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AN INTRODUCTION TO ADJUNCTION

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Abstract. This is an introduction to the notion of adjunction from an abstract point of view. A systematic survey is made of various definitions of this notion, including two definitions not recorded in the literature. A similar survey is also made of the definitions of comonad, which also includes new material. Finally, the relationship between the notion of adjunction and the notion of comonad is explained through two adjunctions involving the category of adjunctions and the category of comonads, where the latter category is isomorphic to a full subcategory of the former. The standard presentation of this relationship, through the category of resolutions of a comonad, is a corollary of this new presentation of the matter.

Adjunction is one of the most important notions of mathematics, which category theory has taught us to recognize everywhere. To put it roughly, adjunction is half of an equivalence of categories, but taken wisely, in a "diagonal" way (cf. 2.2). This means that, though the two categories need not be equivalent—one may be *richer* than the other—something essential is not lost in passing from the richer category to the poorer one: the two categories share a common core. A formal theorem concerning the equivalence of subcategories of categories in an adjoint situation reflects this fact (see [Lambek & Scott 1986, Part 0, sections 3-4] and [Lambek 1981], where rather "obscure" antecedents are found for this important principle).

A typical adjunction is when we have, on the one hand, a category $\mathcal{A}$ whose objects are some algebras, like groups or vector spaces over a fixed field, with arrows being homomorphisms (in the case of vector spaces these are linear transformations), and on the other hand, the category $\mathcal{B}$ whose objects are sets, with functions as arrows. From $\mathcal{A}$ to $\mathcal{B}$ goes a *forgetful* functor G, which assigns to an algebra the underlying set of elements, and to a homomorphism the underlying function. This functor has a left-adjoint functor F from $\mathcal{B}$ to $\mathcal{A}$ that assigns to a set B the free algebra generated by B (with vector spaces, $F(B)$ is the vector space with basis B). In passing with G from the richer category to the poorer one, not all information about the algebras is lost: something essential is preserved. The set $G(A)$ still carries some information about the algebra A from the category $\mathcal{A}$. When we apply next the adjoint functor F to $G(A)$, the algebra $F(G(A))$ is not the same as the initial algebra A, but it is comparable to it: there is a homomorphism φ_A from $F(G(A))$ to A defined by mapping the free generators to themselves, which is a component of a natural transformation called the *counit* of the adjunction. Similarly, for a set B, the set $G(F(B))$ is comparable to B: there is a function γ_B from B to $G(F(B))$ amounting to inclusion, which is a component of a natural transformation called the *unit* of the adjunction. The categories $\mathcal{A}$ and $\mathcal{B}$ would be equivalent if $F(G(A))$ were isomorphic to A, and $G(F(B))$ were isomorphic to B. Of these isomorphisms, we have only halfs, chosen "diagonally", in opposite directions: the homomorphism φ_A from $F(G(A))$ to A and the function γ_B from B to $G(F(B))$. Moreover, arrows derived from the unit composed with arrows derived from the counit give identity arrows: $F(\gamma_B)$ composed with $\varphi_{F(B)}$

is the identity homomorphism on $F(B)$, and $\gamma_{G(A)}$ composed with $G(\varphi_A)$ is the identity function on $G(A)$.

This way, the forgetting of the forgetful functor is controlled. Some conclusions we may reach by reasoning in $\mathcal{B}$ can be transferred back to $\mathcal{A}$. However, it seems that the point of describing an adjoint situation is not so much to provide a tool for proving new theorems, but rather to illuminate, clarify and systematize already known results.

The ability to forget in a controlled manner is an important trait of rationality—perhaps the most important one. We should forget the unessential, so as not to be encumbered by it, and move more easily in our thoughts. But this forgetting should be controlled: what is essential shouldn't be forgotten. There should be a way back: conclusions reached in the simpler context, where the unessential is forgotten, should be applicable to the original, more complicated, context. Controlled forgetting, which exists in abstraction, but not only there, is certainly a major character of mathematical rationality, and an embodiment of it is found in the concept of adjunction.

We can take it as a rule of thumb that behind theorems of the "if and only if" type we should look for adjunctions. In important theorems of this type, where in passing from one side to the other there is a gain, and where, typically, one direction of the theorem is easy to prove and the other difficult, there should be an adjunction that does not amount to equivalence of categories, but obtains between a richer and a poorer category.

We should say, however, that not every adjunction not amounting to equivalence need hold between a richer and a poorer category. Two functors going from a category to this same category may be adjoints without the unit and counit giving isomorphisms. If we have a functor H from a category $\mathcal{A}$ to a category $\mathcal{B}$ that has both a left adjoint F from $\mathcal{B}$ to $\mathcal{A}$ and a right adjoint G from $\mathcal{B}$ to $\mathcal{A}$, then the composite functors FH and GH from $\mathcal{A}$ to $\mathcal{A}$ are adjoints, FH being left-adjoint and GH right-adjoint (analogously, the composite functors HF and HG from $\mathcal{B}$ to $\mathcal{B}$ are adjoints, HF being left-adjoint and HG right-adjoint). Various examples of adjunction may be found in Mac Lane's book [1971].

This introduction to the notion of adjunction will, however, not dwell much on examples. We shall rather try to decipher the abstract, logical, structure of this notion. The work will be divided into five parts. After the first part, devoted to preliminaries of category theory, we shall consider in the second part the adjunction underlying the notion of function. Then in the third part we consider definitions of the notion of adjunction. The fourth part is about the related notion of comonad (we could as well have chosen to deal with monads, also called *triples*). In the final, fifth part, we explain the relationship between the notions of adjunction and comonad.

1. Preliminaries

Before embarking upon our consideration of adjunction, we have to review first some elementary notions of category theory.

1.1. Foundations. Category theory is sometimes taken as providing mathematics with foundations alternative to set theory. This point of view often leads into discussions about the size of classes, i.e., about the distinction between sets and proper classes. Such matters are, otherwise, rather foreign to the spirit of results about categories, which are more about structure than about size. (A presumably germane point is made when, using ancient philosophical terminology, categories are said to be about form, rather than substance; cf. [Lawvere 1964].) So these discussions are usually limited to a preamble of a typical work in category theory (such is the case, too, in the most widely cited text about categories—Mac Lane's book [1971]). In general, they don't leave much trace on the mathematics in the main body of the work, except a tendency to distinguish results that hold only for *small* categories, i.e., those whose objects and arrows, not being too numerous, can be collected into sets. These distinctions often don't have much to do with the import of the results, and can be somewhat distracting.

We are here approaching categories with a logical background, but we shall neglect foundational matters. In fact, this neglect may be explained just by this background. If we were asked about foundations, we would rely on standard set-theoretical foundations, as they have become crystallized within logic. The objects of the category of sets would be for us all the sets that are the elements of the domain of a given model of first-order axiomatic set theory. Since such a domain is itself a set, there is no problem in conceiving of the category of sets as being itself small. So we restrict our attention to small categories only. Bigger categories than these maybe exist, but they shall not be our concern.

1.2. Morphisms and naturalness. The dominant opinion is that the guiding principle of category theory is to look concerning every mathematical object for structure-preserving maps. When the object has no structure, when it is simply a set, then the maps are all functions from sets to sets. When the object has structure, then it may be an algebra, in which case the maps are homomorphisms, or it may be a set with a binary relation, in which case the maps are monotonic functions. Many other sorts of structure can be envisaged.

In model theory, stress is often put on relational or functional structures with a single domain; i.e., relations are defined on a single set and functions are operations on a set. Category theory, on the other hand, is concerned much more with a plurality of domains.

Let us consider the case of relations, and let us generalize monotonicity to relations between two sets. So let A and B be sets and let $R \subseteq A \times B$. If we have another relation $R' \subseteq A' \times B'$, then a structure-preserving map from R to R' would be a pair of functions $f : A \to A'$ and $g : B \to B'$ such that for every a in A and every b in B

$$\text{if } a\,R\,b, \text{ then } f(a)\ R'\ g(b)$$

(of course, $a\,R\,b$ means $(a,b) \in R$). When $A = B$, $A' = B'$ and $f = g$, then we obtain the ordinary monotonicity condition.

The standard approach is to take a function as a special kind of relation, but we may also take the notion of function as being more primitive. Every relation $R \subseteq A \times B$ is associated to a function f from A to the power set of B such that $a\,R\,b$ iff $b \in f(a)$. To understand structure-preserving maps we shall then concentrate on the notion of function.

Let a *function pair* from a pair of sets (A_1, A_2) to a pair of sets (A'_1, A'_2) be a pair of functions (g_1, g_2) such that $g_1 : A_1 \to A'_1$ and $g_2 : A_2 \to A'_2$. A structure-preserving map from a function $f : A_1 \to A_2$ to a function $f' : A'_1 \to A'_2$ is a function pair (g_1, g_2) from (A_1, A_2) to (A'_1, A'_2) such that for every x in A_1 and every y in A_2

$$\text{if } f(x) = y, \text{ then } f'(g_1(x)) = g_2(y).$$

This implication is equivalent to requiring that for every x in A_1

$$g_2(f(x)) = f'(g_1(x)),$$

which means that for the composite functions the following *naturalness* equality holds:

$$g_2 f = f' g_1.$$

We use the term *morphism* for function pairs that satisfy naturalness; so (g_1, g_2) is a morphism from f to f' iff naturalness holds. This defines *morphisms between functions*. (Note that some authors use the term "morphism" for arrows in a category.)

This terminology accords rather well with standard usage. For a binary operation $f : A \times A \to A$ and another binary operation $f' : A' \times A' \to A'$, the function pair that is an obvious candidate for a morphism from f to f' is $(g \times g, g)$ where $g : A \to A'$ and $(g \times g)(x_1, x_2)$ is defined as $(g(x_1), g(x_2))$. Such a function pair $(g \times g, g)$ is a morphism from f to f' iff g is a homomorphism in the ordinary sense.

However, we shall speak of morphisms in other situations, too, where the structure mapped is not only that of a function, but something more complicated, involving several functions, which are moreover of a special kind. Then morphisms will not be simply function pairs, but something more involved, though analogous. In particular cases, we shall introduce special names for the morphisms in question. The guiding idea will always be to impose the naturalness condition for every function involved. Since many, if not all, important structures of mathematics can be expressed in terms of functions, and often gain in clarity by being expressed so, we shall find the notion of structure-preserving map appropriate to these structures by looking for naturalness conditions.

1.3: Graphs, graph-morphisms and transformations. A graph is a function pair (S, T) from (X, X) to (Y, Y). So, S and T are both functions from X to

Y. To help imagination, we call X the set of *arrows*, Y the set of *objects*, S the *source* function and T the *target* function. With that terminology, the denomination "graph" becomes justified. (In graph theory, the corresponding notion is sometimes called "directed multigraph with loops".)

For objects of graphs we use the letters $A, B, C, \ldots$, and for arrows $f, g, h, \ldots$, with indices if needed. We write $f : A \to B$ to indicate that the source of the arrow f is A and its target B; we say that $A \to B$ is the *type* of f. For graphs we use the script letters $\mathcal{G}, \mathcal{H}, \ldots$ A *hom-set* $\mathcal{G}(A, B)$ in a graph $\mathcal{G}$ is $\{f \mid f : A \to B$ is an arrow of $\mathcal{G}\}$.

An alternative way to define a graph is to identify it with a single function $\mathcal{F}$ from X to $Y \times Y$. To pass from a graph (S, T) to a graph $\mathcal{F}$, we have the definition

$$\mathcal{F}_{S,T}(f) \overset{\text{def}}{=} (S(f), T(f)).$$

Conversely, if we are given $\mathcal{F}$, and p^1 and p^2 are, respectively, the first and second projection function, then we define S and T by

$$S_{\mathcal{F}}(f) \overset{\text{def}}{=} p^1(\mathcal{F}(f)), \qquad T_{\mathcal{F}}(f) \overset{\text{def}}{=} p^2(\mathcal{F}(f)).$$

It is clear that if we start from a graph (S, T), define $\mathcal{F}_{S,T}$, and then define $S_{\mathcal{F}_{S,T}}$ and $T_{\mathcal{F}_{S,T}}$,we obtain that S is equal to $S_{\mathcal{F}_{S,T}}$ and T is equal to $T_{\mathcal{F}_{S,T}}$. Analogously, $\mathcal{F}_{S_{\mathcal{F}},T_{\mathcal{F}}}$ is equal to $\mathcal{F}$.

We shall say that the two notions of graph, the (S, T) notion and the $\mathcal{F}$ notion, are *equivalent*. (This we do because there is an equivalence, actually an isomorphism, between the category of (S, T) graphs and the category of $\mathcal{F}$ graphs, as we shall see in 1.5.) The equivalence of two notions does not always mean that the two notions are *coextensive*, i.e., that they cover exactly the same objects, as the notions of equilateral and equiangular triangles are coextensive. The (S, T) graphs and the $\mathcal{F}$ graphs are strictly speaking different objects, though they are in one-to-one correspondence. On the other hand, equivalence is more than just this one-to-one correspondence. The concept of equivalence of notions will be explained in detail in 1.5 (after we have introduced the notion of equivalence of categories).

A binary relation on Y may be identified with a graph $\mathcal{F}$ that is a *one-one* function. We can then forget about X, and consider just the image of $\mathcal{F}$, i.e., a subset of $Y \times Y$. If a binary relation is a *set* of ordered pairs, a graph is a family of ordered pairs indexed by the arrows, a family where the same ordered pair may occur several times with different indices. In other words, a graph is a *multiset* of ordered pairs.

If a graph is a function pair (S, T), then the appropriate notion of morphism is the following. Suppose S and T are functions from X to Y, while S' and T' are functions from X' to Y'. Then as a morphism from $\mathcal{G} = (S, T)$ to $\mathcal{H} = (S', T')$ we can take a function pair (M_X, M_Y) from (X, Y) to (X', Y') such that naturalness is satisfied, i.e.

$$M_Y(S(f)) = S'(M_X(f)), \qquad M_Y(T(f)) = T'(M_X(f)).$$

This means that arrows $f : A \to B$ of $\mathcal{G}$ are mapped to arrows $M_X(f) : M_Y(A) \to MY(B)$ of $\mathcal{H}$. As usual, we shall omit the subscripts from M_X and M_Y, referring to both by M. We shall also find it handy to omit parentheses from $M(A)$ and $M(f)$; instead we write MA and Mf.

So a *graph-morphism* M from $\mathcal{G}$ to $\mathcal{H}$ will be a pair of functions, both written M, assigning, respectively, to every object A of $\mathcal{G}$ an object MA of $\mathcal{H}$, and to every arrow $f : A \to B$ of $\mathcal{G}$ an arrow $Mf : MA \to MB$ of $\mathcal{H}$.

