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INTRODUCTION TO THE THEORY OF THE ITÔ-TYPE STOCHASTIC INTEGRALS AND STOCHASTIC DIFFERENTIAL EQUATIONS

CONTENTS

Preface

The theory of stochastic differential equations, as a part of the general theory of stochastic processes, began to develop in the fifties in the discussions of I.I. Gikhman, and independently of him, K. Itô. The accepted terminology, however, derived from Itô. In his papers [**15**], [**16**], [**17**], for special classes of stochastic processes he introduced the notion of the stochastic integral and of the stochastic differential equation with respect to a Wiener process. Following the classical theory of ordinary differential equations, he proved the fundamental theorem of the existence and uniqueness of solutions and also the Markov property of solutions. From then on this theory has developed in several aspects, mostly induced by mathematical abstractions or by many applications in technical practice, having in mind that a Gaussian white noise could be mathematically interpreted by a Wiener process.

One of the most important moments in the development of the theory of stochastic integrals and stochastic differential equations was the introduction of the notion of a martingale by Doob [**7**] and the subsequent establishment of the notion by Meyer [**34**], [**35**], [**36**]. In this way the fundamental supermartingale-decomposition theorem of Doob–Meyer [**36**] and the basic inequalities for martingales were established.

It is necessary to emphasize the notion of a stochastic integral with respect to a second-order martingale, introduced and studied by Kunita and Watanabe [**27**], which generalizes the stochastic Itô's integral. In fact, many properties of the Itô's integrals remain valid for this class of stochastic processes.

Later on, the theory of stochastic integrals and stochastic differential equations relative to other types of martingales and stochastic measures was developed ([**6**], [**29**]). Concurrently with it, the appropriate theory for a larger class of stochastic processes-semimartingales was introduced by Doléan-Dade and Meyer [**6**], and later essentially studied by Jacod [**22**] and Gikhman and Skorokhod [**11**].

The theory of stochastic differential equations had a permanent development with a large number of innovations, including some nonstandard constructions of stochastic integrals [**12**]. However, the Itô-calculus remains essential because several phenomena in technical, biological and social sciences can be modeled and described by stochastic differential equations of the Itô type. In fact, this theory is now applied in many diverse fields, which proves the flexibility of its application.

This paper represents an introduction to the study of the Itô-type calculus, as the initial information about the general theory of stochastic differential equations. Section 1 contains the basic theory of stochastic Itô's integrals, stating some important properties of stochastic indefinite integrals, introducing the stochastic differential and giving a differential formula known as the Itô's formula. In Section 2 the basic theory of Itô-type stochastic differential equations is established. The basic existence and uniqueness theorem, the Markov property and the continuous dependence on parameters of solutions are considered. Some simple examples are given to illustrate the preceding theoretical considerations.

There is a number of papers about stochastic differential equations. In the References some monographs and historically important papers are also given.

We shall restrict ourselves to the one-dimensional case for notational simplicity. The extension to the multidimensional case is not difficult in itself and it can be treated analogously.

1. Itô-type stochastic integrals

1.1. Definition of the Itô-type stochastic integral. Throughout the paper we suppose that all random variables and processes considered here are defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Let $w = (w_t, t \in R)$ be an one-dimensional standard Wiener process, adapted to the increasing family of sub-σ-algebras $(\mathcal{F}_t, t \in R)$, i.e., for all $s \leq t$ random variables w_s are $\mathcal{F}_t$-measurable, $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$ for $t_1 < t_2$ and $w_t - w_s$ is independent on $\mathcal{F}_s$ for all $t \geq s$. Further on, from this fact w is usually marked with $w = ((w_t, \mathcal{F}_t), t \in R)$.

Having in mind the definition of w, let us recall some of its more important properties: $w(0) = 0$ a.s.; $w_t - w_s : \mathcal{N}(0, |t-s|)$; it has independent increments; sample functions are continuous, but nowhere differentiable and they are of unbounded variation in every finite interval; it is a Markov process; $w_t - w_s = w_{t-s}$; $w = ((w_t, \mathcal{F}_t), t \in R)$ is a martingale, i.e., $E\{w_t | \mathcal{F}_s\} = w_s$ a.s. for $t \geq s$.

Moreover, because $((w_t^2 - t, \mathcal{F}_t), t \in R)$ is a martingale, w is a second-order martingale with the quadratic variation t. Indeed, for all $t \geq s$,

$$\begin{aligned} E\{w_t^2 - t | \mathcal{F}_s\} &= E\{w_s^2 + (w_t - w_s)^2 + 2w_s(w_t - w_s) | \mathcal{F}_s\} - t \\ &= w_s^2 + E(w_t - w_s)^2 - t = w_s^2 - s. \end{aligned}$$

Exactly, the martingale characteristics of the Wiener process play an important role in the construction of the Itô-type stochastic integrals.

In this section we shall define the Itô-type stochastic integral

$$I(\varphi) = \int_a^b \varphi(t)\, dw(t) \tag{1}$$

where $\varphi = (\varphi(t), t \in R)$ is a stochastic process, and we study its basic properties.

Since w is neither differentiable nor of the bounded variation, it is impossible to define (1) as an integral in the ordinary sense, i.e., as a Riemann–Stieltjes or a Lebesgue–Stieltjes integral. Recall that if φ is a nonrandom function, then (1) can be treated as a second-order stochastic integral (see, for example, [27], [28], [30], [31], [39], [45]). In this case only the fact that the Wiener process has orthogonal increments is used. The problem arises if φ is a random function, i.e., a stochastic process. Then the construction of the integral (1) depends on martingale properties of the Wiener process.

Furthermore, we shall suppose that the stochastic processes φ and w are independent.

Denote by $\mathcal{M}_2$ the class of stochastic processes with following properties: if $\varphi \in \mathcal{M}_2$, then

(i) φ is a measurable process, i.e., the function $(\omega, t) \to \varphi(\omega, t)$ is measurable with respect to $\mathcal{F}$ in ω and Lebesgue measurable in t;

(ii) φ is adapted to the family of sub-σ-algebras $(\mathcal{F}_t, t \in R)$, i.e., for each t, $\varphi(\omega, t)$ is measurable with respect to $\mathcal{F}_t$;

(iii) $\int_a^b E|\varphi(t)|^2\, dt < \infty$.

When (i) and (ii) hold, we say that φ is *nonanticipating* with respect to $(\mathcal{F}_t, t \in R)$.

Note that every deterministic function φ is a nonanticipating function. Also, if φ is a nonanticipating function, every product-measurable function $g(t, \varphi)$, defined on $(R \times C)$ into C, is nonanticipating.

We are now in a position to define the stochastic integral of a process $\varphi \in \mathcal{M}_2$ relative to w, following the ideas of Itô [15]. We shall do this gradually, in two phases. In the first phase we define the stochastic integral for step functions in $\mathcal{M}_2$; in the second phase we extend this definition to the entire set $\mathcal{M}_2$, approximating an arbitrary process from $\mathcal{M}_2$ with the sequence of step functions (see [1], [8], [9], [10], [28], [30], [45]).

Definition 1. A stochastic process $\varphi \in \mathcal{M}_2$ is called a step function if there exists a decomposition $a = t_0 < t_1 < t_2 \cdots < t_k = b$, independent of ω, such that

$$\varphi(\omega, t) = \varphi(\omega, t_\nu) \text{ a.s.}, \qquad t_\nu \le t < t_{\nu+1}, \quad \nu = 0, 1, \dots, k-1.$$

Definition 2. The stochastic integral of the step function $\varphi \in \mathcal{M}_2$ with respect to w is the random variable

$$\int_a^b \varphi(\omega, t)\, dw(\omega, t) := \sum_{\nu=0}^{k-1} \varphi(\omega, t_\nu)\, [w(\omega, t_{\nu+1}) - w(\omega, t_\nu)].$$

The following theorem makes it possible to define the stochastic integral for every $\varphi \in \mathcal{M}_2$.

Theorem 1. *Let $w = ((w_t, \mathcal{F}_t), t \in R)$, be a standard Wiener process and $\varphi \in \mathcal{M}_2$. Then:*

(a) There exists a sequence of step functions $\{\varphi_n, n \in N\}$ *such that*

$$\|\varphi - \varphi_n\|^2 = \int_a^b E|\varphi(t) - \varphi_n(t)|^2 \, dt \to 0 \quad \text{as} \quad n \to \infty;$$

(b) If a sequence of step functions $\{\varphi_n, n \in N\}$ *approximates* φ *in the sense* $\|\varphi - \varphi_n \to 0$ *as* $n \to 0$ *and if* $I(\varphi_n)$ *is defined as in Definition 1, then the sequence of random variables* $\{I(\varphi_n), n \in N\}$ *converges in q.m. as* $n \to \infty$;

(c) If $\{\varphi_n, n \in N\}$ *and* $\{\varphi_n', n \in N\}$ *are two sequences of step functions such that* $\|\varphi - \varphi_n\| \to 0$, $\|\varphi - \varphi_n'\| \to 0$ *as* $n \to \infty$, *then*

$$\text{q.m.} \lim_{n\to\infty} I(\varphi_n) = \text{q.m.} \lim_{n\to\infty} I(\varphi_n').$$

Proof. (a) If $\varphi \in \mathcal{M}_2$, but not obviously bounded a.s., we form the sequence of stochastic processes $\{f_n, n \in N\}$ such that

$$f_n(t) = \begin{cases} \varphi(t), & |\mathrm{Re}\,\varphi| \le n, \ |\mathrm{Im}\,\varphi| \le n, \\ n, & \text{otherwise} \end{cases}$$

By the dominated convergence theorem it follows that $\int_a^b E|\varphi(t) - f_n(t)|^2 \, dt \to 0$ as $n \to \infty$. So, further on we can always assume φ to be bounded a.s..

Suppose that φ is q.m. continuous. Then an approximating sequence of step functions $\{\varphi_n, n \in N\}$ can be constructed by an arbitrary decomposition of the segment $[a, b]$: $a = t_0^{(n)} < t_1^{(1)} < \cdots < t_k^{(n)} = b$, such that for $t_\nu^{(n)} \le t < t_{\nu+1}^{(n)}$ we have $\varphi_n(t) = \varphi(t_\nu^{(n)})$ a.s. and $\max_\nu [t_{\nu+1}^{(n)} - t_\nu^{(n)}] \to 0$ as $n \to \infty$. Since φ is q.m. continuous, then $E|\varphi(t) - \varphi_n(t)|^2 \to 0$ as $n \to \infty$ for every $t \in [a, b]$. By the dominated convergence theorem it follows

$$\int_a^b E|\varphi(t) - \varphi_n(t)|^2 \, dt \to 0 \quad \text{as} \quad n \to \infty.$$

If φ is bounded a.s., but not obviously q.m. continuous, we shall define the sequence of stochastic processes $\{g_n, n \in N\}$, where

$$g_n(t) = \int_0^\infty e^{-\tau} \varphi\Big(t - \frac{\tau}{n}\Big) \, d\tau.$$

It is easy to conclude that $g_n \in \mathcal{M}_2$, $n \in N$ and q.m. continuous on $[a, b]$. Since

$$\begin{aligned} \int_a^b E|\varphi(t) - g_n(t)|^2 \, dt &= \int_a^b E\Big| \int_0^\infty e^{-\tau} \Big[\varphi(t) - \varphi\Big(t - \frac{\tau}{n}\Big)\Big] \, d\tau \Big|^2 dt \\ &\le \int_a^b \int_0^\infty e^{-\tau} \, d\tau \int_0^\infty e^{-\tau} E\Big|\varphi(t) - \varphi\Big(t - \frac{\tau}{n}\Big)\Big|^2 \, d\tau \, dt \end{aligned}$$

and since $\int_a^b |\varphi(t) - \varphi\left(t - \frac{\tau}{n}\right)|^2 \, dt \to 0$ as $n \to \infty$ whenever φ is a.s. bounded and Lebesgue measurable, then

$$\int_a^b E|\varphi(t) - g_n(t)|^2 \, dt \to 0 \quad \text{as} \quad n \to \infty.$$

Now it is clear that we can construct indirectly a sequence of step functions approximating φ by sampling q.m. continuous stochastic processes $g_n, n \in N$ at the partition points of the segment $[a, b]$, such that the partitions go to zero as $n \to \infty$.

