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CURVATURE AND SHAPE

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Abstract. The simple but very powerful idea of comparing the geometry of a given curve, surface or Riemannian manifold with the geometry of a space of constant curvature is exemplified and illustrated. Applications to small scale structures of biology (bio-membranes) and large scale structures of the universe (singularities) are briefly mentioned.

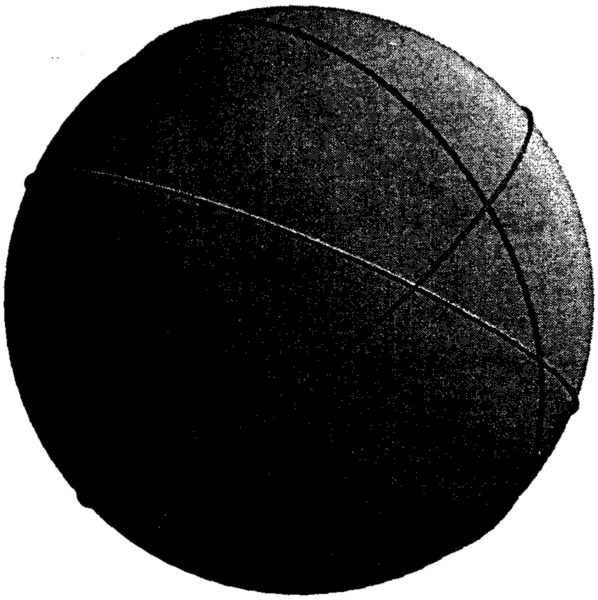

Figure 1. Spherical triangles on a unit sphere.

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These notes cover three lectures given at the Fall School of Differential Geometry, which took place on the mountain Divčibare, Yugoslavia, October 13–17, 1996.

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1. The Area of Spherical Polygons

We begin in the year 1603 when Thomas Harriot proved a very nice theorem about spherical triangles. We present his result and proof in Section 1.2. It is quite explicit and may thus serve as a concrete introduction to the idea of detecting curvature via measurements on triangles.

However, before triangles we must study biangles:

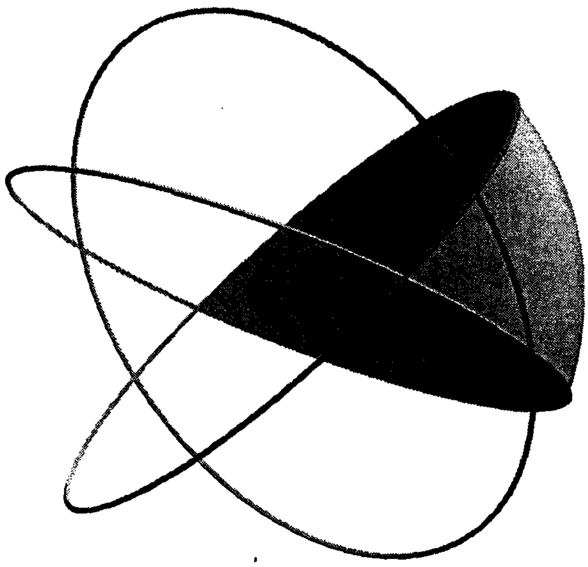

Figure 2. A spherical biangle is bounded by two half great circles

1.1. Spherical Biangles. On a sphere of radius 1 we consider a domain which is bounded by two half great circles. Such a domain is called a *spherical biangle* – see Figure 2.

A spherical biangle is – modulo its position on the sphere – uniquely determined by the inner vertex angle, i.e., the angle between the two defining great circle segments.

The area A of a spherical biangle is therefore a function of the vertex angle α *alone*, i.e., $A = A(\alpha)$, $\alpha \in [0, 2\pi]$. Since the area of the surface of the unit sphere is 4π, we have $A(2\pi) = 4\pi$.

The area is furthermore clearly an additive function of the vertex angle.

The following claim is thus appropriate for our purpose:

LEMMA 1. *Let f be a real, positive and additive function on $]0, 2\pi]$ with*

$f(2\pi) = 4\pi$. *Then*

$$f(x) = 2x. \tag{1.1}$$

1.1.1. *Remarks.* (i) This Lemma is proved in Section 2. (ii) If we drop the condition that f is positive, then the conclusion does not hold!