A graph-morphism M from $\mathcal{G}$ to $\mathcal{H}$ is *faithful* iff for every pair (A, B) of objects of $\mathcal{G}$ and for every pair $(f : A \to B,\ g : A \to B)$ of arrows of $\mathcal{G}$ if $Mf = Mg$ in $\mathcal{H}$, then $f = g$ in $\mathcal{G}$; this means that M restricted to the hom-sets $\mathcal{G}(A, B)$ and $\mathcal{H}(MA, MB)$ is one-one. A graph-morphism M from $\mathcal{G}$ to $\mathcal{H}$ is *full* iff for every pair (A, B) of objects of $\mathcal{G}$ and for every arrow $g : MA \to MB$ of $\mathcal{H}$ there is an arrow $f : A \to B$ of $\mathcal{G}$ such that $g = Mf$; this means that M restricted to the hom-sets $\mathcal{G}(A, B)$ and $\mathcal{H}(MA, MB)$ is onto. Note that if a graph-morphism is one-one on objects, then it is faithful iff it is one-one on arrows, and if it is onto on objects, then it is full iff it is onto on arrows.

A graph-morphism is an *embedding* iff it is one-one both on objects and on arrows, and it is an *isomorphism* iff it is a bijection both on objects and on arrows.

A graph $\mathcal{G}$ is a *subgraph* of a graph $\mathcal{H}$ iff there is a graph-morphism M from $\mathcal{G}$ to $\mathcal{H}$ that is the inclusion function both on objects and on arrows; M is called the *inclusion graph-morphism* from $\mathcal{G}$ to $\mathcal{H}$. This means that the objects of $\mathcal{G}$ are included among the objects of $\mathcal{H}$ and the arrows of $\mathcal{G}$ among the arrows of $\mathcal{H}$, and for every object A of $\mathcal{G}$ the object MA of $\mathcal{H}$ is A, while for every arrow f of $\mathcal{G}$ the arrow Mf of $\mathcal{H}$ is f. Moreover, since M is a graph-morphism, the arrows of $\mathcal{G}$ have in $\mathcal{H}$ the same sources and targets as in $\mathcal{G}$. The inclusion graph-morphism M is an embedding, and a fortiori it is faithful. A subgraph is *full* iff the inclusion graph-morphism is full.

The *identity graph-morphism* $I_\mathcal{G}$ from a graph $\mathcal{G}$ to $\mathcal{G}$ is the identity function both on objects and on arrows. If we have a graph-morphism M from a graph $\mathcal{G}$ to a graph $\mathcal{H}$ and a graph-morphism N from a graph $\mathcal{H}$ to a graph $\mathcal{J}$, then we have the *composite graph-morphism* NM from $\mathcal{G}$ to $\mathcal{J}$ obtained by composing the functions M and N, on objects and on arrows.

Let M and N be graph-morphisms from a graph $\mathcal{G}$ to a graph $\mathcal{H}$. A *transformation* from M to N is a family τ of arrows $\tau_A : MA \to NA$ of $\mathcal{H}$, indexed by the objects A of $\mathcal{G}$. More precisely, a transformation τ is a function from the set of objects of $\mathcal{G}$ to the set of arrows of $\mathcal{H}$, with values $\tau(A)$, which is written τ_A, of type $MA \to NA$. Note that a transformation need not be one-one (i.e., for different objects A and B of $\mathcal{G}$, the arrows τ_A and τ_B may be equal, provided MA is MB and NA is NB).

A slightly more general notion than transformation is obtained by assuming that M and N are only functions from the objects of $\mathcal{G}$ to the objects of $\mathcal{H}$, everything else being as for transformations. We shall have two occasions to rely on this notion of *objectual transformation* (see 3.6 and 4.5).

1.4. Deductive systems, functors, natural transformations and categories. An *identity* $\mathbf{1}$ in a graph $\mathcal{G}$ is a family of arrows $\mathbf{1}_A : A \to A$ of $\mathcal{G}$, indexed by the objects A of $\mathcal{G}$. In other words, $\mathbf{1}$ is a transformation from $I_{\mathcal{G}}$ to $I_{\mathcal{G}}$. The arrows $\mathbf{1}_A$ are called *identity arrows.*

A *composition* $\circ$ in $\mathcal{G}$ is a function that to every pair $(f : A \to B,\ g : B \to C)$ of arrows of $\mathcal{G}$ assigns an arrow $g \circ f : A \to C$ of $\mathcal{G}$.

A *deductive system* is a triple $\langle \mathcal{D}, \mathbf{1}, \circ \rangle$ where $\mathcal{D}$ is a graph, $\mathbf{1}$ is an identity in $\mathcal{D}$ and $\circ$ is a composition in $\mathcal{D}$. The identity and composition of different deductive systems will always be denoted by the same symbols $\mathbf{1}$ and $\circ$, assuming it is clear from the context to which deductive system they belong. (The term "deductive system" was introduced by Lambek because of an obvious analogy with logical consequence. This analogy, which is not superficial, is at the base of *categorial proof theory*; see [Lambek & Scott 1986] and [D. 1996, 1997].)

A *functor* F from a deductive system $\langle \mathcal{D}, \mathbf{1}, \circ \rangle$ to a deductive system $\langle \mathcal{E}, \mathbf{1}, \circ \rangle$ is a graph-morphism from $\mathcal{D}$ to $\mathcal{E}$ that satisfies

$$(fun1) \qquad F\mathbf{1}_A = \mathbf{1}_{FA},$$
$$(fun2) \qquad F(g \circ f) = Fg \circ Ff.$$

These two conditions are just naturalness conditions for morphisms of identities (where identities are understood as functions) and morphisms of compositions.

An *embedding of deductive systems* is a graph-morphism that is a functor and an embedding, and an *isomorphism of deductive systems* is a graph-morphism that is a functor and an isomorphism. A deductive system $\langle \mathcal{D}, \mathbf{1}, \circ \rangle$ is a *subsystem* of a deductive system $\langle \mathcal{E}, \mathbf{1}, \circ \rangle$ iff there is a functor from $\langle \mathcal{D}, \mathbf{1}, \circ \rangle$ to $\langle \mathcal{E}, \mathbf{1}, \circ \rangle$ that is an inclusion graph-morphism from $\mathcal{D}$ to $\mathcal{E}$. As for subgraphs in general, a subsystem is *full* iff the inclusion graph-morphism is full.

It is clear that the identity graph-morphism $I_{\mathcal{D}}$ on the graph $\mathcal{D}$ of a deductive system $\langle \mathcal{D}, \mathbf{1}, \circ \rangle$ is a functor; it is called the *identity functor*. It is also clear that the composite graph-morphism GF is a functor when F and G are functors.

Let M and N be graph-morphisms from a graph $\mathcal{G}$ to a graph $\mathcal{H}$. If $\mathcal{H}$ has a composition $\circ$, and, a fortiori, if $\mathcal{H}$ is the graph of a deductive system $\langle \mathcal{H}, \mathbf{1}, \circ \rangle$, then a transformation from M to N is *natural* iff the following equality holds for every arrow $f : A \to B$ of $\mathcal{G}$:

$$(nat) \qquad \tau_B \circ Mf = Nf \circ \tau_A.$$

If Mf, Nf, τ_A and τ_B are functions and $\circ$ is functional composition, (nat) is the naturalness condition for the morphism (τ_A, τ_B) from Mf to Nf.

A deductive system is a *category* iff the following equalities hold between its arrows:

$$\text{(cat1right)} \qquad f \circ \mathbf{1}_A = f,$$
$$\text{(cat1left)} \qquad \mathbf{1}_B \circ f = f,$$
$$\text{(cat2)} \qquad (h \circ g) \circ f = h \circ (g \circ f).$$

A *subcategory* is a subsystem of a category.

Often, we denote a deductive system $\langle \mathcal{D}, \mathbf{1}, \circ \rangle$ simply by $\mathcal{D}$, taking the identity and composition for granted, provided it is clear from the context that we have in mind a deductive system, rather than simply a graph. We do the same for categories. If, however, we need to emphasize the difference between a deductive system and its graph, we use the notation $\langle \mathcal{D}, \mathbf{1}, \circ \rangle$.

Note that our notion of functor is slightly more general than the usual notion, which is given for categories only, whereas ours apply to arbitrary deductive systems. Note also that our notion of natural transformation is likewise more general than the usual notion, which is given for functors M and N from a category $\mathcal{G}$ to a category $\mathcal{H}$.

1.5. Equivalence of categories. If a graph is a function pair (S, T), then a possible notion of morphism between graphs is not only our notion of graph-morphism, but also a more general notion, which we shall now introduce.

Let (f, h) be a function pair from (A_1, B_1) to (A_2, B_2) and (f', h') a function pair from (A_1', B_1') to (A_2', B_2'). A morphism from (f, h) to (f', h') is then simply two function pairs, (g_1, g_2) from (A_1, A_2) to (A_1', A_2'), which is a morphism from f to f', and (k_1, k_2) from (B_1, B_2) to (B_1', B_2'), which is a morphism from h to h'. If (f, h) and (f', h') are graphs, then $A_1 = B_1 = X$, $A_2 = B_2 = Y$, $A_1' = B_1' = X'$, $A_2' = B_2' = Y'$, but we could keep the same notion of morphism. Let us call these morphisms of graphs *double morphisms*.

A graph-morphism as we have defined it in 1.3 is a double morphism where $g_1 = k_1$ and $g_2 = k_2$. With double morphisms in general we would have a function pair (M_X, M_Y) that in virtue of naturalness preserves sources, i.e., $M_Y(S(f)) = S'(M_X(f))$, and another function pair (N_X, N_Y) that in virtue of naturalness preserves targets, i.e. $N_Y(T(f)) = T'(N_X(f))$.

On the other hand, if a graph is a function $\mathcal{F}$ from X to $Y \times Y$, then a possible notion of morphism is not only our notion of graph-morphism, but also another generalization of this notion. Namely, we would have a function pair $(M_X, M_{Y\times Y})$, where M_X is, as before, a function from X to X', but $M_{Y\times Y}$ is a function from $Y \times Y$ to $Y' \times Y'$. So pairs of objects are mapped to pairs of objects. The required naturalness condition is

$$M_{Y\times Y}(\mathcal{F}(f)) = \mathcal{F}'(M_X(f)).$$

Let us call these morphisms of graphs *single morphisms*. A graph-morphism is a single morphism where $M_{Y\times Y}$ is defined as $M_Y \times M_Y$ in terms of a function M_Y from Y to Y; for $M_Y \times M_Y$ we have

$$(M_Y \times M_Y)(A, B) = (M_Y(A), M_Y(B)).$$

The notion of graph-morphism is a common denominator of double and single morphisms, which can serve for either notion of graph.

An arrow $f : A \to B$ in a category is an ***isomorphism*** iff there is an arrow $g : B \to A$, called the ***inverse*** of f, such that $g \circ f = \mathbf{1}_A$ and $f \circ g = \mathbf{1}_B$. Two objects A and B are ***isomorphic*** iff there is an isomorphism f of the type $A \to B$. A natural transformation τ is a ***natural isomorphism*** iff τ_A is an isomorphism for every A.

Two categories $\mathcal{A}$ and $\mathcal{B}$ are ***equivalent*** iff there is a functor F from $\mathcal{B}$ to $\mathcal{A}$ and a functor G from $\mathcal{A}$ to $\mathcal{B}$ such that there is in $\mathcal{A}$ a natural isomorphism from FG to $I_{\mathcal{A}}$ and there is in $\mathcal{B}$ a natural isomorphism from GF to $I_{\mathcal{B}}$. An equivalence of categories where these natural isomorphisms are identities boils down to isomorphism of categories as we have defined it in the preceding section.

It is easy to show that the category of graphs in the (S, T) sense (i.e., the category whose objects are these graphs) with graph-morphisms as arrows is isomorphic to the category of $\mathcal{F}$ graphs with graph-morphisms as arrows. Hence, these categories are also equivalent. This justifies our saying that the two notions of graph are equivalent. In general, two notions are to be called ***equivalent*** iff they cover objects of two categories that are equivalent.

When two notions are equivalent, it is common to say that we have just two *formulations* of the same notion, or that the same notion is defined in alternative ways. Formulations are then called equivalent, rather than notions. We will often speak in this less formal way, too.

Consider, now, the category of (S, T) graphs with double morphisms as arrows and the category of $\mathcal{F}$ graphs with single morphisms as arrows. These two categories are not equivalent, and neither of them is equivalent to the category of (S, T) graphs with graph-morphisms as arrows, or the category of $\mathcal{F}$ graphs with graph-morphisms as arrows. So, to determine whether two notions are equivalent, it is not enough to find a bijection between the objects that fall under these notions. We also have to find the appropriate morphisms, and prove an equivalence of categories.

With the notions that will be found equivalent later in this work we will find mostly isomorphisms of categories, rather than simply equivalences. We stick, however, to the terminology of "equivalent notions", because this way of speaking is more common ("isomorphic notions" would be a neologism), and because equivalence of categories catches well the intuitive idea of equivalence of notions.

2. Functions redefined

The notion of adjunction presupposes the more elementary notion of function, whose importance and ubiquity in mathematics are, of course, not necessary to mention, let alone justify. We want to show, however, that underlying the notion of function there is an adjunction, and that this adjunction characterizes completely the notion of function. This will serve as another corroboration of the slogan that adjointness arises everywhere.

The standard definitions of the general notions of function, *onto* function and *one-one* function don't exhibit clearly the regularities and symmetries of these notions. It is not immediately clear from these definitions, without some deducing, that

(1) the property of being a function is made of two components exactly dual to the *onto* and *one-one* properties (they go in the opposite direction),
(2) the *onto* and *one-one* properties are dual to each other.

There are definitions of these notions that exhibit immediately (1) and (2), but these definitions are rarely and cryptically mentioned (the earliest reference for them I know of is [Riguet 1948, p. 127]). On their own, these definitions are quite simple. I believe that their ingredients belong to the folklore and sometimes crop up as exercises in textbooks. However, the general picture they provide seems to be missing in the standard textbook approach. Many students of mathematics probably stay pretty much in the dark about (2), and perhaps even (1); many are probably surprised when, after having known for some time about onto functions and one-one functions, they learn about (2) via the cancellation properties of epi and mono arrows in category theory.

I don't wish to suggest that these nonstandard definitions should supplant the standard ones—especially not for a first exposure to the defined notions. I suppose, however, that at some point in the study of mathematics one should get a systematic picture such as will occupy us here.

2.1. The standard definition of function. A *binary relation* is a set of ordered pairs R together with some specified domain D and codomain C such that $R \subseteq D \times C$. We speak here only about "relations", the epithet "binary" being tacitly presupposed, and, as usual, we write $x\,R\,y$ for $(x,y) \in R$.

