B. Stanković

HYPERFUNCTIONS

Contents

Introduction

M. Sato ([27], [28]) introduced a new class of generalized functions, called hyperfunctions, as the n-th derived sheaf of the sheaf of holomorphic functions. He left without proof many details in these papers. To this day, subsequent papers of mathematicians, especially Japanese, completed these "gaps" ([3], [10], [13], [15], [18], [20], [30]).

Hyperfunctions have many important properties which are indispensable for an exquisite theory of partial differential equations, microfunctions, micro-local analysis, Fourier transform (cf. [13]). They became a major tool of several areas of analysis and applications.

The set of hyperfunctions forms a flabby sheaf on $\mathbf{R}^n$ [20]. Schwartz's space $\mathbf{D}'(\Omega)$ (Ω is an open set in $\mathbf{R}^n$) of distributions and the dual space of Gevrey class of functions on Ω are naturally contained in the space $\mathbf{B}(\Omega)$, of hyperfunctions on Ω (cf. [13]). For the relations between hyperfunctions and other generalized functions we refer to [4], [5], [19], [22] and [23].

Since Sato's theory utilizes the most advanced concept of sheaf cohomologies, it is not so popular as Schwartz distributions or Beurling and Roumieu ultradistributions. Also, there are a lot of introductory books on different types of generalized functions, but very few on Sato hyperfunctions. However there is a number of different approaches to hyperfunctions. Some of them are based on the same idea as Schwartz's distributions. Martineau [13] started with the space $\mathbf{A}'(\mathbf{R}^n)$ of analytic functionals carried by compact subsets of $\mathbf{R}^n$. For any open set $\Omega \subset \mathbf{R}^n$ the space of hyperfunctions on Ω is defined so that its elements are locally equal to those in $\mathbf{A}'(\mathbf{R}^n)$. A topology of hyperfunctions, has many exceptional features. (see also [1], [4], [13]). In the book [6] Imai introduced hyperfunctions from the viewpoint of applied mathematics.

In 1988 appeared Kaneko's book [7] (English edition) which is intended to be the first easily accessible introduction to Sato's hyperfunctions. Kaneko defines hyperfunctions using boundary value representation ("intuitive" method). Such an approach has been used from the very beginning only as an illustration. But after progress in the theory of Radon transform this approach has claimed its own place

AMS Subject Classification (1991): Primary 46F15; Secondary 32A45 in the foundation of hyperfunctions as a precise mathematical theory. The first rigorous proofs in this sense have been given by Morimoto [20]. There exist many papers on this subject. Kaneko's book is the first monograph with a systematic elaborated theory of hyperfunctions defined by boundary value representation.

Our aim is to draw attention, especially of young mathematicians, to hyperfunctions and to Kaneko's book which is the main reference in this text and can be the next step to make acquaintance of hyperfunction.

1. PRELIMINARIES

We repeat some standard part of the theory of sheaves and sheaf cohomology we need to introduce hyperfunctions. For this part one can consult any book on algebraic analysis and sheaves theory, for example [10].

1.1. Notation and notions

By $\mathbf{X}$ we denote a topological space and by S a locally closed set in $\mathbf{X}$. S is **locally closed set* in $\mathbf{X}$ if it can be written as the intersection of an open and an closed set in $\mathbf{X}$. Thus there exists an open set $U \subset \mathbf{X}$ containing S as relatively closed subset. In $\mathbf{R}$ every interval is locally closed.

A cone in $\mathbf{R}^n$ will be denoted by Γ or by Δ; $\operatorname{pr}\Gamma = \{x \in \Gamma;\ \|x\| = 1\}$; $\Gamma' \subset\subset \Gamma$ means that $\operatorname{pr}\overline{\Gamma'} \subset \operatorname{int}\Gamma$; $\Gamma^0 = \{\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n; \xi x = \xi_1 x_1, \dots, \xi_n x_n \geq 0$ for every $x = (x_1, \dots, x_n) \in \Gamma\}$ is called the **dual cone* to Γ.

$\{F_\alpha;\ \alpha \in A\}$ is a **locally finite family of subset* of F if for every $x \in F$ and every neighbourhood $V(x)$ of x. $V(x) \cap F_\alpha \neq \emptyset$ only for a finite number $\alpha \in A$.

$E = \bigoplus_{\alpha \in A} E_\alpha$ is the **direct sum* of vector spaces E_α, $\alpha \in A$, if every $x \in E$ can be given in a unique way as the finite sum Σx_α, $x_\alpha \in E_\alpha$.

Let $\mathbf{U} = \{U \subset \mathbf{X};\ U \supset A\}$ be the set of open sets containing $A \subset \mathbf{X}$. To each $U \in \mathbf{U}$ there is associated a $\mathbf{C}$-vector space E_U and to each pair $U, V, \in \mathbf{U}$, $U \supset V$, there is associated a $\mathbf{C}$-linear mapping $\rho_{V,U} : E_U \to E_V$ (restriction) in such a way that: i) $\rho_{UU} = \mathrm{id}$; ii) $\rho_{WU} = \rho_{WV} \circ \rho_{VU}$, whenever $U \supset V \supset W$. Then $\{E_U;\ U \in \mathbf{U}\}$ is an **inductive system of* $\mathbf{C}$*-vector spaces.* Let $E = \bigsqcup_{U \in \mathbf{U}} E_U$ ($\sqcup$ is formed by taking the union of E_U's regarding the E_U's as mutually unrelated). Introduce an equivalence relation $\sim$ in E as follows: $F \sim G$ ($F \in E_U$, $G \in E_V$) means that $\rho_{WU} F = \rho_{WV} G$ in E_W for some $W \subset U \cap V$. The **inductive limit* is

$$\varinjlim_{U \in \mathbf{U}} E_U = E/\sim .$$

Tere exists a natural mapping $\rho_U : E_U \to \varinjlim_{U \in \mathbf{U}} E_U$.

Example. Let Ω be an open set in $\mathbf{R}$ and U an open set in $\mathbf{C}$, a neighbourhood of Ω in $\mathbf{C}$. By $\mathbf{O}(U)$ is denoted the set of holomorphic functions on U. Then $A(\Omega) = \varinjlim_{U \supset \Omega} \mathbf{O}(U)$ is the set of **real analytic functions* on Ω.

1.2. Presheaves and sheaves

We say that a **presheaf* $\mathbf{F}$ of $\mathbf{C}$-vector spaces on $\mathbf{X}$ is given if: i) to each open set $V \subset \mathbf{X}$ there is associated a $\mathbf{C}$-vector space $\mathbf{F}(V)$ and ii) to each pair (V, W), $V \supset W$, there is associated a $\mathbf{C}$-linear mapping $\rho_{W,V} : \mathbf{F}(V) \to \mathbf{F}(W)$ such that: a) $\rho_{VV} = \mathrm{id}$; b) $\rho_{ZW} \circ \rho_{WV} = \rho_{ZV}$, $Z \subset V \subset W$. Every element f of $\mathbf{F}(V)$ is called a **section of* $\mathbf{F}$ *on* U. We also write $\rho_{WV}(f) = f_{|W}$ (**restriction of* $f \in \mathbf{F}(V)$ on $W, W \subset V$).

A presheaf $\mathbf{F}$ is a **sheaf on* $\mathbf{X}$ if for any open covering $\{U_\lambda : \lambda \in \Lambda\}$ of an open set $V \subset \mathbf{X}$ we have the following properties: iii) if $f \in \mathbf{F}(V)$ and $f|_{U_\lambda} = 0$ for every U_λ, $\lambda \in \Lambda$, then $f \equiv 0$ (0 is the zero element of $\mathbf{F}(V)$); iv) for a family $\{f_\lambda;\, \lambda \in \Lambda\}$; $f_\lambda \in \mathbf{F}(U_\lambda)$, such that $f_\lambda|_{U_\lambda \cap U_\eta} = f_\eta|_{U_\lambda \cap U_\eta}$, $U_\lambda \cap U_\eta \neq \emptyset$, there exists $f \in \mathbf{F}(V)$ which has the property $f|_{U_\lambda} = f_\lambda$, $\lambda \in \Lambda$.

$A \subset V$ is the **support* of $f \in \mathbf{F}(V)$ if $V \setminus A$ is the largest open set contained in V on which f is zero.

Remark. Usually presheaves and sheaves are defined for Abelian groups with ρ_{WV} Abelian group homomorphism.

Examples 1. The sheaf $\mathbf{O}$ of holomorphic functions on $\mathbf{C}^n$; to each open set $V \subset \mathbf{C}^n$ there is associated $\mathbf{O}(V)$.

2. The presheaf $\mathbf{L}_1$ on $\mathbf{R}$ (Lebesgue integrable functions). $\mathbf{L}_1$ is not a sheaf because iii) is not satisfied. Let $U_\lambda = (-\lambda, \lambda)$ and $f_\lambda = 1$ for $\lambda \in \mathbf{R}_+$. We can not find an $f \in \mathbf{L}_1(\mathbf{R})$ such that $f|_{U_\lambda} = 1$ for every $\lambda \in \mathbf{R}_+$.

3. The sheaf $\mathbf{A}$ of real analytic functions on $\mathbf{R}^n$.

Let $\mathbf{F}$ and $\mathbf{G}$ be two (pre)sheaves on $\mathbf{X}$. A family $h = \{h_V\}$ of $\mathbf{C}$-linear mappings, $h_V : \mathbf{F}(V) \to \mathbf{G}(V)$ is **a (pre)sheaf homomorphism* if the following diagram commutes:

$$\begin{array}{ccc} \mathbf{F}(V) & \xrightarrow{h_V} & \mathbf{G}(V) \\ {\scriptstyle \rho^{\mathbf{F}}_{WV}}\downarrow & & \downarrow{\scriptstyle \rho^{\mathbf{G}}_{WV}} \\ \mathbf{F}(W) & \xrightarrow[h_W]{} & \mathbf{G}(W) \end{array}$$

Sheaf homomorphisms do not enlarge the support of a section.

The linear differential operator with real analytic coefficients is a homomorphism of the sheaf $\mathbf{A}$ of real analytic functions.

$\mathbf{F}$ is said to be a **subsheaf* of the sheaf $\mathbf{G}$ if for every open set $V \subset \mathbf{X}$ there is associated the inclusion $i_V : \mathbf{F}(V) \to \mathbf{G}(V)$ such that $i = \{i_V\}$ constitutes a sheaf homomorphism. We write in short $\mathbf{F} \subset \mathbf{G}$.

The restriction of the sheaf $\mathbf{F}$ to the open set $V \subset \mathbf{X}$ is the sheaf defined by: $W \to \mathbf{F}(W)$ for every open set $W \subset V$; we denote it by $\mathbf{F}/_V$ (attention, $\mathbf{F}/_V$ is a sheaf and $\mathbf{F}(V)$ is a vector space).

A sheaf $\mathbf{F}$ on $\mathbf{X}$ is **flabby* if for every open set $V \subset \mathbf{X}$, $\rho_{V\mathbf{X}} : \mathbf{F}(\mathbf{X}) \to \mathbf{F}(V)$ is surjective.

Proposition 1.1. *If* $\mathbf{F}$ *is flabby, then for every pair of open sets* (U, V), $U \supset V$, *the restriction* $\rho_{VU} : \mathbf{F}(U) \to \mathbf{F}(V)$ *is surjective.*

Proof. For a given $v \in \mathbf{F}(V)$ there exists $x \in \mathbf{F}(\mathbf{X})$ such that $\rho_{V\mathbf{X}}(x) = v$; let $\rho_{U\mathbf{X}}(x) = u$, then $v = \rho_{V\mathbf{X}}(x) = \rho_{VU} \circ \rho_{U\mathbf{X}}(x) = \rho_{VU}(u)$, where $u \in \mathbf{F}(U)$. □

Let S be a locally closed set in $\mathbf{X}$ and U an open neighbourhood of it containing S as a relatively closed subset. Denote by $\Gamma_S(\mathbf{X}, \mathbf{F}) = \{s \in \mathbf{F}(U); \operatorname{supp} s \subset S\}$, where $\mathbf{F}$ is a sheaf on $\mathbf{X}$.

Proposition 1.2. *The definition of the* $\mathbf{C}$*-vector space* $\Gamma_S(\mathbf{X}, \mathbf{F})$ *does not depend on the choice of* U.

Proof. Let U_1 and U_2 be two such open neighbourhoods of S. Then $U_1 \cap U_2$ is again such an open neighbourhood of S. Hence, it suffices to show that the restriction

$$i : \{s \in \mathbf{F}(U_1);\ \operatorname{supp} s \subset S\} \to \{s \in \mathbf{F}(U_2); \operatorname{supp} s \subset S\}$$

is an isomorphism when $U_1 \supset U_2$. But this is obvious because if $s \in \mathbf{F}(U_2)$, $\operatorname{supp} s \subset S \subset U_2 \subset U_1$, then s can be extended to

$$s' \in \mathbf{F}(U_1),\ \ s'|_{U_2} = s,\ \ s'|_{U_1 \setminus U_2} = 0. \qquad \square$$

A direct consequence of Proposition 1.2 is

Proposition 1.3. $\Gamma_U(\mathbf{X}, \mathbf{F}) = \mathbf{F}(U)$; $\Gamma_S(\mathbf{X}, \mathbf{F}) = \Gamma_S(U, \mathbf{F}|_U)$, *where* S *is relatively closed subset of the open set* U*; if* S *is closed, then* $\Gamma_S(\mathbf{X}, \mathbf{F}) = \{s \in \mathbf{F}(\mathbf{X}), \operatorname{supp} s \subset S\}$.

Proposition 1.4. *Let* V *be an open set in* $\mathbf{X}$ *and* S *a locally closed set in* $\mathbf{X}$. *The correspondence* $V \to \Gamma_{S \cap V}(\mathbf{X}, \mathbf{F})$ *constitutes a sheaf on* $\mathbf{X}$ *denoted by* $\mathbf{T}_S(\mathbf{F})$. *It may also be regarded as a sheaf on* S.

Proof. It is obvious that $\mathbf{T}_S(\mathbf{F})$ is a presheaf. Also iii) and iv) follow from the fact that $\mathbf{F}$ is a sheaf.

Taking S as a topological space with the topology induced by $\mathbf{X}$, then $\mathbf{T}_S(\mathbf{F})(V) = \Gamma_{S \cap V}(\mathbf{X}, \mathbf{F})$ and $V \to \mathbf{T}_S(\mathbf{F})(V)$, $V \cap S \neq \emptyset$, where V is any open set in $\mathbf{X}$, defines a sheaf on S. □

Remark. If U is an open set in $\mathbf{X}$, then $\mathbf{T}_U(\mathbf{F}) = \mathbf{F}|_U$ and $\mathbf{T}_S(\mathbf{F})(\mathbf{X}) = \Gamma_S(\mathbf{X}, \mathbf{F})$.

Proposition 1.5. *If* $\mathbf{F}$ *is flabby, then* $\mathbf{T}_S(\mathbf{F})$ *is flabby, as well.*

Proof. Let U be an open set in $\mathbf{X}$ containing S as a relatively closed subset. We will prove that for any open set $V \subset \mathbf{X}$, $V \cap S$ is relatively closed subset of $V \cap U$. By definition of a locally closed set we have $S = O_S \cap Z_S$, where O_S is an open set in $\mathbf{X}$ and Z_S is a closed set in $\mathbf{X}$. Then $S \cap V = (Z_S \cap O_S) \cap V = Z_S \cap (O_S \cap V)$. Hence, $S \cap V$ is locally closed in $\mathbf{X}$.

To prove that $S \cap V$ is relatively closed in $V \cap U$ take an $x \in V \cap U$ and $x \notin S \cap V$. Since S is a relatively closed in U, there exists an open set $O \ni x$, $O \subset U$ such that $S \cap O = \emptyset$. The open set $O \cap V \ni x$ and $O \cap V \subset U \cap V$. Also,

$$(O \cap V) \cap (V \cap S) = (O \cap V) \cap S = (O \cap S) \cap V = \emptyset.$$

Consequently $S \cap V$ is relatively closed in $V \cap U$.

By Proposition 1.2, $\mathbf{T}_S(\mathbf{F})(V) = \Gamma_{S \cap V}(\mathbf{X}, \mathbf{F}) = \{s \in \mathbf{F}(V \cap U);\ \operatorname{supp} s \subset S \cap V\}$. Therefore, for $s \in \mathbf{T}_S(\mathbf{F})(V)$, $s|_{(U \setminus S) \cap (U \cap V)} = 0$. By Proposition 1.1 there exists an $s' \in \mathbf{F}((U \setminus S) \cup (U \cap V))$ such that $s'|_{U \setminus S} = 0$, $s'|_{V \cap U} = s$. By the same Proposition, s' can be extended to $\tilde{s} = \mathbf{F}(U)$, $\tilde{s}|_{U \setminus S} = 0$. Consequently $\tilde{s} \in \Gamma_S(\mathbf{X}, \mathbf{F}) = \mathbf{T}_S(\mathbf{F})(\mathbf{X})$. □

Let $\mathbf{F}$ be a (pre)sheaf on the topological space $\mathbf{X}$. For an $x \in \mathbf{X}$ and any open neighbourhood V of x,

$$\mathbf{F}_x = \varinjlim_{x \in V} \mathbf{F}(V),$$

is called the **stalk of* $\mathbf{F}$ *at* x. An element of $\mathbf{F}_x$ is called **a germ of sections* of $\mathbf{F}$ at x. A germ consists of local sections of $\mathbf{F}$, defined in a neighbourhood of x, which coincide on a smaller neighbourhood of x. A section $s \in \mathbf{F}(V)$ defines a germ $s_x \in \mathbf{F}_x$ at every point $x \in V$.

Proposition 1.6. *If* $\mathbf{F}$ *is a sheaf and* $s \in \mathbf{F}(V)$, *then* $s = 0$ *if and only if* $s_x = 0$ *for all* $x \in V$.

The proof is a direct consequence of the definition of a sheaf (see property iii)).

Attention. Make a distinction of s_x and $s(x)$; $s_x = 0$ means that $s(y) = 0$ for y belonging to a neighbourhood of x.

For a presheaf $\mathbf{F}$ on $\mathbf{X}$ and for every open set $V \subset \mathbf{X}$ we construct the vector space $\overline{\mathbf{F}}(V) = \{\bar{s} : V \to \bigsqcup_{x \in V} \mathbf{F}_x$, such that for each $x \in V$ there exists an open set $W \subset V$, $W \ni x$ and $t \in \mathbf{F}(W)$, with the property that $\bar{s}(y) = t(y)$ for every $y \in W\}$.