1.2. Spherical Triangles. A domain on a sphere, which is bounded by 3 great circle segments is called a *spherical triangle*, see Figure 3.

We are now in position to state and prove Harriots theorem:

THEOREM. (Harriot, 1603) *On a unit sphere, the area A of a geodesic triangle $\triangle$ is equal to the angular excess of the triangle:*

$$A(\triangle) = (\alpha_1 + \alpha_2 + \alpha_3) - \pi \tag{1.2}$$

$$= 2\pi - \sum_{i=1}^{3}(\pi - \alpha_i). \tag{1.3}$$

where α_i denote the inner angles at the vertices of the triangle.

1.2.1. *Remark.* Note that Lemma 1 may be viewed as a Harriot theorem for biangles.

Proof. The two *figures* in Figure 3 and Figure 4 are reflections of each other through the center of the sphere. Therefore they have the same area. This area must be 2π because together the two figures cover the full sphere without overlapping (except at the boundaries).

On the other hand, the area of Figure 3 is equal to the sum of the areas of the three biangles minus 2Area($\triangle$), since otherwise Area($\triangle$) would be counted three times. Since the biangles according to (1.1) have areas Area(α_i) $= 2\alpha_i$, $i = 1, 2, 3$, we get:

$$2\pi = 2\alpha_1 + 2\alpha_2 + 2\alpha_3 - 2\text{Area}(\triangle).$$

And that is what should be proved.

1.3. The n-gon case. In the general case of an n-gon we simply subdivide the polygon into spherical triangles, use Harriots theorem for each and sum the areas:

COROLLARY 3. *For a geodesic n-gon P_n on a unit sphere the area is*

$$A(P_n) = 2\pi - \sum_{i=1}^{n}(\pi - \alpha_i).$$

where α_i denote the inner angles at the vertices of the polygon.

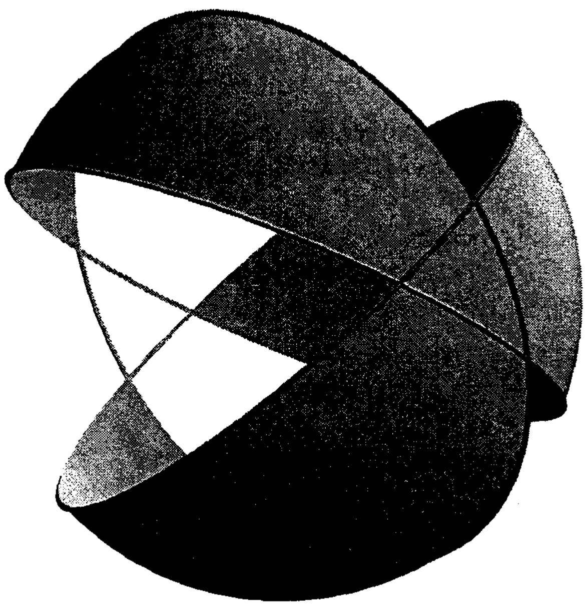

Figure 3. A spherical triangle is the intersection of three biangles

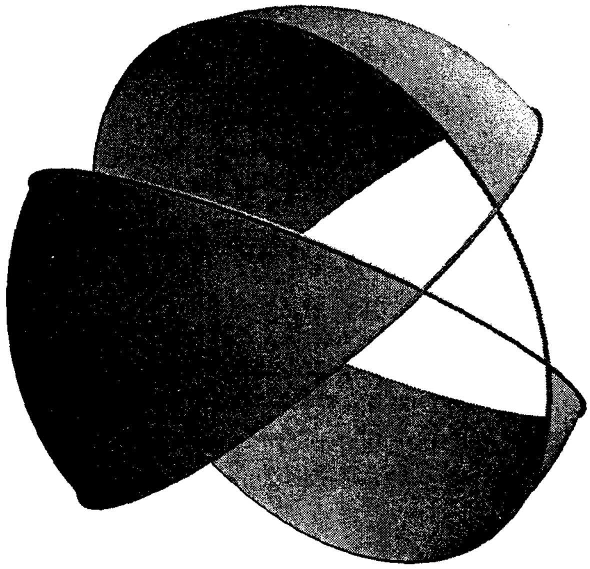

Figure 4. The sphere minus Figure 3 (i.e., the reflection of Figure 3 through the center of the sphere

2. Interlude

Although Harriot probably did not worry at all about proving Lemma 1, the proof has nevertheless some instructive elements, whence we present it below.