A *function* from D to C is a relation $R \subseteq D \times C$ such that for every x in D there is *exactly one* y in C for which $x\,R\,y$. It is easy to deduce that $R \subseteq D \times C$ is a function iff

(*left-total*) for every x in D there is *at least one* y in C such that $x\,R\,y$,

(*right-unique*) for every x in D there is *at most one* y in C such that $x\,R\,y$.

A function $R \subseteq D \times C$ is *onto* iff

(*right-total*) for every y in C there is *at least one* x in D such that $x\,R\,y$,

and it is *one-one* iff

(*left-unique*) for every y in C there is *at most one* x in D such that $x\,R\,y$.

For a relation $R \subseteq D \times C$ the conjunction of (*right-total*) and (*left-unique*) is equivalent to asserting that for every y in C there is *exactly one* x in D such that $x\,R\,y$. So, after a little bit of deducing, we obtained (1): the *onto* and *one-one* properties are the two components of functionality, but going from the codomain to the domain; functionality in the direction from the domain to the codomain is made of two completely analogous, dual, components.

What is still not quite evident is (2); namely, that the *onto* and *one-one* properties are also dual to each other. That "at least one" is dual to "at most one" may be gathered from the fact we can express that a set A is a singleton by the conjunction of "for some x_1 and x_2 in A, $x_1 = x_2$" (which amounts to "there is at least one member of A") and "for every x_1 and x_2 in A, $x_1 = x_2$" (which amounts to "there is at most one member of A"). When we deal specifically with functions, the duality between the *onto* and *one-one* properties is exhibited in category theory by showing that the first property amounts to cancellability on the right in functional composition, while the second property amounts to cancellability on the left. However, as we shall see in 2.3, if we assume functionality neither for $R \subseteq D \times C$ nor for the converse set of ordered pairs, we cannot exhibit in this manner the duality between (*right-total*) and (*left-unique*), or between (*left-total*) and (*right-unique*).

2.2. The square of functions. The definitions below will enable us to see the duality mentioned at the end of the preceding section in a different, more basic, manner—without extra assumptions concerning $R \subseteq D \times C$. They will also display clearly the connection between the *onto* and *one-one* properties and functionality.

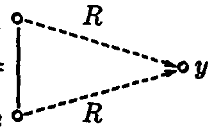
Let $R^{\leftarrow} \subseteq C \times D$ be the relation converse to $R \subseteq D \times C$, i.e., $R^{\leftarrow}$ is $\{(y,x) \mid x\,R\,y\}$, and let $R_2 \circ R_1$ be $\{(x,y) \mid \text{for some } z,\ x\,R_1\,z \text{ and } z\,R_2\,y\}$. (For the composition of R_1 with R_2 we write $R_2 \circ R_1$, rather than $R_1 \circ R_2$, so as not to deviate from standard usage when we come to functional composition. This standard usage is unfortunate—it clashes with our inclination to read other things from left to right—but it is hard to fight against. Anyway, what we have to say about functions does not depend upon reforming the notation for functional composition.) Next, for every set A, let 1_A be $\{(x,x) \mid x \in A\}$.

Then consider the following properties a relation $R \subseteq D \times C$ might have:

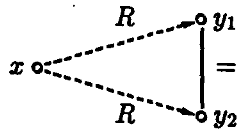
(*left-total*)	$1_D \subseteq R^{\leftarrow} \circ R$	(*right-total*)	$R \circ R^{\leftarrow} \supseteq 1_C$
(*left-unique*)	$1_D \supseteq R^{\leftarrow} \circ R$	(*right-unique*)	$R \circ R^{\leftarrow} \subseteq 1_C$

We pass from left to right in this square by replacing R by its converse $R^{\leftarrow}$ (of course, $R^{\leftarrow\leftarrow}$ is equal to R). We pass from the upper row to the lower row by replacing an inclusion by the converse inclusion.

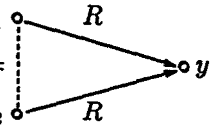
The diagrams in the figure below illustrate the four properties in the square. Solid lines are in the antecedents and dotted lines in the consequents. For example, the upper left diagram is read as follows: "If x_1 is equal to x_2, then we have arrows going from them to the right towards a point y." Since every point is equal to itself, this means that for every point in the domain we have at least one arrow going towards the codomain whose source is this point—the two dotted R arrows become one. The lower right diagram is read: "If we have arrows with the same source x, then their targets y_1 and y_2 are equal." So at most one arrow can start from a point of the domain—the two solid R arrows become one. It is easy to check with the help of these diagrams that the properties in the square are equivalent to the previously introduced properties that bear the same names.



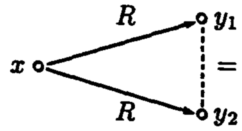
(*left-total*)



(*right-total*)



(*left-unique*)



(*right-unique*)

Functions are defined by the properties in the upper left and lower right corners of our square. With *onto* functions we cover the upper row and the lower right corner, and with *one-one* functions the lower row and the upper left corner. A relation $R \subseteq D \times C$ satisfies the properties in the upper right and lower left corner iff the converse relation $R^{\leftarrow} \subseteq C \times D$ is a function. Our square displays the duality between the *onto* and *one-one* properties, as well as the way how these properties are connected with functionality.

Each corner of the square is "one quarter" of a bijection, i.e., *one-to-one* correspondence. The notion of function involves half of these corners in a diagonal way. An explanation for this judicious choice is given in 2.4 below, when we talk of adjunction.

2.3. Cancellability of relations. Let us now consider how the properties from the square are connected with cancellation properties for relations in relational composition. A relation $R \subseteq D \times C$ may satisfy the property

(*right-cancellable*) for every S_1 and S_2, if $S_1 \circ R \subseteq S_2 \circ R$, then $S_1 \subseteq S_2$,

where $S_1 \subseteq C \times A$ and $S_2 \subseteq C \times A$ for some set A, or the property

(*left-cancellable*) for every S_1 and S_2, if $R \circ S_1 \subseteq R \circ S_2$, then $S_1 \subseteq S_2$,

where $S_1 \subseteq A \times D$ and $S_2 \subseteq A \times D$ for some set A. Note that (*right- cancellable*) and (*left-cancellable*) are equivalent, respectively, to the properties obtained by replacing $\subseteq$ in them by $=$ (to show that, we may use $(S_1 \cup S_2) \circ R = (S_1 \circ R) \cup (S_2 \circ R)$ and $R \circ (S_1 \cup S_2) = (R \circ S_1) \cup (R \circ S_2)$; with $\cup$ replaced by $\cap$ we have the inclusions from left to right of these two distributions, but the converse inclusions may fail).

Since for every relation R we have $R \subseteq R \circ R^{\leftarrow} \circ R$, it is easy to verify that (*right-cancellable*) implies (*right-total*), but for the converse implication we only have that the conjunction of (*right-unique*) and (*right-total*) implies (*right-cancellable*); neither (*right-unique*) alone nor (*right-total*) alone does so. (Let $D = \{d\}$, $C = \{c_1, c_2\}$ and $A = \{a\}$; then for $R = \{(d, c_2)\}$, $S_1 = \{(c_1, a)\}$ and $S_2 = \emptyset$, we have that (*right-unique*) holds, while neither (*right-total*) nor (*right-cancellable*) does, and for $R = \{(d, c_1), (d, c_2)\}$, $S_1 = \{(c_1, a)\}$ and $S_2 = \{(c_2, a)\}$, we have that (*right-total*) holds, while neither (*right-unique*) nor (*right-cancellable*) does hold.) We also have that the conjunction of (*right-unique*) and (*left-cancellable*) implies (*left-unique*), whereas (*left-cancellable*) alone does not (provided A is allowed to be empty). Of course, we obtain something quite analogous if in all these implications we replace everywhere "*right*" by "*left*" and "*left*" by "*right*".

So if R is a function, then (*right-cancellable*) is equivalent to (*right-total*) and (*left-cancellable*) is equivalent to (*left-unique*), but if R is not a function, these equivalences may fail.

2.4. Function and adjunction. Finally, let us try to justify the choice of properties from the square that enter into the definition of function. For $R \subseteq D \times C$ a relation, A a subset of D and B a subset of C, let $\mathbf{R}(A)$ be the set $\{y \in C \mid \text{for some } x \in A,\ x\,R\,y\}$ and $\mathbf{R}^{\leftarrow}(B)$ the set $\{x \in D \mid \text{for some } y \in B,\ x\,R\,y\}$. If $\mathcal{P}(X)$ is the power set of a set X, then for every relation $R \subseteq D \times C$, we have two functions $\mathbf{R} : \mathcal{P}(D) \to \mathcal{P}(C)$ and $\mathbf{R}^{\leftarrow} : \mathcal{P}(C) \to \mathcal{P}(D)$, monotonic with respect to $\subseteq$. We can easily verify that (*left-total*) is equivalent to

$$(\gamma) \qquad \text{for every } A \subseteq D,\ A \subseteq \mathbf{R}^{\leftarrow}(\mathbf{R}(A)),$$

while (*right-unique*) is equivalent to

$$(\varphi) \qquad \text{for every } B \subseteq C,\ \mathbf{R}(\mathbf{R}^{\leftarrow}(B)) \subseteq B.$$

On the other hand, (γ) is equivalent to the left-to-right implication and (φ) to the right-to-left implication of the equivalence

$$(*) \qquad \text{for every } A \subseteq D \text{ and every } B \subseteq C,\ \mathbf{R}(A) \subseteq B \text{ iff } A \subseteq \mathbf{R}^{\leftarrow}(B).$$

So, $\mathbf{R}$ and $\mathbf{R}^{\leftarrow}$ establish a covariant Galois connection between $\langle \mathcal{P}(D), \subseteq \rangle$ and $\langle \mathcal{P}(C), \subseteq \rangle$ iff $R \subseteq D \times C$ is a function. In more general terms, for the preorders $\langle \mathcal{P}(D), \subseteq \rangle$ and $\langle \mathcal{P}(C), \subseteq \rangle$ understood as categories (objects are subsets of D and C, and arrows exist between these objects whenever inclusion obtains), the functors $\mathbf{R}$ and $\mathbf{R}^{\leftarrow}$ together with the natural transformations induced by (γ) and (φ) make an adjunction, where $\mathbf{R}$ is left-adjoint and $\mathbf{R}^{\leftarrow}$ right-adjoint, the natural transformations of (γ) and (φ) being, respectively, the unit and counit of the adjunction. We have this adjunction if and only if $R \subseteq D \times C$ is a function. (The "if" part of this equivalence is stated in [Mac Lane 1971, p. 94].)

For every relation $R \subseteq D \times C$ we have that

$$(**) \qquad \text{for every } A \subseteq D \text{ and every } B \subseteq C,\ \mathbf{R}(A) \subseteq B \text{ iff } A \subseteq D - \mathbf{R}^{\leftarrow}(C - B).$$

(I am indebted to Aleksandar Lipkovski for having drawn my attention to (**) with his note [1995], where it appears in the equivalent form

$$\text{for every } A \subseteq D \text{ and every } B \subseteq C,\ \mathbf{R}^{\leftarrow}(B) \subseteq A \text{ iff } B \subseteq C - \mathbf{R}(D - A).)$$

We also have that $R \subseteq D \times C$ is a function iff

(***) for every $B \subseteq C$, $\mathbf{R}^{\leftarrow}(B) = D - \mathbf{R}^{\leftarrow}(C - B)$.

So, underlying the Galois connection of (*) there is a Galois connection of wider scope, but less pleasing. (The equivalence (**) is implicitly present in temporal logic through the connection between future necessity and past possibility. The equality (***) is also to be found in modal logic, when the functionality of the accessibility relation of Kripke models makes necessity and possibility coincide; see, for example, [Hughes & Cresswell 1996].)

The equivalence "$R \subseteq D \times C$ is a function iff (*)" may hardly serve as an alternative definition of the notion of function, since this notion is presupposed in the definitions of the mappings, or functors, $\mathbf{R}$ and $\mathbf{R}^{\leftarrow}$. However, the adjunction in this equivalence may help to explain why the notion of function, rather than some other notion (for example, the notion of *partial* function, without left totality, or the notion of *onto* function, with right totality), is so important in mathematics. Conversely, if we are already convinced of the importance of the notion of function—as we should be—our equivalence may explain why Galois correspondence and adjointness are important.

3. Definitions of adjunction

We shall now survey the standard definitions of adjunction. However, rather than simply rehash familiar matters, we present also two presumably new definitions of this notion.

One is a definition that does not economize on primitives. It takes as primitive notions the two adjoint functors, F and G, and both the natural transformations that are the counit and unit of the adjunction and the two bijections between the hom-sets $\mathcal{A}(FB, A)$ and $\mathcal{B}(B, GA)$. Usually, if the counit and unit are primitive, the bijections are defined, and vice versa. Having both kinds of notions primitive, together with the adjoint functors, enables us to formulate the specific equalities between arrows one finds in adjointness as a series of equalities defining one of these notions in terms of two remaining notions. These definitional equalities make a regular pattern, which should clarify standard definitions of adjunction.

We shall compare this uneconomical, but regular and simple, definition to standard definitions of adjunction (like those that may be found in MacLane's book [1971, IV]), and show that the notions defined are equivalent. Among the standard definitions we favour those that, like the uneconomical definition, are equationally presented. We also envisage defining adjunction in a more general kind of context—in particular, a context where F and G may fail to be functors because they don't satisfy (fun1), but only (fun2). That is, F and G are only *semifunctors* (cf. 3.4 below).

In 3.7 we consider the other nonstandard definition of adjunction. This one is, on the contrary, an economical definition, where only the functions F and G

on objects and the bijections between the hom-sets $\mathcal{A}(FB, A)$ and $\mathcal{B}(B, GA)$ are primitive. So neither of the adjoint functors F and G is taken as primitive. This economical definition simplifies one of the standard definitions.

3.1. Primitive notions in adjunction. Let $\mathcal{A}$ and $\mathcal{B}$ be two graphs. The objects of $\mathcal{A}$ will be designated by $A, A_1, A_2, \ldots$, and the arrows of $\mathcal{A}$ by $f, f_1, f_2, \ldots$, while the objects of $\mathcal{B}$ will be designated by $B, B_1, B_2, \ldots$, and the arrows of $\mathcal{B}$ by $g, g_1, g_2, \ldots$

Let F be a graph-morphism from $\mathcal{B}$ to $\mathcal{A}$ and G a graph-morphism from $\mathcal{A}$ to $\mathcal{B}$. When we need it for emphasis, we shall write F^a and G^a for the functions on arrows, and F^o and G^o for the functions on objects, of the graph-morphisms F and G. However, in most cases we will, as usual, omit these superscripts.