(b) Suppose that $\{\varphi_n, n \in N\}$ is a sequence of step functions such that $\|\varphi - \varphi_n\| \to 0$ as $n \to \infty$. Using Definition 2, let us define $I(\varphi_n)$ by

$$I(\varphi_n) = \int_a^b \varphi_n(t)\, dw(t) := \sum_\nu \varphi(t_\nu^{(n)})[\, w(t_{\nu+1}^{(n)}) - w(t_\nu^{(n)})\,].$$

Denote by $\Delta_\nu^{(n)} w = w(t_{\nu+1}^{(n)}) - w(t_\nu^{(n)})$. Then

$$E|I(\varphi_n)|^2 = \sum_\nu \sum_\mu E[\, \varphi(t_\nu^{(n)})\overline{\varphi(t_\mu^{(n)})}\, \Delta_\nu^{(n)} w \Delta_\mu^{(n)} w\,].$$

Since $E[\,\Delta_\nu^{(n)} w\, \Delta_\mu^{(n)} w\,] = E\Delta_\nu^{(n)} w\, E\Delta_\mu^{(n)} w = 0$ if $\nu \neq \mu$, we get

$$\begin{aligned} E|I(\varphi_n)|^2 &= \sum_\nu E|\varphi(t_\nu^{(n)})|^2\, E|\Delta_\nu^{(n)} w|^2 \\ &= \sum_\nu E|\varphi(t_\nu^{(n)})|^2 (\, t_{\nu+1}^{(n)} - t_\nu^{(n)}\,) = \int_a^b E|\varphi_n(t)|^2\, dt. \end{aligned}$$

Also, $E|I(\varphi_n)|^2 < \infty$. Since $I(\varphi_{n+m}) - I(\varphi_n) = I(\varphi_{n+M} - \varphi_n)$ and $\varphi_{n+m} - \varphi_n$ is again a step function, it follows

$$\begin{aligned} E|I(\varphi_{n+m}) - I(\varphi_n)|^2 &= E|I(\varphi_{n+m} - \varphi_n)|^2 = \int_a^b E|\varphi_{n+m}(t) - \varphi_n(t)|^2\, dt \\ &\leq 2\int_a^b E|\varphi_{n+m}(t) - \varphi(t)|^2\, dt + 2\int_a^b E|\varphi(t) - \varphi_n(t)|^2\, dt \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Hence, $\{I(\varphi_n), n \in N\}$ converges in q.m. because every Cauchy sequence of random variables is also q.m. convergent. It means that there exists a random variable $I(\varphi)$ such that $E|I(\varphi)|^2 < \infty$ and

$$E|I(\varphi) - I(\varphi_n)|^2 \to 0 \quad \text{as} \quad n \to \infty. \tag{2}$$

(c) Let $\{\varphi_n, n \in N\}$ i $\{\varphi_n', n \in N\}$ be two sequences of step functions approximating φ, i.e., $\|\varphi - \varphi_n\| \to 0$, $\|\varphi - \varphi_n'\| \to 0$ as $n \to \infty$. Because

$$\|\varphi_n - \varphi_n'\| \leq \sqrt{2}\,(\,\|\varphi_n - \varphi\|^2 + \|\varphi - \varphi_n'\|^2\,)^{\frac{1}{2}} \to 0 \quad \text{as} \quad n \to \infty,$$

then

$$E|I(\varphi_n) - I(\varphi_n')|^2 = \int_a^b E|\varphi_n(t) - \varphi_n'(t)|^2\, dt \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, q.m. $\lim_{n\to\infty} I(\varphi_n) =$ q.m. $\lim_{n\to\infty} I(\varphi_n')$. Thus the theorem is completely proved. □

Summarizing the results of the preceding theorem, we conclude that the stochastic integral $I(\varphi)$ can be defined as q.m. limit of the sequence of random variables $\{I(\varphi_n), n \in N\}$, i.e.,

$$I(\varphi) = \int_a^b \varphi(t)\,dw(t) := \text{q.m.} \lim_{n\to\infty} \int_a^b \varphi_n(t)\,dw(t).$$

This limit is in q.m. sense uniquely determined and independent of the choice of the sequence of step functions $\{\varphi_n, n \in N\}$ for which (2) holds.

Note that if a and b are not finite, the stochastic integral is defined as q.m. limit as $a \to \infty$ or $b \to \infty$.

The next theorem summarizes some of the more important properties of the stochastic integral.

Theorem 2. *Let $\varphi, \psi \in \mathcal{M}_2$ and α, β be arbitrary numbers. Then:*

(a) $I(\alpha\varphi + \beta\psi) = \alpha I(\varphi) + \beta I(\psi)$;

(b) $EI(\varphi) = 0$;

(c) $EI(\varphi)\overline{I(\psi)} = \int_a^b E\varphi(t)\overline{\psi(t)}\,dt$.

Proof. (a) This part follows immediately from the construction of the stochastic integral of step functions.

(b) The proof is obvious if $\varphi \in \mathcal{M}_2$ is a step function. If not, let $\{\varphi_n\}$ be a sequence of step functions approximating φ in q.m., i.e., $E|\varphi(t) - \varphi_n(t)|^2 \to 0$ as $n \to \infty$ on $[a, b]$. Since by Theorem 1b

$$0 \le (EI(\varphi) - EI(\varphi_n))^2 \le E|I(\varphi) - I(\varphi_n)|^2 \to 0 \quad \text{as} \quad n \to \infty,$$

then $EI(\varphi) = 0$.

(c) It is enough to prove that $E|I(\varphi)|^2 = \int_a^b E|\varphi(t)|^2\,dt$ because

$$\begin{aligned} EI(\varphi)\overline{I(\psi)} &= \frac{1}{4}[\,E|I(\varphi+\psi)|^2 - E|I(\varphi-\psi)|^2\,] \\ &\quad + \frac{i}{4}[\,E|I(-i\varphi+\psi)|^2 - E|I(-i\varphi-\psi)|^2\,]. \end{aligned}$$

If φ is a step function, the proof directly follows from the proof of Theorem 1b. If not, let $\{\varphi_n, n \in N\}$ be a sequence of step functions approximating φ in q.m. Then

$$\begin{aligned} E|I(\varphi)|^2 &= E|I(\varphi - \varphi_n) + I(\varphi_n)|^2 \\ &= E|I(\varphi - \varphi_n)|^2 + 2\mathrm{Re}\,EI(\varphi - \varphi_n)\overline{I(\varphi_n)} + E|I(\varphi_n)|^2, \end{aligned}$$

and therefore

$$\begin{aligned} E|I(\varphi)|^2 &= \lim_{n\to\infty} E|I(\varphi_n)|^2 = \lim_{n\to\infty} \int_a^b E|\varphi_n(t)|^2\,dt \\ &= \int_a^b \lim_{n\to\infty} E|\varphi_n(t)|^2\,dt = \int_a^b E|\varphi(t)|^2\,dt. \quad \square \end{aligned}$$

The notion of the stochastic integral of the Itô type can be introduced under some weaker conditions (see, for example, [1], [8], [10], [28], [30], [39], [45]). Thus, denote by $\mathcal{P}$ a class of stochastic processes, measurable and adapted to the family of sub-σ-algebras $(\mathcal{F}_t, t \in R)$, satisfying the condition

$$P\Big\{ \int_a^b |\varphi(t)|^2 dt < \infty \Big\} = 1.$$

Clearly, $\mathcal{M}_2 \subset \mathcal{P}$.

Theorem 3. *Let $((w_t, \mathcal{F}_t), t \in R)$ be a standard Wiener process and let $\varphi \in \mathcal{P}$. Let also φ_n be defined by*

$$\varphi_n(t) = \begin{cases} \varphi(t), & \int_a^b |\varphi(t)|^2 \, dt \le n, \\ 0, & \textit{otherwise} \end{cases}$$

and let $I(\varphi_n)$ denote the stochastic integral $I(\varphi_n) = \int_a^b \varphi_n(t) \, dw(t)$. Then $\{I(\varphi_n), n \in N\}$ converges in probability as $n \to \infty$.

Proof. Let φ_n be defined as the above. Then $\varphi_n \in \mathcal{M}_2$ and $I(\varphi_n)$ is well defined. Now for arbitrary $m, n \in N$ and for any $\omega \in \Omega$, such that

$$\int_a^b |\varphi(t)|^2 \, dt \le \min\{m, n\},$$

we obtain $\sup_{t \in [a,b]} |\varphi_n(t) - \varphi_m(t)| = 0$. So, $\int_a^b \varphi_n(t) \, dt = \int_a^b \varphi_m(t) \, dt$ a.s. For every $\epsilon > 0$ it follows that

$$P\{ |I(\varphi_n) - I(\varphi_m)| \ge \epsilon \} \le P\Big\{ \int_a^b |\varphi(t)|^2 dt > \min\{m, n\} \Big\} \to 0 \text{ as } m, n \to \infty,$$

which implies in turn that $\{I(\varphi_n), n \in N\}$ converges in probability since every Cauchy sequence of random variables also converges in probability. □

Therefore, under the conditions of the preceding theorem there exists a random variable $I(\varphi)$ such that $I(\varphi_n) \to I(\varphi)$ in probability as $n \to \infty$. In other words, we can define the stochastic integral

$$I(\varphi) := \text{p.} \lim_{n \to \infty} I(\varphi_n).$$

The notion of the Itô-type stochastic integral can be analogously generalized to the $(n \times m)$-matrix valued stochastic process $\varphi = [\varphi_{ij}]_{n \times m}$, where $\varphi_{ij} \in \mathcal{M}_2$ or $\varphi_{ij} \in \mathcal{P}$, with respect to the m-dimensional standard Wiener process $w = ((w_t, \mathcal{F}_t), t \in R)$, $w_t - w_s : \mathcal{N}(0, |t - s| I)$. The matrix φ has the norm

$$|\varphi| = \Big(\sum_{i=1}^n \sum_{j=1}^m |\varphi_{ij}|^2 \Big)^{1/2} = (\operatorname{tr} \varphi \overline{\varphi'})^{1/2}.$$

Clearly, in this case $I(\varphi)$ is the n-dimensional random variable.

1.2. The stochastic indefinite integral. Denote by $I_{\{s<t\}}$, $a \le s < t < b$, an indicator of the set $[a,t]$ which is obviously $\mathcal{F}_t$-measurable. This fact gives a possibility to introduce a notion of a stochastic indefinite integral.

Definition 3. The stochastic indefinite integral of the process $\varphi \in \mathcal{M}_2$ is the stochastic process $x = (x(t), t \in [a,b])$, defined by

$$x(t) := \int_a^b I_{\{s<t\}}\varphi(s)\,dw(s) = \int_a^t \varphi(s)\,dw(s), \quad t \in [a,b].$$

Having in mind the construction of the Itô-type stochastic integral $I(\varphi)$, the indefinite stochastic integral possesses the following important properties:

(i) x is defined uniquely up to the stochastic equivalence with its separable and measurable modification (Doob's theorem - see [7], [45]);

(ii) $x(t)$ is $\mathcal{F}_t$-measurable for every $t \in [a,b]$;

(iii) $x(a) = 0$ a.s.;

(iv) $x(t) - x(s) = \int_s^t \varphi(u)\,dw(u)$, $t, s \in [a,b]$.

Using the results of Theorem 2, for every $t \in [a,b]$ it follows:

(v) $Ex(t) = 0$;

(vi) $E|x(t)|^2 = \int_a^t E|\varphi(s)|^2\,ds$.

Theorem 4. *If $\varphi \in \mathcal{M}_2$, then $((x_t, \mathcal{F}_t), t \in [a,b])$, is a martingale.*

Proof. Let φ be a step function and $s < t_1 < t_2 < \cdots < t_n < t$. Then

$$\begin{aligned} x(t) - x(s) &= \int_s^t \varphi(u)\,dw(u) \\ &= \varphi(s)[w(t_1) - w(s)] + \varphi(t_1)[w(t_2) - w(t_1)] + \cdots + \varphi(t_n)[w(t) - w(t_n)]. \end{aligned}$$

Therefore, by successively taking conditional expectations, we obtain

$$\begin{aligned} E\{x(t) - x(s)|\mathcal{F}_s\} &= E\{E\{\cdots E\{x(t) - x(s)|\mathcal{F}_{t_n}\}|\mathcal{F}_{t_{n-1}}\}|\cdots \mathcal{F}_{t_1}\}|\mathcal{F}_s\} \\ &= \cdots = E\{x(t_1) - x(s)|\mathcal{F}_s\} = E\varphi(s)\,E(w(s) - w(s)) = 0. \end{aligned}$$

In the following part of the proof we use the well-known convergence property of conditional expectation (see [45]): for $\nu \ge 1$ if the sequence of stochastic variables $X_n \overset{\nu.m.}{\to} X$ as $n \to \infty$, then $E(X_n|\mathcal{F}) \overset{\nu.m.}{\to} E(X|\mathcal{F})$ as $n \to \infty$.

If φ is not a step function, let $\{\varphi_n, n \in N\}$ be a sequence of step functions approximating φ. Denote by $x_n(t) = \int_a^t \varphi_n(s)\,dw(s)$. Then for every $t \in [a,b]$ we have $E|x(t) - x_n(t)|^2 \to 0$ as $n \to \infty$, and therefore $E\{x(t) - x_n(t)\,|\mathcal{F}_s\} \to 0$ as $n \to \infty$. Now for all $t > s$

$$\begin{aligned} &E\{x(t) - x(s)|\mathcal{F}_s\} \\ &\qquad = E\{x(t) - x_n(t)|\mathcal{F}_s\} + E\{x_n(t) - x(s)|\mathcal{F}_s\} \to 0 \quad \text{as} \quad n \to \infty. \quad \square \end{aligned}$$

Moreover, one can be show that $((x_t, \mathcal{F}_t), t \in [a,b])$, is *a second-order martingale* with the quadratic variation

$$u(t) = \int_a^t |\varphi(u)|^2 \, du.$$

In order to do this, recall an important property of second order martingales.