So we consider $I\ =]0, 2\pi]$ and a *positive*, real function f on I, with the properties

$$f(x+y) = f(x) + f(y) \tag{2.1}$$

for all x and y in I with $x + y \in I$ and

$$f(2\pi) = 4\pi. \tag{2.2}$$

We must then prove that

$$f(x) = 2x \qquad \text{for all } x \in I. \tag{2.3}$$

Proof. Firstly, (2.2) gives (2.3) directly for $x = 2\pi$. In the following we therefore assume, that x is a given real value in the interval $J\ =]0, 2\pi[$.

Since $x = p\frac{x}{p}$ for every $p \in \mathbb{N}$, we get by repeated use of (2.1), that $f(x) = pf(\frac{x}{p})$ and hence $f(\frac{x}{p}) = \frac{1}{p}f(x)$. For every $m \in \mathbb{N}$ and $p \in \mathbb{N}$ with $\frac{m}{p}x \in J$ we get similarly: $f(\frac{m}{p}x) = f(m\frac{x}{p}) = mf(\frac{x}{p}) = \frac{m}{p}f(x)$. In other words, whenever q is a rational number so that $qx \in J$, then

$$f(qx) = qf(x). \tag{2.4}$$

We want to show that (2.4) in fact is also true for 'real values' of q. So we let s be a given real number satisfying $0 < sx < 2\pi$. For every positive ε there exist *rational* numbers r and t which 'ε-approximate' s in the following sense:

$$0 < r < s < t, \tag{2.5}$$

$$t - r < \varepsilon, \qquad \text{and} \tag{2.6}$$

$$0 < rx < sx < tx < 2\pi. \tag{2.7}$$

The function f is increasing, since

$$f(x + \text{positive}) = f(x) + f(\text{positive}) > f(x)\ .$$

Hence we have from (2.7):

$$f(rx) < f(sx) < f(tx)$$

and thus from (2.4), since r and t are *rationals*:

$$rf(x) < f(sx) < tf(x).$$

Multiplying (2.5) with the *positive* number $f(x)$ we get for comparison:

$$rf(x) < sf(x) < tf(x).$$

and hence

$$| f(sx) - sf(x) | < tf(x) - rf(x) = (t-r)f(x) < \varepsilon f(x).$$

This is satisfied for every positive ε if and only if $f(sx) - sf(x) = 0$. Therefore

$$f(sx) = sf(x)$$

– which is the 'real version' of (2.4). Since $f(\pi) = \frac{1}{2}f(2\pi) = 2\pi$ we get in particular:

$$f(x) = f(\frac{x}{\pi}\pi) = \frac{x}{\pi}f(\pi) = \frac{x}{\pi}2\pi = 2x.$$

And this was to be proved.

3. About Thomas Harriot (1560–1621)

Harriot wass born in Oxford, where he also went to University. During the years 1585–1590 he was scientific and mathematical advisor for Sir Walter Ralegh, one of the great explorers of that time. Presumably Harriot participated in the colonialization of Virginia, already from 1584, see [44].

Thus Harriot is quite naturally mainly concerned with the mathematics of navigation, instrumentation and map projections.

At around 1590 Harriot discovered the first nontrivial plane curves whose arclength can be determined by elementary methods - namely the logarithmic spirals, which in polar coordinates are given by the parametrization: $r_k(\theta) = e^{k\theta}$, $\theta \in \mathbb{R}$, $k \in \mathbb{R}$. Harriot had neither the exponential function nor analytic tools such as differentiation and integration at his disposal. The logarithmic spirals were only known at that time by the property that they intersect the bundle of half lines issuing from a point under constant angle. Harriot's determination of the arclengths is thus quite shrewd – see [45] or [35], [36]. At the same time Harriot discovered that the stereographic projection (of the unit sphere onto the plane) is conformal – a fact he then puts into immediate use to illustrate his result about the area of spherical triangles.

