Congress, Proc. 13th. Biennial Seminar **1** (1971), 1–40.

[36] S.S. Chern, *On the curvatura integra in a Riemannian manifold*, Ann. of Math. **46** (1945), 674–684.

[37] S. Chern and J. Simons, *Characteristic forms and geometric invariants*, Ann. of Math. **99** (1974), 48–69.

[38] F. Dilen, L. Verstraelen and L. Vrancken, *Complex affine differential geometry*, Preprint Reihe Mathematik FB 3 TU Berlin, No. **153**, 1987.

[39] J. Dodziuk, *Eigenvalues of the Laplacian and the heat equation*, Amer. Math. Monthly **88** (1981), 686–695.

[40] T. Eguchi, P.B. Gilkey, and A.J. Hanson, *Gravitation, gauge theory, and differential geometry*, Physics Reports **6G** (1980), 213–393.

[41] L. Einsenhart, *Non-Riemannian Geometry*, 5th ed., Colloq. Publ. **8**, AMS, Providence, RI, 1964.

[42] G.B. Folland, *Weyl manifolds*, J. Diff. Geom. **4** (1970), 145–153.

[43] P.B. Gilkey, *The spectral geometry of a Riemannian manifold*, J. Diff. Geom. **10** (1975), 601–618.

[44] P.B. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah–Singer Index Theorem*, CRC Press, 1994.

[45] S.I. Goldberg, *Curvature and Homology*, Academic Press, New York, 1962.

[46] A. Gray and L.M. Hervella, *The sixteen classes of almost Hermitian manifolds*, Ann. Mat. Pura Appl. **123** (1980), 35–58.

[47] A. Gray, *Invariants of curvature operators of four-dimensional Riemannian manifolds*, Proc. Thirteenth Biennial Seminar Canad. Math. Congres, Halifax **2** (1971), 42–63.

[48] A. Gray and I.M. Hervella, *The sixteen classes of almost Hermitian manifolds*, Ann. Mat. Pura Appl. **123** (1980), 35–58.

[49] G.S. Hall, *Weyl manifolds and connections*, J. Math. Phys. **33** (1992), 2633–2638.

[50] G.S. Hall and Barry M. Haddow, *Some remarks on metric and Weyl connections*, J. Math. Phys. **36** (1995), 5938–5948.

[51] G.S. Hall, B.M. Haddow and J.R. Pulham, *Weyl connections, curvature and generalized gauge changes*, submitted.

[52] T. Higa, *Weyl manifolds and Einstein-Weyl manifolds*, Comm. Math. Univ. Sancti Pauli **42** (1993), 143–160.

[53] T. Higa, *Curvature tensors and curvature conditions in Weyl geometry*, Comment. Math. Univ. Sancti Pauli **43** (1994), 139–153.

[54] N.S. Hitchin, *Complex manifolds and Einstein's equation*, Lecture Notes in Math. 970, Springer-Verlag, Berlin (1982), 73–99.

[55] S. Ishihara, *Holomorphically projective changes and their groups in an almost complex manifold*, Tohoku Math. J. **9** (1957), 273–297.

[56] S. Ishihara, *Groups of projective transformations and groups of conformal transformations*, J. Math. Soc. Japan **9(2)** (1957), 195–227.

[57] D. I. Jonson, *Sectional curvature and curvature normal forms*, Michigan Math. J. **27** (1980), 275–294.

[58] M. Kac, *Can one hear the shape of a drum?* Amer. Math. Monthly **73** (1966), 1–23.

[59] S. Kobayashi and T. Ochiai, *Holomorphic projective structures on compact complex surfaces*, Math. Ann. **249** (1980) 75–94.

[60] S. Kobayashi and T. Nagano, *On Projective Connections*, J. Math. Mech. **13(2)** (1964), 215–235.

[61] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, vol. 1,2, Interscience, 1969.

[62] S. Kobayashi and T. Ochiai, *Holomorphic projective structures on compact complex surfaces*, Math. Ann. **249** (1980), 75–94.

[63] S. Kobayashi and T. Ochiai, *Holomorphic projective structures on compact complex surfaces.* II Math. Ann. **255** (1981), 519–521.

[64] S. Kobayashi, *Transformation Groups in Differential Geometry*, Springer-Verlag, Berlin, 1972.