Proposition 1.7. *Let* V *be any open set in* $\mathbf{X}$. *The correspondence:* $V \to \overline{\mathbf{F}}(V)$ *with canonical restriction gives a sheaf on* $\mathbf{X}$ *and* $\overline{\mathbf{F}}_x = \mathbf{F}_x$.

Proof. It is obvious that $\overline{\mathbf{F}}$ is a presheaf. First the verification of iii). Let $\{U_\lambda\}$ be an open covering of the open set $V \subset \mathbf{X}$ and let $\bar{s} \in \overline{\mathbf{F}}(V)$, $\bar{s}|_{U_\lambda} = 0$. There exists an open set W, $x \in W \subset U_\lambda$, and $t \in \mathbf{F}(W)$ such that $\bar{s}(y) = t(y) = 0$ for every $y \in W$. It follows that $\bar{s}(x) = 0$ as an element of $\mathbf{F}_x$ for every $x \in U_\lambda$ and for every $U_\lambda \in \{U_\lambda\}$. By definition of $\bar{s}$, $\bar{s} = 0$.

Verification of iv). Given $\{\bar{s}_\lambda\}$, $\bar{s}_\lambda \in \overline{\mathbf{F}}(U_\lambda)$ with the property $\bar{s}_\lambda|_{U_\lambda \cap U_\eta} = \bar{s}_\eta|_{U_\lambda \cap U_\eta}$, where $U_\lambda \cap U_\eta \neq \emptyset$. We construct $\bar{s} \in \overline{\mathbf{F}}(V)$ such that $\bar{s}|_{U_\lambda} = \bar{s}_\lambda$ in the following way: if $x \in V$, then there exists U_λ, $x \in U_\lambda$; now $\bar{s}(x) = \bar{s}_\lambda(x)$.

At the end we prove that $\overline{\mathbf{F}}_x = \mathbf{F}_x$ (These two spaces are isomorphic). Let $\bar{s}_x \in \overline{\mathbf{F}}_x$, then $\bar{s}_x$ is given by an element $\bar{t} \in \overline{\mathbf{F}}(V)$, where V is an open set containing

x. We can take a smaller open set $W \ni x$ such that $\bar{t}|_W = t \in \mathbf{F}(W)$. Then t determines an element of $\mathbf{F}_x$. Hence we constructed a mapping $\overline{\mathbf{F}}_x \to \mathbf{F}_x$. By the construction, it is surjective and an isomorphism. □

The constructed sheaf $\overline{\mathbf{F}}$ is called **the sheaf associated with the presheaf* $\mathbf{F}$.

Example. Let $\mathbf{X} = \mathbf{R}$ and V be an open set in $\mathbf{R}$. By $V \to \mathbf{L}_1(V)$ is defined the presheaf of Lebesgue integrable functions. This is not a sheaf. The sheaf associated with this presheaf is the **sheaf of locally integrable functions* on $\mathbf{R} : V \to \mathbf{L}_{\text{loc}}(V)$.

Let G be a fixed vector space associated to every open set $V \subset \mathbf{X}$, $V \to \mathbf{F}(V) = G$. Take the identity mapping of G as the restriction. Then $V \to G$ defines a presheaf $\mathbf{F}$ on $\mathbf{X}$. It is not a sheaf in general. The property iv) is not always satisfied. Suppose that V is not connected, namely that $V = U_1 \cup U_2$ where U_1 and U_2 are open set and $U_1 \cap U_2 = \emptyset$. Let $g_1 \in \mathbf{F}(U_1) = G$ and $g_2 \in \mathbf{F}(U_2) = G$, $g_1 \neq g_2$. We can not find a $g \in \mathbf{F}(V) = G$ such that $g|_{U_1} = g_1$ and $g|_{U_2} = g_2$.

The sheaf associated to this presheaf $\mathbf{F}$ is called the **constant sheaf* $\mathbf{G_X}$. The difference between $\mathbf{F}(V)$ and $\overline{\mathbf{F}}(V)$ appears when V is not connected.

If $G = \{0\}$, $\mathbf{G_X}$ is a sheaf for each $\mathbf{X}$; it is denoted by 0.

Given a sheaf $\mathbf{G}$ on $\mathbf{X}$ and its subsheaf $\mathbf{F}$. The correspondence: $V \to \mathbf{G}(V)/\mathbf{F}(V)$ (the quotient space) for open sets $V \subset \mathbf{X}$ gives a presheaf on $\mathbf{X}$ (property iv) is not always satisfied). The sheaf associated with this presheaf is called the **quotient sheaf of* $\mathbf{G}$ *by* $\mathbf{F}$ and denoted by $\mathbf{G}/\mathbf{F}$.

Let $h : \mathbf{F} \to \mathbf{G}$ be a sheaf homomorphism and V an open set in $\mathbf{X}$. $V \to \ker h_V$ determines a subsheaf of $\mathbf{F}$ denoted by $\mathbf{Ker}\, h$ (kernel of h). We shall prove that $\mathbf{Ker}\, h$ is a sheaf.

$\ker h_V = \{f \in \mathbf{F}(V);\ h_V(f) = 0\}$ is a vector space. With restrictions $\rho_{WV}, W \subset V$, $V \to \ker h_V$ is a presheaf. Property iii). Let $\{U_\lambda\}$ be an open covering of the open set V and $f \in \ker h_V$, $f|_{U_\lambda} = 0$, $U_\lambda \in \{U_\lambda\}$. Since $\mathbf{F}$ is a sheaf, $f = 0$ on V and $0 \in \ker h_V$. Property iv). With the same open covering $\{U_\lambda\}$ of V let $f_\lambda \in \ker h_{U_\lambda} = \{f \in \mathbf{F}(U_\lambda);\ h_{U_\lambda}(f) = 0\}$. If $U_\lambda \cap U_\eta \neq \emptyset$, then by supposition, $f_\lambda = f_\eta$ on $U_\lambda \cap U_\eta$. Since $\mathbf{F}$ is a sheaf, there exists $f \in \mathbf{F}(V)$ such that $f|_{U_\lambda} = f_\lambda$. By the property of the sheaf homomorphism we have

$$\rho^{\mathbf{G}}_{U_\lambda V} \circ h_V(f) = h_{U_\lambda} \circ \rho^{\mathbf{F}}_{U_\lambda, V}(f) = h_{U_\lambda}(f_\lambda) = 0.$$

Hence, $h_V(f)|_{U_\lambda} = 0$. Since $\mathbf{G}$ is also a sheaf, $h_V(f) = 0$ and $f \in \ker h_V$.

The correspondence: $V \to \operatorname{im} h_V$ for an open set $V \subset \mathbf{X}$ defines a presheaf. The sheaf associated with it is denoted by $\mathbf{Im}\, h$ (image of h).

Example. Consider the sheaf homomorphism $\dfrac{d}{dz} : \mathbf{O} \to \mathbf{O}$, where $\mathbf{O}$ is the sheaf of holomorphic functions on $\mathbf{C}$. $\mathbf{Ker}\, \dfrac{d}{dz}$ is the constant sheaf $\mathbf{C_C}$. The image of $\left(\dfrac{d}{dz}\right)_V : \mathbf{O}(V) \to \mathbf{O}(V)$, where V is an open set in $\mathbf{C}$, consists of all functions

f whose contour integrals around any "hole" in V, if such a "hole" exists in V, are all zero because in this case

$$F(z) = \int_{z_0}^{z} f(\xi)d\xi \in \mathbf{O}(V) \text{ and } \frac{d}{dz}F(z) = f(z),$$

where $z, z_0 \in V$. The sheaf associated with the presheaf: $V \to \mathrm{im}\left(\frac{d}{dz}\right)_V(V)$ is $\mathbf{O}$ $\left(\mathbf{Im}\ \frac{d}{dz} = \mathbf{O}\right)$.

The presheaf homomorphism $h : \mathbf{F} \to \mathbf{G}$ induces the C-linear mapping $h_x : \mathbf{F}_x \to \mathbf{G}_x$ in the following way: $\mathbf{F}_x \ni s_x \xrightarrow{h_x} (h_V(s))_x$, where $s \in s_x$, $s \in \mathbf{F}(V)$, $V \ni x$. We have to prove that this definition does not depend on the chosen representative of s_x and the open set $V \subset \mathbf{X}$. Let $t \in s_x$, $t \in \mathbf{F}(W), W \ni x$. By definition of s_x there exists $Z \subset V \cap W$ such that $t(y) = s(y)$, $y \in Z$, or

$$\rho^{\mathbf{F}}_{ZW}(t) = \rho^{\mathbf{F}}_{ZV}(s).$$

By the property of homomorphism h we have:

$$\rho^{\mathbf{G}}_{ZV} \circ h_V(s) = h_Z \circ \rho^{\mathbf{F}}_{ZV}(s) = h_Z \circ \rho^{\mathbf{F}}_{ZW}(t) = \rho^{\mathbf{G}}_{ZW} \circ h_W(t).$$

Hence, $h_V(s)(y) = h_W(t)(y)$, $y \in Z$ and $(h_V(s))_x = (h_W(t))_x$.

Proposition 1.8. $(\mathbf{Im}\, h)_x = \mathrm{im}\, h_x$.

Proof. Denote by $\mathbf{H}$ the presheaf $V \to \mathrm{im}\, h_V$, where V is any open set in $\mathbf{X}$. Then by Proposition 1.7, $(\mathbf{Im}\, h)_x = \mathbf{H}_x$ for every $x \in \mathbf{X}$. By definition of h_x, $\mathbf{H}_x = \mathrm{im}\, h_x$ because of $\mathbf{H}_x = \varinjlim_{V \ni x} \mathrm{im}\, h_V$. □

Proposition 1.9. *If* $\mathbf{F}$ *and* $\mathbf{G}$ *are two sheaves and* $\mathbf{F} \subset \mathbf{G}$, *then* $\mathbf{F} = \mathbf{G}$ *is equivalent to* $\mathbf{F}_x = \mathbf{G}_x$ *for all* $x \in \mathbf{X}$.

Proof. Denote by $i = (i_V)$ inclusion: $\mathbf{F} \to \mathbf{G}$. If $\mathbf{F}_x = \mathbf{G}_x$, then i_x is surjective. We have to prove that i_V is surjective for every open set $V \subset \mathbf{X}$. Suppose that $\xi \in \mathbf{G}(V)$, then $\xi \in \xi_x \in \mathbf{G}_x$, $x \in V$. There exists $s_x \in \mathbf{F}_x$ such that $s_x = \xi_x$. Consequently, there exists $s^x \in \mathbf{F}(W_x)$, $W_x \ni x$, $W_x \subset V$ sich that $\xi(y) = s^x(y)$, $y \in W_x$. The family of open sets $\{W_x; x \in V\}$ is an open covering of V. By property iv) there exists $f \in \mathbf{F}(V)$ such that $f|_{W_x} = s^x$ for every $x \in V$. Consequently, $f = \xi$ on V. If $\mathbf{F} = \mathbf{G}$ it is clear that $\mathbf{F}_x = \mathbf{G}_x$ for every $x \in \mathbf{X}$. □

1.3. Sheaf cohomology

Let $\mathbf{F} \xrightarrow{h} \mathbf{G} \xrightarrow{k} \mathbf{H}$ be a sequence of sheaf homomorphisms where $\mathbf{F}, \mathbf{G}, \mathbf{H}$ are sheaves on $\mathbf{X}$. This **sequence* is said to be **exact at* $\mathbf{G}$ if $\mathbf{Im}\, h = \mathbf{Ker}\, k$. (For short, **exact sequence)*. In particular, $0 \to \mathbf{G} \xrightarrow{k} \mathbf{H}$ is exact at $\mathbf{G}$ if and only if k is injective; $\mathbf{F} \xrightarrow{h} \mathbf{G} \to 0$ is exact at $\mathbf{G}$ if and only if h is surjective.

The same definition is for the exact sequence of vector spaces.

Proposition 1.10. *The sequence* $\mathbf{F} \xrightarrow{h} \mathbf{G} \xrightarrow{k} \mathbf{H}$ *is exact at* $\mathbf{G}$ *if and only if the sequence of vector spaces.* $\mathbf{F}_x \xrightarrow{h_x} \mathbf{G}_x \xrightarrow{k_x} \mathbf{H}_x$ *is exact at* $\mathbf{G}_x$ *for every* $x \in \mathbf{X}$.

Proof. According to propositions 1.8 and 1.9 the following three assertions are equivalent: $\mathbf{Im}\, h = \mathbf{Ker}\, k$; $(\mathbf{Im}\, h)_x = (\mathbf{Ker}\, k)_x$; $\operatorname{im} h_x = \ker k_x$ for every $x \in \mathbf{X}$. □

If $\mathbf{F} \xrightarrow{h} \mathbf{G} \xrightarrow{k} \mathbf{H}$ is exact, then the sequence of vector spaces

$$\mathbf{F}(V) \xrightarrow{h_V} \mathbf{G}(V) \xrightarrow{k_V} \mathbf{H}(V)$$

is not necessarily exact ($(\mathbf{Im}\, h)_V$ has not to be equal to $\operatorname{im} h_V$). But if the above sequence of vector spaces is exact at $\mathbf{G}(V)$ for all open sets V which constitute a fundamental system of neighbourhood of x, then $\mathbf{F}_x \xrightarrow{h_x} \mathbf{G}_x \xrightarrow{k_x} \mathbf{H}_x$ is exact in $\mathbf{G}_x$.

Proposition 1.11. *Let* $\mathbf{F}$, $\mathbf{F}'$ *and* $\mathbf{F}''$ *be sheaves on* $\mathbf{X}$, S *be locally closed in* $\mathbf{X}$ *and* V *be an open set in* $\mathbf{X}$.

a) *If* $0 \longrightarrow \mathbf{F}' \xrightarrow{h'} \mathbf{F} \xrightarrow{h} \mathbf{F}''$ *is an exact sequence at* $\mathbf{F}'$ *and* $\mathbf{F}$, *then the following sequences of vector spaces are exact*

(1) $0 \to \mathbf{F}'(V) \xrightarrow{h'_V} \mathbf{F}(V) \xrightarrow{h_V} \mathbf{F}''(V)$;

(2) $0 \to \Gamma_S(\mathbf{X}, \mathbf{F}') \to \Gamma_S(\mathbf{X}, \mathbf{F}) \to \Gamma_S(\mathbf{X}, \mathbf{F}'')$.

b) *If* $0 \to \mathbf{F}' \xrightarrow{h'} \mathbf{F} \xrightarrow{h} \mathbf{F}'' \to 0$ *is an exact sequence and if further* $\mathbf{F}'$ *is flabby, then the following sequences of vector spaces are exact*

(3) $0 \to \mathbf{F}'(V) \xrightarrow{h'_V} \mathbf{F}(V) \xrightarrow{h_V} \mathbf{F}''(V) \to 0$;

(4) $0 \to \Gamma_S(\mathbf{X}, \mathbf{F}') \to \Gamma_S(\mathbf{X}, \mathbf{F}) \to \Gamma_S(\mathbf{X}, \mathbf{F}'') \to 0$.

Proof. a) (1) First we shall show that h'_V is injective. Suppose that $s' \in \mathbf{F}'(V)$ and $h'_V(s') = 0$. The injectivity of h' implies that $h'_x(s'_x) = 0$ (cf. Proposition 1.10) for every $x \in V$. Thus there exists a neighbourhood $W_x \subset V$ of x such that $s'|_{W_x} = 0$. In such a way we constructed an open covering $\{W_x; x \in V\}$ of V. By property iii), $s' = 0$. Therefore h'_V is injective.

Next we will prove that $\operatorname{im} h'_V \subset \ker h_V$. Since by Proposition 1.10, $(h_x \circ h'_x)(s'_x) = 0$ for $s' \in \mathbf{F}(V)$ and for each $x \in V$, one can find a neighbourhood $W(x)$ of x such that $(h_V \circ h'_V)(s')|_{W(x)} = 0$. Since $\mathbf{F}''$ is a sheaf, by property iii) it follows that $(h_V \circ h'_V)(s') = 0$. Consequently, $\operatorname{im} h'_V \subset \ker h_V$.

It remains to prove that $\operatorname{im} h'_V \supset \ker h_V$. Let $s \in \mathbf{F}(V)$ such that $h_V(s) = 0$. Then for each $x \in V$, $h_x(s_x) = 0$ holds. By the exactness of the sequence in $\mathbf{F}_x$ there exists $s'_x \in \mathbf{F}'_x$ such that $h'_x(s'_x) = s_x$. This implies that $h'_{W_x}(s^x)|_{W_x} = s|_{W_x}$ for an open set $W_x \ni x$, $W_x \subset V$, and $s^x \in \mathbf{F}'(W_x)$, $s^x \in s'_x$. Since h'_{W_x} is injective, s^x is unique. Therefore we have $s^x|_{W_x \cap W_y} = s^y|_{W_x \cap W_y}$. By property iv), there exists $s'' \in \mathbf{F}'(V)$ such that $s''_{W_x} = s^x|_{W_x}$ for every $x \in V$. Thus $h'_V s'' = s$ and $\ker h_V \subset \operatorname{im} h'_V$.

a)(2) Let S be relatively closed in the open set U. It is only to be shown that $\operatorname{supp} s'' \subset S$ provided that $\operatorname{supp} s \subset S$, where s and s'' are as in the above.

Note that $h'_{V\setminus S}(s''|_{V\setminus S}) = 0$ and that $h'_{V\setminus S}$ is injective. Therefore $s''_{V\setminus S} = 0$ and $\operatorname{supp} s'' \subset S$.

