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VECTORS OF THE BODY MASS MOMENTS

To the Memory of my professors

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Teachers and Friends

Summary

This monograph paper introduces the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(N)}$ *of the body mass inertia moment at the point* N *for the axis oriented by the unit vector* $\vec{n}$*. The vector is used for interpretation of the rigid body kinetic characteristics. The change of the vector of the rigid body mass inertia moment is determined in the transition from one space point to another when the axis retains its orientation which represents the Huygens-Steiner theorem translated for the defined body mass inertia moment vector. Then the change of the vector of the body mass inertia moment is defined at the given point in the case of the axis changing its orientation in the way analogous to the Cauchy equations in the Elasticity theory. Then the interpretation of the main mass inertia moments asymmetry are defined. The relation between the axis deviation load vector by the body mass inertia moment for the octahedron axis and the inertia mass asymmetry moments axis is analyzed.*

This paper defines three dynamic vectors fixed to a certain point and axis passing through the given rigid body point. These are: the vector $\vec{\mathcal{M}}_{\vec{n}}^{(N)}$ *of the body mass at the point* N *for the axis oriented by the unit vector* $\vec{n}$*; the vector* $\vec{\mathfrak{S}}_{\vec{n}}^{(N)}$ *of the body mass static (linear) moment at the point* N *for the axis oriented by the unit vector* $\vec{n}$*; and the vector* $\vec{\mathfrak{J}}_{\vec{n}}^{(N)}$ *of the body mass inertia moment at the point* N *for the axis oriented by the unit vector* $\vec{n}$*. Also, the paper introduces the vectors:* $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ *of the material particle mass inertia moment for the pole* O *and the axis oriented by the unit vector* $\vec{n}$*, and* $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ *of the rigid body mass inertia moment for the pole* O *and the axis oriented by the unit vector* $\vec{n}$ *at the dimensional curvilinear coordinate system* N*.*

The rigid body kinetic parameters are interpreted by these vectors.

Future interpretation of the rigid body kinetic characteristics by means of the body mass inertia moment vector and by means of the body mass linear moment vector for the axis and the point refers to the description of the linear momentum, as well as angular momentum and kinetic energy as the functions of the body mass moment vectors and the angular velocity and the referential point velocity. The special cases of the rigid heavy body rotation are specially analyzed. The deviation part of the body mass inertia moment vector for the fixed point and for the rotation axis in view of the appearance of the dynamic pressure upon the bearings. The kinematic vector rotator is introduced as well as analyzed.

The spherical and the deviational parts of the mass inertia moment vector and of the mass inertia moment tensor are analyzed.

The conditions for dynamic balancing by means of the static mass moment vector and of the deviation load vector of the rotation axis by the rigid body mass inertia moment are shown.

The kinetic equations of a variable mass object motion rotating around a stationary axis are derived by means of the mass moment vectors for the pole and for the rotation axis: vector $\vec{\mathfrak{S}}_n^{(A)}$ *of the body mass linear moment, vector* $\vec{\mathfrak{J}}_n^{(A)}$ *of the body mass inertia moment for the pole A and for the axis oriented by the unit vector* $\vec{n}$ *and its deviational part of the vector* $\vec{\mathfrak{D}}_n^{(A)}$ *of the deviational load by the body mass inertia moment of the rotation axis through the pole A. The vectors of the reactive forces and resulting moments of the reactive forces due to the drop of the body particles are determined which are involved in the body mass change as the function of the body mass moments vector change: vector* $\vec{\mathfrak{S}}_n^{(A)}$ *of the body mass linear moment and vector* $\vec{\mathfrak{J}}_n^{(A)}$ *of the body mass inertia moment for the pole A and for the axis oriented by the unit vector* $\vec{n}$.

CONTENTS

CHAPTER I

I.1. Vectors of the body mass moments

I.1.1. Introduction. The idea for this monograph paper appeared during my considerations of some analogies between the models in the stress theory and the strain theory of the stressed and strained deformable bodies as they are studied or as they can be studied in Elasticity Theory (see **[15]**, **[14]**, **[28]**, **[25]**, **[34]** and **[23]**). While considering this analogy as well as the analogy between the stress tensor matrix, the relative deformation tensor-strain tensor matrix and the body mass inertia tensor matrix it occurred to me to introduce the concept of the vector $\vec{\delta}_{\vec{n}}^{(N)}$ of the total relative deformation — total strain, at the point N and for the line element drawn from that point and oriented by unit vector $\vec{n}$, as well as the concept of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(N)}$ of the body mass inertia moment at the point N, and for the axis oriented by the unit vector $\vec{n}$ (see **[A1]**, **[A2]**, **[A6]**. For more details see **[24]**, **[30]**, **[31]**, **[A5]**, **[35]**, **[37]**, **[38]**, **[34]** and **[23]**.

In further consideration of the dynamic parameters of the rigid and deformable bodies as well as of the possibility of their interpretation by means of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(N)}$ of the body mass inertia moment at the point N for the axis oriented by the unit vector $\vec{n}$, I came to the ideas and conclusions as well as interpretations given in my papers **[22]**, **[A2]** **[A4]**, **[24]**, **[34]** and **[23]**. The question always asked was if something like that already existed in some classic literature or not? The literature available to me which is quoted in the appendix of this paper contains no such interpretation of the rigid deformable bodies dynamic parameters by means of the mass inertia moment vector fixed to the point and to the axis.

This paper defines three dynamic vectors fixed to a certain point and axis passing through the given rigid body point. These are: the vector $\vec{\mathcal{M}}_{\vec{n}}^{(N)}$ of the body mass at the point N for the axis oriented by the unit vector $\vec{n}$; the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(N)}$ of the body mass static (linear) moment at the point N for the axis oriented by the unit vector $\vec{n}$; and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(N)}$ of the body mass inertia moment at the point N for the axis oriented by the unit vector $\vec{n}$ (see **[A1]**, **[A2]**, **[A6]**, and **[A7]**.

The rigid body kinetic parameters are interpreted by these vectors (see **[25]**, **[26]**, **[27]** and **[41]**).

The change of the mass inertia moment vector in the transition from one rigid body point to another is determined when the axis retains its orientation which represents the modification of the Huygens-Steiner theorem expressed by means of the defined mass inertia moment vector. Then the change of the mass inertia moment vector is determined in the case of the axis changing its orientation in the way analogous to the Cauchy equations for the total stress vector in the elasticity theory. Then the interpretation of the main inertia directions are derived as well as of the main mass inertia moment asymmetry are derived. The relation between the axis deviation load vector by the material body mass inertia moment for the octahedron axis and the mass inertia moments asymmetry axis is analyzed.

Further interpretation of the kinetic parameters of the of the body by means of the body mass inertia moment vector and by means of the body mass linear (static) moment vector for the axis and the point refers to the description of the motion quantity (linear momentum) as well as motion quantity moment (angular momentum) and kinetic energy as the function of the mass moment vectors for the axis and the point and the momentary angular velocity and referential point velocity (see **[A3]**, **[32]**, **[33]**, **[36]**, **[A6]**, **[39]**, **[A7]**, **[42]**, **[43]** and **[A8]**).

I.1.2. Body mass moments vectors at point for the axis. In studying the dynamics of a rigid and solid body, geometry of mass plays an important part. In **[3]** and **[4]** there is a conclusion that it is not necessary to know all the details about the mass distribution and the masses internal structures in order to study the rigid body translatory motion under the action of the force. The properties necessary for the study of the rigid body motion as a material system are the rigid body dynamic properties. The values determining the dynamic properties are called the rigid body dynamic parameters (see **[3]**).

According to the given reference these parameters are taken to be: mass M of the rigid body; position vector $\vec{\rho}_C$ of the body mass center, the point C with respect to a certain point O and $\mathbf{J}^{(C)}$ the body mass inertia moment tensor matrix for the point C which is determined with six scalar dynamic parameters. In this way in the general case the dynamic rigid body characteristic ten independent scalar dynamic parameters are required. By means of these ten dynamic parameters of the rigid body the sixth order matrix of the following shape is formed:

$$\mathbf{J}_{\mathrm{ex}}^{(O)} = \begin{pmatrix} M & 0 & 0 & 0 & Mz_C & -My_C \\ 0 & M & 0 & -Mz_C & 0 & Mx_C \\ 0 & 0 & M & My_C & -Mx_C & 0 \\ 0 & -Mz_C & My_C & J_x & D_{yx} & D_{zx} \\ Mz_C & 0 & -Mx_C & D_{xy} & J_y & D_{zy} \\ -My_c & Mx_C & 0 & D_{xz} & D_{yz} & J_z \end{pmatrix} \tag{1}$$

and this matrix is given in [3] and [4] as the rigid body mass inertia matrix for the given point O and the given trihedron. This is the *matrix of the tensor expanded* in an appropriate way. The mass inertia moment matrix changes its coordinates according to the change of the reference trihedron.

In [1] the mass linear polar moment $\vec{M}^{(O)}$ of the material system or the vector static system mass moment is defined with respect to the pole O in the form:

$$\vec{M}^{(O)} = \iiint\limits_V \vec{\rho}\, dm = \vec{\rho}_C M, \qquad dm = \sigma dV \tag{2}$$

where $\vec{\rho}$ is the vector of the rigid body points position with respect to the common pole O, V is the space region that the observed body occupies and σ is the mass density at all the body points.

There are two important properties of a certain body mass: the mass center position of a material body does not depend on the pole choice but only on the body mass distribution and the mass linear polar moment $\vec{M}^{(C)}$ with respect to the body mass center is equal to zero.

Since our aim is to consider a possibility of the interpretation of the rigid body dynamic parameters in a modified shape we are going to set, as a reference, the pole O as well as the axis oriented by the unit vector $\vec{n}$. Considering that the general case the rigid body motion can be represented by one rotation around momentary axis, that is, by the translation of the mass center velocity and the rotation around the axis through the given center we are led to the idea to define the rigid body dynamic parameters by means of the pole O as the referential point through we position an axis parallel to the momentary rotation axis (see **[41]**.

Therefore we define the following (see Fig. 1a):

1* Vector $\vec{\mathcal{M}}_{\vec{n}}^{(O)}$ of the body mass at the point O for the axis oriented by the unit vector $\vec{n}$ in the form:

$$\vec{\mathcal{M}}_{\vec{n}}^{(O)} \stackrel{\text{def}}{=} \iiint\limits_{V} \vec{n}\, dm = M\vec{n}, \quad dm = \sigma\, dV \tag{3}$$

which does not depend on the mass distribution in the body, that is, on the density. For all the space points and parallel axes it has the same values and it changes only with the axis orientation change. It is determined only with the mass quantity and the axis orientation.

2* Vector $\vec{\mathfrak{S}}_{\vec{n}}^{(O)}$ of the body mass static (linear) moment at the point O for the axis oriented by the unit vector $\vec{n}$ in the form:

$$\vec{\mathfrak{S}}_{\vec{n}}^{(O)} \stackrel{\text{def}}{=} \iiint\limits_{V} [\vec{n}, \vec{\rho}]\, dm, \quad dm = \sigma\, dV \tag{4}$$

where $\vec{\rho}$ is the vector of the rigid body points position of the elementary body mass dm with respect to the common pole O. For the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(O)}$ of the body mass static (linear) moment at the point O for the axis oriented by the unit vector $\vec{n}$ we can write:

$$\vec{\mathfrak{S}}_{\vec{n}}^{(O)} = [\vec{n}, \vec{\rho}_C]M = [\vec{n}, \vec{M}^{(O)}] \tag{5}$$

The illustration is given in the Figure 1a.

3* Vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of he body mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$ in the form (see **[A1]**, **[A2]**, **[A6]** and **[A7]**:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} \stackrel{\text{def}}{=} \iiint\limits_{V} [\vec{\rho}, [\vec{n}, \vec{\rho}]]\, dm \tag{6}$$

It can also be considered the body mass square moment vector at the point O for the axis, through the pole, oriented by the unit vector $\vec{n}$. The vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ at the body mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$ can be decomposed into three components: the collinear with the axis $J_{\vec{n}}^{(O)}$ and the two other ones $D_{nu}^{(O)}$ and $D_{nv}^{(O)}$ in the directions, $\vec{u}$ and $\vec{v}$, normal to the orientation axis $\vec{n}$. The collinear component represents the axial moment of the body mass inertia for the axis oriented by the unit vector $\vec{n}$ through the pole O. The other two components represent the deviational moments of the body mass for a couple of normal axes oriented by unit vectors $\vec{n}$ and $\vec{u}$, that is, $\vec{n}$ and $\vec{v}$:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = J_{n}^{(O)}\vec{n} + D_{nu}^{(O)}\vec{u} + D_{nv}^{(O)}\vec{v} \tag{7}$$

The definition–expression for the body mass inertia moment vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ at the point O for the axis oriented by the unit vector $\vec{n}$ can be obtained starting from the expression for the axial body mass inertia moment $J_{\vec{n}}^{(O)}$ for the axis oriented by unit vector $\vec{n}$ drawn through the point O and for the deviational body mass moments for the couples of the orthogonal axes oriented by unit vectors $(\vec{n},\vec{u})$ and $(\vec{n},\vec{v})$, $D_{nu}^{(O)}$ and $D_{nv}^{(O)}$, according to [25], [38]. By means of them we form the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$ in the form:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{n}\iiint\limits_V [\vec{n},\vec{\rho}]^2\,dm + \vec{u}\iiint\limits_V ([\vec{n},\vec{\rho}],[\vec{u},\vec{\rho}])\,dm + \vec{v}\iiint\limits_V ([\vec{n},\vec{\rho}],[\vec{v},\vec{\rho}])\,dm \tag{8}$$

The rigid body axial mass inertia moment is:

$$J_{\vec{n}}^{(O)} = \iiint\limits_V [\vec{n},\vec{\rho}]^2\,dm \tag{8*}$$

The rigid body mass deviation moment vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ at the point O for the axis oriented by the unit vector $\vec{n}$ is in the following form:

$$\vec{\mathfrak{D}}_{\vec{n}}^{(O)} = \vec{u}\iiint\limits_V ([\vec{n},\vec{\rho}],[\vec{u},\vec{\rho}])\,dm + \vec{v}\iiint\limits_V ([\vec{n},\vec{\rho}],[\vec{v},\vec{\rho}])\,dm = \vec{T}\iiint\limits_V ([\vec{T},\vec{\rho}],[\vec{n},\vec{\rho}])\,dm$$

$$\vec{\mathfrak{D}}_{\vec{n}}^{(O)} = \iiint\limits_V [\vec{v},[[\vec{\rho},[\vec{n},\vec{\rho}]]\vec{n}]]\,dm = [\vec{n},[\vec{\mathfrak{J}}_{\vec{n}}^{(O)},\vec{n}]] \tag{9}$$

By means of the previous expressions (8) for the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$ we can write the expression identical to the expression (6) which has been set as a definition.

Figure 1a shows the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$, the rigid body mass deviation moment vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ at the point O for the axis oriented by the unit vector $\vec{n}$, the axial moment of the body mass inertia $J_{\vec{n}}^{(O)}$ for the axis oriented by the unit vector $\vec{n}$ through the pole O, and the other two components, $D_{nu}^{(O)}$ and $D_{nv}^{(O)}$, the deviational moments of the body mass for a couple of normal axes oriented by unit vectors $\vec{n}$ and $\vec{u}$, that is, $\vec{n}$ and $\vec{v}$, through the pole O.

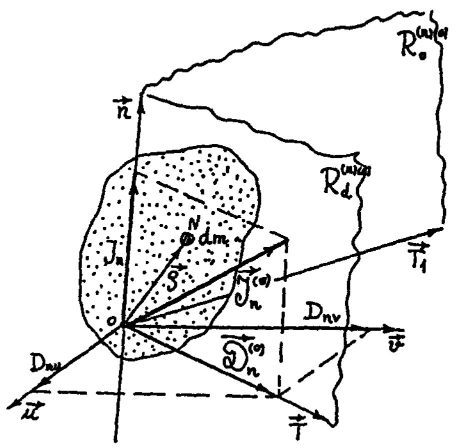

Fig. 1a

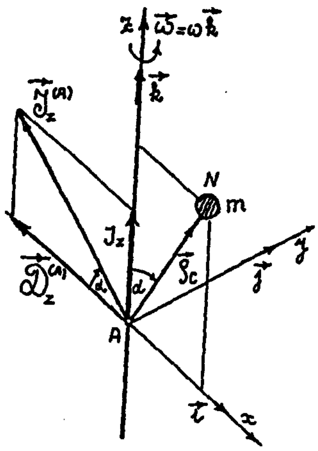

Fig. 1b

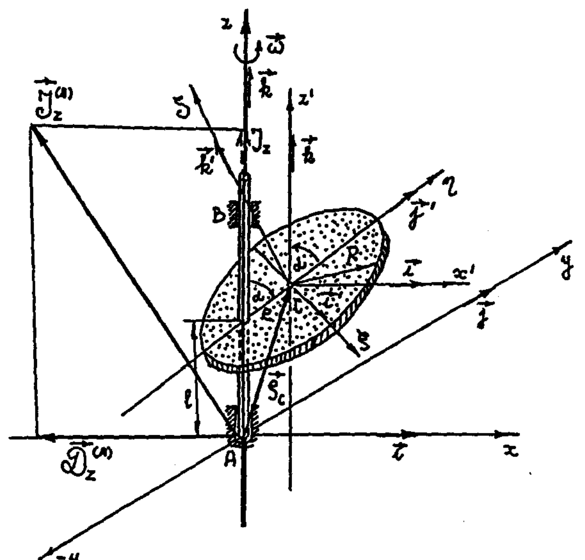

Fig. 1c

Fig. 1b shows the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the material particle mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$, the material particle mass deviation moment vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ at the point O for the axis oriented by the unit vector $\vec{n}$, the axial moment of the material particle mass inertia $J_{\vec{n}}^{(O)}$ for the axis oriented by the unit vector $\vec{n}$ through the pole O.

Fig. 1c shows an eccentrically skewly positioned discus respect to the axis of the shaft, as well as the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the discus mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$, the discus mass deviation moment vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ at the point O for the axis oriented by the unit vector $\vec{n}$, the axial moment of the discus mass inertia $J_{\vec{n}}^{(O)}$ for the axis oriented by the unit vector $\vec{n}$ through the pole O.