4. More Than 200 Years Later

In 1818–1825 C. F. Gauss was very busy working on the geodetic surveying of Hannover. The resulting geodesic triangulation was to be linked up with the corresponding triangulation of Denmark, which had just been completed under the direction of Gauss' good friend and colleague, H. C. Schumacher, who was the director of the Danish astronomical observatory at Altona outside Hamburg.

It was during these years – with a lot of very practical work – that Gauss developed the theory of surfaces and the notion of curvature. These findings he reported a few years later, in 1827, in the booklet [14].

In this section we mention some of the spectacular results from this work. The following presentation could be close to the way Gauss actually thought about curvature, although this is not at all clear from his writings.

4.1. Harriot's result expanded. When approximating a given domain on the unit sphere (or on any other surface) by a geodesic n-gon, the sum of the extrinsic angles approaches the integral of the rate $\frac{d\theta}{ds}$ by which the tangent vector field along the boundary of the domain *rotates* relative to any given parallel vector field along the same curve. Note that for geodesic segments $\frac{d\theta}{ds} = 0$ since geodesics are autoparrallel - they parallel transport their own tangent vector field. The function $\frac{d\theta}{ds}$ is called the (intrinsic) geodesic curvature of the curve in question, and we denote it by κ_g. As a result we therefore get the following for domains on a unit sphere.

THEOREM 4. it For a simple domain Ω on the unit sphere:

$$A(\Omega) = 2\pi - \int_{\partial\Omega} \left(\frac{d\theta}{ds}\right) ds = 2\pi - \int_{\partial\Omega} \kappa_g(s) ds \,.$$

It is now an elementary matter to deduce the following result from the classical definition of Gauss curvature K of a given surface, namely as the determinant of the differential of the Gauss map. The integral of K over a given domain D on a surface M is thus (modulo sign) precisely the area of the image of D by the Gauss mapping. Furthermore, as observed by A. Sengupta (see [40, p. 74–75]), the total geodesic curvature $\int_{\partial D} \kappa_g(s) ds$ of the boundary ∂D within M is also precisely the *same* as the total geodesic curvature of the boundary of the Gauss map image of D within the unit sphere. The previous theorem then gives immediately:

THEOREM 5 (The Gauss–Bonnet Formula). *For a simple domain D on a surface M:*

$$\int_D K \, dA = 2\pi - \int_{\partial D} \kappa_g(s) \, ds \,. \tag{4.1}$$

4.2. Geodesic triangles revisited. In particular we may now, of course, recover the excess formula for the *area* of spherical *geodesic* triangles - for example via the following corollary (of the Gauss–Bonnet formula), which follows directly from the observation that for geodesics the only contribution to $\int_{\partial D} \kappa_g(s) \, ds$ is the sum $\sum_{i=1}^{3} (\pi - \alpha_i)$ of the extrinsic angles at the vertices:

COROLLARY 6. *On a surface M, the total curvature of a geodesic triangle Δ is equal to the angular excess of the triangle:*

$$\int_{\Delta} K\,dA = 2\pi - \sum_{i=1}^{3}(\pi - \alpha_i) \tag{4.2}$$

$$= (\alpha_1 + \alpha_2 + \alpha_3) - \pi\ , \tag{4.3}$$

where α_i denote the inner angles at the vertices of the triangle.

In fact, we may have used this as a *definition* of the function $K(p)$, since the curvature may now be obtained pointwise via the following limit construction.

THEOREM 7 (Gauss). *The Gauss curvature $K(p)$ of the surface M at p is*

$$K(p) = \lim_{\Delta \to p} \left(\frac{\alpha_1 + \alpha_2 + \alpha_3 - \pi}{\text{Area}\,(\Delta)} \right) . \tag{4.4}$$

This shows in particular, that the curvature is really intrinsic: The Gauss map is not needed – we only need geodesic triangles, angles and area to measure curvature.

Furthermore it says the following:
(i) $K(p) \geq 0$ for all p, if and only if every triangle excess is nonnegative.
(ii) If $K(p) \geq 0$ for all p and if a given triangle Δ has excess 0, then $K(q) = 0$ for all $q \in \Delta$.
These observations may be considered as *comparison* statements in a sense that we will now explain:

4.3. General comparison constructions. Let M be an abstract surface with a metric d and let Δ be a geodesic triangle in M, i.e., Δ is a simply connected domain in M whose boundary consists of 3 geodesic segments, i.e., 3 strictly shortest curves connecting the 3 vertices of the triangle. We denote the three interior angles of Δ by α_i, $i = 1, 2, 3$.