In b) it suffices to show that h_V is surjective. We omit this proof. One can find it in [10, Proposition 1.1.2]. $\square$

Corollary 1.1. *Let $0 \to \mathbf{F}' \xrightarrow{h'} \mathbf{F} \xrightarrow{h} \mathbf{F}'' \to 0$ be an exact sequence of sheaves on a topological space $\mathbf{X}$. If $\mathbf{F}'$, $\mathbf{F}$ are flabby, then $\mathbf{F}''$ is also flabby.*

Proof. Since $\mathbf{F}'$ is flabby, by Proposition 1.11b) each row of the following commutative diagram is exact

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{F}'(\mathbf{X}) \arrow[r, "h'_{\mathbf{X}}"] \arrow[d, "\rho'_{V\mathbf{X}}"'] & \mathbf{F}(\mathbf{X}) \arrow[r, "h_{\mathbf{X}}"] \arrow[d, "\rho_{V\mathbf{X}}"'] & \mathbf{F}''(\mathbf{X}) \arrow[r] \arrow[d, "\rho''_{V\mathbf{X}}"'] & 0 \\
0 \arrow[r] & \mathbf{F}'(V) \arrow[r, "h'_V"] & \mathbf{F}(V) \arrow[r, "h_V"] & \mathbf{F}''(V) \arrow[r] & 0
\end{tikzcd}$$

Thus h_V is surjective. Because of the flabbiness of $\mathbf{F}$, $\rho_{V\mathbf{X}}$ is also surjective. Let $s'' \in \mathbf{F}''(V)$, then there exists an element $s \in \mathbf{F}(\mathbf{X})$ such that $h_V \circ \rho_{V\mathbf{X}}(s) = s''$. By the commutativity of the above diagram, $s'' = \rho''_{V\mathbf{X}} \circ h_{\mathbf{X}}(s)$; $h_{\mathbf{X}}(s) \in \mathbf{F}''(\mathbf{X})$ is the desired extension of s''. Hence $\mathbf{F}''$ is flabby. $\square$

Corollary 1.2. *Let $0 \to \mathbf{F}^0 \xrightarrow{h^0} \mathbf{F}^1 \xrightarrow{h^1} \cdots \to \mathbf{F}^r \xrightarrow{h^r} \mathbf{G} \to 0$ be an exact sequence of sheaves on $\mathbf{X}$. If $\mathbf{F}^j$, $0 \le j \le r$ are all flabby, then $\mathbf{G}$ is also flabby. Furthermore, the following sequences are exact*

$$0 \to \mathbf{F}^0(V) \xrightarrow{h^0_V} \cdots \to \mathbf{F}^r(V) \xrightarrow{h^r_V} \mathbf{G}(V) \to 0,$$
$$0 \to \Gamma_S(\mathbf{X}, \mathbf{F}^0) \to \cdots \to \Gamma_S(\mathbf{X}, \mathbf{F}^r) \to \Gamma_S(\mathbf{X}, \mathbf{G}) \to 0.$$

Proof. The given long exact sequence can be decomposed into slanted short exact sequences as follows:

$$\begin{tikzcd}[column sep=small]
 & & 0 \arrow[dr] & & 0 & & & & 0 & & & 0 \\
 & & & \mathbf{G}^1 \arrow[ur] \arrow[dr, "i"] & & & & \mathbf{G}^3 \arrow[ur] & & & \mathbf{G} \arrow[ur] & \\
0 \arrow[r] & \mathbf{F}^0 \arrow[r, "h^0"] & \mathbf{F}^1 \arrow[rr, "h^1"] \arrow[ur, "h^1"] & & \mathbf{F}^2 \arrow[rr, "h^2"] \arrow[dr, "h^2"'] & & \mathbf{F}^3 \arrow[r] \arrow[ur, "h^3"] & \cdots \arrow[rr] & & \mathbf{F}^r \arrow[r] \arrow[ur, "h^r"] & {} & \\
 & \mathbf{G}^0 \arrow[ur, "h^0"] & & & & \mathbf{G}^2 \arrow[ur, "i"] \arrow[dr] & & & \mathbf{G}^{r-1} \arrow[ur] & & & \\
0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 & 0 \arrow[ur] & & & &
\end{tikzcd}$$

Corollary 1.1 to the slanted exact sequences successively from the left, we can see that every $\mathbf{G}^j$, $j = 0, 1, \ldots, r-1$, and $\mathbf{G}$ are flabby. Applying Proposition 1.11b) the corresponding short sequences of vector spaces

$$0 \to \mathbf{G}^{j-1}(V) \to \mathbf{F}^j(V) \to \mathbf{G}^j(V) \to 0, \;\; j = 1, \ldots, r,$$

are all exact. Combining these short sequences into one in the reversed procedure of that applied above, we obtain the first long exact sequence of vector spaces.

For the second long sequence of vector spaces we have only to take care of the support of sections. $\square$

Let $\mathbf{F}$ be a sheaf on $\mathbf{X}$. A **flabby resolution* of $\mathbf{F}$ is an exact sequence

$$0 \xrightarrow{i} \mathbf{F} \to \mathbf{L}^0 \xrightarrow{h^0} \mathbf{L}^1 \xrightarrow{h^1} \cdots$$

with flabby sheaves $\mathbf{L}^j$, $j = 0, 1, \ldots$. The smallest integer r such that $\mathbf{L}^j = 0$, $j > r$ (if it exists) is called **the length of this resolution.* The minimum of the lengths of all flabby resolutions of $\mathbf{F}$ is called **the flabby dimension of* $\mathbf{F}$, denoted by $\operatorname{fl\,dim} \mathbf{F}$. Flabby dimension measures, roughly speaking, how far the sheaf $\mathbf{F}$ is distant from flabbiness.

If $\mathbf{F}$ is flabby, then $r = 0$ since $0 \to \mathbf{F} \xrightarrow{i} \mathbf{F} \to 0$ is an exact sequence ($\mathbf{L}^0 = \mathbf{F}$).

Proposition 1.12. *Every sheaf possesses a flabby resolution.*

Proof. For a sheaf $\mathbf{F}$ on $\mathbf{X}$, we first construct a flabby sheaf $\mathbf{C}^0(\mathbf{F})$ such that $0 \to \mathbf{F} \xrightarrow{i} \mathbf{C}^0(\mathbf{F})$ is exact. Let $\mathbf{C}^0(\mathbf{F})$ be the sheaf constructed in the following way. Let V be an open set in $\mathbf{X}$. To V it corresponds the vector space

$$\mathbf{C}^0(\mathbf{F})(V) = \left\{ s^0 : V \to \bigsqcup_{x \in V} \mathbf{F}_x \text{ such that } s^0(x) \in \mathbf{F}_x \right\}.$$

If $s \in \mathbf{F}(V)$, then s defines an element $s^0 \in \mathbf{C}^0(\mathbf{F})$ where $s^0(x) = s_x \in \mathbf{F}_x$. Thus, inclusion $i : \mathbf{F} \to \mathbf{C}^0(\mathbf{F})$ is a sheaf homomorphism.

The property that $\mathbf{C}^0(\mathbf{F})$ is flabby is obvious. In this way we constructed $0 \to \mathbf{F} \xrightarrow{i} \mathbf{C}^0(\mathbf{F})$.

Next, for the quotient sheaf $\mathbf{C}^0(\mathbf{F})/\mathbf{F}$ we construct $\mathbf{C}^0(\mathbf{C}^0(\mathbf{F})/\mathbf{F})$ in the same way as above and denote it by $\mathbf{C}^1(\mathbf{F})$. Now we construct the following commutative diagram

$$
\begin{tikzcd}
 & & & 0 \arrow[dr] & & 0 \\
 & & & & \mathbf{C}^0(\mathbf{F})/\mathbf{F} \arrow[ur] \arrow[dr, "i_1"] & \\
 & 0 \arrow[r] & \mathbf{F} \arrow[r, "i"] & \mathbf{C}^0(\mathbf{F}) \arrow[ur, "k"] \arrow[rr, "p"] & & \mathbf{C}^1(\mathbf{F}) \\
 & & \mathbf{F} \arrow[ur, "i"] & & & \\
 & 0 \arrow[ur] & & & &
\end{tikzcd}
$$

The sequence $0 \to \mathbf{F} \xrightarrow{i} \mathbf{C}^0(\mathbf{F})$ is exact because $\mathbf{Ker}\, i = \{0\}$. The two slanted sequences: $0 \to \mathbf{C}^0(\mathbf{F})/\mathbf{F} \to \mathbf{C}^1(\mathbf{F})$ and $0 \to \mathbf{F} \xrightarrow{i} \mathbf{C}^0(\mathbf{F}) \xrightarrow{k} \mathbf{C}^0(\mathbf{F})/\mathbf{F}$ are also exact. Hence, $\mathbf{Ker}\, p = \mathbf{Ker}\,(i_1 \circ k) = \mathbf{Im}\, i$ and $0 \to \mathbf{F} \xrightarrow{i} \mathbf{C}^0(\mathbf{F}) \xrightarrow{p} \mathbf{C}^1(\mathbf{F})$ is exact.

If we continue the same procedure, then we obtain a flabby resolution of $\mathbf{F}$ given by the flabby sheaves $\mathbf{C}^j(\mathbf{F})$, $j = 0, 1, \ldots$ The constructed resolution is called the **canonical flabby resolution.* □

Let $\{K^n\}$ be a sequence of $\mathbf{C}$-vector spaces and $\{d_n\}$ be a sequence of $\mathbf{C}$-linear mappings, $d_n : K^n \to K^{n+1}$ such that $d^n \circ d^{n-1} = 0$, $n \in \mathbf{N}$. Then the sequence of pairs $\{(K^n, d^n); n \in \mathbf{N}\}$ is called **a cochain complex of* $\mathbf{C}$*-vector spaces* and is denoted by $K^\bullet$ or $(K^\bullet, d^\bullet)$. An element of K^n is called **an n-cochain.* By definition, $\operatorname{im} d^{n-1} \subset \ker d^n$, $n \in \mathbf{N}$.

An element of $\ker d^n$ is called **an n-cocycle;* an element of $\operatorname{im} d^{n-1}$ is called **an n-coboundary.* The quotient space $\ker d^n / \operatorname{im} d^{n-1}$ is said to be **the cohomology of degree n of the complex* $(K^\bullet, d^\bullet)$ which is denoted by $H^n(K^\bullet)$. $H^n(K^\bullet)$ is a vector space, but according to the traditional terminology (which started with a sequence $\{K^n\}$ of Abelian groups it is called sometimes the n-th **cohomology group.*

If $H^n(K^\bullet) = 0$, then the sequence $\{K^n\}$ is exact at the term K^n. Hence, cohomologies provide the concept for measuring the non-exactness of a sequence of vector spaces.

Let $\mathbf{F}$ be a sheaf on $\mathbf{X}$ and $\{\mathbf{C}^j(\mathbf{F})\}$ be the sequence of flabby sheaves from the canonical flabby resolution. Denote by $\Gamma_S(\mathbf{X}, \mathbf{C}^\bullet(\mathbf{F}))$ the complex of spaces $\{\Gamma_S(\mathbf{X}, \mathbf{C}^j(\mathbf{F})), j = 0, 1, \ldots\}$.

The sequence of vector spaces

$$0 \to \Gamma_S(\mathbf{X}, \mathbf{C}^0(\mathbf{F})) \xrightarrow{d_0} \Gamma_S(\mathbf{X}, \mathbf{C}^1(\mathbf{F})) \xrightarrow{d_1} \cdots$$

is not necessarily exact. The cohomology of degree n of the complex $\Gamma_S(\mathbf{X}, \mathbf{C}^\bullet(\mathbf{F}))$ we denote by $H^n_S(\mathbf{X}, \mathbf{F}) = H^n(\Gamma_S(\mathbf{X}, \mathbf{C}^\bullet(\mathbf{F})))$ and call it **the n-th relative (local)* *cohomology of the pair* $(\mathbf{X}, \mathbf{X} \setminus S)$ *with coefficients in* $\mathbf{F}$ having support in S. If S is an open set U in $\mathbf{X}$, then we denote it by $H^n(U, \mathbf{F}) = H^n(\Gamma(U, \mathbf{C}^\bullet(\mathbf{F})))$ and call it **the n-th absolute (global) cohomology of the open set U* with coefficients in $\mathbf{F}$.

Note that $H^n_S(\mathbf{X}, \mathbf{F})$ can be defined by any flabby resolution of $\mathbf{F}$ (cf. [7, Theorem 1.1.1]).

Proposition 1.13. *For a sheaf* $\mathbf{F}$, $H^0_S(\mathbf{X}, \mathbf{F}) = \Gamma_S(\mathbf{X}, \mathbf{F})$. *If* $\mathbf{F}$ *is flabby, then* $H^n_S(\mathbf{X}, \mathbf{F}) = 0$, $n \geq 1$.

Proof. By Proposition 1.11a)(2) the sequence of vector spaces

$$0 \to \Gamma_S(\mathbf{X}, \mathbf{F}) \to \Gamma_S(\mathbf{X}, \mathbf{C}^0(\mathbf{F})) \xrightarrow{d_0} \Gamma_S(\mathbf{X}, \mathbf{C}^1(\mathbf{F}))$$

is exact. Hence, by the definition of the 0-th cohomology, $H^0_S(\mathbf{X}, \mathbf{F}) = \operatorname{Ker} d^0 = \Gamma_S(\mathbf{X}, \mathbf{F})$.

For the second part of the assertion, let us suppose that $\mathbf{F}$ is flabby. Cut the canonical flabby resolution to a bounded sequence

$$0 \to \mathbf{F} \to \mathbf{C}^0(\mathbf{F}) \xrightarrow{d_0} \mathbf{C}^1(\mathbf{F}) \to \cdots \xrightarrow{d_n} \mathbf{C}^{n+1}(\mathbf{F}) \xrightarrow{d^{n+1}} \operatorname{Im} d^{n+1} \to 0.$$

By Corollary 1.2 after Proposition 1.11 the last term $(\operatorname{Im} d^{n+1})$ is also flabby and

$$0 \to \Gamma_S(\mathbf{X}, \mathbf{F}) \to \Gamma_S(\mathbf{X}, \mathbf{C}^0(\mathbf{F})) \xrightarrow{d_0} \Gamma_S(\mathbf{X}, \mathbf{C}^1(\mathbf{F})) \xrightarrow{d_1} \cdots \xrightarrow{d_n} \Gamma_S(\mathbf{X}, \mathbf{C}^{n+1}(\mathbf{F}))$$

is exact. It follows that $H^n_S(\mathbf{X}, \mathbf{F}) = 0$, $n \geq 1$. □

The **n-th derived sheaf* $\mathbf{H}^n_S(\mathbf{F})$ of $\mathbf{F}$ is the sheaf associated with the following presheaf: $V \to H^n_{S \cap V}(\mathbf{X}, \mathbf{F})$. As we noted in Proposition 1.4 this presheaf can be regarded as the presheaf $S \cap V \to H^n_{S \cap V}(\mathbf{X}, \mathbf{F})$ and $\mathbf{H}^n_S(\mathbf{F})$ can be considered as a sheaf on S.

Since S is a locally closed set in $\mathbf{X}$, there exists an open set $U \subset \mathbf{X}$ containing S as relatively closed subset. Then $H^n_S(\mathbf{X}, \mathbf{F}) = H^n_S(U, \mathbf{F}|_U)$ and $H^n_S(\mathbf{X}, \mathbf{F}) = H^n_S(U, \mathbf{F})$ (cf. Proposition 1.2).

A closed set S in $\mathbf{X}$ is called **purely m-codimensional with respect to a sheaf* $\mathbf{F}$ if $\mathbf{H}^j_S(\mathbf{F}) = 0$ for all $j \neq m$.

Proposition 1.14. (Sato's theorem). $\mathbf{R}^n \subset \mathbf{C}^n$ *is purely n-codimensional relative to the sheaf* $\mathbf{O}$.

Sato's theorem gives a cohomological property of holomorphic functions. We omit the proof. A discussion of this theorem and its proof can be find in [7, Part II, Chapter 6, §5].

We have seen that: $V \to H^n_{S \cap V}(\mathbf{F})$ is only a presheaf. The next proposition gives a sufficient condition that such a presheaf is also a sheaf. First we shall discuss the case $n = 0$ and cite a lemma.

Since $H^0_S(\mathbf{X}, \mathbf{F}) = \Gamma_S(\mathbf{X}, \mathbf{F})$ (Proposition 1.13) and $V \to \Gamma_{S \cap V}(\mathbf{X}, \mathbf{F})$ is the sheaf $\mathbf{T}_S(\mathbf{F})$ (Proposition 1.4), $V \to H^0_{S \cap V}(\mathbf{X}, \mathbf{F})$ defines always a sheaf.

Lemma 1.1. *Let* $0 \to \mathbf{F} \to \mathbf{L}^0 \to \mathbf{L}^1 \to \cdots$ *be a flabby resolution of* $\mathbf{F}$ *and* $\mathbf{T}_S(\mathbf{L}^\bullet)$ *the correspondent sequence of sheaves* $\mathbf{T}_S(\mathbf{L}^j)$, $j = 0, 1, \ldots$:

$$\mathbf{T}_S(\mathbf{L}^\bullet) : 0 \to \mathbf{T}_S(\mathbf{L}^0) \xrightarrow{d^0} \mathbf{T}_S(\mathbf{L}^1) \xrightarrow{d^1} \cdots$$

Then $\mathbf{H}^n_S(\mathbf{F}) = \operatorname{Ker} d^n / \operatorname{Im} d^{n-1}$.

The proof is based on the inductive limit of the family of complexes and we omit it. (cf. [7, Lemma 5.2.8 and the remark after Definition 5.3.4]).

Proposition 1.15. *If* $\mathbf{H}^j_S(\mathbf{F}) = 0$ *for* $0 \leq j \leq n-1$, *then the presheaf:* $V \to H^n_{S\cap V}(V, \mathbf{F})$ *is a sheaf and hence* $\mathbf{H}^n_S(\mathbf{F})(V) = H^n_{S\cap V}(V, \mathbf{F})$.