Let $((z_t, \mathcal{F}_t), t \in [a,b])$ be a sample continuous second-order martingale. Then by the supermartingale-decomposition theorem of Doob and Meyer (see **[36]**, and also **[10]**, **[11]**, [30], [31], [34], [35]), there exist both a sample continuous martingale $((m_t, \mathcal{F}_t), t \in [a,b])$ and a sample continuous increasing process $((u_t, \mathcal{F}_t, t \in [a,b])$ — called *the quadratic variation*, with $u(a) = 0$ a.s. and $Eu(b) < \infty$, such that

$$z^2(t) - u(t) = m(t) \ \text{ a.s.}, \quad t \in [a,b].$$

Also, the following inequality, defined first by Doob [7], and in different variations by Meyer **[36]** and others, holds: for $1 < \alpha < \infty$,

$$E\{\sup_{t\in[a,b]} |z(t)|^\alpha\} \le \Big(\frac{\alpha}{\alpha-1}\Big)^\alpha E|z(b)|^\alpha.$$

For $\alpha = 2$ and $\varphi \in \mathcal{M}_2$ we get

$$\sup_{t\in[a,b]} E|x(t)|^2 \le E\{\sup_{t\in[a,b]} |x(t)|^2\} \le 4\int_a^b E|\varphi(t)|^2 \, dt < \infty.$$

Next, $u(t)$ is $\mathcal{F}_t$-measurable for every $t \in [a,b]$, non-negative and increasing a.s., $u(a) = 0$ a.s. and $Eu(t) \le Eu(b) < \infty$. For all $t \ge s$ we obtain

$$E\{x^2(t) - u(t)|\mathcal{F}_s\} = E\Big\{\Big(\int_a^t \varphi(u)\,dw(u)\Big)^2 \Big|\mathcal{F}_s\Big\} - E\Big\{\int_a^t \varphi^2(u)\,du \Big|\mathcal{F}_s\Big\}$$

$$= E\Big\{\Big(\int_a^s \varphi(u)\,dw(u)\Big)^2 \Big|\mathcal{F}_s\Big\} - E\Big\{\int_a^s \varphi^2(u)\,du \Big|\mathcal{F}_s\Big\}$$

$$+E\Big\{\Big(\int_s^t \varphi(u)\,dw(u)\Big)^2 \Big|\mathcal{F}_s\Big\} - E\Big\{\int_s^t \varphi^2(u)\,du \Big|\mathcal{F}_s\Big\} = x^2(s) - u(s).$$

So, $u(t)$ is the quadratic variation of the martingale $((x_t, \mathcal{F}_t), t \in [a,b])$.

Recall that for $\varphi \in \mathcal{M}_2$ the inequality

$$E\Big\{\sup_{t\in[a,b]} \Big|\int_a^t \varphi(s)ds\Big|\Big\} \le 4\int_a^b E|\varphi(t)|^2 \, dt \tag{3}$$

is also known as Doob's inequality for Itô-type integrals.

Example: The formal application of the classical rules for the integration by parts yields

$$\frac{1}{2}\int_0^t w(s)\,dw(s) = w^2(t).$$

Clearly, it is not correct because for $t > s$

$$E\{w^2(t)\,|\mathcal{F}_s\} = w^2(s) - t + s \neq w^2(s),$$

and therefore $w^2(t)$ is not a martingale.

Also, it can be proved (see, for example, [8], [11], [25]) that if $\varphi \in \mathcal{M}_2$ and τ is a stopping time with respect to $(\mathcal{F}_t, t \in [a,b])$, i.e., $a \le \tau \le b$ a.s. and $\{\tau \le t\} \in \mathcal{F}_t$ for all $t \in [a,b]$, then the process

$$\int_a^{\tau\wedge t} \varphi(s)\, dw(s), \quad a \le t \le b,$$

is a martingale and $E\int_a^{\tau\wedge t} \varphi(s)\, dw(s) = 0$.

Theorem 5. *If $\varphi \in \mathcal{M}_2$, then $x = (x(t), t \in [a,b])$ is a continuous process.*

Proof. Let φ be a step function with a decomposition $a < t_1 < t_2 \cdots < t_n < t$. Then

$$x(t) = \varphi(a)[w(t_1) - w(a)] + \cdots + \varphi(t_n)[w(t) - w(t_n)].$$

Obviously, a.s. continuity of x follows from a.s. continuity of the Wiener process.

If φ is not a step function, let $\{\varphi_n, n \in N\}$ be a sequence of step functions approximating φ, i.e., $\int_a^b E|\varphi(t) - \varphi_n(t)|^2 dt \to 0$ as $n \to \infty$. By Chebyshev's inequality and Doob's inequality (3), it follows that

$$P\Big\{\sup_{t\in[a,b]}\Big|\int_a^t \varphi(s)\, dw(s) - \int_a^t \varphi_n(s)\, dw(s)\Big| > \epsilon\Big\} \le \frac{4}{\epsilon^2}\int_a^b E|\varphi(s) - \varphi_n(s)|^2\, ds.$$

Next, we can choose $\epsilon_k > 0$ such that $\epsilon_k \to 0$ as $n \to \infty$, and $\{n_k, k \in N\}$ in such a way that $n_k \nearrow$ if $k \to \infty$, (for example, $\epsilon_k = 2^{-k}$, $n_k = k^{-2}$), for which

$$\sum_{k=1}^{\infty} \frac{1}{\epsilon_k^2}\int_a^b E|\varphi(s) - \varphi_{n_k}(s)|^2\, ds < \infty.$$

Since

$$\sum_{k=1}^{\infty} P\Big\{\sup_{t\in[a,b]}\Big|\int_a^t \varphi(s)\, dw(s) - \int_a^t \varphi_{n_k}(s)\, dw(s)\Big| > \epsilon_k\Big\} \le \infty,$$

the Borel–Cantelli's lemma implies that

$$\sup_{t\in[a,b]}\Big|\int_a^t \varphi(s)\, dw(s) - \int_a^t \varphi_{n_k}(s)\, dw(s)\Big| \le \epsilon_k \quad \text{a.s.}$$

for all $t \in [a,b]$ if $k \ge k_0(\omega)$, i.e.,

$$\sup_{t\in[a,b]}\Big|\int_a^t \varphi(s)\, dw(s) - \int_a^t \varphi_n(s)\, dw(s)\Big| \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, the integral $\int_a^t \varphi(s)\,dw(s)$ is a.s. uniform limit on $[a,b]$ of the sequence of a.s. continuous stochastic processes $\left\{\int_a^t \varphi_n(s)\,dw(s), t \in [a,b], n \in M\right\}$ and, because of that, it is itself a.s. continuous.

Moreover, x has a.s. continuous sample functions (see [1], [8], [10]). □

Analogously to the stochastic indefinite integral for a process $\varphi \in \mathcal{M}_2$, it is possible to define the Itô-type indefinite integral for $\varphi \in \mathcal{P}$ with

$$x(t) := \int_a^b \varphi(s) I_{\{s<t\}}\,dw(s) = \int_a^t \varphi(s)\,dw(s).$$

In this case the process $x = (x(t), t \in [a,b])$ is measurable, adapted to the family of sub-σ-algebras $(\mathcal{F}_t, t \in [a,b])$, a.s. continuous, but in general *it is not a martingale.* It can be shown that it is *a local martingale* (see, for example, [25], [30], [31], first of all [27]). Remember that if we denote by τ_n the stopping time

$$\tau_n = \min_t \left\{ \int_a^t |\varphi(s)|^2 ds \geq n \right\},$$

then since $\tau_n \nearrow b$ as $n \to \infty$, it can be proved that $((x_n(t \wedge \tau_n), \mathcal{F}_t), t \in [a,b])$ is a martingale for every $n \in N$. By definition $((x_t, \mathcal{F}_t), t \in [a,b])$ is said to be a local martingale.

1.3. The Itô's formula.

In order to determine effectively some classes of stochastic indefinite integrals and to obtain explicit solutions of some types of stochastic differential equations, it is necessary to use *the Itô's formula*, so called *the Itô's differential rule.*

Let $(a(t), t \in [a,b])$ and $(b(t), t \in [a,b])$ be measurable processes adapted to the family of sub-σ-algebras $(\mathcal{F}_t, t \in [a,b])$, such that

$$\int_a^b |a(t)|\,dt < \infty \quad \text{a.s.}, \qquad \int_a^b |b(t)|^2\,dt \leq \infty \quad \text{a.s.}.$$

Then the stochastic process

$$x(t) = x(a) + \int_a^t a(u)\,du + \int_a^u b(u)\,dw(u)$$

is called *the Itô's process.* It is measurable, adapted to $(\mathcal{F}_t, t \in [a,b])$ and a.s. continuous. Here $x(a)$ is a random variable, $\mathcal{F}_a$-measurable and independent of $w(t) - w(a)$ for all $t \geq a$.

Definition 4. If for every s, t such that $a \leq s < t \leq b$,

$$x(t) - x(s) = \int_s^t a(u)\,du + \int_s^t b(u)\,dw(u) \quad \text{a.s.},$$

then the stochastic process x has *the stochastic differential $dx(t)$* on $[a,b]$, given by

$$dx(t) = a(t)\,dt + b(t)\,dw(t).$$

One can easily conclude that $x(t)$ is measurable, adapted to $(\mathcal{F}_t, t \in [a,b])$ and a.s. continuous.

Theorem 6. (The Itô's formula) *Let* $dx(t) = a(t)\,dt + b(t)\,dw(t)$ *and let* $f(t,x)$ *be a nonrandom function defined on* $[a,b] \times R$*, continuous together with its derivatives* f'_t, f'_x, f''_{xx}. *Then the process* $f(t,x(t))$ *has the stochastic differential, given by*

$$df(t,x(t)) = f'_t(t,x(t))\,dt + f'_x(t,x(t))\,dx(t) + \frac{1}{2} f''_{xx}(t,x(t))\,b^2(t)\,dt.$$

For the proof see [1], [8], [9], [11], [28], [30], for example, and first of all [17].

In this formula the surprise is the last term because by the standard calculus formula for total derivatives the term $\frac{1}{2} f''_{xx}(t,x(t))\,b^2(t)\,dt$ would not appear. This correction term arises from the nondifferentiability of the Wiener process. Since

$$\begin{aligned} df(t,x) &\approx f(t+dt, x+dx) - f(t,x) \\ &\approx f'_t(t,x)\,dt + f'_x(t,x)\,dx + \frac{1}{2} f''_{xx}(t,x)\,(dx)^2, \end{aligned}$$

and $Ew^2(t) = t$, we obtain $(dw(t))^2 \approx dt$. So,

$$(dx(t))^2 = [\,a(t)\,dt + b(t)\,dw(t)\,]^2 \approx b^2(t)\,dt.$$

Note that the Itô's formula asserts the two processes: $f(t,x(t)) - f(a,x(a))$ and

$$\int_a^t [\,f'_s(s,x(s)) + f'_x(s,x(s))\,a(s) + \frac{1}{2} f''_{xx}(s,x(s))\,b^2(s)\,]\,ds + \int_a^t f'_x(s,x(s))\,b(s)\,dw(s),$$

which are stochastically equivalent.

Now we are in a position to find the integral $\int_0^t w(s)\,dw(s)$. Since $w(t)$ has the stochastic differential for $a \equiv 0$, $b \equiv 1$, applying the Itô's formula to the function $f(x) = x^2$, we have $dw^2(t) = dt + 2w(t)\,dw(t)$. Thus we obtain

$$\int_0^t w(t)\,dw(t) = \frac{1}{2} w^2(t) - \frac{1}{2} t,$$

which is a martingale.

The Itô's formula can be used to estimate some types of stochastic integrals. Thus, for a process $((\varphi_t, \mathcal{F}_t), t \in [0,T])$, such that $|\varphi(t)| \leq K$ a.s. for all $t \in [a,b]$, by applying the Itô's formula to the function $f(x) = x^{2m}$, $m \in N$, we obtain (see [8], [30])

$$E\Big(\int_0^t \varphi(s)\,dw(s)\Big)^{2m} \leq K^{2m}(2m-1)!!\,t^m.$$

If φ is unbounded a.s., but $\int_0^T E\varphi^{2m}(t)\,dt < \infty$, then (see [8], [28], [30])

$$E\Big(\int_0^t \varphi(s)\,dw(s)\Big)^{2m} \leq [m(2m-1)]^m t^{m-1} \int_0^t E\varphi^{2m}(s)\,ds.$$

The Itô's formula can be easily generalized to a function $f(t, x_1, x_2, \ldots, x_n)$, defined on $[a,b] \times R^n$, continuous together with its derivatives $f'_t, f'_{x_k}, f''_{x_k x_j}$, $1 \le k, j \le n$. If $dx_k(t) = a_k(t)\,dt + b_k(t)\,dw(t)$, $1 \le k \le n$, then the process $f(t, x_1(t), \ldots, x_n(t))$ has the stochastic differential

$$df(t,x(t)) = f'_t(t,x(t))\,dt + \sum_{k=1}^{n} f'_{x_k}(t,x(t))\,dx_k(t) + \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} f''_{x_k x_j}(t,x(t))\,b_k(t)\,b_j(t)\,dt,$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ (see earlier cited references).

