M. Nedeljkov, D. Perišić
and S. Pilipović

PSEUDODIFFERENTIAL OPERATORS

Contents

Part I. CLASSICAL THEORY

The aim of lecture notes is to present the basic facts of the theory of pseudodifferential operators and to give sufficiently enough motivations for further study of this very important theory. Also, in the notes authors develop the theory of pseudodifferential operators within Colombeau's new generalized functions.

Pseudodifferential operators are generalization of differential operators. They form the minimal algebra of operators in which each elliptic operator has the inverse up to a smoothing operator. Thus, the roots of the theory of pseudodifferential operators are in the theory of elliptic operators. This theory is used for microlocal analysis of equations, the hypoellipticity for example. In the second part we show this for the (hypo)elliptic pseudodifferential equations with coefficients in the space of Colombeau's generalized functions.

Part I of the notes was written when the first two authors had studied the classical theory of pseudodifferential operators, as a part of their doctoral studies, under the coordination of the third author, who prepared a seminar on that topic at the Institute of Mathematics of Novi Sad University during 1988/89 and 1990/91. The authors documented their work, writing down an extensive paper (in Serbian), proving the theorems, explaining in details various examples etc. Some parts of this unpublished material constitute these notes. The main references for Part I are monographs **[10]**, **[19]** and **[20]**.

Part II is devoted to the pseudodifferential calculus within Colombeau's space od generalized functions, $\mathcal{G}$. The idea was established by the authors during the seminar on Colombeau's theory which took place in 1989/1990. The third author made a coherent theory on pseudodifferential operators in Colombeau's sense of new generalized functions **[16]**, during his stay in Japan at the Tokyo University in the winter of 1992/1993.

It was not an easy job to present so large theory on around sixty pages, the number which was predicted by the editor. Because of that our exposition is of fragmented character in some parts. We think that the reader can find in the notes enough information for further study of pseudodifferential and Fourier integral operators.

We assume that the reader is familiar with the basic notions of functional analysis, distribution theory and the theory of partial differential equations. For further study we refer to **[10]**, **[11]**, **[15]**, **[19]** **[20]**.

1. Introduction

If K is a compact subset of an open set Ω, $\Omega \subset \mathbb{R}$, and ϕ is C^∞ function, then

$$\|\phi\|_{\alpha,K} = \sup_{\substack{\|\beta\| \leq \alpha \\ x \in K}} |\partial^\beta \phi(x)|.$$

Denote by $\mathcal{D}_{\alpha,K}$ the Banach space of C^∞ functions ϕ on Ω such that $\operatorname{supp} \phi \subset K$ and $\|\phi\|_{\alpha,K} < \infty$. The projective limit of $\mathcal{D}_{\alpha,K}$, as $\|\alpha\| \to \infty$, is denoted by $\mathcal{D}_K$. The Schwartz's space of test functions $\mathcal{D}(\Omega)$ is defined as the inductive limit of spaces $\mathcal{D}_K$ as $K \subset\subset \Omega$ and the union of K's exhaust Ω. We will use the notation $C_0^\infty = \mathcal{D}(\Omega)$. (The notation $K \subset\subset \mathbb{R}$ or $K \subset\subset \Omega$ means that K is compact in $\mathbb{R}$ or $\mathbb{C}$.) The strong dual of the spaces $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ is called the Schwartz space of distributions. The space of distributions with compact supports is denoted by $\mathcal{E}(\Omega)'$. It is the strong dual of the space smooth functions on Ω with the uniform convergence of all the derivatives on compact subsets.

Schwartz's space of rapidly decreasing functions is defined by

$$\mathcal{S} = \mathcal{S}(\mathbb{R}^n) = \{u \in C^\infty(\mathbb{R}^n),\ (\forall \alpha, \beta \in \mathbb{N}_0^n)\ (\exists c \in \mathbb{R})\ (\sup |x^\alpha (\partial^\beta u)(x)| \leq c)\}.$$

Its strong dual is the space of tempered distributions $\mathcal{S}'$.

The Fourier transformation of a function $u \in L^1$ is defined by

$$\mathcal{F}(u)(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} u(x)\, dx,\ \xi \in \mathbb{R}^n,$$

and the inverse transformation by

$$\mathcal{F}^{-1}(u)(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} u(x)\, dx,\ \xi \in \mathbb{R}^n.$$

If u is supported by a compact set, then the Fourier–Laplace transformation is defined as above with ξ substituted by $\zeta \in \mathbb{C}^n$.

The Fourier transformation is an isomorphism of $\mathcal{S}$ (resp. $\mathcal{S}'$) onto the same space.

The Sobolev space $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$ consists of tempered distributions f which Fourier transform $\hat{f}$ satisfies the following condition

$$(1 + |\xi|^2)^{s/2} \hat{f} \in L^2(\mathbb{R}^n).$$

We shall give Palley-Wiener theorem which will be used often in this work.

"Let K be a convex compact subset of $\mathbb{R}^n$ and let H be its characteristic function. If u is a distribution of order N supported by K, then for its Fourier–Laplace transformation satisfies

$$|\hat{u}(\zeta)| \leq C(1 + |\zeta|)^N e^{H(\operatorname{Im} \zeta)}, \quad \zeta \in \mathbb{C}^n. \tag{1.1}$$

Every entire function on $\mathbb{C}^n$ which satisfies (1.1) is a Fourier–Laplace transformation of a distribution with the support contained in K.

If $u \in C_0^\infty(K)$, then for every $N \in \mathbb{N}$

$$|\hat{u}(\zeta)| \le C_N(1+|\zeta|)^{-N} e^{H(\operatorname{Im}\zeta)}, \quad \zeta \in \mathbb{C}^n. \tag{1.2}$$

Conversely, if (1.2) holds for an entire function and for every N, then it is a Fourier–Laplace transformation of some function $u \in C_0^\infty(K)$.

2. Elliptic operators with constant coefficients

As a motivation for the theory of pseudodifferential operators we give the construction of a parametrix for elliptic operators.

2.1. Parametrix of elliptic operator with constant coefficients. Let us consider the following equation in $\mathcal{S}'$

$$P(D)u = \sum_{|\alpha|\le m} c_\alpha D^\alpha u = f, \tag{2.1}$$

where $f \in \mathcal{E}'$ is given $D = (D_1, D_2, \dots, D_n)$, $D_j = -\sqrt{-1}\frac{\partial}{\partial x_j}$, $c_\alpha \in \mathbb{C}$, $|\alpha| \le m$. If a solution exists, then

$$P(\xi)\hat{u}(\xi) = \hat{f}(\xi),\ \xi \in \mathbb{R}^n,$$

and formally, $\hat{u}(\xi) = \hat{f}(\xi)/P(\xi)$. Therefore, a formal solution to problem (2.1) is given by

$$u(x) = \mathcal{F}^{-1}\Big(\frac{\hat{f}(\xi)}{P(\xi)}\Big)(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\xi}\frac{\hat{f}(\xi)}{P(\xi)}\,d\xi,\ x \in \mathbb{R}^n. \tag{2.2}$$

The integral on the right-hand side in (2.2) is not defined in general because of zeros of $P(\xi)$ and the behavior of $\hat{f}(\xi)$ in infinity. There are some special cases in which a modification of (2.2) gives the solution to (2.1). We will discuss one of such cases.

Let $P(D)$ be a differential operator of order m, (i.e. the corresponding polynomial $P(\xi)$ is of order m) and let

$$P(\xi) = P_m(\xi) + Q(\xi),$$

where $P_m = \sum_{|\alpha|=m} a_\alpha D^\alpha$ and $Q(\xi)$ is polynomial of order not greater than $(m-1)$. The operator $P_m(D)$ is called the principal symbol of $P(D)$.

Note $P_m(\lambda\xi) = \lambda^m P_m(\xi)$, for every $\lambda > 0$ and $\xi \in \mathbb{R}^n$, i.e. the polynomial $P_m(\xi)$ is a positive homogeneous function of order m. This implies that the set of zeros of the polynomial $P_m(\xi)$ (the variety of P_m), for $m > 0$ is a cone and it is called the characteristic cone.

Definition 2.1. A differential operator $P(D)$ of order m is elliptic if $P_m(\xi) \ne 0$, for every $\xi \in \mathbb{R}^n\backslash\{0\}$, where $P_m(D)$ is the principal symbol of the operator $P(D)$.

Example 2.1. If the dimension of the space equals one, then all the differential operators with constant coefficients are elliptic.

Example 2.2. The Laplace operator

$$\Delta = \Big(\frac{\partial}{\partial x_1}\Big)^2 + \Big(\frac{\partial}{\partial x_2}\Big)^2 + \cdots + \Big(\frac{\partial}{\partial x_n}\Big)^2$$

is elliptic. Its principal symbol is $-|\xi|^2 = -\xi_1^2 - \cdots - \xi_n^2$.

Example 2.3. For $n = 2$, the Cauchy-Riemann operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)$$

is elliptic, and its principal symbol is $i(\xi + i\mu)/2$.

Lemma 2.2. *Let $P(D)$ be an elliptic differential operator. Then the set of zeros of the polynomial $P(\xi)$ is compact in $\mathbb{R}^n$.*

Proof. If $P(D) = P_m(D) + Q(D)$ as above, then $P_m(\xi) \neq 0$, for $\xi \in S^{n-1}$, where S^{n-1} is the closed unit sphere in $\mathbb{R}^n$. Because of that

$$|P_m(\xi)| \geq c > 0, \quad \xi \in S^{n-1}.$$

If $0 \neq \xi \in \mathbb{R}^n$, then $\xi/|\xi| \in S^{n-1}$. This implies $|P_m(\xi/|\xi|)| \geq c$ and because of the positive homogeneity of $P_m(\xi)$ we have

$$|P_m(\xi)| \geq c|\xi|^m, \quad \xi \in \mathbb{R}^n.$$

The order of polynomial $Q(\xi)$ is not greater than $m-1$, and therefore,

$$|Q(\xi)| \leq c_1|\xi|^{m-1}, \quad \xi \in \mathbb{R}^n, \quad |\xi| > 1.$$

Let $\xi \in \mathbb{R}^n$ satisfy $P(\xi) = 0$ and $|\xi| > 1$. Then we have

$$c|\xi|^m \leq |P_m(\xi)| = |Q(\xi)| \leq c_1|\xi|^{m-1}.$$

This implies $|\xi| \leq c_1/c$. Thus the set of zeros of $P(\xi)$ is bounded. □

Let $P(D)$ be an elliptic operator such that its variety is contained in the ball $L(0,\rho)$, with the center at zero and radius ρ and let $\kappa(\xi) \in C^\infty(\mathbb{R}^n)$ be such that $\kappa(\xi) = 0$ for $|\xi| < \rho$ and $\kappa(\xi) = 1$ for $|\xi| > \rho' > \rho$. Denote

$$v(x) = \mathcal{F}^{-1}(\hat{f}(\xi)\kappa(\xi)/P(\xi))(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\xi}\hat{f}(\xi)\kappa(\xi)/P(\xi)\,d\xi,\ x \in \mathbb{R}^n.$$

This formal integral makes sense within the space of tempered distributions. It is the Fourier transformation of a tempered distribution.

In the sequel we will use the notation which have to be understood in the distributional sense.

It will be shown that $v(x)$ is not the solution of equation (2.1), but it differs from it only by a smooth function.

Formally (in fact in the sense of the tempered distributions)

$$\begin{aligned}P(D)v(x) &= \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{ix\xi}\hat{f}(\xi)\kappa(\xi)\,d\xi = \mathcal{F}^{-1}(\hat{f}(\xi)\kappa(\xi))(x)\\ &= (\mathcal{F}^{-1}(\hat{f}(\xi)) - \mathcal{F}^{-1}(\hat{f}(\xi)(1-\kappa(\xi)))(x) = f(x) - Rf(x),\end{aligned}$$

where $Rf = \mathcal{F}^{-1}(\hat{f}(\xi)(1-\kappa(\xi)))$.

Note that $\kappa(\xi)/P(\xi)$ is a tempered distribution on $\mathbb{R}^n$ since it is a bounded smooth function. Since

$$|P(\xi)| \geq |P_m(\xi)| - |Q(\xi)| \geq (c|\xi| - c_1)|\xi|^{m-1} > 1,$$

for large enough $|\xi|$, it follows that $\mathcal{K} = \mathcal{F}^{-1}(\kappa(\xi)/P(\xi))$ is a tempered distribution, and

$$\begin{aligned}v(x) &= \mathcal{F}^{-1}(\hat{f}(\xi)\kappa(\xi)/P(\xi))(x)\\ &= (\mathcal{F}^{-1}(\kappa(\xi)/P(\xi)) * \mathcal{F}^{-1}(\hat{f}(\xi)))(x) = (\mathcal{K} * f)(x).\end{aligned}$$

Since the function $(1-\kappa) \in C_0^\infty$, the Palley-Wiener theorem implies that its Fourier transform $h = \mathcal{F}^{-1}(1-\kappa)$ can be extended on $\mathbb{C}^n$ as an analytic function of exponential type, such that its restriction on $\mathbb{R}^n$ belongs to $\mathcal{S}$. Then $Rf = h * f$ which implies

$$P(D)(\mathcal{K} * f)(x) = f(x) - h * f(x). \tag{2.3}$$

Let us define operators $\mathbf{R}$ and $\mathbf{K}$ by

$$\begin{aligned}&\mathbf{R} : \mathcal{E}' \to C^\infty, \quad \mathbf{R} : f \to Rf,\\ &\mathbf{K} : \mathcal{E}' \to \mathcal{S}', \quad \mathbf{K} : f \to Kf := \mathcal{K} * f.\end{aligned}$$

Then, $\mathbf{R}$ is a smoothing operator i.e. a linear and continuous mapping from $\mathcal{E}'$ to C^∞.

Using this notation we write (2.3) as $P(D) = \mathbf{K} = I - \mathbf{R}$. The operator $\mathbf{K}$ is called the parametrix of the differential operator $P(D)$. If it is known, then the solution of equation

$$P(D)E = \delta \tag{2.4}$$

(the fundamental solution for $P(D)$) exists, and $u = E * f$ is the solution to problem (2.1). By the classical theory, equation $P(D)w = h$ has a solution which is an analytic function on $\mathbb{C}^n$. Solution to equation (2.4) is $E = \mathcal{K} + w$ (because $P(D)\mathcal{K} = \delta - h$ and $P(D)w = h$).

3. Integral operators

3.1. Kernel theorem. Schwartz's kernel theorem is the basis one for the theory of integral operators is based on it.

Definition 3.1. Let X_i be open subsets of $\mathbb{R}^{n_i}$, and let $u_i \in C(X_i)$, $i \in \{1,2\}$. Then the continuous function $u_1 \otimes u_2$ on $X_1 \times X_2$ defined by

$$(u_1 \otimes u_2)(x_1, x_2) = u_1(x_1)u_2(x_2), \quad x_i \in X_i,$$

is called the tensor product u_1 and u_2.

Proposition 3.2. *Let $u_i \in \mathcal{D}'(X_i)$, $i = 1,2$. Then there exists a distribution $u \in \mathcal{D}'(X_1 \times X_2)$ such that*

$$u(\phi_1 \otimes \phi_2) = u_1(\phi_1)u_2(\phi_2), \ \phi_i \in C_0^\infty(X_i), \quad i = 1,2.$$

Proof. Let us define

$$u(\phi) = u_1(u_2(\phi(x_1, x_2))), \ \phi \in C_0^\infty(X_1 \times X_2),$$

(where u_i depends only on x_i). It is clear that the assertion of the proposition holds for u and $u(\phi) = u_2(u_1(\phi))$. $\square$

Note, if $u_i \in \mathcal{E}'$, $i = 1,2$, then $u(\phi) = u_2(u_1(\phi))$, $\phi \in C^\infty(X_1 \times X_2)$.

The distribution u is called the tensor product of u_1 and u_2 and it is denoted by $u = u_1 \otimes u_2$.

Definition 3.3. A linear and continuous operator $A : \mathcal{D}(X_2) \to \mathcal{D}'(X_1)$ is called integral operator.

Theorem 3.4. *Let $K \in \mathcal{D}'(X_1 \times X_2)$. By*

$$\langle A\phi, \psi \rangle = K(\psi \otimes \phi), \ \psi \in C_0^\infty(X_1), \ \phi \in C_0^\infty(X_2) \tag{3.1}$$

is determinated a linear operator $A : \mathcal{D}(X_2) \to \mathcal{D}'(X_1)$. It is continuous, in the sense that $A\phi_j \to 0$ in the space $\mathcal{D}'(X_1)$, when $\phi_j \to 0$ in $C_0^\infty(X_2)$, i.e. it determines an integral operator.

Conversely, for every integral operator A there exists one and only one distribution K such that (3.1) holds. It is called the kernel of the operator A.

We refer to **[10]** for the proof.

Example 3.1. The kernel of the identity operator $\mathcal{D}(X) \to \mathcal{D}'(X)$, $A\varphi = \varphi$, where X is an open set in $\mathbb{R}^n$, is given by

$$\langle K, \phi \rangle = \int_X \phi(x,x)\, dx, \quad \phi \in C_0^\infty(X \times X),$$

i.e. $K(x,y) = \delta(x-y)$. It has the support on the diagonal.

We will use the following notation. If $A \subset X$ and $B \subset X \times Y$ then

$$A \circ B := \{y \in Y, (\exists x \in A)((x,y) \in B)\}.$$

If $A \subset Y$ and $B \subset X \times Y$, then

$$B \circ A := \{x \in X, (\exists y \in A)((x,y) \in B)\}. \tag{3.2}$$

Note that if A is a compact set and B is closed, then $B \circ A$ is a closed set.

In the following proposition we assume that $\operatorname{supp} K = B \subset X_1 \times X_2$, $A = \operatorname{supp} u \subset X_2$.

Proposition 3.5. *If $K \in \mathcal{D}'(X_1 \times X_2)$ is the kernel of the integral operator $A : \mathcal{D}(X_1) \to \mathcal{D}'(X_2)$, then $\operatorname{supp} Au \subset \operatorname{supp} K \circ \operatorname{supp} u$, $u \in C_0^\infty(X_2)$.*

Proof. Let us suppose that $x_1 \notin (\operatorname{supp} K \circ \operatorname{supp} u)$. Then there exists a neighborhood V of x_1 such that $V \cap (\operatorname{supp} K \circ \operatorname{supp} u) = 0$ because the set $\operatorname{supp} K \circ \operatorname{supp} u$ is closed. If $v \in C_0^\infty(V)$, then

$$(\operatorname{supp}(v \otimes u)) \cap \operatorname{supp} K = 0,$$

and therefore $\langle Au, v\rangle = 0$, i.e. $Au = 0$ on V, and $x_1 \notin \operatorname{supp} Au$. □

3.2. Proper integral operators. Let E and F be topological spaces and f be a continuous mapping of E into F. The mapping f is proper if for every compact set $K \subset F$ the set $f^{-1}(K)$ is compact in E.

Definition 3.6. Let X and Y be open sets in $\mathbb{R}^n$. An integral operator $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ is proper if the mappings $\pi_1 : \operatorname{supp} K_A(x, y) \to X$ and $\pi_2 : \operatorname{supp} K_A(x, y) \to Y$ are proper, where $K_A(x, y)$ is the kernel of A and π_1 and π_2 are the first and the second projection, respectively.

Proposition 3.7. *An integral operator $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ is proper if and only if distributions $K_A(x, y)\varphi(y)$ and $K_A(x, y)\phi(x)$ have compact supports in $X \times Y$ for arbitrary functions $\phi \in C_0^\infty(Y)$ and $\varphi \in C_0^\infty(X)$.*

Proof. Let A be a proper integral operator, $\phi \in C_0^\infty(Y)$ and $\varphi \in C_0^\infty(X)$. Since

$$\operatorname{supp} K_A(x, y)\varphi(y) \subset \operatorname{supp} K_A(x, y) \cap \pi_2^{-1}(\operatorname{supp} \varphi(y)),$$

it follows that $\operatorname{supp} K_A(x, y)\varphi(y)$ is a compact set. Analogously $K_A(x, y)\phi(y) \in \mathcal{E}'(X \times Y)$.

Assume that for every $\phi \in C_0^\infty(Y)$ and $\varphi \in C_0^\infty(X)$ the distributions $K_A(x, y)\varphi(y)$ and $K_A(x, y)\phi(y)$ belong to $\mathcal{E}'(X \times Y)$. We will show that for arbitrary compact sets K_1 and K_2 of X and Y, respectively, the sets

$$\operatorname{supp} K_A \cap \pi_2^{-1}(K_2) \text{ and } \operatorname{supp} K_A \cap \pi_1^{-1}(K_1)$$

are compact in $X \times Y$. Let $\phi \in C_0^\infty(Y)$ and $\phi(y) = 1$ in some neighborhood of the set K_2. It follows

$$\operatorname{supp} K_A \cap \pi_2^{-1}(K_2) \subset \operatorname{supp} K_A(x, y)\phi(y),$$

which implies the compactness of the set $\operatorname{supp} K_A \cap \pi_2^{-1}(K_2)$. Analogously one can prove the compactness of the set $\operatorname{supp} K_A \cap \pi_1^{-1}(K_1)$. □

Proposition 3.8. *If an integral operator A is proper, then its transpose operator ${}^tA$ is proper, as well.*

Proof. Theorem 3.4 implies that there exists $K_A(x,y) \in \mathcal{D}'(X \times Y)$ and $K_{{}^tA}(y,x) \in \mathcal{D}'(Y \times X)$, such that

$$\langle Au, v\rangle = \langle K_A(x,y), u(y)v(x)\rangle$$
$$\langle {}^tAv, u\rangle = \langle K_{{}^tA}(y,x), v(x)u(y)\rangle,$$

for every $u \in C_0^\infty(Y)$ and $v \in C_0^\infty(X)$

Since $\langle Au, v\rangle = \langle u, {}^tAv\rangle$, it follows

$$\langle K_A(x,y), u(y)v(x)\rangle = \langle K_{{}^tA}(y,x), v(x)u(y)\rangle,$$

i.e. $K_A(x,y) = K_{{}^tA}(y,x)$ in $\mathcal{D}(X,Y)$. Thus it follows that ${}^tA$ is a proper operator if A is a proper operator. □

Example 3.2. Let $P : C_0^\infty(Y) \to \mathcal{D}'(X)$ be a continuous linear operator. Let $(\phi_j)_{j\in J}$, and $(\varphi_i)_{i\in I}$ be sequences in $C_0^\infty(X)$ and $C_0^\infty(Y)$ respectively. Let the families of sets $(\operatorname{supp}\phi_j)_{j\in J}$ and $(\operatorname{supp}\varphi_i)_{i\in I}$ be locally finite. (A family $(A_\alpha)_{\alpha\in\Lambda}$ of subsets of $\mathbb{R}^n$ is locally finite if for every $x \in \mathbb{R}^n$ and a bounded neighbourhood B of X, $B \cap A_\alpha \neq \emptyset$ only for finitely many $\alpha \in \Lambda$.) The mapping $u \mapsto Qu$, where

$$(Qu)(x)=\sum_{j\in J}\phi_j(x)P(\varphi_j(y)u(y))(x),\ u \in C_0^\infty(Y),\ x \in X \tag{3.3}$$

is a proper integral operator.

Because of the local finiteness of the family $(\phi_j)_{j\in J}$ the above sum is finite for every fixed x. One can simply check that $Q : C_0^\infty(Y) \to \mathcal{D}'(X)$ is an integral operator. Let us show that it is proper. Let $\psi \in C_0^\infty(X)$. Since P is an integral operator, Theorem 3.4 implies that there exists a kernel $K_P(x,y) \in \mathcal{D}'(X \times Y)$, such that

$$\begin{aligned}\langle (Qu)(x), \psi(x)\rangle &= \Big\langle \sum_{j\in J}\phi_j(x)\langle K_P(x,y), \varphi_j(y)u(y)\rangle, \psi(x)\Big\rangle\\ &= \sum_{j\in J}\langle\langle K_P(x,y), \varphi_j(y)u(y)\rangle, \phi_j(x)\psi(x)\rangle\\ &= \sum_{j\in J}\langle K_P(x,y), \varphi_j(y)u(y)\phi_j(x)\psi(x)\rangle\\ &= \Big\langle \sum_{j\in J}K_P(x,y)\varphi_j(y)\phi_j(x), u(y)\psi(x)\Big\rangle.\end{aligned}$$

Here we have used the fact that the sums are finite. The kernel of the integral operator Q equals

$$\sum_{j\in J}K_P(x,y)\varphi_j(y)\phi_j(x).$$

As $\varphi \in C_0^\infty(Y)$ (analogously $\phi \in C_0^\infty(X)$) the set $\operatorname{supp}\sum_{j\in J}K_P(x,y)\varphi_j(y)\phi_j(x)\varphi(y)$ $(\operatorname{supp}\sum_{j\in J}K_P(x,y)\varphi_j(y)\phi_j(x)\phi(x))$ is compact, since the sum is finite. From Theorem 3.7 it follows that Q is a proper integral operator.

Note that (3.3) is well defined for $u \in C^\infty(Y)$.

Proposition 3.9. *If $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ is a proper integral operator, with the kernel K_A and if $u \in C_0^\infty(Y)$, then*

$$\operatorname{supp}(Au) \subset (\operatorname{supp} K_A) \circ (\operatorname{supp} u) \tag{3.4}$$

and $(\operatorname{supp} K_A) \circ (\operatorname{supp} u)$ is compact.

Proof. By Proposition 3.5, $\operatorname{supp} Au \subset \operatorname{supp} K_A \circ \operatorname{supp} u$. We have

$$\langle Au, \psi\rangle = \langle K_A(x,y), u(y)\psi(x)\rangle = \langle K_A(x,y)u(y), \psi(x)\rangle.$$

Let us denote $T = \operatorname{supp} K_A$, $R = \operatorname{supp} u$. Since R is a compact set, it follows that $T \circ R$ is a closed set. Let $W = Y \setminus (T \circ R)$ and assume $\psi \in C_0^\infty(W)$. This means that $T \cap (\operatorname{supp}\psi \times R) = \emptyset$. The kernel theorem and the fact $\langle Au, \psi\rangle = 0$ imply that (3.4). Let us prove that $T \circ R$ is a compact set. From (3.2) it follows

$$(\operatorname{supp} K_A) \circ (\operatorname{supp} u) = \pi_1(\operatorname{supp} K_A \cap \pi_2^{-1}(\operatorname{supp} u)).$$

The set $\operatorname{supp} K_A \cap \pi_2^{-1}(\operatorname{supp} u)$ is compact, since $\operatorname{supp} u$ is a compact set and $\pi_2 : \operatorname{supp} K_A \to Y$ is a proper mapping. Therefore $\pi_1(\operatorname{supp} K_A \cap \pi_2^{-1}(\operatorname{supp} u))$ is a compact set as a continuous image of a compact set. □

Theorem 3.10. *If $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ is a proper integral operator, then it can be continuously and linearly extended to an operator $A : C^\infty(Y) \to \mathcal{D}'(X)$.*

Proof. Let $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ be a proper integral operator, $u \in C_0^\infty(Y)$, $v \in C_0^\infty(X)$, by Theorem 3.4, there exists $K_A(x,y) \in \mathcal{D}'(X \times Y)$ such that

$$\langle Au, v\rangle = \langle K_A(x,y), u(y)v(x)\rangle.$$

Let $\{\varphi_j\}_{j\in J\subset\mathbf{N}}$ be a partition of unity with the properties

(1) $\varphi_j \subset C_0^\infty(X \times Y)$, $j \in J$, and the collection of supports $\{\operatorname{supp}\varphi_j\}_{j\in J}$ is locally finite,

(2) $\sum_{j\in J} \varphi_j(x,y) = 1$ for every$(x,y) \in X \times Y$,

(3) $\varphi_j(x,y) \geq 0$ for every$(x,y) \in X \times Y$ and $j \in J$.

Let

$$\kappa(x,y) = \sum_{j\,:\,\operatorname{supp}\varphi_j\cap\operatorname{supp} K_A\neq\emptyset} \varphi_j(x,y)$$

Clearly, $\kappa(x,y) \in C^\infty(X \times Y)$. Define the operator $\tilde{A} : C^\infty(Y) \to \mathcal{D}'(X)$ by

$$\langle \tilde{A}u(x), v(x)\rangle = \langle K_A(x,y), \kappa(x,y)u(y)v(x)\rangle, \ u \in C^\infty(Y), v \in C^\infty(X).$$

The set $\operatorname{supp} \kappa(x,y)u(y)v(x)$ is compact. Namely $\operatorname{supp} K_A(x,y)v(x)$ is compact and it implies that a family of functions φ_j such that $\operatorname{supp}\varphi_j \cap \operatorname{supp} K_A \neq \emptyset$, is finite. Therefore $\tilde{A}$ is well defined. From the definition it follows that $\tilde{A}$ is a continuous linear operator. Also, if $u \in C_0^\infty(Y)$, then

$$\langle \tilde{A}u(x), v(x)\rangle = \langle K_A(x,y)\kappa(x,y), u(y)v(x)\rangle = \langle K_A(x,y), u(y)v(x)\rangle,$$

since $\kappa = 1$ on $\operatorname{supp} K_A$. We conclude that $\tilde{A}$ is a linear continuous extension of the operator A. □

Theorem 3.11. *An integral operator $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ is proper if and only if:*

(1) *For every compact subset M of Y there exists a compact subset M_1 of X such that if $\operatorname{supp} u \subset M$ then $\operatorname{supp} Au \subset M_1$, where $u \in C_0^\infty(Y)$.*
(2) *For every compact subset L of X there exists a compact subset S of Y such that if $\operatorname{supp} v \subset L$, then $\operatorname{supp} {}^t\!Au \subset S$, where $v \in C_0^\infty(X)$.*

Proof. Let us prove that condition (2) is equivalent with the following one

(2*) For every compact subset L of X there exists a compact subset S of Y such that if $u = 0$ on S, then $Au = 0$ on L for $u \in C_0^\infty(Y)$.

Assume that (2*) does not hold, i.e. there exists a compact set L_0 such that for every compact set $\tilde{S}$ there exits $u \in C_0^\infty(Y)$ such that $\operatorname{supp} u \subset Y \setminus \tilde{S}$, $\langle Au, \tilde{v}\rangle \neq 0$ for some $\tilde{v}$ with $\operatorname{supp} \tilde{v} \subset L_0$. Let (2) holds and let S_1 be related to the set L_0 by condition (2). For every $v \in C_0^\infty(X)$, with $\operatorname{supp} v \subset L_0$, it follows that support of ${}^t\!Av$ is in S_1. Let $u \in C_0^\infty(Y)$ and let the support of u be in the complement of S_1. We should have that $\langle \tilde{v}, Au\rangle \neq 0$ for some $\tilde{v} \in C_0^\infty$ with support in L_0, but it is not true, since $\langle \tilde{v}, Au\rangle = \langle {}^t\!A\tilde{v}, u\rangle$ and $\langle {}^t\!A\tilde{v}, u\rangle = 0$, for every $\tilde{v}$ with $\operatorname{supp} \tilde{v} \subset L_0$.

Analogously one can prove that (2*) implies (2).

Let us suppose (1) and (2*). We will show that the mapping $\pi_2 : \operatorname{supp} K_A(x,y) \to Y$ is proper. Suppose that M is an arbitrary compact subset of Y and N is a compact subset of X, which is related to the first one by (1). Then we will prove

$$\pi_2^{-1}(M) \cap \operatorname{supp} K_A \subset N \times M. \tag{3.5}$$

Let $(x_0, y_0) \in (X \setminus N) \times M$, and let a function $\omega(x,y) = v(x)u(y)$ be such that $\operatorname{supp} v \subset X \setminus N$, $\operatorname{supp} u \subset M$, $\omega \neq 0$ in some neighborhood of the point (x_0, y_0) and $\omega \in C_0^\infty(X \times Y)$. We have $\langle K_A, \omega\rangle = \langle Au(x), v(x)\rangle = 0$ which implies that $(x_0, y_0) \notin \pi_2^{-1}(M) \cap (\operatorname{supp} K_A)$. This implies (3.5). The proof that the mapping $\pi_1 : \operatorname{supp} K_A(x,y) \to X$ is proper is similar

Let A be a proper integral operator. Condition (1) follows immediately from the properties of a proper integral operator and condition (2) follows from the fact that ${}^t\!A$ is a proper integral operator. □

3.3 Smoothing operators.

Definition 3.12. A continuous linear operator $A : \mathcal{E}'(X_2) \to C^\infty(X_1)$, X_1 and X_2 are open in $\mathbb{R}^n$, is called a smoothing operator.

If a distribution $K(x_1, x_2)$ belongs to the space $C^\infty(X_1 \times X_2)$, then the operator A defined on $\mathcal{E}'(X_2)$ by

$$(A(u(x_2)))(x_1) = \langle K(x_1, x_2), u(x_2)\rangle, \quad x_1 \in X_1,\ u \in \mathcal{E}'(X_2)$$

is a smoothing operator. To prove it we need the following lemma.