Proof. By Lemma 1.1, the complex of sheaves $\mathbf{T}_S(\mathbf{L}^\bullet)$ given above

$$0 \to \mathbf{T}_S(\mathbf{L}^0) \xrightarrow{d^0} \mathbf{T}_S(\mathbf{L}^1) \xrightarrow{d^1} \cdots$$

is exact up to the $(n-1)$-st term. Then

$$0 \to \mathbf{T}_S(\mathbf{L}^0) \xrightarrow{d^0} \mathbf{T}_S(\mathbf{L}^1) \xrightarrow{d^1} \cdots \mathbf{T}_S(\mathbf{L}^{n-1}) \xrightarrow{d^{n-1}} \operatorname{Im} d^{n-1} \to 0$$

is an exact sequence. By Proposition 1.5 every $\mathbf{T}_S(\mathbf{L}^j)$, $j = 1, \ldots$ is flabby. By Corollary 1.2 the sheaf $\operatorname{Im} d^{n-1}$ is also flabby and for any open set $V \subset \mathbf{X}$

$$0 \to \Gamma_{S\cap V}(V, \mathbf{L}^0) \xrightarrow{d^0_V} \Gamma_{S\cap V}(V, \mathbf{L}^1) \xrightarrow{d^1_V} \cdots \to \Gamma_{S\cap V}(V, \mathbf{L}^{n-1}) \xrightarrow{d^{n-1}_V} (\operatorname{Im} d^{n-1})(V) \to 0$$

is exact. Now, we can construct the commutative diagram:

$$\begin{array}{ccc} \Gamma_{S\cap V}(V, \mathbf{L}^{n-1}) & \xrightarrow{d^{n-1}_V} & \Gamma_{S\cap V}(V, \mathbf{L}^n) \\ & \searrow^{d'^{n-1}} \quad \nearrow_{i} & \\ & (\operatorname{Im} d^{n-1})(V) & \\ & \nearrow \quad \searrow & \\ & 0 \qquad\quad 0 & \end{array}$$

From this diagram it follows that $(\operatorname{Im} d^{n-1})(V) = \operatorname{im} d^{n-1}_V$. The sequence

$$0 \to \operatorname{Im} d^{n-1} \to \operatorname{Ker} d^n \to \mathbf{H}^n_S(\mathbf{F}) \to 0$$

is exact. Since $\operatorname{Im} d^{n-1}$ is a flabby sheaf, by Proposition 1.11 b) (1),

$$0 \to (\operatorname{Im} d^{n-1})(V) \to (\operatorname{Ker} d^n)(V) \to \mathbf{H}^n_S(\mathbf{F})(V) \to 0$$

is exact. Consequently

$$\mathbf{H}^n_S(\mathbf{F})(V) = (\operatorname{Ker} d^n)/(\operatorname{Im} d^{n-1})(V) = \ker d^n_V / \operatorname{im} d^{n-1}_V = H^n_{S\cap V}(V, \mathbf{F}). \qquad \square$$

1.4. Čech cohomology

Let $\mathbf{F}$ be a sheaf on a topological space $\mathbf{X}$ and $\mathbf{U} = \{U_\lambda;\ \lambda \in \Lambda\}$ be an open covering of $\mathbf{X}$. Denote by $\sigma = (\sigma(0), \dots, \sigma(n))$ a permutation of the set $\{0, 1, \dots, n\}$. Denote by $\operatorname{sgn} b_{\lambda_0 \dots \lambda_n}$ the equivalence class related to the intersection $U_{\lambda_0} \cap \dots \cap U_{\lambda_n}$ as follows: Classify all the symbols $b_{\lambda_0 \dots \lambda_n}$ into two sets by the relation: $\operatorname{sgn} b_{\lambda_0 \dots \lambda_n} = \operatorname{sgn} \sigma \ \operatorname{sgn} b_{\lambda_{\sigma(0)} \dots \lambda_{\sigma(n)}}$. In particular, if in $(\lambda_0, \dots, \lambda_n)$ two elements are equal, then the expression $\operatorname{sgn} b_{\lambda_0 \dots \lambda_n} \doteq 0$.

Consider the set of formal expressions

$$\sum_{(\lambda_0, \dots, \lambda_n) \in \Lambda^{n+1}} \operatorname{sgn} b_{\lambda_0 \dots \lambda_n} \varphi_{\lambda_0 \dots \lambda_n}, \quad \varphi_{\lambda_0 \dots \lambda_n} \in \mathbf{F}(U_{\lambda_0} \cap \dots \cap U_{\lambda_n}).$$

for a fixed $n \in \mathbf{N}_0$ and with the above convention on $\operatorname{sgn} b_{\lambda_0 \dots \lambda_n}$. This set constitutes a $\mathbf{C}$-vector space with the $\mathbf{C}$-linear operations and it is denoted by $C^n(\mathbf{U}, \mathbf{F})$.

We also define a subspace of $C^n(\mathbf{U}, \mathbf{F})$. Let S be a closed set in $\mathbf{X}$ and $\mathbf{U}' = \{U_\lambda;\ \lambda \in \Lambda'\}$, $\Lambda' \subset \Lambda$, be an open covering of $\mathbf{X} \setminus S$. Then by definition

$$C^n(\mathbf{U} \bmod \mathbf{U}', \mathbf{F}) = \\ = \Big\{ \sum_{(\lambda_0, \dots, \lambda_n) \in \Lambda^{n+1}} \operatorname{sgn} b_{\lambda_0 \dots \lambda_n} \varphi_{\lambda_0 \dots \lambda_n} \in C^n(\mathbf{U}, \mathbf{F}); \varphi_{\lambda_0 \dots \lambda_n} = 0 \\ \text{if } (\lambda_0, \dots, \lambda_n) \in (\Lambda')^{n+1} \Big\}.$$

Furthermore, let $\{\delta^n\}$ be a sequence of $\mathbf{C}$-linear mappings which map $C^n(\mathbf{U}, \mathbf{F}) \to C^{n+1}(\mathbf{U}, \mathbf{F})$ as follows:

$$\delta^n \Big(\sum_{(\lambda_0, \dots, \lambda_n) \in \Lambda^{n+1}} \operatorname{sgn} b_{\lambda_0 \dots \lambda_n} \varphi_{\lambda_0 \dots \lambda_n} \Big) \\ = \sum_{(\lambda_0, \dots, \lambda_n, \lambda_{n+1}) \in \Lambda^{n+2}} \operatorname{sgn} b_{\lambda_0 \dots \lambda_n \lambda_{n+1}} \varphi_{\lambda_0 \dots \lambda_n}|_{U_{\lambda_{n+1}}},$$

where

$$\varphi_{\lambda_0 \dots \lambda_n} \in \mathbf{F}(U_{\lambda_0} \cap \dots \cap U_{\lambda_n}) \quad \text{and}$$
$$\varphi_{\lambda_0 \dots \lambda_n}|_{U_{\lambda_{n+1}}} \equiv \varphi_{\lambda_0 \dots \lambda_n}|_{U_{\lambda_0} \cap \dots \cap U_{\lambda_n} \cap U_{\lambda_{n+1}}} \in \mathbf{F}(U_{\lambda_0} \cap \dots \cap U_{\lambda_n} \cap U_{\lambda_{n+1}}).$$

We shall prove that $\delta^{n+1} \circ \delta^n = 0$, $n = 0, 1, \dots$

$$\delta^{n+1} \circ \delta^n \Big(\sum_{(\lambda_0, \dots, \lambda_n) \in \Lambda^{n+1}} \operatorname{sgn} b_{\lambda_0 \dots \lambda_n} \varphi_{\lambda_0 \dots \lambda_n} \Big) = \\ = \sum_{(\lambda_0, \dots, \lambda_n, \lambda_{n+1}, \lambda_{n+2}) \in \Lambda^{n+3}} \operatorname{sgn} b_{\lambda_0 \dots \lambda_n \lambda_{n+1} \lambda_{n+2}} \varphi_{\lambda_0 \dots \lambda_n}|_{U_{\lambda_{n+1}} \cap U_{\lambda_{n+2}}}.$$

Because of $\varphi_{\lambda_0 \dots \lambda_n}|_{U_{\lambda_{n+1}} \cap U_{\lambda_{n+2}}} = \varphi_{\lambda_0 \dots \lambda_n}|_{U_{\lambda_{n+2}} \cap U_{\lambda_{n+1}}}$ and $\operatorname{sgn} b_{\lambda_0 \dots \lambda_{n+1} \lambda_{n+2}} = -\operatorname{sgn} b_{\lambda_0 \dots \lambda_{n+2} \lambda_{n+1}}$ the correspondent terms cancel each other in pairs. Consequently, $\delta^{n+1} \circ \delta^n = 0$, $n = 0, 1, \dots$

It is clear that δ^n maps $C^n(\mathbf{U} \bmod \mathbf{U}', \mathbf{F})$ into $C^{n+1}(\mathbf{U} \bmod \mathbf{U}', \mathbf{F})$. In such a way we have two cochain complexes of C-vector spaces, $C^\bullet(\mathbf{U}, \mathbf{F}) = \{C^n(\mathbf{U}, \mathbf{F}), \delta^n\}$ and $C^\bullet(\mathbf{U} \bmod \mathbf{U}', \mathbf{F}) = \{C^n(\mathbf{U} \bmod \mathbf{U}', \mathbf{F}), \delta^n\}$. Let us denote by $H^n(\mathbf{U}, \mathbf{F}) = H^n(C^\bullet(\mathbf{U}, \mathbf{F}))$ and by $H^n(\mathbf{U} \bmod \mathbf{U}', \mathbf{F}) = H^n(C^\bullet(\mathbf{U} \bmod \mathbf{U}', \mathbf{F}))$ and call them the n-th **(absolute) cohomology group of the covering* $\mathbf{U}$ *with coefficients in* $\mathbf{F}$ and the n-th **relative cohomology group of the relative covering* $\mathbf{U} \bmod \mathbf{U}'$ *with coefficients in* $\mathbf{F}$, respectively.

We shall cite two theorems without proofs.

Proposition 1.16. (Leray's theorem). *Let* $\mathbf{X}$ *be a topological space and* $F \subset \mathbf{X}$ *be a closed set. Let* $\mathbf{V} = \{V_\lambda,\ \lambda \in \Lambda\}$ *be a covering of* $\mathbf{X}$ *and suppose that its part* $\mathbf{V}' = \{V_\lambda;\ \lambda \in \Lambda'\}$, $\Lambda' \subset \Lambda$, *is a covering of* $\mathbf{X} \setminus F$. *Then, for a sheaf* $\mathbf{F}$ *on* $\mathbf{X}$, *there exist canonical mappings as follows:*

$$\tilde{c}^n_{\mathbf{V}} : H^n(\mathbf{V} \bmod \mathbf{V}', \mathbf{F}) \to H^n_F(\mathbf{X}, \mathbf{F}).$$

In addition, if $H^n(V_{\lambda_0} \cap ... \cap V_{\lambda_k}, \mathbf{F}) = 0$, $n \geq 1$, *holds for any family of indices, then the above mappings are isomorphisms. (The covering* $\{V_\lambda;\ \lambda \in \Lambda\}$ *satisfying this condition is called the Leray covering for the sheaf* $\mathbf{F}$*).*

For the proof see for example [7, p. 268].

Before we cite the next theorem we shall recall some notions of complex analysis of several variables.

A domain $U \subset \mathbf{C}^n$ (an open and connected set in $\mathbf{C}^n$) is said to be **a domain of holomorphy* if for every boundary point $z \in \partial U$ there exists a function $f \in \mathbf{O}(U)$ such that it cannot be analytically continued to any neighbourhood of z. An open set $V \subset \mathbf{C}^n$ is called a **Stein open set* if each connected component of it is a domain of holomorphy. The intersection of Stein open sets is also a Stein open set.

Proposition 1.17. (Oka–Cartan–Serre theorem). *Let* $V \subset \mathbf{C}^n$ *be a Stein open set. Then* $H^n(V, \mathbf{O}) = 0$, $n \geq 1$.

For the proof see for example [7, pp. 307–308].

2. HYPERFUNCTIONS OF SEVERAL VARIABLES

First we give a cohomological definition of the sheaf $\mathbf{B}$ of hyperfunctions following Sato's approach [28]. Secondly we pass to the "intuitive" definition and elaborate it following Kaneko's ideas and results [7].

2.1. Cohomological definitions of hyperfunctions

Definition 2.1. (Sato). $\mathbf{B} = \mathbf{H}^n_{\mathbf{R}^n}(\mathbf{O})$ (regarded as a sheaf on $\mathbf{R}^n$).

Proposition 2.1. *Let* Ω *be an open set in* $\mathbf{R}^n$ *and let* U *be an open set in* $\mathbf{C}^n$ *such that* $\Omega = \mathbf{R}^n \cap U$ *and that* Ω *is relatively closed in* U. *Then* $\mathbf{B}(\Omega) = H^n_\Omega(U, \mathbf{O}) = H^n_\Omega(\mathbf{C}^n, \mathbf{O})$.

Proof. By Proposition 1.2, $H^n_\Omega(U,\mathbf{O}) = H^n_\Omega(\mathbf{C}^n,\mathbf{O})$. Now by definition of $\mathbf{H}^n_{\mathbf{R}^n}$ and by propositions 1.14 and 1.15

$$\mathbf{B}(\Omega) = \mathbf{H}^n_{\mathbf{R}^n}(\mathbf{O})(\Omega) = \mathbf{H}^n_{\mathbf{R}^n}(\mathbf{O})(U) = H^n_{\mathbf{R}^n\cap U}(U,\mathbf{O}) = H^n_\Omega(U,\mathbf{O}). \qquad \square$$

We can relate $\mathbf{B}(\Omega)$ with the n-th relative cohomology group.

Let Ω be an open set in $\mathbf{R}^n$. By Grauert's theorem (cf. [7, p. 311]) there exists a Stein open set $U \subset \mathbf{C}^n$ such that $\Omega = \mathbf{R}^n \cap U$ and that Ω is relatively closed in U. Denote by:

$$\begin{gathered} U_j = U \cap \{z = (z_1,\dots,z_n) \in \mathbf{C}^n;\ Imz_j \neq 0\},\ j = 1,\dots,n; \\ \mathbf{U} = \{U, U_1,\dots,U_n\};\quad \mathbf{U}' = \{U_1,\dots,U_n\}; \\ U\#\Omega = U_1 \cap \dots \cap U_n = \{z \in U; Imz_j \neq 0,\ j = 1,\dots,n\}; \\ U\#_j\Omega = U_1 \cap \dots \cap U_{j-1} \cap U_{j+1} \cap \dots \cap U_n = \\ = \{z \in U;\ \operatorname{Im} z_k \neq 0,\ k = 1,\dots,j-1,\ j+1,\dots,n\}. \end{gathered} \tag{2.1}$$

Proposition 2.2. *Let Ω be an open set in $\mathbf{R}^n$ and let U be a Stein open set in $\mathbf{C}^n$ such that $\Omega = \mathbf{R}^n \cap U$ and that Ω is relatively closed in U. Then* $\mathbf{B}(\Omega) \cong H^n(\mathbf{U} \bmod \mathbf{U}',\mathbf{O})(\mathbf{B}(\Omega)$ *is isomorphic to* $H^n(\mathbf{U} \bmod \mathbf{U}',\mathbf{O}))$, *where the families of covering* $\mathbf{U}$ *and* $\mathbf{U}'$ *are as above.*

Proof. Let U be taken as a topological space and Ω as the closed subset of U, then $\mathbf{U}$ is a covering of U and $\mathbf{U}'$ a covering of $U \setminus \Omega$. If U is a Stein open set, then $U_j = U \cap \{z \in \mathbf{C}^n; \operatorname{Im} z_j \neq 0\}$, $j = 1,\dots,n$, is also a Stein open set because $\{z \in \mathbf{C}^n;\ \operatorname{Im} z_j \neq 0\}$ is Stein. Also $U_{k_1} \cap \dots \cap U_{k_i}$ for any set of indices which belong to $\{1,\dots,n\}$ is Stein. By Proposition 1.17, $H^n(U_{k_1} \cap \dots \cap U_{k_i},\mathbf{O}) = 0$, $n \geq 1$ for any set of indices which belong to $\{1,\dots,n\}$. By Proposition 1.16 and Proposition 2.1

$$H^n(\mathbf{U} \bmod \mathbf{U}',\mathbf{O}) \cong H^n_\Omega(U,\mathbf{O}) = \mathbf{B}(\Omega). \tag{2.2}$$

Corollary 2.1. *Let $\Omega, U, \mathbf{U}$ and $\mathbf{U}'$ be as in Proposition 2.2; then*

$$\mathbf{B}(\Omega) \cong \mathbf{O}(U\#\Omega)\Big/\sum_{j=1}^n \mathbf{O}(U\#j\Omega). \tag{2.3}$$

Proof. By Proposition 2.2, $\mathbf{B}(\Omega) = H^n(\mathbf{U} \bmod \mathbf{U}',\mathbf{O})$. We have to construct $H^n(\mathbf{U} \bmod \mathbf{U}',\mathbf{O})$ when $U, \mathbf{U}$ and $\mathbf{U}'$ are given as in Proposition 2.2.

A relative n-cochain is only of the form $\operatorname{sgn} b_{0\dots n}\varphi_{0\dots n}$, $\varphi_{0\dots n} \in \mathbf{O}(U_0 \cap \dots \cap U_n) = \mathbf{O}(U_1 \cap \dots \cap U_n)$, where $U_0 \equiv U$. This n-cochain is in the same time the n-cocycle.

A relative $(n-1)$-cochain has the form

$$\sum_{j=1}^n \operatorname{sgn} b_{0\dots j-1\, j+1\dots n}\varphi_{0\dots j-1\, j+1\dots n},\ \varphi_{0\dots j-1 j+1\dots n} \in \mathbf{O}(U\#_j\Omega).$$

Its boundary is

$$\sum_{j=1}^{n}(-1)^j \operatorname{sgn} b_{0\ldots n}\varphi_{0\ldots j-1j+1\ldots n}.$$

Consequently (2.3) is true.

Corollary 2.2. *In one-dimensional case* (2.3) *has the form*

$$\mathbf{B}(\Omega) \cong \mathbf{O}(U \setminus \Omega)/\mathbf{O}(U). \tag{2.4}$$

Proof. In this case $\Omega = \mathbf{R} \cap U$ and $\mathbf{U}'$ consists of only one element $U_1 = \{z \in U, \operatorname{Im} z \neq 0\}$. Then $U\#\Omega = U_1 = U \setminus \Omega$ and $U\#_1\Omega = U$. With this notation (2.3) gives (2.4). □

Consequently, in one-dimensional case, $\mathbf{B}(\Omega)$ is given by the quotient space $\mathbf{O}(U \setminus \Omega)/\mathbf{O}(U)$. Every equivalence class $[F]$, where $F \in \mathbf{O}(U \setminus \Omega)$, is considered to be a hyperfunction f on $\Omega \subset \mathbf{R}$; the function F is called a **defining function of* f.

In many-dimensional case we have the same situation. Every equivalence class $[F]$ where $F \in \mathbf{O}(U\#\Omega)$ is considered to be a hyperfunction $f \in \mathbf{B}(\Omega)$, where Ω is an open set belonging to $\mathbf{R}^n$. F is called the **defining function* of f and we write $f = [F]$.

Proposition 2.3. *The sheaf* $\mathbf{B}$ *is flabby.*

For the proof see [7, pp. 350–351].

$f \in \mathbf{B}(\Omega)$ is said to be 0 on an open set $\Omega' \subset \Omega$ if $f|_{\Omega'} = 0$. **The support* of $f \in \mathbf{B}(\Omega)$ (for short $\operatorname{supp} f$) is the complement in Ω of the largest open subset of Ω on which f equals zero.

Between different operations on hyperfunctions we define some of them. Denote by Ω an open set in $\mathbf{R}^n$.