[65] S. Kobayashi and C. Horst, *Complex differential geometry*, (DMV seminar: band 3) Birkhäuser, Basel, 1983.

[66] O. Kowalski, S. Wegrzynowski, *A classification of five-dimensional φ-symmetric spaces*, Tensor N.S. **46** (1987).

[67] O. Kowalski, *Partial curvature structures and conformal geometry of submanifolds*, J. Diff. Geom. (1) **8** (1973) 53–70.

[68] M. Kozlowski, *Affine maximal surfaces*, Anz. Österreich. Akad. Wiss. Math.-Natur. Kl. **124** (1987), 137–139.

[69] M. Kozlowski, *Surfaces defined by solutions of the Euler-Lagrange-Equation.* Anz. Österreich. Akad. Wiss. Math.-Natur. Kl. **125** (1988), 71–73.

[70] M. Kozlowski, *Improper affine spheres.* Anz. Österreich. Akad. Wiss. Math.-Natur. Kl. **125** (1988), 95–96.

[71] M. Kozlowski, *A class of affine maximal hypersurfaces.* Rend. Circ. Mat. Palermo **38** (1989), 444–448.

[72] M. Kozlowski, *The Monge-Ampère equation in affine differential geometry.* Anz. Österreich. Akad. Wiss. **126** (1989), 21–24.

[73] R.S. Kulkarni, *On the Bianchi identities*, Math. Ann. **199** (1972), 175–204.

[74] A.M. Li, G. Penn, *Uniqueness theorems in affine differential geometry*, Part II. Results in Mathematics **13** (1988), 308–317.

[75] M. Magid, P. Ryan, *Flat affine spheres*, Geom. Dedicata **33** (1990), 277–288.

[76] M. Magid, P. Ryan, *Affine 3-Spheres with Constant Affine curvature*, Transactions AMS (to appear).

[77] P. Matzeu, *Chern forms and C-projective curvature of normal almost contact manifolds*, An. Stiint. Univ. "Al. I. Cuza" Iasi, Sect. Ia Mat. **42** (1996), 193–200.

[78] P. Matzeu, *Decomposition of curvature tensors on normal almost contact manifolds*, Rend. Sem. Mat. Univ. Politecn. Torino **47(2)** (1989), 179–210.

[79] P. Matzeu, *Su una caratterizzazione delle varieta normali*, Bull. Unione Mat. Ital. VI **4** (1985), 153–160.

[80] P. Matzeu, *On the Geometry of Almost Contact Structures*, Univ. "Al. I. Cuza", doctoral thesis, 1990.

[81] P. Matzeu and S. Nikčević, *Linear algebra of curvature tensors on Hermitian manifolds*, Annal. Sciint. Univ. "Al. I. Cuza", **37** (1991), 71–86.

[82] A.S. Mischenko and A.T. Fomenko, *Kurs differencialnoj geometrii i topologii*, MGU, Moskva, 1980.

[83] J. Микеш, *Геодезические F-планарные и голоморфно-проективные отображения римановых пространств*, диссерт., Оломоуц, 1995.

[84] R.S. Millman, *Manifolds with the same spectrum*, Amer. Math. Monthly **90** (1983), 553–555.

[85] R. Milman, G. Parker, *Elements of Differential geometry*, Englewood Cliffs, Prentice Hall, 1997.

[86] J. Milnor, *Eigenvalues of the Laplace operator on certain manifolds*, Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 542.

[87] J. Milnor, *On the existence of a connection with curvature zero*, Comment. Math. Helv. **32** (1958), 215–223.

[88] H. Mori, *On the decomposition of generalized K-curvature tensor fields*, Tohoku Math. J. **25** (1973), 225–235.

[89] T. Nagano, *The projective transformation on a space with parallel Ricci tensor*, Kodai Math. Sem. Rep. **11** (1959), 131–138.

[90] A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. Math. **65** (1954), 391–404.

[91] S. Nikčević, *Conformal and projective invariants in Riemannian geometry*, preprint.

[92] S. Nikčević, *On the decomposition of curvature tensor fields on Hermitian manifolds*, Colloq. Math. Soc. Janos Bolyai **56** (1992), 555–568.