Suppose that we would like to know whether M has curvature greater than or equal to 1.

A comparison construction will tell us:

Let $\mathbb{S}^2_1$ denote the sphere of radius 1 in $\mathbb{R}^3$, and let Δ^* denote the comparison triangle on $\mathbb{S}^2_1$, i.e., a triangle with edgelengths equal to the edgelengths of the given Δ in M (the triangle Δ^* is – modulo isometries of $\mathbb{S}^2_1$ – uniquely determined by these lengths, and – most importantly – it exists if and only if the sum of the edgelengths does not exceed 2π, which is therefore implicitly assumed when referring to comparison constructions.) We denote the interior angles of Δ^* by α^*_i, $i = 1, 2, 3$.

THEOREM 8 (Alexandrov–Toponogov Triangle Comparison, [34], [43]) *A surface M has Gauss curvature $K \geq 1$ if and only if*

$$\alpha_i \geq \alpha_i^*, \qquad i = 1, 2, 3, \tag{4.5}$$

for every triangle Δ in M and its comparison triangle Δ^* in the unit sphere $\mathbb{S}_1^2$.

If $K \geq 1$ and if $\alpha_i = \alpha_i^*$ for some α_i for some triangle Δ, then equality also occurs for the other two angles in that triangle, and the triangular domain Δ has constant curvature $K = 1$, so that Δ and Δ^* are in fact isometric.

An alternative (equivalent) to this triangle comparison construction is the *hinge* comparison construction:

A *hinge* $\measuredangle$ is a point (the vertex of the hinge), and two geodesic segments (the sides of the hinge) emanating from p. The three pieces of information about a given hinge that we will focus upon are the lengths of the sides and the angle at p between the sides of the hinge. Suppose the largest side of a given hinge is less than or equal to π. A comparison hinge $\measuredangle^*$ can then be constructed in $\mathbb{S}_1^2$, and the respective distances d and d^* between the two "feet" of the hinges can be compared:

THEOREM 9 (Alexandrov–Toponogov Hinge Comparison, [34], [43]) *A surface M has Gauss curvature $K \geq 1$ if and only if*

$$d \leq d^* \tag{4.6}$$

for every hinge $\measuredangle$ in M and its comparison hinge $\measuredangle^*$ in the unit sphere $\mathbb{S}_1^2$.

If $K \geq 1$ and if $d = d^*$ for some $\measuredangle$, then the triangular domain Δ spanned by $\measuredangle$ in M has constant curvature $K = 1$, so that Δ and $\Delta^* = span(\measuredangle^*)$ are isometric.

By suitable scaling, i.e., using the sphere $\mathbb{S}_k^2$ of constant curvature k as comparison space, we get analogous results for $K \geq k > 0$, and in fact for any lower curvature bound $k \in \mathbb{R}$ by using comparison-triangles and -hinges resp., in flat space $\mathbb{R}^2$ for $k = 0$ and in hyperbolic space, $\mathbb{H}_k^2$, of constant curvature k for $k < 0$.

Furthermore, the comparison still makes sense and the two theorems above still hold true for Riemannian manifolds M^n of any dimension if we replace the Gauss curvature assumption by the corresponding assumption – i.e., a lower curvature bound – on the sectional curvatures of M. The comparison spaces are always the 2–dimensional simply connected spaces of constant curvature!

5. Counting the total curvature of a compact surface

Suppose that we partition a given compact surface M without boundary into geodesic triangles such that two triangles have at most one (and if so, then the full) edge in common. The total curvature of the surface is then the sum of the total curvatures of the triangles. That sum turns out to be purely combinatorial:

If the number of vertices in the triangulation is denoted by v, the number of edges by e, and the number of faces (i.e., the number of triangles) by f, then the combination: $\chi(M) = v - e + f$ is called the Euler characteristic of the triangulation, and this number measures the total curvature of the surface:

THEOREM 10 (Gauss–Bonnet, [11]).