Let $f = [F]$ and $g = [G]$ be elements of $\mathbf{B}(\Omega)$ and λ, η be two complex numbers. Then $\lambda f + \eta g = [\lambda F + \eta G] \in \mathbf{B}(\Omega)$; thus $\mathbf{B}(\Omega)$ has a $\mathbf{C}$-vector space structure.

For a real analytic function $\varphi \in \mathbf{A}(\Omega)$ there exists an open set $U \subset \mathbf{C}^n$ such that $\Omega \subset U$ and $\varphi \in \mathbf{O}(U)$. Therefore we can define the multiplication by $\varphi \in \mathbf{A}(\Omega) : \varphi f = [\varphi F]$, where $f = [F] \in \mathbf{B}(\Omega)$.

Every $f = [F] \in \mathbf{B}(\Omega)$ has all derivatives. If we adopt the abbreviation: $D_x^\alpha = D_1^{\alpha}\ldots D_n^{\alpha_n}$, $D_j = \partial/\partial x_j$, $j = 1, \ldots, n$, then $D_x^\alpha f = [D_z^\alpha F]$. Moreover, the linear partial differential operator with real analytic coefficients $P(x, D) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$ acts as a sheaf homomorphism on the sheaf $\mathbf{B}$, ($|\alpha| = \alpha_1 + \cdots + \alpha_n$).

The sheaf $\mathbf{A}$ of real analytic functions: $\Omega \to \mathbf{A}(\Omega)$ is a subsheaf of $\mathbf{B}$. To define this natural mapping $\mathbf{A} \xrightarrow{i} \mathbf{B}$, let us start with an element $\varphi \in \mathbf{A}(\Omega)$ and let U be an open set in $\mathbf{C}^n$ such that φ is holomorphic on U. Introduce the function ϕ such that

$$\phi(z) = \varphi(z), \;\; z \in (\Omega + i\Gamma_\sigma); \;\; \phi(z) = 0, \;\; z \in (U\#\Omega) \setminus (\Omega + i\Gamma_\sigma)$$

where Γ_σ is any orthant in $\mathbf{R}^n$. Then the looked-for mapping i is: $\varphi \to [\phi]$. The defined mapping i does not depend on the chosen Γ_σ.

*The *singular support* of $f \in \mathbf{B}(\Omega)$ (for short sing supp f) is the complement in Ω of the largest open set $\Omega' \subset \Omega$ such that $f|_{\Omega'}$ is real analytic.

The next proposition shows an important property of the sheaf $\mathbf{B}$ and also that many properties of this sheaf can be obtained from properties of the holomorphic functions.

Proposition 2.4. *Let Ω be an open set in $\mathbf{R}^n$. If $g \in \mathbf{B}(\Omega)$, then the equation $(\partial/\partial x_1)f(x) = g(x)$ admits a solution $f \in \mathbf{B}(\Omega)$ and every solution $(\partial/\partial x_1)f(x) = 0$ is a hyperfunction depending only on the variables $(x_2, \dots, x_n)$.*

Proof. Since $\mathbf{B}$ is flabby, g can be extended to an element belonging to $\mathbf{B}(\mathbf{R}^n)$. Thus we can take $\Omega = \mathbf{R}^n$ and $g \in \mathbf{B}(\mathbf{R}^n)$. Let G be a defining function of g, $G \in \mathbf{O}(\mathbf{C}^n \# \mathbf{R}^n)$. From the theory of holomorphic functions there exists a function $F \in \mathbf{O}(\mathbf{C}^n \# \mathbf{R}^n)$ such that $(\partial/\partial z_1)F(z) = G(z)$. Then the sought hyperfunction is $f = [F]$.

The second part of the proof is not so easy because the hyperfunction zero is defined by any element of the vector space $\sum_{j=1}^{n} \mathbf{O}(U \# j\Omega)$.

By the same reason as in the first part of the proof we can take $\Omega = \{x \in \mathbf{R}^n; |x_j| < q,\ j = 1, \dots, n\}$. Denote by U the convex open set in $\mathbf{C}^n$, $U = \Omega + i\mathbf{R}^n$, and by F the defining function of f which satisfies the equation $(\partial/\partial x_1)f(x) = 0$. Then F satisfies

$$(2.5) \qquad (\partial/\partial z_1)F(z) = \sum_{j=1}^{n} G_j(z)|_{U\#\Omega},\ G_j \in \mathbf{O}(U \# j\Omega),\ j = 1, \dots, n.$$

By the same property of holomorphic functions, we used in the first part of the proof, there exist $H_j \in \mathbf{O}(U \# j\Omega)$, $j = 1, \dots, n$, such that $(\partial/\partial z_1)H_j(z) = G_j(z)$, $j = 1, \dots, n$, because $U \# j\Omega$ is an open set in $\mathbf{C}^n$ consisting of convex components. Consequently (2.5) has now the form

$$\frac{\partial}{\partial z_1}(F(z) - \sum_{j=1}^{n} H_j(z)|_{U\#\Omega}) = 0.$$

It follows that $F(z) - \sum_{j=1}^{n} H_j(z)|_{U\#\Omega} \in \mathbf{O}(U\#\Omega)$ and depends on $(z_2, \dots, z_n)$ only.

Denote by Γ_σ^n the σ-th orthant in $\mathbf{R}^n$ and by $V_\sigma = (\Omega + i\Gamma_\sigma) \cap U$, then $U\#\Omega = \bigcup_\sigma V_\sigma$. If by Ω_1 is denoted the set $\Omega_1 = \{|x_j| < q;\ j = 2, \dots, n\}$, then the function $F(z) - \sum_{j=1}^{n} H_j(z)|_{U\#\Omega}$ can be continued to $(\{|x_1| < q\} + i\mathbf{R}) \times (\Omega_1 + i\Gamma_\sigma^{n-1})$, being constant in z_1.

This shows that f is a hyperfunction which depends on $(x_2, \dots, x_n)$ only. □

A more general assertion can be proved. Let $P(D)$ be a differential operator with constant coefficients of the elliptic type. Denote by $\mathbf{A}^P = \{u \in \mathbf{A};\ P(D)u = 0\}$ then $0 \to \mathbf{A}^P \to \mathbf{B} \xrightarrow{P(D)} \mathbf{B} \to 0$ is a flabby resolution of $\mathbf{A}^P$ [30].

Definition 2.2. An infinite-order differential operator

$$J(D) = \sum_{|\alpha| \geq 0} b_\alpha D^\alpha, \quad (|\alpha| = \alpha_1 + \ldots + \alpha_n),$$

with coefficients satisfying $\lim_{|\alpha| \to \infty} \sqrt[|\alpha|]{b_\alpha \alpha!} = 0$, is called a local operator with constant coefficients.

By properties of holomorphic functions the series

$$J(D)F = \sum_{|\alpha| \geq 0} b_\alpha D^\alpha F, \quad F \in \mathbf{O}(U)$$

converges locally uniformly in U. Hence a local operator is an endomorphism of the sheaf $\mathbf{O}$ and induces also an endomorphism of the sheaf $\mathbf{B}$.

Moreover, a hyperfunction f with support only at the origin is uniquely expressible as

$$f = J(D)\delta = \sum_{|\alpha| \geq 0} b_\alpha D^\alpha \delta,$$

where $J(D)$ is an appropriate local operator (see [7, p. 156]).

2.2. Hyperfunctions defined by boundary value representation

2.2.1. Definition and main properties. In the next definition of hyperfunctions we need the notion of infinitesimal wedge.

Definition 2.3. Let Ω be an open set in $\mathbf{R}^n$ and Γ an open cone in $\mathbf{R}^n$. An open set $W \subset \mathbf{C}^n$ is called an infinitesimal wedge (for short i.w.) of type $\Omega + i\Gamma 0$ if it satisfies the following conditions:

a) $W \subset \Omega + i\Gamma$;

b) For every proper subcone $\Gamma', \Gamma' \subset\subset \Gamma$ and for every $\epsilon > 0$, there exists $\delta > 0$ such that $W \supset \Omega_\epsilon + i(\Gamma' \cap \{y; ||y|| < \delta\})$, where $\Omega_\epsilon = \{x \in \Omega; d(x, \partial\Omega) > \epsilon\}$; Ω is the edge of this i.w.

There are infinitely many infinitesimal wedges of type $\Omega + i\Gamma 0$; such an i.w. we denote by the same symbol $\Omega + i\Gamma 0$ or by $\Omega + iI$. We also express by $F \in \mathbf{O}(\Omega + i\Gamma 0)$ the fact that F is holomorphic on one of such i.w. of type $\Omega + i\Gamma 0$.

Consider $X(\Omega) = \oplus_\Gamma \mathbf{O}(\Omega + i\Gamma 0)$, where Γ ranges over all open cones V in $\mathbf{R}$. By the local Bochner theorem, if F is holomorphic on an i.w. $\Omega + iI$ of the type $\Omega + i\Gamma 0$, then it is also holomorphic on $\Omega + i\tilde{I}$, where $\tilde{I}$ is the convex hull of I. Thus we can assume, without loss of generality, that every Γ is convex.

$X(\Omega)$ is a $\mathbf{C}$-vector space with the $\mathbf{C}$-linear operation: $\lambda \bigoplus_{i=1}^n F_i + \eta \bigoplus_{j=1}^m G_j = \lambda F_1 \oplus \cdots \oplus \lambda F_n \oplus \eta G_1 \oplus \ldots \oplus \eta G_m$, where $F_i \in \mathbf{O}(\Omega + i\Gamma_i 0), i = 1, \ldots, n$, and

$G_j \in \mathbf{O}(\Omega + i\Gamma'_j 0)$, $j = 1, \dots, m$. Using the notation $+$ in place of $\oplus$, consider the C-vector space $Y(\Omega)$ generated by the elements of $X(\Omega)$ of the following form: $F'_1 + F'_2 - F'_3$, where $F'_j \in \mathbf{O}(\Omega + i\Gamma'_j 0)$, $j = 1, 2, 3$ and $\Gamma'_1 \cap \Gamma'_2 \supset \Gamma'_3$; $F'_1(z) + F'_2(z) = F'_3(z)$ holds on the common domain. In particular if $F \in \mathbf{O}(\Omega + i\Gamma 0)$ and $\Gamma' \subset \Gamma$, then the difference of F and its restriction on i.w. of type $\Omega + i\Gamma' 0$ also belong to $Y(\Omega)$.

Definition 2.4. The mapping

$$\Omega \to X(\Omega)/Y(\Omega), \tag{2.6}$$

where Ω is an open set in $\mathbf{R}^n$, defines a presheaf on $\mathbf{R}^n$; we denote it by $\tilde{\mathbf{B}}$. (If $\Omega' \subset \Omega$, then the restriction $r_{\Omega'\Omega} : \tilde{\mathbf{B}}(\Omega) \to \tilde{\mathbf{B}}(\Omega')$ is defined as usually via restriction of functions).

Denote by $F(x + i\Gamma 0)$ an element of the quotient space $X(\Omega)/Y(\Omega)$ determined by $F \in \mathbf{O}(\Omega + iI)$, where $\Omega + iI$ is an i.w. of the type $\Omega + i\Gamma 0$. Any element of $\tilde{\mathbf{B}}(\Omega)$ is represented by

$$f(x) = \sum_{j=1}^{m} F_j(x + i\Gamma_j 0) \tag{2.7}$$

where $\{F_j; j = 1, \dots, m\}$ is the set which gives the defining function of f.

To prove the next proposition we need the assertions of a lemma cited below. The proof of this lemma is easy and one can find it in [7, p. 332].

Lemma 2.1. *Suppose that the vectors $\eta^0, \eta^1, \dots, \eta^n$ belong to $\mathbf{R}^n$ and that the open half spaces determined by them: $E_{\eta^i} = \{y \in \mathbf{R}^n; (\eta^i, y) > 0\}$, $i = 0, 1, \dots, n$ satisfy*

$$E_{\eta^0} \cup E_{\eta^1} \cup \dots \cup E_{\eta^n} = \mathbf{R}^n \setminus \{0\}. \tag{2.8}$$

Then the following statements hold:

a) $E_{\eta^0} \cap E_{\eta^1} \cap \dots \cap E_{\eta^n} = \emptyset$

b) Any n vectors of $\eta^0, \eta^1, \dots, \eta^n$ are linearly independent. Hence the intersection of half spaces corresponding to them is a proper open convex cone.

c) Denote by $\Gamma_j = E_{\eta^0} \cap \dots \cap \hat{E}_{\eta^j} \cap .. \cap E_{\eta^n}$. Let $j, k \in \{0, 1, .., n\}$. Then $\Gamma_j + \Gamma_k = E_{\eta^0} \cap \dots \cap \widehat{E}_{\eta^j} \cap \dots \cap \widehat{E}_{\eta^k} \cap \dots \cap E_{\eta^n}$, where the notation $\widehat{}$ denotes suppression of the factor under it.

Proposition 2.5. *The presheaf $\tilde{\mathbf{B}}$ defined by* (2.6) *is isomorphic to the n-th derived sheaf $\mathbf{H}^n_{\mathbf{R}^n}(\mathbf{O})$ as a presheaf and hence it is actually a sheaf.*

Proof. Let $\eta^0, \eta^1, \dots, \eta^n \in \mathbf{R}^n$ be such that (2.8) holds, where $E_{\eta^i} = \{y \in \mathbf{R}^n; (\eta^i, y) > 0\}$, $i = 0, 1, \dots, n$, are the open half spaces determined by η^i. Set $U_j = (\mathbf{R}^n + iE_{\eta^j}) \cap U$, $j = 0, 1, \dots, n$, and $U_{n+1} \equiv U$. $\mathbf{U} = \{U_0, U_1, \dots, U_n, U_{n+1}\}$, $\mathbf{U}' = \{U_0, U_1, \dots, U_n\}$ give a relative Stein covering of the pair of open sets $(U, U \setminus \Omega)$, where U is a Stein open set in $\mathbf{C}^n$ such that $U \cap \mathbf{R}^n = \Omega$ and Ω is relatively closed in U. Now we can follow the idea of the proof of Corollary 2.1. Just by the same reasons as in the proof of Proposition 2.2, (2.3) holds. Thus a relative n-cochain with respect to the constructed covering is of the form

$$(2.9)\quad \sum_{j=0}^{n} \operatorname{sgn} b_{0\dots\hat{j}\dots n+1} F_j(z),\quad F_j \in \mathbf{O}(U_0 \cap \dots \cap \widehat{U}_j \cap \dots \cap U_{n+1}),\quad j = 0,1,\dots,n.$$

(The notation $\widehat{\ }$ denotes suppression of the factor under it).

By Lemma 2.1 a), $E_{\eta^0} \cap E_{\eta^1} \cap \dots \cap E_{\eta^n} = \emptyset$. It follows that there exist no relative $(n+1)$-cochains and (2.9) is necessarily a relative cocycle.

A relative $(n-1)$-cochain is of the form

$$\sum_{j<k} \operatorname{sgn} b_{0\dots\hat{j}\dots\hat{k}\dots n+1} F_{jk}(z),$$
$$F_{jk} \in \mathbf{O}(U_0 \cap \dots \cap \widehat{U}_j \cap \dots \cap \widehat{U}_k \cap \dots \cap U_{n+1}),\quad j,k = 0,\dots,n,$$

and its boundary is

$$\sum_{j=0}^{n} b_{0\dots\hat{j}\dots n+1}\Big(\sum_{k>j}(-1)^k F_{jk}(z) + \sum_{k<j}(-1)^{k+1}F_{kj}(z)\Big).$$

Denote by $\Gamma_j = E_{\eta^0}\cap\dots\cap\widehat{E}_{\eta^j}\cap\dots\cap E_{\eta^n}$. By Lemma 2.1 b) and c), Γ_j is a proper cone in $\mathbf{R}^n$ and $U_0\cap\dots\cap\widehat{U}_j\cap\dots\cap U_{n+1} = (\mathbf{R}^n + i\Gamma_j)\cap U$; $U_0\cap\dots\cap\widehat{U}_j\cap\dots\cap\widehat{U}_k\cap\dots\cap U_{n+1} = (\mathbf{R}^n + i(\Gamma_j+\Gamma_k))\cap U$.

As in Proposition 2.2 and Corollary 2.1 we conclude that

$$(2.10)\qquad \mathbf{B}(\Omega) \cong \sum_{j=0}^{n} \mathbf{O}((\mathbf{R}^n + i\Gamma_j)\cap U)\Big/\sum_{j<k}\mathbf{O}((\mathbf{R}^n + i(\Gamma_j+\Gamma_k))\cap U).$$

Now we can define a C-linear mapping $\mathbf{B}(\Omega) \to \tilde{\mathbf{B}}(\Omega)$ which is consistent with restrictions so that it is a presheaf homomorphism: Suppose that the functions

$$F_j \in \mathbf{O}(U_0 \cap \dots \cap \widehat{U}_j \cap \dots \cap U_n),\quad j = 0,1,\dots,n.$$

We associate with the element $f \in \mathbf{B}(\Omega)$, given by $(F_0,\dots,F_n)$, the element

$$(2.11)\qquad \sum_{j=0}^{n} (-1)^j F_j(x + i\Gamma_j 0) \in \tilde{\mathbf{B}}(\Omega).$$

We have to construct the inverse correspondence to this one. Take an element $F(x + i\Gamma 0) \in \tilde{\mathbf{B}}(\Omega)$ given by $F \in \mathbf{O}(\Omega + i\Gamma 0)$. Determine $n+1$ vectors $\eta^0, \eta^1, \dots, \eta^n \in \mathbf{R}^n$ in such a way that $E_{\eta^1}\cap\dots\cap E_{\eta^n} \subset\subset \Gamma$ and that (2.8) holds. We also assume that the n-simplex formed by $\eta^1,\dots,\eta^n$ is compatible with the orientation of $\mathbf{R}^n$. Choose a Stein open set $U \subset \mathbf{C}^n$, $U \cap \mathbf{R}^n = \Omega$ such that Ω is relatively closed in U and that $F(z)$ is holomorphic on the i.w. $(\Omega + i(E_{\eta^1}\cap\dots\cap E_{\eta^n}))\cap U$. Now we can construct the relative covering $\mathbf{U} = \{U_0, \dots, U_{n+1}\}$, $\mathbf{U}' = \{U_0, \dots, U_n\}$ of the pair $(U, U \setminus \Omega)$, where $U_j = (\Omega + iE_{\eta^j}) \cap U$, $j = 0, 1, \dots, n$, and $U_{n+1} \equiv U$. With this relative covering the function F defines an element of $H^n(\mathbf{U} \bmod \mathbf{U}', \mathbf{O})$ and an element of $H^n_\Omega(U, \mathbf{O}) = \mathbf{B}(\Omega)$ as in the first part of the proof. Morimoto (see [7, p. 335]) proved that this element does not depend on the choice of the vectors $\eta^0, \eta^1, \dots, \eta^n$. To the obtained element, by C-linear mapping $\mathbf{B}(\Omega) \to \tilde{\mathbf{B}}(\Omega)$ defined in the second part of the proof, it corresponds $F(x + i\Gamma_0 0)$, where $\Gamma_0 = E_{\eta^1} \cap \dots \cap E_{\eta^n} \subset\subset \Gamma$. By the definition of the equivalence class in $X(\Omega)$, $F(x + i\Gamma_0 0) = F(x + i\Gamma 0)$. Consequently, the composition of homomorphisms just defined, $\tilde{\mathbf{B}}(\Omega) \to \mathbf{B}(\Omega) \to \tilde{\mathbf{B}}(\Omega)$ is the identity mapping. Analogously, it can be proved that the composition $\mathbf{B}(\Omega) \to \tilde{\mathbf{B}}(\Omega) \to \mathbf{B}(\Omega)$ is the identity mapping, as well. □

In one-dimensional case there is only two open cones with vertex at zero: $\Gamma_+ = \mathbf{R}_+$ and $\Gamma_- = \mathbf{R}_-$. If $U \subset \mathbf{C}$ is an open set such that $U \cap \mathbf{R} = \Omega$, Ω is relatively closed in U, then $U_+ = U \cap \{z \in \mathbf{C};\ \operatorname{Im} z > 0\}$ and $U_- = U \cap \{z \in \mathbf{C};\ \operatorname{Im} z < 0\}$ are infinitesimal wedges. Now (2.7) can be given as follows

$$f(x) = F_+(x + i\mathbf{R}_+0) - F_-(x + i\mathbf{R}_-0),$$

where $F_+ \in \mathbf{O}(\Omega + i\mathbf{R}_+0)$ and $F_- \in \mathbf{O}(\Omega + i\mathbf{R}_-0)$. We write for short

$$f(x) = F_+(x + i0) - F_-(x - i0). \tag{2.12}$$

(F_+, F_-) is **the pair of defining functions of* f.