Let φ be a transformation from the composite graph-morphism FG to the identity graph-morphism $I_{\mathcal{A}}$ and γ a transformation from the identity graph-morphism $I_{\mathcal{B}}$ to the composite graph-morphism GF. (Remember that, as defined in 1.3, a transformation is a family of arrows like a natural transformation for which we don't assume (nat).)

Finally, for every pair of objects (A, B) (where, according to our convention, A is from $\mathcal{A}$ and B is from $\mathcal{B}$), let $\Phi_{B,A}$ be a function assigning to an arrow $g : B \to GA$ of $\mathcal{B}$ the arrow $\Phi_{B,A}g : FB \to A$ of $\mathcal{A}$, and let $\Gamma_{B,A}$ be a function assigning to an arrow $f : FB \to A$ of $\mathcal{A}$ the arrow $\Gamma_{B,A}f : B \to GA$ of $\mathcal{B}$. We denote by Φ the family of all the functions $\Phi_{B,A}$ and by Γ the family of all the functions $\Gamma_{B,A}$; we call the functions in these families the *seesaw* functions.

Consider now the following six notions we have just introduced:

the functions on arrows F^a and G^a,

the transformations φ and γ,

the families of seesaw functions Φ and Γ.

If $\langle \mathcal{A}, \mathbf{1}, \circ \rangle$ and $\langle \mathcal{B}, \mathbf{1}, \circ \rangle$ are deductive systems, each of these notions can be defined in terms of other two notions from the list (with the help of the identities and compositions of $\langle \mathcal{A}, \mathbf{1}, \circ \rangle$ and $\langle \mathcal{B}, \mathbf{1}, \circ \rangle$) by the following equalities:

for $g : B_1 \to B_2$

(F^a) $\quad Fg = \Phi_{B_1,FB_2}(\gamma_{B_2} \circ g)$,

(φ) $\quad \varphi_A = \Phi_{GA,A}G\mathbf{1}_A$,

for $g : B \to GA$

(Φ) $\quad \Phi_{B,A}g = \varphi_A \circ Fg$,

for $f : A_1 \to A_2$

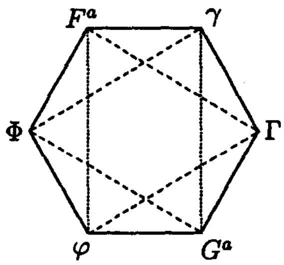
(G^a) $\quad Gf = \Gamma_{GA_1,A_2}(f \circ \varphi A_1)$,

(γ) $\quad \gamma_B = \Gamma_{B,FB}F\mathbf{1}_B$,

for $f : FB \to A$

(Γ) $\quad \Gamma_{B,A}f = Gf \circ \gamma_B$.

The definitional dependences among these notions can be read off from the following hexagonal figure.



The notion in each vertex is definable in terms of the two notions in the neighbouring vertices on the left and on the right. For example, F^a is definable in terms of γ and Φ, while Φ is definable in terms of F^a and φ, etc. On the left-hand side of the hexagon we have F and its Greek correlates, while on the right-hand side we have G with its Greek correlates. Vertices on the big, undrawn, diagonals have labels of the same type: (F^a, G^a), (Φ, Γ) and (φ, γ).

The small, dotted, diagonals are drawn to indicate possible choices of primitives, in terms of which all the six notions can be defined. In the following table we indicate with + the notions taken as primitive by the choice named in the leftmost column.

	F^a	G^a	φ	γ	Γ	Φ
hexagonal	+	+	+	+	+	+
rectangular ||	+	+	+	+		
rectangular \\	+	+			+	+
rectangular //			+	+	+	+
triangular ▷	+		+		+	
triangular ◁		+		+		+

Besides these choices, there are six uneconomical pentagonal choices, with five primitives, and six more uneconomical choices with four primitives, obtained by adding a vertex to one of the triangular choices (so, altogether, we have 18 choices). What can be said about these additional uneconomical choices should be easy to infer from what is said below about the rectangular and triangular choices; so we shall not consider them separately. (In 3.7 below, we shall find one more choice, very economical, with only Φ and Γ primitive; however, this choice is based on slightly different definitional equalities.)

The hexagonal definitional pattern above becomes even more regular if we take into account the identities and compositions of the deductive systems $\mathcal{A}$ and $\mathcal{B}$. For the composition of $\mathcal{A}$, let us introduce the following notation

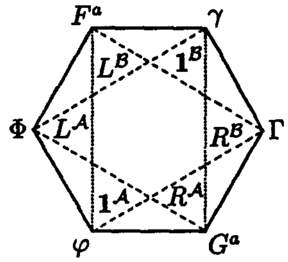
$$L^{\mathcal{A}}_{f_2}(f_1) = R^{\mathcal{A}}_{f_1}(f_2) = f_2 \circ f_1,$$

and analogously for the composition of $\mathcal{B}$. Next, let $\mathbf{1}^{\mathcal{A}}$ and $\mathbf{1}^{\mathcal{B}}$ be the identities of $\mathcal{A}$ and $\mathcal{B}$, respectively.

Then the definitional equalities above become

$$
\begin{array}{llll}
(F^a) & Fg = \Phi L^B_\gamma g, & (G^a) & Gf = \Gamma R^A_\varphi f, \\
(\varphi) & \varphi = \Phi G \mathbf{1}^A, & (\gamma) & \gamma = \Gamma F \mathbf{1}^B, \\
(\Phi) & \Phi g = L^A_\varphi F g, & (\Gamma) & \Gamma f = R^B_\gamma G f,
\end{array}
$$

where, to make matters clearer, we have omitted parentheses and subscripts referring to objects. Our hexagonal figure with these additional notions involved in the definitions looks as follows.



3.2. Hexagonal adjunction. The hexagonal choice of primitives of the preceding section is interesting because we can define adjunction as follows. The conditions

$\langle A, \mathbf{1}, \circ\rangle$ and $\langle B, \mathbf{1}, \circ\rangle$ are categories,
F and G are functors,
φ and γ are natural transformations,
Φ and Γ are families of seesaw functions,
$(F^a), (G^a), (\varphi), (\gamma), (\Phi)$ and (Γ) hold

are satisfied iff the functors F and G are *adjoint*, F being left adjoint and G right adjoint. The natural transformations φ and γ are respectively the *counit* and *unit* of the adjunction (often written ε and η).

In the next sections we shall verify that this notion of adjunction is indeed equivalent to the more usual ones, behind which stand more economical choices of primitives from the table above.

Note that if we replace the equalities (φ) and (γ) by the equalities

$$
(\varphi^\circ) \quad f \circ \varphi_{A_1} = \Phi_{GA_1,A_2} Gf, \qquad\qquad (\gamma^\circ) \quad \gamma_{B_2} \circ g = \Gamma_{B_1,FB_2} Fg,
$$

for $f : A_1 \to A_2$ and $g : B_1 \to B_2$, then the condition that the transformations φ and γ are natural becomes redundant. The equalities (φ°) and (γ°) are an immediate consequence of (Φ), (Γ) and (nat) for φ and γ. On the other hand, these two equalities yield (φ) and (γ) in the presence of (cat1left) and (cat1right). However, (φ°) and (γ°) are not exactly definitions of the transformations φ and γ, but rather definitions of composing with φ on the right and γ on the left (i.e., of R^A_φ and L^B_γ).

3.3. Rectangular || adjunction.

Suppose

$\langle \mathcal{A}, \mathbf{1}, \circ\rangle$ and $\langle \mathcal{B}, \mathbf{1}, \circ\rangle$ are categories,
F^a and G^a satisfy (fun2),
(Φ) and (Γ) hold.

Then the equalities (F^a), (G^a), (φ) and (γ) are interderivable with the equalities

$$(\varphi\gamma F) \quad \varphi_{FB} \circ F\gamma_B = F\mathbf{1}_B, \qquad (\varphi\gamma G) \quad G\varphi_A \circ \gamma_{GA} = G\mathbf{1}_A,$$
$$(\varphi\mathbf{1}) \quad \varphi_A \circ FG\mathbf{1}_A = \varphi_A, \qquad (\gamma\mathbf{1}) \quad GF\mathbf{1}_B \circ \gamma_B = \gamma_B,$$

from which Φ and Γ are absent.

Let us first derive the latter equalities from the former. For $(\varphi\gamma F)$ we have

$$\begin{aligned}\varphi_{FB} \circ F\gamma_B &= \Phi_{B,FB}\gamma_B, \text{ by}(\Phi)\\ &= F\mathbf{1}_B, \text{ by (cat1right) and } (F^a).\end{aligned}$$

For $(\varphi\mathbf{1})$ we have

$$\begin{aligned}\varphi_A \circ FG\mathbf{1}_A &= \Phi_{GA,A}G\mathbf{1}_A, \text{ by } (\Phi)\\ &= \varphi_A, \text{ by } (\varphi).\end{aligned}$$

We proceed analogously for $(\varphi\gamma G)$ and $(\gamma\mathbf{1})$.

Conversely, we derive (F^a) as follows:

$$\begin{aligned}\Phi_{B_1,FB_2}(\gamma_{B_2} \circ g) &= (\varphi_{FB_2} \circ F\gamma_{B_2}) \circ Fg, \text{by } (\Phi), \text{(fun2) and (cat2)}\\ &= F\mathbf{1}_{B_2} \circ Fg, \text{by } (\varphi\gamma F)\\ &= Fg, \text{ by (fun2) and (cat1left).}\end{aligned}$$

For (φ) we use (Φ) and $(\varphi\mathbf{1})$, and we proceed analogously for (G^a) and (γ).

In the standard definition of adjunction with the rectangular || choice of primitives we have that

$\langle \mathcal{A}, \mathbf{1}, \circ\rangle$ and $\langle \mathcal{B}, \mathbf{1}, \circ\rangle$ are categories,
F and G are functors,
φ and γ are natural transformations,
Φ and Γ may be defined by (Φ) and (Γ),
$(\varphi\gamma F)$ and $(\varphi\gamma G)$ hold.

In fact, instead of the equalities $(\varphi\gamma F)$ and $(\varphi\gamma G)$ we usually have the equalities obtained from them by replacing the right-hand sides with $\mathbf{1}_{FB}$ and $\mathbf{1}_{GA}$, respectively. These other equalities clearly amount to $(\varphi\gamma F)$ and $(\varphi\gamma G)$ in the presence of (fun1) for F^a and G^a.

With this standard definition of adjunction, the equalities ($\varphi 1$) and ($\gamma 1$) follow either from (fun1) for F^a and G^a, or from the assumption that φ and γ are natural transformations (together with (cat1right) and (cat1left)). This is enough to conclude that the notion of adjunction of the preceding section is indeed equivalent to the standard notion with the rectangular || choice of primitives.

3.4. Rectangular \\ adjunction. Suppose

$\langle \mathcal{A}, \mathbf{1}, \circ \rangle$ and $\langle \mathcal{B}, \mathbf{1}, \circ \rangle$ are categories,
F^a and G^a satisfy (fun2),
(φ) and (γ) hold.

Then the equalities (F^a), (G^a), (Φ), (Γ) and (nat) for φ and γ are interderivable with the following equalities (in which, since we have (cat2), we don't write parentheses in compositions, and the subscripts of Φ and Γ are omitted so as not to encumber notation excessively; these subscripts can be recovered from the context):

($\Phi\Gamma$) $\Phi(Gf_3 \circ \Gamma f_2 \circ g_1) = f_3 \circ f_2 \circ Fg_1,$ ($\Gamma\Phi$) $\Gamma(f_3 \circ \Phi g_2 \circ Fg_1) = Gf_3 \circ g_2 \circ g_1,$
($\Phi\Phi$) $\Phi(Gf_3 \circ g_2 \circ g_1) = f_3 \circ \Phi g_2 \circ Fg_1,$ ($\Gamma\Gamma$) $\Gamma(f_3 \circ f_2 \circ Fg_1) = Gf_3 \circ \Gamma f_2 \circ g_1,$
(ΦF) $\Phi g \circ F\mathbf{1}_B = \Phi g,$ (ΓG) $G\mathbf{1}_A \circ \Gamma f = \Gamma f.$

In these equalities φ and γ don't occur.

Equalities like these were considered in [Hayashi 1985] and [Hoofman 1993], which deal with notions of adjoint semifunctors, i.e., graph-morphisms satisfying only (fun2), and not necessarily also (fun1). (Note that at the beginning of the preceding section we also didn't assume (fun1) to find equalities without Φ and Γ equivalent to (F^a), (G^a), (φ) and (γ).)

In the standard definition of adjunction with the rectangular \\ choice of primitives we have that

$\langle \mathcal{A}, \mathbf{1}, \circ \rangle$ and $\langle \mathcal{B}, \mathbf{1}, \circ \rangle$ are categories,
F and G are functors,
Φ and Γ are families of seesaw functions,
φ and γ may be defined by (φ) and (γ),
the following equalities hold:

($\Phi\Gamma'$) $\Phi\Gamma f = f,$ ($\Gamma\Phi'$) $\Gamma\Phi g = g,$
($\Phi\Phi'$) $\Phi(g_2 \circ g_1) = \Phi g_2 \circ Fg_1,$
($\Phi\Phi''$) $\Phi(Gf \circ g) = f \circ \Phi g.$

The equalities ($\Phi\Phi'$) and ($\Phi\Phi''$) can be replaced by

($\Gamma\Gamma'$) $\Gamma(f_2 \circ f_1) = Gf_2 \circ \Gamma f_1,$
($\Gamma\Gamma''$) $\Gamma(f \circ Fg) = \Gamma f \circ g.$

It is easy to see that, due to the presence of (fun1) for F^a and G^a, the equalities $(\Phi\Gamma')$, $(\Gamma\Phi')$, $(\Phi\Phi')$ and $(\Phi\Phi'')$ amount to $(\Phi\Phi)$, $(\Gamma\Gamma)$, $(\Phi\Gamma)$, $(\Gamma\Phi)$, (ΦF) and (ΓG).