Thus, if the stochastic processes $x_1(t)$ and $x_2(t)$ have the stochastic differentials $dx_i(t) = a_i(t)\,dt + b_i(t)\,dw(t)$, $i = 1, 2$, then the product $x_1(t)x_2(t)$ has the stochastic differential

$$d(x_1(t)x_2(t)) = x_1(t)\,dx_2(t) + x_2(t)\,dx_1(t) + b_1(t)\,b_2(t)\,dt \qquad (4)$$
$$= [\,x_1(t)\,a_2(t) + x_2(t)\,a_1(t) + b_1(t)\,b_2(t)\,]\,dt + [\,x_1(t)\,b_2(t) + x_2(t)\,b_1(t)\,]\,dw(t).$$

The most important role of the Itô's calculus is that it can be generalized to a stochastic integral, replacing the Wiener process by a more general one. For example, let $((z_t, \mathcal{F}_t), t \in [a,b])$ be a sample-continuous second-order martingale. Then by the supermartingale-decomposition theorem of Meyer (see [**36**]) there exists a sample-continuous a.s. increasing process $((u_t, \mathcal{F}_t), t \in [a,b])$ with $u(a) = 0$ a.s., such that for a stochastic process $(\varphi(t), t \in [a,b])$, measurable, adapted to $(\mathcal{F}_t, t \in [a,b])$ and

$$\int_a^b \varphi^2(t)\,du(t) < \infty \quad \text{a.s.},$$

analogously to the procedure in Theorem 1, the Itô's integral (see [**27**])

$$I(\varphi) = \int_a^b \varphi(t)\,dz(t)$$

can be defined with the help of step functions φ_n, as

$$I(\varphi_n) := \sum_{\nu} \varphi(t_\nu^{(n)})[\,z(t_{\nu+1}^{(n)}) - z(t_\nu^{(n)})\,],$$

where $I(\varphi_n) \xrightarrow{P} I(\varphi)$ as $n \to \infty$.

The stochastic indefinite integral $x(t) = \int_a^t \varphi(s)\,dz(s)$ can be defined adequately.

If the process $x(t)$ has the stochastic differential $dx(t) = a(t)\,dt + b(t)\,dz(t)$, then the analogue Itô's formula, first proved in [**27**], has the form

$$df(t,x(t)) = f'_t(t,x(t))\,dt + f'_x(t,x(t))\,dx(t) + \frac{1}{2}\,f''_{xx}(t,x(t))\,b^2(t)\,du(t).$$

2. Stochastic differential equations

2.1. Definition of the Itô-type stochastic differential equation. The stochastic differential equation (shorter SDE) of an unknown n-dimensional process $x = (x(t), t \in [t_0, T])$ with the initial value η is given by

$$dx(t) = a(t, x(t))\, dt + b(t, x(t))\, dw(t), \quad x(t_0) = \eta \text{ a.s.}, \quad t \in [t_0, T], \tag{5}$$

where $w = (w_t, t \in R)$ is an m-dimensional Wiener process, η is an n-dimensional random variable, stochastically independent of w in the sense that random variables w_t and η are stochastically independent for all t, and $a : [t_0, T] \times R^n \to R^n$, $b : [t_0, T] \times R^n \to R^n \times R^m$ are non-random functions, Borel-measurable on their domains.

Because of simplicity, we shall confine ourselves to the one-dimensional case. So, x, w and η are one-dimensional, and $a : [t_0, T] \times R \to R$, $b : [t_0, T] \times R \to R$.

Denote by $\mathcal{F}_t$ the σ-algebra generated by η and w_t, i.e. the smallest σ-algebra with respect to which η and the random variables $w_s, s \le t$, are measurable, such that $w_t - w_s$ is independent on $\mathcal{F}_s$ for all $t \ge s$. Thus the Wiener process w is adapted with respect to the increasing family of sub-σ-algebras $(\mathcal{F}_t, t \in [t_0, T])$, and η is $\mathcal{F}_{t_0}$-measurable.

Denote by $\mathcal{P}$ the space of stochastic processes $\varphi = (\varphi(t), t \in [t_0, T])$, measurable and adapted to $(\mathcal{F}_t,\ t \in [t_0, T])$, such that

$$P\Big\{ \int_{t_0}^{T} |\varphi(t)|^2 dt < \infty \Big\} = 1.$$

Definition 5. The measurable stochastic process $x = (x(t), t \in [t_0, T])$ is a strong solution of the SDE (5) if:

(i) $x(t)$ is $\mathcal{F}_t$-measurable for each $t \in [t_0, T]$;

(ii) $\overline{a}(t) = a(t, x(t))$, $\overline{b}(t) = b(t, x(t))$, such that

$$\int_{t_0}^{T} |\overline{a}(t)|\, dt < \infty, \qquad \int_{t_0}^{T} |\overline{b}(t)|^2\, dt < \infty \quad \text{a.s.};$$

(iii) $x(t_0) = \eta$ a.s.;

(iv) the equation (5) holds a.s. for each $t \in [t_0, T]$.

Since $dx(t) = \overline{a}(t)\, dt + \overline{b}(t)\, dW(t)$ a.s. for all $t \in [t_0, T]$, this is, therefore, the stochastic differential of the process x.

The SDE (5) has the equivalent integral form

$$x(t) = \eta + \int_{t_0}^{t} a(s, x(s))\, ds + \int_{t_0}^{t} b(s, x(s))\, dw(s), \quad t \in [t_0, T]. \tag{6}$$

Because of (i) and (ii) from Definition 5, the integrals on the right-hand side of (6) are well defined: since $\overline{b} \in \mathcal{P}$, then $\int_{t_0}^{t} \overline{b}(s)\, dW(s)$ is the Itô-type stochastic integral; since $\overline{a}$ is measurable and absolutely integrable random function adapted to $(\mathcal{F}_t, t \in [t_0, T])$, $\int_{t_0}^{t} \overline{a}(s)\, d(s)$ exists as the Lebesgue integral with the parameter ω. Both integrals are defined uniquely up to the stochastic equivalence and therefore the solution x is also determined up to the stochastic equivalence.

Moreover, since both integrals in (6) are a.s. continuous, then x is a.s. continuous. For this, by Doob's theorem [7] we shall always assume that we have chosen a measurable, separable and a.s. continuous version of the strong solution.

Definition 6. The SDE (6) has a unique strong solution if for any two strong solutions x_1 and x_2,

$$P\{\, \omega : x_1(t) = x_2(t),\ t \in [t_0, T]\,\} = 1.$$

This is equivalent to $P\{\sup_{t\in[t_0,T]} |x_1(t) - x_2(t)| > 0\} = 0$.

Example. Solving formally the SDE

$$dx(t) = x(t)\, dw(t), \quad x(0) = \eta \text{ a.s.}, \quad t \geq 0,$$

as an ordinary differential equation, we obtain $x(t) = \eta e^{w(t)}$. By applying the Itô's formula, we get

$$dx(t) = \eta e^{w(t)}\, dw(t) + \frac{1}{2}\eta e^{w(t)}\, dt \neq x(t)\, dw(t).$$

Therefore, the solution must have some other form. We shall express as $x(t) = \eta e^{w(t)+\varphi(t)}$, where φ is an unknown function. Using again the Itô's formula, we obtain

$$\begin{aligned} dx(t) &= \eta e^{w(t)+\varphi(t)}\, \varphi'(t)\, dt + \eta e^{w(t)+\varphi(t)}\, dw(t) + \frac{1}{2}\eta e^{w(t)+\varphi(t)}\, dt \\ &= x(t)[\,\varphi'(t) + 1/2\,]dt + x(t)\, dw(t). \end{aligned}$$

So, $\varphi'(t) + 1/2 = 0$, i.e., $\varphi(t) = -1/2 + c$, $c =$ const. The initial condition easily gives $c = 0$. Thus, $x(t) = \eta e^{w(t)-t/2}$, $t \geq 0$.

2.2. Existence and uniqueness of a solution. Following the ideas of Itô [16] we give the basic existence and uniqueness theorem of a solution of the SDE (6).

Theorem 7. *Let $w = (w_t, t \in R)$ be a standard Wiener process and η be a random variable, independent of w. Let also $a : [t_0, T] \times R \to R$ and $b : [t_0, T] \times R \to R$ be Borel-measurable functions, satisfying the Lipschitz condition and the condition on the restriction on growth on the last argument respectively, i.e. for all $(t, x), (t, y) \in [t_0, T] \times R$ there exists a constant $L > 0$ such that*

$$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \leq L|x - y|, \tag{7}$$

$$|a(t,x)|^2 + |b(t,x)|^2 \leq L^2(1 + x^2). \tag{8}$$

Then there exists a unique a.s. continuous strong solution of the SDE (6).

Proof. The theorem can be proved by Picard-Lindelöf method of iterations, modeled after the corresponding proof for ordinary differential equations (see, for example, [1], [8], [9], [14], [16], [30], [45]). For the proof here we shall apply the Banach fixed point theorem (see [10]).

First, let us suppose that $E|\eta|^2 < \infty$. Denote by $\mathcal{B}$ a space of measurable processes x, defined on $[t_0, T]$, adapted to the nondecreasing family of sub-σ-algebras $(\mathcal{F}_t, t \in [t_0, T])$, satisfying the condition $\sup_{t_0 \le t \le T} E|x(t)|^2 < \infty$. Then $\mathcal{B}$ is the Banach space with the norm

$$\|x\| = \Big(\sup_{t_0 \le t \le T} E|x(t)|^2 \Big)^{1/2}.$$

Let us define an operator S such that for $x \in S$,

$$Sx(t) = \eta + \int_{t_0}^{t} a(s, x(s))\, ds + \int_{t_0}^{t} b(s, x(s))\, dw(s), \quad t \in [t_0, T]. \tag{9}$$

Since a and b are Borel-measurable functions and x is a measurable process, adapted to $(\mathcal{F}_t, t \in [t_0, T])$, it follows that the processes $\overline{a}(t) = a(t, x(t))$ and $\overline{b}(t) = b(t, x(t))$ also have these properties. Moreover, Schwarz inequality and (8) imply

$$E\Big| \int_{t_0}^{T} a(s, x(s))\, ds \Big|^2 \le (T - t_0) \int_{t_0}^{T} E|a(s, x(s))|^2\, ds \le \alpha + \beta \sup_{t_0 \le t \le T} E|x(t)|^2 < \infty;$$

$$\sup_{t_0 \le t \le T} E\Big| \int_{t_0}^{t} b(s, x(s))\, dw(s) \Big|^2 = \sup_{t_0 \le t \le T} \int_{t_0}^{t} E|b(s, x(s))|^2\, ds \le \gamma + \delta \sup_{t_0 \le t \le T} E|x(t)|^2 < \infty,$$

where $\alpha, \beta, \gamma, \delta$ are some constants depending on L, t_0 and T. Accordingly, since $a^{\frac{1}{2}}, b \in \mathcal{P}$, the integrals in (9) are well defined.

Let us prove that $S : \mathcal{B} \to \mathcal{B}$. If $x \in \mathcal{B}$, then $Sx(t)$ is a measurable process, $\mathcal{F}_t$-measurable for every $t \in [t_0, T]$ and a.s. continuous. Also,

$$\begin{aligned} E|Sx(t)|^2 &\le 3E|\eta|^2 + 3(T - t_0) \int_{t_0}^{t} E|a(s, x(s))|^2\, ds + 3 \int_{t_0}^{t} |b(s, x(s))|^2\, ds \\ &\le 3E|\eta|^2 + 3(T - t_0 + 1)L^2 \int_{t_0}^{t} (1 + E|x(s)|^2)\, ds \\ &\le 3E|\eta|^2 + 3(T - t_0 + 1)L^2 (T - t_0)(1 + \|x\|^2) = M. \end{aligned}$$

Thus,

$$\|Sx\| = \Big(\sup_{t_0 \le t \le T} E|Sx(t)|^2 \Big)^{1/2} < \infty,$$

and therefore $S: \mathcal{B} \to \mathcal{B}$.