Lemma 3.13. *The lineal L of the set of translations of δ-distribution ($L = \{\sum_{i=1}^{p} a_i\delta(x - x_i),\ a_i \in \mathbb{C},\ x_i \in X\}$) is dense in the space $\mathcal{E}'(X)$.*

Proof. We will show the assertion for $n = 1$ and $X = \mathbb{R}$. For $n > 1$ and $X \subset \mathbb{R}^n$ the proof is analogous. Let $\phi \in C_0^\infty(\mathbb{R})$. We have

$$\langle \phi, \psi \rangle = \int_\alpha^\beta \phi(x)\psi(x)\,dx,$$

for every $\psi \in C^\infty(\mathbb{R})$. The integral on the right-hand side is equal to the limit value of Riemann's sum i.e.

$$\langle \phi, \psi \rangle = \lim_{n\to\infty} \sum_{i=1}^{n} \phi(x_i)\psi(x_i)\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i\langle \delta(x - x_i), \psi(x)\rangle,$$

where $a_i = \phi(x_i)\Delta x_i$. This implies that $\sum_{i=1}^{n} a_i\delta(x - x_i)$ converges to $\phi \in C_0^\infty(\mathbb{R})$ in $\mathcal{E}'(\mathbb{R})$, i.e. that the set of finite linear combinations of delta distributions is dense in $C_0^\infty(\mathbb{R})$. Since $C_0^\infty(\mathbb{R})$ is dense in $\mathcal{E}'(\mathbb{R})$, it follows that this set is dense in $\mathcal{E}'(\mathbb{R})$. □

Theorem 3.14. *An operator $A : \mathcal{E}'(X_2) \to C^\infty(X_1)$ is a smoothing operator if and only if there exists a distribution $K(x_1, x_2) \in C^\infty(X_1 \times X_2)$ such that*

$$(A(u(x_2)))(\cdot) = \langle K(\cdot, x_2), u(x_2)\rangle,\ u \in \mathcal{E}'(X_2).$$

Proof. Let $A : \mathcal{E}'(X_2) \to C^\infty(X_1)$ be a smoothing operator. Denote

$$K(x_1, a) = A(\delta(\cdot - a))(x_1), a \in X_2,\ x_1 \in X_1.$$

Let a be fixed and $K(x_1, a)$ be a function of x_1. It is an element of $C^\infty(X_1)$. We will show that for every fixed $x_1 \in X_1$, $K(x_1, \cdot)$ is a function in $C^\infty(X_2)$. This will imply $K(x_1, x_2) \in C^\infty(X_1 \times X_2)$. Thus, let x_1 be fixed, $\{a_n\}_{n\in\mathbb{N}} \subset X_2$ and $\lim_{n\to\infty} a_n = a \in X_2$. Then

$$\lim_{n\to\infty} A(\delta(x_2 - a_n))(x_1) = A(\delta(x_2 - a))(x_1)$$

(which is equivalent to $\lim_{n\to\infty} K(x_1, a_n) = K(x_1, a)$), because of the continuity of A and the fact that $\delta(x_2 - a_n) \to \delta(x_2 - a)$ in $\mathcal{E}'(X_2)$ as $n \to \infty$. Therefore, $K(x_1, a)$ is continuous with respect to the variable a. We have

$$\begin{aligned}\frac{K(x_1, a + h) - K(x_1, a)}{h} &= \frac{A(\delta(x_2 - a - h))(x_1) - A(\delta(x_2 - a))(x_1)}{h}\\ &= A\Big(\frac{\delta(x_2 - a - h) - \delta(x_2 - a)}{h}\Big)(x_1).\end{aligned}$$

Since

$$\frac{\delta(x_2 - a - h) - \delta(x_2 - a)}{h} \to \delta'(x_2 - a) \text{ in } \mathcal{E}'(X_2), \quad h \to 0,$$

the continuity of A implies

$$\lim_{h\to 0} \frac{K(x_1, a + h) - K(x_1, a)}{h} = A(\delta'(x_2 - a)).$$

Analogously one can continue the proof for all the derivatives. This means that the mapping

$$(x_1, x_2) \mapsto K(x_1, x_2) = A(\delta(t - x_2))(x_1)$$

is in $C^\infty(X_1 \times X_2)$

It remains to prove that $K_A = K$, where K_A is the kernel of A. Since $\langle K_A(x_1, x_2), u(x_2)\rangle \in C^\infty(X_1)$, it is enough to prove

$$\langle K_A(x_1, x_2), u(x_2)\rangle = \int_{X_2} K(x_1, x_2)u(x_2)\,dx_2, \quad u \in C_0^\infty(X_2).$$

As we have shown in Lemma 3.13, L is dense in $C_0^\infty(X_2)$. Thus, there exists $\sum_{i=1}^{p_r} a_i^r \delta(x_2 - x_{2i}^n)$ in L which converges to u in $\mathcal{E}'(X_2)$ as $r \to \infty$. From above it follows $(x_1 \in X_1)$

$$\begin{aligned}
\langle K_A(x_1, x_2), u(x_2)\rangle &= \lim_{n\to\infty} A\Big(\sum_{i=1}^{p_n} a_i^n \delta(x_2 - x_{2i}^n)\Big)(x_1) \\
&= \lim_{n\to\infty} \sum_{i=1}^{p_n} a_i^n K(x_1, x_{2i}^n) = \lim_{n\to\infty} \Big\langle K(x_1, x_2), \sum_{i=1}^{p_n} a_i^n \delta(x_2 - x_{2i}^n)\Big\rangle \\
&= \Big\langle K(x_1, x_2), \lim_{n\to\infty} \sum_{i=1}^{p_n} a_i^n \delta(x_2 - x_{2i}^n)\Big\rangle = \langle K(x_1, x_2), u(x_2)\rangle \\
&= \int_{X_2} K(x_1, x_2)u(x_2)\,dx_2 \qquad \Box
\end{aligned}$$

4. Oscillatory integrals

The notion of oscillatory integral is the crucial one for the theory of pseudo-differential and Fourier integral operators.

In oder to explain the oscillatory integrals we will consider the definition of generalized Fourier transformation of continuous functions $u(x)$ for which there exists positive real number c and $m \in \mathbb{N}$ such that

$$|u(x)| \le c(1 + |x|)^m, \ x \in \mathbb{R}^n. \tag{4.1}$$

In other words we will give the meaning to the right-hand side of equality

$$\langle \hat{u}, \phi\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\xi}u(x)\phi(\xi)\,dx\,d\xi, \ \phi \in \mathcal{S}(\mathbb{R}^n), \tag{4.2}$$

when a continuous function u satisfies (4.1). Later on we shall give a method which will be applied in the general case.

Let $k \in \mathbb{N}$ and $\psi \in \mathcal{S}$ If $u \in \mathcal{S}$, then the integral (4.2) makes sense, since

$$e^{-ix\xi} = (1 + |x|^2)^{-k}(1 - D_{\xi_1}^2 - \cdots - D_{\xi_n}^2)^k e^{-ix\xi}.$$

Then we have

$$\langle \hat{u}, \psi \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x)\psi(\xi)(1+|x|^2)^{-k}(1 - D^2_{\xi_1} - \cdots - D^2_{\xi_n})^k e^{-ix\xi}\, dx\, d\xi.$$

The integration by parts implies

$$(4.3) \quad \langle \hat{u}, \psi \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\xi}(1+|x|^2)^{-k} u(x)(1 - D^2_{\xi_1} - \cdots - D^2_{\xi_n})^k \psi(\xi)\, dx\, d\xi.$$

The right-hand side of (4.3) is defined not only when $u \in \mathcal{S}(\mathbb{R}^n)$ but as well as when u satisfies (4.1) and $k > m + n$.

Let us suppose (4.1). Since $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is the isomorphism, it follows $\hat{u}(\xi) \in \mathcal{S}'(\mathbb{R}^n)$. Let $\phi \in C_0^\infty(\mathbb{R}^n)$, $\phi(0) = 1$ and

$$I_{\phi,\varepsilon}(\psi) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\xi} \phi(\varepsilon x) u(x) \psi(\xi)\, dx\, d\xi, \ \psi \in \mathcal{S}(\mathbb{R}^n), \ \varepsilon > 0.$$

where the integral on the right-hand side converges because of (4.1). Analogously as above, for $k \in \mathbb{N}_0^n$, we obtain

$$I_{\phi,\varepsilon} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\xi} \phi(\varepsilon x)(1+|x|^2)^{-k} u(x)(1 - D^2_{\xi_1} - \cdots - D^2_{\xi_n})^k \psi(\xi)\, dx\, d\xi.$$

Let $k > m + n$. By the Lebesgue theorem, it follows that there exists $I \in \mathbb{R}$ such that $\lim_{\varepsilon \to 0} I_{\phi,\varepsilon} = I$. Note that the integral in (4.3) does not depend on k for which $k > m + n$. We define the mapping $\mathcal{S}(\mathbb{R}^n) \ni \psi \mapsto \langle \hat{u}, \psi \rangle = I(\psi)$ which gives the definition of $\hat{u}$ as an element of $\mathcal{S}'(\mathbb{R})$.

4.1. Space of symbols $S^m_{\rho,\delta}(X, \mathbb{R}^N)$. Let X be an open set in $\mathbb{R}^n$ and let (formally)

$$(4.4) \qquad I_\phi(au) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} e^{i\phi(x,\xi)} a(x,\xi) u(x)\, dx d\xi, \ u \in C_0^\infty(X),$$

where functions ϕ and a are the phase function and the symbol defined as follows.

Definition 4.1. A real valued function ϕ which is of the class $C^\infty(X \times (\mathbb{R}^N \backslash \{0\}))$ positively homogeneous of order 1 with respect to the variable ξ (i.e. $\phi(x, t\xi) = t\phi(x,\xi)$ for every $x \in \mathbb{R}^n, \xi \in \mathbb{R}^N, t \in \mathbb{R}, \ t > 0$) and which does not have characteristic points on $X \times (\mathbb{R}^N \backslash \{0\})$ (i.e. $0 \neq d\phi(x,\xi) = (\phi_{x_1}, \cdots, \phi_{x_n}, \phi_{\xi_1}, \cdots, \phi_{\xi_N})$ for $\xi \neq 0$), is called a phase function.

Definition 4.2. Let $m, \rho, \delta \in \mathbb{R}$, $0 < \rho \leq 1$, $0 \lesssim \delta < 1$.

Elements of the space $S^m_{\rho,\delta}(X, \mathbb{R}^N)$, which are called symbols, are functions $a(x,\xi) \in C^\infty(X \times \mathbb{R}^N)$ such that for arbitrary multi-indices α and β and arbitrary compact set $K \subset X$ there exists a constant $c_{\alpha,\beta,K} > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq c_{\alpha,\beta,K}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}, \ x \in K, \ \xi \in \mathbb{R}^N$$

Example 4.1. $(1+|\xi|)^m \in S^m_{1,0}$.

We will use the following notations

$$S^m(X,\mathbb{R}^N) = S^m_{1,0}(X,\mathbb{R}^N), \quad S^m_{\rho,\delta} = S^m_{\rho,\delta}(X,\mathbb{R}^N),$$
$$S^{\infty}_{\rho,\delta}(X,\mathbb{R}^N) = \bigcup_m S^m_{\rho,\delta}(X,\mathbb{R}^N), \quad S^{-\infty}_{\rho,\delta}(X,\mathbb{R}^N) = \bigcap_m S^m_{\rho,\delta}(X,\mathbb{R}^N).$$

The space S^m is called the space of standard symbols.

Let us introduce the topology in the space $S^m_{\rho,\delta}$. Suppose that $(K_\nu)_{\nu\in\mathbf{N}}$ is a sequence of compact sets such that

$$K_1 \subset K_2 \cdots \subset K_\nu \subset \cdots \subset X, \quad \bigcup_{\nu=1}^{\infty} K_\nu = X.$$

For $a(x,\xi) \in S^m_{\rho,\delta}$ define

$$\|a(x,\xi)\|_\nu = \sup_{x\in K_\nu,\xi\in\mathbb{R}^N,|\alpha|<\nu,|\beta|<\nu} |\partial^\alpha_\xi \partial^\beta_x a(x,\xi)|(1+|\xi|)^{-m+\rho\alpha-\delta\beta}.$$

It is clear that $\|\cdot\|_\nu$, $\nu \in \mathbb{N}$ is a growing sequence of seminorms; it defines the topology on $S^m_{\rho,\delta}$ such that $S^m_{\rho,\delta}$ is Freshet's space.

One can simply prove:

Proposition 4.3. *If $a \in S^m_{\rho,\delta}(X,\mathbb{R}^n)$, then $\partial^\alpha_\xi\partial^\beta_x a \in S^{m-\rho|\alpha|+\delta|\beta|}_{\rho,\delta}(X,\mathbb{R}^N)$. If $a \in S^m_{\rho,\delta}(X,\mathbb{R}^N)$ and $b \in S^{m'}_{\rho,\delta}(X,\mathbb{R}^N)$ then $a\cdot b \in S^{m+m'}_{\rho,\delta}(X,\mathbb{R}^N)$.*

The right-hand side in (4.4), where $a(x,\xi) \in S^m_{\rho,\delta}(X,\mathbb{R}^N)$ and $\phi(x,\xi)$, is a phase function, is called an oscillatory integral. Our aim will be to give the meaning to the integral, which in the general case does not converge absolutely.

Theorem 4.4. *Let $\phi(x,\xi)$, $(x,\xi) \in X \times \mathbb{R}^N$, be a phase function. There exists an operator*

$$L = \sum_{j=1}^{N} a_j(x,\xi)\frac{\partial}{\partial \xi_j} + \sum_{k=1}^{n} b_k(x,\xi)\frac{\partial}{\partial x_k} + c(x,\xi) \tag{4.5}$$

such that $a_j(x,\xi) \in S^0(X,\mathbb{R}^N)$, $b_k(x,\xi), c(x,\xi) \in S^{-1}(X,\mathbb{R}^N)$ and that for its transpose operator (determined by $\int(L\varphi)\psi = \int \varphi({}^tL\psi)$, $\varphi,\psi \in C_0^\infty$)

$${}^tLu(x,\xi) = -\sum_{j=1}^{N}\frac{\partial}{\partial\xi_j}(a_j u) - \sum_{k=1}^{n}\frac{\partial}{\partial x_k}(b_k u) + c(x,\xi)$$

there holds ${}^tLe^{i\phi} = e^{i\phi}$.

Note that the operator L is not uniquely determined.

Proof. Since $\frac{\partial}{\partial\xi_j}e^{i\phi} = i\frac{\partial\phi}{\partial\xi_j}e^{i\phi}$, $\frac{\partial}{\partial x_k}e^{i\phi} = i\frac{\partial}{\partial x_k}e^{i\phi}$, we have

$$\Big(\sum_{j=1}^{N} -i|\xi|^2\frac{\partial\phi}{\partial\xi_j}\frac{\partial}{\partial\xi_j} + \sum_{k=1}^{n} -i\frac{\partial\phi}{\partial x_k}\frac{\partial}{\partial x_k}\Big)e^{i\phi}$$

$$=\Big(\sum_{j=1}^{N}|\xi|^2\left|\frac{\partial\phi}{\partial\xi_j}\right|^2 + \sum_{k=1}^{n}\left|\frac{\partial\phi}{\partial x_k}\right|^2\Big)e^{i\phi} = \frac{e^{i\phi}}{\psi},$$

where $\psi(x,\xi) \in C^\infty(X \times (\mathbb{R}^n\backslash\{0\}))$ is a positively homogeneous of order -2 as a function of variable ξ. It follows

$$-i\psi\Big(\sum_{j=1}^{N}|\xi|^2\frac{\partial\phi}{\partial\xi_j}\frac{\partial}{\partial\xi_j} + \sum_{k=1}^{n}\frac{\partial\phi}{\partial x_k}\frac{\partial}{\partial x_k}\Big)e^{i\phi} = e^{i\phi},$$

and it remains only to take care of the singularity in $\xi = 0$.

Let $\kappa(\xi) \in C_0^\infty(\mathbb{R}^n)$ be such that $\kappa(\xi) = 1$ for $|\xi| < 1/4$ and $\kappa(\xi) = 0$ for $|\xi| > 1/2$. Let us define

$$M = -i(1-\kappa)\psi\Big[\sum_{j=1}^{N}|\xi|^2\frac{\partial\phi}{\partial\xi_j}\frac{\partial}{\partial\xi_j} + \sum_{k=1}^{n}\frac{\partial\phi}{\partial x_k}\frac{\partial}{\partial x_k}\Big] + \kappa.$$

Note $Me^{i\phi} = e^{i\phi}$. By using Proposition 4.3 one can prove that the coefficients of ${}^tM = L$ satisfy the asserted conditions. Since ${}^tM = L$, it follows ${}^tL = M$. □

For $m' > m$ we have $S^m_{\rho,\delta} \subset S^{m'}_{\rho,\delta}$ and the identity mapping $I : S^m_{\rho,\delta} \to S^{m'}_{\rho,\delta}$ is continuous.

Theorem 4.5. *Let $m' > m$ and let B be a bounded subset in $S^m_{\rho,\delta}$. The topologies in B induced by*

(a) topology of pointwise convergence on $S^{m'}_{\rho,\delta}$,

(b) the topology of the uniform convergence on compact sets (topology from $\mathcal{E}(X,\mathbb{R}^n)$) on $S^{m'}_{\rho,\delta}$ and

(c) the topology of the space $S^{m'}_{\rho,\delta}$ are the same.

Proof. We will give the proof of this assertion from [**15**]. Let us recall that a convergence satisfies the Urysohn condition if the following holds:

A sequence is convergent if and only if its every subsequence has a convergent subsequence.

It is obvious that all of the mentioned topologies are Hausdorff, that they fulfill the Urysohn axiom (because they are topological convergencies) and that the first two are weaker that the third one on B.

We will show that the set B is relatively compact in $S^{m'}_{\rho,\delta}$ (every sequence in B has a convergent subsequence in the sense of the convergence in $S^{m'}_{\rho,\delta}$). Since B is a bounded subset of $S^m_{\rho,\delta}$, a sequence $\{\phi_n\}_{n\in\mathbb{N}} \subset B$ is bounded in the sense of convergence in $\mathcal{E}(X \times \mathbb{R}^N)$. Therefore it has a convergent subsequence ϕ_{k_n} which converges to $\phi \in C^\infty(X \times \mathbb{R}^N)$.

Note that for every compact set K and $\alpha, \beta \in \mathbb{N}_0^n$

$$|\partial_\xi^\alpha \partial_x^\beta \phi_{k_n}(x,\xi)| \le c_{K,\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|},\ x \in K,\ \xi \in \mathbb{R}^N,$$

where $c_{K,\alpha,\beta}$ does not depend on the subsequence. It implies

$$|\partial_\xi^\alpha \partial_x^\beta \phi(x,\xi)| \le c_{K,\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|},\ x \in K,\ \xi \in \mathbb{R}^N.$$

Therefore $\phi \in S^m_{\rho,\delta}$. We have

$$(1+|\xi|)^{-m'+\rho|\alpha|-\delta|\beta|}|\partial_\xi^\alpha \partial_x^\beta(\phi_{k_n}(x,\xi) - \phi(x,\xi))| \le 2c_{K,\alpha,\beta}(1+|\xi|)^{m-m'},$$
$$x \in K,\ \xi \in \mathbb{R}^N,$$

for fixed compact set $K \subset X$, $\alpha \in \mathbb{N}_0^N$, $\beta \in \mathbb{N}_0^n$. Therefore, there exists $a > 0$ such that for $|\xi| > a$ the left-hand side of the inequality is less than $\varepsilon > 0$ independently of k_n.

For $|\xi| \le a$ the set $K \times \{\xi,\ |\xi| \le a\}$ is compact. Since the sequence ϕ_{k_n} converges to ϕ in the sense of convergence in $\mathcal{E}$, it follows

$$(1+|\xi|)^{-m'+\rho|\alpha|-\delta|\beta|}|\partial_\xi^\alpha \partial_x^\beta(\phi_{k_n}(x,\xi) - \phi(x,\xi))| < \varepsilon,$$

for some $n_0 \in \mathbb{N}, k_n > n_0, (x,\xi) \in K \times \{\xi,\ |\xi| \le a\}$. Thus, every sequence in B has a convergent subsequence in $S^{m'}_{\rho,\delta}$.

Now we will prove that (a) implies (c). Let a sequence in B be pointwisely convergent. We have proved that every subsequence of it has a convergent subsequence in $S^{m'}_{\rho,\delta}$. From Urysohn's condition follows the assertion. □

4.2. The oscillatory integral and its properties. Let $u \in C_0^\infty(X), a \in S^m_{\rho,\delta}(X \times \mathbb{R}^N)$, X is open in $\mathbb{R}^n$ and $m < -N$. Note, if $a \in S^m_{\rho,\delta}$ and $s = \min(\rho, 1-\delta)$, then the properties of L (cf. (4.5)) and a imply that there exists $C > 0$ such that

$$|L^k(a(x,\xi)u(x))| \le C(1+|\xi|)^{m-ks}, \quad x \in X, \xi \in \mathbb{R}^N.$$

With the above assumptions the integral on the right-hand side of (4.4) makes sense. Moreover

$$I_\phi(au) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\phi(x,\xi)} L^k(a(x,\xi)u(x))\,dx\,d\xi \tag{4.6}$$

and

$$|I_\phi(ua)| \le c\sup\{(1+|\xi|)^{-m}|a(x,\xi)|,\ x \in \operatorname{supp} u,\ \xi \in \mathbb{R}^N\},$$

where

$$c = \int_X |u(x)|\,dx \int_{\mathbb{R}^N}(1+|\xi|)^m\,d\xi.$$

This implies that $a \mapsto I_\phi(au)$ is a continuous mapping $S^m_{\rho,\delta} \to \mathbb{C}$. In the following theorem we shall show that this mapping has a continuous extension on $S^\infty_{\rho,\delta} = \bigcup_{m>0} S^m_{\rho,\delta}$. This extension is called the oscillation integral and it is denoted by

$$I_\phi(au) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\phi(x,\xi)}a(x,\xi)u(x)\,dxd\xi \text{ [osc]}. \tag{4.7}$$

Theorem 4.6. *Let $\rho \in (0,1]$, $\delta \in [0,1)$ and ϕ be a phase function. For a fixed $u \in C_0^\infty(X)$ define $I_\phi(\cdot u)$ by*

$$a \mapsto I_\phi(au) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\phi(x,\xi)}a(x,\xi)u(x)\,dxd\xi, \quad a \in S^\infty_{\rho,\delta} = \bigcup_{m,\rho,\delta} S^m_{\rho,\delta}(X,\mathbb{R}^N)$$

when that integral is absolutely convergent. Then $I_\phi(\cdot u)$ can be extended uniquely on the whole $S^\infty_{\rho,\delta}$ such that the mapping $u \mapsto I_\phi(au)$, $a \in S^m_{\rho,\delta}(X,\mathbb{R}^N)$, is continuous and linear (i.e. it is a distribution).

Proof. Let $\kappa(\xi) \in C_0^\infty(\mathbb{R}^N)$, $\kappa(\xi) = 1$ in a neighborhood of zero and $\kappa_\nu(\xi) = \kappa(\xi/\nu)$, $\nu \in \mathbb{N}$. The set $\{\kappa_\nu(\xi)a(x,\xi),\ \nu \in \mathbb{N}\}$ is bounded in $S^m_{\rho,\delta}(X,\mathbb{R}^N)$, therefore $\kappa_\nu(\xi)a(x,\xi)$ converges to $a(x,\xi)$ in $S^{m'}_{\rho,\delta}(X,\mathbb{R}^N)$, as $\nu \to \infty$ for $m' > m$. Also it converges pointwise. This follows from Theorem 4.5. The integral is absolutely convergent because κ_ν and u are compactly supported and therefore

$$\begin{aligned} I_\phi(a(x,\xi)\kappa_\nu(\xi)u(x)) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\phi(x,\xi)}a(x,\xi)\kappa_\nu(\xi)u(x)\,dx\,d\xi \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\phi(x,\xi)}L^k(a(x,\xi)\kappa_\nu(\xi)u(x))\,dxd\xi, \quad u \in C_0^\infty(X) \end{aligned} \tag{4.8}$$

(cf. (4.6)). It is clear that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\phi(x,\xi)}L^k(a(x,\xi)\kappa_\nu(\xi)u(x))\,dxd\xi$$

converges to

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\phi(x,\xi)}L^k(a(x,\xi)u(x))\,dxd\xi, \tag{4.9}$$

as $\nu \to \infty$, since $a(x,\xi)\kappa_\nu(\xi)$ converges to $a(x,\xi)$ in $S^{m'}_{\rho,\delta}(X,\mathbb{R}^N)$ and L^k maps $S^{m'}_{\rho,\delta}(X,\mathbb{R}^n)$ continuously in $S^{m'-ks}_{\rho,\delta}$, for $s = \min(\rho, 1-\delta)$. This implies the convergence of the integral in (4.8). Let us denote this limit by

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\phi(x,\xi)}a(x,\xi)u(x)\,dx\,d\xi \text{ [osc]}. \tag{4.10}$$

Since for fixed ν, $\mathcal{D} \ni u \mapsto I_\phi(a\kappa_\nu u)$ defines a distribution and $I_\phi(a\kappa_\nu u)$ converges to $I_\phi(au)$ for every $u \in \mathcal{D}$. By the sequential completeness of $\mathcal{D}'$, it follows that $u \mapsto I_\phi(au)$ is a distribution.

Therefore (4.10) is defined by (4.9). Clearly, in (4.8) the operator L can be substituted by any other one which has the properties as in Theorem 4.4 and we can take any k such that $m - ks < -N$. This implies that (4.10) does not depend on L and k, i.e.

$$\mathcal{D}(X) \ni u \mapsto I_\phi(au) = \int_{\mathbf{R}^n} \int_{\mathbf{R}^N} e^{i\phi(x,\xi)} a(x,\xi) u(x)\, dx d\xi \text{ [osc.]},$$

is an element of the space $\mathcal{D}'(X)$.

The same proof show that (4.8) does not depend on the choice of $\kappa_\nu(\xi)$ with the prescribed properties. □

Example 4.2. Let us show that

$$\delta(x) = (2\pi) \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} \cdot 1\, d\xi \text{ [osc.]}.$$

Note, $1 \in S^0_{\rho,\delta}$. Let $\kappa \in C_0^\infty(X)$, $\kappa(\xi) = 1$ in a neighborhood of zero and $u \in C_0^\infty$. Then $\kappa(\xi/t) \to 1$ in $S^m_{1,0}$ as $t \to \infty$ and

$$\begin{aligned}\Big\langle (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\xi}\, d\xi, u(x) \Big\rangle &= (2\pi)^{-n} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{ix\xi} u(x)\, dx\, d\xi \text{ [osc.]} \\ = \lim_{t\to\infty} (2\pi)^{-n} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{ix\xi} \kappa(\xi/t) u(x)\, dx\, d\xi &= \lim_{t\to\infty} \int_{\mathbf{R}^n} \kappa(\xi/t) \mathcal{F}^{-1}(u)(\xi)\, d\xi \\ &= \kappa(0) \int_{\mathbf{R}^n} \mathcal{F}^{-1}(u)(\xi)\, d\xi = u(0).\end{aligned}$$

We have used $\mathcal{F}(\mathcal{F}^{-1}(u(\xi))(x) = u(x)$, which implies $\mathcal{F}(\mathcal{F}^{-1}(u(\xi))(0) = u(0)$.

4.3. Singularities of an oscillatory integral. Let X be open in $\mathbb{R}^n$ and

$$C_\phi = \{(x,\xi),\ x \in X, \xi \in \mathbb{R}^N \backslash \{0\}, \phi_\xi(x,\xi) = 0\},\ S_\phi = \pi_1 C_\phi,\ R_\phi = X \backslash S_\phi,$$

where $\pi_1 : (X \times \mathbb{R}^N \backslash \{0\}) \to X$ being the first projection of the set $(X \times \mathbb{R}^N \backslash \{0\})$. Since S_ϕ is closed, R_ϕ is open.

The set C_ϕ is a cone with respect to ξ, because $\phi(x,\xi)$ is homogeneous function of ξ of order 1 and $\partial\phi/\partial\xi$ is homogeneous of order 0.

Theorem 4.7. *Denote by A the distribution defined by $\langle A, u\rangle = I_\phi(au)$, $u \in C_0^\infty(X)$. Then* $\operatorname{Sing\,supp} A \subset S_\phi$.

Recall, $\operatorname{Sing\,supp} A$ is the complement of the maximal open set where A is smooth.

Proof. We will show that there exists $A \in C^\infty(R_\phi)$ such that

$$I_\phi(au) = \int_X A(x) u(x)\, dx, \quad u \in C_0^\infty(R_\phi).$$

We shall show that there exists $L = \sum a_j(x,\xi)\frac{\partial}{\partial \xi_j} + c(x,\xi)$, where $a_j \in S^0(R_\phi, \mathbb{R}^N)$, $c \in S^{-1}(R_\phi, \mathbb{R}^N)$ such that $Le^{i\phi} = e^{i\phi}$. Put

$$M = -i(1-\kappa)\psi\Big[\sum_{j=1}^{N} |\xi|^2 \frac{\partial \phi}{\partial \xi_j}\frac{\partial}{\partial \xi_j}\Big] + \kappa.$$

where ψ satisfies $-i\psi\sum_{j=1}^N |\xi|^2 |\frac{\partial \phi}{\partial \xi_j}|^2 e^{i\phi} = e^{i\phi}$, $\xi \neq 0$, $\kappa \in C_0^\infty(\mathbb{R}^N)$ and $\kappa(\xi) = 1$ for $|\xi| < 1$. Then $Me^{i\phi} = e^{i\phi}$ and put $Le^{i\phi} = {}^t\!Me^{i\phi}$. Thus

$$Lu = \sum_{j=1}^{N} \frac{\partial}{\partial \xi_j}\Big(i(1-\kappa)\psi|\xi|^2\frac{\partial \phi}{\partial \xi}u\Big) + \kappa u.$$

Let $\kappa \in C_0^\infty(\mathbb{R}^n)$, $\kappa(0) = 1$ and $\kappa_\nu(\xi) = \kappa(\xi/\nu)$, $\nu \in \mathbb{N}$. Note, for every $K \subset\subset X$

$$|M^k a(x,\xi)| \leq C(1+|\xi|)^{m-k},\ \xi \in \mathbb{R}^n,\ x \in K.$$

$$\begin{aligned}
\langle A, u\rangle &= \lim_{\nu\to\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\phi(x,\xi)} L^k(\kappa_\nu(\xi)a(x,\xi)u(x))\,d\xi\,dx \\
&= \lim_{\nu\to\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^N} e^{i\phi(x,\xi)} (\kappa_\nu(\xi)a(x,\xi)u(x))\,d\xi\,dx \\
&= \int_{\mathbb{R}^n}(\int_{\mathbb{R}^N} e^{i\phi(x,\xi)} L^k a(x,\xi)\,d\xi)u(x)\,dx.
\end{aligned}$$

Therefore

$$A(x) = \int_{\mathbb{R}^N} e^{i\phi(x,\xi)} L^k a(x,\xi)\,d\xi\ [\text{osc}].$$

(It does not depend on k.) For large enough k the integral exists in ordinary sense and the function A is continuous. Moreover, we can differentiate $A(x)$ by differentiating the function under the integral sign. This is the consequence of the fact that $\phi(x,\xi)$ is a homogeneous function of ξ of order 1 as well as all its derivatives with respect to x. Note, if a function $r(\xi)$ is homogeneous of order 1, then

$$|r(\xi)| < \text{const}\cdot(1+|\xi|),\ \xi \in \mathbb{R}^N.$$

This implies that by taking large enough k differentiation under the integral is legitimate. Thus for any $p \in \mathbb{N}_0$ we have $A(x) \in C^p(R_\phi)$. □

Analogously one can prove:

Proposition 4.8. *If $a \in S^m_{\rho,\delta}(X, \mathbb{R}^N)$ and $a = 0$ in some conic neighborhood of the set C_ϕ, then $A \in C^\infty(X)$, where A is defined by $\langle A, u\rangle = I_\phi(au)$.*

5. Fourier integral operators

We shall give some introductory facts which are useful for the theory of pseudodifferential operators.

5.1. Definition and the basic properties. Let X and Y be open sets in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, $\rho > 0$, $\delta < 1$. Let

$$Au(x) = \int_Y \int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} a(x,y,\xi) u(y)\, dy\, d\xi, \ u \in C_0^\infty(Y), \ x \in X, \text{ [osc.]}$$

where $\phi(x,y,\xi)$ is a phase function on $(X\times Y)\times\mathbb{R}^N$ and $a(x,y,\xi) \in S^m_{\rho,\delta}(X\times Y, \mathbb{R}^N)$. Under these conditions the integral

$$(5.1) \qquad \langle Au, v\rangle = \int_X \int_Y \int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} a(x,y,\xi) u(y) v(x)\, dx\, dy\, d\xi, \quad v \in C_0^\infty(X)$$

is defined as an oscillatory integral. For fixed u the right-hand side in (5.1) defines a distribution $Au \in \mathcal{D}'(X)$ (see Theorem 4.6).