$$\int_M K\,dA = \sum_{j=1}^{f} \int_{\Delta_j} K\,dA \tag{5.1}$$

$$= 2\pi v - f\pi \tag{5.2}$$

$$= 2\pi(v - e + f) \tag{5.3}$$

$$= 2\pi \cdot \chi(M). \tag{5.4}$$

Proof. Let Ω_j denote the j'th triangle in the family of f triangles covering the surface, and let A_j, B_j and C_j denote the corresponding angles of that triangle. According to Corollary 6 we then have:

$$\int_M K\,dA = \sum_{j=1}^{f} (A_j + B_j + C_j) - \sum_{j=1}^{f} 3\pi + \sum_{j=1}^{f} 2\pi. \tag{5.5}$$

The first sum on the right-hand side is $2\pi v$, since the sum of the angles appearing at any given vertex is 2π. The second sum is $3\pi f$, but $3f = 2e$, because $3f$ is the total number edges counted twice. The third sum is clearly $2\pi f$. In total we get $2\pi v - 2\pi e + 2\pi f$, which is $2\pi \cdot \chi(M)$ as claimed. □

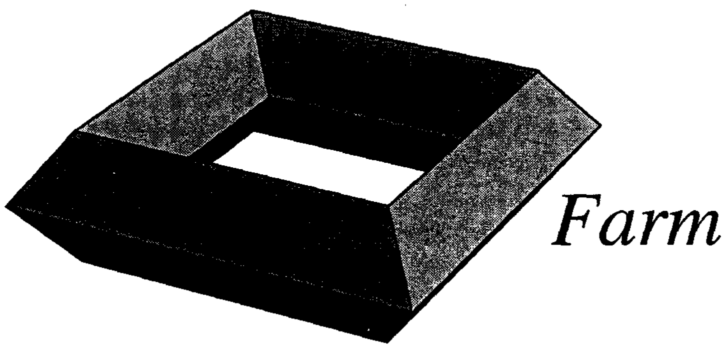

Figure 5. The farm has Euler characteristic $\chi = 0$ (since $v = 20$, $e = 40$ and $f = 20$)

5.1. Polygonalizations. Theorem 10 remains valid if instead of a triangulation we use a partition consisting of simply connected geodesic *polygons* and count the vertices, edges and faces of the 'polygonalization' instead. Namely, each polygon may be triangulated by introducing more vertices, edges and faces, but in doing so, the sum $v - e + f$ is not changed! (For example: A four-gon may be triangulated by adding 1 new vertex in the middle. This gives 3 new faces and 4 new edges, and indeed $1-4+3 = 0$.) The total curvature is clearly also unaffected by this alternative partitioning, and thus the generalized theorem is proved.

5.2. Deformations. Furthermore, if we deform the surface by a piecewise diffeomorphism, i.e., by stretching and warping – in fact, even if we concentrate all the curvature into the vertices of a given polygonalization in such a way that the numbers v, e, f and hence $\chi(M)$ do not change, then we recover Eulers Theorem for convex polyhedra and for polyhedra of higher genus: The total curvature of any sphere is 4π and therefore the Euler characteristic of every polyhedron obtained by deforming a sphere is 2. The information in the Gauss–Bonnet Theorem may of course also be used the other way around: The Euler characteristic of a "Farm" is 0, see Figure 5, and therefore the total curvature of every surface obtained by deforming a torus is 0. The Euler characteristic of the "Estate" is -2, and therefore the total curvature of every surface obtained by deforming a double torus (i.e., the connected sum $T^2 \# T^2$) is -4π, see Figure 6.

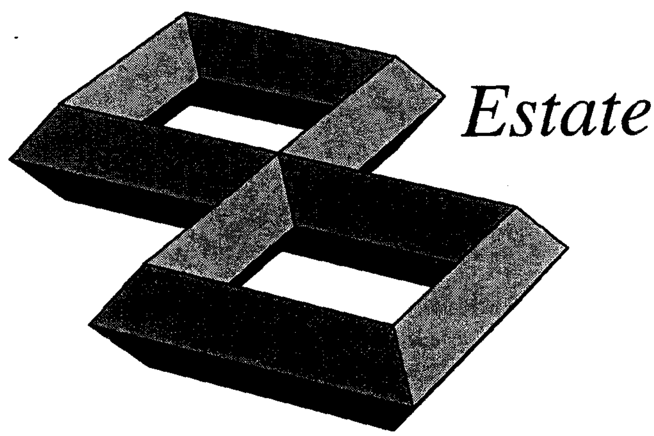

Figure 6. The estate has Euler characteristic $\chi = -2$ (since $v = 39$, $e = 82$ and $f = 41$)