In the next step of the proof we shall show that there exists a unique fixed point of the operator S. Indeed, for every $x_1, x_2 \in \mathcal{B}$ we have

$$\begin{aligned}
&E|Sx_1(t) - Sx_2(t)|^2 \\
&\leq 2E\Big|\int_{t_0}^{t}[a(s,x_1(s)) - a(s,x_2(s))]\,ds\Big|^2 + 2E\Big|\int_{t_0}^{t}[b(s,x_1(s)) - b(s,x_2(s))]\,dws\Big|^2 \\
&\leq 2(T-t_0)L^2\int_{t_0}^{t}E|x_1(s)-x_2(s)|^2\,ds + 2L^2\int_{t_0}^{t}E|x_1(s)-x_2(s)|^2\,ds \\
&\leq K\|x_1 - x_2\|^2\,(t-t_0),
\end{aligned}$$

where $K = 2(T - t_0 + 1)L^2$. Now it is easy to prove by induction that

$$\begin{aligned}
E|S^n x_1(t) - S^n x_2(t)|^2 &\leq K\int_{t_0}^{t}E|S^{n-1}x_1(s) - S^{n-1}x_2(s)|^2 ds \\
&\leq \cdots \leq \frac{K^n(t-t_0)^n}{n!}\|x_1 - x_2\|^2, \quad t \in [t_0, T] \quad n \in N,
\end{aligned}$$

such that

$$\|S^n x_1 - S^n x_2\|^2 \leq \frac{K^n(T-t_0)^n}{n!}\|x_1 - x_2\|^2, \quad n \in N.$$

Since $K^n(T-t_0)^n/n! \to 0$ as $n \to \infty$, then there exists $n_0 \in N$ such that $K^{n_0}(T-t_0)^{n_0}/n_0! = q < 1$. Thus S^{n_0} is a contraction. By one version of the Banach fixed point theorem it follows that the operator S has a unique fixed point $x \in \mathcal{B}$, i.e., $x = Sx$. On the other hand,

$$x(t) = \eta + \int_{t_0}^{t}a(s,x(s))\,ds + \int_{t_0}^{t}b(s,x(s))\,dw(s) \text{ a.s.,} \quad t \in [t_0, T].$$

Since $x(t_0) = \eta$ a.s., from Definition 5 holds that x is a unique strong solution of the SDE (6), moreover satisfying $\sup_{t_0 \leq t \leq T} E|x(t)|^2 < \infty$. Also, it is easy to show that

$$\sup_{t_0 \leq t \leq T} E|x(t)|^2 \leq 3E|\eta|^2 e^{3K(T-t_0)}.$$

Let us prove now the existence of a solution of the SDE (6) without the assumption $E|\eta|^2 < \infty$. Denote by $I_\eta^N = I_{\{|\eta| \leq N\}}$ and $\eta^N = \eta I_\eta^N$. Obviously, η^N is a random variable, independent with respect to w and $\mathcal{F}_{t_0}$-measurable. Since $E|\eta^N|^2 \leq N^2 < \infty$, the SDE

$$x^N(t) = \eta^N + \int_{t_0}^{t}a(s,x^N(s))\,ds + \int_{t_0}^{t}b(s,x^N(s))\,dw(s), \quad t \in [t_0, T] \tag{10}$$

has a unique solution. For $N' > N$ it follows that

$$\begin{aligned}
x^{N'}(t) - x^N(t) = \eta^{N'} - \eta^N &+ \int_{t_0}^{t}[a(s,x^{N'}(s)) - a(s,x^N(s))]\,ds \\
&+ \int_{t_0}^{t}[b(s,x^{N'}(s)) - b(s,x^N(s))]\,dW(s).
\end{aligned}$$

Since $(\eta^{N'} - \eta^N)I_\eta^N = \eta^{N'} I_\eta^N - \eta^N I_\eta^N = 0$, we obtain

$$\begin{aligned}\sup_{t_0\le t\le u}\Big(x^{N'}(t) - x^N(t)\Big)^2 I_\eta^N &\le 2\sup_{t_0\le t\le u}\Big(I_\eta^N\int_{t_0}^t [a(s,x^{N'}(s)) - a(s,x^N(s))]\,ds\Big)^2 \\ &\quad + 2\sup_{t_0\le t\le u}\Big(I_\eta^N\int_{t_0}^t [b(s,x^{N'}(s)) - b(s,x^N(s))]\,dw(s)\Big)^2 \\ &\le 2(T-t_0)\int_{t_0}^u I_\eta^N|a(s,x^{N'}(s)) - a(s,x^N(s))|^2\,ds \\ &\quad + 2\sup_{t_0\le t\le u}\Big|\int_{t_0}^t I_\eta^N[b(s,x^{N'}(s)) - b(s,x^N(s))]\,dw(s)\Big|^2.\end{aligned}$$

By applying the Lipschitz condition (7) and Doob's inequality (3), we finally get

$$E\sup_{t_0\le t\le u}|x^{N'}(t) - x^N(t)|^2 I_\eta^N \le 2(T-t_0+4)L^2\int_{t_0}^u E\sup_{t_0\le v\le s}|x^{N'}(v) - x^N(v)|^2 ds.$$

Now we need the well-known Gronwall's lemma: if $u:[a,b]\to R$ and $v:[a,b]\to R$ are non-negative integrable functions and $L = \text{const} > 0$, then

$$u(t) \le v(t) + L\int_a^t u(s)ds \implies u(t) \le v(t) + L\int_a^t e^{L(t-s)}v(s)ds, \quad t\in[a,b].$$

Especially, if $v(t)\equiv \text{const} = u(a)$, then

$$u(t) \le u(a) + L\int_a^t u(s)ds \implies u(t)\le u(a)e^{L(t-a)}, \quad t\in[a,b].$$

If $u(a) = 0$, then $u(t) = 0$ for all $t\in[a,b]$.

By applying the preceding lemma, it follows that

$$E\sup_{t_0\le t\le T}|x^{N'}(t) - x^N(t)|^2 I_\eta^N = 0,$$

which implies $P\{\sup_{t_0\le T\le t}|x^{N'}(t) - x^N(t)|^2 = 0\} = 0$. Now,

$$P\{\sup_{t_0\le t\le T}|x^{N'}(t) - x^N(t)|^2 > 0\} \le P\{|\eta| > N\} \to 0 \quad \text{as} \quad N', N\to\infty.$$

Therefore, $\{x^N(t)\}$ is a Cauchy sequence converging in probability for all $t\in[t_0,T]$. So, there exists $\mathcal{F}_t$-measurable process $(x(t), t\in[t_0,T])$, such that $\sup_{t_0\le t\le T}|x^N(t) - x(t)| \overset{\text{rm p.}}{\to} 0$ as $N\to\infty$. Since

$$\begin{aligned}\int_{t_0}^T |a(s,x(s)) - a(s,x^N(s))|^2\,ds + \int_{t_0}^T |b(s,x(s)) - b(s,x^N(s))|^2\,ds &\le 2L^2\int_{t_0}^T \sup_{t_0\le u\le s}|x(u) - x^N(u)|^2\,ds \\ &\le 2L^2(T-t_0)\sup_{t_0\le t\le T}|x(t) - x^N(t)|^2 \overset{\text{p.}}{\to} 0 \quad \text{as} \quad N\to\infty,\end{aligned}$$

then for every fixed $t \in [t_0, T]$,

$$\int_{t_0}^{t} a(s, x^N(s))\, ds \xrightarrow{P} \int_{t_0}^{t} a(s, x(s))\, ds,$$

$$\int_{t_0}^{t} b(s, x^N(s))\, dw(s) \xrightarrow{P} \int_{t_0}^{t} b(s, x(s))\, dw(s).$$

holds. The limits in probability on both sides of the equation (10) show that $x(t)$ satisfies the SDE (6) a.s. and, therefore, it is its strong solution.

It remains to prove a uniqueness of a solution of the SDE (6) if $E|\eta|^2 < \infty$ does not hold.

Let x_1 and x_2 be two solutions of this equation. Then for every $t \in [t_0, T]$,

$$x_1(t) - x_2(t) = \int_{t_0}^{t} [a(s, x_1(s)) - a(s, x_2(s))]\, ds + \int_{t_0}^{t} [b(s, x_1(s)) - b(s, x_2(s))]\, dw(s)$$

holds a.s. Denote

$$I_N(t) = \begin{cases} 1, & |x_1(s)| \le N, |x_2(s)| \le N, \quad s \in [t_0, t], \\ 0, & \text{otherwise} \end{cases}$$

Since $I_N(t)I_N(s) = I_N(t)$ for all $s \le t$, then

$$I_N(t)[x_1(t) - x_2(t)] = I_N(t) \int_{t_0}^{t} I_N(s)[a(s, x_1(s)) - a(s, x_2(s))]\, ds$$
$$+ I_N(t) \int_{t_0}^{t} I_N(s)[b(s, x_1(s)) - b(s, x_2(s))]\, dw(s).$$

Thus

$$I_N(s)|a(s, x_1(s)) - a(s, x_2(s))| \le I_N(s) L |x_1(s) - x_2(s)| \le 2LN, \quad \text{a.s.},$$

and analogously for b. If we apply the dominated convergence theorem, we obtain

$$\begin{aligned} &E I_N(t)|x_1(t) - x_2(t)|^2 \\ &\quad \le 2(t - t_0) \int_{t_0}^{t} E\{I_N(s)|a(s, x_1(s)) - a(s, x_2(s))|^2\}\, ds \\ &\qquad + 2 \int_{t_0}^{t} E\{I_N(s)|b(s, x_1(s)) - b(s, x_2(s))|^2\}\, ds \\ &\quad \le 2(T - t_0 + 1)L^2 \int_{t_0}^{t} E\{I_N(s)|x_1(s) - x_2(s)|^2\}\, ds. \end{aligned}$$

Applying now the Gronwall's lemma we get $E\{I_N(t)|x_1(t) - x_2(t)|^2\} = 0$ for all $t \in [t_0, T]$, which implies $P\{ I_N(t)x_1(t) = I_N(t)x_2(t) \} = 1$. From there we easily conclude

$$P\{x_1(t) \ne x_2(t)\} \le P\{ \sup_{t_0 \le s \le t} |x_1(s)| > N\} + P\{ \sup_{t_0 \le s \le t} |x_2(s)| > N\}.$$

Since x_1 i x_2 are a.s. continuous on $[t_0, T]$, they are a.s. bounded. It means that the right-hand side of the last inequality goes to zero by taking $N \to \infty$ and, therefore $P\{x_1(t) \neq x_2(t)\} = 0$ for all $t \in [t_0, T]$, i.e.

$$P\{\sup_{t_0 \le t \le T} |x_1(t) - x_2(t)| > 0\} = 0.$$

Thus the proof is complete. □

Clearly, Theorem 7 gives only sufficient conditions for the existence and uniqueness of a solution of the SDE (6). Note that if the functions a and b are defined on $[t_0, \infty) \times R$ and if the assumptions of Theorem 7 hold on every finite subinterval $[t_0, T] \subset [t_0, \infty)$, then the SDE (6) has a unique solution, defined on the entire half-line $[t_0, \infty)$, called *a global solution*. Naturally, in some cases the SDE (6) could have *a local solution*, particularly if the assumptions of Theorem 7 do not hold, as in the following example.

Indeed, the coefficients od the SDE

$$dx(t) = -\frac{1}{2} e^{-2x(t)} dt + e^{-x(t)} dw(t), \quad x(t_0) = \eta \text{ a.s}, \quad t \ge t_0,$$

do not satisfy any Lipschitz condition or any growth condition for $x < 0$. However, there exists a unique local solution $x(t) = \ln[w(t) - w(t_0) + e^\eta]$, defined on the random interval $[t_0, \tau)$, where the random variable τ is determined with $\tau = \inf\{t : w_t - w_{t_0} + e^\eta < 0\}$ (see [1], [32]). Naturally, we use the Itô's formula to prove that $x(t)$ is the solution of this equation.

The next theorem, known as *the local uniqueness theorem*, plays a very important role in the study of stochastic differential equations (see, for example, [1], [8], [9]).

Theorem 8. *Let the functions a_i and b_i, $i = 1, 2$, satisfy the assumptions of Theorem 7 and let there exist $N > 0$ such that $a_1(t,x) = a_2(t,x)$, $b_1(t,x) = b_2(t,x)$ for all $(t,x) \in [-N, N]$. Let $x_i(t)$, $i = 1, 2$, be a solution of the SDE*

$$dx_i(t) = a_i(t, x_i(t))\, dt + b_i(t, x_i(t))\, dw(t), \quad x_i(t_0) = \eta \text{ a.s.}, \quad t \in [t_0, T].$$

Denote by τ_i the first time, after t_0, such that $x_i(t)$ intersects $R \setminus [-N, N]$ if such time $t \in [t_0, T]$ exists, and $\tau_i = T$ otherwise. Then

$$P\{\tau_1 = \tau_2\} = 1 \quad \text{and} \quad P\{\sup_{t_0 \le t \le \tau_1} |x_1(t) - x_2(t)| = 0\} = 1.$$

Proof: Denote by

$$\psi_1(t) = \begin{cases} 1, & \sup_{t_0 \le t \le t} |x_1(t)| \le N, \\ 0, & \text{otherwise}, \end{cases}$$

Let $\psi_1(t) = 1$. Then $\psi_1(s) = 1$ for all $t_0 \le s \le t \le \tau_1$ and here $a_1(s, x_1(s)) = a_2(s, x_1(s))$ a.s., $b_1(s, x_1(s)) = b_2(s, x_1(s))$ a.s.. From integral form of the SDE-s it is easy to obtain

$$\psi_1(t)[x_1(t) - x_2(t)]^2 \le 2\Big\{\int_{t_0}^t \psi_1(s)[a_2(s, x_1(s)) - a_2(s, x_2(s))]\, ds\Big\}^2 + 2\Big\{\int_{t_0}^t \psi_1(s)[b_2(s, x_1(s)) - b_2(s, x_2(s))]\, dw(s)\Big\}^2.$$

By applying the Lipschitz condition (7), it follows that

$$E\psi_1(t)\,[\,x_1(t) - x_2(t)\,]^2 \le c \int_{t_0}^{t} E\psi_1(s)[\,x_1(s) - x_2(s)\,]^2\, ds,$$

where c is a constant depending on L, T and t_0. Then from Gronwall's lemma

$$E\psi_1(t)\,[\,x_1(t) - x_2(t)\,]^2 = 0, \quad t \in [t_0, \tau_1],$$

holds. From that,

$$P\{\sup_{t_0 \le t \le \tau_1} |\,x_1(t) - x_2(t)\,| = 0\,\} = 1,$$

and therefore $x_1(t) = x_2(t)$ a.s. for $t \in [t_0, \tau_1]$. Consequently, $P\{\tau_2 \ge \tau_1\} = 1$. Analogously we get $P\{\tau_1 \le \tau_2\} = 1$, which completes the proof. □

Theorem 8 makes it possible to express the next stronger existence and uniqueness theorem.