Remark 5.1. In the sequel we will not write explicitly [osc.] for integrals which are defined as oscillatory integrals. It will clear from the context.

Definition 5.1. An operator $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ defined by (5.1) is called a Fourier integral operator with a phase function $\phi(x,y,\xi)$ and an amplitude $a(x,y,\xi)$.

Every smoothing integral operator can be written in the form of a Fourier integral operator:

Theorem 5.2. *An integral operator $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ is a smoothing operator if and only if there exists a phase function $\phi(x,y,\xi)$ and amplitude $\tilde{a}(x,y,\xi) \in S^{-\infty}_{1,0}$ such that*

$$(5.2) \qquad Au(x) = \int_Y \int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} \tilde{a}(x,y,\xi) u(y)\, dy\, d\xi.$$

Proof. Let A be of the form (5.2). If $\tilde{a}(x,y,\xi) \in S^{-\infty}_{1,0}$ it is clear that the kernel of the operator

$$\int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} \tilde{a}(x,y,\xi)\, d\xi$$

is of the class $C^\infty(X \times Y)$.

Conversely, by Theorem 3.14 there exists $K(x,y) \in C^\infty(X \times Y)$ such that

$$\begin{aligned} Au(x) &= \langle K(x,y), u(y)\rangle = \int_Y K(x,y) u(y)\, dy \\ &= \int_Y \int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} (K(x,y) e^{-i\phi(x,y,\xi)} \kappa(\xi)) u(y)\, dy\, d\xi, \ u \in C_0^\infty, \ x \in X, \end{aligned}$$

where ϕ is an arbitrary phase function, $\kappa \in C_0^\infty(\mathbb{R}^N)$, $\int \kappa(\xi)\, d\xi = 1$ and $\kappa(\xi) = 0$ in some neighbourhood of zero.

Since $\tilde{a}(x,y,\xi) = K(x,y) e^{-i\phi(x,y,\xi)} \kappa(\xi) \in S^{-\infty}_{1,0}$, the assertion follows. $\square$

A distribution $K_A \in \mathcal{D}'(X \times Y)$ defined as the oscillatory integral

$$\langle K_A, w\rangle = \int_X \int_Y \int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} a(x,y,\xi) w(x,y)\, dx\, dy\, d\xi\, [\text{osc.}],$$

$w \in C_0^\infty(X \times Y)$, is the kernel of the operator A. It is the kernel of the operator A since

$$\langle Au, v\rangle = \langle K_A(x,y), u(y)v(x)\rangle, u \in C_0^\infty(Y), \quad v \in C_0^\infty(X).$$

Proposition 5.3. *Let A be a Fourier integral operator given by (5.1), and let K_A be its kernel. Then $K_A \in C^\infty(R_\phi)$, where*

$$R_\phi = \{(x,y),\ \forall \xi \in \mathbb{R}^N \backslash \{0\}, \phi'_\xi(x,y,\xi) \neq 0\}.$$

If $a(x,y,\xi) = 0$ in a conic neighbourhood of the set

$$C_\phi = \{(x,y,\xi),\ \phi'_\xi(x,y,\xi) = 0\},$$

then $K_A \in C^\infty(X \times Y)$.

Proof. It follows immediately from Theorem 4.7 and Proposition 4.8. □

Remark 5.2. Different pairs ϕ_1, a_1 and ϕ_2, a_2 may define the same operator A of the form (5.1). Moreover, a function $a(x,y,\xi)$ is not completely determined by the operator A, even when the phase function ϕ is fixed.

Let $A : C_0^\infty(X) \to \mathcal{D}'(X)$ be a Fourier integral operator given by (5.1). We shall evaluate the form of ${}^tA$ and A^*. Recall, ${}^tA : C_0^\infty(X) \to \mathcal{D}'(X)$ such that

$$\langle Au, v\rangle = \langle u, {}^tAv\rangle, \quad u \in C_0^\infty(X),\ v \in C_0^\infty(X)$$

i.e.

$$\langle Au, v\rangle = \int_X \int_Y \int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} a(x,y,\xi) u(y) v(x)\, dx\, dy d\xi = \langle u, {}^tAv\rangle.$$

We have

$$({}^tAv(x))(y) = \int_X \int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} a(x,y,\xi) v(x)\, dx\, d\xi,$$

for $y \in Y = X$. By the change of the variables $x \mapsto y$ and $y \mapsto x$, we obtain

$$({}^tAv(y))(x) = \int_Y \int_{\mathbf{R}^N} e^{i\phi(y,x,\xi)} a(y,x,\xi) v(y)\, dy\, d\xi. \tag{5.3}$$

Therefore, for $x \in X$

$$({}^tAv(y))(x) = \int_X \int_{\mathbf{R}^N} e^{i\tilde{\phi}(x,y,\xi)} \tilde{a}(x,y,\xi) v(y)\, dy\, d\xi. \tag{5.4}$$

(The above integrals are oscillatory integrals.) This proves

Proposition 5.4. *The phase function and the amplitude of ${}^tA$ are defined by $\tilde{\phi}(x,y,\xi) = \phi(y,x,\xi)$ and $\tilde{a}(x,y,\xi) = a(y,x,\xi)$.*

The operator A^* is determined by $\langle Au, v\rangle = (u, A^*v)$, $A^* : C_0^\infty(X) \to \mathcal{D}'(X)$. Therefore

$$\langle u, {}^tAv\rangle = \langle Au, v\rangle = (Au, \overline{v}) = (u, A^*\overline{v}) = \langle u, \overline{A^*\overline{v}}\rangle,$$

for $u \in C_0^\infty(X)$ and $v \in C_0^\infty(X)$ i.e.

$$(5.5)\qquad {}^tAv(x) = \overline{A^*\overline{v}}(x) = \int_Y \int_{\mathbf{R}^N} e^{-i\phi(y,x,\xi)}\overline{a(y,x,\xi)\overline{v(y)}}\, dy\, d\xi,$$

and for $y \in Y = X$

$$(5.6)\qquad (A^*v(x))(y) = \int_X \int_{\mathbf{R}^N} e^{i\varphi(x,y,\xi)} b(x,y,\xi) v(x)\, dx\, d\xi.$$

Proposition 5.5. *The phase function and the amplitude of A^* are given by $\varphi(x,y,\xi) = \phi(y,x,\xi)$ and $b(x,y,\xi) = \overline{a(y,x,\xi)}$.*

5.2. Fourier integral operator with operator phase function.

Definition 5.6. Phase function $\phi(x,y,\xi)$, $x \in X$, $y \in Y$, X, Y are open in $\mathbb{R}^n$, is an operator phase function if the following holds

$$(5.7)\quad \phi'_{y,\xi}(x,y,\xi) = (\phi_{y_1}, \dots, \phi_{y_n}, \phi_{\xi_1}, \dots, \phi_{\xi_n}) \neq 0 \text{ for } \xi \neq 0, x \in X, y \in Y,$$

$$(5.8)\qquad \phi'_{x,\xi}(x,y,\xi) \neq 0 \text{ for } \xi \neq 0, \quad x \in X, y \in Y.$$

Proposition 5.7. *If (5.7) holds then the operator $A : C_0^\infty(Y) \to \mathcal{D}'(X)$, determined by (5.1), continuously map $C_0^\infty(Y)$ into $C^\infty(X)$.*

Proof. From (5.7) it follows that $\phi(x,y,\xi)$, considered as function of (y,ξ), is a phase function (x is a parameter). By Theorem 4.7 there exists an operator L (which does not contain $\partial/\partial x$) such that ${}^tLe^{i\phi} = e^{i\phi}$. Analogously as in the proof of Theorem 4.7 (with operator L instead of M) we obtain

$$\begin{aligned}\langle Au, v\rangle &= \int_X \int_Y \int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} a(x,y,\xi) u(y) v(x)\, dx\, dy\, d\xi \\ &= \int_X \Big(\int_Y \int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} L^k(a(x,y,\xi)u(y))\, dy\, d\xi\Big) v(x)\, dx,\end{aligned}$$

for $u \in C_0^\infty(Y)$ and $v \in C_0^\infty(X)$. Therefore, as in Theorem 4.7

$$(Au(y))(x) = \int_Y \int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} L^k_{y,\xi}(a(x,y,\xi)u(y))\, dy\, d\xi,\ x \in X,$$

we can prove that Au is a smooth function. $\square$

Proposition 5.8. *If (5.8) holds, then the operator $A : C_0^\infty(Y) \to \mathcal{D}'(X)$, given by (5.2), can be linearly and continuously extended to $A : \mathcal{E}'(Y) \to \mathcal{D}'(X)$, where the topologies in $\mathcal{E}'(Y)$ and $\mathcal{D}'(X)$ are weak topologies.*

Proof. The transpose operator ${}^t\!A : C_0^\infty(X) \to \mathcal{D}'(Y)$ of the operator $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ is given by

$$({}^t\!Av(x))(y) = \int_X \int_{\mathbf{R}^N} e^{i\phi(x,y,\xi)} a(x,y,\xi) v(x)\, dx\, d\xi, v \in C_0^\infty(X).$$

From (5.8) by the previous theorem it follows ${}^t\!A : C_0^\infty(X) \to C^\infty(Y)$. Therefore ${}^t({}^t\!A) : \mathcal{E}'(Y) \to \mathcal{D}'(X)$. Since ${}^t({}^t\!A)|_{C_0^\infty(Y)} = A$ and ${}^t({}^t\!A) : \mathcal{E}'(Y) \to \mathcal{D}'(X)$ is a linear and continuous mapping, the assertion of the proposition follows. □

From the previous two proposition it follows

Theorem 5.9. *Let $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ be a Fourier integral operator with an operator phase function ϕ. Then*

a) $A : C_0^\infty(Y) \to C^\infty(X)$,

b) A can be linearly and continuously extended to $A : \mathcal{E}'(Y) \to \mathcal{D}'(X)$,

c) ${}^t\!A : C_0^\infty(X) \to C^\infty(Y)$,

d) ${}^t\!A$ can be linearly and continuously extended to ${}^t\!A : \mathcal{E}'(X) \to \mathcal{D}'(Y)$.

For the singular support the following estimation holds.

Theorem 5.10. *Let $A : \mathcal{E}'(Y) \to \mathcal{D}'(X)$ be a Fourier integral operator with an operator phase function ϕ. Then*

$$\operatorname{Sing\,supp} Au \subset S_\phi \circ \operatorname{Sing\,supp} u, u \in \mathcal{E}'(Y),$$

where $R_\phi = \{(x,y),\ \phi_\xi(x,y) \neq 0$ for every $\xi \in \mathbb{R}^N\backslash\{0\}\}$ and $S_\phi = (X \times Y)\backslash R_\phi$.

Proof. Let $u_1 \in \mathcal{E}'(U)$, where U is fixed neighbourhood of $K = \operatorname{Sing\,supp} u$ such that $u = u_1$ on some neighbourhood of $K \subset U$. Then for $u_2 = u_1 - u$ we have $\operatorname{supp} u_2 \subset Y \setminus K$. Since $u_2 \in C_0^\infty(Y)$ and $A : C_0^\infty(Y) \to C^\infty(X)$, it follows $Au_2 \in C^\infty(X)$. If we show that

$$\operatorname{Sing\,supp} Au_1 \subset M = S_\phi \circ \operatorname{supp} u_1, \tag{5.9}$$

it will means that $\operatorname{Sing\,supp} Au \subset \operatorname{Sing\,supp} Au_1 \subset M \subset S_\phi \circ U$. By letting $U \to K$, we will have

$$\operatorname{Sing\,supp} Au \subset S_\phi \circ K = S_\phi \circ \operatorname{Sing\,supp} u.$$

Let us prove (5.9). Let $K_0 = \operatorname{supp} u_1$, $K' \subset X$ such that $K' \times K_0 \subset R_\phi$ ($K' \subset X \setminus M$)) and let $X' \times X \subset R_\phi$ be a neighbourhood of $K' \times K_0$. We have

$$\langle Ah, k\rangle = I_\phi(ahk),$$

for $h \in C_0^\infty(X)$ and $k \in C_0^\infty(X')$. By Theorem 4.7

$$\operatorname{Sing\,supp} A \subset S_\phi.$$

It follows $A \in C^\infty(X' \times X)$. Therefore $\operatorname{Sing\,supp} Au_1 \subset X \setminus K'$, which implies the theorem. □

Example 5.1. Let
$$A = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha,$$
where $a_\alpha(x) \in C_0^\infty(X)$, $X \subset \mathbb{R}^n$. Using the Fourier transform we obtain
$$D^\alpha u(x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} \xi^\alpha e^{i(x-y)\xi} u(y)\, dy\, d\xi.$$
It implies,
$$Au(x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi} \sigma_A(x,\xi) u(y)\, dy\, d\xi,$$
where $\sigma_A = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$ and is called the symbol of the operator A. Since $\sigma_A(x,\xi) \in S^m(X \times \mathbb{R}^n)$, A is a Fourier integral operator.

Example A solution to the Cauchy problem

$$c^{-2}\frac{\partial^2 E}{\partial t^2} - \Delta E = 0,\ E(0,x) = 0,\ \frac{\partial}{\partial t}E(0,x) = \delta(x), \tag{5.10}$$
$E = E(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, is given by
$$(2\pi)^{-1}E(t,x) = \int \frac{e^{i(ct|\xi|+x\xi)} - e^{-i(ct|\xi|+x\xi)}}{2i|\xi|c}\, d\xi \text{ [osc]}. \tag{5.11}$$
Let us prove it. Applying the Fourier transformation on equation (5.10) we obtain
$$c^{-2}\frac{\partial^2 \tilde{E}}{\partial t^2} + |\xi|^2 \tilde{E}(t,\xi) = 0,$$
where $\tilde{E}(t,\xi) = \mathcal{F}(E(t,x))(\xi)$. Let us fix ξ. We obtain an ordinary differential equation (with respect to the variable t) which solution is $\tilde{E}(t,\xi) = c_1 e^{-itc|\xi|} + c_2 e^{itc|\xi|}$. It follows
$$\begin{aligned} E(0,x) = 0 &\Rightarrow \tilde{E}(0,\xi) = 0 \Rightarrow c_1 + c_2 = 0 \\ &\Rightarrow \frac{\partial}{\partial t}\tilde{E}(0,\xi) = 1 = \mathcal{F}(\delta(x)) \Rightarrow -c_1 + c_2 = 1/ic|\xi|. \end{aligned}$$

Therefore (5.11) holds.

Example 5.3. Pseudodifferential operators.

If $n_1 = n_2 = N = n$ and $X = Y$, then a Fourier integral operator with a phase function $\phi(x,y,\xi) = (x-y)\xi$ is called a pseudodifferential operator (ΨDO).

6. Pseudodifferential operators

Pseudodifferential operators generalizes differential and singular integral operators. In this section we shall analyze the basic properties of pseudodifferential operators.

6.1. Definition and the properties. Let X be an open set in $\mathbb{R}^n$; then a Fourier integral operator $A: C_0^\infty(X) \to \mathcal{D}'(X)$ given by

$$Au(x) = \int_X \int_{\mathbb{R}^n} e^{i(x-y)\xi} a(x,y,\xi) u(y)\, dy\, d\xi \tag{6.1}$$

is called a pseudodifferential operator, for short ΨDO.

Example 6.1. An example of a pseudodifferential operator which is not a differential operator is a singular operator in $\mathbb{R}^n$ given by

$$\begin{aligned} A &= a(x)u(x) + \text{v.p.} \int \frac{L(x,(x-y)/|x-y|)}{|x-y|^n} u(y)\, dy \\ &= a(x)u(x) + \lim_{\varepsilon \to 0} \int_{|y-x| \geq \varepsilon} \frac{L(x,(x-y)/|x-y|)}{|x-y|^n} u(y)\, dy, \end{aligned}$$

where $a \in C^\infty(\mathbb{R}^n)$, $L = L(x,\omega) \in C^\infty(\mathbb{R}^n \times S^{n-1})$ (S^{n-1} is a unit sphere in $\mathbb{R}^n$) such that

$$\int_{S^{n-1}} L(x,\omega) d\omega = 0, \; x \in \mathbb{R}^n.$$

With accuracy up to the operator with a smooth kernel, the operator A has an amplitude $a(x,\xi) = a(x) + \chi(\xi) g(x,y)$, where $\chi \in C^\infty(\mathbb{R}^n)$, $\chi(\xi) = 1$, for $|\xi| \geq 1$, $\chi(\xi) = 0$, for $|\xi| \leq 1/2$ and $g = \frac{1}{|x-y|^n} L\left(x, \frac{x-y}{|x-y|}\right)$.

Theorem 6.1. *Let $A: C_0^\infty(X) \to \mathcal{D}'(X)$ be a ΨDO, K_A be the kernel of the operator A and let Δ be the diagonal in $X \times X$. Then*

a) $K_A \in C^\infty((X \times X) \setminus \Delta)$.

b) *Operator A defines linear and continuous mappings $A: C_0^\infty(X) \to C^\infty(X)$, $A: \mathcal{E}'(X) \to \mathcal{D}'(X)$. If $u \in \mathcal{E}'(X)$, then* $\operatorname{Sing\,supp} Au \subset \operatorname{Sing\,supp} u$. *(This property is called the pseudolocality of the operator A.)*

c) *The operators ${}^tA$ and A^* define linear and continuous mappings*

$$\begin{aligned} &{}^tA : C_0^\infty(X) \to C^\infty(X), \quad {}^tA : \mathcal{E}'(X) \to \mathcal{D}'(X); \\ &A^* : C_0^\infty(X) \to C^\infty(X), \quad A^* : \mathcal{E}'(X) \to \mathcal{D}'(X). \end{aligned}$$

Proof. a) The phase function for a ΨDO A is $\phi(x,y,\xi) = (x-y)\xi$. Therefore $R_\phi = X \times X \setminus \Delta$, since $\phi_\xi = (x-y)$. By putting $X = Y$, Proposition 5.3 immediately implies $K_A \in C^\infty((X \times X)\setminus\Delta)$.

b) The following conditions are fulfilled for phase function of the operator A

$$\begin{aligned} \phi'_{y,\xi}(x,y,\xi) &= (-\xi_1, \dots, -\xi_n, x_1 - y_1, \dots, x_n - y_n) \neq 0, \\ \phi'_{x,\xi}(x,y,\xi) &= (\xi_1, \dots, \xi_n, x_1 - y_1, \dots, x_n - y_n) \neq 0, \end{aligned}$$

for $\xi \neq 0$, $x, y \in X$. Therefore $A: C_0^\infty(X) \to \mathcal{D}'(X)$ is a Fourier integral operator with the operator phase function. The assertions a) and c) follow from Theorem

5.9. Since $S_\phi = \Delta$, where S_ϕ is the set attached to the operator A determined in Theorem 5.10. From this theorem it follows that $\operatorname{Sing\,supp} Au \subset \Delta \circ \operatorname{Sing\,supp} u = \operatorname{Sing\,supp} u$.

c) Let $A : C_0^\infty(X) \to \mathcal{D}'(X)$ be a ΨDO given by (6.1). We shall evaluate the forms of ${}^tA : C_0^\infty(X) \to \mathcal{D}'(X)$ and $A^* : C_0^\infty(X) \to \mathcal{D}'(X)$. Note that these operators are again pseudodifferential operators and the assertion *c)* follows from *b)*

From (5.3) it follows

$$ {}^tAv(x) = (2\pi)^{-n} \int_X \int_{\mathbf{R}^n} e^{i(x-y)(-\xi)} a(y,x,\xi) v(y)\, dy\, d\xi, $$

for $v \in C_0^\infty(X)$. By changing of variables $-\xi \to \xi$, we obtain

$$ {}^tAv(x) = (2\pi)^{-n} \int_X \int_{\mathbf{R}^n} e^{i(x-y)\xi} a(y,x,-\xi) v(y)\, dy\, d\xi, $$

for $v \in C_0^\infty(X)$, i.e.

$$ (6.2) \qquad {}^tAv(x) = (2\pi)^{-n} \int_X \int_{\mathbf{R}^n} e^{i(x-y)\xi} \tilde{a}(x,y,\xi) v(y)\, dy\, d\xi, \quad v \in C_0^\infty(X). $$

where $\tilde{a}(x,y,\xi) = a(y,x,-\xi)$. From (5.5) it follows

$$ A^*v(x) = (2\pi)^{-n} \int_X \int_{\mathbf{R}^n} e^{i(x-y)\xi} \overline{a(y,x,\xi)} v(y)\, dy\, d\xi, \quad v \in C_0^\infty(X), $$

i.e.

$$ (6.3) \qquad A^*v(x) = (2\pi)^{-n} \int_X \int_{\mathbf{R}^n} e^{i(x-y)\xi} b(x,y,\xi) v(y)\, dy\, d\xi, \quad v \in C_0^\infty(X), $$

where $b(x,y,\xi) = \overline{a(y,x,\xi)}$. $\square$

Remark 6.1. Linear differential operators fulfills the condition of locality ($\operatorname{supp} Au \subset \operatorname{supp} u$, $u \in C_0^\infty(X)$), which for ΨDO's in general case do not hold.

6.2. Algebra of pseudodifferential operators and its symbols.

6.2.1. Proper pseudodifferential operators.

Definition 6.2. Pseudodifferential operator $A : C_0^\infty(X) \to \mathcal{D}'(X)$, X is open in $\mathbb{R}^n$, is proper if it is proper as an integral operator.

For example, linear differential operators (5.10) are proper ΨDO.

Theorem 6.3. *Let A be a proper ΨDO. Then, A defines linear and continuous mapping $A : C_0^\infty(X) \to C_0^\infty(X)$ which can be linearly and continually continued to mappings*

$$ A : \mathcal{E}'(X) \to \mathcal{E}'(X), \quad A : C^\infty(X) \to C^\infty(X), \quad A : \mathcal{D}'(X) \to \mathcal{D}'(X). $$

Proof. By Theorem 6.1 $A : C_0^\infty(X) \to C^\infty(X)$ and by Proposition 3.9

$$\operatorname{supp}(Au) \subset (\operatorname{supp} K_A) \circ (\operatorname{supp} u), \quad u \in C_0^\infty, \tag{6.4}$$

where K_A is the kernel of A. The set on the right-hand side of (6.4) is compact. This immediately implies $A : C_0^\infty(X) \to C_0^\infty(X)$. Continuity of the operator $A : C_0^\infty(X) \to C_0^\infty(X)$ easily follows. Since (6.2) holds and ${}^tA$ defines a proper ΨDO, it follows ${}^tA : C_0^\infty(X) \to C_0^\infty(X)$, and

$${}^t({}^tA) : \mathcal{D}'(X) \to \mathcal{D}'(X).$$

Since ${}^t({}^tA)|_{C_0^\infty} = A$, we have that $A : C_0^\infty(X) \to C_0^\infty(X)$ can be linearly and continuously extended to a mapping $A : \mathcal{D}'(X) \to \mathcal{D}'(X)$.

By Theorem 6.1, the operator $A : C_0^\infty(X) \to \mathcal{D}'(X)$ can be linearly and continuously continued to mapping $A : \mathcal{E}'(X) \to \mathcal{D}'(X)$. Then (6.4) holds for $u \in \mathcal{E}'(X)$, as well. The proof follows from the fact that $C_0^\infty(X)$ is dense in $\mathcal{E}'(X)$. This means that the continuation (6.2) maps $\mathcal{E}'(X)$ in $\mathcal{E}'(X)$. □

Proposition 6.4. *Let A be a proper ΨDO. Then ${}^tA : C_0^\infty(X) \to C_0^\infty(X)$ can be linearly and continuously extended to the mappings*

$${}^tA : \mathcal{E}'(X) \to \mathcal{E}'(X), \quad {}^tA : C^\infty(X) \to C^\infty(X), \quad {}^tA : \mathcal{D}'(X) \to \mathcal{D}'(X).$$

Proof. The proof is analogous to the proof of the previous theorem because of the duality of operators A and ${}^tA$. □

We will prove that the space of pseudodifferential operators is an algebra with respect to operation of composition.

From Theorem 6.3 it follows that the composition of two proper ΨDO defines a linear and continuous operator on every one of the spaces $C_0^\infty(X)$, $\mathcal{E}'(X)$, $C^\infty(X)$ or $\mathcal{D}'(X)$.

Definition 6.5. It is said that $a(x,y,\xi) \in S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n)$ is an amplitude with a proper support if the projections

$$\pi_1 : \operatorname{supp}_{x,y} a(x,y,\xi) \to X, \quad \pi_2 : \operatorname{supp}_{x,y} a(x,y,\xi) \to X$$

are proper for every $\xi \in \mathbb{R}^n$.

Theorem 6.6. *Let*

$$Au(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} a(x,y,\xi) u(y)\, dy\, d\xi \text{ [osc]}, \quad u \in C_0^\infty(X)$$

be a proper pseudodifferential operator, where $a(x,y,\xi) \in S^m_{\rho,\delta}$. Then A can be defined by the formula

$$Au(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} b(x,y,\xi) u(y)\, dy\, d\xi \text{ [osc]}, u \in C_0^\infty,$$

where $b(x,y,\xi) \in S^m_{\rho,\delta}$ is an amplitude with a proper support.

Proof. Let the functions $\kappa(x,y)$ and $\varphi_j(x,y)$ be the ones constructed in the proof of Theorem 3.10, K_A be a kernel of the operator A. It can be easily seen that

$$\kappa(x,y) = \sum_{\operatorname{supp}\varphi_j \cap \operatorname{supp} K_A \neq \emptyset} \varphi_j(x,y) \in C^\infty(X \times X)$$

and that $\operatorname{supp}\kappa(x,y)$ is contained in some neighbourhood of $\operatorname{supp} K_A$. Let us show that $\pi_1 : \operatorname{supp}\kappa(x,y) \to X$ is a proper mapping, i.e. that for every compact subset K in X the set $\operatorname{supp}\kappa(x,y) \cap \pi_1^{-1}(K)$ is compact. We have

$$\begin{aligned}\operatorname{supp}\kappa(x,y) \cap \pi_1^{-1}(K) &\subset \operatorname{supp}\Big(\sum_{\operatorname{supp}\varphi_j \cap \operatorname{supp} K_A \neq 0} \varphi_j(x,y)\Big) \cap \pi_1^{-1}(K) \\ &\subset \bigcup_{\operatorname{supp}\varphi_j \cap \operatorname{supp} K_A \neq 0} (\operatorname{supp}\varphi_j(x,y) \cap \pi_1^{-1}(K)).\end{aligned}$$

This union is finite, because A is a proper ΨDO, $\pi_1^{-1}(K) \cap \operatorname{supp} K_A$ is compact and a family $\operatorname{supp}\varphi_j$ is locally finite. Since $\operatorname{supp}\kappa(x,y) \cap \pi_1^{-1}(K)$ is a closed subset of a finite union of compact sets, it is compact, too. In the same way, it can be shown that the mapping $\pi_2 : \operatorname{supp}\kappa(x,y) \to X$ is proper.

We will show that the amplitude $b(x,y,\xi) = \kappa(x,y)a(x,y,\xi)$ belongs to the space of symbols $S^m_{\rho,\delta}$ and that it has a proper support. It has a proper support, because $\operatorname{supp}_{x,y} b(x,y,\xi) \subset \operatorname{supp}\kappa(x,y)$ and the first and second projections of $\operatorname{supp}\kappa$ are proper mappings. From $a(x,y,\xi) \in S^m_{\rho,\delta}$ it follows $b(x,y,\xi) \in S^m_{\rho,\delta}$. For every $u \in C_0^\infty(X)$ and $v \in C_0^\infty(X)$

$$\begin{aligned}\langle Au(x), v(x)\rangle &= \langle K_A(x,y), u(y)v(x)\rangle = \langle K_A(x,y), \kappa(x,y)u(y)v(x)\rangle \\ &= \iiint e^{i(y-x)\xi} a(x,y,\xi)\kappa(x,y)v(x)u(y)\, dx\, dy\, d\xi \ [\text{osc.}].\end{aligned}$$

This proves the last part of the assertion. □

Let us note that if $a(x,y,\xi)$ has a proper support, then the integral (6.1) is defined for every $u \in C^\infty(X)$. More precisely, we have

Theorem 6.7. *A proper pseudodifferential operator continuously and linearly maps $C^\infty(X)$ into $C^\infty(X)$.*

Theorem 6.8. *Every pseudodifferential operator $A : C_0^\infty(X) \to \mathcal{D}'(X)$ is of the form $A = A_1 + A_2$, where A_1 is a proper operator and A_2 is a smoothing one.*

Proof. Let A be an arbitrary pseudodifferential operator and let for $u \in C_0^\infty(X)$

$$Au(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} a(x,y,\xi)u(y)\, dy\, d\xi.$$

Then

$$\begin{aligned}Au(x) &= (2\pi)^{-n} \iint e^{i(x-y)\xi} \kappa(x,y)a(x,y,\xi)u(y)\, dy\, d\xi \\ &\quad + (2\pi)^{-n} \iint e^{i(x-y)\xi}(1 - \kappa(x,y))a(x,y,\xi)u(y)\, dy\, d\xi,\end{aligned}$$

where $\kappa(x,y)$ is smooth and $\kappa(x,y) = 1$ in some neighbourhood of the diagonal Δ such that the both projection $\pi_1 : \operatorname{supp}\kappa(x,y) \to X$ and $\pi_2 : \operatorname{supp}\kappa(x,y) \to X$ are proper mapping. (The construction of a function κ is given in the proof of Theorem 3.10.)

The operator A_1, defined by

$$(A_1u)(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi}\kappa(x,y)a(x,y,\xi)u(y)\,dy\,d\xi.$$

is proper. The proof is analogous to a part of the proof of Theorem 6.6. The function $e^{i(x-y)\xi}(1-\kappa(x,y))\,a(x,y,\xi)$ equals zero in some neighbourhood of the diagonal and out of the diagonal it is C^∞. So the operator A_2 defined by

$$(A_2u)(x) = \iint e^{i(x-y)\xi}(1-\kappa(x,y))a(x,y,\xi)u(y)\,dy\,d\xi$$

for $u \in C_0^\infty(X)$ is a smoothing operator by Theorem 6.1. □

6.2.2. The symbol of a proper pseudodifferential operator.

Definition 6.9 Let A be a proper ΨDO. The function $\sigma_A(x,\xi)$ defined on $X \times \mathbb{R}^n$, X is open in $\mathbb{R}^n$, by

$$\sigma_A(x,\xi) = e^{ix\xi}(Ae^{iy\xi})(x), \tag{6.5}$$

where $e_\xi(x) = e^{ix\xi}$, is called a symbol of the pseudodifferential operator A.

If $\sigma_A(x,\xi)$ is a symbol of a proper ΨDO, then $\sigma_A(x,\xi) \in C^\infty(X\times\mathbb{R}^n)$, because A is a linear and continuous mapping $C^\infty(X) \to C^\infty(X)$ and $\xi \mapsto e^{i\xi x}$ is C^∞-function with respect to ξ with values in $C^\infty(X)$. Let us write $u \in C_0^\infty(X)$ as

$$u(y) = (2\pi)^{-n}\int_{\mathbf{R}^n} e^{iy\xi}\hat{u}(\xi)\,d\xi.$$

The continuity of A and the fact that

$$\sum_\nu e^{i\xi_\nu y}\hat{u}(\xi_\nu)\Delta\xi_\nu \to \int_{\mathbf{R}^n} e^{iy\xi}\hat{u}(\xi)\,d\xi, \quad \nu\to\infty,$$

in $\mathcal{E}(\mathbb{R}^n)$ (where on the left-hand side we have a sequence of integral sums) imply

$$(Au(y))(x) = (2\pi)^{-n}\int_{\mathbf{R}^n}(Ae^{iy\xi})(x)\hat{u}(\xi)\,d\xi = (2\pi)^{-n}\int_{\mathbf{R}^n} e^{ix\xi}\sigma_A(x,\xi)\hat{u}(\xi)\,d\xi,$$

$u \in C_0^\infty(X)$, i.e.

$$(Au(y))(x) = (2\pi)^{-n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{i(x-y)\xi}\sigma_A(x,\xi)u(y)\,dy\,d\xi,\ x \in X. \tag{6.6}$$

From (6.5) and (6.6) it follows that the symbol $\sigma_A(x,\xi)$ determines the operator A.

We shall show in Theorem 7.1 that if A has an amplitude in $S^m_{\rho,\delta}$ and $\delta < \rho$, then $\sigma_A(x,\xi) \in S^m_{\rho,\delta}$, so the integral on the left-hand side in (6.6) can be considered as the oscillating one.