[93] S. Nikčević, *Induced representation of unitary group in tensor spaces of Hermite manifolds*, (preprint).

[94] S. Nikčević, *On the decomposition of curvature tensor*, Collection Sci. Papers Fac. Sci. Kragujevac **16** (1996), 61–68.

[95] K. Nomizu, *A survey of Recent Results in Affine Differential Geometry*, In: L. Verstraelen, A. West (Eds.), *Geometry and Topology of Submanifolds III*, Conf. Leeds 1990, World Scientific, London, Singapore, etc., 1991, pp. 227–256.

[96] K. Nomizu and F. Podestá, *On affine Kähler structures*, Bull. Soc. Math. Belg. **41** (1989), 275–282.

[97] K. Nomizu and T. Sasaki, *Affine Differential Gometry*, Cambridge Univ. Press, 1993.

[98] A.P. Norden, *Spaces of Desquartes composition*, Izvestiya VUZ **4** (1963), 117–128.

[99] A.P. Norden, *On structure of connections on line manifolds of noneuclidean spaces*, Izvestiya VUZ **12** (1972), 84–94.

[100] A.P. Norden, *Prostranstva afinoy svyznosti*, Nauka, Moskva, 1976.

[101] M. Obata, *Affine connections on manifolds with almost complex, quaternion or Hermitian structure*, Japan J. Math. **26** (1956), 43–77.

[102] V. Oliker and U. Simon, *Affine Geometry and Polar Hypersurfaces*, In: *Analysis and Geometry: Trends in Research and Teaching* (B. Fuchssteiner and W.A.J. Luxemburg, eds.), 87–112, Bibl. Inst. Mannheim–Zuerich, 1992.

[103] V. Oproiu, *The C-projective curvature tensor on 3-dimensional normal almost contact manifolds*, Tensor N.S. **46** (1987), 250–257.

[104] T. Parker and S. Rosenberg, *Invariants of conformal Laplacians*, J. Diff. Geom. **25** (1987), 199–222.

[105] E.M. Patterson, *A characterization of Kähler manifolds in terms of parallel fields of planes*, J. London Math. Soc. **28** (1953), 260–269.

[106] H. Pedersen, A. Swann, *Riemannian submersions, four manifolds and Einstein-Weyl geometry*, Proc. London Math. Soc. **66** (1991), 381–399.

[107] R. Penrose, W. Rindler, Spinors and spacetime. Two-spinor calculus and relativistic fields, Cambridge Monographs on Mathematical Physics, vol. 1. Cambridge University Press, 1984.

[108] U. Pinkall, A. Schwenk–Schellschmidt, and U. Simon, *Geometric methods for solving Codazzi and Monge–Ampére equations*, Math. Ann. **298** (1994), 89–100.

[109] W.A. Poor, *Differential Geometric Structures*, McGraw-Hill New York, 1981.

[110] P.K. Rasewskij, *Riemanowa geometria i tenzornij analiz*, Gostehizdat, Moskva, 1953.

[111] P.A. and A.P. Schirokow, *Affine Differentialgeometrie*, Teubner, Leipzig, 1962, (Zentralblatt für Mathematik 106.147; Russ. Original Zbl. 85.367).

[112] J.A. Schouten, *Ricci-Calculus*, 2nd ed., Springer-Verlag, Berlin, 1954.

[113] A. Schwenk, *Eigenwertprobleme des Laplace–Operators und Anwendungen auf Untermannigfaltigkeiten*, Dissertation, TU, Berlin, 1984.

[114] A. Schwenk, U. Simon, *Hypersurfaces with constant equiaffine mean curvature*. Archiv Math. 46 (1986), 85–90.

[115] U. Simon, *Kennzeichnungen von Sphären*, Math. Ann., **175**, (1968). 81–88.

[116] U. Simon, *Dirichlet problems and the Laplacian in affine hypersurface theory*, Lecture Notes Math. **1369**, Springer-Verlag, Berlin, 1989, pp. 243–260.

[117] U. Simon, *Zur Entwicklung der affinen Differentialgeometrie*. In W. Blaschke, *Gesammelte Werke, Vol. 4, Affine Differentialgeometrie. Differentialgeometrie der Kreis- und Kugelgrupen*. Thales Verlag, Essen 1985.