Theorem 9. *Let $a : [t_0, T] \times R \to R$, $b : [t_0, T] \times R \to R$ be measurable functions satisfying the assumptions:*

(i) there exists a constant $K > 0$ such that for all $(t, x) \in [t_0, T] \times R$,

$$|a(t,x)|^2 + |b(t,x)|^2 \le L^2(1 + |x|^2);$$

(ii) for any $N > 0$ there exists a constant $L_N > 0$ such that for all $(t, x), (t, y) \in [t_0, T] \times [-N, N]$,

$$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \le L_N |x - y|.$$

If a standard Wiener process w and a random variable η are independent and $E|\eta|^2 < \infty$, there exists a unique solution $(x(t), t \in [t_0, T])$ of the SDE (6), satisfying the initial value $x(t_0) = \eta$ a.s.

The proof can be found in [9].

Let us give some important notions. Remark that Theorem 7 can be extended to the SDE, similar to the SDE (6), in which the coefficients $a : \Omega \times [t_0, T] \times R \to R$ and $b : \Omega \times [t_0, T] \times R \to R$ are random functions, Borel measurable on their domains, adapted to the family of sub-σ-algebras $(\mathcal{F}_t, t \in [t_0, T])$ generated by w, such that the stochastic integrals in this SDE exist in the sense of Definition 5-(ii).

Theorem 10. *Let $(\eta(t), t \in [t_0, T])$ be a stochastic process, independent of w, adapted to $(\mathcal{F}_t, t \in [t_0, T])$, such that $\sup_{t \in [t_0, T]} E|\eta(t)|^2 < \infty$. Let also the random functions a and b satisfy a.s. the Lipschitz condition (7) and the condition of the restriction on growth (8). Then there exists a unique solution $(x(t), t \in [t_0, T])$ of the SDE*

$$x(t) = \eta(t) + \int_{t_0}^{t} a(\omega, s, x(s))\, ds + \int_{t_0}^{t} b(\omega, s, x(s))\, dw(s), \quad t \in [t_0, T],$$

with $\sup_{t\in[t_0,T]} E|x(t)|^2 < \infty$. *Moreover, if the process* $\eta(t)$ *is a.s. continuous, then the solution* $x(t)$ *is a.s. continuous.*

A theorem analogous to Theorem 9 can also be proved.

Note that the approach given by Theorems 7, 8, 9 and 10 is appropriatelly extended to analyze the existence and uniqueness problem for special classes of stochastic differential equations, stochastic functional differential equations, stochastic integral and integrodifferential equations containing the Itô's integrals (see [3], [4], [5], [9], [11], [25], [26], [30], [37], for example, and many others).

Remember again that Theorem 7 gives only sufficient conditions for the existence and uniqueness of the solution of the SDE (6). In fact, there is a number of papers in which various sufficient conditions, essentially other than the conditions (7) and (8), are considered. Note that many new theorems present a direct extension of the corresponding deterministic results (see, for example, [3], [4], [5], [9], [18], [22], [28], [45], [46]). In many papers different kinds of contractions are used instead of the Lipschitz condition, for example in [24], [38].

Naturally, the permanently current problem is the relationship between ordinary and stochastic differential equations, especially for applications to stochastic control problems and to stochastic filtering problems (see [30], [42], [43], [44], for example).

An important fact is that the problem of the existence and uniqueness of solutions of the Itô-type stochastic differential equations can be extended to stochastic differential equations with respect to martingales and stochastic measures (see, for example, [6], [10], [11], [14], [25], [27], [29], [31], [34], [41], [47]), and also to stochastic differential equations with semimartingales (see [22], [33], [47]).

One of the most important problems in qualitative analysis of solutions for different classes of stochastic differential equations is the stability problem, including the asymptotic behavior of solutions when $t \to \infty$ and the existence of singular solutions (see [1], [2], [3], [4], [5], [13], [14], [37], [46], for example). By using the concept of Lyapunov function and the theory of stochastic and deterministic inequalities, several comparison theorems are developed in many papers and books (see, for example, [9], [13], [14], [28], [46]).

2.3. Stochastic differential equations depending on parameters. Now we give the basic theorem which describes the stochastic differential equation of the Itô type depending on a parameter $\alpha \in A$, where A is a parameter set. This theorem shows that the change in the solution can be made arbitrarily small by making the change in the parameter sufficiently small.

Theorem 11. *Let the random functions* η_α, a_α, b_α *satisfy the assumptions of Theorem 10 for any parameter* $\alpha \in A$, *with the same constant* L *in (7) and (8). Let also the process* $(\eta_\alpha(t), t \in [t_0, T])$ *be a.s. continuous and* $\sup_{t\in[t_0,T]} E|\eta_\alpha(t)|^2 < c$ *for all* $\alpha \in A$, c *=const.. Suppose that for any* $N > 0$, $\alpha_0 \in A$, $\epsilon > 0$ *and* $t \in [t_0, T]$,

$$\lim_{\alpha\to\alpha_0} P\{ \sup_{|x|\le N} [\,|a_\alpha(\omega,t,x) - a_{\alpha_0}(\omega,t,x)| + |b_\alpha(\omega,t,x) - b_{\alpha_0}(\omega,t,x)|\,] > \epsilon\} = 0$$

and

$$\lim_{\alpha\to\alpha_0}\sup_{t\in[t_0,T]} E|\eta_\alpha(t)-\eta_{\alpha_0}(t)|^2=0.$$

If $(x_\alpha(t), t\in[t_0,T])$ *is a solution of the SDE*

$$x_\alpha(t)=\eta_\alpha(t)+\int_{t_0}^t a_\alpha(\omega,s,x_\alpha(s))\,ds+\int_{t_0}^t b_\alpha(\omega,s,x_\alpha(s))\,dw(s),\ t\in[t_0,T],\ ()$$

then $\lim\alpha\to\alpha_0\sup_{t\in[t_0,T]} E|x_\alpha(t)-x_{\alpha_0}(t)|^2=0.$

Proof. Denote

$$x_\alpha(t)-x_{\alpha_0}(t)=\xi_\alpha(t)+\int_{t_0}^t [\,a_\alpha(\omega,s,x_\alpha(s))-a_\alpha(\omega,s,x_{\alpha_0}(s))\,]\,ds$$
$$+\int_{t_0}^t b_\alpha(\omega,s,x_\alpha(s))-b_\alpha(\omega,s,x_{\alpha_0}(s))\,]\,dw(s),$$

where

$$\xi_\alpha(t)=\eta_\alpha(t)-\eta_{\alpha_0}(t)+\int_{t_0}^t [\,a_\alpha(\omega,s,x_{\alpha_0}(s))-a_{\alpha_0}(\omega,s,x_{\alpha_0}(s))\,]\,ds$$
$$+\int_{t_0}^t [\,b_\alpha(\omega,s,x_{\alpha_0}(s))-b_{\alpha_0}(\omega,s,x_{\alpha_0}(s))\,]\,dw(s).$$

Using the Lipschitz condition (7) on the first identity and applying the usual stochastic isometry, we easily obtain

$$E|x_\alpha(t)-x_{\alpha_0}(t)|^2\le 3|\xi_\alpha(t)|^2+K\int_{t_0}^t E|x_\alpha(s)-x_{\alpha_0}(s)|^2\,ds,$$

where $K=3(T-t_0+1)L^2$. By Gronwall's lemma it follows that

$$E|x_\alpha(t)-x_{\alpha_0}(t)|^2\le 3E|\xi_\alpha(t)|^2+K\int_{t_0}^t e^{K(t-s)}E|\xi_\alpha(s)|^2\,ds.$$

Therefore, it follows from the last inequality that the theorem will be proved if we show that $\sup_{t\in[t_0,T]} E|\xi_\alpha(t)|^2\to 0$ as $\alpha\to\alpha_0$.

Since

$$E\left|\int_{t_0}^t [\,a_\alpha(\omega,s,x_{\alpha_0}(s))-a_{\alpha_0}(\omega,s,x_{\alpha_0}(s))\,]\,ds\right|^2$$
$$\le (t-t_0)\int_{t_0}^t E|\,a_\alpha(\omega,s,x_{\alpha_0}(s))-a_{\alpha_0}(\omega,s,x_{\alpha_0}(s))\,|^2\,ds,$$

by applying the condition (8) we obtain that the last integrand is bounded by $2L^2(1+|x_{\alpha_0}(t)|^2)$. Since $E\int_{t_0}^T(1+|x_{\alpha_0}(t)|^2\,dt<\infty$, it follows from the conditions of the theorem that this integrand also converges to zero in probability, as $\alpha \to \alpha_0$. So, by the Lebesgue bounded convergence theorem we conclude

$$\sup_{t\in[t_0,T]} E\Big|\int_{t_0}^{t}[a_\alpha(\omega,s,x_{\alpha_0}(s))-a_{\alpha_0}(\omega,s,x_{\alpha_0}(s))]ds\Big|^2$$
$$\le (T-t_0)\int_{t_0}^{T} E|\,a_\alpha(\omega,s,x_{\alpha_0}(s))-a_{\alpha_0}(\omega,s,x_{\alpha_0}(s))\,|^2 ds \to 0 \quad \text{as} \quad \alpha\to\alpha_0.$$

Similarly, using Doob's inequality (3) and the previous arguments, we have

$$E\sup_{t\in[t_0,T]}\Big\{\int_{t_0}^{t}[b_\alpha(\omega,s,x_{\alpha_0}(s))-b_{\alpha_0}(\omega,s,x_{\alpha_0}(s))]dw(s)\Big\}^2$$
$$\le 4\int_{t_0}^{T} E|\,b_\alpha(\omega,s,x_{\alpha_0}(s))-b_{\alpha_0}(\omega,s,x_{\alpha_0}(s))\,|^2 ds \to 0 \quad \text{as} \quad \alpha\to\alpha_0.$$

This completes the proof, because $\sup_{t\in[t_0,T]} E|\eta_\alpha(t)-\eta_{\alpha_0}(t)|^2 \to 0$ as $\alpha\to\alpha_0$. □

Note that there are suitable versions of the preceding theorem for different classes of stochastic differential equations. So, for the SDE (6) one can state a theorem which ensures the continuous dependence of the solution on the initial value (t_0,η) (see [9], [24]).

The more important application of Theorem 11 is for a discrete parameter set, i.e., if $A=\{\alpha_n, n=0,1,\dots\}$ and $\alpha_n\to\alpha_0$ as $n\to\infty$. Then the following theorem holds:

Theorem 12. *Let the random functions $\eta_n(t)$, $a_n(\omega,t,x)$, $b_n(\omega,t,x)$, $n=0,1,2,\dots$, satisfy all conditions of Theorem 11 for n and 0 instead of α and α_0 respectively. If $(x_n(t), t\in[t_0,T])$ is the solution of the SDE*

$$x_n(t)=\eta_n(t)+\int_{t_0}^{t} a_n(\omega,s,x_n(s))\,ds+\int_{t_0}^{t} b_n(\omega,s,x_n(s))\,dw(s),\ t\in[t_0,T], \quad ()$$

then

$$\lim_{n\to\infty}\sup_{t\in[t_0,T]} E|x_n(t)-x_0(t)|^2=0.$$

From purely theoretical point of view, and much more from the point of view of various applications, this theorem gives a possibility to study the solution $x_0(t)$ of the SDE (12) for $n=0$ by finding at least an approximate solution $x_{n_0}(t)$ of the SDE (12) for $n=n_0$.

This theorem enables the construction of some iterative methods for solving the SDE (6), or the SDE (12) for $n=0$, and to estimate an error of the n-th approximation of the solution of the original equation. There is a number of papers in which various sufficient conditions of closeness of the random or non-random functions η_0, a_0, b_0 with the functions η_n, a_n, b_n respectively, are given, such that $x_n(t)\to x(t)$ as $n\to\infty$ in probability or in p-th mean sense or with probability one (see, for example, [3], [9], [11], [23], [26], [45]).