If A is an arbitrary ΨDO on X, then the function $\sigma_A(x,\xi)$, which is a symbol of a proper ΨDO A_1 on X such that $A - A_1$ is smoothing, is called the symbol of A. In this case a symbol is not uniquely determinated and two symbols differs by a function $r(x,\xi) \in S^{-\infty}$.

6.2.3. Asymptotic decomposition in $S^m_{\rho,\delta}$

Definition 6.10. Let $a_j(x,\xi) \in S^{m_j}_{\rho,\delta}(X \times \mathbb{R}^n)$, $j = 1,2,\ldots$, $\lim_{j\to\infty} m_j = -\infty$, $a(x,\xi) \in C^\infty(X \times \mathbb{R}^n)$. Then a is an asymptotic sum of a_k,

$$a(x,\xi) \approx \sum_{j=1}^{\infty} a_j(x,\xi),$$

if for every integer $r \geq 2$ there holds

$$a(x,\xi) - \sum_{j=1}^{r-1} a_j(x,\xi) \in S^{\overline{m}_r}_{\rho,\delta}(X,\mathbb{R}^N),$$

where $\overline{m}_r = \max_{j\geq r} m_j$. Note $a \in S^{\overline{m}_r}_{\rho,\delta}(X,\mathbb{R}^N)$.

Theorem 6.11. *Let $a_j \in S^{m_j}_{\rho,\delta}(X,\mathbb{R}^N)$, $j \in \mathbb{N}$, $\lim_{j\to\infty} m_j = -\infty$. Then there exists a function $a(x,\xi)$ such that*

$$a(x,\xi) \approx \sum_{j=1}^{\infty} a_j(x,\xi).$$

If there exists another function a' with the same property

$$a'(x,\xi) \approx \sum_{j=1}^{\infty} a_j(x,\xi),$$

then $a - a' \in S^{-\infty}(X \times \mathbb{R}^n)$.

The proof will be given in the case $\rho = 1$, $\delta = 0$. We follow the proof given in [11]. First, we shall prove the following two lemmas.

Lemma 6.12. *Let $\kappa \in C_0^\infty(\mathbb{R}^n)$, $\kappa(\xi) = 1$ in some neighbourhood of $\xi = 0$ and $\kappa_\lambda(\xi) = \kappa(\lambda\xi)$. Then the set $\{\lambda^{-k}(1-\kappa_\lambda)\}_{0<\lambda\leq 1}$ is bounded in $S^k_{10}(X,\mathbb{R}^n)$ for every $k \geq 0$.*

Proof. Let us prove that the functions

$$\mathbb{R}^n \ni \xi \mapsto (1+|\xi|)^{|\beta|-k}\lambda^{-k}\Big(\frac{\partial}{\partial\xi}\Big)^{\beta}(1-\kappa_\lambda(\xi)),\ \beta \in \mathbb{N}_0^n,$$

are bounded independently on $\lambda \in (0,1]$. Since for κ there holds

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\beta}(1-\kappa)\right| \leq c_\beta,$$

we have

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\beta}(1-\kappa_\lambda)\right| \leq \left|\lambda^{|\beta|}\left(\left(\frac{\partial}{\partial\xi}\right)^{\beta}(1-\kappa)\right)_\lambda\right| \leq \lambda^{|\beta|}c_\beta,\ \lambda \in (0,1].$$

First, we shall prove it for $\beta = 0$. Let $R > 1$ be large enough such that $\kappa(\xi) = 0$ for $|\xi| > R$ and $\kappa(\xi) = 1$ for $|\xi| < 1/R$. If $0 < \lambda \leq 1$ and $(1-\kappa_\lambda) \neq 0$, then $|\xi| \geq 1/\lambda R$,

$$((1+|\xi|)\lambda)^{-k} \leq ((1/R\lambda)\lambda)^{-k} \leq R^k$$

and

$$|\lambda^{-k}(1-\kappa_\lambda)| \leq c_0 R^k (1+|\xi|)^k.$$

If $\left(\frac{\partial}{\partial\xi}\right)^{\beta}(1-\kappa_\lambda) \neq 0$ for $\beta \neq (0,0,\ldots 0)$, then $|\xi| \leq R/\lambda$. This implies

$$((1+|\xi|)\lambda)^{|\beta|} \leq ((1+R/\lambda)\lambda)^{|\beta|} \leq (R+1)^{|\beta|}$$

and

$$\left|\lambda^{-k}\left(\frac{\partial}{\partial\xi}\right)^{\beta}(1-\kappa_\lambda)\right| \leq c_\beta R^k (R+1)^{|\beta|}(1+|\xi|)^{k-|\beta|}. \qquad \square$$

Lemma 6.13. *Let $\{F_k\}$ be a sequence of Frechét spaces such that $F_{k+1} \subset F_k$ and the topology in F_{k+1} is stronger than the topology induced by F_k. For every k, let (a_k^m) be a sequence of elements in F_k which converges to 0 as $m \to \infty$. Then there exists a sequence m_k such that for every N the series $\sum_{k\geq N} a_k^{m_k}$ converges in F_N.*

Proof. Let $p_k^l (l \in \mathbb{N})$ be a fundamental sequence of seminorms in F_k, $k \in \mathbb{N}$, such that $p_k^l \leq p_k^{l+1}$, $l \in \mathbb{N}$. By a simple procedure one can substitute a sequence with equivalent one such that there holds $p_k^l \leq p_{k+1}^l$, $k, l \in \mathbb{N}$. For example, p_k^l can be substituted by

$$\sup_{k' \leq k} p_{k'}^l|_{F_k}.$$

Since $\lim_{m\to\infty} a_k^m = 0$, let us chose m_k (increase as k increases) such that $p_k^k(a_k^{m_k}) \leq 2^{-k}$. Then for $l \leq k$ there holds

$$p_k^l(a_k^{m_k}) \leq p_k^k(a_k^{m_k}) \leq 2^{-k},$$

so, for every $l \geq 0$ the series $\sum_{k=N}^{\infty} p_k^l(a_k^{m_k})$ converges. Since F_N is Frechét space it follows that $\sum_{k=N}^{\infty} a_k^{m_k}$ converges in F_N. $\square$

Proof of Theorem 6.11. One can suppose that $a_k \in S_{10}^{-k}(X \times \mathbb{R}^n)$ when $k \geq 1$. This can be achieved by summing elements in the sequences if it is necessary. Let $a_k^m = (1-\kappa_{1/m})a_k$, where $\kappa_{1/m}$ is defined in the proof of Theorem 4.7. The sequence $(1-\kappa_{1/m})$ converges to zero in S^1 and a_k^m converges to zero in S^{-k+1} as $m \to \infty$.

Lemma 6.13 implies that one can chose a sequence m_k such that for every $N \geq 1$ the series $\sum_{k=N}^{\infty} a_k^{m_k}$ converges in S^{-N+1}. Let $a = \sum_{k=0}^{\infty} a_k^{m_k}$. Then a is a symbol and

$$a - \sum_{k<N} a_k = \sum_{k<N} (a_k^{m_k} - a_k) + \sum_{k=N}^{\infty} a_k^{m_k} \in S^{-N+1},$$

because $(a_k^{m_k} - a_k) = -\kappa_{1/m} a_k \in S^{-\infty}$ for every k. So $a \approx \sum a_k$. Second part of the assertion is obvious.

Theorem 6.14. *Let* $a_j \in S^{m_j}_{\rho,\delta}(X \times \mathbb{R}^n), \lim_{j\to\infty} m_j = -\infty, a \in C^\infty(X \times \mathbb{R}^n)$. *Assume:*

1) For every compact set $K \subset X$ and for all multi-indices α, β there exist constants $\mu = \mu(\alpha, \beta, K)$ and $C = C(\alpha, \beta, K)$ such that

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C(1 + |\xi|)^\mu, x \in K. \tag{6.7}$$

2) If for every compact set $K \subset X$ there exists a sequence of real numbers $\mu_l = \mu_l(K), l \in \mathbb{N}$, and a sequence of constants $C_l = C_l(K)$ such that $\mu_l \to -\infty$ for $l \to \infty$ and

$$\left|a(x,\xi) - \sum_{j=1}^{l-1} a_j(x,\xi)\right| \leq C_l(1 + |\xi|)^{\mu_l}, \quad x \in K. \tag{6.8}$$

Then

$$a(x,\xi) \approx \sum_{j=1}^{\infty} a_j(x,\xi).$$

Proof. First we will prove the following assertion. Let the function $f(t)$ has continuous derivatives $f'(t)$ and $f''(t)$ in $[-1,1]$. Let us denote $A_j = \sup_{-1\leq t\leq 1} |f^{(j)}(t)|, j = 0,2$. Then

$$|f'(0)|^2 \leq 4A_0(A_0 + A_2). \tag{6.9}$$

By Lagrange's theorem,

$$|f'(t) - f'(0)| \leq A_2|t|, \quad t \in [-1,1].$$

Because of that,

$$|f'(t)| \geq \frac{1}{2}|f'(0)|, \quad \text{if } A_2|t| \leq \frac{1}{2}|f'(0)|, \ |t| \leq 1.$$

Let us denote $\Delta = \min\left\{\frac{1}{2A_2}|f'(0)|, 1\right\}$. There holds

$$|f'(t)| \geq \frac{1}{2}|f'(0)|, \quad t \in [-\Delta, \Delta]$$

and

$$2A_0 \geq |f(\Delta) - f(-\Delta)| \geq 2\Delta\frac{1}{2}|f'(0)|.$$

It follows

$$|f'(0)| \le 2A_0/\Delta = 2A_0 \max\{2A_2/|f'(0)|, 1\},$$

which implies (6.9). Now by (6.9) we have the estimate needed for the proofs of theorem.

Also, we need the following estimate. Let K_1 and K_2 be compact sets in $\mathbb{R}^n$ such that $K_1 \subset \operatorname{int} K_2$. Then there exists a constant $C > 0$ such that for every C^∞-function f in a neighbourhood of K_2

$$\left(\sup_{x\in K_1} \sum_{|\alpha|=1} |D^\alpha f(x)|\right)^2 \le c \sup_{x\in K_2} |f(x)| + \left(\sup_{x\in K_2} |f(x)| + \sup_{x\in K_2} \sum_{|\alpha|=2} |D^\alpha f(x)|\right). \tag{6.10}$$

Now we give the proof of the assertion in the theorem. Let $b \approx \sum_{j=1}^\infty a_j(x,\xi)$ (such b exists by Theorem 6.11) and let $d(x,\xi) = a(x,\xi) - b(x,\xi)$. By the assumptions, for arbitrary compact set $K \subset X$ there holds

$$|\partial_\xi^\alpha \partial_x^\beta d(x,\xi)| \le C(1+|\xi|)^\mu, x \in K.$$

where C and μ depend on α, β, K and

$$|d(x,\xi)| \le C_r(1+|\xi|)^{-r}, x \in K, r > 0, \tag{6.11}$$

where $C_r = C_r(K)$. Let us denote $d_\xi(x,\vartheta) = d(x,\xi+\vartheta)$. Then

$$\partial_\vartheta^\alpha \partial_x^\beta d_\xi(x,\vartheta)|_{\vartheta=0} = \partial_\xi^\alpha \partial_x^\beta d(x,\xi).$$

By (6.10), for $K_1 = K \times \{0\}, K_2 = \hat{K} \times \{|\xi| \le 1\}$, where $\hat{K}$ is a compact set in X such that $K \subset \operatorname{int} \hat{K}$ and from (6.11) it follows that for $\vartheta = 0$ there holds

$$\sup_{x\in K} \sum_{|\alpha|+|\beta|\le 1} |\partial_\xi^\alpha \partial_x^\beta d(x,\xi)|)^2 \le C(1+|\xi|)^{-r}[(1+|\xi|)^{-r} + (1+|\xi|)^\mu],$$

where r is arbitrary, $\mu = \mu(\alpha,\beta,K)$ and $C = C(\alpha,\beta,K,r)$. Moreover, for $x \in K$ and $|\alpha| + |\beta| \le 1$ the function $\partial_\xi^\alpha \partial_x^\beta d(x,\xi)$ decreases faster than each power of $|\xi|$ as $|\xi| \to \infty$. By induction, it follows that $d \in S^{-\infty}(X, \mathbb{R}^n)$. □

7. Calculus with symbols

The simplicity of the calculus with symbols is the central point of the theory of ΨDO. The main ideas of their calculus are given in Theorems 7.1 and (7.6) below.

7.1. Symbol of a proper ΨDO.

Let $\delta < \rho$. This will be a permanent assumption in the rest of the notes.

Theorem 7.1. *Let A be a proper ΨDO given by (6.1) and $\sigma_A(x,\xi)$ be its symbol. Then*

$$\sigma_A(x,\xi) \approx \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha D_y^\alpha a(x,y,\xi)|_{y=x}, \tag{7.1}$$

where the asymptotic sum is taken over all the multi-indices.

Remark that $\partial_\xi^\alpha D_y^\alpha a(x,y,\xi)|_{y=x} \in S^{m-(\rho-\delta)|\alpha|}$.

Proof. We will apply Theorem 6.14. We can assume that the amplitude $a(x,y,\xi)$ is properly supported. Then by (6.5)

(7.2)
$$\sigma_A(x,\xi) = e^{-i\xi x}A(e^{iy\xi})(x) = (2\pi)^{-n}\iint_{\mathbb{R}^{2n}} a(x,y,\xi)e^{i(x-y)\vartheta}e^{i(y-x)\xi}\,dyd\vartheta\ [\mathrm{osc}],$$

(for fixed x the integration by y is made over a compact set). If K is a compact subset of X, then for $x \in K$ (7.2) determines the oscillating integral depending on the parameter x. Let us change the variables by $z = y - x, \eta = \vartheta - \xi$. Then

$$(2\pi)^n\sigma_A(x,\xi) = \iint_{\mathbb{R}^{2n}} a(x,x+z,\xi+\eta)e^{-iz\eta}\,dzd\eta. \tag{7.3}$$

Expand $a(x,x+z,\xi+\eta)$ into the Taylor series at $\eta_0 = 0$ with the powers of η. Then,

$$a(x,x+z,\xi+\eta) = \sum_{|\alpha|\le N-1} \partial_\xi^\alpha a(x,x+z,\xi)\eta^\alpha/\alpha! + r_N(x,x+z,\xi,\eta), \tag{7.4}$$

where

$$r_N(x,x+z,\xi,\eta) = \sum_{|\alpha|=N} \frac{N\eta^\alpha}{\alpha!}\int_0^1 (1-t)^{N-1}\partial_\xi^\alpha a(x,x+z,\xi+t\eta)\,dt. \tag{7.5}$$

Let us note that for every $\xi \in \mathbb{R}^n$ and $x \in K$, $a(x,x+z,\xi)$ is compactly supported with respect to variable z. By the Fourier transform

$$\begin{aligned}(2\pi)^{-n}&\iint_{\mathbb{R}^{2n}} \partial_\xi^\alpha a(x,x+z,\xi)\eta^\alpha e^{-iz\eta}\,dz\,d\eta \\ &= \mathcal{F}^{-1}(\mathcal{F}(i^{-|\alpha|}\partial_\xi^\alpha\partial_z^\alpha a(x,x+z,\xi))(\eta))(z)|_{z=0} \\ &= \partial_\xi^\alpha D_z^\alpha a(x,x+z,\xi)|_{z=0}.\end{aligned} \tag{7.6}$$

This gives

$$\sigma_A(x,\xi) = \sum_{|\alpha|\le N}\frac{1}{\alpha!}\partial_\xi^\alpha D_y^\alpha a(x,y,\xi)|_{y=x} + (2\pi)^{-n}\iint_{\mathbb{R}^{2n}} e^{-iz\eta}r_N(x,x+z,\xi,\eta)\,dz\,d\eta.$$

Integration by parts gives, from (7.3),

$$\sigma_A(x,\xi) = (2\pi)^{-n}\iint_{\mathbb{R}^{2n}} e^{-iz\eta}(1-D_{z_1}^2-\cdots-D_{z_n}^2)^{\nu/2}a(x,x+z,\xi+\eta)(1+|\eta|)^{-\nu/2}\,dzd\eta,$$

where ν is a even and nonnegative number. By using

$$(1+|\xi+\eta|^2)^{1/2} \le (1+|\xi|^2)^{1/2}(1+|\eta|^2)^{1/2}$$

the above equality implies

$$|\partial_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)| \le C(1+|\xi|^2)^{(p+\delta\nu)/2} \int_{\mathbf{R}^n} (1+|\eta|^2)^{(p-(1-\delta)\nu)/2}\, d\eta,$$

where $p = \max(m - \rho|\alpha| + \delta|\beta|, 0)$, $x \in K$ and ν is large enough. Thus, we obtain estimates of the form (6.7).

Let us estimate the rest of the series. Substitute a and r_N (defined by (7.5) in (7.3). After the change of the order of integration (first by t, then by z and η), let us note that we have to estimate the integral

$$R_{\alpha,t}(x,\xi) = (2\pi)^{-n} \iint_{\mathbf{R}^{2n}} e^{-iz\eta} \eta^\alpha \partial_\xi^\alpha a(x, x+z, \xi + t\eta)\, dz\, d\eta,$$

where $|\alpha| = N$, uniformly over $t \in (0,1]$ and $x \in K$. Integration by parts gives

$$R_{\alpha,t}(x,\xi) = (2\pi)^{-n} \iint_{\mathbf{R}^{2n}} e^{-iz\eta} \partial_\xi^\alpha D_z^\alpha a(x, x+z, \xi + t\eta)\, dz d\eta.$$

Let

$$R_{\alpha,t}(x,\xi) = R'_{\alpha,t}(x,\xi) + R''_{\alpha,t}(x,\xi),$$

where $R'_{\alpha,t}(x,\xi)$ is the integral over the set $\{(z,\eta),\ |\eta| \le |\xi|/2\}$ and $R''_{\alpha,t}(x,\xi)$ over its complement. (Recall, z belongs to a compact set.) If $|\eta| \le |\xi|/2$, then $|\xi|/2 \le |\xi + t\eta| \le 3|\xi|/2$. Since the measure of the domain of the integration of $R'_{\alpha,t}(x,\xi)$ with respect to η variable is less or equal to $C|\xi|^n$, then

$$|R'_{\alpha,t}(x,\xi)| \le C(1+|\xi|^2)^{(m-(\rho-\delta)N+n)/2},$$

where C does not depend on ξ and t. Let us estimate $R''_{\alpha,t}(x,\xi)$. By using

$$(1+|\eta|^2)^{-\nu/2}(1 - D_{z_1}^2 - \cdots - D_{z_n}^2)^{\nu/2} e^{-iz\eta} = e^{-iz\eta},$$

where ν is even positive integer, let us integrate by parts. Then $R''_{\alpha,t}(x,\xi)$ is a finite sum of terms of the form

$$R_{\alpha,\beta,t}(x,\xi) = (2\pi)^{-n} \iint_{|\eta|>|\xi|/2} e^{-iz\eta}(1+\eta^2)^{-\nu/2} \partial_\xi^\alpha D_z^{\alpha+\beta} a(x, x+z, \xi+t\eta)\, dz\, d\eta,$$

where $|\beta| \le \nu$. Since x and z belong to a compact set for $|\eta| > |\xi|/2$ there holds

$$|\partial_\xi^\alpha D_z^{\alpha+\beta} a(x, x+z, \xi+t\eta)| \le C(1+|\eta|^2)^{(m-(\rho-\delta)N+\delta\nu)/2},$$

for $m-(\rho-\delta)N+\delta\nu \ge 0$, i.e. $|\partial_\xi^\alpha D_z^{\alpha+\beta} a(x, x+z, \xi+t\eta)| \le C$ for $m-(\rho-\delta)N+\delta\nu < 0$. In both cases C does not depend on ξ, η and t. For large enough ν there holds

$$|R_{\alpha,\beta,t}(x,\xi)| \le C \int_{|\eta|>|\xi|/2} (1+|\eta|^2)^{(p-(1-\delta)\nu)/2}\, d\eta,$$

where $p = \max(m - (\rho-\delta)N, 0)$. If $p - (1-\delta)\nu + n + 1 < 0$, then

$$\begin{aligned} |R_{\alpha,\beta,t}(x,\xi)| &\le C(1+|\xi|^2)^{(p-(1-\delta)\nu+n+1)/2} \int_{\mathbf{R}^n} (1+|\eta|^2)^{(n+1)/2}\, d\eta \\ &\le C(1+|\xi|^2)^{(p-(1-\delta)\nu+n+1)/2}, \end{aligned}$$

where C does not depend on x, ξ, t if $x \in K$ and $t \in (0,1]$. For ν large enough we have

$$|R_{\alpha,t}(x,\xi)| \leq C(1+|\xi|^2)^{(m-(\rho-\delta)N+n)/2}, \quad x \in K, \ t \in (0,1].$$

By Theorem 6.14 the proof follows. Note that the assumption $\rho > \delta$ is crucial for the proof. □

Proposition 7.2 *Let A be a proper ΨDO, $\sigma_A(x,\xi)$ its symbol and $\sigma'_A(x,\xi)$ a symbol of ${}^tA$. Then,*

$$\sigma'_A(x,\xi) \approx \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \sigma_A(x,-\xi).$$

Proof. By (6.6)

$${}^tAv(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} \sigma_A(y,-\xi) v(y)\, dy\, d\xi.$$

The assertion follows from (7.1). □

Analogously, one can prove the following assertion.

Proposition 7.3. *Let A be a proper ΨDO with a symbol $\sigma_A(x,\xi)$ and A^* its adjoint operator. If $\sigma^*_A(x,\xi)$ is a symbol of adjoint operator, then*

$$\sigma^*_A(x,\xi) \approx \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \overline{\sigma_A}.$$

Definition 7.4 A dual symbol $\tilde{\sigma}_A(x,\xi)$ for A is given by

$$\tilde{\sigma}_A(x,\xi) = \sigma'_A(x,-\xi).$$

By using ${}^t({}^tA) = A$ we obtain

$$Au(x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi} \tilde{\sigma}_A(y,\xi) u(y)\, dy\, d\xi. \tag{7.7}$$

The following proposition follows immediately.

Proposition 7.5. $\tilde{\sigma}_A(x,\xi) \approx \sum (-\partial_\xi)^\alpha D_x^\alpha \sigma_A(x,\xi)/\alpha!$.

7.2. Composition of proper ΨDO's.

Theorem 7.6. *Let A and B be proper ΨDO's in $X \subset \mathbb{R}^n$, $\sigma_A(x,\xi)$, $\sigma_B(x,\xi)$ their symbols and $C = BA$. Then C is a proper ΨDO, with the symbol $\sigma_{BA}(x,\xi)$ which is given by*

$$\sigma_{BA}(x,\xi) \approx \sum \partial_\xi^\alpha \sigma_B(x,\xi) D_x^\alpha \sigma_A(x,\xi)/\alpha!.$$

Recall $D_x^j = \left(\frac{1}{\sqrt{-1}}\frac{\partial}{\partial x}\right)^j$.

Proof. Note, the dual symbol is used for representation (7.7):

$$\widehat{(Au)}(\xi) = \int_{\mathbf{R}^n} e^{-iy\xi}\tilde{\sigma}_A(y,\xi)u(y)\,dy.$$

By (6.6)

$$Bu(x) = \int_{\mathbf{R}^n} e^{ix\xi}\sigma_B(x,\xi)\hat{u}(\xi)\,d\xi,$$

which implies

$$Cu(x) = (2\pi)^{-n}\iint_{\mathbf{R}^{2n}} e^{i(x-y)\xi}\sigma_B(x,\xi)\tilde{\sigma}_A(y,\xi)u(y)\,dy\,d\xi.$$

Clearly, if $\sigma_A \in S^{m_1}_{\rho,\delta}$ and $\sigma_B \in S^{m_2}_{\rho,\delta}$, then $\sigma_B\tilde{\sigma}_A \in S^{m_1+m_2}_{\rho}$ and thus the symbol of C is in $S^{m_1+m_2}_{\rho,\delta}$. Analogously, we have that the symbol of ${}^tC = {}^tA^tB$ is in $S^{m_1+m_2}_{\rho,\delta}$. By Theorem 3.11 it follows that C is proper.

Let us find the symbol for C. By using Theorem 7.1 and Proposition 7.5 we have

$$\begin{aligned}\sigma_{BA}(x,\xi) &\approx \sum_\alpha \partial_\xi^\alpha D_y^\alpha \sigma_B(x,\xi)\tilde{\sigma}_A(y,\xi)/\alpha!|_{y=x} \\ &= \sum_\alpha \partial_\xi^\alpha[\sigma_B(x,\xi)D_x^\alpha\tilde{\sigma}_A(x,\xi)]/\alpha! \\ &\approx \sum_{\alpha,\beta}\partial_\xi^\alpha[\sigma_B(x,\xi)(-\partial_\xi)^\beta D_x^{\alpha+\beta}\sigma_A(x,\xi)]/\alpha!\beta!.\end{aligned} \tag{7.8}$$

Leibnitz formula implies

$$\begin{aligned}\sigma_{BA}(x,\xi) &\approx \sum_{\alpha,\beta}\sum_{\gamma,\delta,\gamma+\delta=\alpha}(-1)^{|\beta|}[\partial_\xi^\gamma\sigma_B(x,\xi)][\partial_\xi^{\beta+\delta}D_x^{\alpha+\beta}\sigma_A(x,\xi)]/\delta!\beta!\gamma! \\ &= \sum_{\beta,\gamma,\delta}(-1)^{|\beta|}[\partial_\xi^\gamma\sigma_B(x,\xi)][\partial_\xi^{\beta+\delta}D_x^{\beta+\gamma+\delta}\sigma_A(x,\xi))]/\delta!\beta!\gamma! \\ &= \sum_\gamma\sum_\kappa\Big(\sum_{\beta+\delta=\kappa}(-1)^{|\beta|}\frac{1}{\beta!\delta!}\Big)[\partial_\xi^\gamma\sigma_B(x,\xi)][\partial_\xi^\kappa D_x^{\kappa+\gamma}\sigma_A(x,\xi))]/\gamma!.\end{aligned} \tag{7.9.}$$

We shall use the following identity

$$(x-y)^\alpha = (x_1-y_1)^{\alpha_1}\dots(x_n-y_n)^{\alpha_n} = \sum_{\beta+\delta=\alpha}\frac{\alpha!}{\beta!\delta!}(-1)^{|\delta|}x^\beta y^\delta$$

for $x=(1,\dots,1)$, $y=(1,\dots,1)$. This gives

$$\sum_{\beta+\delta=\alpha}(-1)^{|\beta|}\frac{1}{\beta!\delta!} = 1, \qquad \sum_{\beta+\delta=\alpha}(-1)^{|\beta|}\frac{1}{\beta!\delta!} = \begin{cases}0, & \alpha\neq 0\\ 1, & \alpha = 0\end{cases}$$

and by substituting this in (7.8), the assertion of theorem follows. □

Proposition 7.7. *Let $\sigma_A \in S^{m_1}_{\rho,\delta}(X,\mathbb{R}^n), \sigma_B \in S^{m_2}_{\rho,\delta}(X,\mathbb{R}^n), 0 \le \delta < \rho \le 1$ and let B be a proper operator. Then the operators AB, BA are determined by the symbols in $S^{m_1+m_2}_{\rho,\sigma}(X,\mathbb{R}^n)$.*

Proof. Let $A = A_1 + R$, where A is a proper ΨDO and R has a kernel $K_R(x,\xi) \in C^\infty(X \times X)$. Then, BR and RB have smooth kernels. Let us prove this for BR. Let $\varphi \in C_0^\infty$. We have

$$\begin{aligned} B(R\varphi)(x) &= (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi}(R\varphi)(y)b(x,y,\xi)\,dy\,d\xi \\ &= (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi}\left(\int_{\mathbb{R}^n} K_R(y,t)\varphi(t)\,dt\right) b(x,y,\xi)\,dy\,d\xi \\ &= \int_{\mathbb{R}^n}\left(\iint_{\mathbb{R}^{2n}} e^{i(x-y)\xi}K_R(y,t)b(x,y,\xi)\varphi(t)\,dy\,d\xi\right)\varphi(t)\,dt. \end{aligned}$$

Thus the kernel of BR equals

$$\begin{aligned} &\iint_{\mathbb{R}^{2n}} e^{i(x-y)}\xi K_R(y,t)b(x,y,\xi)\,dy\,d\xi \\ &\qquad = \iint_{\mathbb{R}^{2n}} \frac{(-1)^r}{|x-y|^{2r}}\Delta^r e^{i(x-y)\xi}K_R(y,t)b(x,y,\xi)\,dy\,d\xi. \end{aligned}$$

Since $|x-y| > d > 0$ by taking enough large r, we obtain that the kernel of BR is smooth with respect to x and t. The same holds for RB. □

7.3. Classical symbols and pseudodifferential operators.

Definition 7.8. A classical symbol is a function $a(x,\xi) \in C^\infty(X \times \mathbb{R}^n)$, X is open in $\mathbb{R}^n$ which has an asymptotic expansion

$$a(x,\xi) \approx \sum_{j=0}^{\infty} a_{m-j}(x,\xi), \tag{7.10}$$

for some complex m, where $a_{m-j}(x,\xi) \in C^\infty(X \times (\mathbb{R}^n \setminus \{0\}))$ are positively homogeneous with respect to ξ of order $m-j$, $j = 0,1,\ldots$. The set of such symbols is denoted by $CS^m(X \times \mathbb{R}^n)$ and the corresponding pseudodifferential operators are called classical pseudodifferential operators. a_m is called the main symbol.

Note a_{m-j} is not smooth for $\xi = 0$ and should be cuted off in an appropriate way.

If $a_k(x,\xi)$ is positive homogeneous with respect to ξ of order k, then $\partial_\xi^\alpha\partial_x^\beta a_k(x,\xi)$ is positive homogeneous with respect to ξ of order $k-|\alpha|$. Because of that,

$$CS^m(X \times \mathbb{R}^n) \subset S^{\mathrm{Re}(m)}(X \times \mathbb{R}^n).$$

The following proposition can be easily proved

Proposition 7.9. a) *If A and B are proper classical pseudodifferential operators determinated by the symbols in CS^{m_1} and CS^{m_2}, then BA is a classical pseudodifferential operator with the symbol in $CS^{m_1+m_2}(X)$.*

b) *If A is a classical operator then ${}^tA$ and A^* are also classical with the symbols in the same class.*

7.4. Hypoellipticity and ellipticity. Parametrix.

As we already said, ΨDO are founded in the development for the theory of elliptic and hypoelliptic operators. The construction of a parametrix for a given hypoelliptic operator which is to follow is the most important application of the pseudodifferential calculus. Note that in the first section we gave the motivation of the whole theory by considering elliptic operators.

Definition 7.10. A function $\sigma(x,\xi) \in C^\infty(X \times \mathbb{R}^n)$, where X is open in $\mathbb{R}^n$, is hypoelliptic symbol if the following holds.

a) There exist reals m and m_0 such that for every compact set $K \subset X$ there exist positive constants R, C_1, C_2 such that

$$C_1|\xi|^{m_0} \le |\sigma(x,\xi)| \le C_2|\xi|^m, |\xi| \ge R, x \in K. \tag{7.11}$$

b) There exist $\rho, \delta, 0 \le \delta < \rho \le 1$ such that for every compact set $K \subset X$ there exists a constant R such that for every pair of multi-indices α, β there exists a constant $C_{\alpha,\beta,K}$ such that

$$|(\partial_\xi^\alpha \partial_x^\beta \sigma(x,\xi))\sigma^{-1}(x,\xi)| \le C_{\alpha,\beta,K}|\xi|^{-\rho|\alpha|+\delta|\beta|}, |\xi| \ge R, x \in K. \tag{7.12}$$

The class of hypoelliptic symbols is denoted by $HS^{m,m_0}_{\rho,\delta}(X \times \mathbb{R}^n)$. From (7.11) and (7.12) it follows $HS^{m,m_0}_{\rho,\delta}(X \times \mathbb{R}^n) \subset S^m_{\rho,\delta}(X \times \mathbb{R}^n)$.

Definition 7.11. ΨDO A is called hypoelliptic if there exists a proper ΨDO A_1 with the symbol $HS^{m,m_0}_{\rho,\delta}(X \times \mathbb{R}^n)$, such that $A = A_1 + R_1$, where R_1 is smoothing.

If $m = m_0$ then σ is called elliptic, i. e. A is called elliptic ΨDO.

Let us note that in the decomposition of a hypoelliptic operator $A = A_1 + R_1$, where R_1 is smoothing and A_1 is a proper ΨDO, it follows that its symbol belongs to $HS^{m,m_0}_{\rho,\delta}(X \times \mathbb{R}^n)$.

Recall, $A = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$ is called elliptic, if its principal symbol satisfies

$$a_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \ne 0, (x,\xi) \in X \times (\mathbb{R}^n \backslash \{0\}). \tag{7.13}$$

Example 7.1. Examples of hypoelliptic operators.