1.1.3. The material body mass inertia moment vectors for the two parallel axes through two referential points theorem. The Figure 2a shows the material body and two referential points - poles O and O_1 and two parallel axes through them oriented by unit vector $\vec{n}$. The same Figure also shows the denoted elementary mass dm at the point N of the rigid body and $\vec{\rho}$ and $\vec{r}$, the position vector of that point with respect to the pole O, that is, pole O_1, as well as the position vectors $\vec{\rho}_0$ of the pole O_1 with respect to pole O.

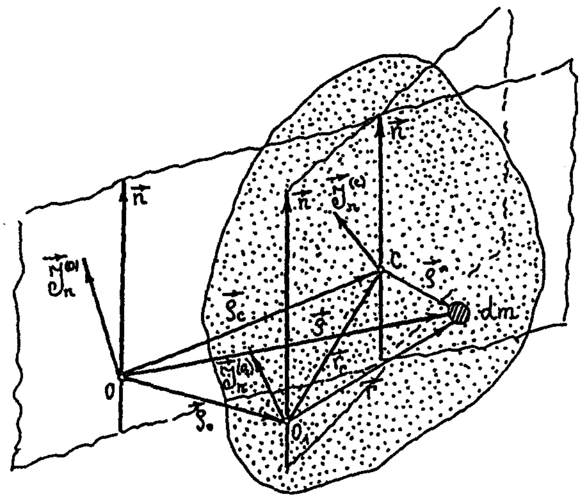

Fig. 2a

Now it is necessary to determine the change of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$ and its relation to the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the point O_1 for the axis oriented by the same unit vector $\vec{n}$.

This means we are interested in the change of the body mass inertia moment vector a certain axis which moves from one point to another retaining its orientation.

By using the expression (6) defining the mass inertia moment vector for a certain point and axis as well as the expression $\vec{\rho} = \vec{\rho}_O + \vec{r}$, we can write the following:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \iiint\limits_V [\vec{\rho}_O + \vec{r}, [\vec{n}, \vec{\rho}_O + \vec{r}]]\, dm =$$
$$= \vec{\mathfrak{J}}_{\vec{n}}^{(O_1)} + [\vec{\rho}_O, \vec{\mathfrak{S}}_{\vec{n}}^{(O_1)}] + [\vec{M}_c^{(O_1)}, [\vec{n}, \vec{\rho}_O]] + [\vec{\rho}_O, [\vec{n}, \vec{\rho}_O]]\, M \quad (10)$$

We see that all the members in the last expression have the same structures. These structures are: $[\vec{\rho}_O, [\vec{n}, \vec{r}_C]]\, M$, $[\vec{r}_C, [\vec{n}, \vec{\rho}_O]]\, M$ and $[\vec{\rho}_O, [\vec{n}, \vec{\rho}_O]]\, M$.

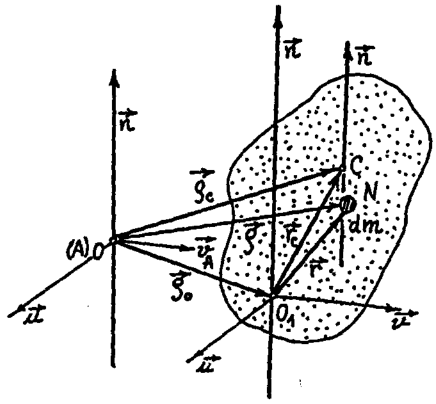

Fig. 2b

The expression (10) is the mathematical form of the theorem for the relation of the material body mass inertia moment vectors, $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ and $\vec{\mathfrak{J}}_{\vec{n}}^{(O_1)}$, for the two parallel axes through two corresponding points, pole O and pole O_1.

In the case when the pole O_1 is the center C of the body mass the vector $\vec{r}_C$ (the position vector of the masses center with respect to the pole O_1) is equal to zero, whereas the vector $\vec{\rho}_O$ turns into $\vec{\rho}_C$ so that the last expression (10) can be written in the following form (see Figure 2b):

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(C)} + [\vec{\rho}_C, [\vec{n}, \vec{\rho}_C]]\, M \quad (11)$$

This expression (11) represents the mathematical form of the theorem of the change of the mass moment vector for the pole and the axis when the axis is translated from the pole in the mass center C to the arbitrary point, pole O.

The Huygens-Steiner theorems (see [11], [1], [3] and [4]) for the axial mass inertia moment as well as for the mass deviational moments came from this theorem (11) about the change of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$ passing trough the mass center C and when the axis translate to the other point O.

The vector $\vec{\mathfrak{J}}_{\vec{n}}^{(C)}$ of the body mass inertia moment for the body mass center C as well as for the axis oriented by unit vector $\vec{n}$ passing trough the mass center C we are going to call *the central or proper (eigen, personal) vector of the body mass inertia moment for the axis oriented by unit vector* $\vec{n}$.

The part $\vec{\mathfrak{J}}_{\vec{n},\text{position}}^{(O)} = [\vec{\rho}_C, [\vec{n}, \vec{\rho}_C]]\, M$ from the expression (11) represents the position part of the body mass inertia moment vector and we going to call it *the body mass inertia position moment vector for the point O and the axis oriented by unit vector* $\vec{n}$ *in relation to the body mass center C.* We can see that the body mass inertia moment vector for the axis trough the mass center C is the *smallest* vector since for all the other parallel axes the position part $\vec{\mathfrak{J}}_{\vec{n},\text{position}}^{(O)} = [\vec{\rho}_C, [\vec{n}, \vec{\rho}_C]]\, M$ has to be taken into consideration. This can be expressed by means of the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(O)}$ of the body mass linear moment for the point O and the axis oriented by unit vector $\vec{n}$ in the form $[\vec{\rho}_C, \vec{\mathfrak{S}}_{\vec{n}}^{(O)}]$.

The vector $\vec{\mathfrak{J}}_{\vec{n},\text{position}}^{(O)} = [\vec{\rho}_C, [\vec{n}, \vec{\rho}_C]]\, M$ is the free vector as the moment of the couple:

$$\begin{aligned}\vec{\mathfrak{J}}_{\vec{n},\text{position}}^{(O)} &= [\vec{\rho}_C, [\vec{n}, \vec{\rho}_C]]\, M = \vec{\mathfrak{J}}_{\vec{n},\text{position}}^{(O\to C)} = [\vec{\rho}_C, \vec{\mathfrak{S}}_{\vec{n}}^{(O)}] = \vec{\mathfrak{J}}_{\vec{n},\text{position}}^{(C\to O)} \\ &= [-\vec{\rho}_C, -\vec{\mathfrak{S}}_{\vec{n}}^{(O)}] = [-\vec{\rho}_C, [\vec{n}, -\vec{\rho}_C]]\, M \end{aligned} \tag{11*}$$

This vector $\vec{\mathfrak{J}}_{\vec{n},\text{position}}^{(O)}$ can be moved from mass center C to arbitrary point O, as well as opposite from O to C, without change. This vector $\vec{\mathfrak{J}}_{\vec{n},\text{position}}^{(O)}$ is the moment of a couple of the mass linear position moment vectors: $-\vec{\mathfrak{S}}_{\vec{n}}^{(O)}$ in the pole O and $\vec{\mathfrak{S}}_{\vec{n}}^{(O)}$ in the pole mass center C.

Two vectors $-\vec{\mathfrak{S}}_{\vec{n}}^{(O)} = [\vec{n}, -\vec{\rho}_C]\, M$ and $\vec{\mathfrak{S}}_{\vec{n}}^{(O)} = [\vec{n}, \vec{\rho}_C]\, M$ having the same magnitude, parallel lines of the orientation, and opposite sense form a couple. Clearly, the sum of the moments of the two vectors about a given points, however, is not zero.

I.1.4. The change of the body mass inertia moment vector for the point and axis orientation change through the referential point. Let us now define the vectors $\vec{\mathfrak{J}}_x^{(O)}$, $\vec{\mathfrak{J}}_y^{(O)}$ and $\vec{\mathfrak{J}}_z^{(O)}$ of the body mass inertia moments at the point O and for the coordinate axes Ox, Oy and Oz. These vectors can be expressed in the form:

$$\vec{\mathfrak{J}}_x^{(O)} = \iiint\limits_V [\vec{\rho}, [\vec{i}, \vec{\rho}]]\, dm, \quad \vec{\mathfrak{J}}_y^{(O)} = \iiint\limits_V [\vec{\rho}, [\vec{j}, \vec{\rho}]]\, dm, \quad \vec{\mathfrak{J}}_z^{(O)} = \iiint\limits_V [\vec{\rho}, [\vec{k}, \vec{\rho}]]\, dm \tag{12}$$

If we denote the senses cosine of the unit vector $\vec{n}$ with $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ when the unit vector defines the orientation of the axis passing though the point O, then we can successively multiply the expressions (12) and we obtain them added:

$$\vec{\mathfrak{J}}_x^{(O)} \cos\alpha + \vec{\mathfrak{J}}_y^{(O)} \cos\beta + \vec{\mathfrak{J}}_z^{(O)} \cos\gamma = \iiint\limits_V [\vec{\rho}, [\vec{i}\cos\alpha + \vec{j}\cos\beta + \vec{k}\cos\gamma, \vec{\rho}]]\, dm$$
$$= \iiint\limits_V [\vec{\rho}, [\vec{n}, \vec{\rho}]]\, dm$$

From the previous expression we conclude that the body mass inertia moment vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ at the point O for the axis oriented by the unit vector $\vec{n}$ is equal to:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_x^{(O)} \cos\alpha + \vec{\mathfrak{J}}_y^{(O)} \cos\beta + \vec{\mathfrak{J}}_z^{(O)} \cos\gamma \tag{13}$$

The last expression is analogous to the equation for determining the total stress vector $\vec{p}_{\vec{n}}^{(O)}$ at the point O of the stressed body for the plane with normal unit vector $\vec{n}$ which is known as the Cauchy equation in the elasticity theory. There fore we are going to call it the Cauchy equation giving the relation of the body mass inertia moment vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ at the point O for the axis oriented by the unit vector $\vec{n}$ and the vectors $\vec{\mathfrak{J}}_x^{(O)}$, $\vec{\mathfrak{J}}_y^{(O)}$ and $\vec{\mathfrak{J}}_z^{(O)}$ of the body mass inertia moments at the point O and for the coordinate axes Ox, Oy and Oz.

I.1.5. Cauchy equations in the matrix form. Now by means of the mass inertia moment tensor matrix $\mathbf{J}^{(O)}$ the Cauchy vector equation (13) can be written in the matrix form:

$$\{\mathfrak{J}_{\vec{n}}^{(O)}\} = (\{\mathfrak{J}_x^{(O)}\}\{\mathfrak{J}_y^{(O)}\}\{\mathfrak{J}_z^{(O)}\})\{n\} = \mathbf{J}^{(O)}\{n\} \tag{14}$$

Now for the body mass axial inertia moment $J_{\vec{n}}^{(O)}$ for the axis oriented by the unit vector $\vec{n}$, as well as for the body mass deviation moment $D_{\vec{n}\vec{v}}^{(O)}$ for the orthogonal axes $\vec{n}$ and $\vec{v}$ we can write the following expressions:

$$J_{\vec{n}}^{(O)} = (n)\{\mathfrak{J}_{\vec{n}}^{(O)}\} = (n)\mathbf{J}^{(O)}\{n\}, \quad D_{\vec{n}\vec{v}}^{(O)} = (v)\{\mathfrak{J}_{\vec{n}}^{(O)}\} = (v)\mathbf{J}^{(O)}\{n\} \tag{15}$$

The invariants of the body mass inertia moment state at a certain point can be determined as the first $J_1^{(O)}$, second $J_2^{(O)}$ and third $J_3^{(O)}$ scalar of the body mass inertia moment tensor matrix.

The rigid body mass inertia moment tensor matrix $\mathbf{J}^{(O)}$ for a certain pole can be separated into two matrices corresponding to the spherical $\mathbf{J}^{(O)\mathrm{sph}}$ and deviational $\mathbf{J}^{(O)\mathrm{dev}} = \mathbf{D}^{(O)\mathrm{dev}}$ part of the rigid body mass inertia moment tensor (which is analogous to the stress tensor matrix and strain (relative deformation) tensor matrix in the elasticity theory):

$$\mathbf{J}^{(O)\mathrm{sph}} = \frac{1}{3} J_1^{(O)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}J_1^{(O)} & 0 & 0 \\ 0 & \frac{1}{3}J_1^{(O)} & 0 \\ 0 & 0 & \frac{1}{3}J_1^{(O)} \end{pmatrix}$$

$$\mathbf{J}^{(O)\text{dev}} = \mathbf{J}^{(O)} - \mathbf{J}^{(O)\text{sph}} = (\{\vec{\mathfrak{J}}_{\vec{n}}^{(O)}\}\{\vec{\mathfrak{J}}_{\vec{u}}^{(O)}\}\{\vec{\mathfrak{J}}_{\vec{v}}^{(O)}\}) - \frac{1}{3}J_1^{(O)}\,I \tag{17}$$

I.1.6. Axial and deviational part of the rigid body mass inertia moment vector. The body mass inertia moment vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ at the point O for the axis oriented by the unit vector $\vec{n}$ can be written in the transformed form in which we separate the part $\vec{\mathfrak{J}}_{\vec{n}}^{(O)\text{aks}}$ collinear with axis oriented by unit vector $\vec{n}$ and the part $\vec{\mathfrak{D}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(O)\text{dev}}$ normal to the axis oriented by unit vector $\vec{n}$ as it is shown in the Figures 1a and 3.

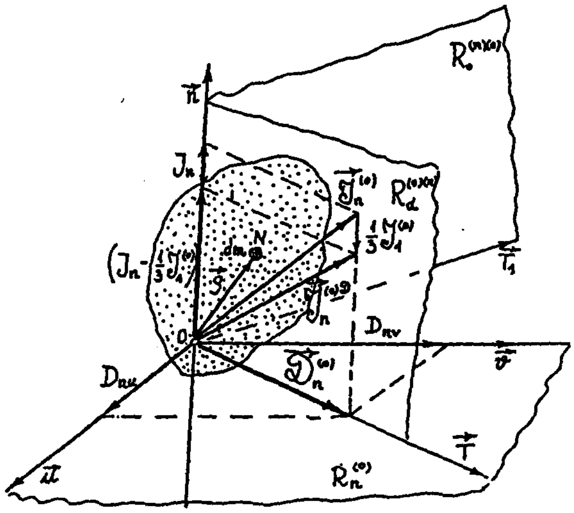

Figure 3

Now the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$ can be transformed to the following form:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(O)\text{aks}} + \vec{\mathfrak{J}}_{\vec{n}}^{(O)\text{dev}} = \vec{n}(\vec{\mathfrak{J}}_{\vec{n}}^{(O)}, \vec{n}) + [\vec{n}, [\vec{\mathfrak{J}}_{\vec{n}}^{(O)}, \vec{n}]] = \vec{\mathfrak{J}}_{\vec{n}}^{(O)\text{aks}} + \vec{\mathfrak{D}}_{\vec{n}}^{(O)} \tag{18}$$

with components:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)\text{aks}} = \vec{n}(\vec{\mathfrak{J}}_{\vec{n}}^{(O)}, \vec{n}) = \vec{n}J_{\vec{n}}^{(O)} \tag{19}$$

$$\vec{\mathfrak{D}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(O)\text{dev}} = [\vec{n}, [\vec{\mathfrak{J}}_{\vec{n}}^{(O)}, \vec{n}]] \tag{20}$$

The first part $\vec{\mathfrak{J}}_{\vec{n}}^{(O)\text{aks}}$ collinear with axis oriented by unit vector $\vec{n}$ given by formula (19) represents body mass axial inertia moment vector at the point and for the axis oriented by unit vector $\vec{n}$, and it does not depend on the pole position on the axis.

The second part $\vec{\mathfrak{D}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(O)\text{dev}}$ normal to the axis oriented by unit vector $\vec{n}$ given by formula (20) lies in the plane formed by the axis oriented by unit vector $\vec{n}$ and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment. This plane is determined by the axis selection and by the body mass distribution with respect to the axis and the pole.

The vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ is the deviation load by the rigid body mass inertia moment at the point O of the axis oriented by the unit vector $\vec{n}$ and it can be defined as the rigid body mass inertia moment vector component normal to the axis and in the plane which is formed by the axis oriented by the unit vector $\vec{n}$ and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment. This can be seen in the Figure 1a and 3. We conclude that the vector magnitude is equal to the deviation moment of the body mass for the axis oriented by the unit vector $\vec{n}$ and the axis oriented by the unit vector $\vec{T}$ normal to the axis oriented by the unit vector $\vec{n}$, in the direction of the cutting line of the plane normal to the axis trough the pole O and of the plane formed by the axis oriented by the unit vector $\vec{n}$ and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the pole and for axis oriented by the unit vector $\vec{n}$. The unit vector of this cutting line is denoted with $\vec{T}$. The unit vector normal to the unit vectors $\vec{n}$ and $\vec{T}$ is denoted with $\vec{T}_1$. We conclude that the body mass deviation moment for the axes $\vec{n}$ and $\vec{T}_1$ passing through the pole O is equal to zero. This means that for an arbitrary axis at the observed point O there can always be found at least one axis normal to it oriented by $\vec{T}_1$ for which, together with the axis oriented by the unit vector $\vec{n}$, the body mass deviation moment is equal to zero. This axis is normal to the axis oriented by the unit vector $\vec{n}$ and to the *deviation plane* formed by the unite vector $\vec{n}$ and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the pole O and for axis oriented by the unit vector $\vec{n}$. The deviation plane we denote by R_d. Only for the mass inertia moment main axis through a retain point-pole the deviation plane is not defined nor it can be said it exists since if the axis oriented by the unit vector $\vec{n}$ through a certain point is the main axis of the body mass inertia moment then for this axis the deviation load to the axis is equal to zero. In this case the body mass inertia moment vector has only one component collinear with the axis. That is, if a certain axis through a certain point-pole is the main mass inertia moment than the vector of its deviation load by the body mass inertia moment is equal to zero.

I.1.7. Spherical and deviatorial part of the rigid body mass moment vector.