In this standard definition of rectangular \\ adjunction, $(\Phi\Phi')$ can be replaced by

$$\Phi g = \Phi 1_{GA} \circ Fg,$$

an equality that in the presence of (φ) and (fun1) for G^a amounts to (Φ). Analogously, $(\Gamma\Gamma')$ can be replaced by an equality that in the presence of (γ) and (fun1) for F^a amounts to (Γ):

$$\Gamma f = Gf \circ \Gamma 1_{FA}.$$

The equalities $(\Phi\Phi')$ and $(\Phi\Phi'')$ can be replaced by the implication

$$\text{if } g_2 \circ g_1 = Gf \circ g, \text{ then } \Phi g_2 \circ Fg_1 = f \circ \Phi g$$

(to show that we use (cat1right), (cat1left) and (fun1) for F^a and G^a). Analogously, $(\Gamma\Gamma')$ and $(\Gamma\Gamma'')$ can be replaced by the implication

$$\text{if } f_2 \circ f_1 = f \circ Fg, \text{ then } Gf_2 \circ \Gamma f_1 = \Gamma f \circ g.$$

With these implications, which are involved in Lawvere's definition of adjunction as an isomorphism of comma categories (see [Mac Lane 1971, p. 84, Exercise 2, and p. 53]), we abandon, however, the equational style of defining adjunction favoured here.

3.5. Rectangular // adjunction. If $\mathcal{A}$ and $\mathcal{B}$ are deductive systems that satisfy (cat1right) and (cat1left), and (F^a) and (G^a) hold, then it is clear that the equalities (φ), (γ), (Φ) and (Γ) are interderivable with the equalities

$$\varphi_A = \Phi_{GA,A}\Gamma_{GA,A}\varphi_A, \qquad \gamma_B = \Gamma_{B,FB}\Phi_{B,FB}\gamma_B,$$
$$\Phi_{B,A}g = \varphi_A \circ \Phi_{B,FGA}(\gamma_{GA} \circ g), \qquad \Gamma_{B,A}f = \Gamma_{GFB,A}(f \circ \varphi_{FB}) \circ \gamma_B,$$

from which the functions F^a and G^a are absent. (The equalities in the first line are instances of $(\Phi\Gamma')$ and $(\Gamma\Phi')$, respectively.) However, there doesn't seem to be a standard definition of adjunction with the rectangular // choice of primitives, which would be based on equalities such as these. Standard definitions take the adjoint functors F and G, or at least one of them, as primitive. In 3.7, we shall consider a definition of adjunction where neither of the functions F^a and G^a is primitive.

3.6. Triangular adjunction. Suppose

$\langle\mathcal{A}, \mathbf{1}, \circ\rangle$ and $\langle\mathcal{B}, \mathbf{1}, \circ\rangle$ are categories,
F^a satisfies (fun2),
φ satisfies (nat),
(G^a), (γ) and (Φ) hold.

Then the equalities (F^a), (φ), (Γ), (fun2) for G^a and (nat) for γ are interderivable with the equalities

$$(\beta) \qquad \varphi_A \circ F\Gamma_{B,A} f = f \circ F\mathbf{1}_B,$$
$$(\Gamma\Gamma'') \qquad \Gamma_{B_1,A}(f \circ Fg) = \Gamma_{B_2,A} f \circ g,$$

from which G^a, γ and Φ are absent. Note that again we have not assumed (fun1) for F^a (nor for G^a).

The equality (β) could be replaced above by

$$\varphi_A \circ F(\Gamma_{B,A} f \circ g) = f \circ Fg,$$

while in the presence of the assumptions that $\mathcal{A}$ is a category, that F^a satisfies (fun2) and that (G^a) and (β) hold, the equality (nat) for φ is replaceable by

$$\varphi_A \circ F\mathbf{1}_{GA} = \varphi_A.$$

This last equality follows, of course, from (cat1right) and (fun1) for F^a.

In the standard definition of adjunction with the triangular choice of primitives we have that

$\langle \mathcal{A}, \mathbf{1}, \circ \rangle$ and $\langle \mathcal{B}, \mathbf{1}, \circ \rangle$ are categories,
F is a functor and G^o is a function on objects,
φ is an objectual transformation,
Γ is a family of seesaw functions,
G^a, γ and Φ may be defined by (G^a), (γ) and (Φ),
the following equalities hold:

$$(\beta') \qquad \varphi_A \circ F\Gamma_{B,A} f = f,$$
$$(\eta) \qquad \Gamma_{B,A}(\varphi_A \circ Fg) = g.$$

Remember that an "objectual transformation", as specified in 1.3, is like a transformation between functions on objects, instead of graph-morphisms. We didn't assume that G^o belongs to a graph-morphism; so, to be precise, we can say only that φ is an objectual transformation from the composite function FG on the objects of $\mathcal{A}$ to the identity function on the objects of $\mathcal{A}$.

Note that in the presence of (Φ), the equalities (β') and (η) can be written as ($\Phi\Gamma'$) and ($\Gamma\Phi'$). (The names "β" and "η" come from the adjunction of cartesian closed categories, where the corresponding equalities are related to β and η conversion in the typed lambda calculus.)

It is clear that with (cat1right) and (fun1) for F^a the equality (β) amounts to (β'). On the other hand, ($\Gamma\Gamma''$), (cat1left) and

$$(\Gamma\varphi) \qquad \Gamma_{GA,A}\varphi_A = \mathbf{1}_{GA}$$

yield (η), while, conversely, from (β'), (η), (cat2) and (fun2) for F^a we obtain $(\Gamma\Gamma'')$, and from (η), (cat1right) and (fun1) for F^a we obtain $(\Gamma\varphi)$. The equality (β') implies in the presence of (G^a) that φ satisfies (nat).

The equality (η) is replaceable by the implication

$$\text{if } \varphi_A \circ Fg = f, \text{ then } g = \Gamma_{B,A}f,$$

which together with (β') is tantamount to asserting that there is a unique g such that $\varphi_A \circ Fg = f$. The definition of adjunction via a solution to a universal arrow problem is based on that (see [Mac Lane 1971, IV.1, p. 81, Theorem 2(iv)]).

Since (β') is replaceable by the converse implication, and since we have (Φ), we could assume instead of (β') and (η) the equivalence

$$g = \Gamma_{B,A}f \quad \text{iff} \quad \Phi_{B,A}g = f,$$

which is another way of assuming $(\Phi\Gamma')$ and $(\Gamma\Phi')$. However, with these implications and this equivalence we abandon the equational style of defining adjunction favoured here.

For the definition of adjunction with the triangular $\triangleleft$ choice of primitives we would have completely analogous considerations.

3.7. Seesaw adjunction. The rectangular \\ and rectangular // choices of primitives are not minimal for defining adjunction if we change slightly the defining equalities (F^a), (G^a), (φ) and (γ). The transformations φ and γ may be defined as follows in terms of Φ and Γ without F^a and G^a:

$$(\varphi') \quad \varphi_A = \Phi_{GA,A}\mathbf{1}_{GA}, \qquad (\gamma') \quad \gamma_B = \Gamma_{B,FB}\mathbf{1}_{FB},$$

which serves to transform (F^a) and (G^a) into the following definitions of F^a and G^a in terms of Φ and Γ without φ and γ:

$$(F^{a\prime}) \quad Fg = \Phi_{B_1,FB_2}(\Gamma_{B_2,FB_2}\mathbf{1}_{FB_2} \circ g),$$
$$(G^{a\prime}) \quad Gf = \Gamma_{GA_1,A_2}(f \circ \Phi_{GA_1,A_1}\mathbf{1}_{GA_1}).$$

We then have a definition of adjunction where

$\langle \mathcal{A}, \mathbf{1}, \circ \rangle$ and $\langle \mathcal{B}, \mathbf{1}, \circ \rangle$ are categories,
F^o and G^o are functions on objects,
Φ and Γ are families of seesaw functions,
F^a, G^a, φ and γ may be defined by $(F^{a\prime})$, $(G^{a\prime})$, (φ') and (γ'),
the following equalities hold:

$$(\Phi\Gamma') \quad \Phi\Gamma f = f, \qquad (\Gamma\Phi') \quad \Gamma\Phi g = g,$$
$$(\Phi\Phi''') \quad \Phi(g_2 \circ g_1) = \Phi g_2 \circ \Phi(\Gamma\mathbf{1} \circ g_1)$$

(with the subscripts of Φ, Γ and $\mathbf{1}$ omitted).

We could replace $(\Phi\Phi''')$ by

$$(\Gamma\Gamma''') \qquad \Gamma(f_2 \circ f_1) = \Gamma(f_2 \circ \Phi\mathbf{1}) \circ \Gamma f_1.$$

To verify that this notion of adjunction is equivalent to the usual ones it suffices to show that it is equivalent to the notion with the rectangular \\ choice of primitives of 3.4. For that we have first to check that F^a and G^a defined by $(F^{a\prime})$ and $(G^{a\prime})$ satisfy (fun1) and (fun2). Next, the equalities $(\Phi\Phi''')$ and $(\Gamma\Gamma''')$ amount to the equalities $(\Phi\Phi')$ and $(\Gamma\Gamma')$ of 3.4 in the presence of $(F^{a\prime})$ and $(G^{a\prime})$, while equalities corresponding to $(\Phi\Phi'')$ and $(\Gamma\Gamma'')$ are now derivable. Here is a derivation of $(\Phi\Phi'')$:

$$\begin{aligned} \Phi(Gf \circ g) &= \Phi\Gamma(f \circ \Phi\mathbf{1}) \circ \Phi(\Gamma\mathbf{1} \circ g), \quad \text{by } (G^{a\prime}) \text{ and } (\Phi\Phi''') \\ &= f \circ \Phi g, \quad \text{by } (\Phi\Gamma'),\ (\Phi\Phi'''),\ (\text{cat1left}) \text{ and } (\text{cat2}) \end{aligned}$$

(cf. [D. 1996, section 3.1]).

This economical definition of adjunction is at the opposite end of the hexagonal definition of 3.2, in which we did not economize on primitives.

To prove strictly the equivalences of various notions of adjunction considered here, we would have to introduce the appropriate morphisms between adjunctions and demonstrate equivalences of categories, which would actually be isomorphisms of categories. We shall not do that, however, since this rather straightforward matter would take too much space. We define morphisms between adjunctions in 5.1 below.

4. Definitions of comonad

We shall now survey definitions of comonad. Besides the standard definition of this notion, we shall present several alternative definitions, of equivalent notions.

The principle guiding this survey will be the adjunction between the category of our comonad and a subcategory of it, equivalent to the Kleisli category, which we will call *the delta category*. This adjunction defines the comonad, and since adjunction can be formulated in various ways, as we saw in the preceding part, we may envisage various definitions of comonad. After extracting as many interesting definitions as we could find, we compare the delta category of a comonad to its Kleisli and Eilenberg-Moore categories. These last categories play an essential role in the adjunctions involving the category of adjunctions and the category of comonads, which we shall consider in 5.3.

Of course, we could as well deal throughout with monads. Our only reason for preferring comonads is that, from a logical point of view, they seem to bear a certain primacy over monads, as the universal quantifier bears a primacy over the existential quantifier. On the other hand, from an algebraic point of view, monads bear a primacy over comonads (see [Mac Lane 1971, VI] and [Manes 1976]).

4.1. Standard definition of comonad. Suppose we are given the following:

a deductive system $\langle \mathcal{A}, \mathbf{1}, \circ \rangle$,

a graph-morphism D from $\mathcal{A}$ to $\mathcal{A}$,

a transformation ε from D to the identity graph-morphism $I_{\mathcal{A}}$,

a transformation δ from D to the composite graph-morphism DD.

So in ε we have the arrows $\varepsilon_A : DA \to A$, and in δ the arrows $\delta_A : DA \to DDA$. Then we say that $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ is a *comonograph*. We may say that this is a comonograph *in* $\mathcal{A}$, and we use sometimes the same form of speaking with comonads, later. To simplify the notation, we don't mention the identity and composition of $\langle \mathcal{A}, \mathbf{1}, \circ \rangle$, taking them for granted.

A *monograph* would be a comonograph with arrows reversed—sources become targets and targets sources. Note that the function D on objects in a comonograph resembles a topological interior operation, while in a monograph it would resemble a closure operation.

The appropriate morphisms between comonographs will be called *comonofunctors*. A comonofunctor from a comonograph $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ to a comonograph $\langle \mathcal{A}', D', \varepsilon', \delta' \rangle$ is a functor N from the deductive system $\mathcal{A}$ to the deductive system $\mathcal{A}'$ such that the following naturalness equalities hold:

$$ND = D'N,$$
$$N\varepsilon_A = \varepsilon'_{NA},$$
$$N\delta_A = \delta'_{NA}.$$

A *comonad* is a comonograph $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ such that

$\langle \mathcal{A}, \mathbf{1}, \circ \rangle$ is a category,

D is a functor,

ε and δ are natural transformations,

the following equalities hold:

$(\varepsilon\delta)$ $\quad \varepsilon_{DA} \circ \delta_A = \mathbf{1}_{DA}$,

$(\varepsilon\delta D)$ $\quad D\varepsilon_A \circ \delta_A = \mathbf{1}_{DA}$,

$(\delta\delta)$ $\quad D\delta_A \circ \delta_A = \delta_{DA} \circ \delta_A$.

A *monad* (also called a *triple*) is a comonad with arrows reversed.

4.2. The delta category. Let $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ be a comonad, and for an arrow $f : DA \to A'$ of $\mathcal{A}$ let the arrow $\Delta f : DA \to DA'$ be defined by

$$\Delta f \overset{\text{def}}{=} Df \circ \delta_A.$$

The operation Δ should be taken as indexed by A, and the same index is inherited by $\circledast$ in 4.5, but we take these indices for granted and omit them.

Then consider the subgraph $\mathcal{A}_\Delta$ of $\mathcal{A}$ whose objects are the objects of $\mathcal{A}$ of the form DA and whose arrows are the arrows of $\mathcal{A}$ of the form Δf. In $\mathcal{A}_\Delta$, there is an identity made of the arrows $\mathbf{1}_{DA}$ of $\mathcal{A}$ and the composition of $\Delta f_1 : DA_1 \rightarrow DA_2$ and $\Delta f_2 : DA_2 \rightarrow DA_3$ is defined as the arrow $\Delta f_2 \circ \Delta f_1$ of $\mathcal{A}$. To ensure that $\mathbf{1}_{DA}$ and $\Delta f_2 \circ \Delta f_1$ are indeed arrows of $\mathcal{A}_\Delta$ we check that the following equalities hold in $\mathcal{A}$:

$$(\Delta\varepsilon) \qquad \Delta\varepsilon_A = \mathbf{1}_{DA},$$
$$(\Delta\circ) \qquad \Delta(f_2 \circ \Delta f_1) = \Delta f_2 \circ \Delta f_1.$$

It is clear that $\mathcal{A}_\Delta$ is a category with this identity and this composition; namely, it is a subcategory of $\mathcal{A}$. We call $\mathcal{A}_\Delta$ the ***delta category*** of the comonad $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$.