Remark. After Proposition 2.5 we can identify $\mathbf{B}$ and $\tilde{\mathbf{B}}$ and we shall write only $\mathbf{B}$ for the both sheaves. The definition of hyperfunctions via $\tilde{\mathbf{B}}$ is said to be "intuitive" definition or definition by boundary value representation. The "intuitive" definition is easier to understand and to apply in solving mathematical models. But theoretically it is in some sense incomplete. First, expression (2.7) is not invariant under coordinate transformations. Secondly, it is not easy to check that a given hyperfunction is zero in a neighbourhood of a point.

The elementary operations, we gave for the elements of $\mathbf{B}(\Omega)$, can be easily transfered if these elements have the form given in (2.7). Let

$$f(x) = \sum_j F_j(x + i\Gamma_j 0) \text{ and } g(x) = \sum_k G_k(x + i\Gamma'_k 0)$$

be elements of $\mathbf{B}(\Omega)$ given in the form as in (2.7) then:

$$\lambda f(x) + \eta g(x) = \sum_j \lambda F_j(x + i\Gamma_j 0) + \sum_k \eta G_k(x + i\Gamma'_k 0), \quad \eta, \lambda, \mu \in \mathbf{C}$$

$$(\varphi f)(x) = \sum_j (\varphi F_j)(x + i\Gamma_j 0), \quad \varphi \in \mathbf{A}(\Omega)$$

$$D^\alpha_x f(x) = \sum_j (D^\alpha_z F_j)(x + i\Gamma_j 0).$$

2.2.2. Microfunctions. When we investigate solutions of mathematical models focusing our attention on points in which these solutions have their singularities, we need not the sheaf $\mathbf{B}$ but the sheaf of microfunctions $\mathcal{R}$. The theory and applications of microfunctions are significantly developed within last years (cf. [13]). For the further study of microfunctions and micro-local operators one can derive a profit from the book [10]; see also [24].

The construction of the sheaf $\mathcal{R}$ we shall give in one-dimensional case because of simplicity, but our intention is to explain the concept and the idea of microfunctions theory in many-dimensional case, too. According to this purpose we shall adapt the more general notation than are really needed in one-dimensional case.

Definition 2.5. Let $S^0 = \{\pm 1\}$ and denote a point (x, ξ) of $\mathbf{R} \times S^0$ by $(x, (\xi/i)dx\infty)$ for convenience. A hyperfunction f is said to be microanalytic at the point $(x, (1/i)dx\infty)$ if a pair of defining functions (F_+, F_-) of f can be both analytically continued to $U_+ = U \cap \{z \in \mathbf{C};\ \operatorname{Im} z > 0\}$, where U is a suitable complex neighbourhood of x. Similarly, f is said to be microanalytic at the point $(x, -(1/i)dx\infty)$ if F_+ and F_- can be both analytically continued to $U_- = U \cap \{z \in \mathbf{C};\ \operatorname{Im} z < 0\}$.

From the definition of the set of points, where f is microanalytic, it follows that this set is an open set in $\mathbf{R} \times S^0$.

Definition 2.6. The set of all points where the hyperfunction f is not microanalytic is called the singular spectrum of f (for short $\operatorname{SS} f$).

If $\pi : \mathbf{R} \times S^0 \to \mathbf{R}$ is a natural projection, then $\pi(\operatorname{SS} f) = \operatorname{sing\,supp} f$. The linear differential operator with real analytic coefficients does not enlarge the singular spectrum of a hyperfunction.

The first idea to investigate local properties of hyperfunctions required the construction of the quotient sheaf $\mathbf{B}/\mathbf{A}$. But for the singular spectrum of a hyperfunction, $\mathbf{B}/\mathbf{A}$ was still incomplete. So we have to introduce an other quotient space.

Definition 2.7. Let $h : \mathbf{X} \to \mathbf{Y}$ be a continuous mapping of a topological space $\mathbf{X}$ into a topological space $\mathbf{Y}$. Let U be an open set in $\mathbf{X}$ and V be any open set belonging to $\mathbf{Y}$ and containing $h(U)$. For a sheaf $\mathbf{G}$ on $\mathbf{Y}$, the correspondence $U \to \varinjlim_{V \supset h(U)} \mathbf{G}(V)$ is a presheaf on $\mathbf{X}$. Its associated sheaf is called the inverse sheaf of $\mathbf{G}$ by h and is denoted by $h^{-1}\mathbf{G}$.

In particular when f is an open function, if for every $y \in Y$ and open set $U \subset \mathbf{X}$, $U \cap f^{-1}(y)$ is connected, then $f^{-1}\mathbf{G}(U) = \mathbf{G}(f(U))$ holds.

Let us apply the construction of the inverse sheaf to the canonical projection $\pi : \mathbf{R} \times S^0 \to \mathbf{R}$. Let $\Omega_1 \times \{idx\infty\} \cup \Omega_2 \times \{-idx\infty\}$ be an open set in $\mathbf{R} \times S^0$ (Ω_1 and Ω_2 are open in $\mathbf{R}$). Then we have

$$\pi^{-1}\mathbf{B}(\Omega_1 \times \{idx\infty\} \sqcup \Omega_2 \times \{-idx\infty\}) = \mathbf{B}(\Omega_1) \oplus \mathbf{B}(\Omega_2)$$

Definition 2.8. We have the following two sheaves over $\mathbf{R} \times S^0$;

1. The subsheaf $\mathbf{A}^*$ of $\pi^{-1}\mathbf{B}$ defined by

$$\begin{aligned}\mathbf{A}^*(\Omega_1 \times \{idx\infty\} \sqcup \Omega_2 \times \{-idx\infty\} = \\ = \{f \in \mathbf{B}(\Omega_1);\ \mathrm{SS}\, f \cap \Omega_1 \times \{idx\infty\} = \emptyset\} \\ \oplus \{f \in \mathbf{B}(\Omega_2);\ \mathrm{SS}\, f \cap \Omega_2 \times \{-idx\infty\} = \emptyset\}.\end{aligned}$$

2. The sheaf of microfunctions: $\mathcal{R} = \pi^{-1}\mathbf{B}/\mathbf{A}^*$.

From 2 we have the exact sequence: $0 \to \mathbf{A}^* \to \pi^{-1}\mathbf{B} \to \mathcal{R} \to 0$.

The sheaf $\mathcal{R}$ has the following main properties:

Proposition 2.6. *1. $\mathcal{R}$ is a flabby sheaf.*

2. For any open set $U \subset \mathbf{R} \times S^0$, $\mathcal{R}(U) = \pi^{-1}\mathbf{B}(U)/\mathbf{A}^*(U)$, or equivalently

$$0 \to \mathbf{A}^*(U) \to \pi^{-1}\mathbf{B}(U) \to \mathcal{R}(U) \to 0$$

is an exact sequence.

3. The linear differential operator with real analytic coefficients induces a sheaf endomorphism $\mathcal{R} \to \mathcal{R}$.

For the proof see for example [7, pp. 53–55].

Let $F \in \mathrm{O}(U)$, where $U \subset \mathbf{C}$ is a domain (open and connected set) and a neighbourhood of a point a. Define D^{-1} by

$$D^{-1}F(z) = \int_a^z F(\zeta)d\zeta \tag{2.13}$$

with an appropriate path connecting a and z. Consider the infinite series of operators

$$Q(z, D_z) = \sum_{k=1}^{\infty} b_k(z) D_z^{-k}. \tag{2.14}$$

Definition 2.9. Operator (2.14) whose coefficients satisfy the following condition

1. $b_k(z)$ are holomorphic in a complex domain $U \subset \mathbf{C}$;
2. $\limsup_{k\to\infty} \sqrt[k]{\sup_{z\in K} |b_k(z)|/k!} < \infty$

holds for every compact set $K \subset U$, is called a **pseudo-differential operator* or a *micro-differential operator of order* ≤ 0.

A pseudo-differential operator of order ≤ 0 defines a sheaf endomorphism of $\mathcal{R}$ in a special way via germs (cf. [7, p. 61]).

2.3. Fourier hyperfunctions and the Fourier transform of them

2.3.1. Mainly used approaches to Fourier hyperfunctions. 1. **Sato's definition* ([27] for the proofs see also [12]). Denote by $\mathbf{D}^n$ the compactification of

$\mathbf{R}^n$, $\mathbf{D}^n = \mathbf{R}^n \sqcup S_\infty^{n-1}$, obtained by adding points at infinity in all directions. A fundamental system of neighbourhoods of a point at infinity $(a\infty)$ is

$$U_{B,r}(a\infty) = \{x \in \mathbf{R}^n; (x/\|x\|) \in B, \|x\| \geq r\} \sqcup \{x\infty; x \in B\},$$

where B is a neighbourhood of the point a in S^{n-1}. $\tilde{\mathbf{O}}$ will be the sheaf on $\mathbf{D}^n + i\mathbf{R}^n$ defined as follows: For any open set $U \subset \mathbf{D}^n + i\mathbf{R}^n$, $\tilde{\mathbf{O}}(U)$ consists of those elements of $\mathbf{O}(U \cap \mathbf{C}^n)$ which satisfy $|F(z)| \leq C_{V,\epsilon} \exp\{\epsilon|\operatorname{Re} z|)$ uniformly for any open set $V \subset \mathbf{C}^n, \bar{V} \subset U$ and for every $\epsilon > 0$, where $\bar{V}$ is the closure of V in $\mathbf{D}^n + i\mathbf{R}^n$. If $U \subset \mathbf{C}^n$, then $\tilde{\mathbf{O}}(U) = \mathbf{O}(U)$. Hence, $\tilde{\mathbf{O}}|_{\mathbf{C}^n} = \mathbf{O}$ It is proved that $\mathbf{D}^n \subset \mathbf{D}^n + i\mathbf{R}^n$ is purely n-codimensional relative to $\tilde{\mathbf{O}}$ ([25]). The n-th derived sheaf $\mathbf{H}^n_{\mathbf{D}^n}(\tilde{\mathbf{O}})$, denoted by $\mathbf{Q}$ and regarded as a sheaf on $\mathbf{D}^n$, is called the sheaf **of Fourier hyperfunctions (of slowly increasing hyperfunctions).* $\mathbf{Q}$ is flabby sheaf on $\mathbf{D}^n$. In particular $\mathbf{Q}|_{\mathbf{R}^n} = \mathbf{H}^n_{\mathbf{R}^n}(\mathbf{O}) = \mathbf{B}$. Hence the sequence

$$\mathbf{Q}(\mathbf{R}^n) \to \mathbf{B}(\mathbf{R}^n) \to 0$$

is exact.

One of the main results on the sheaf $\mathbf{Q}$ is the following proposition.

Proposition 2.7. [7] *Let $U \subset \mathbf{D}^n + i\mathbf{R}^n$ be an open set such that $U \cap \mathbf{C}^n$ is convex and $\operatorname{Im} z$ is bounded on $\partial(U \cap \mathbf{C}^n)$. Then $H^k(U, \tilde{\mathbf{O}}) = 0$ for $k \geq 1$. Hence, in particular if we choose a convex neighbourhood I of $0 \in \mathbf{R}^n$, then $U = \mathbf{D}^n + iI$, $U_j = (\mathbf{D}^n + iI) \cap \{Imz_j \neq 0\}$, $j = 1, \dots, n$, is a relative Leray covering for the pair $(\mathbf{D}^n + iI, (\mathbf{D}^n + iI) \setminus \mathbf{D}^n)$ relative to the sheaf $\tilde{\mathbf{O}}$ and the representation*

$$\mathbf{Q}(\mathbf{D}^n) = \tilde{\mathbf{O}}((\mathbf{D}^n + iI)\#\mathbf{D}^n) \Big/ \sum_{j=1}^{n} \tilde{\mathbf{O}}((\mathbf{D}^n + iI)\#j\mathbf{D}^n)$$

is valid.

This theorem gives a possibility of another approach to the Fourier hyperfunctions. Namely, the set of Fourier hyperfunctions can be defined as

$$\tilde{\mathbf{O}}((\mathbf{D}^n + iI)\#\mathbf{D}^n) \Big/ \sum_{j=1}^{n} \tilde{\mathbf{O}}((\mathbf{D}^n + iI)\#j\mathbf{D}^n).$$

2. **Zharinov's definition* [35]. Denote by $T^M = \mathbf{R}^n + iM$ and by $s_M(\xi) = \sup\{-y\xi; y \in M\}$, where $y\xi = y_1\xi_1 + \cdots + y_n\xi_n$ and $M \subset \mathbf{R}^n$. Let A and B be bounded domains in $\mathbf{R}^n$. We denote by $\Phi(A, B)$ the Banach space of holomorphic functions on T^A with the norm

$$\|\varphi\|^A_{s_B} = \sup\{\exp(s_B(\xi))|\varphi(\xi + i\eta)|;\ \xi + i\eta \in T^A\}.$$

The space $\vec{\Phi}$, defined as the inductive limit over all A and B which contain zero,

$$\vec{\Phi} = \varinjlim_{A \ni 0, B \ni 0} \Phi(A, B)$$

is a DFS space. The dual space, $\vec{\Phi}'$, is an FS space (Fréchet–Schwartz). $\vec{\Phi}'$ is isomorphic to the space of Fourier hyperfunctions. The Fourier transform of $f \in \vec{\Phi}'$ is given by $\langle \mathbf{F}f, \varphi\rangle = \langle f, \mathbf{F}\varphi\rangle$, $\varphi \in \vec{\Phi}$ and is an automorphism on $\vec{\Phi}'$.

3. *"Intuitive" definition of Fourier hyperfunctions.* The systematic exposition of this approach one can find in [7]. We shall follow it in the next part.

2.3.2. "Intuitive" definition of Fourier hyperfunctions.

Let Γ_j be an open cone in $\mathbf{R}^n$ and $\mathbf{D}^n + iI_j$ an infinitesimal wedge of type $\mathbf{D}^n + i\Gamma_j 0$. $F_j \in \tilde{\mathrm{O}}(\mathbf{D}^n + iI_j)$ means that F_j is holomorphic on $\mathbf{R}^n + iI_j$ and for every $\epsilon > 0$, $|F_j(z)| \le C_{V,\epsilon} \exp(\epsilon |\operatorname{Re} z|)$ uniformly for any open set $V \subset \mathbf{C}^n$, $\bar{V} \subset \mathbf{D}^n + iI_j$.

Consider $X = \oplus_\Gamma \tilde{\mathrm{O}}(\mathbf{D}^n + i\Gamma 0)$ where Γ ranges over all open convex cones. X is a C-vector space. We denote by Y the C-vector space generated by the elements of X of the following form: $F_1 + F_2 - F_3$, where $F_j \in \tilde{\mathrm{O}}(\mathbf{D}^n + i\Gamma_j 0)$, $j = 1,2,3$, and $\Gamma_1 \cap \Gamma_2 \supset \Gamma_3$; $F_1(z) + F_2(z) = F_3(z)$ holds on the common domain.

Denote by $\tilde{\mathbf{Q}} = X/Y$. This is a C-vector space too. By $F(x + i\Gamma 0)$ we denote the element of the quotient space $\tilde{\mathbf{Q}}$ determined by $F \in \tilde{\mathrm{O}}(\mathbf{D}^n + iI)$.

If $F_2 = 0$ and F_3 can be extended to $\mathbf{D}^n + iI_1$, then F_3 can be substituted by F_1 in $\oplus_\Gamma$.