If we now follow the idea of the formation of matrices of the spherical and deviatorial part of the mass inertia moment tensor according to the analogy (see [24], [23] and [34]) with the spherical and deviatorial part of the stress tensor, that is, of the relative deformation (strain) tensor we can define two vectors (see Figure 3):

$\vec{\mathfrak{J}}_{\vec{n}}^{(O)\text{sph}}$ the vector spherical part of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment at the pole O and for axis oriented by the unit vector $\vec{n}$:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)\text{sph}} = \frac{1}{3}J_1^{(O)}\vec{n} = \frac{1}{3}J_0^{(O)}\vec{n} \tag{21}$$

$\vec{\mathfrak{J}}_{\vec{n}}^{(O)D}$ the vector deviatorial part of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment at the pole O and for axis oriented by the unit vector $\vec{n}$:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)D} = \vec{n}(\vec{n},\vec{\mathfrak{J}}_{\vec{n}}^{(O)}) - \frac{1}{3}J_1^{(O)}\vec{n} + [\vec{n},[\vec{\mathfrak{J}}_{\vec{n}}^{(O)},\vec{n}]] = \vec{n}((\vec{n},\vec{\mathfrak{J}}_{\vec{n}}^{(O)}) - \frac{1}{3}J_1^{(O)}\vec{n}) + \vec{\mathfrak{D}}_{\vec{n}}^{(O)} \quad (22)$$

Let us now consider the modification of the Huygens-Steiner theorem in its application to the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)\mathrm{dev}} = \vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ the deviation part of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment at the pole O and for axis oriented by the unit vector $\vec{n}$, as well as the vector of the deviation load by the rigid body mass inertia moment on the axis oriented by the unit vector $\vec{n}$ in the transition from the mass center C to the pole O (see Figure 2b). We use the definition of the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ of the deviation load by the mass inertia moment (18) and the formula (11) derived in the paragraph I.1.3. for the Huygens-Steiner formula modified of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment at the pole O and for axis oriented by the unit vector $\vec{n}$ so that:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)\mathrm{dev}} = \vec{\mathfrak{D}}_{\vec{n}}^{(O)} = [\vec{n},[\vec{\mathfrak{J}}_{\vec{n}}^{(O)},\vec{n}]] = \vec{\mathfrak{D}}_{\vec{n}}^{(C)} - (\vec{n},\vec{\rho}_C)[\vec{n},[\vec{\rho}_C,\vec{n}]]\,M \quad (23)$$

The expression (23) represents the Huygens-Steiner Theorem modified to the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ of the deviation load by the mass inertia moment of the axis oriented by the vector $\vec{n}$ connected to the pole O. From this expression we conclude that the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ of the axis deviational load through an arbitrary point O oriented by the unit vector $\vec{n}$ equal to the sum of the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(C)}$ of the axis deviation load through the center C of the body mass for the parallel axis and the position deviation load in the transition of the axis from the pole C-mass center to the pole – arbitrary point O determined from the expression:

$$\vec{\mathfrak{D}}_{\vec{n}}^{(C\to O)} = [\vec{n},[[\vec{\rho}_C,[\vec{n},\vec{\rho}_C]]\vec{n}]]\,M = -(\vec{n},\vec{\rho}_C)[\vec{n},[\vec{\rho}_C,\vec{n}]]\,M \quad (24)$$

If the pole O and the center C of the body mass are located on the same normal to the axis oriented by the unit vector $\vec{n}$ then the position part of the deviation load in the transition from the axis through the mass center C to the parallel axis through the pole O is equal to zero. This means that the deviation load vectors of the axis by the body mass inertia moment for the central plane points corresponding to the given axis are equal to the deviation load belonging to the central axis $\vec{\mathfrak{D}}_{\vec{n}}^{(C)}$.

I.1.8. Main mass inertia moment directions, main mass inertia moment vectors. By means of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment at the pole O and for axis oriented by the unit vector $\vec{n}$ we can introduce a new definition of the main mass inertia momemt axes. Through one pole O we can draw an infinite number of axes of orientations. Among them we are looking for the axis for which the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment had only one component, collinear with the axis, that is, the one for which the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ of the deviation load of the axis by the body mass inertia moment is equal to zero.

Using the analogy given in the papers [245] and [34] as well as the analogy with the matrix interpretation from books [14], [15], [28] and [23] as more appropriate for this case and by denoting the unit vector of the main masss inertia moment axis orientation with $\vec{n}_s$ which is in accordance with the Fig. 3a, we can write:

$$\{\mathfrak{J}_{\vec{n}_s}^{(O)}\} = \mathbf{J}^{(O)}\{n_s\} = J_s^{(O)}\{n_s\} \Rightarrow (\mathbf{J}^{(O)} - J_s^{(O)}I)\{n_s\} = \{0\} \tag{25}$$

so that the Hamilton equation for determining the main mass inertia moments is:

$$f(J_s^{(O)}) = |\mathbf{J}^{(O)} - J_s^{(O)}I| = 0 \tag{26}$$

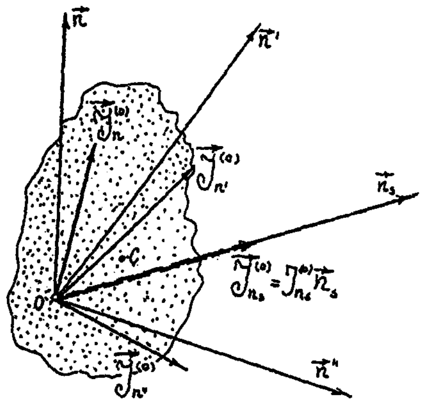

Fig. 3a

while for the senses cosines of the main mass inertia moment axes the following relations are obtained:

$$\frac{\cos\alpha_S}{\mathbf{K}_{31}^{(S)}} = \frac{\cos\beta_S}{\mathbf{K}_{32}^{(S)}} = \frac{\cos\gamma_S}{\mathbf{K}_{33}^{(S)}} = C_S, \quad \cos^2\alpha_s + \cos^2\beta_s + \cos^2\gamma_s = 1 \tag{27}$$

where $\mathbf{K}_{3k}^{(S)}$, $k = 1,2,3$ are co-factors of the third kind elements and the corresponding matrix column, successively for the roots $J_s^{(O)}$, $s = 1,2,3$ of the Hamilton equation (26), which are the main mass inertia moments and which represent the axial mass inertia moments for the main mass inertia moments axes. There are three roots and three orthogonal main axes at every point with respect to which the rigid body mass inertia moment vectors are determined. The Hamilton equation coefficients are the first, second and third invariants of the mass inertia moment state at referent point, and they are the first, second and third scalar of the body mass inertia moment tensor matrix at referent point (see [24] or [23]).

1.1.9. Extreme values of the mass deviation moments. In [24] is given an analogy between the stress state model, the strain state model and the mass inertia moment state of the body at the observed body point. For determining the mass deviation moments extreme values we shall use this analogy which exists between the stress tensor, the strain tensor and the body mass inertia tensor, as well as between the vector $\vec{p}_{\vec{n}}^{(O)}$ of the total stress at a certain body point for the plane with the normal oriented by unit vector $\vec{n}$, the vector $\vec{\delta}_{\vec{n}}^{(O)}$ of the total strain (relative deformation) of the line element drawn from the observed point in the direction of the unit vector $\vec{n}$ and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at the observed pole for the axis oriented by unit vector $\vec{n}$.

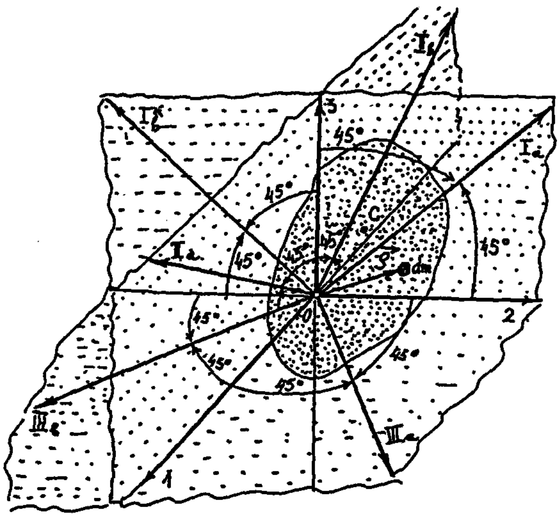

Figure 4a

On the basis of the given analogy in [24] and [23], the following conclusions are drawn, though without proofs: on the basis of the analogy between the mass deviation moments extreme values for a couple of orthogonal axes (that is, of the mass centrifugal moments) and yield stress extreme values in the orthogonal planes that pass in pair through one main stress direction and form an angle of 45° with the other two main stress direction, we conclude that the mass deviation moments extreme values appear for the axes pairs I_a and I_b, II_a and II_b, III_a and III_b that pass in pairs through the main body mass inertia moment axis trough the given point and form angles of 45° with the other two main mass inertia moment axes (see Figures 4a and 4b). For these pairs of the defined axes the mass deviation moments (the mass centrifugal moments) are equal to the semi-difference between the two main (axial) body mass inertia moment and for each axis in the corresponding pair the axial inertia moments are equal to the semi-sum of the two corresponding main moments of the body mass inertia for the given point.

The pairs of these coupled axes are the body mass inertia moments asymmetry axes since for them the mass centrifugal moments are extreme values and the axial mass inertia moments for both the axes in pair are mutually equal. The concept of "asymmetry" can be accepted since for symmetry axes the body mass centrifugal moment is equal to zero and for these axes the body mass centrifugal moment is of extreme value so that this leads to the conclusion about the asymmetry of the material body mass inertia moment properties. On the basis of the given analogy we cam write the values of the mass deviation moments and the body mass axial inertia moments of these axes (see Figures 4a and 4b):

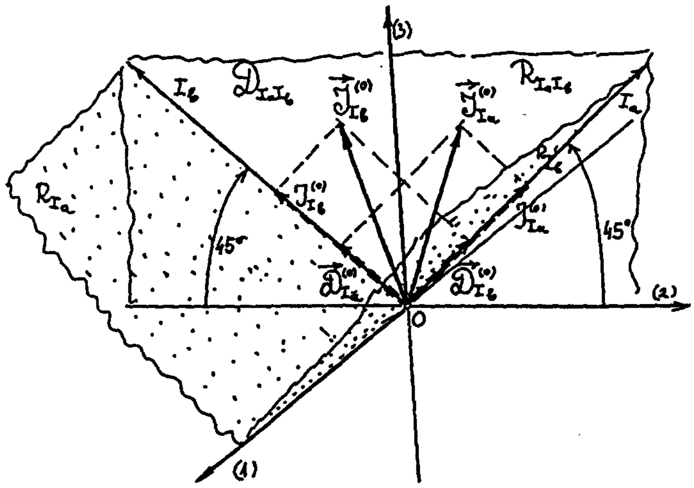

Figure 4b

$$
\begin{aligned}
D^{(O)}_{I_aI_b} &= \pm\frac{1}{2}(J^{(O)}_2 - J^{(O)}_3), & J^{(O)}_{I_a} &= J^{(O)}_{I_b} = \frac{1}{2}(J^{(O)}_2 + J^{(O)}_3),\\
D^{(O)}_{II_aII_b} &= \pm\frac{1}{2}(J^{(O)}_1 - J^{(O)}_3), & J^{(O)}_{II_a} &= J^{(O)}_{II_b} = \frac{1}{2}(J^{(O)}_1 + J^{(O)}_3), \qquad (28)\\
D^{(O)}_{III_aIII_b} &= \pm\frac{1}{2}(J^{(O)}_1 - J^{(O)}_2), & J^{(O)}_{III_a} &= J^{(O)}_{III_b} = \frac{1}{2}(J^{(O)}_1 + J^{(O)}_2).
\end{aligned}
$$

In the coordinate system of the main body mass inertia directions $\vec{n}_s$, $s = 1,2,3$ the vectors $\vec{\mathfrak{J}}^{(O)}_{\vec{n}_s}$, $s = 1,2,3$ for the referential point as the pole are the body mass inertia moment vectors for the main mass inertia moment axes and we see that they have only the components collinear with the corresponding main mass inertia moment axes $\vec{\mathfrak{J}}^{(O)}_{\vec{n}_s} = J^{(O)}_s\vec{n}_s$, $s = 1,2,3$.

Let's now define the vectors $\vec{\mathfrak{J}}^{(O)}_{I_a}$, $\vec{\mathfrak{J}}^{(O)}_{II_a}$ and $\vec{\mathfrak{T}}^{(O)}_{III_a}$ of the body mass inertia moment at the observed point for the axis oriented by the unit vector $\vec{n}_{I_a}$, or $\vec{n}_{II_a}$ or $\vec{n}_{III_a}$ of the mass inertia moment asymmetry axis I_a or II_a or III_a by using the definition of this vector so that we have (see Figure 4b):

$$\begin{aligned}
\vec{\mathfrak{J}}^{(O)}_{I_a} &= \frac{\sqrt{2}}{2}(\vec{\mathfrak{J}}^{(O)}_{n_2} + \vec{\mathfrak{J}}^{(O)}_{n_3});\\
\vec{\mathfrak{J}}^{(O)}_{II_a} &= \frac{\sqrt{2}}{2}(\vec{\mathfrak{J}}^{(O)}_{n_1} + \vec{\mathfrak{J}}^{(O)}_{n_3});\\
\vec{\mathfrak{J}}^{(O)}_{III_a} &= \frac{\sqrt{2}}{2}(\vec{\mathfrak{J}}^{(O)}_{n_1} + \vec{\mathfrak{J}}^{(O)}_{n_2});
\end{aligned} \tag{29}$$

Let's now define the vectors $\vec{\mathfrak{J}}^{(O)}_{I_b}$, $\vec{\mathfrak{J}}^{(O)}_{II_b}$ and $\vec{\mathfrak{J}}^{(O)}_{III_b}$ of the body mass inertia moment at the observed point for the axis oriented by the unit vector $\vec{n}_{I_b}$ or $\vec{n}_{II_b}$ or $\vec{n}_{III_b}$ of the mass inertia moment asymmetry axis I_b or II_b or III_b by using the definition of this vector so that we have:

$$\begin{aligned}
\vec{\mathfrak{J}}^{(O)}_{I_b} &= \frac{\sqrt{2}}{2}(-\vec{\mathfrak{J}}^{(O)}_{n_2} + \vec{\mathfrak{J}}^{(O)}_{n_3});\\
\vec{\mathfrak{J}}^{(O)}_{II_b} &= \frac{\sqrt{2}}{2}(-\vec{\mathfrak{J}}^{(O)}_{n_1} + \vec{\mathfrak{J}}^{(O)}_{n_3});\\
\vec{\mathfrak{J}}^{(O)}_{III_b} &= \frac{\sqrt{2}}{2}(-\vec{\mathfrak{J}}^{(O)}_{n_1} + \vec{\mathfrak{J}}^{(O)}_{n_2});
\end{aligned} \tag{30}$$

Now we define the components of the vector $\vec{\mathfrak{J}}^{(O)}_{I_b}$. The collinear one with the body mass inertia moments symmetry axis I_a:

$$(\vec{\mathfrak{J}}^{(O)}_{I_a}, \vec{n}_{I_a}) = \frac{J^{(O)}_2 + J^{(O)}_3}{2} = J^{(O)}_{I_a} = J^{(O)}_{I_b} \tag{31}$$

The component normal to the body mass inertia moment asymmetry axis lying in the deviation plane representing the vector $\vec{\mathfrak{D}}^{(O)}_{I_a}$ of the deviation load by the body mass inertia moment of the mass inertia moment asymmetry axis according to the previously given definition in the form:

$$\vec{\mathfrak{D}}^{(O)}_{I_a} = [\vec{n}_{I_a}, [\vec{\mathfrak{J}}^{(O)}_{I_a}, \vec{n}_{I_a}]] = \frac{J^{(O)}_2 - J^{(O)}_3}{2}\vec{n}_{I_b} = D^{(O)}_{I_a I_b}\vec{n}_{I_b} \tag{32}$$

Analysis the expressions from (28) to (32) we conclude the following:

1* The expressions given in (28) on the analogy basis are correct;

2* Both the vectors $\vec{\mathfrak{J}}^{(O)}_{I_a}$ and $\vec{\mathfrak{J}}^{(O)}_{I_b}$ of the rigid body mass inertia moments for the pole O and the axis of the pair I of the mass inertia moment asymmetry, I_a and I_b are normal to the main mass inertia moment axis (1) and they lie in the plane $R_{I_a I_b}$ which is their mutual deviation plane. This plane is normal to the main mass inertia moment axis (1) and contains the other two main mass inertia moment directions (2) and (3);

3* The vector $\vec{\mathfrak{D}}^{(O)}_{I_a}$ of the deviation load by the body mass inertia moment of the mass inertia moment asymmetry axis oriented by unit vector $\vec{n}_{I_a}$ at given point lie in the direction of the second mass inertia moment asymmetry axis oriented by unit vector $\vec{n}_{I_b}$ of the pair I which is normal to the main mass inertia moment direction (1) and to the axis the mass inertia moment asymmetry I_a and vice versa. These two vectors, that is, $\vec{\mathfrak{D}}_{I_a}^{(O)}$ and $\vec{\mathfrak{D}}_{I_b}^{(O)}$, are the same magnitude and of the same components, of axial and deviational, and they have the same axial mass inertia moments. In a similar way the calculation can be applied to the other two pairs of the mass inertia moment asymmetry axes and the corresponding conclusions can be drawn in accordance with the expressions (28) and the previous conclusions.

1.1.10. Mass inertia moment vectors for the octahedron directions in the referential point. In analogy with defining the octahedron directions a certain point of the stressed and strained body as it is done in the elasticity or plasticity theory we shall define the octahedron directions at a certain point of the rigid body form the viewpoint of the body mass inertia moment state with respect to this pole as the direction that forms the same angles with the main inertia axes, that is, with the main inertia directions. There are eight such octahedron directions.

The vector $\vec{\mathfrak{J}}_{\text{oct}}^{(O)}$ of the mass inertia moment at the point O for the octahedron direction by using the basic definition is calculated as:

$$\vec{\mathfrak{J}}_{\text{oct}}^{(O)} = \iiint_V [\vec{\rho}, [\vec{n}_{\text{oct}}, \vec{\rho}_C]]\, dm = \frac{\sqrt{3}}{3}(\vec{\mathfrak{J}}_{\vec{n}_1}^{(O)} + \vec{\mathfrak{J}}_{\vec{n}_2}^{(O)} + \vec{\mathfrak{J}}_{\vec{n}_3}^{(O)}) \tag{33}$$

and we can decompose it into two components.