2.4. The Markov property. Now we describe in short one of the most important properties of the solutions of the SDE (6), known as *the Markov property.*

Having in mind that a solution $x(t)$ of the SDE (6) must be $\mathcal{F}_t$-measurable, it can be interpreted as a stochastic process determined by non-random functions a and b and by random elements η and $w_s, s \leq t$. So, $x(t)$ depends on η and $w_s, s \leq t$. Moreover, the construction of $x(t)$, especially the construction of a solution by Picard–Lindelöf method of iterations, shows that it depends only on $w_s - w_{t_0}$ for $t_0 \leq s \leq t$ (see [1], [8]). Thus, $x(t)$ can be expressed as a functional

$$x(t) = f(\eta; w_s - w_{t_0}, t_0 \leq s \leq t).$$

This fact makes possible a description of the Markov property of the solution of the SDE (6).

Definition 7. The stochastic process $(x(t), t \in [t_0, T])$ is said to be a Markov process with respect to $(\mathcal{F}_t, t \in [t_0, T])$ if for all $t_0 \leq s \leq t \leq T$ and for any set $A \in \mathcal{B}$

$$P\{\, x(t) \in A | \mathcal{F}_s\} = P\{\, x(t) \in A | x(s)\} \quad \text{a.s.}$$

holds.

Theorem 13. *Let the conditions of Theorem 7 hold with $E|\eta|^2 < \infty$ and let $(\mathcal{F}_t, t \in [t_0, T])$ be the increasing family of the sub-σ-algebras generated by η and w. Then the unique solution $(\, x(t), t \in [t_0, T])$ of the SDE (6) is a Markov process with respect to $(\mathcal{F}_t, t \in [t_0, T])$.*

For a detailed proof see [8], for example. We give only a short survey of the proof.

Together with the SDE (6) we consider the same equation, now on an interval $[s, T] \subset [t_0, T]$, i.e., for $t \in [s, T]$ we have

$$x(t) = x + \int_s^t a(u, x(u))\, du + \int_0^t b(u, x(u))\, dw(u), \quad x(s) = x \;\text{ a.s.} \qquad ()$$

For the given initial value $x(s) = x$ a.s., let $(\, x_{s,x}(t), t \in [s, T]\,)$ be a solution of the SDE (13). From the fact that the SDE (6) has a unique solution $(x(t), t \in [t_0, T])$, it follows that $x(t) = x_{s,x}(t)$ a.s. for all $t \in [s, T]$. Also, for $t \in [s, T]$, $x_{s,x}(t)$ is completely determined as a functional $x_{s,x}(t) = f(x; w_u - w_s, u \in [s, T])$. Moreover, since $x(s)$ is $\mathcal{F}_s$-measurable and increments $w_u - w_s, u \in [s, t]$, are independent on $\mathcal{F}_s$, for any set $A \in \mathcal{B}$ it follows that $P\{x(t) \in A | \mathcal{F}_s\} = P\{x(t) \in A | x(s)\}$ a.s.. Therefore, the solution of the SDE (6) is a Markov process.

For $t_0 \leq s \leq t \leq T$ and for any set $A \in \mathcal{B}$, the function

$$p(s, x; t, A) = P\{x(t) \in A | x(s) = x\}$$

is called *the transition probability function.* Clearly, considering s and x fixed, $p(s, x; t, A)$ is precisely the distribution of the solution $x_{s,x}(t)$ of the equation (13). Also, $p(s, x; t, A)$ has the following properties: it is Borel measurable in x for fixed

s, t, A; it is a probability measure in A for fixed s, x, t; the function p satisfies the Chapman–Kolmogorov equation: for all $x \in R$ and $s < u < t$,

$$p(s,x;t,A) = \int_{-\infty}^{\infty} p(s,x;u,dy)\, p(u,y;t,A)$$

holds.

Recall that a Markov process is said to be *homogeneous* if the transition probability functions are stationary, i.e., $p(s,x;t,A) = \phi(t-s,x,A)$.

It is easy to see that if the SDE (6) is *autonomous*, that is $a(t,x) = a(x)$, $b(t,x) = b(x)$, then its solution will be a homogeneous Markov process.

Moreover, in addition to the conditions of Theorem 7, if the functions $a(t,x)$ and $b(t,x)$ are supposed to be continuous, then a solution of the SDE (6) is *a diffusion process*, i.e., a stochastic process with continuous sample functions whose transition probability functions $p(s,x;t,A)$ have certain infinitesimal properties as $t \to s$ (see, for example, [1], [8], [9], [10], [45]).

The density function of the transition probability function is called *the transition density function.* Under some very strict conditions of differentiability of the functions a and b, beginning from the Chapman–Kolmogorov equations one comes to the well-known *backward and forward parabolic equations*, alternatively called *diffusion equations*, whose solutions are transition density functions. Note that the forward equation is also known as *the Fokker–Planck equation.* Naturally, the solution of the SDE (6) is completely described if the transition probability functions, i.e., the transition density functions, are known.

Emphasize an important fact that the theory of diffusion processes is applied to study several phenomena in physics, astronomy, biology, etc. The modern theory of the Markov processes, primarily a semigroup theory, is engaged in the studies of the solutions of diverse classes of stochastic differential equations, which are diffusion processes.

2.5. Solvable stochastic differential equations. We say that the SDE (6) is *explicitly solvable* if its solution can be represented by quadratures, i.e., in terms of ordinary (Lebesgue) and Itô's stochastic integrals.

I. Just as with ordinary differential equations, a lot of theory is developed to describe solutions of linear Itô-type stochastic differential equations, first of all analytic properties of the solutions, including the overall behavior of sample functions on the interval $[t_0, \infty)$. Now we give the procedure to obtain explicit solutions of homogeneous and non-homogeneous linear stochastic differential equations.

Let $a : [t_0,\infty) \to R$ and $b : [t_0,\infty) \to R$ be Borel-measurable bounded functions. Then the equation

$$dx(t) = a(t)x(t)\,dt + b(t)x(t)\,dw(t), \quad x(t_0) = \eta = \text{const. a.s.}, \quad t \geq t_0.$$

is said to be the homogeneous linear SDE. If $\eta = 0$ a.s., this equation has a trivial solution $x(t) = 0$ a.s. Since the conditions of Theorem 7 hold, then there exists a unique solution such that $x(t) > 0$ a.s. for $\eta > 0$ a.s.; $x(t) < 0$ a.s. for $\eta < 0$ a.s.

If we put $y(t) = \ln x(t)$ for $\eta > 0$ a.s., or $y(t) = \ln(-x(t))$ for $\eta < 0$ a.s., by Itô's formula we have

$$dy(t) = \frac{1}{x(t)}\,dx(t) + \frac{1}{2}\Big(-\frac{1}{x(t)}\Big)b^2(t)x(t)\,dt$$

i.e.,

$$dy(t) = \Big[a(t) - \frac{1}{2}b^2(t)\Big]\,dt + b(t)\,dw(t), \quad y(t_0) = \ln\eta \ \text{ a.s.}$$

Thus we obtain the stochastic differential of the process $y(t)$ and, therefore

$$y(t) = \ln\eta + \int_{t_0}^{t}\Big[a(s) - \frac{1}{2}b^2(s)\Big]\,ds + \int_{t_0}^{t} b(s)\,dw(s), \quad t \geq t_0.$$

From that the homogeneous linear SDE has the solution

$$x(t) = \eta\exp\Big\{\int_{t_0}^{t}\Big[a(s) - \frac{1}{2}b^2(s)\Big]\,ds + \int_{t_0}^{t} b(s)\,dw(s)\Big\}, \quad t \geq t_0.$$

Especially, the Langevin SDE

$$dx(t) = -\alpha x(t)\,dt + \beta\,dw(t), \quad x(0) = \eta \text{ a.s.}, \quad t \geq 0,$$

where $\alpha > 0$ and β are constants, has the solution

$$x(t) = e^{-\alpha t}\Big[\eta + \int_0^t e^{\alpha s}\beta\,dw(s)\Big], \quad t \geq 0.$$

For normally distributed or constant η, the solution $x(t)$ is a Gaussian process, the so-called Ornstein–Uhlenbech velocity process (see [1], [8]).

The non-homogeneous linear SDE

$$dx(t) = [\,\alpha(t) + a(t)x(t)\,]\,dt + [\,\beta(t) + b(t)x(t)\,]\,dw(t), \qquad (14)$$
$$x(t_0) = \eta \ \text{ a.s.}, \quad t \geq t_0,$$

can be solved analogously, putting $y(t) = \Phi^{-1}(t)\,x(t)$, where $\Phi^{-1}(t)$ is a particular solution of the corresponding homogeneous linear SDE with the initial value $\Phi(t_0) = 1$. So,

$$\Phi^{-1}(t) = \exp\Big\{-\int_{t_0}^{t}[\,a(s) - \frac{1}{2}b^2(s)\,]\,ds - \int_{t_0}^{t} b(s)\,dw(s)\Big\}.$$

Applying the Itô's formula we have

$$d\Phi^{-1}(t) = \Phi^{-1}(t)\Big\{[\,a(s) - \frac{1}{2}b^2(s)\,]\,ds - b(s)\,dw(s)\Big\}.$$

Applying again the Itô's formula on the product $\Phi^{-1}(t)x(t)$, from (4) we obtain

$$\begin{aligned} dy(t) &= d\,(\,\Phi^{-1}(t)\,x(t)\,) \\ &= \Phi^{-1}(t)\,dx(t) + x(t)d\Phi^{-1}(t) - [\,\beta(t) + b(t)x(t)\,]\,\Phi^{-1}(t)b(t)\,dt. \end{aligned}$$

By replacing $dx(t)$ and $d\Phi^{-1}(t)$ with the corresponding differentials, we obtain finally

$$dy(t) = \Phi^{-1}(t)\{[\alpha(t) - \beta(t)b(t)] + \beta(t)\,dw(t)\}$$

and, therefore

$$y(t) = \eta + \int_{t_0}^{t} \Phi^{-1}(s)\,[\alpha(s) - \beta(s)b(s)]\,ds + \int_{t_0}^{t} \Phi^{-1}(s)\beta(s)\,dw(s).$$

Thus the explicit solution of the non-homogeneous linear SDE (14) is given as

$$x(t) = \Phi(t)\Big[\eta + \int_{t_0}^{t} \Phi^{-1}(s)\,[\alpha(s) - \beta(s)b(s)]\,ds + \int_{t_0}^{t} \Phi^{-1}(s)\beta(s)\,dw(s)\Big].$$

II. In general, in order to transform the SDE (6) on a solvable form, we introduce a change of variables $y = h(t,x)$, where a smooth function $h(t,x)$ has an inverse $k(t,y)$, such that $h(t,k(t,y)) \equiv y$, $k(t,h(t,x)) \equiv x$.

According to the Itô's formula, the process $y(t) = h(t,x(t))$ satisfies the SDE

$$dy(t) = f(t,y(t))\,dt + g(t,y(t))\,dw(t), \quad y(t_0) = h(t_0,\eta) \text{ a.s.},$$

where

$$f(t,y) = \Big[h'_t + a\,h'_x + \frac{1}{2}\,b^2\,h''_{xx}\Big](t,k(t,y)), \tag{15}$$

$$g(t,y) = [\,b\,h'_x\,](t,k(t,y)). \tag{16}$$

The SDE (6) is said to be *reducible* if such a function h can be found so that the functions f and g, given by (15) and (16) respectively, are independent of y. Thus, the change of variables $y = h(t,x)$ permits the explicit representation of the solution $x(t)$ of the SDE (6) as

$$x(t) = k(t,y(t)),$$

where

$$y(t) = h(t_0,\eta) + \int_{t_0}^{t} f(s)\,ds + \int_{t_0}^{t} g(s)\,dw(s).$$

In other words, the SDE (6) is reducible if a sufficiently smooth invertible function $h(t,x)$ and functions $f(t)$ and $g(t)$, exist, such that

$$\left[\frac{\partial h}{\partial t} + a\,\frac{\partial h}{\partial x} + \frac{1}{2}\,b^2\,\frac{\partial^2 h}{\partial x^2}\right](t,x) \equiv f(t), \tag{17}$$

$$\left[b\,\frac{\partial h}{\partial x}\right](t,x) \equiv g(t). \tag{18}$$

Under the assumptions that $b \neq 0$ and a and b possess the indicated derivatives, one can obtain the necessary and sufficient conditions so that the SDE (6) be reducible. Indeed, differentiating (17) with respect to x gives

$$\frac{\partial^2 h}{\partial x \partial t} + \frac{\partial}{\partial x}\left\{a\,\frac{\partial h}{\partial x} + \frac{1}{2}\,b^2\,\frac{\partial^2 h}{\partial x^2}\right\} \equiv 0. \tag{19}$$

Since from (18) we get

$$\frac{\partial h(t,x)}{\partial x} \equiv \frac{g(t)}{b(t,x)}, \tag{20}$$

then the following derivatives hold

$$\frac{\partial^2 h}{\partial t \partial x} \equiv \frac{b(t,x)g'(t) - g(t)\,\partial b(t,x)/\partial t}{b^2(t,x)}, \quad \frac{\partial^2 h}{\partial x^2} \equiv -\frac{g(t)\,\partial b(t,x)/\partial x}{b^2(t,x)}.$$

By substituting the appropriate derivatives into (19), we obtain finally

$$g' = g\,b\left[\frac{1}{b^2}\frac{\partial b}{\partial t} - \frac{\partial}{\partial x}\left(\frac{a}{b}\right) + \frac{1}{2}\frac{\partial^2 b}{\partial x^2}\right] \equiv 0. \tag{21}$$

Since the left side of this identity is independent of x, then

$$\frac{\partial}{\partial x}\left\{b\left[\frac{1}{b^2}\frac{\partial b}{\partial t} - \frac{\partial}{\partial x}\left(\frac{a}{b}\right) + \frac{1}{2}\frac{\partial^2 b}{\partial x^2}\right]\right\} \equiv 0. \tag{22}$$

If (22) holds, the function g, $g \not\equiv 0$, can be found as a solution to the ordinary differential equation (21). The function h, which is at least locally invertible since $\partial h/\partial x \neq 0$, can be determined from (20), and the function f from (17). Then (21) is equivalent with (19) and thus the functions f and g are independent of x. Therefore, *the SDE* (6) *is reducible if and only if f and g satisfy* (22).