Corollary 8.5.4 in the book of Kaneko [7] asserts that $\mathbf{Q}(\mathbf{D}^n) = \tilde{\mathbf{Q}}$. The proof is just the same as the proof for Proposition 2.5. We shall prove only that there exists a homomorphism $\mathbf{Q}(\mathbf{D}^n) \to \tilde{\mathbf{Q}}$. Notice that Proposition 2.7 asserts that

$$\mathbf{Q}(\mathbf{D}^n) = \tilde{\mathrm{O}}((\mathbf{D}^n + iI)\#\mathbf{D}^n) \Big/ \sum_{j=1}^{n} \tilde{\mathrm{O}}((\mathbf{D}^n + iI)\#j\mathbf{D}^n).$$

Then every element of $\mathbf{Q}(\mathbf{D}^n)$ is represented by $F \in \tilde{\mathrm{O}}((\mathbf{D}^n + iI)\#\mathbf{D}^n)$ and F consists of 2^n independent holomorphic functions F_σ, $F_\sigma \in \tilde{\mathrm{O}}(\mathbf{D}^n + iI_\sigma)$ where $\mathbf{D}^n + iI_\sigma$ is an infinitesimal wedge of the form $\mathbf{D}^n + i\Gamma_\sigma 0$, Γ_σ is the σ-th orthant in $\mathbf{R}^n$. To F we associate the following element of $\tilde{\mathbf{Q}}$:

$$\sum_\sigma \operatorname{sgn} \sigma F_\sigma(x + i\Gamma_\sigma 0).$$

Any element $G_j \in \tilde{\mathrm{O}}((\mathbf{D}^n + iI)\#j\mathbf{D}^n)$ is holomorphic across the interface $Im z_j = 0$. The pairs given by G_j in the sum $\sum_\sigma \operatorname{sgn}\sigma F_\sigma(x + i\Gamma_\sigma 0)$ cancel each other because of the definition of Y in $\tilde{\mathbf{Q}}$. Thus the mapping $\mathbf{Q}(\mathbf{D}^n) \to \tilde{\mathbf{Q}}$ is well defined and it is C-linear. □

In $\mathbf{Q}(\mathbf{D}^n)$ is defined a topology. First, we define a family of seminorms $\|\cdot\|_{K,\epsilon}$ in $\tilde{\mathrm{O}}((\mathbf{D}^n + iI)\#\mathbf{D}^n) \equiv E$: For every compact set $K \subset\subset I \setminus \{0\}$ and $\epsilon > 0$

$$\|F\|_{K,\epsilon} = \sup_{z \in \mathbf{R}^n + iK} |F(z)| e^{-\epsilon |\operatorname{Re} z|}, \quad F \in E.$$

The set of all such seminorms reduces essentially to a countable family and $\tilde{\mathrm{O}}((\mathbf{D}^n + iI)\#\mathbf{D}^n)$ turns out to be a Fréchet space. It is also a Montel space. Since the space

$H \equiv \sum_{j=1}^{n} \tilde{O}((D^n + iI)\#jD^n)$ is a closed subspace of $\tilde{O}((D^n+iI)\#D^n)$, the quotient space E/H admits the structure of a Fréchet and Montel space. If π is the canonical mapping: $E \to E/H$, then π is an open mapping. A family of seminorms on $\mathbf{Q}(D^n)$ is given by

$$p_{K,\epsilon}(\widehat{F}) = \inf_{h\in H} \|F+h\|_{K,\epsilon}, \quad F \in \widehat{F} \in \mathbf{Q}(D^n).$$

Since the space $\tilde{\mathbf{Q}}$ is isomorphic to the space $\mathbf{Q}(D^n)$, this isomorphism induces a topology on $\tilde{\mathbf{Q}}$. In this way the construction of $\tilde{\mathbf{Q}}$ gives an approach to the Fourier hyperfunctions, easier then the classical one, given by Sato which uses the cohomology theory. Every element $f \in \tilde{\mathbf{Q}}$ is given by

$$f(x) = \sum_{j=1}^{N} F_j(x + i\Gamma_j 0), \tag{2.15}$$

where every $F_j(x+i\Gamma_j 0)$ denotes the element of the quotient space $\tilde{\mathbf{Q}}$ determined by $F_j \in \tilde{O}(D^n + iI_j)$, $j = 1,\dots,N$. The functions F_j, $j = 1,\dots,N$ define a function F and we write $f = [F]$.

The relation between Fourier hyperfunctions and hyperfunctions is unexpected. Namely, we have a well defined mapping $\tilde{\mathbf{Q}} \to \mathbf{B}(\mathbf{R}^n)$: given $f \in \tilde{\mathbf{Q}}$ by (2.15), it can be regarded as a hyperfunction in the form (2.3) with the same defining function. Theorem 8.4.4 in Kaneko's book [7] asserts that this is a surjective mapping.

Let φ be a real analytic function such that it can be analytically continuable to a complex neighbourhood $U \subset D^n + i\mathbf{R}^n$ of D^n and such that $\varphi(z) \in \tilde{O}(U)$. If $f \in \tilde{\mathbf{Q}}$, then the multiplication is defined by: $\varphi f = [\varphi F]$, where $f = [F]$.

2.3.3. Fourier transform of Fourier hyperfunctions. Kaneko [7] has explained Sato's fundamental ideas concerning the Fourier transform as follows. Denote by $\mathcal{F}$ the Fourier transform. Let $f \in \tilde{\mathbf{Q}}$, where $f(x) = F_+(x+i\mathbf{R}_+0) - F_-(x+i\mathbf{R}_-0)$ then $\mathcal{F}(f) = [\phi]$, where

$$\phi_+(\zeta) = \int_{-\infty}^{0} e^{-i(x+iy_+)\zeta} F_+(x+iy_+)dx - \int_{-\infty}^{0} e^{-i(x+iy_-)\zeta} F_-(x+iy_-)dx, \ \operatorname{Im}\zeta > 0,$$

$$\phi_-(\zeta) = \int_{0}^{\infty} e^{-i(x+iy_+)\zeta} F_+(x+iy_+)dx - \int_{0}^{\infty} e^{-i(x+iy_-)\zeta} F_-(x+iy_-)dx, \ \operatorname{Im}\zeta < 0,$$

where $y_+ > 0$ and $y_- < 0$ are fixed belonging to the infinitesimal wedges $R + iR_+0$ and $R + iR_-0$, respectively.

All the integrals have a meaning because of: $-i(x+iy)(\xi+i\eta) = x\eta + \xi y - i(x\xi - \eta y)$.

To give a precise definition of the Fourier transform of elements belonging to $\tilde{\mathbf{Q}}$ we need the following proposition.

Let $F \in \tilde{\mathrm{O}}(\mathbf{D}^n+iI)$, where $\mathbf{R}^n+iI$ is an infinitesimal wedge of type $\mathbf{R}^n+i\Gamma 0$. It is said that F **decreases exponentially* outside a closed convex proper cone Δ^0 if restricting $\operatorname{Re} z$ outside any cone containing Δ^0 as a proper subcone, then F satisfies the estimate $|F(z)| = O(\exp(-\delta|\operatorname{Re} z|))$ for a suitable $\delta > 0$ and locally uniformly for $y \in I$.

Proposition 2.8. *Suppose that for an infinitesimal wedge $\mathbf{R}^n + iI$ of type $(\mathbf{R}^n + i\Gamma 0)$ the function $F \in \tilde{\mathrm{O}}(\mathbf{D}^n + iI)$ and decreases exponentially outside a closed convex proper cone Δ^0 (Δ^0 is the dual cone to the cone Δ). Set*

$$G(\zeta) = \int\limits_{\operatorname{Im} z=y} e^{-iz\zeta}F(z)dx$$

for any $y \in I$. Then it converges locally uniformly in ζ ranging over an infinitesimal wedge $\mathbf{R}^n-iJ$ of type $\mathbf{R}^n-i\Delta 0$ and $G \in \tilde{\mathrm{O}}(\mathbf{D}^n-iJ)$. Furthermore, $G(\zeta)$ decreases exponentially outside Γ^0. Hence $\mathcal{F}[F(x+i\Gamma 0)] = G(\xi - i\Delta 0)$, where $G \in \tilde{\mathbf{Q}}$, as well.

Proof. Let K be a fixed compact set belonging to $-\Delta$. Choose the cone Δ'_K containing Δ^0 such that $\operatorname{Re}(-iz\zeta) = x\eta + y\xi \le -c_K|x| + y\xi$ for $\eta \in K$, $x \in \Delta'_K$, where $c_K > 0$. We can now analyse the function $G(\zeta)$.

$$G(\zeta) = \int\limits_{\operatorname{Im} z=y\in I} e^{-iz\zeta}F(z)dx =$$
$$= \int\limits_{\Delta'_K} e^{-i(x+iy)\zeta}F(x+iy)dx + \int\limits_{\mathbf{R}^n\setminus\Delta'_K} e^{-(x+iy)\zeta}F(x+iy)dx.$$

The first integral converges locally uniformly in $\zeta \in \mathbf{R}^n+iK$ because $F \in \tilde{\mathrm{O}}(\mathbf{D}^n+iI)$ and $\operatorname{Re}(-iz\zeta) \le -c_K|x| + y\xi$. For the second integral we can use that $|F(z)| = O(e^{-\delta|x|})$ locally uniformly for $y \in I$. If we suppose that $\eta \in K \cap \{|\eta| < \delta_K\}$ for a suitable chosen δ_K, then the second integral converges locally uniformly on $\mathbf{R}^n + i(K \cap \{|\eta| < \delta_K\})$. Hence, $G(\zeta)$ is a holomorphic function in ζ on an infinitesimal wedge of type $\mathbf{R}^n - i\Delta 0$. From the both integrals we can draw out the factor $e^{y\xi}$. Consequently $G \in \tilde{\mathrm{O}}(\mathbf{D}^n - iJ)$ and if ξ moves outside a cone containing Γ^0 as a proper subcone, we have $y\xi \le -\delta_y|\xi|$, $\delta_y > 0$. Thus $G(\zeta)$ decreases exponentially outside Γ^0. □

In order to define the Fourier transform of an element $f \in \tilde{\mathbf{Q}}$, $f = [F] = \sum_{m=1}^{M} F_m(x + i\Gamma_m 0)$ we shall first prove that F_m, $m = 1,\dots,M$, can be made decomposed into a finite sum of functions decreasing exponentially outside a closed convex cone. One of such decomposition can be in the following way:

Let $\sigma_k = \pm 1$, $k = 1,\dots,n$; the multi signature $\sigma = (\sigma_1,\dots,\sigma_n)$ determines the cone Γ_σ as the σ-th orthant in $\mathbf{R}^n$. Put $\chi_+(t) = e^t/(1+e^t)$, $\chi_-(t) = 1/(1+e^t)$ and $\chi_\sigma(z) = \chi_{\sigma_1}(z_1)\dots\chi_{\sigma_n}(z_n)$. Every $\chi_\sigma(z)$ decreases exponentially along the real axis outside any cone containing the closed σ-th orthant as a proper subcone and $\sum_\sigma \chi_\sigma(z) = 1$. These properties of χ_σ make possible the decomposition of $F_m, F_m(z) = \sum_\sigma \chi_\sigma(z)F_m(z)$, where each term $\chi(z)F_m(z)$ decreases exponentially outside the closed σ-th orthant. Consequently, the Fourier hyperfunction $f = [F]$ can be given in the form

$$f(x) = \sum_{k=1}^{N} U_k(x + i\Gamma_k 0), \tag{2.16}$$

where $U_k \in \tilde{\mathrm{O}}(\mathbf{D}^n + iI_k)$, $\mathbf{D}^n + iI_k$ is an infinitesimal wedge of the form $\mathbf{R}^n + i\Gamma_k 0$ and $|U_k(z)| = O(\exp(-\delta|\operatorname{Re} z|))$ for a $\delta > 0$ when restricting $\operatorname{Re} z$ outside any cone containing a fixed cone Δ_k^0 but locally uniformly for $\operatorname{Im} z \in I_k$.

Definition 2.10. The Fourier transform of $f = [F]$ given by (2.16) is

$$\mathcal{F}[f] = \sum_{k=1}^{N} \mathcal{F}[U_k(x + i\Gamma_k 0)].$$

By Proposition 2.8 it maps $\tilde{\mathbf{Q}}$ into $\tilde{\mathbf{Q}}$. One can prove (Lemma 8.3.3 in [7]) that $\mathcal{F}[f]$ does not depend on the decomposition of the defining function F into finite sums of hyperfunctions decreasing exponentially outside a closed convex cone.

By Proposition 2.8 it is easy to define the inverse Fourier transform $\mathcal{F}^{-1}$:

$$\mathcal{F}^{-1}[G](z) = \frac{1}{(2\pi)^n} \int_{\operatorname{Im}\zeta = \eta \in -J} e^{iz\zeta} G(\zeta) d\xi \equiv F(z).$$

The properties of F and G given in Proposition 2.8 make elementary the proof that $\mathcal{F}^{-1}\mathcal{F} = \mathcal{F}\mathcal{F}^{-1} = \mathrm{id}$. Hence this holds for any Fourier hyperfunction and the Fourier transform is an automorphism of $\tilde{\mathbf{Q}}$.

We saw that the mapping $\tilde{\mathbf{Q}} \to \mathbf{B}(\mathbf{R}^n)$ is surjective. In this sense every hyperfunction has the Fourier transform.

2.3.4. An other definition of the Fourier transform of Fourier hyperfunctions.

First we shall define the space $\mathbf{P}_*$. Let δ be a positive constant and I an open set in $\mathbf{R}^n$ containing 0. Then $\tilde{\mathrm{O}}^{-\delta}(\mathbf{D}^n + iI)$ is defined as the set of holomorphic functions F on $\mathbf{R}^n + iI$ such that for every compact set $K \subset\subset I$ and every $\epsilon > 0$ there exists $C_{K,\epsilon} > 0$, $|F(z)| \leq C_{K,\epsilon} \exp(-(\delta - \epsilon)|\operatorname{Re} z|)$ uniformly for $z \in R^n + iK$. Then

$$\mathbf{P}_* = \varinjlim_{I \ni 0} \varinjlim_{\delta \downarrow 0} \tilde{\mathrm{O}}^{-\delta}(\mathbf{D}^n + iI)$$

with the topology of inductive limit.

It is easy to prove that if $f \in \tilde{\mathrm{O}}^{-\delta}(\mathbf{D}^n + i\{|y| < \gamma\})$, the Fourier transform

$$\mathcal{F}f = \hat{f}(\zeta) = \int_{\operatorname{Im} z = y} e^{-iz\zeta} f(z) dx \in \tilde{\mathrm{O}}^{-\gamma}(\mathbf{D}^n + i\{|\eta| < \delta\}), |y| < \gamma.$$

The Fourier transform is an automorphism of $\mathbf{P}_*$. $\mathbf{P}_*$ is called the space of **rapidly decreasing real analytic functions.*

By Theorem 8.6.2 in [7], $\mathbf{P}_*$ and $\tilde{\mathbf{Q}}$ are topological dual to each other. The inner product is given by

$$\langle f, \varphi\rangle = \int_{\mathbf{R}^n} f(x)\varphi(x)dx = \sum_{j=1}^{n} \int_{\mathrm{Im}\, z=y^{(j)}} (\varphi F_j)(z)dx,$$

where

$$y^{(j)} \in I_j,\ \varphi \in \mathbf{P}_*,\ f = [F] = \sum_{j=1}^{N} F_j(x + i\Gamma_j 0) \in \tilde{\mathbf{Q}}.$$

The Fourier transform acts as a topological automorphism on $\tilde{\mathbf{Q}}$ and $\langle \mathcal{F}f, \varphi\rangle = \langle f, \mathcal{F}\varphi\rangle$ is valid.

Let us remark that the space $\vec{\Phi}$ in Zharinov's approach is just the space $\mathbf{P}_*$. This gives a connection between Zharinov's approach and the other two. Also the three different definitions of the Fourier transform give the same operation.

Remark. The proof that $\vec{\Phi}'$ is isomorphic to $\mathbf{Q}(\mathbf{D}^n)$ can be find in [12].

2.4. Asymptotic behaviour of Fourier hyperfunctions and its applications

Asymptotic behaviour of generalized functions has an important role in the analysis of solutions to mathematical models, to precise the asymptotics of integral transforms or to discuss some problems in the theoretical phisics.

2.4.1. Quasiasymptotics. As we cited in 2.3.1, Zharinov [35] defined the space $\vec{\Phi}'$ which is isomorphic to $\tilde{\mathbf{Q}}$ or $\mathbf{Q}(\mathbf{D}^n)$. But in the same paper he constructed the space $\vec{\Lambda}'(\mathcal{O}) \subset \vec{\Phi}'$, where $\mathcal{O}$ is a domain in $\mathbf{R}^n$. For an element of $\vec{\Lambda}'(\mathcal{O})$, he defined the quasiasymptotics.

Let Γ be a convex closed acute cone in $\mathbf{R}^n$. We denote by $\Sigma = \operatorname{int}\Gamma^0$, where Γ^0 is the dual cone to Γ. We will follow Zharinov's definitions and results given in [34] and [35].

Let A and B be two bounded domains in $\mathbf{R}^n$. Denote by $s_B(\xi) = \sup\{-y\xi;\ y \in B\}$ and by $\Lambda(A, B)$ the Banach space of functions holomorphic on $\mathbf{R}^n + iA$ and such that

$$\|\varphi\|^A_{-s_B} = \sup\{e^{-s_B(\xi)}|\varphi(\xi + i\eta)|;\ \zeta \in \mathbf{R}^n + iA\} < \infty$$

with the topology given by the norm $\|\cdot\|^A_{-s_B}$. It is easy to see that $\Lambda(A, B) \subset \Lambda(A', B')$, when $A' \subset A$ and $B \subset B'$. With the inclusion mapping $\rho_{AB,A'B'} : \Lambda(A, B) \to \Lambda(A', B')$ we can define

$$\overrightarrow{\Lambda}(\Sigma) = \varinjlim_{A\ni 0,\ B\subset\subset\Sigma} \Lambda(A, B);\quad \overleftarrow{\Lambda}(\Sigma) = \varprojlim_{B\subset\subset\Sigma,\ 0\in A} \Lambda(B, A).$$

The space $\overrightarrow{\Lambda}(\Sigma)$ is a DFS space and its dual space $\overrightarrow{\Lambda}'(\Sigma)$ is an FS space. But $\overleftarrow{\Lambda}(\Sigma)$ is an FS space. Zharinov (cf. [35]) proved that $\vec{\Phi}'_\Gamma \subset \overrightarrow{\Lambda}'(\Sigma) \subset \vec{\Phi}'$, where $\vec{\Phi}'$ is defined in 2.3.1 and $\vec{\Phi}'_\Gamma = \{g \in \vec{\Phi}';\operatorname{supp} g \subset \Gamma\}$.

Now we can cite the definition of the quasiasymptotics (cf. [34]).

Definition 2.11. Suppose that $g \in \vec{\Lambda}'(\Sigma)$ and that ρ is a positive and continuous function on $(0,\infty)$. If there exists

$$\lim_{t\to\infty} g(t\zeta)/\rho(t) = h(\zeta) \quad \text{in } \vec{\Lambda}'(\Sigma), \quad h \neq 0,$$

then it is said that g has the quasiasymptotics related to ρ.