1* The axial component in the octahedron direction in the form:

$$J_{\vec{n}_{\text{oct}}}^{(O)} = (\vec{n}_{\text{oct}}, \vec{\mathfrak{J}}_{\text{oct}}^{(O)}) = \frac{1}{3}J_1^{(O)} = \frac{2}{3}J_O^{(O)} \tag{34}$$

which represents the axial moment of the rigid body mass inertia moment for the octahedron direction axis for the given pole and it is equal to one third of the first mass inertia moment invariant or one third of the first scalar of the mass inertia polar moment for the pole O.

2* Normal component to the octahedron direction which is equal to the vector $\vec{\mathfrak{D}}_{\text{oct}}^{(O)}$ of the octahedron axis deviation load, by the body mass inertia moment and has the form:

$$\vec{\mathfrak{D}}_{\text{oct}}^{(O)} = -\frac{2\sqrt{6}}{9}(\vec{\mathfrak{D}}_{I_a}^{(O)} + \vec{\mathfrak{D}}_{II_a}^{(O)} + \vec{\mathfrak{D}}_{III_a}^{(O)}) \tag{35}$$

The vector $\vec{\mathfrak{D}}_{\text{oct}}^{(O)}$ of the deviation load by the body mass inertia moment of the octahedron axis can be expressed as the linear combination of the vectors $\vec{\mathfrak{D}}_{I_a}^{(O)}$, $\vec{\mathfrak{D}}_{II_a}^{(O)}$, $\vec{\mathfrak{D}}_{III_a}^{(O)}$ of the deviation load of the mass inertia moments asymmetry axes when it is related to one of the pair.

The intensity square of the vector $\vec{\mathfrak{D}}_{\text{oct}}^{(O)}$ of the deviation load by the body mass inertia moment of the octahedron axis can be defined by the following expression:

$$|\vec{\mathfrak{D}}_{\text{oct}}^{(O)}|^2 = \frac{4}{9}(|\vec{\mathfrak{D}}_{I_a}^{(O)}|^2 + |\vec{\mathfrak{D}}_{II_a}^{(O)}|^2 + |\vec{\mathfrak{D}}_{III_a}^{(O)}|^2) \tag{36}$$

It should be noted that there are eight axes (or four axes) at each point of the rigid body for which the mass inertia axial moments are equal to a third of the first mass inertia moment invariant and they are the octahedron directions determined with respect to the main mass inertia moment axes. The question should be asked about what sort of motion the body performs while rotating around the octahedron axis and if the conclusions can be generalized to hold for the bodies with different mass inertia moment characteristics.

If this conclusion is related to the previous section we can conclude that is: one set of eight (or four) axes for which the inertia axial moments of the body mass are mutually equal and equal to a third of the first mass inertia moment invariant: Three sets of two pairs of orthogonal axes of the inertia asymmetry for the axial inertia moments are also equal to the semi-sum of two main inertia moments each. The same stand for each body and for each pole chosen within the space or outside the space of the rigid body. Only the spherical body as the pole of all fourteen axes the axial mass inertia moment is the same and the deviation load is equal to zero.

I.2. The mass moment vectors at the dimensional coordinate system N

I.2.1. Introduction. This part introduces the vectors: $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the material particle mass inertia moment for the pole O and the axis oriented by the unit vector $\vec{n}$, and $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment for the pole O and the axis oriented by the unit vector $\vec{n}$ at the dimensional curvilinear coordinate system N. The vectors can be used for the interpretation of the rigid body kinetic characteristics for the interpretation of the body dynamics at the dimensional curvilinear coordinate system N.

The change of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body or particle mass inertia moment for the pole O and the axis oriented by the unit vector $\vec{n}$, is determined in the transition from one space point to another when the axis retains its orientation which represents Huygens-Steiner theorem generalized for the defined mass inertia moment vector at the dimensional curvilinear coordinate system N.

This part gives the interpretation of the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ of the deviation load by the material particles mass inertia moment at the point O of the axis oriented by the unit vector $\vec{n}$ at dimensional curvilinear coordinate system N as well as by body mass inertia moment at the point O of the axis oriented by the unit vector $\vec{n}$ at dimensional curvilinear coordinate system N.

I.2.2. The dimensional curvilinear coordinate system N. According to the notation in the Fig. 5 the material point position vector $\vec{\rho}$, at the dimensional coordinate system n, can be written in the form:

$$\vec{\rho} = x^k \vec{g}_k \tag{37}$$

while unit vector $\vec{n}$ of the axis orientation can be written in the form:

$$\vec{n} = \lambda^k \vec{g}_k \tag{38}$$

In the previous expression $\vec{g}_k$ the basic vectors of the dimensional N of the curvilinear coordinates $\vec{g}_k = \dfrac{\partial \vec{\rho}}{\partial x^k}$ for these vectors it stands that:

$$(\vec{g}_k, \vec{g}_l) = g_{kl} \tag{39}$$

their product represents the metric tensor coordinates of the defined curvilinear coordinates system space. The position vector $\vec{\rho}$ magnitude squared is:

$$(\vec{\rho}, \vec{\rho}) = (\vec{g}_k, \vec{g}_l)\, x^l x^k = g_{kl} x^k x^l \tag{40}$$

while for the axis orientation unit vector $\vec{n}$:

$$(\vec{n}, \vec{n}) = (\vec{g}_k, \vec{g}_l)\, \lambda^l \lambda^k = g_{kl} \lambda^l \lambda^k = 1 \tag{41}$$

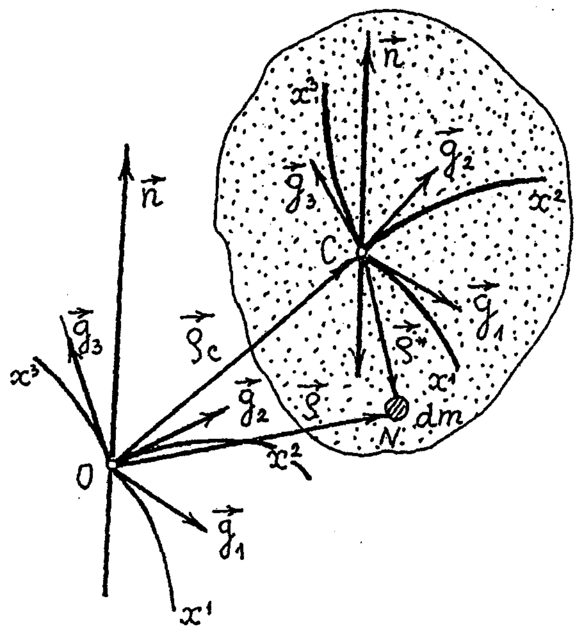

Figure 5

I.2.3. The material particle mass inertia moment vector for the pole and the axis.

By introducing the expression (37) and (38) into expression (6) for the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ definition of the material particle mass inertia moment for the pole O and the axis oriented by the unit vector $\vec{n}$, we obtain that:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = [\vec{g}_k, [\vec{g}_l, \vec{g}_p]] x^k x^p \lambda^l m \tag{42}$$

If we have in mind that the double vector product can be written in the transformed shape, the previous expression (42) can be write in the following form:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = (g_{kp}\vec{g}_l - g_{kl} g_p) x^k x^p \lambda^l m \tag{43}$$

If we multiply scalarly the previous expression (43) with the unit vector $\vec{n}$, we obtain:

$$J_{\vec{n}}^{(O)} = (\vec{\mathfrak{J}}_{\vec{n}}^{(O)}, \vec{n}) = (g_{kp}g_{li} - g_{kl}g_{pi})x^k x^p \lambda^l \lambda^i m \tag{44}$$

which represent the material particle mass axial inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$. This formula is same as the formula (2.3) in [6] written by Vujičić.

If we now multiply the expression (43) twice vectorly with the unit vector $\vec{n}$, that is, according to [40], we separate the material particle mass inertia moment vector deviational part for the pole O and the axis oriented by the unit vector $\vec{n}$ we obtain:

$$\vec{\mathfrak{D}}_{\vec{n}}^{(O)} = [[\vec{n}, [\vec{\mathfrak{J}}_{\vec{n}}^{(O)}, \vec{n}]] = \{g_{kp}g_{lj}\vec{g}_i - g_{ki}g_{lj}\vec{g}_p + (g_{ki}g_{lp} - g_{kp}g_{li})\vec{g}_j\}x^k x^p \lambda^l \lambda^i \lambda^j m \tag{45}$$

The last expression represents the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ of the deviation load by the material particles mass inertia moment at the point O of the axis oriented by the unit vector $\vec{n}$ at dimensional coordinate system N.

By introducing the expressions (37) and (38) into the expression (4) for the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(O)}$ definition of the material particle mass linear moment for the pole O and the axis oriented by the unit vector $\vec{n}$ we obtain that:

$$\vec{\mathfrak{S}}_{\vec{n}}^{(O)} = [\vec{g}_i, \vec{g}_k]x^k \lambda^i m \tag{46}$$

I.2.4. The rigid body mass inertia moment vector for the pole and the axis. By introducing the expression (37) and (38) into expression (6) for the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ definition of the rigid body mass inertia moment for the pole O and the axis oriented by the unit vector $\vec{n}$, we obtain that:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \iiint\limits_V [\vec{g}_k, [\vec{g}_l, \vec{g}_p]]x^k x^p \lambda^l dm \tag{47}$$

If we have in mind that the double vector product can be written in the transformed shape, the previous expression (47) can be written in the following form:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \iiint\limits_V (g_{kp}\vec{g}_l - g_{kl}\vec{g}_p)x^k x^p \lambda^l dm \tag{47*}$$

If we multiply scalarly the previous expression (48) with the unit vector $\vec{n}$, we obtain:

$$J_{\vec{n}}^{(O)} = (\vec{\mathfrak{J}}_{\vec{n}}^{(O)}, \vec{n}) = \iiint\limits_V (g_{kp}g_{li} - g_{kl}g_{pi})x^k x^p \lambda^l \lambda^i dm \tag{48}$$

which represent the body mass axial inertia moment at the point O for the axis oriented by the unit vector $\vec{n}$.

If now we multiply the expression (48) twice vectorly with the unit vector $\vec{n}$, that is, according to [40], we separate the body mass inertia moment vector deviational part for the pole O and the axis oriented by the unit vector $\vec{n}$ we obtain:

$$\vec{\mathfrak{D}}_{\vec{n}}^{(O)} = [\vec{n}, [\vec{\mathfrak{J}}_{\vec{n}}^{(O)}, \vec{n}]] = \iiint\limits_V \{g_{kp}g_{lj}\vec{g}_i - g_{ki}g_{lj}\vec{g}_p + (g_{ki}g_{lp} - g_{kp}g_{li})\vec{g}_j\}x^k x^p \lambda^l \lambda^i \lambda^j dm \tag{49}$$

The last expression represents the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ of the deviation load by the body mass inertia moment at the point O of the axis oriented by the unit vector $\vec{n}$ at dimensional coordinate system N.

By introducing the expressions (37) and (38) into the expression (4) for the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(O)}$ definition of the body mass linear moment for the pole O and the axis oriented by the unit vector $\vec{n}$ we obtain that:

$$\vec{\mathfrak{S}}_{\vec{n}}^{(O)} = \iiint\limits_V [\vec{g}_i, \vec{g}_k] x^k \lambda^i dm \tag{46*}$$

1.2.5. The Huygens-Steiner theorem. Following previous expression (11) for the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the rigid body mass inertia moment for the pole O and the axis oriented by the unit vector $\vec{n}$, the Huygens-Steiner theorem is derived which can be written in the following form for the curvilinear coordinate system (see Fig. 2a):

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(C)} + [\vec{\rho}_C, [\vec{n}, \vec{\rho}_C]]M = \vec{\mathfrak{J}}_{\vec{n}}^{(C)} + [\vec{g}_k, [\vec{g}_l, \vec{g}_p]] x_C^k x_C^p \lambda^l M \tag{47*}$$

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(C)} + (g_{kp}\vec{g}_l - g_{kl}\vec{g}_p) x_C^k x_C^p \lambda^l M \tag{47**}$$

Following previous expression (23) for the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ of the deviation load by the rigid body mass inertia moment for the pole O and the axis oriented by the unit vector $\vec{n}$, the Huygens-Steiner theorem can be written in the following form in the curvilinear coordinate system:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)\mathrm{dev}} = \vec{\mathfrak{D}}_{\vec{n}}^{(O)} = \vec{\mathfrak{D}}_{\vec{n}}^{(C)} - g_{ij}[\vec{g}_k, [\vec{g}_l, \vec{g}_p]] x_C^j x_C^l \lambda^i \lambda^k \lambda^p M \tag{49*}$$

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)\mathrm{dev}} = \vec{\mathfrak{D}}_{\vec{n}}^{(O)} = \vec{\mathfrak{D}}_{\vec{n}}^{(C)} - g_{ij}(g_{kp}\vec{g}_l - g_{kl}\vec{g}_p) x_C^j x_C^l \lambda^i \lambda^k \lambda^p M \tag{49**}$$

which represents the expression of the Huygens-Steiner generalized to the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$.

CHAPTER II

II.1. Vector interpretations of the rigid bodies kinetic parameters

II.1.1. Rigid body kinetic energy. We shall consider the kinetic energy (see [**13**], [**7**], [**10**], and [**21**])a little with a slight modification due to the interpretation of the rigid body dynamic parameters by means of the introduced vectors $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the body mass linear moment at the pole A for the axis oriented by the unit vector $\vec{n}$ and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(A)}$ of the body mass inertia moment at the pole A for the axis oriented by the unit vector $\vec{n}$. Since the velocity of each body point (see [**12**])can be defined by the two kinematic parameters of the translation velocity $\vec{v}_A$ of the referential point A and the angular velocity $\vec{\omega}$ and the unit vector $\vec{n}$ of the momentary rotation axis orientation we shall define the kinetic energy in relation to the body mass state properties with respect to the referential point translation velocity and the body mass moments state for the pole at the referential point A and for the axis oriented by the momentary rotation axis unit vector.

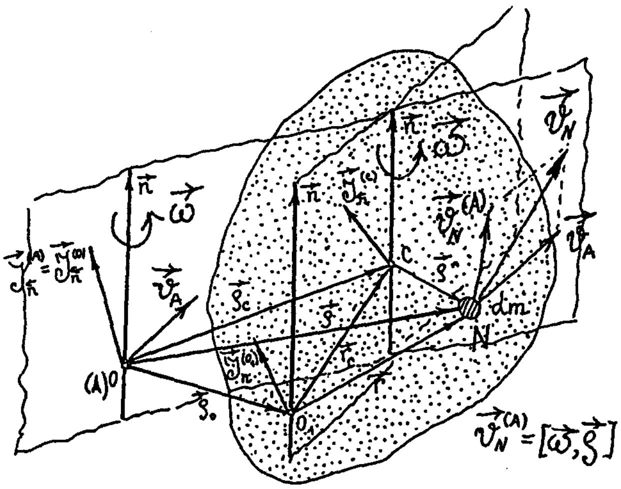

Figure 6

Using the notation in the Figures 2b and 6, for the rigid body kinetic energy we can write:

$$2E_k = \iiint\limits_V \vec{v}_N^2 dm \iiint\limits_V (\vec{v}_A + [\vec{\omega}, \vec{\rho}])^2 \, dm \tag{50}$$

that is, according to our idea the double kinetic energy is only expressed by means of the masses center velocity, body mass, momentary angular velocity and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(C)}$ of the body mass inertia moment for the axis through the body mass center at the pole C and oriented by the momentary angular velocity unit vector $\vec{n}$:

$$2E_k = M(\vec{v}_C, \vec{v}_C) + \omega(\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)}) \tag{51}$$

In the case when the referential point A is not the mass center the kinetic energy can be expressed in the form:

$$2E_k = M(\vec{v}_A, \vec{v}_A) + 2\omega(\vec{v}_A, \vec{\mathfrak{S}}_{\vec{n}}^{(A)}) + \omega(\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(A)}) \tag{52}$$

which is expressed by means of the velocity $\vec{v}_A$ of the arbitrary referential point A, angular velocity $\vec{\omega}$ of the rotation around the axis through the point A and mass moment vectors as the mass inertia moment properties of the rigid body mass distribution with respect to the pole at the referential point A and for axis oriented by the momentary angular velocity $\vec{\omega}$.

We can see that the kinetic energy has a part which corresponds to the body translation of the velocity $\vec{v}_A$ of the referential point A, that is, the part corresponding to the pure rotation around the relative rotation axis that passes through the referential point A and is oriented by the vector $\vec{\omega}$ of the momentary angular velocity, as well as the mixed member which represents the coupling of the translation and rotation and can be called *"Coriolis member"* representing the double scalar products of the velocity $\vec{v}_A$ of the referential point translation and the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the body mass linear moment at the referential point A for the axis oriented by the unit vector $\vec{n}$ multiplied by the angular velocity vector magnitude. This third member represents the kinetic energy of the coupling of the translation motion of the referential pole velocity and the rotation motion around the axis through this referential point.

This *"Coriolis member"* which represents the kinetic energy of the rotatory and translatory motion coupling with respect to the referential pole is equal zero in the following cases:

1* when the translatory velocity of the referential point A is orthogonal to the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the body mass linear moment at the referential point A for the axis oriented by the unit vector $\vec{n}$, that is when the velocity $\vec{v}_A$ of the referential point A is parallel to the plane formed by the rotation axis through referential pole A and the body mass center;

2* when the referential point A is on the momentary rotation axis or at the momentary rotation pole; and

3* when the referential point A is at the body mass center C.

The expression (51) represents the modified expression of the Samuel Köning theorem for the kinetic energy, that is, the Samuel Köning theorem in new interpretation, which states that the rigid body kinetic energy is equal to the sum of the kinetic energy of its translator motion with mass center velocity and the kinetic energy of its rotation motion around the axis oriented by the momentary angular velocity through the body mass center.

If the referential point is at the momentary pole all the time, or the momentary rotation axis then the kinetic energy can be expressed as:

$$2E_k = \omega(\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(P)}) \tag{53}$$

and it has only the member corresponding to the rotation around the momentary rotation axis and is equal to the half of the product of the momentary angular velocity squared and the axial inertia moment for the momentary rotation axis as it is known.

II.1.2. Linear momentum and angular momentum of the body motion.