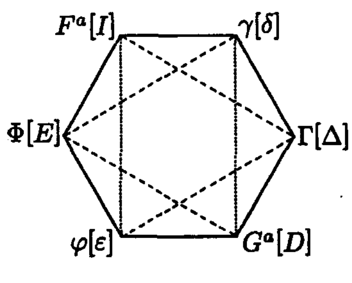
Between $\mathcal{A}$ and $\mathcal{A}_\Delta$ there is an adjunction, where the left-adjoint functor F from $\mathcal{A}_\Delta$ to $\mathcal{A}$ is inclusion I and the right-adjoint functor G from $\mathcal{A}$ to $\mathcal{A}_\Delta$ is D. To show that Df is of the form $\Delta f'$ we check that in $\mathcal{A}$ for every $f : A \rightarrow A'$ we have

$$Df = \Delta(f \circ \varepsilon_A).$$

The counit φ of this adjunction is just ε, where φ_A is ε_A, and the unit γ is δ, but with γ_{DA} being δ_A. That this adjunction obtains indeed will be shown in the next three sections.

Later, in 4.6 and 4.7, we shall compare the delta category to the Kleisli category and to the category of free coalgebras of a comonad. Before that, in the next three sections, we find the delta category handy to survey various possibilities of defining a comonad.

4.3. Primitive notions in comonad. Let us now consider how one could express the adjunction between $\mathcal{A}$ and $\mathcal{A}_\Delta$ in various ways according to the definitions of adjunction in 3. First, the primitive notions we might have to express this adjunction are displayed in square brackets in our hexagonal figure.



Besides the notions we have already encountered, we find in square brackets the seesaw functions E, corresponding to Φ, which will be defined below. The six definitional equalities of 3.1 connecting these notions would now read:

$$\begin{array}{llll}
\text{for } \Delta f : DA' \to DA & & \text{for } f : A \to A' & \\
(F_I^a) & \Delta f = E_{DA}(\delta_A \circ \Delta f), & (G_D^a) & Df = \Delta(f \circ \varepsilon_A), \\
(\varphi_\varepsilon) & \varepsilon_A = E_A D\mathbf{1}_A, & (\gamma_\delta) & \delta_A = \Delta \mathbf{1}_{DA}, \\
\text{for } \Delta f : DA' \to DA & & \text{for } f : DA' \to A & \\
(\Phi_E) & E_A \Delta f = \varepsilon_A \circ \Delta f, & (\Gamma_\Delta) & \Delta f = Df \circ \delta_{A'}.
\end{array}$$

The subscripts in Δ are unimportant now, because FA is A, but the second subscript of E, understood as Φ, matters, and this is the one we note above.

We must first settle what E stands for. The equality (Φ_E) would permit us to get rid of E in (F_I^a) and (φ_ε) if $\delta_A \circ \Delta f$ and $D\mathbf{1}_A$ were equal to arrows of the form $\Delta f'$. Now, for $D\mathbf{1}_A$ this follows immediately from (G_D^a), while for $\delta_A \circ \Delta f$ we have

$$\begin{aligned}
\delta_A \circ \Delta f &= (\delta_A \circ Df) \circ \delta_{A'}, \quad \text{by } (\Gamma_\Delta) \text{ and (cat2)} \\
&= DDf \circ (\delta_{DA'} \circ \delta_{A'}), \quad \text{by (nat) for } \delta \text{ and (cat2)} \\
&= (DDf \circ D\delta_{A'}) \circ \delta_{A'}, \quad \text{by } (\delta\delta) \text{ and (cat2)} \\
&= \Delta\Delta f, \quad \text{by (fun2) and } (\Gamma_\Delta).
\end{aligned}$$

So we may take that E is defined by (Φ_E).

The possible choices of primitives for our adjunction would now be the following, taking into account that F is now inclusion and doesn't figure anywhere:

hexagonal:	$\langle D, \varepsilon, \delta, E, \Delta \rangle$
rectangular | |:	$\langle D, \varepsilon, \delta \rangle$
rectangular \\\\:	$\langle D, E, \Delta \rangle$
rectangular //:	$\langle \varepsilon, \delta, E, \Delta \rangle$
triangular $\rhd$:	$\langle \varepsilon, \Delta \rangle$
triangular $\lhd$:	$\langle D, \delta, E \rangle$

The rectangular | | choice is the choice of the standard definition. The rectangular \\ choice boils down to $\langle \varepsilon, \Delta \rangle$, since ε can be defined in terms of D and E, while D can be defined in terms of ε and Δ, and E can be defined in terms of ε alone. The rectangular // choice boils down to $\langle \varepsilon, \Delta \rangle$, too, since δ can be defined in terms of Δ alone, and E can be defined in terms of ε alone. Finally, the triangular $\lhd$ choice boils down to $\langle D, \varepsilon, \delta \rangle$, since ε can be defined in terms of D and E, while E can be defined in terms of ε alone.

We should mention also the seesaw choice $\langle E, \Delta \rangle$. This boils down to $\langle \varepsilon, \Delta \rangle$, since ε_A can be defined as $E_A \mathbf{1}_{DA}$, and E is definable in terms of ε alone.

The hexagonal choice is of course full of redundances, but we shall nevertheless consider this choice in the next section. Besides that, we are left with only two interesting choices: the standard choice $\langle D, \varepsilon, \delta \rangle$ and $\langle \varepsilon, \Delta \rangle$.

4.4. Hexagonal comonads. With the hexagonal choice of primitives, we assume for a comonad $\langle \mathcal{A}, D, \varepsilon, \delta, E, \Delta \rangle$ that

$\langle \mathcal{A}, \mathbf{1}, \circ \rangle$ is a category,

D is a functor,

ε and δ are natural transformations,

the equalities (F_I^a), (G_D^a), (φ_ε), (γ_δ), (Φ_E) and (Γ_Δ) hold,

and, moreover, the equality $(\delta\delta)$ holds.

The equality $(\delta\delta)$ is assumed not because of the adjunction, but in order to insure that $\mathcal{A}_\Delta$ is closed under composition. It is also used in order to guarantee that E can be defined by (Φ_E) in (F_I^a), as we have shown above.

Let us show now that this hexagonal notion of comonad is equivalent to the standard $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ notion. With (Φ_E), the equality (F_I^a) reads

$$\Delta f = \varepsilon_{DA} \circ (\delta_A \circ \Delta f).$$

This equality clearly follows from $(\varepsilon\delta)$, (cat1left) and (cat2). Conversely, $(\varepsilon\delta)$ follows from this equality as follows. Since from (G_D^a) with (fun1) and (cat1left) we have $\Delta \varepsilon_A = \mathbf{1}_{DA}$ (i.e., the equality $(\Delta\varepsilon)$ mentioned above), our equality with (cat1right) will give $(\varepsilon\delta)$. Therefore, (F_I^a) amounts to $(\varepsilon\delta)$.

With (Γ_Δ), the equality (G_D^a) reads

$$Df = D(f \circ \varepsilon_A) \circ \delta_A.$$

This equality follows from $(\varepsilon\delta D)$, (fun2), (cat2) and (cat1right). Conversely, $(\varepsilon\delta D)$ immediately follows from this equality with (cat1left) and (fun1). Therefore, (G_D^a) amounts to $(\varepsilon\delta D)$. The equalities (F_I^a) and (G_D^a) are more important than the remaining four equalities (φ_ε), (γ_δ), (Φ_E) and (Γ_Δ), which boil down to definitions.

So, our hexagonal notion of comonad is equivalent to the standard $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ notion. To prove quite strictly the equivalence of these two notions, we would have to demonstrate an equivalence of categories, which would actually be an isomorphism of categories.

Note that in the hexagonal definition a comonad is defined by assuming that $\mathcal{A}$ and $\mathcal{A}_\Delta$ are categories and that the functors I and D are adjoints, I being left-adjoint and D right-adjoint. An adjunction between $\mathcal{A}$ and $\mathcal{B}$ where the left adjoint F is the inclusion functor from $\mathcal{B}$ into $\mathcal{A}$ is called a *coreflection* of $\mathcal{A}$ in its subcategory $\mathcal{B}$. So a comonad in $\mathcal{A}$ is defined by assuming that there is a coreflection of a category $\mathcal{A}$ in its subcategory $\mathcal{A}_\Delta$.

The standard $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ notion of comonad of 4.1 corresponds to the rectangular $||$ notion of adjunction of 3.3. The equality $(\varepsilon\delta)$ corresponds to $(\varphi\gamma F)$ and $(\varepsilon\delta D)$ to $(\varphi\gamma G)$, while $(\delta\delta)$ is related to (nat) for γ.

4.5. Triangular comonads. With the $\langle \varepsilon, \Delta \rangle$ choice of primitives, we can imitate the definition of triangular adjunction of 3.6 to define comonads. We define a *triangular* comonad $\langle \mathcal{A}, \varepsilon, \Delta \rangle$ by assuming that

$\mathcal{A}$ is a category,

D is a function from the objects of $\mathcal{A}$ to the objects of $\mathcal{A}$,

ε is an objectual transformation from D to the identity function on the objects of $\mathcal{A}$,

Δ is a function mapping the arrows $f : DA \to A'$ of $\mathcal{A}$ to the arrows $\Delta f : DA \to DA'$ of $\mathcal{A}$,

the following equalities hold:

$$(\varepsilon\Delta) \qquad \varepsilon_A \circ \Delta f = f, \text{ i.e., } E_A \Delta f = f,$$
$$(\Delta\circ) \qquad \Delta(f_2 \circ \Delta f_1) = \Delta f_2 \circ \Delta f_1,$$
$$(\Delta\varepsilon) \qquad \Delta\varepsilon_A = \mathbf{1}_{DA}.$$

These three equalities correspond to the equalities that were mentioned in 3.6 as a possible choice for defining triangular adjunction: $(\varepsilon\Delta)$ corresponds to (β'), while $(\Delta\circ)$ corresponds to $(\Gamma\Gamma'')$ and $(\Delta\varepsilon)$ to $(\Gamma\varphi)$. The new notion of comonad is equivalent to the standard $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ notion, via the definitions (G^a_D), (γ_δ) and (Γ_Δ). (A definition of monad analogous to this triangular notion of comonad may be found in [Manes 1976, 1.3, Exercise 12, p. 32].)

The triangular notion of comonad becomes more transparent if for $f_1 : DA_1 \to A_2$ and $f_2 : DA_2 \to A_3$ we introduce the definition given by the equality

$$(\circledcirc) \qquad f_2 \circledcirc f_1 = f_2 \circ \Delta f_1.$$

We call $\circledcirc$ *delta composition*. With delta composition, $(\Delta\circ)$ reads

$$(\Delta\circledcirc) \qquad \Delta(f_2 \circledcirc f_1) = \Delta f_2 \circ \Delta f_1.$$

Conversely, we may define Δ in terms of delta composition by the equality

$$(\Delta) \qquad \Delta f = \mathbf{1}_{DA} \circledcirc f.$$

With delta composition primitive, a comonad could be defined as being $\langle \mathcal{A}, \varepsilon, \circledcirc \rangle$, where $\mathcal{A}$, D and ε are as for the triangular $\langle \mathcal{A}, \varepsilon, \Delta \rangle$ notion above, $\circledcirc$ is a function that assigns to a pair $(f_1 : DA_1 \to A_2, f_2 : DA_2 \to A_3)$ of arrows of $\mathcal{A}$ the arrow $f_2 \circledcirc f_1 : DA_1 \to A_3$ of $\mathcal{A}$, and the following equalities hold:

$$(\text{cat1right}\circledcirc) \qquad f \circledcirc \varepsilon_A = f,$$
$$(\text{cat1left}\circledcirc) \qquad \varepsilon_A \circledcirc f = f,$$
$$(\text{cat2}\circledcirc) \qquad (f_3 \circledcirc f_2) \circledcirc f_1 = f_3 \circledcirc (f_2 \circledcirc f_1),$$
$$(\text{shift}) \qquad (f_3 \circ f_2) \circledcirc f_1 = f_3 \circ (f_2 \circledcirc f_1).$$

The first three equalities are clearly analogous to the corresponding categorial equalities, ε behaving as identity. The fourth equality can be replaced by either of the following two equalities:

$$(\text{shift1}) \qquad f_3 \circ (\mathbf{1}_{DA} \circledcirc f_1) = f_3 \circledcirc f_1,$$
$$(\text{shift}\varepsilon) \qquad (f_3 \circ \varepsilon_A) \circledcirc f_1 = f_3 \circledcirc f_1.$$

(With (shiftε), the equality (cat1left$\circledast$) becomes superfluous.) The $\langle \mathcal{A}, \varepsilon, \circledast \rangle$ notion of comonad and the triangular $\langle \mathcal{A}, \varepsilon, \Delta \rangle$ notion are equivalent, via the definitions (Δ) and ($\circledast$). (A definition of monad analogous to the (shiftε) variant of our $\langle \mathcal{A}, \varepsilon, \circledast \rangle$ notion may be found in [Manes 1976, 1.3, Definition 3.2, p. 24]; the other variants are from [D. 1996, section 4.1].)

If we don't economize on primitives, and take both Δ and delta composition as primitives, then an equivalent notion of comonad is obtained by defining it as $\langle \mathcal{A}, \varepsilon, \Delta, \circledast \rangle$, where $\mathcal{A}$, D, ε, Δ and $\circledast$ are as before and the equalities ($\varepsilon\Delta$), ($\Delta\circledast$) and ($\Delta\varepsilon$) hold. Now the defining equalities (Δ) and ($\circledast$) become derivable (this definition is in [D. 1996, section 4.1]).

Note that we are certainly not allowed to suppose that we have now exhausted all possible ways of defining comonads. But the definitions through the adjunction between $\mathcal{A}$ and $\mathcal{A}_\Delta$ are well covered, and among these definitions we find the standard definition and other definitions mentioned in the literature.

4.6. The Kleisli category. Let $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ be a comonad. Then consider the graph $\mathcal{A}_D$ whose objects are all the objects of $\mathcal{A}$, while its arrows are obtained by taking that for every object A of $\mathcal{A}$ and every arrow $f : DA \to A'$ of $\mathcal{A}$, the pair (A, f), which we abbreviate by f^A, is an arrow of $\mathcal{A}_D$ of type $A \to A'$. (Formally, we need a bijection κ that assigns to the pairs (A, f) the arrows $\kappa(A, f) : A \to A'$ of $\mathcal{A}_D$. So, $\kappa(A, f)$ may be identified with the ordered pair (A, f). We cannot identify $\kappa(A, f)$ just with f instead of (A, f), because, if D is not one-one on objects, then f could have more than one source in $\mathcal{A}_D$. Definitions of Kleisli category in the literature, including Kleisli's own definition of [1965], usually don't make this clear.)