Let us suppose that (22) holds. Then:

(i) If $g \equiv 1$, then $h(t,x) = \int_{x_0}^{x} \frac{du}{b(t,u)}$, x_0 =const.;

(ii) If $f \equiv 0$, then h must be a solution of the partial differential equation $h'_t + ah'_x + \frac{1}{2}b^2 h''_{xx} = 0$;

(iii) If the SDE (6) is autonomous, i.e., $a(t,x) = a(x)$, $b(t,x) = b(x)$, then it is reducible if and only if

$$b\left[\frac{1}{2}b'' - \left(\frac{a}{b}\right)'\right] = c, \quad c = \text{const.}$$

From (21) and (18) we obtain $g(t) = e^{ct}$, $h(t,x) = e^{ct}\int_{x_0}^{x} \frac{du}{b(u)}$ respectively.

Note that, in general, linear SDE-s are not reducible. For the SDE (14) the reducibility condition becomes

$$\beta(t)b'(t) - [\,\alpha(t)b(t) - a(t)\beta(t) + \beta'(t)\,]\,b(t) \equiv 0,$$

until the homogeneous linear SDE is always reducible.

III. Let us present now a very strict type of reducibility, illustrated by the autonomous SDE. The fact that the linear SDE (14) is solvable motivates us to find an invertible transformation $y = h(x)$, such that the transformed equation be linear with constant coefficients. In other words, we require the existence of the constants $\alpha, \beta, \gamma, \delta$, $\delta \neq 0$, such that the conditions (15) and (16) become

$$a(x)h'(x) + \frac{1}{2}b^2(x)h''(x) \equiv \alpha + \beta h(x), \qquad b(x)h'(x) \equiv \gamma + \delta h(x). \tag{23}$$

If we assume $b \neq 0$, then $h(x)$ is a solution of the linear ordinary differential equation $b(x)h' - \delta h = \gamma$. Thus,

$$h(x) = c\,e^{\delta B(x)} - \gamma/\delta,$$

where $B(x) = \int_{x_0}^{x} \frac{du}{b(u)}$ and x_0 and c are some constants. The substitution of $h(x)$ into (23) gives finally

$$\left\{ \left[\frac{a(x)}{b(x)} - \frac{1}{2} b'(x) \right] \delta + \frac{1}{2}\delta^2 - \beta \right\} e^{\delta B(x)} \equiv \frac{\alpha\gamma - \beta\delta}{c\gamma}.$$

Replacing $A(x) = \frac{a(x)}{b(x)} - \frac{1}{2} b'(x)$ in the last identity and differentiating results, we have

$$\left\{ \left[A(x)\delta + \frac{1}{2}\delta^2 - \beta \right] \frac{1}{b(x)} + A'(x) \right\} \delta\, e^{\delta B(x)} \equiv 0.$$

Differentiating again we finally obtain

$$\delta\, A'(x) + (\, b(x)\, A'(x)\,)' \equiv 0.$$

From that

$$A'(x) \equiv 0 \quad \text{or} \quad \left(\frac{(\, b(x)\, A'(x)\,)'}{A'(x)} \right)' \equiv 0 \tag{24}$$

follows. Conversely, if the last condition in (24) is satisfied, then the transformation

$$h(x) = c e^{\delta B(x)}, \quad \text{where} \quad \delta = -\frac{(\, b(x)\, A'(x)\,)'}{A'(x)},$$

reduces the autonomous SDE to the linear form. Also, for $\delta = 0$ the simple choice $h(x) = \gamma B(x) + c$ leads to the reducibility condition

$$(\, b(x)\, A'(x)\,)' \equiv 0.$$

At the end, let us indicate briefly how to apply the foregoing results to find the explicit solution of the autonomous nonlinear SDE

$$dx(t) = \lambda x(t)\Big(1 - \frac{x(t)}{k}\Big)\, dt + \mu x(t)\, dw(t), \quad x(0) = \eta \text{ a.s.}, \quad t \geq 0,$$

where $\lambda,\ k, \mu$ are constants. This equation is reducible in the previous sense, because the condition (24) is valid. It is easy to conclude that $\delta = -\mu$, $h(x) = 1/x$, and from (23) that $\alpha = \lambda/k$, $\beta = -\lambda + \mu^2$, $\gamma = 0$. So, the original SDE is transformed to the linear form

$$dy(t) = \Big[\frac{\lambda}{k} + (-\lambda + \mu^2) y(t)\Big] dt - \mu y(t)\, dw(t), \quad y(0) = \eta^{-1} \text{ a.s.}, \quad t \geq 0.$$

Now it is easy to obtain the explicit solution of the original equation,

$$x(t) = \frac{1}{y(t)} = \frac{\exp\left\{ (\lambda - \mu^2/2)t + \mu w(t) \right\}}{\eta^{-1} + \frac{\lambda}{k} \int_0^t \exp\left\{ (\lambda - \mu^2/2)s + \mu w(s) \right\} ds}, \quad t \geq 0.$$

References

[1] L. Arnold, *Stochastic Differential Equations; Theory and Applications*, Wiley, New York, 1973.

[2] M.T. Barlow, E. Perkins, *One-dimensional stochastic differential equations involving a singular increasing process*, Stochastics **12** (1984), 229–249.

[3] M. Berger, V. Mizel, *Volterra equations with Itô integrals–I*, J. Integral Equations **2** (1980), 187–245.

[4] M. Berger, V. Mizel, *Volterra equations with Itô integrals–II*, J. Integral Equations **4** (1980), 319–337.

[5] A.T. Bharucha-Reid, *Random Integral Equations*, Academic Press, New York, 1972.

[6] C. Doléans-Dade, P.A. Meyer, *Integrales stochastiques par rapport aux martingales locales*, Sém. Probab. IV, Lect. Notes Math. **124** (1970), 77–107.

[7] J.L. Doob, *Stochastic Processes*, Wiley, New York, 1953.

[8] A. Friedman, *Stochastic Differential Equations and Applications*, Academic Press, New York, 1975.

[9] I.I. Gikhman, A.V. Skorokhod, *Stochastic Differential Equations*, Naukova Dumka, Kiev, 1968. (In Russian)

[10] I.I. Gikhman, A.V. Skorokhod, *Introduction to the Theory of Random Processes*, Nauka, Moscow 1977. (In Russian)

[11] I.I. Gikhman, A.V. Skorokhod, *Stochastic Differential Equations and Applications*, Naukova Dumka, Kiev, 1982. (In Russian)

[12] D.N. Hower, E. Perkins, *Nonstandard construction of the stochastic integral and applications to stochastic differential equations, I, II*, Trans. Amer. Math. Soc. **275** (1983), 1–58.

[13] N. Ikeda, Sh. Watanabe, *A comparison theorem for solutions of stochastic differential equations and its applications*, Osaka J. Math. **14** (1977), 619–633.

[14] N. Ikeda, Sh. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North Holland, Amsterdam, 1981.

[15] K. Itô, *Stochastic Integrals*, Proc. Imp. Acad. Tokyo **20** (1944), 519–524.

[16] K. Itô, *On stochastic differential equations*, Mem. Amer. Math. Soc. **4** (1951), 1–51.

[17] K. Itô, *On a formula concerning stochastic differentials*, Nagoya Math. J. **3** (1951), 55–65.

[18] K. Itô, M. Nisio, *On stationary solutions of a stochastic differential equation*, J. Math. Kyoto Univ. **4** (1964), 1–75.

[19] K. Itô, Sh. Watanabe, *Transformation of Markov processes by multiplicative functionals*, Ann. Inst. Fourier **15** (1965), 15–30.

[20] K. Itô, *Stochastic differentials*, Appl. Math. Optim. **1**(4) (1975), 374–381.

[21] K. Itô, Sh. Watanabe, *Introduction to stochastic differential equations*, Proc. of Intern. Symp. SDE, Kyoto, 1978, 1–20.

[22] J. Jacod, *Calcul stochastique et problems de martingales*, Lect. Notes Math. **714** Springer-Verlag, 1979.

[23] Sv. Janković, *Iterative procedure for solving stochastic differential equations*, Math. Balkanica, New Series **1**(1) (1987), 64–71.

[24] M. Jovanović, Sv. Janković, *On a class of nonlinear stochastic hereditary integrodifferential equations*, Indian J. Pure Appl. Math. **28** (1997), 1061–1082.

[25] G. Kallianpur, *Stochastic Filtering Theory*, Springer-Verlag, New York, 1980.

[26] S. Kanagawa, *The rate of convergence for approximate solutions of stochastic differential equations*, Tokyo J. Math. **12**(1) (1989), 31–48.

[27] H. Kunita, Sh. Watanabe, *On square integrable martingales*, Nagoya Math. J. **30** (1967), 209–245.

[28] G.S. Ladde, V. Lakshmikantham, *Random Differential Inequalities*, Academic Press, New York, 1980.

[29] V.A. Lebedev, *On the existence of a solution of the stochastic equations with respect to a martingale and a stochastic measure*, Internat. Sump. on Stoch. Diff. Equations, Vilnius, 1975, 69–89.

[30] R.Sh. Liptzer, A.N. Shiryayev, *Statistics of Random Processes*, Nauka, Moscow, 1974. (In Russian)

[31] R.Sh. Liptzer, A.N. Shiryayev, *Theory of Martingales*, Nauka, Moscow, 1986. (In Russian)

[32] H.P. McKean, *Stochastic Integrals*, Academic Press, New York, 1969.

[33] A.V. Mel'nikov, *On properties of strong solutions of stochastic equations with respect to semimartingales*, Stochastics **8** (1982), 103–120.

[34] P.A. Meyer, *A decomposition theorem for supermartingales*, Illinois J. Math. **2** (1962), 193–205.

[35] P.A. Meyer, *Decompositions supermartingales; the uniqueness theorem*, Illinois J. Math. **7** (1963), 1–17.

[36] P.A. Meyer, *Probability and Potentials*, Blaisdell, Waltham, 1966.

[37] V. Mizel, V.Trutzer, *Stochastic hereditary equations: Existence and asymptotic stability*, J. Integral Equations **7** (1984), 1–72.

[38] W.J. Padgett, *The method of random contractors and its applications to random nonlinear equations*, Probab. Analysis and Related Topics, A.T. Bharucha-Reid, Ed. **3** (1983), 195–255.

[39] J. Stoyanov, *Stochastic Processes*, Nauka, Sofia, 1978. (In Bulgarian)

[40] A.Yu. Veretenikov, *On the strong solutions of stochastic differential equations*, Theory Probab. Appl. **24** (1979), 354–366.

[41] D. Williams, *Diffusions, Markov Processes and Martingales*, (Vol. I), Wiley, New York, 1979.

[42] E. Wong, M. Zakai, *On the relationship between ordinary and stochastic differential equations*, Int. J. Engng. Sci. **3** (1965), 213–229.

[43] E. Wong, M. Zakai, *On the convergence of ordinary integrals to stochastic integrals*, Ann. Math. Stat. **36** (1965), 1560–1564.

[44] E. Wong, M. Zakai, *On the relationship between ordinary and stochastic differential equations and applications to stochastic problems in control theory*, Proc. 3rd IFAC Congress, 1966, paper 313.

[45] E. Wong, *Stochastic Processes in Information and Dynamical Systems*, McGraw Hill, New York, 1971.

[46] K. Yamada, *A stability theorem for stochastic differential equations and applications to stochastic control problems*, Stochastics **13** (1984), 257–297.

[47] J.A. Yan, *Martingale and Stochastic Integral Theory*, Shangai Science and Technology Press, 1981.