The classic literature (see [10], [7], [11]) gives a very well known definition of the rigid body linear momentum (motion quantity) and angular momentum (motion quantity moment). We shall consider it a little with a slight modification due to the interpretation of the rigid body dynamic parameters by means of the introduced body mass moment vectors. We are following the classic definition by using the prepositions from previous paragraph, as well as Fig. 6, so that we write for the linear momentum following expression:

$$\vec{\mathfrak{K}} = \iiint\limits_V \vec{v}_N dm = \iiint\limits_V (\vec{v}_A + [\vec{\omega}, \vec{\rho}])dm = M\vec{v}_A + \omega\vec{\mathfrak{S}}_{\vec{n}}^{(A)} \tag{54}$$

The expression (54) of the linear momentum $\vec{\mathfrak{K}}$ of the rigid body whose points have the translation velocity $\vec{v}_A$ of the referential point A and the relative velocity $[\vec{\omega}, \vec{\rho}]$ due to the rotation around the axis oriented by the vector $\vec{\omega} = \omega\vec{n}$ through the point A has two parts: 1* the translatory one equal to the product of the referential point velocity and the body mass—the linear momentum due to the translation motion with the velocity of the referential point A; and 2* the rotatory one equal to the product of the magnitude ω of the angular velocity $\vec{\omega} = \omega\vec{n}$ and the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the body mass linear moment at the referential point A for the axis oriented by the unit vector $\vec{n}$.

If the pole A is the body mass center C then the linear momentum is equal only in the translatory part since the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the body mass linear moment for the pole in the body mass center is equal to zero regardless of its orientation so that the linear momentum is equal to the product of this velocity $\vec{v}_C$ of the body mass center and the rigid body mass: $\vec{\mathfrak{K}} = M\vec{v}_C$. The same stands for if the pole A is not the body mass center but if the axis oriented with $\vec{\omega} = \omega\vec{n}$ trough pole A passes trough the mass center.

The second kinetic vector connected to the referential point which plays an important part (role) in the rigid body dynamics is the rigid body angular momentum (motion quantity moment) for the given pole, $\vec{\mathfrak{L}}_O$. Following the classic definition according to [1], [3] and [11] and according to the notation given in the Fig. 6 the rigid body angular momentum is calculated by means of the following expression:

$$\vec{\mathfrak{L}}_O = \iiint_V [\vec{r}, \vec{v}_N] dm = \iiint_V [\vec{r}_A + \vec{\rho}, \vec{v}_A + [\vec{\omega}, \vec{\rho}]] \, dm \tag{55}$$

Following the idea of this paper that at the basis of the rigid body motion interpretation there are rigid body dynamic parameters which express the mass inertia moment properties and the kinematic parameters, translation velocity $\vec{v}_A$ of the rigid body referential point and the angular velocity $\vec{\omega}$ of the relative momentary rotation around the axis oriented with $\vec{\omega}$ and through the referential point A then the angular momentum for the point A, $\vec{\mathfrak{L}}_A$ is connected not only to the pole but to the axis oriented by the momentary angular velocity vector to which we connect the vectors $\vec{\mathcal{M}}^{(A)}$ and $\vec{\mathfrak{J}}_{\vec{n}}^{(A)}$ of the rigid body mass linear and inertia moments by connecting the body mass to the translation velocity of the referential point A. Therefore we write that it is:

$$\vec{\mathfrak{L}}_A = [\vec{\mathcal{M}}^{(A)}, \vec{v}_A] + \omega \vec{\mathfrak{J}}_{\vec{n}}^{(A)}, \quad \vec{\mathcal{M}}^{(A)} = \vec{\rho}_C \mathcal{M} \tag{56}$$

that is,

$$\vec{\mathfrak{L}}_O = [\vec{\mathcal{M}}^{(A)}, \vec{v}_A] + \omega \vec{\mathfrak{J}}_{\vec{n}}^{(A)} + [\vec{r}_A, M\vec{v}_A + \omega \vec{\mathfrak{S}}_{\vec{n}}^{(A)}] \tag{57}$$

If the referential point A is in the body mass center than the angular momentum for the pole O is equal to:

$$\vec{\mathfrak{L}}_O = [\vec{\mathcal{M}}^{(O)}, \vec{v}_C] + \omega \vec{\mathfrak{J}}_{\vec{n}}^{(C)}, \quad \vec{\mathcal{M}}^{(O)} = \vec{v}_C \mathcal{M} \tag{58}$$

while the angular momentum for the pole in the mass center C is:

$$\vec{\mathfrak{L}}_C = \omega \vec{\mathfrak{J}}_{\vec{n}}^{(C)} \tag{59}$$

and it is equal to the product of the magnitude of the momentary angular velocity ω and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(C)}$ of the rigid body mass inertia moment for the central axis oriented by the vector of the momentary angular velocity $\vec{\omega}$.

The Ref. [3] has the deviation center of the body for the given direction for the material particles system and the deviation load by the linear momentum analysis. Considering that we have introduced the deviation load vector by the analysis of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(A)}$ of the body mass inertia moment as its component normal to the axis for which it is determined we can see that the deviational part of the angular momentum vector proportional to the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(A)}$ of the deviational load the body mass inertia moment of the axis around which the rigid body rotates since it is:

$$\vec{\mathfrak{L}}_A = [\vec{\mathcal{M}}^{(A)}, \vec{v}_A] + \vec{\omega}(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(A)}) + \omega[\vec{n}[\vec{\mathfrak{J}}_{\vec{n}}^{(A)}, \vec{n}]] = [\vec{\mathcal{M}}^{(A)}, \vec{v}_A] + \vec{\omega}(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(A)}) + \omega \vec{\mathfrak{D}}_{\vec{n}}^{(A)} \tag{60}$$

If the point A is the mass center then it stands for:

$$\vec{\mathfrak{L}}_C = \vec{\omega}(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)}) + \omega \vec{\mathfrak{D}}_{\vec{n}}^{(C)} \tag{61}$$

If the rotation axis is the main mass inertia moment axis then the angular momentum does not have any deviational part since the rotation axis is not subjected to the deviation load by the rigid body mass inertia moment and the angular momentum vector for the mass center is collinear with the rotation axis.

II.1.3. Some interpretations for the case of the rigid body rotation around the fixed axis.

Figure 7 shows the rigid body with the rotation axis around which it rotates with the angular velocity $\vec{\omega}$ which changes in time so that there appears the angular acceleration $\dot{\vec{\omega}}$ (see **[A3]**, **[32]**). The kinetic energy is expressed as $2E_k = \omega(\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(A)}) == \omega^2 J_{\vec{n}}^{(A)}$. The linear momentum and angular momentum are:

$$\vec{\mathfrak{K}} = [\vec{\omega}, \vec{\rho}_C]M = \omega \vec{\mathfrak{S}}_{\vec{n}}^{(A)} \tag{62}$$

$$\vec{\mathfrak{L}}_A = \vec{\omega}(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(A)}) + \omega[\vec{n}[\vec{\mathfrak{J}}_{\vec{n}}^{(A)}, \vec{n}]] = \vec{\omega}(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(A)}) + \omega \vec{\mathfrak{D}}_{\vec{n}}^{(A)} \tag{63}$$

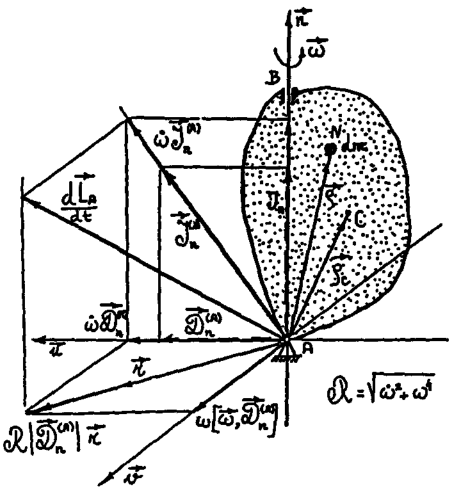

Figure 7a

Since the velocity $\vec{v}$ and the acceleration $\vec{a}$ of the body elementary mass at the point N are (see [31], [12]):

$$\vec{v} = [\vec{\omega}, \vec{\rho}], \quad \vec{a} = [\dot{\vec{\omega}}, \vec{\rho}] + [\vec{\omega}, [\vec{\omega}, \vec{\rho}]] \tag{64}$$

then for the main vector $\vec{F}_{rj}$ of the inertia force of the overall rigid body rotating around the axis with the angular velocity $\vec{\omega}$ we obtain:

$$\vec{F}_{rj} = -\iiint_V \vec{a}\, dm = -\dot{\omega}\vec{\mathfrak{S}}_{\vec{n}}^{(A)} - \omega[\vec{\omega}, \vec{\mathfrak{S}}_{\vec{n}}^{(A)}] \tag{65}$$

For the main moment of the inertia forces of the overall rigid body rotating around the axis and for the point A we calculate the following:

$$\vec{\mathfrak{M}}_{Aj} = \iiint\limits_V [\vec{\rho}, d\vec{F}_{rj}] = -\dot{\omega}\vec{\mathfrak{J}}_{\vec{n}}^{(A)} - \omega[\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(A)}] \tag{66}$$

as well as:

$$\vec{\mathfrak{M}}_{Aj} = \iiint\limits_V [\vec{\rho} d\vec{F}_{rj}] = -\frac{\dot{\omega}}{\omega}\vec{\mathfrak{L}}_A - [\vec{\omega}, \vec{\mathfrak{L}}_A] \tag{66*}$$

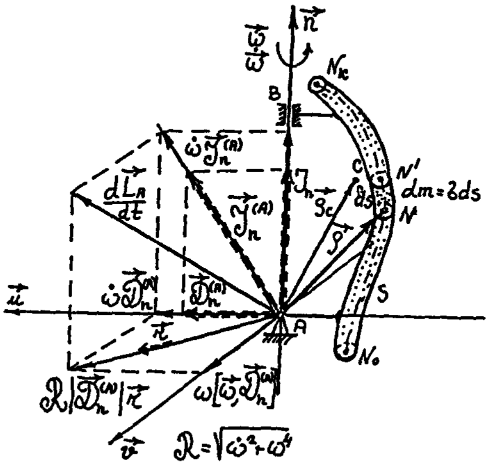

Figure 7b

The dynamic equations of the body rotation around fixed axis can be obtained by differentiating in time the expression (62) for the linear momentum and expression (54) for angular momentum on the basis of which we obtain:

$$1^* \qquad \frac{d\vec{\mathfrak{K}}}{dt} = \dot{\omega}\vec{\mathfrak{S}}_{\vec{n}}^{(a)} + \omega[\vec{\omega}, \vec{\mathfrak{S}}_{\vec{n}}^{(A)}] = -\vec{F}_{rj} = \vec{F}_r \tag{67}$$

$$\frac{d\vec{\mathfrak{K}}}{dt} = |\vec{\mathfrak{S}}_{\vec{n}}^{(A)}|(\dot{\omega}\vec{u}_1 + \omega^2\vec{v}_1) = \vec{\mathfrak{R}}\,|\vec{\mathfrak{S}}_{\vec{n}}^{(A)}| = \mathfrak{R}|\vec{\mathfrak{S}}_{\vec{n}}^{(A)}|\vec{r}_1 \tag{68}$$

$$\vec{\mathfrak{R}}_1 = \mathfrak{R}\vec{r}_1, \quad \mathfrak{R} = \sqrt{\dot{\omega}^2 + \omega^4} \tag{69}$$

The rotator $\vec{\mathfrak{R}} = \mathfrak{R}\vec{r}_1$ is normal to the rotation axis and the deviation plane through the axis.

The equation (67) for the linear momentum change which is equal to the main vector (resultant) of the active and reactive forces shows that the motion linear momentum changes the vector normal to the rotation axis and has two components: one due to the angular velocity change which is normal to the rotation axis and the plane which contains the body mass center and the rotation axis, and the other component which depends on the angular velocity square which is normal to the rotation axis and lie in the plane formed by rotation axis and the rigid body mass center doing rotation.

$$2^* \qquad \frac{d\vec{\mathfrak{L}}_A}{dt} = \dot{\omega}\vec{\mathfrak{J}}_{\vec{n}}^{(A)} + \omega[\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(A)}] = -\vec{\mathfrak{M}}_{Aj} = \vec{\mathfrak{M}}_A \tag{70}$$

$$\frac{d\vec{\mathfrak{L}}_A}{dt} = \dot{\vec{\omega}}J_{\vec{n}}^{(A)} + \dot{\omega}\vec{\mathfrak{D}}_{\vec{n}}^{(A)} + \omega[\vec{\omega}, \vec{\mathfrak{D}}_{\vec{n}}^{(A)}] = \dot{\vec{\omega}}J_{\vec{n}}^{(A)} + |\vec{\mathfrak{D}}_{\vec{n}}^{(A)}|\vec{\mathfrak{R}} \tag{71}$$

$$\vec{\mathfrak{R}}_1 = \mathfrak{R}\vec{r}_1, \quad \mathfrak{R} = \sqrt{\dot{\omega}^2 + \omega^4} \tag{72}$$

The rotator $\vec{\mathfrak{R}} = \mathfrak{R}\vec{r}_1$ which is rotating and increasing by the angular velocity and by the angular acceleration at the same causes the inertia forces deviation moment to increase.

The equation (70) which is written on the basis of the law of the body angular momentum change which is says that the derivative in time of the body angular momentum for a certain pole in stationary bearing, equal to the moment of the active and reactive forces acting on the body for the same pole.

This form (71) immediately shows that the first component depending on the angular acceleration is collinear with the rotation axis; the second component which also depends on the angular acceleration is normal to the rotation axis and the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(A)}$ of the rigid body mass inertia moment for the pole in the fixed bearing A and for the rotation axis, that is, it is proportional to the magnitude of the angular acceleration $\dot{\vec{\omega}}$ and the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(A)}$ of the rotation rigid body mass deviation moment of the rotation axis in the stationary bearing A and for the rotation axis; the third component is proportional to the square of the angular velocity ω^2 and to the magnitude of the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(A)}$ of the rotation rigid body mass deviation moment of the rotation axis in the stationary bearing A and for the rotation axis, whereas it is like a vector normal to the rotation axis and the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(A)}$ of the deviation load to the rotation axis which means it is normal to the deviation plane. In the case it is the rotation with a constant angular velocity the stroke derivative components in time do not appear in the deviation plane; there is only a component normal to the deviation plane $\omega[\vec{\omega}, \vec{\mathfrak{D}}_{\vec{n}}^{(A)}]$.

Figure 7 shows the characteristic vectors, the rigid body mass moment vectors and the rigid body dynamics kinetic vectors in the rotation around fixed axis.

If we now return to the expressions (65) and (66) for the inertia force main vector and the inertia force main moment for the pole at the stationary bearing A we come to the following conclusion: 1* the expression (65) is equal to the one for the rigid body linear momentum derivative in time a changed sigh, while the expression (66) is equal to the angular momentum for the pole at the stationary bearing A, derivative in time, with a changed sigh so that the conclusions drawn to the expressions (67) and (53) also stand for the expression (65) and (66). These conclusions can also be defined in another way: we conclude from expression (66) that the inertia forces main moment for the rigid body rotation around the fixed axis has three components: the first one is purely rotatory around the rotation axis collinear with it if the angular acceleration is different from zero and it is proportional to the angular acceleration $\dot{\omega}$ and the body mass axial inertia moment for the rotation axis, $J_{\vec{n}}^{(A)}$; and the second deviational component is normal to the rotation axis which also depends on the angular acceleration and the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(A)}$ of the deviation load of the rotation axis; and third component depending on the angular velocity squared of the rigid body rotation around the fixed axis and on the magnitude of the mass deviation moment vector of the rotation axis at the pole in the stationary bearing.

The derivative in time of the body angular momentum for a certain pole in stationary bearing normal to the rotation axis is:

$$\frac{d\vec{\mathfrak{L}}_A^d}{dt} = \dot{\omega}\vec{\mathfrak{D}}_{\vec{n}}^{(A)} + \omega[\vec{\omega}, \vec{\mathfrak{D}}_{\vec{n}}^{(A)}] = |\vec{\mathfrak{D}}_{\vec{n}}^{(A)}|\mathfrak{R} \tag{73}$$

By expressions (66), (68) and (73) we can write following relations:

$$\frac{|\vec{F}_{rj}|}{|\vec{\mathfrak{M}}_{Aj}|} = \frac{\left|\frac{d\vec{\mathfrak{R}}}{dt}\right|}{\left|\frac{d\vec{\mathfrak{L}}_A^d}{dt}\right|} = \frac{|\vec{\mathfrak{S}}_{\vec{n}}^{(A)}|}{|\vec{\mathfrak{D}}_{\vec{n}}^{(A)}|} = \text{constant} \tag{74}$$

II.1.4. Conditions for the dynamic balance of the rotor rotating around the fixed axis. Figure 7 shows the rotor with the main forces vector components denoted, that is, the motion linear momentum derivative in time and the inertia forces resulting moment components, that is, motion angular momentum derivative in time. In order that the effects of the dynamic balancing can appear it is necessary that bearings do not bear dynamic pressure which means that the deviational components should be equal to zero, that is, the components of the main force vector and the inertia forces resulting moment. Hence we draw the following conclusions:

1* Condition for the dynamic balancing exclusively and primarly depends on the dynamic, that is, kinetic properties of the rigid body with respect to the pole in the stationary bearing and to the rotation axis, but they do not depend on the angular velocity and the character of the acceleration;

2* Rotation axis should be the gravitational axis which is expressed by the condition that the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the rigid body mass linear moment for the rotation axis and the stationary bearing should be equal to zero;

$$|\vec{\mathfrak{S}}_{\vec{n}}^{(A)}| = 0 \tag{75}$$

3* Deviational part magnitude of the motion angular momentum derivative in time is equal to zero, that is, that the magnitude of the vector $\vec{\mathfrak{D}}_{\vec{n}}^{(A)}$ of the deviation load by the body mass inertia moment for the rotation axis is equal to zero:

$$|\vec{\mathfrak{D}}_{\vec{n}}^{(A)}| = 0 \tag{76}$$

which can be reduced to the condition that the rotation axis is the main central mass inertia moment axis or that it is the symmetry axis or that it is the axis which for the point at stationary bearing represents one main direction of the rotor mass inertia moment.