Since $\vec{\Lambda}'(\Sigma)$ is an FS space, the limit in Definition 2.11 is equivalent with

$$\lim_{t\to\infty} \langle g(t\xi)/\rho(t), \varphi(\xi)\rangle = \langle h, \varphi\rangle, \; h \neq 0$$

for every $\varphi \in \vec{\Lambda}(\Sigma)$.

Similarly as for the quasiasymptotics of distributions (cf. [33]) one can prove that ρ and h in Definition 2.11 have the following properties:

1) ρ has the form $\rho(t) = t^{\alpha}L(t)$, $\alpha \in \mathbf{R}$ and L is Karamata's slowly varying function [9];

2) h is homogeneous of degree α.

The defined quasiasymptotic behaviour of Fourier hyperfunctions can be used to precise properties of solutions to mathematical models (partial differential equations, integral equations,...) as it is done by means of the quasiasymptotics of distributions (cf. [33]). Applications of the quasiasymptotic behaviour of Fourier hyperfunctions are not yet developed but one can expect interesting results of such investigations.

To illustrate the applications of the quasiasymptotics we cite an Abelian type theorem for the Laplace transform of Fourier hyperfunctions (cf. [34]). But first we have to define the Laplace transform of elements belonging to $\vec{\Lambda}'(\Sigma)$.

For a fixed $z \in \mathbf{R}^n + iB$, where B is a bounded subset of Σ, $e^{iz} \in \Lambda(A,B)$ for every bounded set A and $\|e^{iz}\|^{A}_{-s_B} = e^{s_A(x)}$, $z = x + iy$. Thus for every fixed $z \in \mathbf{R}^n + i\Sigma$, $e^{iz\xi} \in \vec{\Lambda}(\Sigma)$.

Definition 2.12. The Laplace transform of $g \in \vec{\Lambda}'(\Sigma)$, $\mathcal{L}g$, is defined by

$$\mathcal{L}g(z) = \langle g(\xi), e^{iz\xi}\rangle, \quad z \in \mathbf{R}^n + i\Sigma.$$

In [35] Zharinov have proved that the Laplace transform defines an isomorphism $\vec{\Lambda}'(\Sigma)$ onto $\overleftarrow{\Lambda}(\Sigma)$. With this property and the cited properties of the family of functions $\{e^{iz\xi};\ z \in \mathbf{R}^n + i\Sigma\}$ it is easy to prove the following proposition of the Abelian type.

Proposition 2.9. *Suppose that $g, h \in \vec{\Lambda}'(\Sigma)$ and $\rho(t) = t^{\alpha}L(t)$, $\alpha \in \mathbf{R}$. Denote by $G = \mathcal{L}g$ and $H = \mathcal{L}h$, then $G, H \in \overleftarrow{\Lambda}(\Sigma)$. If*

$$g(t\xi)/\rho(t) \to h(\xi), \; t \to \infty, \; \text{in } \vec{\Lambda}'(\Sigma),$$

then

$$G(z/t)/t^n\rho(t) \to H(z), \; t \to \infty, \; \text{in } \overleftarrow{\Lambda}(\Sigma).$$

In [34] one can find other properties of the quasiasymptotics of Fourier hyperfunctions.

Let us remark that Komatsu in [16] has also defined the Laplace transform of a subspace of hyperfunctions, denoted by $\mathbf{B}^{\exp}_{[a,\infty]}$, and in [17] he has related his theory with other theories of the Laplace transform of generalized functions.

2.4.2. S-asymptotics. An other asymptotic behaviour has been defined for distributions (ultradistributions) and has been applied in the quantum field theory (cf. [25], [26]). It is called the S-asymptotics. It is easy to extend it to Fourier hyperfunctions.

Definition 2.13. Suppose that c is a positive function defined on $\mathbf{R}^n$ and $f \in \mathbf{Q}(\mathbf{D}^n)$. f is said to have the S-asymptotics related to c in the cone Γ if there exists

$$\lim_{k\in\Gamma,\,\|k\|\to\infty} \frac{f(\cdot+k)}{c(k)} = h \text{ in } \mathbf{Q}(\mathbf{D}^n),\ h \neq 0. \tag{2.17}$$

Since $\mathbf{Q}(\mathbf{D}^n)$ is a Montel space, (2.17) can be given in the form:

$$\lim_{k\in\Gamma,\,\|k\|\to\infty} \langle \frac{f(x+k)}{c(k)}, \varphi(x)\rangle = \langle h, \varphi\rangle,\ \ h \neq 0, \tag{2.18}$$

for every $\varphi \in \mathbf{P}_*$.

The next examples shows that Definition 2.13 is not a trivial extension of the S-asymptotics of distributions. Let $P(D)$ be a local operator $\sum_{|\alpha|\geq 0} b^\alpha D^\alpha$, $b^\alpha \neq 0$. The Fourier hyperfunction $f = 1 + P(D)\delta$ has the S-asymptotics related to $c = 1$ in any cone Γ and with the limit $h = 1$ but f is not a distribution. For the S-asymptotics of f it is enough to prove that

$$\lim_{k\in\Gamma,\,\|k\|\to\infty} \langle P(D)\delta(x+k), \varphi(x)\rangle = 0,\ \ \varphi \in \mathbf{P}_*.$$

Since $P(D)$ maps $\mathbf{P}_*$ into $\mathbf{P}_*$,

$$\langle P(D)\delta(x+k), \varphi(x)\rangle = \langle \delta(x+k),\ P(-D)\varphi(x)\rangle = \psi(k),$$

where $\psi = P(-D)\varphi$. By the property of elements belonging to $\mathbf{P}_*$ (see 2.3.4) $\lim_{k\in\Gamma,\,\|k\|\to\infty} \psi(k) = 0$ for every cone Γ.

A hyperfunction g supported by the origin is uniquely expressible as $g = \tilde{P}(D)\delta$, where $\tilde{P}(D)$ is a local operator. In such a way, with the above, we proved that every Fourier hyperfunction with support $\{0\}$ has the limit, given in (2.17) and (2.18), equal zero.

Since $P(D)\delta = \sum_{|\alpha|\geq 0} b_\alpha D^\alpha \delta$ is a distribution if and only if $b_\alpha \neq 0$ for a finite number of α, the Fourier hyperfunction $1 + P(D)\delta$ is not a distribution, but it has the S-asymptotics related to $c = 1$.

We can also find such coefficients b_α of the local operator $P(D)$ such that $f = 1 + P(D)\delta$ is not defined by an ultradistribution belonging to the Gevrey class

$D^{(s)'}$ or $D^{\{s\}'}$, $s > 1$. Because of simplicity, we shall consider one-dimensional case. Choose $P(D)$ such that the coefficients of $P(D)$ are: $b_n = (n!)^{-(1+c_n)}$, $n \in N$, where $c_n = (10 \log n)^{-1}$. With these coefficients, $P(D)$ is a local operator. Namely,

$$\lim_{n\to\infty} \sqrt[n]{b_n n!} = \lim_{n\to\infty} (n!)^{-1/(n \log\log n)} = 0.$$

Also any ultradistribution in Gevrey class $s > 1$, supported by $\{0\}$, is of the form

$$J(D)\delta = \sum_{n=0}^{\infty} a_n D^n \delta, \ |a_n| \le Ck^n/(n!)^s \tag{2.19}$$

with some constants k and C (Beurling's type) or for any $k > 0$ with a constant C (Roumieu's type). But $b_n = (n!)^{-(1+c_n)}$ does not satisfy condition for coefficients of $J(D)$ in (2.19). Namely, since $c_n \to 0$ when $n \to \infty$, for any $s > 1$, there exists n_0 such that $1 < 1 + c_n < s$, $n \ge n_0$. Thus,

$$(n!)^{-(1+c_n)} > Ck^n/(n!)^s, \quad n \ge n_0, \ k > 0.$$

Consequently, $P(D)\delta$ does not represent an ultradistribution.

However we can suppose that $P(D)\delta$ is an ultradistribution g with support $\{0\}$ in Gevrey class $s > 1$. Then we would have an ultradifferential operator $J_1(D)$ such that

$$g = J_1(D)\delta = \sum_{n=0}^{\infty} e_n D^n \delta, \ |e_n| \le Ck^n/(n!)^s.$$

But in this case $J_1(D)$ would be a local operator, $J_1(D) \ne P(D)$. This contradicts the fact that a hyperfunction with support at $\{0\}$ is given by a unique local operator.

The defined S-asymptotics can be also used in order to precise the behaviour of solutions to mathematical models as it is done with the S-asymptotics of distributions (cf. [26]). We shall illustrate this with the problem of asymptotic behaviour of solutions to equations given by local operators.

Since a local operator maps continuously $\mathbf{Q}(\mathbf{D}^n)$ into $\mathbf{Q}(\mathbf{D}^n)$, we have:

Proposition 2.10. *Suppose that $f \in \mathbf{Q}(\mathbf{D}^n)$ and has the S-asymptotics related to c and to the cone Γ with the limit h. Then*

$$\lim_{k\in\Gamma,\, \|k\|\to\infty} \frac{P(D)f(x+k)}{c(h)} = P(D)h \ \ in \ \mathbf{Q}(\mathbf{D}^n).$$

Corollary. *A necessary condition that a solution of the equation $P(D)x = f$ has the S-asymptotics related to c and to the cone Γ with the limit u is that f has the limit (2.16) with $h = P(D)u$.*

If $P(D)$ fulfils some additional properties, we would have in the Corollary not only necessary, but necessary and sufficient condition. Such a case is if $P(D)y = \delta$ has a solution in $\mathbf{Q}^{-\gamma}(\mathbf{D}^n)$, $\gamma > 0$.

Let us remark that we have only first results concerning the asymptotics of Fourier hyperfunctions. Regarding the definition of the asymptotic behaviour of hyperfunction in general case we do not know that such a definition exists.

2.4.3. Asymptotics Taylor expansion. Estrada and Kanwal [2] elaborated a method of asymptotic expansions for distributions quite different in relation to the methods which can be find for distributions in [21], [26], [31] and [33]. The results of Estrada and Kanwal gave a nice confirmation that the asymptotic expansions have arisen in several fields of applications as a powerful technique. They started by considering the asymptotic Taylor expansion for distributions, its application and generalizations.

Definition 2.14. If $f \in \mathbf{D}'$, then for a fixed $\xi \in \mathbf{R}^n$ and $\epsilon \in \mathbf{R}$

$$f(x+\epsilon\xi) \sim \sum_{|k|=0}^{\infty} \frac{D^k f(x)}{k!}(\epsilon\xi)^k, \quad \text{as } \epsilon \to 0, \tag{2.20}$$

which means that for any function $w \in \mathbf{D}$ and for any $N \in \mathbf{N}$

$$\langle f(x+\epsilon\xi), w(x)\rangle = \sum_{|k|=0}^{N} \frac{\langle D^k f(x), w(x)\rangle}{k!}(\epsilon\xi)^k + O(\epsilon^{N+1}),$$

as $\epsilon \to 0$. The formal series in (2.20) is called the asymptotic Taylor expansion for f (on the straight line $\{h\xi; h \in \mathbf{R}\}$).

For any $f \in \mathbf{D}'$, (2.20) holds. Also, Definition 2.14 can be applied to any space of generalized functions defined as the dual space $\mathbf{A}'$ of a basic space $\mathbf{A}$ of smooth functions. Since the space of Fourier hyperfunctions is a space of this type, Definition 2.14 can be repeated with the space $\mathbf{Q}(\mathbf{D}^n)$ instead of $\mathbf{D}'$.

Concerning this definition a natural question arises: *What are necessary and sufficient conditions that the asymptotic Taylor expansion for a generalized function f is in the same time the Taylor series for f, convergent in the space of generalized functions.*

The answer on this question for distributions and ultradistributions one can find in [32]. For the Fourier hyperfunction we can prove

Proposition 2.11. *The asymptotic Taylor expansion* (2.20) *for $u \in \mathbf{Q}(\mathbf{D}^n)$ on the straight line $\{h\xi; h \in \mathbf{R}\}$, where $\xi_i \neq 0$, $i = 1, \dots, n$, is the Taylor series convergent in $\mathbf{Q}(\mathbf{D}^n)$ when $\eta\xi \in B(0, \eta_0\xi)$ for an $\eta_0 > 0$ if and only if there exists an $r = (r_1, \dots, r_n)$, $r_i > 0$, $i = 1, \dots, n$, such that u is determined by a real analytic function which can be extended as a holomorphic function on $\{z \in \mathbf{C}^n;\ |\operatorname{Im} z_i| < r_i,\ i = 1, \dots, n\}$.*

The proof is based on two Kaneko's results. First every Fourier hyperfunction $u \in \mathbf{Q}(\mathbf{D}^n)$ can be given in the form $u = P_1(D)f$, where $P_1(D)$ is an elliptic local operator and f is an infinitely differentiable function of infra exponential growth [7]. Second, there exist an elliptic local operator $P_2(D)$ and an infinitely differentiable function g *rapidly decreasing* ($|g(x)| \leq C\exp(-\alpha\|x\|)$, $x \in \mathbf{R}^n$ for some $\alpha > 0$) such that $\delta = P_2(D)g$ (δ is the delta distribution) [8].

REFERENCES

1 A. Cerezo, A. Piriou and J. Chazarain, *Introduction aux hyperfunctions,* Lectures Notes in Mathematics 449, Springer-Verlag, Berlin, 1975, 1–53.

2 R. Estrada and R. P. Kanwal, *Asymptotic Analysis, A distributional Approach,* Bikhäser, Basel, 1994.

3 R. Harvey, *Hyperfunctions and Linear Partial Differential Equations,* Thesis, Stanford University, 1966.

4 L. Hörmander, *The Analysis of Linear Partial Differential Operators I, II,* Springer-Verlag, Berlin, 1983.

5 L. Hörmander, *Between Distributions and Hyperfunctions,* Astérisque **131** (1985), 89–106.

6 I. Imai, *Applied Hyperfunction theory,* Kluwer, Dordrecht, 1992.

7 A. Kaneko, *Introduction to Hyperfunctions,* Kluwer, Dordrecht, 1988.

8 A. Kaneko, *Representation of hyperfunctions by measures and some of its applications,* J. Fac. Sci. Univ. Tokyo 1A **19** (1972), 321–352.

9 J. Karamata, *Sur an mode de croissance régulière des functions,* Mathematica (Cluj) **4** (1930), 38–53.

10 M. Kashiwara, T. Kawai and T. Kimura, *Foundations of Algebraic Analysis,* Princeton University Press, Princeton, 1986.

11 K. Kataoka, *Radon Decomposition of Hyperfunctions and its Applications,* Master's thesis, Univ. of Tokyo, 1976, (in Japanese).

12 T. Kawai, *On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients,* J. Fac. Sci. Univ. Tokyo, Sec. IA **17** (1970), 467–517.

13 H. Komatsu, *An introduction to the theory of hyperfunctions,* Lecture Notes in Mathematics 287, Springer-Verlag, Berlin, 1973, 3–40.

14 H. Komatsu, *Hyperfunction and linear partial differential equations,* Conference on Generalized Functions, Katowice, 1966.

15 H. Komatsu, *Relative cohomology of sheaves of solutions of differential equations,* Lecture Notes in Mathematics 287, Springer-Verlag, Berlin 1973, 192–261.

16 H. Komatsu, *Laplace transform of hyperfunctions-A new fondation of the Heaviside calculus,* J. Fac. Sci. Univ. Tokyo, Sec. IA **34** (1987), 805–820.

17 H. Komatsu, *Generalized Functions Convergent structures and Their Applications,* Proceedings, Dubrovnik, June 23–27, 1987, Plenum Press, 1988.

18 A. Martineau, *Les hyperfonctions de M. Sato,* Séminaire Bourbaki, 13, (1960–61), No. 214.

19 T. Matsuzawa, *A calculus approach to hyperfunctions I,* Nagoya Math. J. **108** (1987), 53–66.

20 M. Morimoto, *Introduction to Sato Hyperfunctions,* Kyoritsu, 1976 (in Japanese).

21 D. Nikolić-Despotović, S. Pilipović, *The quasiasymptotic expansion at the origin,* Math. Nachr. **174** (1995), 231–238.

22 H. Petzshe, *Generalized functions and the boundary values of holomorphic functions,* J. Fac. Sci. Univ. Tokyo Sec. IA **31** (1984), 391–431.

23 H. J. Petzshe, D. Vogt, *Almost Analytic Extension of ultradifferentiable Functions and the Boundary Values of Holomorphic Functions,* Math. Ann. **267** (1984), 17–35.

24 S. Pilipović, *Pseudodifferential operators and the microlocalization in the space of Colombeau's generalized functions,* Bull. Acad. Serbe Sci. Arts **109** Sci. Math. 20 (1995), 13–27.

25 S. Pilipović, B. Stanković, *Quasi-asymptotics and S-asymptotics of ultradistribuions,* Bull. Acad. Serbe Sci. Arts **109** Sci. Math. **20** (1995), 61–74.

26 S. Pilipović, B. Stanković, A. Takači, *Asymptotic behaviour and Stieltjes transformation of distributions,* Teubner Verlagsgesellschaft, Leipzig, 1990.

27 M. Sato, *On a generalization of the concept of functions,* Proc. Japan Acad. **34** (1958), 126–130 and 604–608.

28 M. Sato, *Theory of hyperfunctions,* J. Fac. Sci. Univ. Tokyo Sec. I **8** (1959-1960), 139–193 and 387–436.

29 M. Sato, T. Kawai and M. Kashiwara, *Microfunctions and Pseudo-Differential Equations, Part II*, Lecture Notes in Mathematics 287, Springer-Verlag, Berlin, 1973, 263–529.

30 P. Schapira, *Théorie des Hyperfunctions,* Lecture Notes in Mathematics 126, Springer-Verlag, Berlin, 1970.

31 B. Stanković, *S-asymptotic expansion of distributions,* Internat. J. Math. Sci. **11** (1988), 449–456.

32 B. Stanković, *Asymptotic Taylor expansion for Fourier hyperfunctions,* J. Math. Anal. Appl. **203** (1996), 31–37.

33 V. S. Vladimirov, Yu. N. Drozhinov, B. I. Zavyalov, *Multidimensional Tauberian theorems of generalized functions,* Nauka, Moscow, 1986 (in Russian).

34 V. V. Zharinov, *On the quasiasymptotics of Fourier hyperfunctions,* Theor. Math. Phys. **43**:1 (1980), 32–38 (in Russian).

35 V. V. Zharinov, *Laplace transform of Fourier hyperfunctions and other close classes of analytic functionals,* Teor. Math. Phys. **33**:3 (1977), 291–309 (in Russian).

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