The Heat operator $\partial_t - \sum_{i=1}^n \partial_{x_i}^2$ is an example of a hypoelliptic and not elliptic operator.

(2) Differential operator $D_y^2 + y^2 D_x^2 + \lambda D_x$, $\operatorname{Re}\lambda = 0$ is hypoelliptic if and only if $\lambda \ne 2k+1$, $k \in \mathbb{Z}$, while

$$D_y + iay^r D_x, \ \operatorname{Re} a \ne 0,$$

is hypoelliptic if and only if $r = 2k$, $k \in \mathbb{N}$.

(3) Pseudodifferential operator $D_y + iay^r\sqrt{D_x^2 + D_y^2}$, $\operatorname{Re} a \neq 0$, is hypoelliptic if r is even or r is odd and $\operatorname{Re} a > 0$, and the pseudodifferential operator given by the symbol $P(x,\xi) = 1 + |x|^{2\nu}|\xi|^{2\mu}$ is hypoelliptic for $\mu/\nu < 1$.

Remark 7.1. The change of the variables does not preserve the hypoellipticity. For example, the change of variables

$$y_i = x_i,\ i = 1,\dots,n,\ \tau = t + x_1^2/2$$

in the heat operator, gives a non-hypoelliptic operator.

Proposition 7.12. *For a differential operator A the following two conditions are equivalent*

a) *A is elliptic.* b) *The symbol of A is in $HS_{1,0}^{m;m}(X \times \mathbb{R}^n)$.*

Proof. The implication *b)* $\Rightarrow$ *a)* is obvious. For the another part of the proof we note that the symbol of A is

$$a(x,\xi) = \sum_{|\alpha| \le m} a_\alpha(x)\xi^\alpha. \tag{7.14}$$

If a) holds, then

$$a(x,\xi)/a_m(x,\xi) = 1 + b_{-1}(x,\xi) + \dots + b_{-m}(x,\xi),$$

where $b_{-j}(x,\xi) \in C^\infty(X \times (\mathbb{R}^n\backslash\{0\}))$ are homogeneous in respect to ξ of order $-j$. This implies (7.11), while (7.12) follows in the same manner. $\square$

Definition 7.13. A classical operator A is called elliptic if its main symbol $a_m(x,\xi) \in CS^m(X \times \mathbb{R}^n)$ satisfies (7.13).

Proposition 7.12 holds for a classical ΨDO. More precisely, if a symbol of A satisfies (7.11) for $m = m_0$ then it satisfies (7.12), too. This means that in the case of the symbols of elliptic operators we can omit the condition (7.12) for them. This follows from the following proposition.

Proposition 7.14. *Let $\sigma(x,\xi) \in HS_{\rho,\delta}^{m,m_0}(X \times \mathbb{R}^n)$. Then*

$$\sigma^{-1}(x,\xi) \in HS_{\rho,\delta}^{-m,-m_0}(X \times \mathbb{R}^n)$$

for ξ large enough, $|\xi| > \xi_0 > 0$. Further on, for any pair of multi-indices $\alpha,\beta \in \mathbb{N}_0^n$,

$$\partial_\xi^\alpha \partial_x^\beta \sigma(x,\xi)/\sigma(x,\xi) \in S_{\rho,\delta}^{-\rho|\alpha|+\delta|\beta|}, \tag{7.15}$$

for ξ large enough.

Proof. One can simply prove (7.15) for $|\alpha| = |\beta| = 1$. Let $p \in \mathbb{N}_0^{2n}$. By induction with respect to $|p|$, it can be shown that

$$\partial^p \frac{\partial^{(\alpha,\beta)}\sigma(x,\xi)}{\sigma(x,\xi)} = \sum_{k=0}^{|p|} \sum_{p_0+\dots+p_k=p} \frac{\partial^{p_0+(\beta,\alpha)}\sigma(x,\xi)}{\sigma(x,\xi)} \prod_{l=1}^{k} \frac{\partial^{p_l}\sigma(x,\xi)}{\sigma(x,\xi)}. \tag{7.16}$$

Now (7.15) follows from (7.16) by induction. □

Theorem 7.15. *Let A be a proper pseudodifferential operator with a symbol in $HS^{m,m_0}_{\rho,\delta}(X \times \mathbb{R}^n)$, $\delta < \rho$. Then there exists a proper pseudodifferential operator B with a symbol in $HS^{-m,-m_0}_{\rho,\delta}(X \times \mathbb{R}^n)$ such that*

$$BA = I + R_1, \quad AB = I + R_2, \tag{7.17}$$

where R_1, R_2 are smoothing operators and I is the unity operator.

If B' is an operator with the same property, then $B - B'$ is a smoothing operator.

Proof. Let σ_A be the symbol of the operator A. Chose $b_0(x,\xi) \in HS^{-m,-m_0}_{\rho,\delta}(X \times \mathbb{R}^n)$ such that $b_0(x,\xi) = \sigma_A^{-1}(x,\xi)$ for large enough ξ and a proper pseudodifferential operator B_0 with a symbol in $HS^{-m,-m_0}_{\rho,\delta}(X \times \mathbb{R}^n)$ such that $\sigma_{B_0} - b_0 \in S^{-\infty}(X \times \mathbb{R}^n)$. Let us show that

$$B_0 A = I - R_0,$$

where the symbol of R_0 is in $S^{-(\rho-\delta)}_{\rho,\delta}(X \times \mathbb{R}^n)$. By Theorem 7.6 it follows that

$$\sigma_{B_0 A}(x,\xi) \approx 1 + \sum_{|\alpha|\geq 1} \partial_\xi^\alpha \sigma_A^{-1} D_x^\alpha \sigma_A/\alpha! = 1 + \sum_{|\alpha|\geq 1} \partial_\xi^\alpha \sigma_A^{-1} D_x^\alpha \sigma_A/(\alpha! \sigma_A^{-1}\sigma_A)$$

for large enough ξ. Proposition 7.14 implies that R_0 has the symbol in $S^{-(\rho-\delta)}_{\rho,\delta}$. Let C_0 be a proper ΨDO which satisfies

$$C_0 \approx \sum_{j=0}^{\infty} (-1)^j R_0^j, \quad \text{i.e.} \tag{7.18}$$

$$\sigma_{C_0} \approx \sum_{j=0}^{\infty} (-1)^j \sigma_R^j. \tag{7.19}$$

From (7.18) immediately follows that the operator $C_0(I + R_0) - I$ is smoothing, so, if we put $B_1 = C_0 B_0$ we obtain

$$B_1 A = I + R_1, \tag{7.20}$$

where R_1 is smoothing. It is clear from the construction that the symbol of B_1 belongs to $HS^{-m,-m_0}_{\rho,\delta}(X \times \mathbb{R}^n)$. Analogously, we obtain that the symbol of the operator B_2 is in $HS^{-m,-m_0}_{\rho,\delta}(X \times \mathbb{R}^n)$ for which

$$AB_2 = I + R_2, \tag{7.21}$$

where R_2 is smoothing.

Let B_1 and B_2 be a pair of ΨDO's for which (7.20) and (7.21) hold. We can suppose that they are proper operators. By multiplying the right-hand side of

(7.20) with B_2 (in fact by applying B_2) and by using (7.21) we obtain $B_1 - B_2 = R_1 B_2 - B_1 R_2$ and $R_1 B_2 - B_1 R_2$ is smoothing. □

Definition 7.16. The operator B satisfying (7.17) is called a parametrix of the operator A.

Note that an elliptic operator A with a symbol belonging to $S^m_{\rho,\sigma}(X \times \mathbb{R}^n)$ has a parametrix B with a symbol in $HS^{-m,-m}_{\rho,\delta}(X \times \mathbb{R}^n)$.

Proposition 7.17. *Let A be a hypoelliptic ΨDO. Then*

$$\text{Sing supp}\, Au = \text{Sing supp}\, u, \quad u \in \mathcal{E}'(X). \tag{7.22}$$

If A is also a proper operator, then (7.22) holds for every $u \in \mathcal{D}'(X)$.

Proof. The relation $\text{Sing supp}\, Au \subset \text{Sing supp}\, u$ follows from the pseudolocality of the operator A. Let B be a proper ΨDO which is a parametrix of the operator A. Then from the equation $u = B(Au) - R_1 u$ and pseudolocality of the operator B it follows that

$$\text{Sing supp}\, u \subset \text{Sing supp}\, Au \cup \text{Sing supp}\, R_1 u.$$

Since $R_1 u \in C^\infty(X)$ (and $\text{Sing supp}\, R_1 u = \emptyset$) the assertion follows. □

This was a global aspect of hypoellipticity of ΨDO's. Now, we shall give few assertions about a local hypoellipticity.

Definition 7.18. A class of symbols in $HS^{m;m_0}_{\rho,\sigma}(x_0, \xi_0)$ consists of symbols in $S^m_{\rho,\delta}$, which are hypoelliptic at (x_0, ξ_0), i.e. which satisfies the conditions of Definition 7.10 in the set of the form $U \times \Gamma_{R,\eta}$ where U is a neighbourhood of the point x_0 and $\Gamma_{R,\eta} = \left\{\xi, \left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right| < \eta, |\xi| > R\right\}$.

A ΨDO A is called hypoelliptic at x_0 (locally hypoelliptic at x_0) if there exists a proper ΨDO A_1 with a symbol in $HS^{m;m_0}_{\rho,\delta}(x_0, \xi)$ for every $\xi \in \mathbb{R}^n$ such that $A = A_1 + R_1$, where R_1 is smoothing in a neighbourhood of x_0. Locally elliptic ΨDO are analogously defined.

The following assertion can be proved in the same way as in Theorem 7.15.

Proposition 7.19. *Let an operator A be hypoelliptic at x_0 (and proper). Then there exists an operator B, hypoelliptic at x_0 (and proper) such that*

$$BA = I + R_1, \quad AB = I + R_2, \tag{7.23}$$

where R_1, R_2 are smoothing operators in a neighbourhood of x_0, and I denotes identity operator. If B' is an operator with the same property as B, then $B - B'$ is smoothing in a neighbourhood of x_0.

Let A be a classical elliptic ΨDO with a symbol $a(x, \xi)$ such that

$$a(x, \xi) \approx \sum_{j=0}^{\infty} a_{m-j}(x, \xi),$$

where $a_{m-j}(x,\xi) \in C^\infty(X \times (\mathbb{R}^n \setminus \{0\}))$, $a_{m-j}(x,\xi)$ is positively homogeneous with respect to ξ of order $m-j$, $j \in \mathbb{N}$, and $a_m(x,\xi) \neq 0$, $x \in X$, $\xi \neq 0$.

Let B be a parametrix of A given by the symbol $b(x,\xi)$. We shall prove that $b(x,\xi)$ has an asymptotic expansion

$$b(x,\xi) \approx \sum_{j=0}^{\infty} b_{-m-j}(x,\xi),$$

where $b_{-m-j}(x,\xi) \in C^\infty(X \times (\mathbb{R}^n \setminus \{0\}))$, $b_{-m-j}(x,\xi)$ is positively homogeneous with respect to ξ of order $-m-j$, $j \in \mathbb{N}$. The formula for composition implies

$$\sum_{\alpha} \partial_\xi^\alpha a(x,\xi) D_x^\alpha b(x,\xi)/\alpha! \approx 1 \quad \text{or}$$

$$\sum_{\alpha,k,j} \partial_\xi^\alpha a_{m-k}(x,\xi) D_x^\alpha b_{-m-j}(x,\xi)/\alpha! \approx 1. \tag{7.24}$$

By factoring the expression with respect to the degree of homogeneity we obtain the following system of equations

$$a_m b_{-m} = 1, a_m b_{-m-j} + \sum_{\substack{k+l+|\alpha|=j \\ l<j}} (\partial_\xi^\alpha a_{m-k})(D_x^\alpha b_{-m-l})/\alpha! = 0, \tag{7.25}$$

$$j = 1, 2, \ldots .$$

The functions $b_{-m-j}(x,\xi)$ in (7.25) are uniquely determinated and we have to find a proper ΨDO B such that $\sigma_B(x,\xi) - b(x,\xi) \in S^{-\infty}(X \times \mathbb{R}^n)$. Such B is the solution to the system.

8. Wave front sets and ΨDO

The notion of the wave front set was introduced by Hörmander **[10]** and, independently, by Sato (he called it singular spectrum). It is a basic notion of microlocal analysis.

Pseudodifferential operators do not increase the wave front set and this is one of the most important property of this class of operators. For example, if we apply the method of parametrix on elliptic operators, then the set of microlocal singularities will not be changed.

8.1. Sobolev spaces and the wave front set. First we recall some properties of Sobolev spaces.

A distribution f belongs to $H^s(\mathbb{R}^n)$ if and only if $(1-\Delta)^{s/2} f \in L^2(\mathbb{R}^n)$.

Note that $(1-\Delta)^{s/2}$ is an elliptic ΨDO of order s. (Note in this section we deal with operators with symbols in $S^s = S^s_{1,0}$, $s \in \mathbb{R}$.)

Let X be an open set in $\mathbb{R}^n$. Then $H^s_{loc}(X)$ is the space of distributions $f \in \mathcal{D}(X)$ such that $Af \in L^2_{loc}(X)$ where A is proper elliptic pseudodifferential operator of order s. Note, $f \in L^2_{loc}(X)$ if and only if for every $\varphi \in C_0^\infty(X)$, $f\varphi \in L^2(X)$.

Proposition 8.1. *(1) Let $f \in \mathcal{D}'(X)$. Then $f \in H^s_{loc}(X)$ if and only if $Af \in L^2_{loc}(X)$ for every proper ΨDO A of order s.*

(2) $A(H^s_{loc}(X)) \subset H^{s-m}_{loc}(X)$ for every ΨDO A of order m.

Proof. (1) Let A be a proper ΨDO. Then $Sf \in L^2_{loc}$ since $Af = AB^{-1}Bf$, where B is a proper elliptic operator of order s. Thus the assertion follows.

(2) Since the composition of two proper operators of orders m_1 and m_2 is a proper one of order $m_1 + m_2$, Part 1 implies that $A(H^s_{loc}) \subset H^{s-m}_{loc}$, where A is a proper pseudodifferential operator of order m. □

Note that $U \to H^s_{loc}(U)$ is a sheaf with respect to the restrictions. (For the definition of a sheaf we refer to next section)

Definition 8.2 Let K be a compact subset of X. Define $H^s_K = H^s_{loc}(X) \cap \mathcal{D}'_K$ (where $\mathcal{D}'_K$ denotes the space of distributions with supports in K).

With the appropriate scalar product, $H^s_K(X)$ is a Hilbert space ($H^s_{loc}(X)$ is a Frechét space).

The following assertion is important for the microlocal analysis of distributions.

Theorem 8.3. *Let A be a proper elliptic pseudodifferential operator of order m on X and $f \in \mathcal{D}'(X)$. If $Af|_{X'} \in H^s_{loc}(X')$, then $f|_{X'} \in H^{s+m}_{loc}(X')$, where $X' \subset X$, X is an open set.*

Proof. Let B be a proper operator in $HS^{-m_1,-m}$ which is a parametrix for A ($BA = I + R$, where R is a smoothing operator). We have shown in Proposition 7.9 that for every $f \in \mathcal{D}'(X)$, $BAf - f \in C^\infty(X)$. Let $x \in X$ and $g = \phi Af$, where $\phi \in C_0^\infty(X)$, and $\phi = 1$ in a compact neighbourhood of x. Then $g \in H^s_{loc}(X)$ and $g - Af|_V = 0$, where $V = \operatorname{int} K$. Moreover, $(Bg - BAf)|_V, (Bg - f)|_V \in C^\infty(V)$ and since B is of the order $-m$, by Proposition 8.1, (2) it follows that $Bg \in H^{s+m}_{loc}(X)$. So $f|_V \in H^{s+m}(V)$. This holds for every $x \in X'$ and this implies $f|_{X'} \in H^{s+m}_{loc}(X')$. □

Definition 8.4. Let X be open in $\mathbb{R}^n$, $(x_0, \xi_0) \in X \times (\mathbb{R}^n \backslash \{0\})$ and $u \in \mathcal{D}'(X)$. Then (x_0, ξ_0) is not in $\mathrm{WF}(u)$ if there exists $v \in \mathcal{E}'(X)$ such that $u = v$ in a neighbourhood of x_0 and there exists $\varepsilon > 0$ such that for every $N > 0$ there exists $C_N > 0$ such that

$$|\hat{v}(\xi)| \le C_N(1 + \xi^2)^{-N/2} \text{ for} \left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right| < \varepsilon, \tag{8.1}$$

that is, $\hat{v}(\xi)$ rapidly decreases in a conic neighbourhood of ξ_0. In this case it is said that u is microlocally regular in (x_0, ξ_0).

The closed conic set $\mathrm{WF}(u) \subset X \times (\mathbb{R}^n \backslash \{0\})$ (closure in $X \times \mathbb{R}^n \setminus \{0\}$ of the complement of the set of all microlocally regular points) is called the wave front set of the distribution u.

Theorem 8.5. *If (x_0, ξ_0) is not in $\mathrm{WF}(u)$, then (x_0, ξ_0) is not in $\mathrm{WF}(\varphi u)$ for every $\varphi \in C_0^\infty$.*

Proof. Let Γ_ε be an open cone of the form $\Gamma_\varepsilon = \{\eta \mid \frac{\eta}{|\eta|} - \frac{\xi_0}{|\xi_0|} < \varepsilon\}$ and $v \in \mathcal{E}'$, $u = v$ in a neighbourhood of x_0. Then

$$\widehat{(\varphi v)}(\xi) = \int_{|\eta| \le R} \hat{v}(\xi - \eta)\hat{\varphi}(\eta)\, d\eta + \int_{|\eta| > R} \hat{v}(\xi - \eta)\hat{\varphi}(\eta)\, d\eta,$$

for $\xi \in \Gamma_\varepsilon$, where R will be determined later. We have

$$|\widehat{(\varphi v)}(\xi)| \le \bar{C} \sup_{|\eta| \le R} |\hat{v}(\xi - \eta)| + CC_L \int_{|\eta| > R} (1 + |\xi - \eta|)^p (1 + |\eta|)^{-L}\, d\eta$$

where we have used the Paley-Wiener theorem for $v \in \mathcal{E}'$ and $\varphi \in C_0^\infty$ $\left(|\hat{v}(\xi)| \le C(1 + |\xi|)^p,\ |\hat{\varphi}(\xi)| \le \frac{C_L}{(1+|\xi|)^L}\right)$. This implies

$$\begin{aligned} |\hat{\varphi v}(\xi)| &\le \bar{C} \sup_{|\eta| \le R} |\hat{v}(\xi - \eta)| + CC_L(1 + |\xi|)^p \int_{|\eta| > R} (1 + |\eta|)^{p-L}\, d\eta \\ &\le \bar{C}C \sup_{|\eta| \le R} |\hat{v}(\xi - \eta)| + CC_L(1 + |\xi|)^p R^{n+p-L}. \end{aligned}$$

Put $R = |\xi|^{1/2}$. If ξ belongs to a cone $\Gamma_{\varepsilon'}$, $\varepsilon' < \varepsilon$, then $\xi - \eta \in \Gamma_\varepsilon$ for large enough ξ and $|\eta| < R$. Beside that, $|\xi - \eta| \approx |\xi|$ and $R^{n+p-L} \approx |\xi|^{(n+p-L)/2}$. For large enough L we obtain that $\widehat{(\varphi v)}(\xi)$ rapidly decreases when $|\xi| \to \infty, \xi \in \Gamma_{\varepsilon'}$. □

By this theorem it follows that in Definition 7.10 we can take $v = \varphi u, \varphi \in C_0^\infty(X), \varphi = 1$ in a neighbourhood of x_0.

Example 8.1. 1. $\mathrm{WF}(\delta(x)) = \{(0, \xi),\ \xi \in \mathbb{R}^n \setminus \{0\}\}$. 2. Since $\delta(x_1) = \delta(x_1) \otimes \kappa_{\mathbb{R}^{n-1}}$, where $\mathbb{R}^{n-1} = \{x' = (x_2, \dots, x_n)\}$ and $\kappa_{\mathbb{R}^{n-1}} = 1$ for $x' \in \mathbb{R}^{n-1}$, it follows that $\mathrm{WF}(\delta(x_1)) = \{((0, x'), (\xi_1, 0)),\ x' \in \mathbb{R}^{n-1},\ \xi_1 \in \mathbb{R} \setminus \{0\}\}$.

Proposition 8.6. *Let $\pi : X \times (\mathbb{R}^n \setminus \{0\}) \to X$ be the natural projection and let $u \in \mathcal{D}'(X)$. Then $\pi\, \mathrm{WF}(u) = \operatorname{Sing\,supp} u$.*

Proof. If x_0 is not element of $\operatorname{Sing\,supp} u$, then by taking $\varphi \in C_0^\infty(X), \varphi(x) = 1$ in a neighbourhood of x_0, and $\varphi(x) = 0$ in a neighbourhood of $\operatorname{Sing\,supp} u$, we obtain that $\varphi u \in C_0^\infty(X)$. This implies $\widehat{(\varphi v)} \in \mathcal{S}(\mathbb{R}^n)$ and thus x_0 is not in $\pi\, \mathrm{WF}(u)$.

Let $x_0 \notin \pi\, \mathrm{WF}(u)$. For every $\xi_0 \in S^{n-1}$ there exist $\varphi_{\xi_0} \in C_0^\infty(X)$ and a conic neighbourhood Γ_{ξ_0} of ξ_0 such that $\varphi_{\xi_0}(x) = 1$ in a neighbourhood of x_0 and $\widehat{(\varphi_{\xi_0} u)}(\xi)$ rapidly decreases in Γ_{ξ_0}. Since S^{n-1} is compact there exist finitely many points $\xi_1, \dots, \xi_N$ such that S^{n-1} is covered by $\Gamma_{\xi_1} \cap S^{n-1}, \dots, \Gamma_{\xi_N} \cap S^{n-1}$. Thus, $\Gamma_{\xi_1}, \dots, \Gamma_{\xi_N}$ cover $\mathbb{R}^n \setminus \{0\}$. Then, by putting $\varphi = \prod_{j=1}^N \varphi_{\xi_j}$, we obtain that $\widehat{(\varphi u)}$ rapidly decreases, and this means $\varphi u \in C_0^\infty(X)$, i.e. $u \in C^\infty$ in a neighbourhood of x_0. So, x_0 is not in $\operatorname{Sing\,supp} u$. □

Proposition 8.7. *Let $u \in \mathcal{D}'(X)$ and $(x_0, \xi_0) \notin \mathrm{WF}(u)$. Then there exists a classical ΨDO A of order 0 such that $\sigma_A = 1(\operatorname{mod} S^{-\infty})$ in a conic neighbourhood of (x_0, ξ_0) and $Au \in C_0^\infty(X)$.*

(Recall that a conic neighbourhood of (x_0, ξ_0) is of the form $U \times \Gamma_{R,\xi_0}$, where U is a neighbourhood of x_0 and Γ_{R,ξ_0} a cone around ξ_0 (cf. Definition 7.18)

Proof. Let $\varphi \in C_0^\infty(X), \varphi = 1$ around x_0. Then $\widehat{(\varphi u)}(\xi)$ rapidly decreases in a conic neighbourhood of ξ_0. Let $\chi(\xi) \in C^\infty(\mathbb{R}^n), \chi(t\xi) = \chi(\xi)$ for $t \geq 1$, $|\xi| \geq 1$ (χ is homogeneous of order 0 for $|\xi| \geq 1$.), $\chi(\xi) = 1$ in some small enough neighbourhood of ξ_0. This means that $\chi(\xi)\widehat{(\varphi u)}(\xi)$ rapidly decreases, so $\chi(D)(\varphi(x)u(x)) \in C^\infty(X)$. But then $\psi(x)\chi(D)(\varphi(x)u(x)) \in C_0^\infty(X)$ if $\psi(x) \in C_0^\infty(X)$. We can take ψ such that $\psi(x) = 1$ in a neighbourhood of x_0. Then $A = \psi(x)\chi(D)\varphi(x)$ satisfies all assertions of the proposition. $\square$

Note that the operator $A = \psi(x)\chi(D)\varphi(x)$ from the previous proposition is locally elliptic (see Definition 7.18).

Theorem 8.8. *Let $u \in \mathcal{D}'(X), (x_0, \xi_0) \in X \times (\mathbb{R}^n \backslash \{0\})$ be given as well as the classical operator A defined by the principal symbol $a_m(x, \xi) \in CS^m(X \times \mathbb{R}^n)$. Let either $u \in \mathcal{E}'(X)$ or A be proper. Suppose that $a_m(x_0, \xi_0) \neq 0$ and $Au \in C^\infty(X)$. Then $(x_0, \xi_0) \notin \mathrm{WF}(u)$.*

Proof. By Proposition 7.19 and Section 7.4, we can make the parametrix for a classical elliptic operators. So there exists a classical pseudodifferential operator B with the symbol in $CS^{-m}(X \times \mathbb{R}^n)$, such that $\sigma_{BA} = 1(\operatorname{mod} S^{-\infty})$. Since $BAu \in C^\infty(X)$ we can assume that $\sigma_A = 1(\operatorname{mod} S^{-\infty})$ in a conic neighbourhood of (x_0, ξ_0).

Let $\chi(\xi) = 1$ in a neighbourhood of ξ_0, $\chi(\xi) \in C^\infty(\mathbb{R}^n), \chi(\xi)$ is homogeneous of zero order with respect to ξ for $|\xi| \geq 1$ and let $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$, $\varphi = 1$ in a neighbourhood of x_0. Let the supports of φ, χ be chosen such that

$$\chi(\xi)\varphi(x)\sigma_A(x, \xi) = \chi(\xi)\varphi(x) \quad (\operatorname{mod} S^{-\infty}).$$

Then $\chi(D)\varphi(x)A - \chi(D)\varphi(x)$ is smoothing operator, and since $\chi(D)\varphi(x)Au \in C^\infty(X)$, it follows

$$\chi(D)\varphi(x)u \in C^\infty(\mathbb{R}^n). \tag{8.2}$$

If we prove that

$$\chi(D)\varphi(x)u \in \mathcal{S}(\mathbb{R}^n), \tag{8.3}$$

then it would follow that $\chi(\xi)\widehat{(\varphi u)}(\xi) \in \mathcal{S}(\mathbb{R}^n)$, and specially, $\widehat{(\varphi u)}(\xi)$ would rapidly decrease in a conic neighbourhood of ξ_0, what we are aimed to prove.

The implication (8.2) $\Leftarrow$ (8.3) follows from the following lemma, which is formulated separately because it has a more general meaning. $\square$

Lemma 8.9. *Let $v \in \mathcal{E}'(\mathbb{R}^n), \chi(\xi) \in S^m_{\rho,0}$, $\rho > 0$. Then for every $N > 0$ and $\alpha \in \mathbb{N}_0^n$ there exists $C_{\alpha,N}$ such that*

$$|D^\alpha \chi(D)v(x)| \le c_{\alpha,N}|x|^{-N},\ x \in \mathbb{R}^n,\ d(x, \operatorname{supp} v) \ge 1. \tag{8.4}$$

Proof. We can consider only the case $\alpha = 0$, because $\xi^\alpha \chi(\xi) \in S^{m+|\alpha|}_{\rho,0}$. Also, we may assume that v is continuous because every element $v \in \mathcal{E}'(\mathbb{R}^n)$ is of the form $v = \sum_{|\gamma|\le p} D^\gamma v_\gamma, v_\gamma \in C(\mathbb{R}^n)$. We have

$$\chi(D)v(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\xi}\chi(\xi)v(y)\, dyd\xi. \tag{8.5}$$

Integration by parts gives

$$|x-y|^{-2N}(-\Delta_\xi)^N e^{i(x-y)\xi} = e^{i(x-y)\xi}.$$

From (8.5), with $d(x, \operatorname{supp} v) \ge 1$, we have

$$\chi(D)v(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\xi}|x-y|^{-2N}((-\Delta_\xi)^N\chi(\xi))v(y)\, dyd\xi. \tag{8.6}$$

By choosing large enough N, such that $(-\Delta_\xi)^N\chi(\xi) \in S^{-n-1}_{\rho,0}$, one can see that the integral in (8.6) converges absolutely and satisfies $C(1+x^2)^{-N}$. □

Definition 8.10. Let A be a classical pseudodifferential operator with the symbol in $CS^m(X \times \mathbb{R}^n)$. Then

$$\operatorname{char}(A) = \{(x,\xi) \in X \times (\mathbb{R}^n \backslash \{0\}),\ a_m(x,\xi) = 0\}.$$

Theorem 8.8 directly implies the following important (and practical) characterization of the wave front set.

Theorem 8.11. (1) *Let $u \in \mathcal{E}'(X)$ and A be a classical ΨDO with a symbol in $CS^m(X \times \mathbb{R}^n)$. If $Au \in C^\infty(X)$, then* $\mathrm{WF}(u) \subset \operatorname{char}(A)$.

(2) *Let $u \in \mathcal{E}'(X)$. Then* $\mathrm{WF}(u) = \bigcap \operatorname{char}(A)$, *where the intersection is taken over all classical operators of the order zero (with the symbols in $CS^0(X \times \mathbb{R}^n)$) for which $Au \in C^\infty(X)$.*

(3) *Let $u \in \mathcal{D}'(X)$. Then* $\mathrm{WF}(u) = \bigcap \operatorname{char}(A)$, *where the intersection is taken over all proper classical operators of the order zero for which $Au \in C^\infty(X)$.*

(4) *Let A be a proper ΨDO with the symbol in $CS^m(X \times \mathbb{R}^n)$, $u \in \mathcal{D}'$, or $u \in \mathcal{E}'(X)$. If $a_m(x_0,\xi_0) \ne 0$ and $(x_0,\xi_0) \notin \mathrm{WF}(Au)$, then $(x_0,\xi_0) \notin \mathrm{WF}(u)$. This means*

$$\mathrm{WF}(u) \subset \operatorname{char}(A) \cup \mathrm{WF}(Au). \tag{8.7}$$

The importance of the second assertion is that the definition of $\mathrm{WF}(u)$ makes sense if X is a manifold (see Section 9.1). This theorem gives us the estimate of the propagation of singularities of a pseudodifferential equation.

Theorem 8.12. (Microlocallity of ΨDO's) *Let $u \in \mathcal{D}'$, A be a ΨDO with symbol in $S^m_{\rho,\delta}(X \times \mathbb{R}^n)$, $0 \le \delta < \rho \le 1$ and let A be proper or $u \in \mathcal{E}'(X)$. If* $(x_0, \xi_0) \notin \mathrm{WF}(Au)$. *In other words*

$$\mathrm{WF}(Au) \subset \mathrm{WF}(u). \tag{8.8}$$

Proof. The condition $(x_0, \xi_0) \notin \mathrm{WF}(u)$ is equivalent to the existence of a proper classical ΨDO of order 0 such that $Pu \in C^\infty(X)$ and $\sigma_P = 1 (\mathrm{mod}\, S^{-\infty})$ in a conic neighbourhood of (x_0, ξ_0). Let Q be a proper classical ΨDO of order zero such that $q_0(x_0, \xi_0) \neq 0$ (q_0 is the main symbol of Q) and $\sigma_Q \in S^{-\infty}$ outside some small conic neighbourhood of (x_0, ξ_0) and

$$PQ = Q \text{ and } QP = Q \quad \text{(mod smoothing operators).}$$

We shall show that $QAu \in C^\infty(X)$ because $\sigma_P = 1$ (mod $S^{-\infty}$) in a conic neighbourhood of (x_0, ξ_0) and $\sigma_{QA} - \sigma_{QAP} \in S^{-\infty}$ in this neighbourhood, and $\sigma_Q \in S^{-\infty}$ out of it. We have that $QA - QAP$ is smoothing. So, it is enough to verify that $QAPu \in C^\infty(X)$. But this follows immediately, because $Pu \in C^\infty(X)$. The fact that $(x_0, \xi_0) \notin \mathrm{WF}(Au)$ follows from the previous theorem. □

From the two previous theorems we have the following theorem.

Theorem 8.13. *If $u \in \mathcal{E}'(X)$ and A is a classical ΨDO with a symbol in $CS^m(X \times \mathbb{R}^n)$, then*

$$\mathrm{WF}(Au) \subset \mathrm{WF}(u) \subset \mathrm{WF}(Au) \cup \mathrm{char}(A).$$

With the assumption that A is proper, the assertion holds for $u \in \mathcal{D}'(X)$. Specially, if the operator A is elliptic, then $\mathrm{WF}(Au) = \mathrm{WF}(u)$.

8.2. Microfunctions. In this section we shall present the notion of a microfunction by following **[11]**. Microfunctions are the equivalence classes in the space of distributions whose representatives are determined only with their singularities.

First, we shall present some of the basic facts of sheaf theory.