II.1.5. Interpretation of the kinetic pressures on bearing by means of the mass moment vectors for the pole and the axis. In this part the kinetic pressures of shaft bearings are expressed by means of the mass moment vectors: $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the body mass linear moment and $\vec{\mathfrak{D}}_{\vec{n}}^{(A)}$ of the deviation load by the body mass inertia moment of the rotation axis for the pole in the stationary bearing.

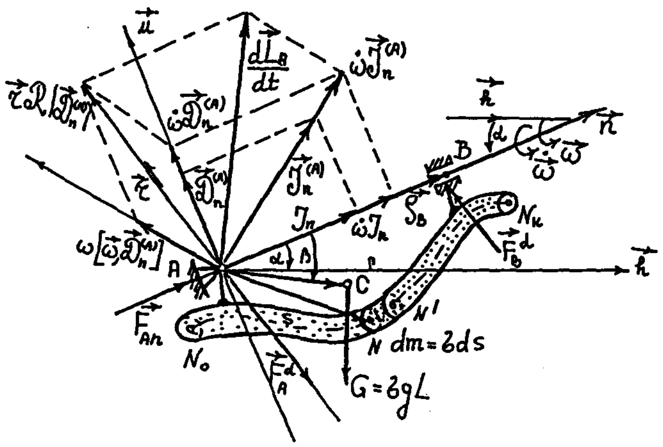

Figure 8

Figure 8 shows a rigid body that can rotate around a stationary axis is like a rigid shaft without mass supported on the stationary bearing A and on the moveable sliding one along the rotation axis. In the general case let a rigid body be subjected to a system of forces $\vec{F}_k$ whose points application N_{ko} are determined by the position vectors $\vec{\rho}_k$ with respect to the pole in the stationary bearing A.

Let's denote the rotation angle of the body around the stationary axis oriented by unit vector $\vec{n}$ with $\vec{\varphi} = \varphi\vec{n}$.

Following the expressions (67) and (70), as well as expression (68) and (71) we can write the following two vector equations:

$$\frac{d\vec{\mathfrak{K}}}{dt} = |\vec{\mathfrak{S}}_{\vec{n}}^{(A)}|(\dot{\omega}\vec{u}_1 + \omega^2\vec{v}_1) = \vec{\mathfrak{R}}|\vec{\mathfrak{S}}_{\vec{n}}^{(A)}| =$$

$$= \mathfrak{R}|\vec{\mathfrak{S}}_{\vec{n}}^{(A)}|\vec{r}_1 = \sum_{k=1}^{k=N} \vec{F}_k + \vec{F}_A + \vec{F}_B \tag{77}$$

$$\frac{d\vec{\mathfrak{L}}_A}{dt} = \dot{\vec{\omega}} J_{\vec{n}}^{(A)} + \dot{\omega}\vec{\mathfrak{D}}_{\vec{n}}^{(A)} + \omega[\vec{\omega}, \vec{\mathfrak{D}}_{\vec{n}}^{(A)}] =$$

$$= \dot{\vec{\omega}} J_{\vec{n}}^{(A)} + |\vec{\mathfrak{D}}_{\vec{n}}^{(A)}|\vec{\mathfrak{R}} = \sum_{k=1}^{k=N} [\vec{\rho}_k, \vec{F}_k] + [\vec{\rho}_B, \vec{F}_B] \tag{78}$$

These two vectorial equations are kinetic equations of dynamic equilibrium in motion-rotation of the body around the stationary axis under the action of the active force system $\vec{F}_k$.

If we now multiply scalarly and vectorly these equations (77) and (78) by the unit vector $\vec{n}$ and having in mind that the $\vec{\rho}_B = \rho_B\vec{n}$, we obtain:

1* the rotation equation around the axes oriented by unit vector $\vec{n}$ in the form:

$$(\vec{\mathfrak{J}}_{\vec{n}}^{(A)}, \dot{\vec{\omega}}) = \sum_{k=1}^{k=N} ([\vec{\rho}_k, \vec{F}_k], \vec{n}) \tag{79}$$

2* the equations for determining the bearings kinetic pressures, that is pressures upon the bearings, $\vec{F}_A$ and $\vec{F}_B$, that is, their components in the axis direction $\vec{n}$ and normal to the rotation axis:

$$\vec{F}_{A\vec{n}} = (\vec{F}_A, \vec{n})\vec{n} = -\vec{n}\sum_{k=1}^{k=N} (\vec{F}_k, \vec{n}) \tag{80}$$

$$\vec{F}_{AT} = -\vec{F}_B + \vec{\mathfrak{R}}_1|\vec{\mathfrak{S}}_{\vec{n}}^{(A)}| - \sum_{k=1}^{k=N} [\vec{n}, [\vec{F}_k, \vec{n}]] \tag{82}$$

$$\vec{F}_B = \frac{1}{\rho_B}\vec{\mathfrak{R}}|\vec{\mathfrak{D}}_{n}^{(A)}| - \frac{1}{\rho_B}\sum_{k=1}^{k=N} [\vec{n}, [[\vec{\rho}_k, \vec{F}_k], \vec{n}]] \tag{83}$$

From the expression for the bearings pressures (resistance) we select a part which is the result of the action of an external active forces and the influence of which upon the bearings resistances in possible variable in time is only due to the change of their line of application as well as the point of application with respect to the configuration of the body which is rotating such as in the case when the force of the body's own weight which retains the application line direction in relation to the rotation axis, and thus its position with respect to the body configuration, although in doing this it retains the application point constantly in the body mass center which rotates around the rotation axis together with body. The body mass center describes a circle or an arc in the plane through the mass center normal to the rotation axis.

Other part of the bearing kinetic resistance (pressures) in the body rotation around the stationary axis is the result exclusively of the kinetic-inertial body properties with respect to the rotation axis and the rotation kinematics and rigid body rotation kinematics around the stationary axis. These parts appear as parameters depending on the rotator vector $\vec{\mathfrak{R}}$ which in itself contains the angular velocity and the angular acceleration of the body rotation around the rotation axis and the rigid body mass moment properties with respect to the pole A at stationary bearing and the rotation axis expressed by the mass moment vectors: $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the body mass linear moment and $\vec{\mathfrak{D}}_{\vec{n}}^{(A)}$ of the deviation load by the body mass inertia moment of the rotation axis for the pole A in the stationary bearing.

In order to discuss the rotor effect on the kinetic pressure upon the bearings in which the rigid body shaft axis is rotating it is necessary to know the angular acceleration $\dot{\vec{\omega}}$ and the angular velocity $\vec{\omega}$ and in order to do this it is necessary to solve the body rotation/oscillation equation around the axis (79), namely, to determine $\vec{\varphi}(t)$ and $\vec{\omega}(t)$ as well as $\omega(\varphi)$.

If the rotation axis is the central and main mass inertia moment axis and for the pole in the stationary bearing then it is a rigid body which is dynamically balanced and the member in the kinetic pressures depending on the vectors $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the body mass linear moment and $\vec{\mathfrak{D}}_{\vec{n}}^{(A)}$ of the deviation load by the body mass inertia moment of the rotation axis for the pole A in the stationary bearing are equal to zero and are not influenced by the rotator change. Then there are only the components of the bearing resistance arising from the bearings "kvazi-static" resistances in the definite position of the active forces system and the reactive forces system during the body rotation.

If the rotation axis is the axis of the mass inertia moment asymmetry for the referential point in the stationary bearing then the kinetic pressures are the greatest both on moveable and stationary bearing. Since at each point on the rigid body there are three pairs of such mutually perpendicular axes which are in pair perpendicular to one main mass inertia moment direction and they form with the others an angle of $\frac{\pi}{4}$ each so that the mass inertia moment asymmetry axes which are perpendicular to the second main mass inertia moment direction forming angle of $\frac{\pi}{4}$ each with the first and third main mass inertia moment directions as the rotation axes will be the greatest vector of the deviation load and at the same time the greatest kinetic pressures on both the bearings.

The kinetic pressure on the stationary bearing depends on the body mass center position with respect to the rotation axis and this can be adjusted by the choice of the inertia asymmetry axes in pair as well as by the choice of the moveable bearing position with respect to the stationary one on the definite axis of mass inertia moment asymmetry. The body mass inertia moment asymmetry axes should be avoided as the rotation axis in order to reduce the dynamic pressures upon the bearings.

For a pair of the mass inertia moment asymmetry axes as the rotation axes the axial mass inertia moment of the rotatory body is identical so that depending on the body mass center position with respect to one axis or another and on the choice of the moveable bearing an increase, that is, decrease of the kinetic pressure at a given constant value of the initial energy communicated to the rotating body.

There are four (that is, eight) axes through each point of the body which we have chosen as a stationary bearing for which the axial mass inertia moments are the same value and the vectors $\vec{\mathfrak{D}}_{\vec{n}}^{(A)}$ of the deviation load by the body mass inertia moment of the rotation axis for the pole A in the stationary bearing are proportional to the sum of the three mass deviation load vectors by the body mass inertia moment of the mass inertia moment asymmetry axes. For these octahedral axes the dynamic pressures on both the stationary and moveable bearings are the same while the pressures on the stationary bearing are different and by choosing one of the octahedral axes minimization of maximization of their value can be performed. By displacing the moveable bearing from one to another octahedral axis through the stationary bearing the kinetic pressure on the stationary bearing can be adjusted while retaining the share in the pressure on both the bearing of the part that corresponds to the deviation load vector although the rotator is going to change as well (but this can also be adjusted). The smallest pressures would appear an octahedral axis is chosen so that the body mass center is closest to the rotation axis, that is, the most favorable of all the octahedral axes for the rotation axes is the one which body mass center is closest to.

A general conclusion would be that if we cannot select in the design way the rotation axis as the rigid body main central mass inertia moment axis when the system is dynamically balanced and analysis of the mass inertia moment state should be performed at each of the possible points of the stationary bearing positioning and according to the design requirements the selection should be done of both the stationary bearing and of the rotation axis according to the analysis.

These conclusions are very important if the designer cannot change the stationary bearing but if we can change the moveable one and chose it freely in the rigid body then his choose is important since the dynamic pressures should be as small as possible (see [**33**], [**32**]).

II.2. Interpretation of the motion equations of a variable mass object rotating around a stationary axis by means of the mass moment vector for the pole and the axis.

In this part the kinetic equations of a variable mass object motion rotating around a stationary axis are derived by means of the mass moment vectors for the pole and for the rotation axis: vector $\vec{\mathfrak{S}}_n^{(A)}$ of the body mass linear moment, vector $\vec{\mathfrak{J}}_n^{(A)}$ of the body mass inertia moment for the pole A and for the axis oriented by the unit vector $\vec{n}$ and its deviational part of the vector $\vec{\mathfrak{D}}_n^{(A)}$ of the deviational load by the body mass inertia moment of the rotation axis through the pole A. The vectors of the reactive forces and resulting moments of the reactive forces due to the drop of the body particles are determined which are involved in the body mass change as the function of the body mass moments vector change: vector $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the body mass linear moment and vector $\vec{\mathfrak{J}}_{\vec{n}}^{(A)}$ of the body mass inertia moment for the pole A and for the axis oriented by the unit vector $\vec{n}$ (see [49], [45], [27]).

The bearings resistances of the shaft on which an object of the variable mass is rotation and the analysis of the kinetic pressures is performed.

II.2.1. Introduction. In the last fifty years the equations of Meschersky given in his M.Sc. theses in 1897 [9] have obtained a wide theoretical consideration and the practical application in scientific centers of many countries. Meschersky has introduced the notion of the reactive force whereas Newton has defined the dynamic object properties by means of the kinetic properties of the matter quantity as the inertia measure. In the context of the Meschersky theory [9] the object are discussed as the dynamic variable objects. If the object is subjected to the dynamic change (see [17], [18]) (change of its own mass inertia moments) then it is the dynamic variable object whose rotation around the stationary axis is discussed in this paper.

The reactive force acts on a body in motion whose mass changes in time (due to the mechanical wasting – rejection or adhesion) in the sense of action and reaction. This motion is described by the Meschersky equation and gives an expression for the reactive force while the Ciolkovsky formula (see [5], [2], [11]) determines the motion velocity due to such a force and the dependence of the mass separation velocity in the case of the mass rejection.

Beside the papers quoted above relating to the mechanics of the variable mass body and rocket-dynamics which began to develop between the two World Wars there are other publications of a famous Italian scientist Tullio Levy-Civita (1873–1941) (see [8]) who discovered these laws 31 years after Meschersky and independently of him.

In engineering practice, especially in Mechanical Engineering, an important role is played by the rotor of the variable mass so that it is of greatest to consider the dynamic equations of the motion of the variable mass rotor as well as the dynamic resistances of the shaft bearings which these rotors are rotating upon.

II.2.2. Main vector of the reactive forces and the reactive forces resulting moment. By means of the previous introduced mass moment vectors here we are going to the interpret the kinetic equations of the variable mass body rotation.

Figure 9 shows a rigid body of a variable mass rotating around the axis oriented by the unit vector $\vec{n}$ by the angular velocity $\vec{\omega}$.

We introduce the hypothesis about the knowledge of the of the law on the mass separation from the body as the absolute velocity $\vec{w}_N$ of the particles falling off which create the reactive force. Let's assume that the absolute velocity $\vec{w}_N$ of the body particles falling of is equal to the velocity of the body point which rotates around the axis by the angular velocity $\vec{\omega}$, that is, that it is: $\vec{w}_N = \lambda \vec{v}_N = \lambda[\vec{\omega}, \vec{\rho}]$, where λ is a scalar, the proportionality coefficient (see [19], [27]). The reactive force $d\vec{\mathfrak{F}}_r$, due to the elementary particle falling off is: $d\vec{\mathfrak{F}}_r = \vec{w}_N d\dot{m} = \lambda[\vec{\omega}, \vec{\rho}]d\dot{m}$.

Due to the falling off of all the particles which are involved in the body mass change the main vector of the reactive forces is:

$$\vec{\mathfrak{F}}_r = \iiint\limits_V \vec{w}_N d\dot{m} = \lambda \iiint\limits_V [\vec{\omega}, \vec{\rho}]\, d\dot{m} = \lambda\omega \frac{\overset{*}{d}}{dt} \iiint\limits_V [\vec{n}, \vec{\rho}]\, dm = \lambda\omega \frac{\overset{*}{d}\vec{\mathfrak{S}}_{\vec{n}}^{(A)}}{dt} \tag{84}$$

we see that it is proportional to the body rotation angular velocity and to the derivative in time of the body mass static moment vector in the case when the body changes its mass in rotation. In the formula (84) the differential operator $\frac{\overset{*}{d}}{dt}$ is a derivative in the time of the body mass linear moment vector for the body mass change:

$$\frac{\overset{*}{d}\vec{\mathfrak{S}}_{\vec{n}}^{(A)}}{dt} = \frac{\overset{*}{d}}{dt} \iiint\limits_V [\vec{n}, \vec{\rho}]\, dm = \iiint\limits_V [\vec{\omega}, \vec{\rho}] d(\frac{dm}{dt}) \tag{84*}$$

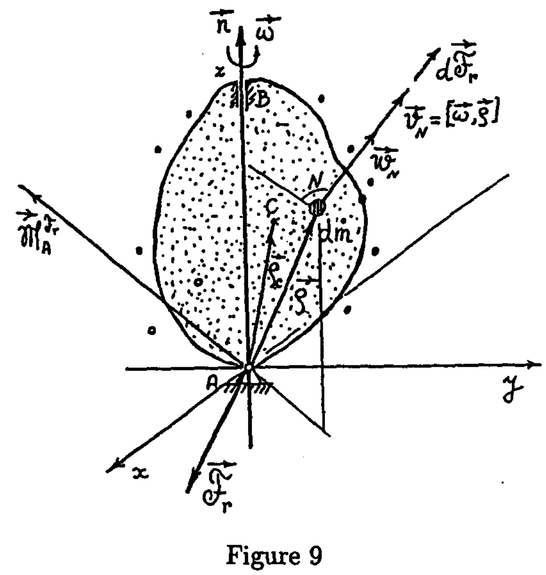

Figure 9

Due to the falling off of all the body particles which are involved in the body mass change the resulting moment of the reactive forces:

$$\vec{\mathfrak{M}}_r^{\vec{\mathfrak{F}}} = \iiint\limits_V [\vec{\rho}, d\vec{\mathfrak{F}}_r] = \lambda \iiint\limits_V [\vec{\rho}, [\vec{\omega}, \vec{\rho}]]\, d\dot{m} = \lambda\omega \frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(A)}}{dt} \tag{85}$$

We see that the resulting moment of the reactive forces due to the body particles falling off, that is, of the particles involved in the body mass change for the case of the rotation is proportional to the body rotation angular velocity and to the derivative in the time of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(A)}$ of the vector of the body mass inertia moment for the pole at A and for the rotation axis.