6. A "Biophysics" Corollary

In a living cell there are a lot of membranes formed by so called lipid bilayers. If we model these bilayers by regular 2-dimensional surfaces, their Gauss curvature is bounded in numerical value and also the area is naturally bounded. The purpose of some of these membranes is to form fully 2D highways for the transportation of proteins from one location in the cell to almost any other place in the cell. Therefore these membranes are shaped with a high degree of topological complexity – in other words, their Euler characteristic is very negative. It cannot be arbitrarily negative, however, as the following corollary to the Gauss–Bonnet theorem states.

COROLLARY 11. *A compact surface with bounded curvature cannot be arbitrarily topologically complicated if it also has bounded area or bounded diameter.*

Proof. Suppose $|K| \leq 1$. Then

$$|\chi(M)| \leq \frac{1}{2\pi}\int_M |K| dA \tag{6.1}$$

$$\leq \int_0^{\mathrm{diam}(M)} \sinh(r) dr \tag{6.2}$$

$$= \cosh(\mathrm{diam}(M)) - 1. \tag{6.3}$$

Therefore, if $\mathrm{diam}(M) \leq \mathrm{Areacosh}(1+q)$, then $|\chi(M)| \leq q$. □

Within the area of this observation we must also mention a yet very open problem, namely the variational problem for bio-membranes:

PROBLEM 12 (Canham–Helfrich–Willmore). *Given 3 positive (structure) constants λ_1, λ_2 and λ_3. Minimize the following functional F on the set of all smooth, compact surfaces M^2 in $\mathbb{R}^3$ which bound a fixed volume V:*

$$F(M^2) = \lambda_1 \cdot \mathrm{Area}(M) + \lambda_2 \cdot \int_M H^2 dA + \lambda_3 \cdot \chi(M),$$

where H denotes the mean curvature of M.

7. Curvature, Diameter and Area

PROBLEM 13 (Alexandrovs Isodiametric Problem). *Consider a convex surface M without boundary in $\mathbb{R}^3$. Assume that* $\mathrm{diam}(M) = \pi$. *What is the largest area that M can have under these conditions, and which surface(s) realize(s) the maximum?*

The sphere does not solve this problem. Indeed, consider the limit space of the sequence of deformations shown in Figure 7, namely the *double disk* of diameter π, i.e., the surface obtained by identifying the boundaries of two disks of radius $\pi/2$. The sphere has area 4π, whereas the double disk with diameter π has area $2\pi \cdot (\pi/2)^2 = \pi^3/2 > 4\pi$.

It has been conjectured by Alexandrov in [1, p. 417], that the double disk is actually the solution to the isodiametric problem. This conjecture is still open. Several interesting ramifications of the problem are found in [28].

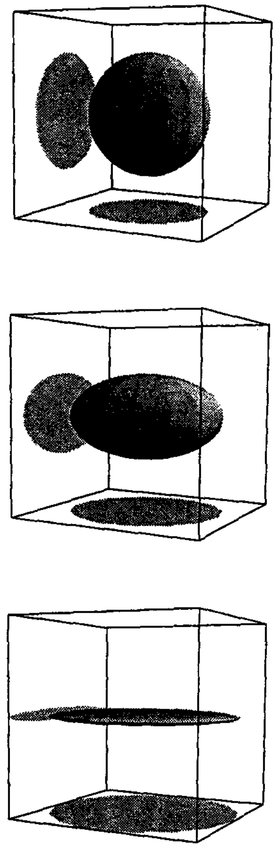

Figure 7. A collapse of a unit sphere to a double disk preserving convexity and diameter π

8. The recognition Program

It is apparent that information about the most basic geometric invariants such as curvature, diameter and volume alone does not suffice to characterize Riemannian manifolds in general.

For this reason we have introduced and investigated in [25] new families of metric invariants such as q-extents and q-packing radii, which seem to pick up global shape in the most general sense, not only for Riemannian manifolds but for metric spaces in general.