Let X be a topological space, U be an open set in X. Let $\{\mathcal{F}(U)\}_{U \text{open set in} X}$ be a family of vector spaces. For U such that $V \subset U$ there exists a linear mapping $\rho_{VU} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\mathcal{F}(U)$ is a vector space of the functions on U and

$$\rho_{UU} = \mathrm{id} \text{ and } \rho_{WV} \circ \rho_{VU} = \rho_{WU},$$

for $W \subset V \subset U$.

The family $\{\mathcal{F}(U), \rho_{U,V}, U, V \subset X\}$ is a presheaf. $\mathcal{F}(U)$ is called the set of sections. In the sequel we shall consider the case when $\mathcal{F}(U)$ is a subspace of $\mathcal{F}(V)$ and if $\rho_{U,V}$ is a restriction of $f \in \mathcal{F}(U)$, then $\rho_{U,V} f = f_{|V}$ is a restriction of f to V for $V \subset U$.

Presheaf is a sheaf if the following two conditions are satisfied.

(i) Let $U = \bigcup_{\lambda \in \Lambda} U_\lambda$ (all sets are open) and $f \in \mathcal{F}(U)$. If for every $\lambda \in \Lambda$ $f|_{U_\lambda} = 0$, then $f|_U = 0$.

(ii) Let $f_\lambda \in \mathcal{F}(U_\lambda)$ and let for every $\lambda, \mu \in \Lambda$, $f_\lambda = f_\mu$ in $U_\lambda \cap U_\mu$. Then there exists $f \in \mathcal{F}(\bigcup_{\lambda\in\Lambda} U_\lambda)$ such that $f|_{U_\lambda} = f_\lambda$.

Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves or sheaves on a topological space X. The family $h = \{h_U\}$ of linear mappings $\mathcal{F}(U) \to \mathcal{G}(U)$ is a (pre)sheaf homomorphism if the following diagram commutes

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "h_U"] & \mathcal{G}(U) \\ \mathcal{F}(V) \arrow[u, "\rho^{\mathcal{F}}_{VU}"] \arrow[r, "h_V"'] & \mathcal{G}(V) \arrow[u, "\rho^{\mathcal{G}}_{VU}"'] \end{tikzcd}$$

Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on a topological space X. Then $\mathcal{F}$ is a subpresheaf of $\mathcal{G}$ if for every open set U there exists associated inclusion $i_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that the family $i = \{i_U\}$ is a presheaf homomorphism. In the same way we define a subsheaf.

Let $\mathcal{F}$ be a (pre)sheaf on X and $x \in X$. Then $\mathcal{F}_x = \operatorname{limind}_{x\in U} \mathcal{F}(U)$ is called a stalk in x. An element in $\mathcal{F}_x$ is called a section germ or a germ of $\mathcal{F}$ in x.

For a presheaf $\mathcal{F}$ one can construct a sheaf $\overline{\mathcal{F}}$ with the same stalks as in $\mathcal{F}$. This sheaf is called the associated sheaf for presheaf $\mathcal{F}$. If a presheaf $\mathcal{F}$ satisfies condition (i) for sheaves, then its associated sheaf is simply defined:

$$\overline{\mathcal{F}}(U) = \operatorname{limind}_{\{U_\lambda\}} \{(s_\lambda) | s_\lambda \in \mathcal{F}(U_\lambda), s_\lambda|_{U_\lambda \cap U_\mu} = s_\mu|_{U_\lambda \cap U_\mu}\},$$

where U_λ are open subsets of U.

Now we shall present the definition of a microfunction.

Let X be an open set in $\mathbb{R}^n$ and $SX = X \times S^{n-1}$. Let U be an open set in SX and CU be a cone generated by U in $X \times \mathbb{R}^n$:

$$CU = \{(x, \lambda\xi) | (x, \xi) \in U, \lambda > 0\}.$$

Let us define

$$O^m(U) = S^m(CU)/S^{-\infty}(CU) \text{ and } O(U) = \bigcup_{m\in\mathbb{N}} O^m(U) = S^\infty(CU)/S^{-\infty}(CU).$$

The elements of these sets are called classes of pseudodifferential operators (of order m) on U. If there are no misunderstandings, we shall omit the word "class".

Let us define

$$\operatorname{Sing}(X) = \mathcal{D}'(X)/C^\infty(X).$$

This is a space of singularities on X. The family $\operatorname{Sing}(X)$, $X \subset \mathbb{R}$, is a sheaf. For $f \in \mathcal{D}'$, the support of f in $\operatorname{Sing}(X)$ is $\operatorname{Sing\,supp} f$ in $\mathcal{D}'$. ΨDO acts as a local operator on the space of singularities, which means that it does nod increase the singular support of the distribution (pseudolocality).

Definition 8.14. Let $f \in \mathcal{D}'(X)$ and $(x, \xi) \in X \times \mathbb{R}^n \setminus \{0\}$. It is said that f is a C^∞-function in (x, ξ) if there exists a proper ΨDO A, elliptic in (x, ξ), such that $Af \in C^\infty(X)$. Singular spectrum of f, $\operatorname{Sing}\operatorname{Sp} f$, is the closure of the set of all points (x, ξ) in $X \times \mathbb{R}^n$ in which f is not C^∞.

Definition 8.15. Let $f \in \mathcal{D}'(X)$ and U be an open set in SX. We say that $f \in C^\infty(U)$ if $\operatorname{Sing}\operatorname{Sp} f \cap U = \emptyset$. The microfunction defined by f in U is a class of f modulo the space $C^\infty(U)$.

By Section 8.2, one can see that the notions of WF and Sing Sp are equivalent, so in the sequel we shall use only the notion of the wave front set instead of singular spectrum.

We shall define the sheaf of the microfunctions in one dimension.

Recall, $\mathcal{D}'(X), X \subset \mathbb{R}$ is a sheaf of the distributions and $S^0 = \{-1, 1\}$ is the unit circle in $\mathbb{R}$.

We say that $u \in \mathcal{D}'(X)$ is microanalytical in $(x, 1)$ (resp. $(x, -1)$), $x \in X$ if there exists a neighbourhood U of x and $v \in \mathcal{E}'$, $u = v$ on U such that for every $N \in \mathrm{N}$ there exists a constant C_N such that

$$|\mathcal{F}(v)(\xi)| < C_N(1 + \xi^2)^{-N/2}, \ \xi > 0 \text{ (resp. } \xi < 0).$$

The point (x, ξ_0) (where $\xi_0 = 1$ or -1) is in $\mathrm{WF}\, u$ if and only if it is not microanalytical in (x, ξ_0).

Let us define a subsheaf $C^{\infty *}$ of the sheaves $\mathcal{D}'(X) \times \{-1\} \oplus \mathcal{D}'(X) \times \{1\}$ in the following way. *Definition* 8.16. Let

$$\begin{aligned} C^{\infty *} &= \{f \in \mathcal{D}'(X);\ \mathrm{WF}(u) \cap X \times \{-1\} = \emptyset\} \\ &\oplus \{f \in \mathcal{D}'(X);\ \mathrm{WF}(u) \cap X \times \{1\} = \emptyset\}. \end{aligned}$$

The associated sheaf for a presheaf $\mathcal{D}'(X) \times \{-1\} \oplus \mathcal{D}'(X) \times \{1\}/C^{\infty *}$ is denoted by $\mathcal{C}$ and it is called the sheaf of microfunctions.

Intuitively, $f \in \mathcal{D}'(X)$ defines a germ in (x, ξ_0) ($\xi_0 = \pm 1$) modulo germs of any $C^\infty_{(x,\xi_0)}$-function which are microlocal in (x, ξ_0).

The support of a microfunction is a wave front set of a distribution which defines it.

9. Change of variables

Let $(y, \eta) \to (x, \xi)$, $(y, \eta) \in V$, $(x, \xi) \in U$, be a diffeomorphism where U and V are conic regions in $\mathbb{R}^n \times \mathbb{R}^N$ and $\mathbb{R}^{n_1} \times \mathbb{R}^N$, respectively, $x = x(y, \eta)$, $\xi = \xi(y, \eta)$, where $x(y, \eta)$ is positively homogeneous of order 0 and $\xi(y, \eta)$ positively homogeneous of order 1 with respect to η. Let $b(y, \eta) = a(x(y, \eta), \xi(y, \eta))$.

Theorem 9.1. *Let $a(x, \xi) \in S^m_{\rho,\delta}(U)$. Assume that one of conditions*
a) $\rho + \delta = 1$; b) $\rho + \delta \geq 1$ and $x = x(y)$; c) $x = x(y)$, $\xi = \xi(\eta)$;
holds. Then $b \in S^m_{\rho,\delta}(V)$.

Let us consider the oscillating integral

$$I_\phi(au) = \iint e^{i\phi(x,\xi)} a(x, \xi) u(x)\, dx\, d\xi \text{ [osc]} = \langle A(x), u(x) \rangle,$$

where

$$A(x) = \int e^{i\phi(x,\xi)} a(x,\xi)\, d\xi \text{ [osc]}$$

and where we use the same notions as in the Section 5.

A phase function $\phi(x,\xi)$ is called regular if $d(\partial\phi/\partial\xi_j)$, $j = 1,\dots,N$, is a linearly independent set in C_ϕ, i. e. if the range of the matrix $(\phi_{\xi\xi}\phi_{\xi x})_{N\times(N+n)}$ equals to N.

(We shall use the notation

$$d(\partial\phi/\partial\xi_j) = \sum_{k=1}^{N} \frac{\partial\phi}{\partial\xi_k\partial\xi_j} d\xi_k + \sum_{k=1}^{n} \frac{\partial\phi}{\partial x_k\partial\xi_j} dx_k,$$

and let us remind that $C_\phi = \{(x,\xi),\ \phi_\xi(x,\xi) = 0\}$ and $R_\phi = X \setminus \mathrm{C}$.)

Let $a \in S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ and $a = 0$ in a conic neighbourhood of C_ϕ. Then $A \in C^\infty(\mathbb{R}_\phi)$ and one can simply prove that $A \in C^\infty(X)$.

The following lemma is interesting in its own. It is called Hadamard's lemma.

Lemma 9.2. *Let $\phi_1(x,\xi),\dots,\phi_k(x,\xi)$ be in $C^\infty(U)$ and let them be positively homogeneous of order 0 with respect to ξ. Let $d\phi_1,\dots,d\phi_k$ be linearly independent on the set $C = \{(x,\xi) \in U | \phi_j(x,\xi) = 0,\ j = 1,\dots,k\}$ and $a \in S^m_{\rho,\delta}(U)$, $a|_C = 0$ and $\rho + \delta = 1$. Then there exists $a_j(x,\xi) \in S^{m+\delta}_{\rho,\delta}(U)$, $j = 1,\dots,k$ such that*

$$a = \sum_{j=1}^{k} a_j\phi_j. \tag{9.1}$$

If $a(x,\xi)$ has a zero of infinite order on C, then the same holds for all $a_j(x,\xi)$ on C as well.

Theorem 9.3. *Let ϕ be a regular phase function, $a \in S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ and let one of the following conditions hold:*

1) $\rho > \delta$ and $\rho + \delta = 1$, 2) $\rho > \delta$ and ϕ is linear with respect to ξ.

Then: a) If a has a zero of infinite order in C_ϕ, then $A(x) \in C^\infty(X)$.

b) If $a = 0$ in C_ϕ, then there exists $b \in S^{m-(\rho-\delta)}_{\rho,\delta}(X \times \mathbb{R}^N)$ such that

$$I_\phi(au) = I_\phi(bu) \text{ for every } u \in C_0^\infty(X).$$

Proof. Suppose that 1) holds. If $a|_C = 0$, by previous lemma, we can write

$$a = \sum_{j=1}^{N} a_j\phi_j,\ a_j \in S^{m+\delta}_{\rho,\delta}(U), \tag{9.2}$$

where $\phi_j = \partial\phi/\partial\xi_j$. By using the fact that $\phi_j e^{i\phi} = -i\frac{\partial}{\partial\xi_j}e^{i\phi}$ and integrating by parts, we obtain

$$I_\phi(au) = \sum_{j=1}^{N} I_\phi\Big(i\frac{\partial a_j}{\partial\xi_j}u\Big).$$

Since $\frac{\partial a_j}{\partial \xi_j} \in S^{m+\delta-\rho}_{\rho,\delta}(U)$, we have proved the assertion a). This implies that if a has an infinite order zero in C_ϕ, then b can be chosen such that it has the same property. Thus, we can transfer the assertion a) to the case $a(x,\xi) \in S^{-M}_{\rho,\delta}(X \times \mathbb{R}^N)$, where M is arbitrary large. But, then the integral

$$A(x) = \int_{\mathbb{R}^N} e^{i\phi(x,\xi)} a(x,\xi)\, d\xi$$

absolutely uniformly converges with respect to x, as well as all the integrals which can be obtained from it by differentiating the integrand with respect to x of order up to $l(M)$, where $l(M) \to \infty$ as $M \to \infty$, which implies the smoothness of $A(x)$. □

Let $\kappa : X \to X_1$ (X and X_1 are open), $x = \kappa(t), x \in X_1 \subset \mathbb{R}^n, t \in X \subset \mathbb{R}^n$ be a diffeomorphism. Then the induced mapping, the pull back, $\kappa^* : C^\infty(X_1) \to C^\infty(X)$ is defined by $(\kappa^*\psi)(t) = (\psi \circ \kappa)(t) = \psi(\kappa(t))$.

Let A be a ΨDO on X. We define $A_1 : C_0^\infty(X_1) \to C^\infty(X_1)$ by the diagram

$$\begin{array}{ccc} C_0^\infty(X) & \xrightarrow{A} & C^\infty(X) \\ {\scriptstyle\kappa^*}\uparrow & & {\scriptstyle\kappa^*}\uparrow\downarrow{\scriptstyle\kappa_1^*} \\ C_0^\infty(X_1) & \xrightarrow[A_1]{} & C^\infty(X_1) \end{array}$$

where $\kappa_1 = \kappa^{-1}$. Then

$$A_1 u = (A(u \circ \kappa)(x)) \circ \kappa_1, \text{ i.e.}$$

$$A_1 u(x) = (2\pi) \iint_{\mathbb{R}^{2n}} e^{i(\kappa_1(x)-p)\xi} a(\kappa_1(x), p, \xi) u(\kappa(p))\, dp\, d\xi.$$

If we change the variables by $p = \kappa_1(y)$, then

$$A_1 u(x) = (2\pi) \iint_{\mathbb{R}^{2n}} e^{i(\kappa_1(x)-\kappa_1(y))\xi} a(\kappa_1(x), \kappa_1(y), \xi) u(y) \Big| \frac{\partial \kappa_1}{\partial y} \Big|\, dy\, d\xi, \tag{9.3}$$

where $\partial p/\partial y = \partial \kappa_1/\partial y$ and $|\partial \kappa_1/\partial y|$ is Jacobian.

This means that A_1 is a Fourier integral operator with the phase function $\phi(x,y,\xi) = (\kappa_1(x) - \kappa_1(y))\xi$.

Theorem 9.4. *With the above notation, A_1 is a pseudodifferential operator for $1 - \rho \le \delta < \rho$.*

This will be a special case of the following theorem.

Theorem 9.5. *Let ϕ be a phase function on $X \times X \times \mathbb{R}^n$ such that*

1) $\phi(x,y,\xi)$ is linear with respect to ξ.

2) $\phi'_\xi(x,y,\xi) = 0$ if and only if $x = y$.

Let A be a Fourier integral operator

$$Au(x) = \iint_{\mathbb{R}^{2n}} e^{i\phi(x,y,\theta)} a(x,y,\theta) u(y)\, dy d\theta, \tag{9.4}$$

where $a \in S^m_{\rho,\delta}$ and $1-\rho \le \delta < \rho$. Then A is a pseudodifferential operator with an amplitude in $S^m_{\rho,\delta}$.

We need the following lemma for the proof.

Lemma 9.6. *Let assumptions 1) and 2) of Theorem 9.5 hold. Then there exists a neighbourhood X of the diagonal Δ and a C^∞ mapping $\psi : X \to \mathrm{Gl}\,(n,\mathbb{R})$ (regular matrices of order n on $\mathbb{R}$) such that:*

a) $\phi(x,y,\psi(x,y)\xi) = \langle (x-y),\xi\rangle, (x,y) \in X^2$.

b) $\det\psi(x,x)\cdot\det\phi''_{x\xi}(x,y,\xi)|_{y=x} = 1$.

Proof. By 1),

$$\phi(x,y,\theta) = \sum_{j=1}^n \phi_j(x,y)\theta_j.$$

Now, by 2) we have $\phi_j(x,x) = 0$ and if $\phi_j(x,y) = 0$ for $j = 1,\dots,n$, then $x = y$. Note,

$$(\phi'_x,\phi'_y) = \phi'_{x,\theta} = \Big(\sum_{j=1}^n \frac{\partial\phi}{\partial x_1}\theta_j,\dots,\sum_{j=1}^n \frac{\partial\phi}{\partial x_n}\theta_j,\phi_1,\dots,\phi_n\Big).$$

By differentiating the expression $\phi(x,x,\theta) = 0$ with respect to x, it follows

$$\phi'_x(x,y,\theta)_{|x=y} + \phi'_y(x,y,\theta)_{|x=y} = 0, \quad \text{i.e.}$$

$$\phi'_x(x,x,\theta) = -\phi'_y(x,x,\theta). \tag{9.5}$$

From $\phi'_\theta(x,x,\theta) = 0$ and $\phi'_{x,y,\theta}(x,y,\theta)_{|x=y} \neq 0$ it follows $\phi'_x(x,y,\theta)_{|x=y} \neq 0$. If this is not true, then (9.5) implies $\phi'_y(x,y,\theta)_{|x=y} = 0$, i. e. $\phi'_{x,y,\theta}(x,y,\theta)_{|x=y} = 0$, which gives a contradiction. This means that there exists $k \in \{1,\dots,n\}$ such that $\sum_{j=1}^n \frac{\partial\phi_j}{\partial x_k}\theta_{j|x=y} \neq 0$, so

$$\det\Big(\frac{\partial\phi_j}{\partial x_k}(x,y)\Big)\Big|_{x=y} \neq 0. \tag{9.6}$$

By Hadamard's lemma (Lemma 9.2), for close enough x and y we have

$$\phi_j(x,y) = \sum_{k=1}^n \phi_{kj}(x,y)(x_k - y_k),$$

where $\phi_{kj} \in C^\infty(X'), X'$ is some neighbourhood of the diagonal in $X\times X$. We also have

$$\phi_{kj}(x,x) = \frac{\partial\phi_j(x,y)}{\partial x_k}\Big|_{x=y}. \tag{9.7}$$

Denote by $\phi(x,y)$ the matrix $(\phi_{kj}(x,y))$. From (9.6) and (9.7) it follows that there exists a neighbourhood Ω of the diagonal in $X\times X$ such that $\det\phi(x,y)\neq 0$ for $(x,y)\in\Omega$. Let

$$\psi(x,y) = \phi(x,y)^{-1} \quad \text{(the inverse of } \phi) \tag{9.8}$$

Since

$$\phi(x,y,\theta) = \sum_{j,k=1}^{n} \phi_{kj}(x,y)\theta_j(x_k - y_k) = \langle (x-y), \phi(x,y)\theta\rangle,$$

by putting $\phi(x,y)\theta = \xi$ we obtain a), while b) follows from (9.7) and (9.8). □

Proof of Theorem 9.5. We assume that $a(x,y,\theta)$ equals 0 for $(x,y) \in X \times X \setminus \Omega'$, where $\Omega' \subset \Omega$ and Ω' is a neighbourhood of Δ. By putting $\theta = \phi(x,y)^{-1}\xi$ in (9.4) we obtain

$$Au(x) = \iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} a(x,y,\psi(x,y)\xi)|\det\psi(x,y)|u(y)\,dy\,d\xi.$$

From Theorem 9.1 it follows that $a_1(x,y,\xi) = a(x,y,\psi(x,y)\xi)$ is in $S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n)$.

9.1. Pseudodifferential operators on C^∞-manifolds. We will give the definition of pseudodifferential operators on a manifold, but before that we shall recall the definitions of the theory of generalized functions on a manifold. Let us remind that Hausdorff topological space M is locally Euclidean of dimension n if every point in M has a neighbourhood which is homeomorphic to an open subset of $\mathbb{R}^n$.

If φ is a homomorphism of an open set $U \subset M$ on an open subset of $\mathbb{R}^n$, φ is called the coordinate mapping and (U,φ) is called the coordinate system or coordinate section. Recall, a differentiable structure $\mathcal{F}$ of the class C^k, $k \in [1,\infty]$, on a locally Euclidean space M is a collection of coordinate systems $\{(U_\alpha, \varphi_\alpha), \alpha \in A\}$ which satisfies:

(i)] $\bigcup_{\alpha\in A} U_\alpha = M$.

(ii) $\varphi_\alpha \circ \varphi_\beta^{-1}$ is of the class C^k in $\varphi_\beta(U_\alpha \cap U_\beta)$ for every $\alpha,\beta \in A$.

(iii) The collection $\mathcal{F}$ is maximal with respect to (ii) which means that if (U,φ) is a coordinate system such that $\varphi \circ \varphi_\alpha^{-1}$ and $\varphi_\alpha \circ \varphi^{-1}$ are of the class C^k for every $\alpha \in A$, then $(U,\varphi) \in \mathcal{F}$.

If $\mathcal{F}_0 = \{(U_\alpha, \varphi_\alpha), \alpha \in A\}$ is an arbitrary collection of coordinate systems satisfying (i) and (ii), then there exists a unique differentiable structure $\mathcal{F}$ containing $\mathcal{F}_0$. $\mathcal{F}_0$ is called the atlas of a manifold M.

In the sequel we shall consider only C^∞-manifolds. Let M and N be C^∞-manifolds.

Let $O \subset M$ be open. Then $F : O \to \mathbb{R}$ is a C^∞-function on O, $(f \in C^\infty(O))$ if $f \circ \varphi^{-1}|_{\varphi(U\cap O)}$ is a C^∞-function for every coordinate section (U,φ).

A mapping $\psi : M \to N$ is of the class C^∞ if for every two coordinate sections (U,φ) on M and (U_1,φ_1) on N, $\varphi_1 \circ \psi \circ \varphi^{-1}|_{\varphi(U)}$ is a C^∞-function.

The important construction in the analysis on manifolds is the partition of unity. Let M be a manifold and $\mathcal{U} = \{U_\alpha,\ \alpha \in A\}$ be a cover of M. Then there exists a C^∞ partition of unity $\{\varphi_i,\ i \in \mathbb{N}\}$ corresponding to the cover $\mathcal{U}$ such that $\operatorname{supp}\varphi_i$ is compact for every $i \in \mathbb{N}$ and $\operatorname{supp}\varphi_i \subset U_\alpha$ for some $\alpha \in$A.

If $v \in C_0^\infty(\tilde{U})$ (where $\tilde{U} = \varphi(U)$), then we define

$$u = \begin{cases} v \circ \varphi, & \text{in } U, \\ 0, & \text{otherwise.} \end{cases}$$

The definition is the same if $v \in C^k(\tilde{U})$ or $v \in L^p(\tilde{U})$. We shall use the notation $u = v \circ \varphi$.

Let $u \in C^k(M)$ and $u_k = u \circ \varphi_k^{-1}$, where (U_k, φ_k) is an arbitrary coordinate section. There holds:

(a) $u = u_k \circ \varphi_k = u_{k'} \circ \varphi_{k'}$ on $U_k \cap U_{k'}$.

(b) $u_k = u_{k'} \circ (\varphi_{k'} \circ \varphi_k^{-1})$, which is denoted by $(\varphi_{k'} \circ \varphi_k^{-1})^* u_{k'} = u_k$.

Conversely, if (b) holds for arbitrary sections (U_k, φ_k) and $(U_{k'}, \varphi_{k'})$, then there exists a unique function $u \in C^k(M)$ satisfying (a).

Definition 9.7. Let $\mathcal{F} = \{(U_k,\ \varphi_k), k \in A\}$ be a differentiable structure of a manifold M. If there exists a distribution u_k in $\mathcal{D}'(\varphi_k(U_k))$ for every coordinate section (U_k, φ_k) and if

(c) $u_k = u_{k'} \circ (\varphi_{k'} \circ \varphi_k^{-1})$ on $\varphi_k(U_k \cap U_{k'})$,

then $\{u_k,\ k \in A\}$ is a distribution in M. We shall denote it by $u \in \mathcal{D}'(M)$, and that is in fact the notation for the family $\{u_k,\ k \in A\}$. We shall use the notation $u_k = u \circ \varphi_k^{-1}$.

This definition generalizes the definition of a function in $C^k(M)$. The proof of the next theorem is omitted.

Theorem 9.8. *Let $\mathcal{F} = \{(U_k,\ \varphi_k),\ k \in A_0\}$ be an atlas for M. If $\{u_k,\ k \in A_0\}$ is a family of distributions in $\mathcal{D}'(\varphi_k(U_k))$ satisfying (c) for every k, k' in A_0, Then there exist one and only one distribution $u \in \mathcal{D}'(M)$ such that*

$$u \circ \varphi_k^{-1} = u_k \text{ for every } k \in A_0.$$

There appears a natural question: Why one can not define the distribution on a manifold M as a continuous linear function on $C_0^\infty(M)$? The reason is that there does not exist an invariant procedure for the definition of the integral $\int f\phi, f \in C(M), \phi \in C_0^\infty(M)$ such that f can be identified with a continuous linear functional.

Let u be a continuous linear functional on $C_0^\infty(M)$. For every (U_k, φ_k), by

$$u_k(\phi) = u(\phi \circ \varphi_k), \phi \in C_0^\infty(\tilde{U}_k)$$

is defined an element in $\mathcal{D}'(\varphi_k(U_k))$. But $\{u_k,\ k \in A\}$ does not satisfy condition (c).

Let $\phi \in C_0^\infty(\varphi_k(U_k \cap U_{k'}))$. Then

$$\langle u_k, \phi\rangle = \langle u, \phi \circ \varphi_k\rangle = \langle u, \phi \circ \varphi_k \circ \varphi_{k'}^{-1} \circ \varphi_{k'}\rangle = \langle u_{k'}, \phi \circ \varphi_k \circ \varphi_{k'}^{-1}\rangle.$$

By the change of variables: $t = \varphi_k \circ \varphi_{k'}^{-1}(x)$ we obtain

$$\langle u_k(t), \phi(t)\rangle = \langle u_{k'}(x), \phi \circ \varphi_k \circ \varphi_{k'}^{-1}(x)\rangle = \Big\langle u_{k'}(\varphi_{k'} \circ \varphi_k^{-1}(t)), \phi(t)\Big|\frac{\partial \varphi_{k'} \circ \varphi_k^{-1}(x)}{\partial t}\Big|\Big\rangle,$$

i.e.

$$u_k = \Big|\frac{\partial \varphi_{k'} \circ \varphi_k^{-1}(x)}{\partial t}\Big| u_{k'} \circ \varphi_{k'} \circ \varphi_k^{-1}. \tag{9.9}$$

This is similar to condition (c), but now we have an additional multiplication by Jacoby's determinant which equals $|\partial \varphi_k \circ \varphi_{k'}^{-1}(x)/\partial t|$.

A family $\{u_k,\ k \in A\}$ of elements in $\mathcal{D}'(\varphi_k(U_k))$ satisfying (9.9) is called a distributional density.

In the same way we define a C^k-density by (9.9).

If a is a strictly positive C^∞-density on M, and $u \in \mathcal{D}'(M)$, then au is the distributional density, and the mapping $u \to au$ is a bijection of the space of the distributions to the space of distributional densities.

Let u be a distributional density and $x = \varphi(y)$. There holds

$$\begin{aligned}\langle \varphi_* u(x), \psi(x)\rangle &= \langle u(y), \varphi^*\psi(y)\rangle = \langle u(y), \psi(\varphi(y))\rangle \\ &= \langle u(\varphi^{-1}(x)), |J|\psi(x)\rangle,\end{aligned} \tag{9.10}$$

where $|J|$ is a Jacoby's determinant. This formula will be useful for the definition of a pseudodifferential operator on a manifold which acts on distributions with compact supports.

Let A be a linear operator, $A : C_0^\infty(M) \to C^\infty(M)$, where M is an n-dimensional C^∞-manifold. Let (U, φ) be a coordinate section of the manifolds. Then the commutative diagram

$$\begin{array}{ccc} C_0^\infty(U) & \xrightarrow{A} & C^\infty(U) \\ {\scriptstyle \varphi^*}\uparrow & & \uparrow{\scriptstyle \varphi^*} \\ C_0^\infty(\tilde U) & \xrightarrow[A_1]{} & C^\infty(\tilde U) \end{array}$$

uniquely defines the operator A_1.

Definition 9.9. $A : C_0^\infty(M) \to C^\infty(M)$ is a ΨDO on M if for every coordinate section the operator A_1 defined above, is a ΨDO on U_1.

By using (9.10) and the analogous procedure as in the case of ordinary ΨDO's and like in the previous definition we have that A is a ΨDO on a manifold if A_1 is a ΨDO on U, where A_1 is defined by the following commutative diagram

$$\begin{array}{ccc} \mathcal{E}'(U) & \xrightarrow{A} & \mathcal{D}'(U) \\ {\scriptstyle \varphi_*^{-1}}\uparrow & & \downarrow{\scriptstyle \varphi_*} \\ \mathcal{E}'(\tilde U) & \xrightarrow[A_1]{} & \mathcal{D}'(\tilde U) \end{array}$$

Theorem 9.5 ensures that a ΨDO on an open set $X \subset \mathbb{R}^n$ can be considered as ΨDO on manifold X. Theorem 9.1 shows that $S^m_{\rho,\delta}(T^*M)$ is well defined for $1-\rho \leq \delta < \rho$.

Part II. COLOMBEAU GENERALIZED FUNCTIONS AND ΨDO

In this part we present the basic concept of the pseudodifferential calculus in the frame of Colombeau's generalized functions. It is developed in **[16]**, **[17]**, **[18]**, **[12]** as well as by Oberguggenberger **[14]**.

10. Basic notions

We recall in this section the notation and notions in Colombeau's theory.

$\mathcal{A}_0(\mathbb{R}^n)$ denotes the set of the functions ϕ in $C_0^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \phi(t)\,dt = 1$, $\mathcal{A}_q(\mathbb{R}^n) = \{\phi \in \mathcal{A}_0,\ \int_{\mathbb{R}^n} t^i\phi(t)\,dt = 0,\ 0 < |i| \leq q\}$, $q \in \mathbb{N}$, where $t^i = t_1^{i_1}\cdots t_n^{i_n}$.

Obviously, if $\phi \in \mathcal{A}_q$, $q \in \mathbb{N}_0$, then for every $\varepsilon > 0$, $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$, $x \in \mathbb{R}^n$, belongs to $\mathcal{A}_q$.

If $\phi \in \mathcal{A}_0$, then it's support number $d(\phi)$ is defined by

$$d(\phi) = \sup\{|x|,\ \phi(x) \neq 0\}.$$

In the sequel we assume that ϕ in $\mathcal{A}_0$ has the support number equals one, $d(\phi) = 1$, i.e. $\operatorname{supp}\phi \subset B(0,1)$.

Denote by $\mathcal{E}[\Omega]$ the set of the functions $R : \mathcal{A}_0 \times \Omega \to \mathbb{C}$, $(\phi, x) \mapsto R(\phi, x)$, which are in $C^\infty(\Omega)$ for every fixed ϕ. Note that for any $f \in C^\infty(\Omega)$, the mapping $(\phi, x) \mapsto f(x)$, $(\phi, x) \in \mathcal{A}_0 \times \Omega$, defines an element in $\mathcal{E}[\Omega]$ which does not depend on ϕ.

The space of functions $R : \mathcal{A}_0 \to \mathbb{C}$ (resp.$\mathbb{R}$) is denoted by $\mathcal{E}_0(\mathbb{C})$ (resp. $\mathcal{E}_0(\mathbb{R})$). It is an algebra and it is subalgebra of $\mathcal{E}[\Omega]$ in the sense of natural identification of $R \in \mathcal{E}_0(\mathbb{C})$ (resp. $\mathcal{E}_0(\mathbb{R})$), $R : (\phi, x) \mapsto C(\phi) \in \mathbb{C}$ (resp. $\mathbb{R}$).

A function $R \in \mathcal{E}[\Omega]$ is called moderate if for every $K \subset\subset \Omega$ and $\alpha \in \mathbb{N}_0^n$ there exists $N \in \mathbb{N}_0$ such that, for every $\phi \in \mathcal{A}_N$, there exist $\eta > 0$ and $C > 0$ such that

$$|\partial^\alpha R(\phi_\varepsilon, x)| \leq C\varepsilon^{-N},\ x \in K,\ 0 < \varepsilon < \eta.$$

The set of all moderate elements is denoted by $\mathcal{E}_M[\Omega]$.