In (85) the differential operator $\frac{\overset{*}{d}}{dt}$ is a derivative in the time of the body mass inertia moment vector for the body mass change:

$$\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(A)}}{dt} = \frac{\overset{*}{d}}{dt}\iiint\limits_V [\vec{\rho},[\vec{\omega},\vec{\rho}]]\,dm = \iiint\limits_V [\vec{\rho},[\vec{\omega},\vec{\rho}]]\,d\dot{m} \tag{85*}$$

II.2.3. Linear momentum and angular momentum of the body rotation around the stationary axis. Following the idea of this part the linear momentum $\vec{\mathfrak{K}}$ and the angular momentum $\vec{\mathfrak{L}}_A$ for the pole in the stationary A bearing for the case of the body rotation around the stationary axis can be written by means of the previously defined vectors of the body mass moments by the expressions (4) and (6), as well as by the expressions (62) and (63), in the following form: $\vec{\mathfrak{K}} = \omega\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$, $\vec{\mathfrak{L}}_A = \omega\vec{\mathfrak{J}}_{\vec{n}}^{(A)}$. Since for the formation of the dynamic equations is necessary to determine the derivatives in the time of the linear momentum and of the angular momentum of the body rotation, we write that it is:

$$\frac{d\vec{\mathfrak{K}}}{dt} = \dot{\omega}\vec{\mathfrak{S}}_{\vec{n}}^{(A)} + \omega[\vec{\omega},\vec{\mathfrak{S}}_{\vec{n}}^{(A)}] + \omega\frac{\overset{*}{d}\vec{\mathfrak{S}}_{\vec{n}}^{(A)}}{dt} \tag{86}$$

$$\frac{d\vec{\mathfrak{L}}_A}{dt} = \vec{\dot{\omega}}J_{\vec{n}}^{(A)} + \dot{\omega}\vec{\mathfrak{D}}_{\vec{n}}^{(A)} + \omega[\vec{\omega},\vec{\mathfrak{D}}_{\vec{n}}^{(A)}] + \omega\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(A)}}{dt} \tag{87}$$

II.2.4. Kinetic equations of a variable mass body rotation around a stationary axis. By using the basic laws of the dynamics that the linear momentum derivative in time is equal to the sum of all the active and reactive forces and that the angular momentum derivative in time for the pole in the stationary bearing is equal to the sum of all the active and reactive moments for the same pole, we can write the following two vector equations by means of the expressions (86) and (87) as well as of the expressions (84) and (85):

$$\frac{d\vec{\mathfrak{K}}}{dt} = \dot{\omega}\vec{\mathfrak{S}}_{\vec{n}}^{(A)} + \omega[\vec{\omega},\vec{\mathfrak{S}}_{\vec{n}}^{(A)}] + \omega\frac{\overset{*}{d}\vec{\mathfrak{S}}_{\vec{n}}^{(A)}}{dt} = -\vec{F}_{rj} = \vec{F}_r + \lambda\omega\frac{\overset{*}{d}\vec{\mathfrak{S}}_{\vec{n}}^{(A)}}{dt} \tag{88}$$

$$\frac{d\vec{\mathfrak{L}}_A}{dt} = \vec{\dot{\omega}}J_{\vec{n}}^{(A)} + \omega[\vec{\omega},\vec{\mathfrak{D}}_{\vec{n}}^{(A)}] + \omega\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(A)}}{dt} = -\vec{\mathfrak{M}}_{Aj} = \vec{\mathfrak{M}}_{Ar} + \lambda\omega\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(A)}}{dt} \tag{89}$$

These two vector-equations are the motion kinetic ones-of the rotation of a variable mass body around the stationary axis. In these equations $\vec{F}_r$ and $\vec{\mathfrak{M}}_{Ar}$ are the main vector of the active and passive forces acting on the body as well as these forces' resulting moments for the pole at A.

If we multiply these equations (88) and (89) first scalarly and then from different sides by the vector $\vec{n}$ of the rotation having in view that we obtain:

a) Rotation equation around the axis oriented by the unit vector $\vec{n}$:

$$(\vec{\mathfrak{J}}_{\vec{n}}^{(A)}, \dot{\vec{\omega}}) = ([\vec{\rho}_C, \vec{G}], \vec{n}) + \sum_{k=1}^{k=N}([\vec{\rho}_k, \vec{F}_k], \vec{n}) + (\lambda - 1)\left(\vec{\omega}, \frac{d\overset{*}{\vec{\mathfrak{J}}}_{\vec{n}}^{(A)}}{dt}\right) \tag{90}$$

b) Equations for the bearings kinetic resistances:

$$(\vec{F}_A, \vec{n}) + (\vec{G}, \vec{n}) + (\lambda - 1)\left(\frac{d\overset{*}{\vec{\mathfrak{S}}}_{\vec{n}}^{(A)}}{dt}, \vec{n}\right) + \sum_{k=1}^{k=N}(\vec{F}_k, \vec{n}) = 0 \tag{91}$$

$$\begin{aligned}\vec{\mathfrak{R}}_1|\vec{\mathfrak{S}}_{\vec{n}}^{(A)}| = [\vec{n}, [\vec{F}_A, \vec{n}]] + [\vec{n}, [\vec{F}_B, \vec{n}]] + \sum_{k=1}^{k=N}[\vec{n}, [\vec{F}_k, \vec{n}]] + \\ + [\vec{n}, [\vec{G}, \vec{n}]] + (\lambda - 1)\left[\vec{n}, \left[\frac{d\overset{*}{\vec{\mathfrak{S}}}_{\vec{n}}^{(A)}}{dt}, \vec{\omega}\right]\right]\end{aligned} \tag{92}$$

$$\begin{aligned}\vec{\mathfrak{R}}|\vec{\mathfrak{D}}_{\vec{n}}^{(A)}| = [\vec{n}, [[\vec{\rho}_B, \vec{F}_B], \vec{n}]] + [\vec{n}, [[\vec{\rho}_B, \vec{G}]\vec{n}]] + \\ + (\lambda - 1)[\vec{n}, [\frac{\overset{*}{\vec{\mathfrak{J}}}_{\vec{n}}^{(A)}}{dt}, \vec{\omega}]] + \sum_{k=1}^{k=N}[\vec{n}, [[\vec{\rho}_k, \vec{F}_k], \vec{n}]]\end{aligned} \tag{93}$$

II.2.5. Shaft bearings resistances carried by the variable mass body. From the equations (91), (92) and (93) we determine the bearings resistances components in the form:

The stationary bearing resistance components A are:

1* The axial components in the rotation axis direction is:

$$\vec{F}_{An} = \vec{n}\left\{(\vec{G}, \vec{n}) - (\lambda - 1)[\vec{n}, \left(\frac{d\overset{*}{\vec{\mathfrak{S}}}_{\vec{n}}^{(A)}}{dt}, \vec{n}\right) \sum_{k=1}^{k=N}(\vec{F}_k, \vec{n})\right\} \tag{94}$$

2* The deviational components perpendicular to the rotation axis are:

2.1* The component coming from the body mass center eccentricity is:

$$\vec{F}_{AN}^{*} = \vec{F}_{A*}^{(\mathrm{dev})} = \vec{\mathfrak{R}}_1|\vec{\mathfrak{S}}_{n}^{(A)}| - \sum_{k=1}^{k=N}[\vec{n}, [\vec{F}_k, \vec{n}]] - [\vec{n}, [\vec{G}, \vec{n}]] - (\lambda - 1)\left[\vec{n}, \left[\frac{d\overset{*}{\vec{\mathfrak{S}}}_{\vec{n}}^{(A)}}{dt}, \vec{\omega}\right]\right] \tag{95}$$

2.2* The component coming from the deviational couple:

$$\vec{F}_{AN}^{**} = \vec{F}_{A**}^{(\text{dev})} = -\vec{F}_B = -\frac{1}{\rho_B}\vec{\mathfrak{R}}\,|\vec{\mathfrak{D}}_{\vec{n}}^{(A)}| + \frac{1}{\rho_B}[\vec{n},[[\vec{\rho}_C,\vec{G}],\vec{n}]] +$$

$$+\frac{(\lambda-1)}{\rho_B}\left[\vec{n},\left[\frac{d\overset{*}{\vec{\mathfrak{J}}}_{\vec{n}}^{(A)}}{dt},\vec{\omega}\right]\right] + \frac{1}{\rho_B}\sum_{k=1}^{k=N}[\vec{n},[[\vec{\rho}_k,\vec{F}_k],\vec{n}]] \quad (96)$$

3* The moveable bearing resistance - sliding in the axis direction is of the deviational character:

$$\vec{F}_B = \frac{1}{\rho_B}\vec{\mathfrak{R}}\,|\vec{\mathfrak{D}}_{\vec{n}}^{(A)}| - \frac{1}{\rho_B}[\vec{n},[[\vec{\rho}_C,\vec{G}],\vec{n}]]$$

$$-\frac{(\lambda-1)}{\rho_B}\left[\vec{n},\left[\frac{d\overset{*}{\vec{\mathfrak{J}}}_{\vec{n}}^{(A)}}{dt},\vec{\omega}\right]\right] - \frac{1}{\rho_B}\sum_{k=1}^{k=N}[\vec{n},[[\vec{\rho}_k,\vec{F}_k],\vec{n}]] \quad (97)$$

in which the rotator $\vec{\mathfrak{R}}$ is determined by the formula:

$$\vec{\mathfrak{R}} = \mathfrak{R}\vec{n} = \dot{\omega}\vec{u} + \omega^2\vec{v}\,(=)\,\frac{\vec{a}}{r}; \quad \vec{u}^2 = \vec{v}^2 = 1; \quad \vec{u}\perp\vec{v}\perp\vec{n}; \quad \mathfrak{R} = \sqrt{\dot{\omega}^2+\omega^4} \quad (98)$$

From the expressions for the bearings resistances we select the part which is the result of the direct "static-dynamic" action of the active forces and a part which is the result of the rotating variable mass body kinetic properties.

We see that as the result of the rotor kinetic properties the deviational couple appears which is equal to the product of the rotator vector $\vec{\mathfrak{R}}$ and of the vector intensity $\vec{\mathfrak{D}}_{\vec{n}}^{(A)}$ of the deviation load by the body mass inertia moment of the rotation axis and it directly depends on the axis selection in the variable mass rotating body. This deviational couple causes a part of the kinetic pressures of the same intensity and perpendicular to the rotation axis in both the bearings, the stationary and the moveable one.

In the case that the rotation axis is always the main inertia axis for the pole in the stationary axis this deviational couple is equal to zero and it does not cause any pressure upon the bearings.

An additional pressure only upon the stationary bearing is formed when the masses center is outside the rotation axis and this part is proportional to the rotator vector $\vec{\mathfrak{R}}$ and to the vector intensity $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the mass linear moment for the pole in the stationary bearing and for the rotation axis $\vec{n}$.

Due to the mass changeability the kinetic pressures are formed in both the stationary and moveable bearings and they depend on the character of the body mass inertia vector change for the pole at A and for the rotation axis and they also make another deviational couple.

An additional pressures on the stationary bearing is formed due to the change of the vector $\vec{\mathfrak{S}}_{\vec{n}}^{(A)}$ of the mass linear moment and the angular velocity. A part of the kinetic pressures due to the reactive effect of the mass falling off from the rotator in the fact depends on the falling-off masses kinetic properties.

II.2.6. Special case of self-rotation. To illustrate let's observe a special case when there are no active forces but the rotor is only under the action of the reactive forces due to the masses separation (for instances, rotor with nozzles through which the particles are falling). Then the self- rotation equation is:

$$(\vec{\mathfrak{J}}_{\vec{n}}^{(A)}, \dot{\vec{\omega}}) = (\lambda - 1)\left(\vec{\omega}, \frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(A)}}{dt}\right) \tag{99}$$

whose one first integral is: $(\vec{\mathfrak{J}}_{\vec{n}}^{(A)}, \vec{\omega}) = \text{const.}$

If the rotation axis is the central rotation axis and the main inertia axis for the pole in the stationary bearing then the dynamic pressures do not effect the bearings. Then we can conclude that due to the reactive forces the body rotates around a free axis which retains its orientation. This would be a case of the body self-rotation around the central axis. In [**20**] the motion integral of the form is given which according to the Savić-Kašanin theory [**16**] represents the motion integral, that is, the self-rotation equations of celestial bodies (of the Earth, of the Sun).

II.3. Vectorial equations for the self induced rotations

Starting from the idea of Savić and Kašanin [**16**] and from idea of Vujičić [**18**], as well as from an analogy with paper of Vujičić [**20**] and idea of [**23**], a new form of the vectorial equation for the self-induced rotations of a rigid body is derived. That equation is:

$$\dot{\omega}\vec{\mathfrak{J}}_{\vec{n}}^{(C)} + \omega[\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)}] + \omega(1-\lambda)\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)}}{dt} = 0 \tag{100}$$

where $\vec{\mathfrak{J}}_{\vec{n}}^{(C)}$ is the vector of body mass inertia moment at the point C center of mass, for the instantaneous rotation axis oriented by the unit vector $\vec{n}$ and $\vec{\omega}$ is the instantaneous angular velocity vector of the self-induced rotation, where $\omega = |\vec{\omega}|$.

II.3.1. Introduction. In the monograph [**16**] it is supposed that the rotation of a celestial body result from the expulsion of electrons from atoms: *"The expulsion of electrons from an atom has as its consequence the rotation of a celestial body, this rotation occuring at the instant in which the magnetic moment occurs-both phenomena occur concurrently with one another; both of them are the consequences of the expulsion of electrons from atoms, without which there would be neither a magnetic moment nor a rotation"*. The authors of this theory in their monograph, starting from the relation of the rotation of the plane rigid body derive a formula for calculating the angular velocity of a celestial body (see [**16**, p. 75]).

In [20] a new form of the tensorial equations for the self-rotation of a celestial body is derived by Vujičić. In [20] author have the following view: *"The classical mechanics have never succeeded neither could to explain the origin of rotation of celestial bodies by material point model and rigid body. The sum of interior force moments have disappeared during any analysis, and therefore the dynamics have to account the exterior forces as the cause of rotation. However the celestial mechanics have not accounted for electromagnetic forces although they are predominant in comparison with gravitational in microstructure. The gravitational forces became predominant within the large mass bodies. But the evolutional processes are much more complex the later mechanical model. So far the scientific opinion as that the formulation of stars-starts with gravitational condensation of low density hydrogen".*

II.3.2. Vectorial equations for the self-induced-rotations of bodies. According to [25] we shall introduce the notation of the mass inertia moment vector $\vec{\mathfrak{J}}_{\vec{n}}^{(C)}$ for the pole in the mass center C and for the axis oriented by the unit vector $\vec{n}$, defined by:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(C)} = \sum_{\nu=1}^{\nu=N} [m_\nu[\vec{r}_\nu, [\vec{n}, \vec{r}_\nu]] \tag{101}$$

where $\vec{r}_\nu$ is a position vector of mass particle m_ν, $v = 1, 2, \ldots N$, relative to a fixed pole (in the mass center C. This vector is connected for the pole in the mass enter and for the self-rotation axis.

The vector $\vec{\mathfrak{S}}_{\vec{n}}^{(C)}$ for the pole in the mass center C and for the axis of the self-induced rotation, oriented by the unit vector $\vec{n}$, defined by:

$$\vec{\mathfrak{S}}_{\vec{n}}^{(C)} = \sum_{\nu=1}^{\nu=N} m_\nu[\vec{n}, \vec{r}_\nu] = 0 \tag{102}$$

is equal to zero. In [20] author wrote: *"If we have in mind very complicated structure of celestial bodies, these results, as the one concerning the magnetic moment (see [16]), very sufficient stimulus for further work on this theory. For the purpose of mathematical generalization, it is always possible to consider any part of the body as the material points with the mass m_i, $i = 1, 2, \ldots N$, if its own rotation is considered. If we separate any part of the body, even one single electron, from the original body, the mass of the body m_i changes for the mass Δm_i of the separated particles. If the mass Δm_i is separated, with the velocity $\vec{u}_i$, from the body with mass m_i, there appears a reactive impulse:*

$$\Delta m_i \vec{u}_i = \frac{\Delta m_i}{\Delta t} \vec{u}_i \Delta t \tag{103}$$

and it provokes the change of impulse $m_i \vec{v}_i$ in the original with mass m_i. Naturally if the separated particle, for example an electron, takes with itself an electrical charge it induced also the electromagnetic field, and the occurrence of a magnetic moment".

In [20] it was assumed that the observed object was composed of the set of N parts of masses m_ν, $v = 1, 2, \ldots N$. Starting from the theory of separation under the pressure, we can accept the assumption that the mass m_ν of the body changes for a differentially small amount of mass Δm_ν. At the moment of expulsion of masses dm_ν, $v = 1, 2, \ldots M$, with the corresponding absolute velocities $\vec{u}_\nu$, there appear the reactive forces $\vec{u}_\nu \frac{dm_\nu}{dt}$ which perform the work:

$$\delta \mathcal{A}_\nu = \frac{dm_\nu}{dt}(\vec{u}_\nu, \delta\vec{r}_\nu) \tag{104}$$

on virtual displacements $\delta\vec{r}_\nu$.

The perturbation in the state of the j-th particle provokes (causes) a change in the impulse of the motion of all other particles. For such a dynamical system, the general classical principle of mechanics should be valid, and according to it, we can write:

$$\delta \int_0^1 \frac{1}{2} \sum_{\nu=1}^{\nu=N} m^v(\vec{v}_\nu, d\vec{r}_\nu) = -\int_0^1 \sum_{\nu=1}^{\nu=M} \dot{m}^v(\vec{u}_\nu, \delta\vec{r}_\nu) \tag{105}$$

where $M < N$ and $\vec{r}_\nu$ are the radius vectors of observed material points with the assumption that the eventual displacements $\delta\vec{r}_{v_0}$ and $\delta\vec{r}_{v_1}$ are equal to zero, and with the validity of the relations $\delta d\vec{r}_\nu = d\delta\vec{r}_\nu$. Now, for left-hand side of (105) we can write:

$$\begin{aligned}\frac{1}{2}\sum_{\nu=1}^{\nu=N} \int_0^1 \{(\delta(m_\nu, \vec{v}_\nu), d\vec{r}_\nu) + m_\nu(\vec{v}_\nu, \delta d\vec{r}_\nu)\} &= \sum_{\nu=1}^{\nu=N} \int_0^1 m_\nu(\vec{v}_\nu, d\delta\vec{r}_\nu) = \\ &= -\sum_{\nu=1}^{\nu=N} \int_0^1 (d(m_\nu \vec{v}_\nu), \delta\vec{r}_\nu)\end{aligned} \tag{106}$$

because

$$\sum_{\nu=1}^{\nu=N} m_\nu(\vec{v}_\nu, \delta\vec{r}_\nu)|_0^1 = 0 \tag{107}$$

Introducing the time t, the last integral (106) transforms in the form:

$$-\int_{t_0}^{t_1} \sum_{\nu=1}^{\nu=N} \Big(\frac{d(m_\nu \vec{v}_\nu)}{dt}, \delta\vec{r}_\nu\Big)\, dt \tag{108}$$

If we introduced the time also in the right-hand side of the relation (105), by means of $dm_\nu = \dot{m}_\nu\, dt$, where obviously $\dot{m}_\nu = \frac{dm_\nu}{dt}$ is the mass velocity (secondary change of mass) if will be

$$\int_{t_0}^{t_1} \sum_{\nu=1}^{\nu=N} \Big(\frac{d(m_\nu \vec{v}_\nu)}{dt}, \delta\vec{r}_\nu\Big)\, dt = \int_{t_0}^{t_1} \sum_{\nu=1}^{\nu=M} \dot{m}_\nu(\vec{u}_\nu, \delta\vec{r}_\nu)\, dt \tag{109}$$

Due to the arbitrariness in the choice of the pole of the position vector $\vec{r}_\nu$, the mass center C can be taken as the pole.