The graph $\mathcal{A}_D$ has an identity whose arrows $\mathbf{1}_A : A \to A$ are defined as ε_A^A and composition in $\mathcal{A}_D$ is defined as follows in terms of the delta composition of $\mathcal{A}$:

$$f_2^{A_2} \circ f_1^{A_1} \overset{\text{def}}{=} (f_2 \circledast f_1)^{A_1}.$$

Let us call the graph $\mathcal{A}_D$ with this identity and this composition the *Kleisli deductive system* of the comonad $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$. It is clear that due to (cat1right$\circledast$), (cat1left$\circledast$) and (cat2$\circledast$) of the preceding section, this deductive system is a category. This category is called the *Kleisli category* of the comonad $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$.

A category isomorphic to $\mathcal{A}_D$ is a category $\mathcal{A}'_\Delta$ related to the delta category $\mathcal{A}_\Delta$, which is defined as follows. Its objects are again the objects of $\mathcal{A}$, while its arrows are obtained by taking that for every pair (A_1, A_2) of objects of $\mathcal{A}$ and every arrow $h : DA_1 \to DA_2$ of $\mathcal{A}$ such that

$$(\text{homo } \delta) \qquad Dh \circ \delta_{A_1} = \delta_{A_2} \circ h,$$

the triple $\langle A_1, A_2, h \rangle$, which we abbreviate by h^{A_1, A_2}, is an arrow of $\mathcal{A}'_\Delta$ of type $A_1 \to A_2$. The identity arrows $\mathbf{1}_A : A \to A$ of $\mathcal{A}'_\Delta$ are defined as $\mathbf{1}_{DA}^{A,A}$ and composition is defined by

$$h_2^{A_2, A_3} \circ h_1^{A_1, A_2} \overset{\text{def}}{=} (h_2 \circ h_1)^{A_1, A_3}.$$

The equality (homo δ), which is a kind of naturalness condition, could alternatively be written as

$$\Delta h = \Delta \mathbf{1}_{DA_2} \circ h.$$

Other conditions equivalent to (homo δ) are

$$\Delta(\varepsilon_{A_2} \circ h) = h, \text{ i.e., } \Delta E_{A_2} h = h,$$
$$\exists f(\Delta f = h).$$

The isomorphism between the categories $\mathcal{A}_D$ and $\mathcal{A}'_\Delta$ is obtained by the functor K from $\mathcal{A}_D$ to $\mathcal{A}'_\Delta$ such that $KA = A$ and for $f : DA_1 \to A_2$

$$Kf^{A_1} = (\Delta f)^{A_1, A_2}.$$

The inverse K^{-1} of K is defined by $K^{-1}A = A$ and for $h : DA_1 \to DA_2$

$$K^{-1}h^{A_1,A_2} = (\varepsilon_{A_2} \circ h)^{A_1} = (E_{A_2} h)^{A_1}.$$

If D is one-one on objects, then it is clear that the category $\mathcal{A}'_\Delta$ is isomorphic to the delta category $\mathcal{A}_\Delta$, which we have considered in 4.2. Without supposing that D is one-one on objects, we can ascertain only that $\mathcal{A}'_\Delta$ and $\mathcal{A}_\Delta$ are equivalent categories (see 1.5).

The $\langle \mathcal{A}, \varepsilon, \circledS \rangle$ definition of comonad from the preceding section shows that we could define a comonad by assuming that its Kleisli deductive system is a category and by the the (shift) equality. This equality expresses the adjunction between $\mathcal{A}$ and $\mathcal{A}_D$, which we shall examine in 5.

4.7. The Eilenberg-Moore category. Let $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ be a comonad. Then consider the graph $\mathcal{A}^D$ whose objects are arrows $d : A \to DA$ of $\mathcal{A}$ such that

(ob1)	$\varepsilon_A \circ d = \mathbf{1}_A,$
(ob2)	$\delta_A \circ d = Dd \circ d.$

An arrow of $\mathcal{A}^D$ with source $d_1 : A_1 \to DA_1$ and target $d_2 : A_2 \to DA_2$ is made of an arrow $h : A_1 \to A_2$ of $\mathcal{A}$ such that

(homo) $\qquad Dh \circ d_1 = d_2 \circ h.$

To prevent the same arrow from having more then one source or more than one target, the arrow h in $\mathcal{A}^D$ should be indexed by d_1 and d_2. Formally, the arrows of $\mathcal{A}^D$ will be triples $\langle d_1, d_2, h \rangle$, but we shall take the indices d_1 and d_2 for granted and omit them (usually, they are not even mentioned).

The identity arrows of $\mathcal{A}^D$ are just $\mathbf{1}_A : A \to A$ and composition is defined as composition in $\mathcal{A}$. We can check that the equality (homo) holds when d_1 and d_2 are equal and for h we put an identity arrow; it holds also for $h_2 \circ h_1$ if it holds for h_1 and h_2. So $\mathcal{A}^D$ is a category, which is called the *Eilenberg-Moore category* of the comonad $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$.

For $d : A \to DA$ and $f : A \to A'$ let

$$\Delta_d f \overset{\text{def}}{=} Df \circ d.$$

It is clear that for $f : DA \to A'$ the arrow $\Delta_{\delta_A} f$ is Δf. To define the Eilenberg-Moore category of a comonad we can assume

$$\begin{array}{ll} (\text{ob1}') & \varepsilon_A \circ \Delta_d f = f, \\ (\text{ob2}') & \delta_A \circ \Delta_d f = \Delta_d \Delta_d f, \\ (\text{homo}') & \Delta_{d_1} h = \Delta_{d_2} \mathbf{1}_{A_2} \circ h \end{array}$$

instead of (ob1), (ob2) and (homo).

The full subcategory $\mathcal{A}^D_{\text{free}}$ of $\mathcal{A}^D$ whose objects are all the arrows $\delta_A : DA \to DDA$ of $\mathcal{A}$ is called the category of *free coalgebras* of the comonad. This category is isomorphic to the delta category $\mathcal{A}_\Delta$ when there is a bijection between the objects of $\mathcal{A}$ of the form DA and the arrows δ_A of $\mathcal{A}$. This bijection exists when D is one-one on objects. When D is not such, we may still have this bijection, provided that if DA_1 is the same object as DA_2, then $\delta_{A_1} = \delta_{A_2}$ (the converse implication obtains anyway). But the bijection may also fail. (In [D. 1996, section 4.2] it is stated that it can be shown without the supposition that D is one-one on objects that $\mathcal{A}_\Delta$ and $\mathcal{A}^D_{\text{free}}$ are isomorphic. What should have been said is that this can be shown *sometimes* even without making this supposition.)

We obtain a category isomorphic to the Kleisli category $\mathcal{A}_D$ (and to $\mathcal{A}'_\Delta$) by replacing the objects δ_A of $\mathcal{A}^D_{\text{free}}$ with pairs (A, δ_A), and the arrows $h : DA_1 \to DA_2$ of $\mathcal{A}^D_{\text{free}}$ with triples $\langle A_1, A_2, h \rangle$. (In the usual presentation of Eilenberg-Moore categories, objects are said to be pairs (A, d) where A is the source of $d : A \to DA$ and d satisfies (ob1) and (ob2). These pairs are in one-to-one correspondence with the arrows d. Mentioning the source of d in the pair is not essential: it seems to be there for heuristical reasons. However, introducing A into (A, δ_A) makes a difference. Note that A is not the source DA of δ_A.)

In general, we can assert only that $\mathcal{A}^D_{\text{free}}$ is equivalent to $\mathcal{A}_\Delta$ and $\mathcal{A}_D$, without necessarily being isomorphic.

5. Adjunction between adjunctions and comonads

We shall now try to clarify the relationship between the notions of comonad and adjunction. It will appear that comonads may be understood as a special kind of adjunction, since the category of comonads (with comonofunctors as arrows) is isomorphic to a full subcategory of the category of adjunctions (with appropriate morphisms, which we shall call *junctors*, as arrows). Moreover, there are two adjunctions involving these two categories.

First, we have a functor that associates in a standard manner a comonad to an adjunction. After investigating some aspects of this functor, we show that it has a left adjoint, which associates to a comonad the adjunction with the Kleisli category, and a right adjoint, which associates to a comonad the adjunction with the Eilenberg-Moore category. At the end (5.4), we show how the usual presentation of these matters, via the category of resolutions of a comonad, where the Kleisli category is tied to the initial object and the Eilenberg-Moore category to the terminal object, is a simple corollary of our presentation.

5.1. The comonad of an adjunction. We shall first introduce the notions of *junction* and *junctor* in the rectangular || style of 3.3. A junction is a structure like an adjunction, but without the corresponding equalities between arrows. So a junction is to an adjunction what a deductive system is to a category and what a comonograph is to a comonad. A junctor is a morphism of junctions, and also a morphism of adjunctions.

Suppose we are given the following:

two deductive systems, $\langle \mathcal{A}, \mathbf{1}, \circ\rangle$ and $\langle \mathcal{B}, \mathbf{1}, \circ\rangle$,
a graph-morphism F from $\mathcal{B}$ to $\mathcal{A}$ and a graph-morphism G from $\mathcal{A}$ to $\mathcal{B}$,
a transformation φ from FG to $I_\mathcal{A}$ and a transformation γ from $I_\mathcal{B}$ to GF.

Then $\langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma\rangle$ is a *junction*.

A *junctor* from a junction $\langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma\rangle$ to a junction $\langle \mathcal{A}', \mathcal{B}', F', G', \varphi', \gamma'\rangle$ is a pair $(N_\mathcal{A}, N_\mathcal{B})$ such that $N_\mathcal{A}$ is a functor from the deductive system $\mathcal{A}$ to the deductive system $\mathcal{A}'$, and $N_\mathcal{B}$ a functor from the deductive system $\mathcal{B}$ to the deductive system $\mathcal{B}'$; moreover, the following naturalness equalities hold:

$$N_\mathcal{A} F = F' N_\mathcal{B}, \qquad N_\mathcal{B} G = G' N_\mathcal{A},$$
$$N_\mathcal{A} \varphi_A = \varphi'_{N_\mathcal{A} A}, \qquad N_\mathcal{B} \gamma_B = \gamma'_{N_\mathcal{B} B}.$$

An *adjunction* is a junction $\langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma\rangle$ such that

$\langle \mathcal{A}, \mathbf{1}, \circ\rangle$ and $\langle \mathcal{B}, \mathbf{1}, \circ\rangle$ are categories,
F and G are functors,
φ and γ are natural transformations,
the equalities $(\varphi\gamma F)$ and $(\varphi\gamma G)$ hold (see 3.3).

To every adjunction $\langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma\rangle$ we may associate the comonad $\langle \mathcal{A}, FG, \varphi, F\gamma_G\rangle$, where the composite functor FG is the functor D of the comonad, φ_A is ε_A and $F\gamma_{GA}$ is δ_A. (We may analogously associate to the adjunction a monad in $\mathcal{B}$.) It is routine to check that $\langle \mathcal{A}, FG, \varphi, F\gamma_G\rangle$ is indeed a comonad. It is called *the comonad of the adjunction* $\langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma\rangle$.

5.2. Reflections and coreflections in comonads. An adjunction between $\mathcal{A}$ and $\mathcal{B}$ where the right adjoint G is the inclusion functor from $\mathcal{A}$ into $\mathcal{B}$ is called a *reflection* of $\mathcal{B}$ in its subcategory $\mathcal{A}$. We have seen in 4.4 that a comonad in a category $\mathcal{A}$ is defined by a coreflection of $\mathcal{A}$ in its subcategory $\mathcal{A}_\Delta$, the delta category of the comonad. However, with comonads of adjunctions we may have in some interesting (and in logic rather common) cases also a reflection of a category isomorphic to $\mathcal{A}_\Delta$ in its subcategory $\mathcal{A}$. We shall now consider this matter.

Let us first prove the following proposition.

Proposition 1. *Let $\langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma \rangle$ be an adjunction where G is one-one on objects. Then the Kleisli category $\mathcal{A}_{FG}$ of the comonad $\langle \mathcal{A}, FG, \varphi, F\gamma_G \rangle$ of the adjunction is isomorphic to the full subcategory $G(\mathcal{A})$ of $\mathcal{B}$ whose objects are all the objects of $\mathcal{B}$ of the form GA.*

Proof: First we show that for $f_1 : FGA_1 \to A_2$ and $f_2 : FGA_2 \to A_3$ in the comonad $\langle \mathcal{A}, FG, \varphi, F\gamma_G \rangle$ we have

$$(\circledast\Phi\Gamma) \qquad f_2 \circledast f_1 = \Phi_{GA_1,A_3}(\Gamma_{GA_2,A_3} f_2 \circ \Gamma_{GA_1,A_2} f_1).$$

Indeed,

$$\begin{aligned} f_2 \circledast f_1 &= f_2 \circ (FGf_1 \circ F\gamma_{GA_1}), \text{ by definition} \\ &= f_2 \circ F\Gamma_{GA_1,A_2} f_1, \text{ by (fun2) and } (\Gamma) \text{ of 3.1,} \end{aligned}$$

and we obtain $(\circledast\Phi\Gamma)$ by applying $(\Phi\Gamma')$ and $(\Gamma\Gamma'')$ from 3.4.

We now define a functor N from $\mathcal{A}_{FG}$ to $G(\mathcal{A})$ in the following way. For every object A of $\mathcal{A}_{FG}$, which is by definition an object of $\mathcal{A}$, let NA be GA. For every arrow $f^{A_1} : A_1 \to A_2$ of $\mathcal{A}_{FG}$, for which, by definition, we have an arrow $f : FGA_1 \to A_2$ of $\mathcal{A}$, let Nf^{A_1} be $\Gamma_{GA_1,A_2} f : GA_1 \to GA_2$. To check that N is a functor we have

$$\begin{aligned} N\varphi_A^A &= \Gamma_{GA,A}\varphi_A = \mathbf{1}_{GA}, \quad \text{by } (\varphi) \text{ of 3.1, (fun1) and } (\Gamma\Phi') \text{ of 3.4,} \\ N(f_2 \circledast f_1)^{A_1} &= \Gamma_{GA_1,A_3}(f_2 \circledast f_1) \\ &= \Gamma_{GA_2,A_3} f_2 \circ \Gamma_{GA_1,A_2} f_1, \quad \text{by } (\circledast\Phi\Gamma) \text{ and } (\Phi\Gamma') \text{ of 3.4} \\ &= Nf^{A_2} \circ Nf^{A_1}. \end{aligned}$$

Relying on the fact that G is one-one on objects, we define the functor N^{-1} from $G(\mathcal{A})$ to $\mathcal{A}_{FG}$ by taking that $N^{-1}GA$ is A and that for $g : GA_1 \to GA_2$ the arrow $N^{-1}g$ is $(\Phi_{GA_1,A_2} g)^{A1}$. It remains to use the equalities $(\Phi\Gamma')$ and $(\Gamma\Phi')$ to verify that $N^{-1}Nf^{A_1} = f^{A_1}$ and $NN^{-1}g = g$.