The set of all moderate elements in $\mathcal{E}_0(\mathbb{C})$ (resp. $\mathcal{E}_0(\mathbb{R})$), denoted by $\mathcal{E}_{0M}(\mathbb{C})$ (resp. $\mathcal{E}_{0M}(\mathbb{R})$), consists of elements $R \in \mathcal{E}_0(\mathbb{C})$ (resp. $\mathcal{E}_0(\mathbb{R})$) which satisfy: There exists $N \in \mathbb{N}_0$ such that for every $\phi \in \mathcal{A}_N$ there exist $\eta > 0$ and $C > 0$ such that

$$|R(\phi_\varepsilon)| < C\varepsilon^{-N},\ \ 0 < \varepsilon < \eta.$$

Clearly $\mathcal{E}_M[\Omega]$ and $\mathcal{E}_{0M}(\mathbb{C})$ (resp. $\mathcal{E}_{0M}(\mathbb{R})$) are associative subalgebras of $\mathcal{E}[\Omega]$ and $\mathcal{E}_0(\mathbb{C})$ (resp. $\mathcal{E}_0(\mathbb{R})$).

Denote by Γ the set of sequences $\{a_q\}$ of positive numbers which strictly increase to infinity.

An element $R \in \mathcal{E}_M[\Omega]$ is called a null element if for every $K \subset\subset \Omega$ and every $\alpha \in \mathbb{N}_0^n$ there exist $N \in \mathbb{N}_0$ and $\{a_q\} \in \Gamma$ such that for every $q \geq N$ and every $\phi \in \mathcal{A}_q$ there exists $\eta > 0$ and $C > 0$ such that

$$|\partial^\alpha R(\phi_\varepsilon, x)| \leq C\varepsilon^{a_q - N}, \ x \in K, \ 0 < \varepsilon < \eta.$$

The space of null elements is denoted by $\mathcal{N}[\Omega]$.

The space of null elements of $\mathcal{E}_0(\mathbb{C})$ (resp. $\mathcal{E}_0(\mathbb{R})$) denoted by $\mathcal{N}_0(\mathbb{C})$ (resp. $\mathcal{N}_0(\mathbb{R})$) consists of all the elements $R \in \mathcal{E}_{0M}(\mathbb{C})$ (resp. $\mathcal{E}_{0M}(\mathbb{R})$) with the following property: There exist $N \in \mathbb{N}_0$ and $\{a_q\} \in \Gamma$ such that for every $q \geq N$ and every $\phi \in \mathcal{A}_q$ there exists $\eta > 0$ and $C > 0$ such that

$$|R(\phi_\varepsilon)| \leq C\varepsilon^{a_q - N}, \ \ 0 < \varepsilon < \eta.$$

Clearly, $\mathcal{N}[\Omega]$ and $\mathcal{N}_0(\mathbb{C})$ (resp. $\mathcal{N}_0(\mathbb{R})$) are ideals of $\mathcal{E}_M[\Omega]$ and $\mathcal{E}_{0M}(\mathbb{C})$ (resp. $\mathcal{E}_{0M}(\mathbb{R})$).

The spaces of generalized functions on Ω, $\mathcal{G}(\Omega)$, generalized complex numbers $\overline{\mathbb{C}}$ and generalized real numbers $\overline{\mathbb{R}}$ are defined by

$$\mathcal{G}(\Omega) = \mathcal{E}_M[\Omega]/\mathcal{N}[\Omega], \ \ \overline{\mathbb{C}} = \mathcal{E}_{0M}(\mathbb{C})/\mathcal{N}_0(\mathbb{C}), \ \ \overline{\mathbb{R}} = \mathcal{E}_{0M}(\mathbb{R})/\mathcal{N}_0(\mathbb{R}).$$

$\Omega \mapsto \mathcal{G}(\Omega)$ is a sheaf. This implies the natural definition of the support, $\operatorname{supp}_g \mathcal{G}$.

Note that $\overline{\mathbb{C}}$ and $\overline{\mathbb{R}}$ are not fields and $\overline{\mathbb{C}} = \overline{\mathbb{R}} + i\overline{\mathbb{R}}$. Because of that, from now on, we shall use the symbols $\mathcal{E}_{0M} = \mathcal{E}_{0M}(\mathbb{C})$ and $\mathcal{N}_0 = \mathcal{N}_0(\mathbb{C})$.

The classical complex numbers are embedded in $\overline{\mathbb{C}}$ by

$$\mathbb{C} \ni z \mapsto R(\phi) = z, \ \phi \in \mathcal{A}_0.$$

Let $g \in \mathcal{D}'$. Then $\mathrm{Cd}(g) \in \mathcal{G}$ is given by the representative $g * \check{\phi}_\varepsilon$, where $\check{\phi}(y) = \phi(-y)$.

$\mathcal{E}_t$ is the set of all elements $G \in \mathcal{E}$ with the following property: For every $\beta \in \mathbb{N}_0^n$ there exist $N \in \mathbb{N}$, $a \in \mathbb{R}$ and $\gamma > 0$ such that for every $\phi \in \mathcal{A}_N$ there exist $C > 0$ and $\eta > 0$ such that

$$|\partial^\beta G(\phi_\varepsilon, x)| \leq C(1 + |x|)^\gamma \varepsilon^a, \text{ for } \varepsilon < \eta, \ x \in \mathbb{R}^n.$$

$\mathcal{N}_t$ is the set of elements $G \in \mathcal{E}_t$ with the property: For every $\beta \in \mathbb{N}_0^n$ there exist $\gamma > 0$, $N \in \mathbb{N}$ and $g \in \Gamma$ such that for every $\phi \in \mathcal{A}_q$, $q \geq N$, there exist $C > 0$ and $\eta > 0$ such that

$$|\partial^\beta G(\phi_\varepsilon, x)| \leq C(1 + |x|)^\gamma \varepsilon^{g(q) - N}, \text{for} \varepsilon < \eta, \ x \in \mathbb{R}^n.$$

It is an ideal of $\mathcal{E}_t$.

Colombeau's space of tempered generalized functions is defined by $\mathcal{G}_t = \mathcal{E}_t/\mathcal{N}_t$.

It is said that $G \in \mathcal{G}$ ($G \in \mathcal{G}_t$) is equal to $H \in \mathcal{G}$ ($H \in \mathcal{G}_t$) in generalized distribution sense, $G = H(g.d.)$ (in generalized tempered distribution sense, $G = H(g.t.d.)$), if for every $\psi \in \mathcal{D}$ ($\psi \in \mathcal{S}$)

$$\langle G - H, \psi \rangle = 0 \text{ in } \overline{\mathbb{C}}.$$

$A \in \overline{\mathbb{C}}$ is associated to $c \in \mathbb{C}$, $A \approx c$, if there exists $N \in \mathbb{N}$ such that $\lim_{\varepsilon\to 0} A_{\phi,\varepsilon} = c$ for every $\phi \in \mathcal{A}_N$.

$G \in \mathcal{G}$ is associated to $H \in \mathcal{G}$, $G \approx H$, if for every $\psi \in \mathcal{D}$ there exists $N \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_N$

$$\lim_{\varepsilon\to 0} \langle G(\phi_\varepsilon, \cdot) - H(\phi_\varepsilon, \cdot), \psi \rangle = 0.$$

The definition of t-association is obtained if one takes $\psi \in \mathcal{S}$ instead of $\psi \in \mathcal{D}$ above.

For the microlocal analysis of Colombeau's generalized functions we shall define a subalgebra $\mathcal{G}^\infty(\Omega)$ by following Oberguggenberger [13].

$\mathcal{G}^\infty(\Omega)$ is the set of all $G \in \mathcal{G}(\Omega)$ which have representatives $G(\phi, x) \in \mathcal{E}_M[\Omega]$ with the property: For every $K \subset\subset \Omega$ there is $N \in \mathbb{N}$ such that for every $\alpha \in \mathbb{N}_0$, there is $M \in \mathbb{N}_0$ such that for every $\phi \in \mathcal{A}_N$ there are $C > 0$ and $\eta > 0$ such that

$$\sup_{x\in K} |G^{(\alpha)}(\phi_\varepsilon, x)| \leq C\varepsilon^{-N}, \ 0 < \varepsilon < \eta.$$

One can prove that $\mathcal{G}^\infty(\Omega)$ is a subalgebra of $\mathcal{G}(\Omega)$.

Proposition 10.1. *1. $\mathcal{G}^\infty(\Omega) \bigcap \mathcal{D}'(\Omega) = C^\infty(\Omega)$ ([**13**]).*

2. $\mathcal{G}^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$ is a sheaf.

3. $G \in \mathcal{G}(\Omega)$ is $\mathcal{G}^\infty$ in $\Omega_1 \subset \Omega$ if it is $\mathcal{G}^\infty$ at every point of Ω_1.

The last assertion means: For every $x \in \Omega_1$ there are open sets U and V such that

$$x \in U, \ \bar{U} \subset\subset V, \ V \subset\subset \Omega_1$$

and a function $\psi \in C_0^\infty(V)$, $\psi \equiv 1$ on $\bar{U}$, such that $\psi G \in \mathcal{G}^\infty(\Omega_1)$.

Definition 10.2. Let $G \in \mathcal{G}(\Omega)$. The complement of the largest open set of Ω in which G is $\mathcal{G}^\infty$ is called the singular support of G, $\operatorname{Sing\,supp}_g G$.

Recall, it is said that G is $\mathcal{G}^\infty$ in $\Omega_1 \subset \Omega$ if $G|_{\Omega_1} \in \mathcal{G}^\infty(\Omega_1)$.

The set $\operatorname{Sing\,supp}_g G$, $G \in \mathcal{G}(\Omega)$ is defined to be the complement of the largest open set $\Omega' \subset \Omega$ such that $G|_{\Omega'} = 0$.

From Proposition 10.1 we have that for distributions

$$\operatorname{Sing\,supp} f = \operatorname{Sing\,supp}_g \operatorname{Cd} f, \ f \in \mathcal{D}'(\Omega).$$

Denote by $\mathcal{G}_c(\Omega)$ the set of all elements in $\mathcal{G}(\Omega)$ which have compact supports.

If $G \in \mathcal{G}_c(\Omega)$, then it belongs to $\mathcal{G}_c(\mathbb{R}^n)$ by defining

$$G(\phi, x) = 0,\ \phi \in \mathcal{A}_0,\ x \in \mathbb{R}^n \setminus \operatorname{supp}_g G$$

(and by using the sheaf property of $\mathcal{G}(\mathbb{R}^n)$).

For every ψ_1 and ψ_2 in $C_0^\infty(\mathbb{R}^n)$, where ψ_1 and ψ_2 equals one on corresponding neighborhoods of $\operatorname{supp}_g G$,

$$\psi_1(\cdot)(G(\phi_\varepsilon,\cdot) + \mathcal{N}[\mathbb{R}^n]) \text{ and } \psi_2(\cdot)(G(\phi_\varepsilon,\cdot) + \mathcal{N}[\mathbb{R}^n])$$

determine the same element in $\mathcal{G}_t(\mathbb{R}^n)$. They are equal in $\mathcal{G}(\mathbb{R}^n)$.

Thus the mapping $\mathcal{M} : \mathcal{G}_c(\mathbb{R}^n) \to \mathcal{G}_t(\mathbb{R}^n)$, symbolically written by

$$G(\phi_\varepsilon,\cdot) + \mathcal{N}(\mathbb{R}^n) \mapsto \psi(\cdot)(G(\phi_\varepsilon,\cdot) + \mathcal{N}[\mathbb{R}^n]) + \mathcal{N}_t[\mathbb{R}^n],$$

is linear, multiplicative and injective, which will enable us to consider $\mathcal{G}_c(\mathbb{R}^n)$ as a subspace of $\mathcal{G}_t(\mathbb{R}^n)$.

If $G \in \mathcal{G}_t(\mathbb{R}^n)$, then $G|_\omega \in \mathcal{G}(\omega)$ is defined by a representative $G(\phi_\varepsilon,\cdot)|_\omega$, where $G(\phi_\varepsilon,\cdot)$ is a representative of G.

If $f \in S'(\mathbb{R}^n)$ then $\mathrm{Cd}_t\, f$ denotes the corresponding element in $\mathcal{G}_t(\mathbb{R}^n)$ defined by

$$(f * \check{\phi})(x) + G(\phi, x),\ \text{ where } G(\phi, x) \in \mathcal{N}_t[\mathbb{R}^n].$$

Let $G \in \mathcal{G}_t(\mathbb{R}^n)$ and ω be an open set. If $G|_\omega$ determines an element in $\mathcal{G}^\infty(\omega)$, then we say that it is $\mathcal{G}_t^\infty$ in ω, where we use this notation to emphasize that the generalized function in consideration is from $\mathcal{G}_t(\mathbb{R}^n)$.

Let $G \in \mathcal{G}(\Omega)$ and if $G|_\omega \in \mathcal{G}^\infty(\omega)$ where ω is a bounded open set in Ω. Then $\mathcal{M}(\kappa G)$, where $\kappa \in C_0^\infty(\mathbb{R}^n)$ is equal to 1 on $\overline{\omega}$, is $\mathcal{G}_t^\infty$ in ω.

(Recall $\mathcal{M}(\kappa G) = \kappa_1(x)(\kappa(x)G(\phi, x) + \mathcal{N}[\Omega]) + \mathcal{N}_t[\mathbb{R}^n]$.)

Thus the singular support of $G \in \mathcal{G}_t(\mathbb{R}^n)$ is the singular support of G considered as an element of $\mathcal{G}(\mathbb{R}^n)$. We define the subalgebra $\mathcal{G}_t^\infty(\mathbb{R}^n)$ as follows.

$\mathcal{G}_t^\infty(\mathbb{R}^n)$ is the set of all $U \in \mathcal{G}_t(\mathbb{R}^n)$ which have a representative $G(\phi, x) \in \mathcal{E}_t[\mathbb{R}^n]$ with the property: There is $N \in \mathbb{N}$ such that for every $\alpha \in \mathbb{N}_0^n$ there is $M \in \mathbb{N}_0$ such that for every $\phi \in \mathcal{A}_M$ there are $C > 0$ and $\eta > 0$ such that

$$|G^{(\alpha)}(\phi, x)| \le C(1 + |x|)^M \varepsilon^{-N},\ 0 < \varepsilon < \eta.$$

Note, if $G \in \mathcal{G}_t^\infty(\mathbb{R}^n)$ then $G \in \mathcal{G}^\infty(\mathbb{R}^n)$.

Let $\mu \in C_0^\infty(\mathbb{R}^n)$ such that $\mu = 1$ in some neighborhood of zero. Then $\mu_\varepsilon(x) = \mu(x\varepsilon)$, $\varepsilon \in (0,1)$, is called a unit net.

Let μ_ε be a unit net, B a measurable subset of $\mathbb{R}^n$ and $G \in \mathcal{G}_t$. Then we define

$$\int_B^{t,\mu} G(x)\,dx \in \overline{\mathbb{C}} \text{ by its representative } \int_B G(\phi_\varepsilon, x)\mu_\varepsilon(x)\,dx \in \mathcal{E}_{0,M}.$$

If $B = \mathbb{R}^n$ then the symbol $\int^{t,\mu}$ is used. One can easily prove that $G(\phi_\varepsilon, \cdot) \in \mathcal{N}_t$ implies $\int_B G(\phi_\varepsilon, x)\mu_\varepsilon(x)\,dx \in \mathbb{C}_0$. Thus the definition of the integral in $\mathcal{G}_t$ makes sense.

11. Pseudodifferential operators

We will give the simplest definitions of an amplitude of type $\rho = 1$, $\sigma = 0$.

Definition 11.1. The set of amplitudes $S_g^m = S_g^m(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times (0,1])$, $m \in \mathbb{R}$, is the set of functions $a(x,y,\xi,\varepsilon)$, smooth in $(x,y,\xi) \in (\mathbb{R}^n)^3$ for every $\varepsilon \in (0,1]$, continuous in $\varepsilon \in (0,1]$ for every $(x,y,\xi) \in (\mathbb{R}^n)^3$, such that there exists $N \in \mathbb{N}_0$ such that for every $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ there exists $C = C(\alpha,\beta,\gamma) > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x,y,\xi,\varepsilon)| \le \frac{C}{\varepsilon^N}(1+|\xi|)^{m-|\alpha|},\ (x,y,\xi) \in (\mathbb{R}^n)^3, \varepsilon \in (0,1]. \tag{11.1}$$

If there exists $N \in \mathbb{N}_0$ such that for every $m \in \mathbb{R}$ and every $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ there exists $C = C(\alpha,\beta,\gamma,m) > 0$ such that (11.1) holds, then $a(x,y,\xi,\varepsilon) \in S_g^{-\infty}$.

The following set of amplitudes is suitable for the calculus in the frame of Colombeau's generalized functions.

Definition 11.2. The set of amplitudes $S_{gt}^m = S_{gt}^m(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times (0,1])$, $m \in \mathbb{R}$, is the set of functions a with the same regularity properties as in Definition 11.1 but which satisfies the following:

There exists $N \in \mathbb{N}_0$ such that for every $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ there exist $C = C(\alpha,\beta,\gamma)$ and $k = k(\alpha,\beta,\gamma)$ such that

$$|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x,y,\xi,\varepsilon)| \le \frac{C}{\varepsilon^N}(1+|x|)^k(1+|\xi|)^{m-|\alpha|},\ (x,y,\xi) \in (\mathbb{R}^n)^3, \varepsilon \in (0,1].$$

Elements of $S_{gt}^{-\infty}$ are appropriately defined. In this case constants C and k depend also on m.

We will use Definition 11.1 in Section 11 and later in order to avoid a lot of technical difficulties which may appear.

Definition 11.3. Let $a \in S_{gt}^m$, $r \in \mathbb{N}_0$ and $\mu_{1\varepsilon}(\xi)$, $\mu_{2\varepsilon}(y)$ be unit nets from $C_0^\infty(\mathbb{R}_\xi^n)$ and $C_0^\infty(\mathbb{R}_y^n)$, respectively. Let $G \in \mathcal{G}_t(\mathbb{R}^n)$.

We define $A_{\mu_2 r}$ and $A_{\mu_1\mu_2}$ on $\mathcal{G}_t(\mathbb{R}^n)$ by

$$\begin{aligned} A_{\mu_2 r}G(\phi_\varepsilon, x) = \frac{1}{(2\pi)^n}\iint_{\mathbb{R}^{2n}} &\frac{e^{i\langle x-y,\xi\rangle}}{(1+|\xi|^2)^{[(|m|+n)/2]+r}}(1-\Delta_y)^{[(|m|+n)/2]+r} \\ &\times (a(x,y,\xi,\varepsilon)\mu_{2\varepsilon}(y)G(\phi_\varepsilon,y)\,dy\,d\xi,\ (\phi,x) \in \mathcal{A}_0 \times \mathbb{R} \end{aligned} \tag{11.2}$$

where $[(|m|+n)/2]$ is the integer part of $(|m|+n)/2$, and by

$$\begin{aligned} &A_{\mu_1\mu_2}G(\phi_\varepsilon, x) \\ &= \frac{1}{(2\pi)^n}\iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} a(x,y,\xi,\varepsilon)\mu_{1\varepsilon}(\xi)\mu_{2\varepsilon}(y)G(\phi_\varepsilon,y)\,dy\,d\xi. \end{aligned} \tag{11.3}$$

Note, if $m < -n$ then we take $r = 0$ in (11.2) and

$$A_{\mu_2}G(\phi_\varepsilon, x) = A_{\mu_2}G(\phi_\varepsilon, x) = \frac{1}{(2\pi)^n}\iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle}a(x,y,\xi,\varepsilon)\mu_{2\varepsilon}(y)G(\phi_\varepsilon,y)\,dy\,d\xi.$$

Theorem 11.4. *1. $A_{\mu_2 r}$ and $A_{\mu_1\mu_2}$ are linear mappings from $\mathcal{G}_t(\mathbb{R}^n)$ to $\mathcal{G}_t(\mathbb{R}^n)$.*

2. For every $\mu_{1\varepsilon}(\xi), \mu_{2\varepsilon}(y), r$ and $G \in \mathcal{G}_t(\mathbb{R}^n)$, $A_{\mu_2 r}G$ and $A_{\mu_1\mu_2}G$ are equal in (g.t.d.) sense.

3. For every $\mu_{1\varepsilon}(\xi), \tilde{\mu}_{1\varepsilon}(\xi), \mu_{2\varepsilon}(y), \tilde{\mu}_{2\varepsilon}(y)$ and $G \in \mathcal{G}_t(\mathbb{R}^n)$, $A_{\mu_1\mu_2}G$ and $A_{\tilde{\mu}_1\tilde{\mu}_2}G$ are equal in (g.t.d.) sense.

Proof. The proof of 1 is obvious. Note that (11.3) is equal to

$$A_{\mu_1\mu_2}G(\phi_\varepsilon, x) = \frac{1}{(2\pi)^n}\iint_{\mathbb{R}^{2n}} \frac{e^{i\langle x-y,\xi\rangle}}{(1+|\xi|^2)^{[(|m|+n)/2]+r}}(1-\Delta_y)^{[(|m|+n)/2]+r} \times (a(x,y,\xi,\varepsilon)\mu_{1\varepsilon}(\xi)\mu_{2\varepsilon}(y)G(\phi_\varepsilon,y))\,dy\,d\xi.$$

Since the proof of 3 is typical for the calculus we will collect here the equalities and the estimations which will be used in the sequel.

There holds

$$\begin{gathered}(1-\Delta_x)^s e^{i\langle x-y,\xi\rangle} = (1+|\xi^2|)^s e^{i\langle x-y,\xi\rangle},\\ \frac{(1-\Delta_\xi)^s(1-\Delta_x)^p e^{i\langle x-y,\xi\rangle}}{(1+|y^2|)^s(1+|\xi^2|)^p} = e^{i\langle x-y,\xi\rangle},\\ (1-\Delta_x)^s e^{i\langle x-y,\xi\rangle} = (1+\Delta_y)^s e^{i\langle x-y,\xi\rangle}.\end{gathered} \tag{11.5}$$

A unit net $\mu_\varepsilon(\xi)$, $\varepsilon \in (0,1]$, where $\mu(\xi) = 1$, $|\xi| \le A$, $\mu(\xi) = 0$, $|\xi| \ge B > A$, satisfies the following estimation. Let $\alpha \in \mathbb{N}_0^n$. Since

$$|\partial^\alpha \mu_\varepsilon(\xi)| = |\varepsilon^{|\alpha|}\partial^\alpha \mu(\varepsilon\xi)|, \ \ A \le |\varepsilon\xi| \le B$$

it follows

$$|\partial^\alpha\mu_\varepsilon(\xi)| \le C_\alpha\varepsilon^{|\alpha|} \le \frac{\beta^{|\alpha|}C_\alpha}{|\xi|^{|\alpha|}}, \ \ |\xi| > A/\varepsilon, \ |\partial^\alpha\mu_{\varepsilon(\xi)}| = 0, \ |\xi| \le A/\varepsilon. \tag{11.6}$$

If μ_ε and $\tilde{\mu}_\varepsilon$ are unit nets determined by different functions μ_1 and μ_2 then, by the above notation,

$$|\mu_\varepsilon(\xi) - \tilde{\mu}_\varepsilon(\xi)| = 0, \ \text{ for } |\xi| \le \frac{\min\{A,\tilde{A}\}}{\varepsilon} \text{ and } |\xi| \ge \frac{\max\{B,\tilde{B}\}}{\varepsilon}. \tag{11.7}$$

Now, we will give the proof of 3. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $I = \int_{\mathbb{R}}(A_{\mu_1\mu_2}G(\phi_\varepsilon,x) - A_{\tilde{\mu}_1\tilde{\mu}_2}G(\phi_\varepsilon,x))\psi(x)\,dx$. By (11.5), for enough large s and p, we have

$$I = \iiint_{\mathbb{R}^{3n}} e^{i\langle x-y,\xi\rangle}\frac{1}{(1+|y|^2)^s}(1-\Delta_\xi)^s(1-\Delta_x)^p \times \left(\frac{a(x,y,\xi,\varepsilon)}{(1+|\xi|^2)} + G(\phi_\varepsilon,y)\,(\mu_{1\varepsilon}(\xi)\mu_{2\varepsilon}(y) - \tilde{\mu}_{1\varepsilon}(\xi)\tilde{\mu}_{2\varepsilon}(y))\,\psi(x)\right)\,dx\,dy\,dz.$$

Note that the differentiation with respect to y is changed by differentiation with respect to x. By using the identity

$$\mu_{1\varepsilon}(\xi)\mu_{2\varepsilon}(y) - \tilde{\mu}_{1\varepsilon}(\xi)\tilde{\mu}_{2\varepsilon}(y) = (\mu_{1\varepsilon}(\xi) - \tilde{\mu}_{1\varepsilon}(\xi))\mu_{2\varepsilon}(y) + \tilde{\mu}_{1\varepsilon}(\xi)(\mu_{2\varepsilon}(y) - \tilde{\mu}_{2\varepsilon}(y))$$

we have that I is smaller than the linear combination of factors of the form

$$\iiint_{\mathbf{R}^{3n}} \frac{1}{(1+|y|^2)^s}\left|\partial_x^q\partial_\xi^r \frac{a(x,y,\xi)}{(1+|\xi|^2)^p}\right| \cdot |\partial_\xi^t(\mu_{1\varepsilon}(\xi) - \tilde{\mu}_{1\varepsilon}(\xi))| \\ \times |\mu_{2\varepsilon}(y)| \cdot |G(\phi_\varepsilon, y)| \cdot |\partial^h\psi(x)|\, dx\, dy\, dz$$

$$\iiint_{\mathbf{R}^3} \frac{1}{(1+|y|^2)^s}\left|\partial_x^q\partial_\xi^r \frac{a(x,y,\xi)}{(1+|\xi|^2)^p}\right| \cdot |\partial_\xi^t\tilde{\mu}_{1\varepsilon}(\xi)| \\ \times |\mu_{2\varepsilon}(y) - \tilde{\mu}_{2\varepsilon}(y)| \cdot |G(\phi_\varepsilon, y)| \cdot |\partial^h\psi(x)|\, dx\, dy\, dz,$$

where $|q|, |h| \leq 2p$, $|r|, |t| \leq 2s$. The properties of $a(x, y, \xi, \varepsilon)$ imply that for suitable constants

$$\left|\partial_x^q\partial_\xi^r \frac{a(x,y,\xi)}{(1+|\xi|^2)^p}\right| \cdot |\partial_\xi^t(\mu_{1\varepsilon}(\xi) - \tilde{\mu}_{1\varepsilon}(\xi))| \leq \\ C(1+|x|)^k(1+|\xi|^2)^{-p+m-s} \leq C_1\varepsilon^{p+s-m}(1+|x|)^k.$$

since the left side is equal to 0 for $|\xi| <$ const$/s$. Note that

$$\int_{\mathbf{R}^n} |G(\phi_\varepsilon, y)\mu_{2\varepsilon}(y)|\, dy \leq \frac{C}{\varepsilon^{N_G}}\int_{\mathbf{R}^n}(1+|y|)^{p_G}\mu_{2\varepsilon}(y)\, dy \leq C\varepsilon^{-N_G-p_G-n}.$$

By choosing enough large p and s, this implies that for every $d > 0$ the members of the form (11.8) are $o(\varepsilon^d)$, $\varepsilon \to 0$.

To prove that the members of the form (11.8) are $o(\varepsilon^d)$, $\varepsilon \to 0$, for every $d > 0$, we have to estimate the factor

$$\frac{1}{(1+|y|^2)^s|\mu_{2\varepsilon}(y) - \tilde{\mu}_{2\varepsilon}(y)|},$$

which is different from zero if $|y| > \frac{\text{const}}{\varepsilon}$, and to take sufficiently large s.

This proves 3. The proof of 2 is almost the same. □

The relation $\overset{\text{g.t.d.}}{=}$ is the relation of equivalence in $\mathcal{G}_t(\mathbb{R}^n)$. So, the mappings $A_{\mu_2 r}$ and $A_{\mu_1\mu_2}$ are equal if they are considered as the mappings from $\mathcal{G}_t(\mathbb{R}^n)$ into $\mathcal{G}_t(\mathbb{R}^n)/\overset{\text{g.t.d.}}{=}$.

Definition 11.5. The mappings $A_{\mu_2 r}$ and $A_{\mu_1\mu_2}$ are the representatives of the mapping

$$A : \mathcal{G}_t(\mathbb{R}^n) \to \mathcal{G}_t(\mathbb{R}^n)/\overset{\text{g.t.d.}}{=}$$

which is called the pseudodifferential operator which corresponds to $a \in S^m_{gt}$.

Proposition 11.6. *If $a \in S_{gt}^{-\infty}$, then for every $\mu_{1\varepsilon}(\xi)$, $\mu_{2\varepsilon}(y)$ and $r \in \mathbb{N}_0$ the operators $A_{\mu_2 r}G(\phi_\varepsilon, x)$ and $A_{\mu_1\mu_2}G(\phi_\varepsilon, x)$ are in $\mathcal{G}_t^\infty(\mathbb{R}^n)$.*

Proof. Let $r = 0$. We will prove that

$$A_{\mu_2}G(\phi_\varepsilon, x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} a(x,y,\xi,\varepsilon) G(\phi_\varepsilon, y)\mu_{2\varepsilon}(y)\, dy\, d\xi$$

is in $\mathcal{G}_t^\infty(\mathbb{R}^n)$. Other parts of the proposition may be proved in a similar way by using (11.2).

Let $\alpha \in \mathbb{N}_0^n$. Then

$$|\partial^\alpha(A_{\mu_2}G(\phi_\varepsilon, x))| =$$
$$\left|\frac{1}{(2\pi)^n} \iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} \sum_{j\le\alpha} \binom{\alpha}{j} i^{|\alpha-j|}\xi^{\alpha-j}\partial_x^\alpha a(x,y,\xi,\varepsilon) G(\phi_\varepsilon, y)\mu_{2\varepsilon}(y)\, dy\, d\xi\right|.$$

By using

$$|\partial_x^\alpha a(x,y,\xi,\varepsilon)| \le C_{m,\alpha}\varepsilon^{-N}(1+|x|)^k(1+|\xi|)^m, \quad x,y,\xi \in \mathbb{R}^n,\ \varepsilon \in (0,1],$$

which holds for enough large $-m$ (where N does not depend on m and α),

$$|G(\phi_\varepsilon, y))| \le \tilde{C}\varepsilon^{-N_1}(1+|y|)^{N_1}, \quad y \in \mathbb{R}^n,\ \varepsilon \in (0,\eta),\ \phi \in \mathcal{A}_N$$

and

$$\left|\int_{\mathbb{R}^n} (1+|y|)^{N_1}\mu_{2\varepsilon}(y)\, dy\right| \le \tilde{\tilde{C}}\varepsilon^{-N_1-n}, \quad \varepsilon \in (0,1],$$

we obtain:

If $N_2 = N + N_1 + n$, then for every $\alpha \in \mathbb{N}_0^n$ there is $M \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_M$ there are $C > 0$ and $\eta > 0$ such that

$$|\partial^\alpha(A_{\mu_2}G(\phi_\varepsilon, x))| \le C\varepsilon^{-N_2}(1+|x|))^M, \quad 0 < \varepsilon < \eta. \qquad \square$$

If an amplitude $a \in S_{gt}^m$ does not depend on ε, i.e. $a = a(x,y,\xi)$, then it determines a convenient pseudodifferential operator which will be denoted by $\mathbb{A}$:

$$\mathbb{A}\varphi(x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} a(x,y,\xi)\varphi(y)\, dy\, d\xi,\ \varphi \in \mathcal{S}(\mathbb{R}^n).$$

It can be extended on $\mathcal{S}'(\mathbb{R}^n)$ to be linear and continuous mapping from $\mathcal{S}'(\mathbb{R}^n)$ into itself.

In fact,

$$({}^t\mathbb{A}\varphi)(x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} \tilde{a}(x,y,\xi)\varphi(y)\, dy\, d\xi$$

where $\tilde{a}(x,y,\xi) = a(y,x,\xi)$ is continuous and linear: $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ and $\mathbb{A} = {}^t({}^t\mathbb{A})$ is continuous and linear: $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.

We will compare A and A but before that we need the following definition and proposition.

Definition 11.7. If $a \in S^m_{gt}$ and $G \in \mathcal{G}_c(\mathbb{R}^n)$, then $AG = A(\mathcal{M}G)$. Let $G(\phi_\varepsilon, x), (\phi_\varepsilon, x) \in \mathcal{A}_0 \times \mathbb{R}^n$, be a representative of G and $\kappa \in C_0^\infty(\mathbb{R}^n), \kappa \equiv 1$ on $\mathrm{supp}_g G$. Then,

$$\kappa(x)G(\phi_\varepsilon, x), x \in \mathbb{R}^n, \phi \in \mathcal{A}_0,$$

is a representative of $\mathcal{M}G \in \mathcal{G}_t(\mathbb{R}^n)$. $AG = A(\mathcal{M}G)$ is defined by 2 in Theorem 11.4

$$A_{\mu_1\mu_2}(\mathcal{M}G)(\phi_\varepsilon, x) = (2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} a(x,y,\xi,\varepsilon)\mu_{1\varepsilon}(\xi)\kappa(y)G(\phi_\varepsilon, y)\, dy\, d\xi.$$

From the next proposition it follows that this definition does not depend on κ.