Setting the origin of an inertial reference system at the center of mass, which is always possible due to the arbitrariness of the choice of the reference point, the velocity of v-th point can be determined approximately by the relation $\vec{v}_\nu = [\vec{\omega}, \vec{r}_\nu]$. In the model of the body, the angular velocities $\vec{\omega}_\nu$, are equal to the instantaneous angular velocity vectors of a body-fixed, non inertial reference system with vector base $\vec{e}_\nu$, that is $\vec{\omega}_\nu \approx \vec{\omega}$. For the particles of a fluid medium, its velocity can be considered as an average angular velocity, for which $\vec{\omega}_\nu = \vec{\omega}$ so that the angular displacement $\delta\vec{\varphi} \approx \vec{\omega}\, dt$, within the limits of such an approximations, can connect the velocity $\vec{v}_\nu$ of the point of mass m_ν with the velocity $\vec{u}_\nu$ of an expulsive particle of mass dm_ν, that is $\vec{v} = \lambda\vec{u}$, where λ is an unknown scalar multiplier. Consequently, from the equation (105) [**20**] we can write:

$$\int_{t_0}^{t_1} \sum_{\nu=1}^{\nu=N} \frac{d}{dt}(m_\nu[\vec{\omega}, \vec{r}_\nu], [\delta\vec{\varphi}, \vec{r}_\nu])\, dt = \int_{t_0}^{t_1} \lambda \sum_{\nu=1}^{\nu=N} \frac{dm_\nu}{dt}([\vec{\omega}, \vec{r}_\nu], [\delta\vec{\varphi}, \vec{r}_\nu])\, dt \tag{110}$$

Integration of the left-hand side of relation (110) can be transformed to (see [**20**]):

$$\frac{d}{dt}\sum_{\nu=1}^{\nu=N} m_\nu([\vec{r}_\nu, [\vec{\omega}, \vec{r}_\nu]], \delta\vec{\varphi}) = \frac{d}{dt}(\omega\vec{\mathfrak{J}}_{\vec{n}}^{(C)}, \delta\vec{\varphi}) =$$

$$= \dot{\omega}(\vec{\mathfrak{J}}_{\vec{n}}^{(C)}, \delta\vec{\varphi}) + \omega([\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)}], \delta\vec{\varphi}) + \omega\left(\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)}}{dt}, \delta\vec{\varphi}\right) \tag{111}$$

where

$$\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)}}{dt} = \overset{*}{\vec{\mathfrak{J}}}{}_{\vec{n}}^{(C)} = \sum_{\nu=1}^{\nu=N} \dot{m}_\nu[\vec{r}_\nu, [\vec{n}, \vec{r}_\nu]] \tag{112}$$

The left-hand side of relation (110) can be transformed into the following form:

$$\int_{t_0}^{t_1} \left\{ \dot{\omega}(\vec{\mathfrak{J}}_{\vec{n}}^{(C)}, \delta\vec{\varphi}) + \omega([\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)}], \delta\vec{\varphi}) + \omega\left(\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)}}{dt}, \delta\vec{\varphi}\right) \right\} dt \tag{113}$$

Similarly, the right-hand side of the relation (110) can be transformed and it will have following form:

$$\int_{t_0}^{t_1} \lambda \sum_{\nu=1}^{\nu=N} \dot{m}_\nu)[\vec{r}_\nu, [\vec{\omega}, \vec{r}_\nu]], \delta\vec{\varphi})\, dt = \int_{t_0}^{t_1} \lambda\omega\left(\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)}}{dt}, \delta\vec{\varphi}\right) dt \tag{114}$$

Due to the transformed expressions (113) and (114), the relation (110) can be written in the form:

$$\int_{t_0}^{t_1} \left\{ \left(\dot{\omega}\vec{\mathfrak{J}}_{n}^{(C)} + \omega[\vec{\omega}, \vec{\mathfrak{J}}_{n}^{(C)}] + \omega(1-\lambda)\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)}}{dt}, \delta\vec{\varphi} \right) \right\} dt = 0 \tag{115}$$

Hence, from here, the vectorial equation of the self-induced rotation of the body has the following form:

$$\dot{\omega}\vec{\mathfrak{J}}_{n}^{(C)} + \omega[\vec{\omega}, \vec{\mathfrak{J}}_{n}^{(C)}] + \omega(1-\lambda)\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)}}{dt} = 0 \tag{116}$$

i.e.,

$$\dot{\omega}(\vec{\mathfrak{J}}_{\vec{n}}^{(C)}, \vec{n}) + \dot{\omega}\vec{\mathfrak{D}}_{\vec{n}}^{(C)} + \omega[\vec{\omega}, \vec{\mathfrak{D}}_{n}^{(C)}] = \omega(\lambda - 1)\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)}}{dt} \tag{117}$$

On the right-hand side of this vectorial equation there is the vector $\vec{\mathfrak{M}}_{C(R)}$ of the reactive moment:

$$\vec{\mathfrak{M}}_{C(R)} = \omega(\lambda - 1)\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)}}{dt} \tag{118}$$

on which the change in the body motions-rotation begins. In the case of appearing of a total (complete) central symmetry of expulsion of parts of mass, the sum of all components of moments of all reactive forces is equal to zero, because the moment vectors (torques) in pairs probably act in opposite directions.

From the vectorial equation (117) it follows that:

$$(\vec{\mathfrak{J}}_{\vec{n}}^{(C)}, \dot{\vec{\omega}}) = (\lambda - 1)\left(\vec{\omega}, \frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)}}{dt}\right) \tag{119}$$

$$\dot{\omega}\vec{\mathfrak{D}}_{\vec{n}}^{(C)} + \omega[\vec{\omega}, \vec{\mathfrak{D}}_{\vec{n}}^{(C)}] = (\lambda - 1)\left[\vec{n}, \left[\frac{\overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)}}{dt}, \vec{\omega}\right)\right]\right] \tag{120}$$

Now, Equation (119) can be written in the following form:

$$\frac{d\omega}{\omega} = (\lambda - 1)\frac{(\vec{n}, \overset{*}{d}\vec{\mathfrak{J}}_{\vec{n}}^{(C)})}{(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)})} \tag{121}$$

This last equation is equivalent to the relation (2.1), which appears in [**16**, p. 75], or to the relation (2.14) which appears in [**20**, p. 99]. Thus, with the integration we will have:

$$(\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)})(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)})^{\lambda-2} = \text{const} \tag{122}$$

i.e.

$$(\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)}) = \text{const} \tag{123}$$

where the constant of integration is to be determined from chosen initial conditions. This formula (123) is analogous with corresponding result of Savić-Kašanin from [**16**].

According to the theory applied here, at the initial time t_0, the vector of the body mass inertia moment, for the pole C and for the axis oriented by the unit vector $\vec{n}$, is $\vec{\mathfrak{J}}_{\vec{n}_0}^{(A)}$ and the instantaneous angular velocity of particles is $\vec{\omega}_0$, so we can write:

$$(\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)})(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)})^{\lambda-2} = (\vec{\omega}_0, \vec{\mathfrak{J}}_{\vec{n}_0}^{(C)})(\vec{n}_0, \vec{\mathfrak{J}}_{\vec{n}_0}^{(C)})^{\lambda-2} \tag{124}$$

Therefore

$$(\vec{\omega}, \vec{\mathfrak{J}}_{\vec{n}}^{(C)}) = (\vec{\omega}_0, \vec{\mathfrak{J}}_{\vec{n}_0}^{(C)}) \tag{124*}$$

For the classical case when the mass of the body is constant, the right-hand side of the equation (121) is equal to zero, so that the vectorial equation is reduced to the equation of the rotation of a body by inertia $\vec{\omega} = \vec{\omega}_0 = \vec{c}\mathrm{onst}$.

II.3.3. Concluding remarks. The exposed analysis of the bodies self-rotation does not aim to explain finally and describe fully the appearance of its induced self-rotation. This is only a contribution to the attempt for the mathematical vectorial descriptions of the law of motion - self-rotation by a new form of the vectorial differential equation, which are typical to the motion of rotor under the action of the reactive forces due to the masses separation (for instances, rotor with nozzles through which the particles are falling). If the rotation axis is the central rotation axis and the main inertia axis for the pole in the stationary bearing then the dynamic pressures do not effect the bearings. Then we can conclude that due to the reactive forces the body rotates around a free axis which retains its orientation. This would be a case of self-rotation of a body around the central axis.

8 * *

Main results of this monograph paper were presented on various seminars, congresses and other scientific meetings, as follows:

A1. Hedrih (Stevanović), K., *On Some interpretations of the rigid bodies kinetic parameters*, XVIII ICTAM Haifa, 1992, Abstracts.

A2. Hedrih (Stevanović), K., *New interpretation of the rigid bodies kinetic parameters*, Abstracts of 2-nd International Symposium of Ukrainian Mechanical Engineers in Lviv, State University "Lvivska Politechnika", Ukainian engineerr's Society in Lviv and Ukainian engineer's Society of America, 1995, p. 51

A3. Hedrih (Stevanović), K., *On rotation of a heavy body around a stationary axis in the fields with turbulent damping and dynamic pressures on bearings*, Abstract of lectures YUCNP Niš, 1991, pp. 38–39.

A4. Hedrih (Stevanović), K., *Vektori vezani za polove i pravu i pojmovi vektora sektorskih momenata masa i površina za pol i osu i sektorski pol*, Apstrakt, Sažeci, PRIM 94, Primenjena analiza, IX Seminar primenjene matematike, Budva, 30. maj – 1. jun 1994, Novi Sad, p. 44.

A5. Hedrih (Stevanović), K., *Energijska analiza kinetike konstrukcija za različite modele materijala*, Kratki prikazi radova naučnog skupa "Mehanika, materijali i konstrukcije", Srpska Akademija Nauka i Umetnosti, Odelenje tehničkih nauka, 1995, 106–107.

A6. Hedrih (Stevanović), K., *On New Interpretations of the rigid Bodies Kinetic Parameters and on Rotation of a Heavy body around a Stationary Axis in the Foeld* *with Turbulent Damping and Dynamic Pressures on Bearings*, Conference on Differential Geometry and Applications, Masaryk University, Brno 1995, Zbornik Apstrakata.

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Index of symbols

x^i curvilinear coordinate 68, 69, 70, 71

$\vec{\omega}$ angular velocity of the rotation around the axis 72, 73, 74, 75, 76, 77, 78, 79, 81, 82, 84, 85, 86, 87, 88, 89, 92, 93, 94

$\vec{\mathfrak{K}}$ vector of the linear momentum of the rigid body dynamic 74, 76, 77

$\vec{\mathcal{L}}_A$ vector of the angular momentum for the point A 75, 76, 78

$\vec{F}_{rj}$ the main vector of the inertia force of the rigid body rotating around the axis with the angular velocity $\vec{\omega}$ 76, 77, 79

$\vec{\mathfrak{M}}_{Aj}$ the main moment of the inertia forces of the rigid body rotating around the axis and for the point A 76, 77, 79

$\vec{\mathfrak{R}} = \mathfrak{R}\vec{r}_1$ rotator is normal to the rotation axis 77, 78, 79, 81, 87, 88

$\vec{w}_N$ absolute velocity of the body particles falling of 85

λ a scalar, the proportionality coefficient 85, 88

$d\vec{\mathfrak{F}}_r$ reactive force due to the elementary particle falling off 85

$\vec{\mathfrak{F}}_r$ main vector of the reactive forces 84, 85, 86

$\vec{\mathfrak{M}}_r^{\vec{\mathfrak{F}}}$ resulting moment of the reactive forces due the body particles falling off 85, 86

$\vec{r}_v$ a position vector of mass particle $m_v, v - 1, 2, \dots N$ 90, 91, 92

$m_v, v = 1, 2, \dots N$ mass particles 90, 91, 92

$\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ body mass deviation moment vector at the point O for the axis oriented by the unit vector $\vec{n}$ 48, 54, 56, 69, 71, 75, 76, 78, 79, 80, 81, 82, 83, 86, 87, 88

$\vec{\mathfrak{S}}_{\vec{n}}^{(N)}$ vector of the body mass linear moment 47, 48, 51, 53, 58, 69, 71, 73, 74, 76, 77, 81, 84, 85, 86, 87, 90

$\vec{\mathcal{M}}_{\vec{n}}^{(N)}$ vector of the body mass at the point N for the axis oriented by the unit vector $\vec{n}$ 47, 51, 53

$\vec{\delta}_{\vec{n}}^{(N)}$ vector of the total relative deformation - total relative strain, at the point N and for the line element drawn from point N and oriented by unit vecto $\vec{n}$ 51

$\vec{p}_{\vec{n}}^{(O)}$ vector of the total stress at a certain body point for the plane with the normal oriented by unit vector $\vec{n}$ 59

$\mathbf{J}^{(O)}$ body mass inertia moment matrix 52 59 69

$\vec{M}^{(O)}$ the mass linear polar moment of the material system 52

$J_{\vec{n}}^{(O)}$ axial mass inertia moment 54, 59

$D_{nu}^{(O)}, D_{nv}^{(O)}$ the deviational moments of the body mass for a coulpe of normal axes orianted by unit vectors $\vec{n}$ and $\vec{u}$, thet is, $\vec{n}$ and $\vec{v}$ 54, 55

$J_1^{(O)}, J_1^{(O)}, J_1^{(O)}$ first, second and third scalar of the body mass inertia moment tensor matrix 59

$\mathbf{J}^{(O)\mathrm{sph}}, \mathbf{J}^{(O)\mathrm{dev}} = \mathbf{D}^{(O)\mathrm{dev}}$ two matrices corresponding to the spherical and deviational part of the rigid body mass inertia moment tensor 59, 71

$\vec{\mathfrak{J}}_{\vec{n}}^{(O)\mathrm{aks}}$ body mass axial inertia moment vector at the point on for the axis orianted by unit vector $\vec{n}$ 60

$\vec{\mathfrak{D}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(O)\mathrm{dev}}$ body mass deviation moment vector at the point and for the axis oriented by unit vector $\vec{n}$; vector of the axis deviation load 60, 61, 62, 71

$\vec{n}_S$ unit vector of the main mass inertia moment axis orientation 62, 63

$\mathrm{K}_{3k}^{(S)}, k = 1,2,3$ are co-factors of the third kind elements and the corresponding matrix column, successively for the roots $J_s^{(O)}, s = 1,2,3;$ 63

I_a and I_b, II_a and II_b, III_a and III_b the axes pairs of the mass deviation moments extreme values 63, 64, 65, 66

$\vec{\mathfrak{J}}_{\vec{N}_s}^{(O)}, s = 1,2,3$ the body mass inertia moment vectors for the main mass inertia moment axis for the referential point 65, 66

ВЕКТОРИ МОМЕНАТА МАСА ТЕЛА

Овај монографски чланак уводи вектор $\vec{\mathfrak{J}}_{\vec{n}}^{(N)}$ момента инерције масе тела за тачку N и осу орјентисану јединичним вектором $\vec{n}$. Вектор момента инерције масе крутог тела је коришћен за анализу стања момената инерције масе тела за одређену конфигурацију масе тела, као и за интерпретацију кинетичких параметера материјалног система у кретању. Промена вектора момента инерције масе тела при промени пола када оса задржава своју орјентацију је одређена и представља уопштење Huygens-Steiner-ове теореме на уведене векторе момената инерције масе тела. Изведен је израз за одређивање промене вектора момената инерције масе тела када оса мења орјентацију, што је једначина аналогна Cauchy-јевим једначинама из теорије еластичности. Показано је како се помоћу вектора момената маса одређују главни правци момената инерције маса као и правци инерционе асиметрије. Одређени су вектори момената инерције маса за октаедарске правце. Указано је на аналогије модела стања момената инерције маса тела, стања напона и стања деформације помоћу вектора везаних за тачку и осу, односно раван. Анализирани су сферна и девијациона својства вектора момената маса.

Овим чланком су уведени следећи вектори везани за тачку и осу: вектор $\vec{\mathcal{M}}_{\vec{n}}^{(N)}$ масе тела у тачки N за осу орјентисану јединичним вектором $\vec{n}$; вектор $\vec{\mathfrak{S}}_{\vec{n}}^{(N)}$ линеарног (статичког) момента масе тела у тачки N за осу орјентисану јединичним вектором $\vec{n}$; и вектор $\vec{\mathfrak{J}}_{\vec{n}}^{(N)}$ момента инерције масе тела у тачки N за осу орјентисану јединичним вектором $\vec{n}$. Изведени су изрази за векторе момената маса у n-димензионалном криволинијском систему координата.

Затим су помоћу уведених вектора момената маса изражени кинетички параметри кретања крутог тела. Даље интерпретације су одредиле изразе за кинетичку енергију, количину кретања и момент количине кретања крутог тела помоћу уведених вектора момената маса тела. Специјално, за случај обртања тела око непомичне осе, одређени су изводи количине кретања и момента количине кретања у функцији тих вектора момената маса и написане кинетичке једначине ротације у векторском облику. Одређени су изрази за кинетичке притиске и уведен кинематички вектор ротатор. Показује се да коришћење вектора момената маса и вектора ротатора даје сасвим једноставне изразе за кинетичке притиске који зависе од девијационих делова вектора момената маса у односу на осу ротације и од кинематичког вектора ротатора. Услови динамичког балансирања се такође једноставно изражавају у услову да су девијациони делови вектора момената маса једнаки нули.

У чланку су изведени изрази за промене вектора момената маса при ротацији тела и за случај крутог тела променљиве масе. Изведена је векторска једначина самоaccording ротације крутог тела променљиве масе.

Овај монографски чланак представља преглед научних резултата које је аутор публиковано у научним часописима и/или саопштио на научним когресима и конференцијама међународног или националног значаја што се види из списка литературе која садржи ауторових 30 библиографских јединица.

Овај монографско прегледни чланак представља целину по векторској методи коју је аутор засновао на векторима везаним за пол и осу увођењем вектора момената маса тела за пол и осу којима се изражавају геометријско конфигурациона својства маса тела и кинематичких вектора ротатора који су везани за пол и осу и ротирају око ње одговарајућом угаоном брзином и убрзањем. Такође, чланак представља целину и по садржајима: комплетном интерпретацијом анализе стања момената маса тела у односу на пол уведеним векторима момената маса и комплетном интерпретацијом кинетичких параметара кретања ротора.

САДРЖАЈ

ГЛАВА I

ГЛАВА II