[118] U. Simon, *Connections and conformal structure in affine differential geometry*. In: D. Krupka, A. Svec (Eds); *Differential Geometry and its Applications*, J.E. Pyrkine Univ., CSSR, 1987, 315–327.

[119] U. Simon, *Dirichlet problems and the Laplacian in affine hypersurface theory*. Lecture Notes Math. **1369**, Springer Verlag, Berlin, 1989, 243–260.

[120] U. Simon, *Local classification of two-dimensional affine spheres with constant curvature metric*, Diff. Geom. Appl. 1, (1991), 123–132.

[121] U. Simon, *Hypersurfaces in equiaffine differential geometry*, Geom. Dedicata **17** (1984), 157–168.

[122] U. Simon, C.P. Wang, *Local Theory of Affine 2-Spheres*, Proc. 1990 Summer Inst. Diff. Geom (R.E. Greene, S.T. Yau; eds). Proc. Symp. Pure Math., **54**, part 3, (1993), 585–598.

[123] U. Simon, *Connections and conformal structure in affine differential geometry*, in *Differential geometry and its applications*, Proc. of the Conference, Brno, August 24-30, (D. Reidel, Dordrecht, 1986, pp. 315–328.

[124] U. Simon, A. Schwenk-Schellschmidt and H. Viesel, *Introduction to the Affine Differential Geometry of Hypersurfaces*, Lecture Notes, Science University Tokyo, 1991.

[125] I.M. Singer and J.A. Thorpe, *The curvature of 4-dimensional Einstein space*, Global Analysis (paper in honour of K. Kodaira), Univ. of Tokyo Press, Tokyo (1969), 355–365.

[126] M. Sitaramayya, *Curvature tensors in Kähler manifolds*, Tran. Amer. Math. Soc. **183** (1973), 341–351.

[127] R.S. Strichartz, *Linear algebra of curvature tensors and their covariant derivatives*, Canad. J. Math. **40** (1988), 1105–1143.

[128] A. Schwenk, *Affinsphären mit ebenen Schattengrenzen.* In: D. Ferus, R.B. Gardner, S. Helgason, U. Simon (Eds), *Global Differential Geometry and Global Analysis 1984*, Lecture Notes Math. **1156**, Springer-Verlag, 1985, pp. 296–315.

[129] Y. Tashiro, *On a holomorphically projective correspondences in an almost complex space*, Math. J. Okayama Univ. **6** (1957), 147–152.

[130] K.P. Tod, *Compact three-dimensional Einstein-Weyl structures*, J. London Math. Soc. **45** (1993), 341–351.

[131] F. Tricerri and L. Vanhecke, *Curvature tensors on almost Hermitian manifolds*, Trans. Amer. Math. Soc. (2) **267** (1981), 365–398.

[132] F. Tricerri and L. Vanhecke, *Homogeneous structures on Riemannian manifolds*, London Math. Soc., Lect. Notes **83**, Cambridge Univ. Press, 1983.

[133] L. Vrancken, A.-M. Li, U. Simon, *Affine spheres with constant sectional curvature*. Math. Z. **206** (1991), 651–658.

[134] Ch. Wang, *Some examples of complete hyperbolic affine 2-spheres in*$\mathbb{R}^3$. In: *Global Differential Geometry Global Analysis*, Proceedings, Berlin 1990. Lecture Notes Math. **1481**, Springer Verlag, Berlin etc. 1991, 272–280.

[135] H. Weyl, *Classical Groups, Their Invariants and Representations*, Princenton Univ. Press, Princeton, N. J., 1946.

[136] H. Weyl, *Zur Infinitesimalgeometrie, Einordnung der projetiven und der konformen Auffassung*, Gottinger Nachrichten, (1921), 99–112.

[137] H. Weyl, *Über die Asymptötische Verteilung der Eigenwerte*, Nachr. Königl. Ges. Wiss. Göttingen (1911), 110–117.

[138] H. Weyl, *Space-time Matter*, Dover, 1922.

[139] K. Yano, *Differential Geometry of Complex and Almost Complex Spaces*, Pergamon Press, New York, 1965.

[140] K. Yano and S. Bochner, *Curvature and Betti Numbers*, Princeton University Press, Pricenton, New Yersey, 1953.