Proposition 11.8. *If $\kappa_1, \kappa_2 \in C_0^\infty$ are equal to 1 on $\mathrm{supp}_g G$ then*

$$(2\pi)^{-n} \iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} a(x,y,\xi,\varepsilon)\mu_{1\varepsilon}(\xi)(\kappa_1(y) - \kappa_2(y))G(\phi_\varepsilon, y)\, dy\, d\xi = 0$$

in $\mathcal{G}_t(\mathbb{R}^n)$.

The following proposition also can be proved.

Proposition 11.9. *Let $a \in S^m_{gt}$ be independent on ε and let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $A(\mathrm{Cd}_t f) = \mathrm{Cd}_t(\mathrm{A}f)$ in $\mathcal{G}_t(\mathbb{R}^n)/ \overset{\text{g.t.d.}}{=}$.*

12. Pseudolocal property and the microlocalization

Denote $\mathcal{G}_c^\infty(\Omega) = \mathcal{G}^\infty(\Omega) \cap \mathcal{G}_c(\Omega)$. Clearly, if $G \in \mathcal{G}_c^\infty(\Omega)$ then $\mathcal{M}G \in \mathcal{G}_t^\infty(\mathbb{R}^n)$. In the sequel we will consider $\mathcal{G}_c(\Omega)$ and $\mathcal{G}_c^\infty(\Omega)$ as subspaces of $\mathcal{G}_t(\mathbb{R}^n)$.

Without the proof we give the following theorem.

Theorem 12.1. *Let $a \in S^m_{gt}$ and $G \in \mathcal{G}_c^\infty(\mathbb{R}^n)$. Then, $AG \in \mathcal{G}_t^\infty(\mathbb{R}^n)/ \overset{\text{g.t.d.}}{=}$. More precisely, for every $\mu_{1\varepsilon}(\xi)$, $\mu_{2\varepsilon}(y)$ and $r \in \mathbb{N}_0$, $A_{\mu_2 r}G(\phi_\varepsilon, x)$ and $A_{\mu_1\mu_2}G(\phi_\varepsilon, x)$ are in $\mathcal{G}_t^\infty(\mathbb{R}^n)$ and they are equal in (g.t.d.) sense.*

Definition 12.2. Let $G \in \mathcal{G}_t(\mathbb{R}^n)$ and A be a pseudodifferential operator. It is said that AG is regular at $x \in \mathbb{R}^n$ if there exists an open set $\omega \ni x$ such that for every unit nets $\mu_{1\varepsilon}$, $\mu_{2\varepsilon}$ and $r \in \mathbb{N}_0$,

$$A_{\mu_1\mu_2}(G)|_\omega \text{ and } A_{\mu_2 r}(G)|_\omega \text{ belong to } \mathcal{G}^\infty(\omega).$$

The singular support of AG, $\mathrm{Sing\,supp}_g AG$, is the complement of a set of points in which AG is regular. If x (resp. any point of ω) does not belong to $\mathrm{Sing\,supp}_g AG$, then it is said that AG is $\mathcal{G}_t^\infty/ \overset{\text{g.t.d.}}{=}$ in x (resp. in ω).

Proposition 12.3. *Let $G \in \mathcal{G}_c(\mathbb{R}^n)$, $a \in S^m_{gt}$. Then,*

$$\mathrm{Sing\,supp}_g AG \subset \mathrm{Sing\,supp}_g G.$$

More precisely, for every $\mu_{1\varepsilon}(\xi), \mu_{2\varepsilon}(y)$ *and* $r \in \mathbb{N}_0$,

$$\operatorname{Sing\,supp}_g A_{\mu_{2\varepsilon} r} G \subset \operatorname{Sing\,supp}_g G,$$
$$\operatorname{Sing\,supp}_g A_{\mu_{1\varepsilon}\mu_{2\varepsilon}} G \subset \operatorname{Sing\,supp}_g G.$$

Proof. Let G be $\mathcal{G}^\infty$ in a neighborhood ω of x_0. We shall show that $AG = A(\mathcal{M}G)$ is in $\mathcal{G}_t^\infty / \overset{\text{g.t.d.}}{=}$ in some open set $\omega_1 \ni x_0$, such that $\overline{\omega}_1 \subset\subset \omega$.

Let $\kappa_1 \in C_0^\infty(\omega)$ such that $\kappa_1 \equiv 1$ on $\overline{\omega}_1$ and let $\kappa_2 \in C_0^\infty(\omega)$ such that $\kappa_2 \equiv 1$ on $K_1 = \operatorname{supp} \kappa_1$.

For small enough ε, we have

$$\begin{aligned}
&\kappa_1(x) A_{\mu_1\mu_2} G(\phi_\varepsilon, x) = \\
&\quad \frac{1}{(2\pi)^n} \iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} a(x,y,\xi,\varepsilon)\mu_{1\varepsilon}(\xi)\kappa_1(x)\kappa(y)\kappa_2(y) G(\phi_\varepsilon, y)\, dy\, d\xi \\
&\quad + \frac{1}{(2\pi)^n} \iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} a(x,y,\xi,\varepsilon)\mu_{1\varepsilon}(\xi)\kappa_1(x)(1-\kappa_2(y))\kappa(y) G(\phi_\varepsilon, y)\, dy\, d\xi \\
&\quad = I_1 + I_2.
\end{aligned}$$

As earlier we have that I_1 is $\mathcal{G}_t^\infty$ in $\mathbb{R}^n$. So we have to prove the same for I_2. Let $k \in \mathbb{N}$. Then

$$I_2 = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int \frac{e^{i\langle x-y,\xi\rangle}}{|x-y|^{2k}} (-\Delta_\xi)^k (a(x,y,\xi,\varepsilon)\mu_{1\varepsilon}(\xi))\kappa_1(x)(1-\kappa_2(y))\kappa(y) G(\phi_\varepsilon, y)\, dy\, d\xi.$$

By using (11.3) and Leibniz's rule one can prove that

$$\begin{aligned}
|\Delta_\xi^k(a(x,y,\xi,\varepsilon)\mu_{1\varepsilon}(\xi))| &\le \frac{C_k}{\varepsilon^N}(1+|x|)^{r_k}(1+|\xi|)^{m-2nk} \\
&\le \frac{\tilde{C}_k}{\varepsilon^N}(1+|\xi|)^{m-2nk}, \quad y,\xi \in \mathbb{R}^n, \ x \in \operatorname{supp}\kappa_1,
\end{aligned}$$

where C_k and $\tilde{C}_k$ are suitable constants. By taking large enough k we can apply the same procedure as in the proof of Proposition 11.9. This implies that $I_2 \in \mathcal{G}_t^\infty(\mathbb{R}^n)$. □

The notion of the wave front for Colombeau's generalized functions has been introduced by Scarpalezos [18] as a natural generalization of the wave front for distributions.

Definition 12.4. A tempered generalized function G is called $\mathcal{G}^\infty$-rapidly decreasing if it has a representative $G(\phi_\varepsilon, x)$ with the following property. There exists $N \in \mathbb{N}$ such that for every $\alpha \in \mathbb{N}_0^n$ and $p \in \mathbb{N}$ there is $n_0 \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_{n_0}$ there are $C > 0$ and $\delta > 0$ such that

$$|D^\alpha G(\phi_\varepsilon, x)| \le C\varepsilon^{-N}(1+|x|^2)^{-p/2}, \ x \in \mathbb{R}^n.$$

Clearly, if $G \in \mathcal{G}^\infty(\mathbb{R}^n) \cap \mathcal{G}_c(\mathbb{R}^n)$, then $\mathcal{M}G$ is $\mathcal{G}^\infty$-rapidly decreasing. If $G \in \mathcal{G}(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$, then we will denote $\mathcal{M}(\varphi G)$ simply by φG.

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $G(\phi_\varepsilon, \cdot)$ be a representative of G. We define $\mathcal{F}(\varphi G) \in \mathcal{G}_t(\mathbb{R}^n)$ by a representative

$$\mathcal{F}_t(\varphi G)(\phi_\varepsilon, \xi) = \mathcal{F}(\varphi(x)G(\phi_\varepsilon, x))(\xi), \quad \xi \in \mathbb{R}^n, \tag{12.1}$$

where $\mathcal{F}$ denotes the Fourier transformation in $L^1(\mathbb{R}^n)$. One can prove easily that this definition makes sense. Also, the following proposition is simple.

Proposition 12.5. *If representative (12.1) has the properties given in Definition 12.4, then $\varphi G \in \mathcal{G}_c^\infty$.*

We denote by Γ a convex open cone in $\mathbb{R}^n$ which does not contain a straight line.

Let $(x_0, \xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$. The following functions will be used.

$$\left\{ \begin{array}{l} \text{(a) } \varphi \in C_0^\infty(\Omega),\ \varphi = 1 \text{ in a neighbourhood of } x_0; \\ \operatorname{supp} \psi \subset \Gamma,\ \psi \text{ is positive-homogeneous} \\ \text{of degree zero in } \Gamma \text{ and } \psi = 1 \text{ in a neighbourhood of} \xi_0. \end{array} \right. \tag{12.2}$$

Definition 12.6. It is said that $G \in \mathcal{G}(\mathbb{R}^n)$ is $\mathcal{G}^\infty$-rapidly decreasing in a cone Γ if for every $\xi_0 \in \Gamma$ there is ψ with the properties in (12.2)(b) such that ψG is $\mathcal{G}^\infty$-rapidly decreasing.

The cone $\sum_g(G)$ is the set of all $\eta \in \mathbb{R}^n \setminus \{0\}$ for which does not exist ψ with the properties in (12.2)(b) such that ψG is $\mathcal{G}^\infty$-rapidly decreasing.

Definition 12.7. It is said that $G \in \mathcal{G}(\Omega)$ is microlocally regular in an open conic set $\gamma \subset \Omega \times \mathbb{R}^n$ (conic in the second variable) if for every $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ there exist an open neighborhood Ω_0 of x_0, a conic neighborhood Γ_0 of ξ_0, and functions φ and ψ with the properties in (12.2) (with Ω_0 and Γ_0 instead Ω and Γ) such that $\psi(\xi)\mathcal{F}_t(\varphi G)(\xi)$ is $\mathcal{G}^\infty$-rapidly decreasing. The wave front of $G \in \mathcal{G}$ denoted by $\mathrm{WF}_g\, G$, is the complement of the union of all conic open sets γ where G is microlocally regular.

By using functions φ and ψ satisfying (12.2) and a unit net μ_ε we define operator $\psi(D)_\mu\varphi$ on $\mathcal{G}_t(\mathbb{R}^n)$ by $G \to \psi(D)_\mu(\varphi G)$, where

$$\psi(D)_\mu(\varphi G)(\phi_\varepsilon, x) = (2\pi)^{-n} \int\int e^{i\langle x-y,\xi\rangle}\psi(\xi)\varphi(y)G(\phi_\varepsilon, y)\mu_\varepsilon(\xi)d\mu\, d\xi.$$

Clearly $\psi(D)_\mu\varphi(\cdot)$ maps $\mathcal{G}_t(\mathbb{R}^n)$ into itself and it defines a pseudodifferential operator. Because of (11.2), (11.3), (11.4) and the estimate

$$|\partial^\alpha\psi(\xi)| \le C_\alpha|\xi|^{-\alpha}, |\xi| > R,$$

one can prove that $\psi(D)_{\mu_1}(\varphi G)$ and $\psi(D)_{\mu_2}(\varphi G)$ are equal in (g.t.d.) sense for every unit nets $\mu_{1\varepsilon}$ and $\mu_{2\varepsilon}$. The amplitude of $\psi(D)\varphi$ is $a(x, y, \xi, \varepsilon) = \psi(\xi)\varphi(y)$.

Proposition 12.8. *A point* $(x_0, \xi_0) \notin \mathrm{WF}_g G$, $G \in \mathcal{G}(\Omega)$, *if and only if there exist smooth functions* φ, ψ *with the properties in (12.2) and a unit net* μ_ε *such that* $\psi(D)_\mu(\varphi G) \in \mathcal{G}_{\mathrm{t}}^\infty$.

The proofs of the following propositions are similar to the classical one in distribution theory and because of that they are omitted.

Proposition 12.9 *If* $h \in C_0^\infty(\mathbb{R}^n)$ *and* $G \in \mathcal{G}(\mathbb{R}^n)$, *then* $\mathrm{WF}_g(hG) \subset \mathrm{WF}_g(G)$.

This proposition implies

Corollary 12.10. $\mathrm{WF}_g G = \{(x,\xi),\ \xi \in \Sigma_{gx}(G) = \bigcap_h \Sigma_g(hG)\}$, *where the intersection is taken over all* $h \in C_0^\infty$.

Denote $T^*\Omega = \Omega \times \mathbb{R}^n$ and $\pi : T^*\Omega \to \Omega$ the first projection.

Proposition 12.11. $\pi\, \mathrm{WF}_g G = \mathrm{Sing\,supp}_g G$

Proposition 12.12. *Let* $f \in \mathcal{D}'(\Omega)$. *Then* $\mathrm{WF}\, f = \mathrm{WF}_g \,\mathrm{Cd}\, f$.

For the propagation of singularities of a pseudodifferential operator we need the following definition.

Definition 12.13. $\mathrm{WF}_g AG$, $G \in \mathcal{G}_{\mathrm{t}}$, is the complement of the set of points $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ such that for every unit nets $\mu_{1\varepsilon}$, $\mu_{2\varepsilon}$ and $r \in \mathbb{N}_0$,

$$A_{\mu_1\mu_2}(G)|_\omega \text{ and } A_{\mu_2 r}(G)|_\omega$$

are microlocally regular at (x_0, ξ_0).

Proposition 12.14. *Let* $G \in \mathcal{G}_c(\Omega)$ *and* A *be a pseudodifferential operator. Then*

$$\mathrm{WF}_g AG \subset \mathrm{WF}_g G.$$

13. Composition of pseudodifferential operators

The results of the sections which are to follow are proved in **[12]**. We shell present only the definitions and assertions without proofs.

First, we define properly supported pseudodifferential operators.

Let $a \in S_g^m$ and $h \in C_0^\infty(\mathbb{R})$ such that $h(t) = 1$, $|t| \le t_0$, $h(t) = 0$, $|t| > t_1 > t_0$. We decompose a representative $A_{\mu_1\mu_2}$ of A as follows:

$$A_{\mu_1\mu_2}G(\phi_\varepsilon, x) = \dot{A}_{\mu_1\mu_2}G(\phi_\varepsilon, x) + \tilde{A}_{\mu_1\mu_2}G(\phi_\varepsilon, x),\ G \in \mathcal{G}_{\mathrm{t}}(\mathbb{R}^n),$$

where

$$\dot{A}_{\mu_1\mu_2}G(\phi_\varepsilon, x) = \frac{1}{(2\pi)^n}\iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle} h(|x-y|) a(x,y,\xi,\varepsilon)\eta_{1\varepsilon}(\xi)\eta_{2\varepsilon}(y) G(\phi_\varepsilon, y)\, dy\, d\xi$$

and $\tilde{A}_{\mu_1\mu_2}$ has $(1 - h(|x-y|))$ instead of $h(|x-y|)$ in the double integral.

Let (ξ, ε) be arbitrary, but fixed. Then the function

$$(x,y) \mapsto h(|x-y|)a(x,y,\xi,\varepsilon)$$

is properly supported which means that the inverses for the first and second projection of a compact set in $\mathbb{R}^n$ intersect the support of this function over the compact sets.

One can easily prove that $h(|x-y|)a(x,y,\xi,\varepsilon) \in S_g^m$.

Definition 13.1. Pseudodifferential operator corresponding to $a \in S_g^m$ satisfying the property that for every $(\xi,\varepsilon) \in \mathbb{R}^n \times (0,1]$,

$$(\mathbb{R}^n)^2 \ni (x,y) \mapsto a(x,y,\xi,\varepsilon),$$

is properly supported, is called a properly supported pseudodifferential operator.

Pseudodifferential operator which maps $\mathcal{G}_c(\mathbb{R}^n)$ into $\mathcal{G}_t^\infty(\mathbb{R}^n)/ \overset{\text{g.t.d}}{=}$ is called the smoothing pseudodifferential operator.

As in Proposition 11.6 one can prove

Proposition 13.2. $\tilde{A}_{\mu_1\mu_2} : \mathcal{G}_c(\mathbb{R}^n) \to \mathcal{G}_t^\infty(\mathbb{R}^n)$.

So, for every pseudodifferential operator

$$A : \mathcal{G}_t(\mathbb{R}^n) \to \mathcal{G}_t(\mathbb{R}^n)/ \overset{\text{g.t.d}}{=},$$

there exists a properly supported pseudodifferential operator

$$\dot{A} : \mathcal{G}_t(\mathbb{R}^n) \to \mathcal{G}_t(\mathbb{R}^n)/ \overset{\text{g.t.d}}{=}$$

such that $A - \dot{A}$ is a smoothing pseudodifferential operator.

Remark The extension of a properly supported pseudodifferential operator on $\mathcal{G}(\mathbb{R}^n)$ may be done as follows. Let $\dot{A}$ be properly supported with the properly supported amplitude $a \in S_g^m$ and let $\{\kappa_i,\ i \in \mathbb{N}\}$ be a partition of unity with elements in $C_0^\infty(\mathbb{R}^n)$.

Let $G \in \mathcal{G}(\mathbb{R}^n)$. Put

$$\dot{A}G(\phi_\varepsilon, x) = \sum_{i\in\mathbb{N}} \dot{A}(\kappa_i G)(\phi_\varepsilon, x).$$

Since $\kappa_i G \in \mathcal{G}_c(\mathbb{R}^n)$, any member in the sum is well defined.

One can prove easily that

$$\dot{A}_{\mu_1\mu_2}G(\phi_\varepsilon, x) \in \mathcal{E}_M[\mathbb{R}^n]$$

for every unit nets $\mu_{1\varepsilon}$ and $\mu_{2\varepsilon}$, and that for different unit nets the corresponding elements are equal in (g.t.d.) sense.

Let $a \in S_g^m$, $b \in S_g^{m'}$ determine operators A and B by representatives $A_{\mu_1\mu_2}$ and $B_{\bar{\mu}_1\bar{\mu}_2}$, where $\mu_{1\varepsilon}, \mu_{2\varepsilon}, \bar{\mu}_{1\varepsilon}$ and $\bar{\mu}_{2\varepsilon}$ are unit nets. Put

$$(A_{\mu_1\mu_2} \circ B_{\bar{\mu}_1\bar{\mu}_2})G = A_{\mu_1\mu_2}((B_{\bar{\mu}_1\bar{\mu}_2})G), \ G \in \mathcal{G}_t(\mathbb{R}^n).$$

The following proposition shows that the composition of properly supported pseudodifferential operators AB defined by a representative given above is well defined and it can be proved by a direct calculation.

Proposition 13.3. *Let $a \in S_g^m$. For every eight unit nets $\mu_{1\varepsilon}(\xi)$, $\mu_{2\varepsilon}(y)$, $\bar{\mu}_{1\varepsilon}(\xi)$, $\bar{\mu}_{2\varepsilon}(y)$, $\mu_{3\varepsilon}(\xi)$, $\mu_{4\varepsilon}(y)$, $\bar{\mu}_{3\varepsilon}(\xi)$, $\bar{\mu}_{4\varepsilon}(y)$ and $G \in \mathcal{G}_t(\mathbb{R}^n)$*

$$A_{\mu_1\mu_2}(B_{\bar{\mu}_1\bar{\mu}_2}G) \ \text{ and } A_{\mu_3\mu_4}(B_{\bar{\mu}_3\bar{\mu}_4}G)$$

are equal in (g.t.d.) sense in $\mathcal{G}_t(\mathbb{R}^n)$.

From now on we shall assume that amplitudes are defined by Definition 11.1.

Properly supported amplitudes will be indicated by $\dot{a}$. By $\dot{A}$ is denoted the corresponding pseudodifferential operator.

Theorem 13.4 *Let $\dot{a} \in S_g^m$, $\dot{b} \in S_g^{m'}$. Then the composition of $\dot{A}$ and $\dot{B}$ is represented by*

$$\text{(13.1)} \qquad A_{\mu_1\mu_2}B_{\bar{\mu}_1\bar{\mu}_2}G(\phi_\varepsilon, x) = \iint_{\mathbb{R}^{2n}} e^{i\langle x-y,\xi\rangle}\dot{k}(x,y,\xi,\varepsilon)\mu_{1\varepsilon}(\xi)G(\phi_\varepsilon, y)\,dy\,d\xi,$$

where

$$\text{(13.2)} \quad \dot{k}(x,y,\xi,\varepsilon) = \frac{1}{(2\pi)^n}\iint_{\mathbb{R}^{2n}} e^{i\langle y-z,\xi-\eta\rangle}a(x,z,\xi,\varepsilon)b(z,y,\eta,\varepsilon)\bar{\mu}_{2\varepsilon}(\eta)\,dz\,d\eta,$$

$$x,y,z,\xi,\eta \in \mathbb{R}^n, \ \ \varepsilon \in (0,\eta_0), \ \ (\eta_0 = \eta_0(\phi)).$$

Moreover, $\dot{k}(x,y,\xi,\varepsilon) \in S_g^{m+m'}$ and it is properly supported.

14. Calculus with symbols. Hypoelliptic operator

Definition 14.1. By $S_{sg}^m = S^m(\mathbb{R}^n \times \mathbb{R}^n \times [0,1))$ is denoted the subspace of $S_g^m(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [0,1))$ consisting of amplitudes $a(x,\xi,\varepsilon)$ independent of y for which (11.1) holds. By $S_{sg}^{-\infty}$ is denoted the set of elements from $S_g^{-\infty}$ which do not depend on y. Elements of S_{sg}^m are called the symbols of degree m.

As before, it can be proved that every $a \in S_{sg}^m$ defines a pseudodifferential operator $A : \mathcal{G}_t(\mathbb{R}^n) \to \mathcal{G}_t(\mathbb{R}^n)/ \overset{\text{g.t.d.}}{=}$.

Definition 14.2 A formal symbol is a sequence of symbols $a_j \in S_{sg}^{m_j}$, $j \in \mathbb{N}_0$, such that $m_j \to -\infty$ strictly, and $N_j \leq N < \infty$ (N_j are exponents of ε for a_j). It is denoted by

$$\sum_{j=0}^{\infty} a_j(x,\xi,\varepsilon).$$

As in standard theory one can make the construction of the true symbol:

Proposition 14.3. *There exists $a \in S_{sg}^{m_0}$ such that for every $j_0 \in \mathbb{N}_0$,*

$$a - \sum_{j<j_0} a_j \in S_{sg}^{m_{j_0}}.$$

It is determinated uniquely modulo $S_{sg}^{-\infty}$.

Theorem 14.4. *For every amplitude $\tilde{a}(x, y, \xi, \varepsilon) \in S_g^m$ there exists the symbol $a(x, \xi, \varepsilon) \in S_{sg}^m$ which determinates the same pseudodifferential operator $A : \mathcal{G}_t(\mathbb{R}^n) \to \mathcal{G}_t(\mathbb{R}^n)/\overset{\text{g.t.d.}}{=}$ modulo the smoothing pseudodifferential operator $\tilde{A}$.*

Thus,

$$\sum_{\alpha\in\mathbb{N}_0^n} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_y^\alpha \tilde{a}(x, y, \xi, \varepsilon)_{|y=x}$$

determines $a \in S_{sg}^m$.

For example,

$$\sum_{\alpha\in\mathbb{N}_0^n} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \psi(\xi) \partial_x^\alpha \varphi(x)$$

is the symbol for $\psi(D)\varphi$.

Theorem 14.5. *Let A and B be pseudodifferential operators with the symbols $a \in S_{sg}^m$ and $b \in S_{sg}^{m'}$ and let $\dot{A}$ and $\dot{B}$ be the corresponding properly supported operators. The symbol of the properly supported operator $\dot{A}\dot{B}$ is given by*

$$\sum_{\alpha\in\mathbb{N}_0^n} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a(x, \xi, \varepsilon) \partial_x^\alpha b(x, \xi, \varepsilon).$$

We are going to give the microlocal analysis of solutions of a pseudodifferential equation. For this we need the next definition.

Definition 14.6. A pseudodifferential operator A in Ω is smoothing in $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ if there exist $\varphi \in C_0^\infty(\Omega)$, $\varphi = 1$ in a neighborhood of x_0, and a convex open cone Γ, a neighborhood of ξ_0, such that the symbol $a(x, \xi, \varepsilon)$ of A has the following property:

There is $N > 0$ such that for every $\alpha, \beta \in \mathbb{N}_0^n$ and $M \in \mathbb{N}_0$ there is $C_{\alpha,\beta,M} \geq 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta \varphi(x) a(x, \xi, \varepsilon)| \leq C_{\alpha,\beta,M} \varepsilon^{-N} (1 + |\xi|)^{-M}, \ x \in \Omega, \ \xi \in \Gamma_R, \ |\xi| > R.$$

A pseudodifferential operator A in Ω is said to be smoothing in a conic open subset γ of $\Omega \times (\mathbb{R}^n \setminus \{0\})$ if it is smoothing in every point of γ.

The complement in $\Omega \times (\mathbb{R}^n \setminus \{0\})$ of the union of all conic open sets in which A is regularizing is called the microsupport of A and it is denoted by $\mu \operatorname{supp}_g A$.

Proposition 14.7. *Let $G \in \mathcal{G}_c(\Omega)$ and $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$. Then $(x_0, \xi_0) \notin \mathrm{WF}_g\, G$ if and only if there is a conic open neighborhood γ of (x_0, ξ_0) in $\Omega \times (\mathbb{R}^n \setminus \{0\})$ such that $B_{\mu_1 \mu_2} G \in \mathcal{G}_t^\infty(\Omega)$ $(B_{\mu_1 r} G \in \mathcal{G}_t^\infty(\Omega))$ for any pseudodifferential operator B in Ω whose microsupport is contained in γ.*

Theorem 14.8. *Let A be a pseudodifferential operator which is smoothing in a conic open set γ of $\Omega \times (\mathbb{R}^n \setminus \{0\})$. If the wave front of $G \in \mathcal{G}_c(\Omega)$ is contained in $\gamma' \subset \gamma$, then $\mathrm{Sing\,supp}_g\, A(G)$ is empty.*

The previous two propositions simply imply the following important assertion.

Proposition 14.9. *Let A be a properly supported pseudodifferential operator in Ω and $G \in \mathcal{G}(\Omega)$. Then*

$$\mathrm{WF}_g(AG) \subset (\mathrm{WF}_g\, G) \cap \mu\,\mathrm{supp}_g\, A.$$

Definition 14.10. A proper pseudodifferential operator P with a symbol $[p(x, \xi, \varepsilon)]$ is called hypoelliptic if the following holds:

(1) There exists $N \in \mathbb{N}$ such that for every compact set $K \subset \mathbb{R}^n$ there exist $\xi_0 > 0$ and $M > 0$ such that for every $\phi \in \mathcal{A}_N$ there exist $C > 0$ and $\eta > 0$ such that

$$C^{-1}(1 + |\xi|)^{-M} \varepsilon^N \le |p(x, \xi, \varepsilon)| \le C(1 + |\xi|)^M \varepsilon^{-N}, \tag{14.1}$$

for $x \in K$, $|\xi| \ge \xi_0$, $\varepsilon < \eta$.

(2) There exists $N \in \mathbb{N}$ such that for every compact set $K \subset \mathbb{R}^n$ there exists $\xi_0 > 0$ such that for every $\phi \in \mathcal{A}_N$ there exist $C_{\alpha,\beta} > 0$ and $\eta > 0$ such that

$$\left| \frac{D_\xi^\alpha D_x^\beta p(x, \xi, \varepsilon)}{p(x, \xi, \varepsilon)} \right| \le C_{\alpha,\beta}(1 + |\xi|)^{-|\alpha|}, \ x \in K, \ |\xi| \ge \xi_0, \ \varepsilon < \eta. \tag{14.2}$$

Without a proof we give

Theorem 14.11. *(i) Let P be a proper pseudodifferential operator with symbol $p(x, \xi, \varepsilon)$ which satisfies Definition 14.6. Then the following holds: There exists $N \in \mathbb{N}$ such that for every compact set $K \subset \mathbb{R}^n$ there exists $\xi_0 > 0$ such that for every $\phi \in \mathcal{A}_N$ there exist $C'_{\alpha,\beta} > 0$ and $\eta > 0$ such that*

$$\left| \frac{D_\xi^\alpha D_x^\beta p(x, \xi, \varepsilon)^{-1}}{p(x, \xi, \varepsilon)^{-1}} \right| \le C_{\alpha,\beta}(1 + |\xi|)^{-|\alpha|}, \ x \in K, \ |\xi| \ge \xi_0, \ \varepsilon < \eta. \tag{14.3}$$

(ii) For every hypoelliptic pseudodifferential operator P there exists a proper pseudodifferential operator Q such that $PQ - I \in S^{-\infty}$, and $QP - I \in S^{-\infty}$.

Proposition 14.12. *Let P be a hypoelliptic pseudodifferential operator. Then*

$$\mathrm{WF}_g(PG) = \mathrm{WF}_g\, G$$

for every $G \in \mathcal{G}$.

Pseudodifferential operator P is called elliptic with a classical amplitude if its symbol $p(x,\xi,\varepsilon)$ satisfies the following inequality

$$C^{-1}(1+|\xi|)^{-M} \le |p(x,\xi,\varepsilon)| \le C(1+|\xi|)^{M} \tag{14.4}$$

instead of (14.1). One can prove that (14.4) implies (14.2) and that means that there exist a parametrix for such pseudodifferential operators, too.

Pseudodifferential operator is called elliptic if

$$C^{-1}(1+|\xi|)^{-M}\varepsilon^{-N} \le |p(x,\xi,\varepsilon)| \le C(1+|\xi|)^{M}\varepsilon^{-N} \tag{14.5}$$

holds instead of (14.1). As in the previous case, one can prove that then (14.5) implies (14.2), and this implies the existence of the parametrix for A.

References

1. H. A. Biagoni, *A Nonlinear Theory of Generalized Functions,* Lecture Notes Math. 1421, Springer-Verlag, Berlin, 1990.
2. J. F. Colombeau, *New Generalized Functions and Multiplication of the Distributions,* North Holland, Amsterdam, 1983.
3. J. F. Colombeau, *Elementary Introduction in New Generalized Functions,* North Holland, Amsterdam, 1985.
4. J. F. Colombeau, *A Multiplications of Distributions,* Bull. Amer. Math. Soc. **23** (1990), 251–268.
5. J. J. Duistermaat, L. Hörmander, *Fourier integral operators II,* Acta Math. **128** (1972), 183–269
6. Yu. V. Egorov, *A contribution to the theory of generalized functions,* Russian Math. Surveys **45**:5 (1990), 1–49; translated from: Uspehi Mat. Nauk 45:5 (1990), 3–40.
7. T. Gramchev, *Nonlinear maps in spaces of distributions,* Math. Zeitschr. **209** (1992), 101–114.
8. T. Gramchev, M. Oberguggenberger, *Regularity theory and pseudodifferential operators in algebras of generalized functions,* in preparation.
9. L. Hörmander, *Fourier integral operators I,* Acta Math. **127** (1971), 79–183
10. L. Hörmander, *The Analysis of Linear Partial Differential Operators III,* Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985.
11. B. de Monvel, *Pseudodifferential Operators and their Applications,* Mathematic Series II, Duke University, Durham, 1976.
12. M. Nedeljkov, S. Pilipović, *Pseudodifferental calculus in Colombeau's algebras,* Preprint
13. M. Oberguggenberger, *Multiplication of Distributions and Applications to Partial Differential Equations,* Pitman Res. Not. Math. 259, Longman Sci. Techn., Essex, 1992.
14. M. Oberguggenberger, *Lecture Notes on Pseudodifferential calculus,* Preprint.
15. B. E. Petersen, *Introduction to the Fourier Transform and Pseudodifferential Operators,* 1983.
16. S. Pilipović, *Colombeau's generalized functions and the pseudodifferential operators,* Lecture Notes of Tokyo University, Tokyo, 1993.
17. S. Pilipović, *Pseudodifferential operators and the microlocalization in the space of Colombeau's generalized functions,* Bull. Acad. Serb. Sci. Arts **20** (1995), 13–27.
18. D. Scarpalézos, *Colombeau's generalized functions,* topological structures, microlocal properties. A simplified point of view, preprint.
19. F. Treves, *Introduction to Pseudodifferential and Fourier Integral Operators,* Plenum Press, New York, London, 1980.
20. M. Shubin, *Pseudodifferential Operators and Spectral Theory,* Nauka, Moscow, 1978.