This is an immediate corollary of Proposition 1:

Proposition 2. *Let $\langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma \rangle$ be an adjunction where G is a bijection on objects. Then the categories $\mathcal{A}_{FG}$ and $\mathcal{B}$ are isomorphic.*

We know from 4.6 that if in a comonad $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ we have that D is one-one on objects, then the Kleisli category $\mathcal{A}_D$ of the comonad is isomorphic to the subcategory $\mathcal{A}_\Delta$ of $\mathcal{A}$, the delta category of the comonad. With the comonad $\langle \mathcal{A}, FG, \varphi, F\gamma_G \rangle$ of an adjunction, for $f : FGA_1 \to A_2$ we have

$$\Delta f = F\Gamma_{GA_1,A_2} f.$$

So $\mathcal{A}_\Delta$ will be denoted in this case by $\mathcal{A}_{F\Gamma}$. We can then state the following as a corollary of Proposition 1:

Proposition 3. *Let $\langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma\rangle$ be an adjunction where both F are G are one-one on objects. Then the categories $\mathcal{A}_{F\Gamma}$ and $G(\mathcal{A})$ are isomorphic.*

The point of this proposition is that $\mathcal{A}_{F\Gamma}$ is a subcategory of $\mathcal{A}$. So in all adjunctions $\langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma\rangle$ where F is one-one on objects and G is a bijection on objects, $\mathcal{B}$ is isomorphic to a subcategory of $\mathcal{A}$. Note that in such an adjunction $\mathcal{A}$ may actually be a subcategory of $\mathcal{B}$, so that the adjunction is a reflection of $\mathcal{B}$ in its subcategory $\mathcal{A}$. But we can assert that $\mathcal{B}$ is also isomorphic to a subcategory of $\mathcal{A}$, namely $\mathcal{A}_{F\Gamma}$, and that there is a coreflection of $\mathcal{A}$ in this subcategory.

(The situation we have just described obtains sometimes in the adjunction of *deductive completeness*, a strengthening of the deduction theorem, originally called *functional completeness* in [Lambek 1974]; see also [Lambek & Scott 1986, I.6-7] and [D. 1996]. Then $\mathcal{B}$ is the polynomial category generated by $\mathcal{A}$ and an indeterminate arrow.)

It is instructive to see that the isomorphism from $\mathcal{B}$ to $\mathcal{A}_{F\Gamma}$ above is the functor F, the left adjoint in the adjunction.

5.3. The adjunctions involving the categories of adjunctions and comonads. Let **Adj** be the category whose objects are adjunctions, with arrows being junctors (this category should not be confused with the category bearing the same name in [Mac Lane 1971, IV.8], where arrows are adjunctions), and let **Com** be the category whose objects are comonads, with arrows being comonofunctors.

Consider now the functor **C** from **Adj** to **Com** that assigns to an adjunction $\langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma\rangle$ the comonad $\langle \mathcal{A}, FG, \varphi, F\gamma_G\rangle$ of the adjunction, and to a junctor $(N_\mathcal{A}, N_\mathcal{B})$ the comonofunctor $N_\mathcal{A}$ (we may readily check that $N_\mathcal{A}$ is indeed a comonofunctor).

The functor **C** has a left adjoint **F** that assigns to a comonad $\langle \mathcal{A}, D, \varepsilon, \delta\rangle$ the adjunction between $\mathcal{A}$ and the Kleisli category $\mathcal{A}_D$ of this comonad, namely the adjunction $\langle \mathcal{A}, \mathcal{A}_D, F_D, G_D, \varphi_D, \gamma_D\rangle$, which is defined as follows:

$$F_D A \stackrel{\text{def}}{=} DA, \qquad G_D A \stackrel{\text{def}}{=} A,$$

$$F_D f^A \stackrel{\text{def}}{=} \Delta f, \qquad G_D f \stackrel{\text{def}}{=} (f \circ \varepsilon_A)^A,$$

$$\varphi_{D_A} \stackrel{\text{def}}{=} \varepsilon_A, \qquad \gamma_{D_A} \stackrel{\text{def}}{=} (\mathbf{1}_{DA})^A.$$

If $N_\mathcal{A}$ is a comonofunctor from a comonad $\langle \mathcal{A}, D, \varepsilon, \delta\rangle$ to a comonad $\langle \mathcal{A}', D', \varepsilon', \delta'\rangle$, then $\mathbf{F}N_\mathcal{A}$ is the junctor $(N_\mathcal{A}, N_{\mathcal{A}_D})$ from the adjunction between $\mathcal{A}$ and $\mathcal{A}_D$ to the adjunction between $\mathcal{A}'$ and $\mathcal{A}'_{D'}$, where $N_{\mathcal{A}_D}$ is defined as follows:

$$N_{\mathcal{A}_D} A \stackrel{\text{def}}{=} N_\mathcal{A} A, \qquad N_{\mathcal{A}_D} f^A \stackrel{\text{def}}{=} (N_\mathcal{A} f)^{N_\mathcal{A} A}.$$

For an adjunction $J = \langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma \rangle$ let φ_J be the junctor $(N_{\mathcal{A}}, N_{\mathcal{B}})$ from $\mathbf{FC}J$ to J where $N_{\mathcal{A}}$ is the identity functor $I_{\mathcal{A}}$ and the functor $N_{\mathcal{B}}$ is defined by

$$N_{\mathcal{B}}A \stackrel{\text{def}}{=} GA, \qquad N_{\mathcal{B}}f^A \stackrel{\text{def}}{=} Gf \circ \gamma_{GA} = \Gamma_{GA,A'}f.$$

The arrows φ_J of **Adj** make a natural transformation φ from **FC** to $I_{\mathbf{Adj}}$. It is easy to check that for every comonad $S = \langle \mathcal{A}, D, \varepsilon, \delta \rangle$ the comonad $\mathbf{CF}S$ is identical to S; so the identity comonofunctor $I_{\mathcal{A}}$ is an arrow from S to $\mathbf{CF}S$ in **Com**. It is trivial that the arrows $I_{\mathcal{A}}$ make a natural transformation **I** from $I_{\mathbf{Com}}$ to **CF**.

That **F** is left adjoint to **C** means that $\langle \mathbf{Adj}, \mathbf{Com}, \mathbf{F}, \mathbf{C}, \varphi, \mathbf{I} \rangle$ is an adjunction. In this adjunction, the unit is the identity of the category **Com**. We can infer that **Com** is isomorphic by **F** to a full subcategory of **Adj** (cf. [Mac Lane 1971, IV.4, pp. 92–93]).

The functor **C** has also a right adjoint **G** that assigns to a comonad $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ the adjunction between $\mathcal{A}$ and the Eilenberg-Moore category $\mathcal{A}^D$ of this comonad, namely the adjunction $\langle \mathcal{A}, \mathcal{A}^D, F^D, G^D, \varphi^D, \gamma^D \rangle$, which is defined as follows:

$$F^D d \stackrel{\text{def}}{=} \text{source}(d), \qquad G^D A \stackrel{\text{def}}{=} \delta_A,$$

$$F^D h \stackrel{\text{def}}{=} h, \qquad G^D f \stackrel{\text{def}}{=} Df,$$

$$\varphi^D_A \stackrel{\text{def}}{=} \varepsilon_A, \qquad \gamma^D_d \stackrel{\text{def}}{=} d.$$

If $N\mathcal{A}$ is a comonofunctor from a comonad $\langle \mathcal{A}, D, \varepsilon, \delta \rangle$ to a comonad $\langle \mathcal{A}', D', \varepsilon', \delta' \rangle$, then $\mathbf{G}N_{\mathcal{A}}$ is the junctor $(N_{\mathcal{A}}, N_{\mathcal{A}_D})$ from the adjunction between $\mathcal{A}$ and $\mathcal{A}^D$ to the adjunction between $\mathcal{A}'$ and $\mathcal{A}'^{D'}$, where $N_{\mathcal{A}_D}$ is defined as follows:

$$N_{\mathcal{A}^D} d \stackrel{\text{def}}{=} N_{\mathcal{A}} d, \qquad N_{\mathcal{A}^D} h \stackrel{\text{def}}{=} N_{\mathcal{A}} h.$$

For an adjunction $J = \langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma \rangle$ let now γ_J be the junctor $(N_{\mathcal{A}}, N_{\mathcal{B}})$ from J to $\mathbf{GC}J$ where $N_{\mathcal{A}}$ is the identity functor $I_{\mathcal{A}}$ and the functor $N_{\mathcal{B}}$ is defined by

$$N_{\mathcal{B}}B \stackrel{\text{def}}{=} F\gamma_B, \qquad N_{\mathcal{B}}g \stackrel{\text{def}}{=} Fg.$$

The arrows γ_J of **Adj** make a natural transformation γ from $I_{\mathbf{Adj}}$ to **GC**. It is easy to check that for every comonad $S = \langle \mathcal{A}, D, \varepsilon, \delta \rangle$ the comonad $\mathbf{CG}S$ is identical to S; so the identity comonofunctor $I_{\mathcal{A}}$ is an arrow from $\mathbf{CG}S$ to S in **Com**. It is trivial that the arrows $I_{\mathcal{A}}$ make a natural transformation **I** from **CG** to $I_{\mathbf{Com}}$.

That **G** is right adjoint to **C** means that $\langle \mathbf{Com}, \mathbf{Adj}, \mathbf{C}, \mathbf{G}, \mathbf{I}, \gamma \rangle$ is an adjunction. In this adjunction, the counit is the identity of the category **Com**. We can infer that **Com** is isomorphic by **G** to a full subcategory of **Adj** (following the terminology of [Mac Lane 1971, IV.4, pp. 92–93], the functor **C** is a left-adjoint-left-inverse of **G**; the category **Com** is isomorphic to a full reflective subcategory of **Adj**).

One could expect that adjunctions similar to those with **C**, **F** and **G** treated in this section may be obtained by taking instead of **Adj** the category of junctions (with junctors as arrows) and instead of **Com** the category of comonographs (with comonofunctors as arrows).

5.4. The category of resolutions. Take a functor **C** from a category $\mathcal{A}$ to a category $\mathcal{B}$, and for a given object B of $\mathcal{B}$ consider the set of objects A of $\mathcal{A}$ such that $\mathbf{C}A = B$ and the set of arrows f of $\mathcal{A}$ such that $\mathbf{C}f = \mathbf{1}_B$. These two sets make the graph of a subcategory $\mathcal{A}_B$ of $\mathcal{A}$.

An object A is *initial* in a graph iff from A to every object in the graph there is exactly one arrow; A is *terminal* iff from every object to A there is exactly one arrow.

If **C** has a left adjoint **F** such that the unit of the adjunction is the identity of $\mathcal{B}$, then $\mathcal{A}_B$ has an initial object $\mathbf{F}B$, and if **C** has a right adjoint **G** such that the counit of the adjunction is the identity of $\mathcal{B}$, then $\mathcal{A}_B$ has a terminal object $\mathbf{G}B$.

To show that $\mathbf{F}B$ is initial, take an object A of $\mathcal{A}_B$; then it can be shown that $\varphi_A : \mathbf{FC}A \to A$ is the unique arrow of $\mathcal{A}_B$ from $\mathbf{F}B$ to A. For suppose there is another arrow $f : \mathbf{FC}A \to A$ in $\mathcal{A}_B$; since

$$\mathbf{C}f \circ \gamma_B = \mathbf{C}f = \mathbf{1}_B,$$

because γ_B is an identity arrow and f is in $\mathcal{A}_B$, and since

$$\mathbf{C}\varphi_A \circ \gamma_B = \mathbf{1}_B, \text{ by the equality } (\varphi\gamma G) \text{ of 3.3},$$

we obtain

$$\varphi_{\mathbf{F}B} \circ \mathbf{F}(\mathbf{C}f \circ \gamma_B) = \varphi_{\mathbf{F}B} \circ \mathbf{F}(\mathbf{C}\varphi_A \circ \gamma_B),$$

from which with (fun2), (nat) and the equality $(\varphi\gamma F)$ of 3.3, the equality $f = \varphi_A$ follows. Analogously, in the other adjunction, the one with **G**, the arrow $\gamma_A : A \to \mathbf{GC}A$ is the unique arrow of $\mathcal{A}_B$ from A to $\mathbf{G}B$.

So by taking the functor **C** from **Adj** to **Com** and by fixing a comonad S in **Com** we obtain a subcategory $\mathbf{Adj}_S$ of **Adj**. We may call the category $\mathbf{Adj}_S$ the category of *resolutions* of S, by analogy with the terminology usual when one deals with monads instead of comonads. For a comonad $S = \langle \mathcal{A}, D, \varepsilon, \delta \rangle$, the adjunctions in $\mathbf{Adj}_S$ are all between the category $\mathcal{A}$ and a category $\mathcal{B}$, and the junctors $(N_{\mathcal{A}}, N_{\mathcal{B}})$ in $\mathbf{Adj}_S$ all have for $N_{\mathcal{A}}$ the identity functor on $\mathcal{A}$.

The category $\mathbf{Adj}_S$ has an initial object $\mathbf{F}S$ and a terminal object $\mathbf{G}S$, according to what we have said above. The arrow $\varphi_J : \mathbf{FC}J \to J$ is the unique arrow of $\mathbf{Adj}_S$ from $\mathbf{F}S$ to an adjunction J of $\mathbf{Adj}_S$, and $\gamma_J : J \to \mathbf{GC}J$ is the unique arrow of $\mathbf{Adj}_S$ from J to $\mathbf{G}S$. These arrows correspond to what in the case of monads is called *comparison functors.*

Suppose a functor **C** from a category $\mathcal{A}$ to a category $\mathcal{B}$ has both a left adjoint **F** and a right adjoint **G**. Then the functors **FC** and **GC** from $\mathcal{A}$ to $\mathcal{A}$ are adjoint, **FC** being left adjoint and **GC** right adjoint. (Analogously, **CF** and **CG** from $\mathcal{B}$ to $\mathcal{B}$ are adjoint, **CF** being left adjoint and **CG** right adjoint.) This is a consequence of the fact that two successive adjunctions compose to give a single adjunction (see [Mac Lane 1971, IV.8, p. 101]).

By taking that $\mathcal{A}$ is **Adj** and $\mathcal{B}$ is **Com**, we obtain that the functors **FC** and **GC** from **Adj** to **Adj** are adjoint. (The functors **CF** and **CG** are uninteresting, since they are the identity functor from **Com** to **Com**.)

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