In particular, when the spaces in question carry a good notion of curvature, then a set of well chosen conditions on a few metric invariants will balance each other in such a way, that the manifold is recognizable – either topologically or isometrically, i.e., up to homeomorphism, diffeomorphism or isometry. The full recognition program is of course then mainly concerned with the enlargening of the set of manifolds that can be so recognized.

The following theorems are by now classical and celebrated results of this recognition type:

THEOREM 14 (Rauch–Berger–Klingenberg Diameter Theorem, [10]). *Let M^n be a complete simply connected Riemannian manifold whose sectional curvature satisfies*

$$1 \leq \sec(M) \leq 4. \tag{8.1}$$

Then either M^n is homeomorphic to the sphere $\mathbb{S}^n$ or M^n is a compact rank one symmetric space.

THEOREM 15 (Bonnet–Myers). *Let M^n be a Riemannian manifold with* $\sec(M) \geq 1$. *Then* $\operatorname{diam}(M) \leq \pi$.

THEOREM 16 (Grove–Shiohama Diameter Sphere Theorem). *Let M^n be a Riemannian manifold with* $\sec(M) \geq 1$ *and* $\operatorname{diam}(M) > \pi/2$. *Then M is homeomorphic to the sphere $\mathbb{S}^n$.*

THEOREM 17 (Toponogov's Maximal Diameter Theorem). *Let M^n be a Riemannian manifold with* $\sec(M) \geq 1$ *and* $\operatorname{diam}(M) = \pi$. *Then M is isometric to the unit sphere $\mathbb{S}^n_1$.*

Proof of the Maximal Diameter Theorem. We let γ denote a geodesic segment connecting two diametrically oppsosite points p and q, i.e., Length$(\gamma) = \pi$. Let $x \in M$ and let r denote the point of γ closest to x. Then $\pi \leq d(p,x)+d(x,q)$ by the triangular inequality, but also $d(p,x) \leq d^*(p^*,x^*)$ and $d(x,q) \leq d^*(x^*,q^*)$ by the Hinge Comparison Theorem applied to the two hinges with common vertex r. In the Comparison sphere we have $d^*(p^*,x^*)+d^*(x^*,q^*) = \pi$, whence $d(p,x)+d(x,q) = \pi$, so that $d(p,x) = d^*(p^*,x^*)$ and $d(x,q) = d^*(x^*,q^*)$. But then from the rigidity part of the Hinge Comparison Theorem the triangles $\Delta_{p,x,r}$ and $\Delta_{q,x,r}$ both are isometric to their respective comparison triangles with constant curvature $K = 1$. By moving the point x and the diameter–realizing segment γ in M we thus construct a global isometry between M and $\mathbb{S}^n_1$. □

It is clear that to proceed in this programme, it is necessary to introduce many more metric invariants whose combined interactions then hopefully will mold more and more complicated topologies and geometries (if not all?).

We envision that the techniques of (finite) distance geometry as developed by Menger, Schoenberg and Blumenthal in the 30'ies in conjunction with the relatively new powerful tools from the comparison geometric study of Alexandrov spaces, will provide further families of such invariants.

For example, in [30] we have introduced the invariant notion of *strictly negative type* for finite metric spaces and show that this property implies unique realization of the so called infinity-extent (i.e., of the transfinite diameter). Finite metric spaces that have this property include all trees and all finite subsets of the Euclidean spaces. We show that the strictly negative type finite subspaces of spheres are precisely those which do not contain two or more pairs of antipodal points.

To be specific about the relevance of the notion of q-extents mentioned above we briefly recall the precise definition and some of the results from [25]:

Definition 18. The q-extent, $\mathrm{xt}_q(X)$ of a compact metric space X is the *maximal average distance* between the points in q-tuples in X:

$$\mathrm{xt}_q(X) = \max_{x_1, \ldots, x_q} \mathrm{xt}_q(x_1, \ldots, x_q),$$

where $\mathrm{xt}_q : X^q \longmapsto \mathbb{R}$ is the *q-extent function*:

$$\mathrm{xt}_q(x_1, \ldots, x_q) = \binom{q}{2}^{-1} \cdot \sum_{i<j} \mathrm{dist}(x_i, x_j).$$

Any q-tuple $(x_1, \ldots, x_q) \in X^q$, which realizes the q-extent of X is called a *q-extender* of X.

THEOREM 19 (Fary, Nielsen, Grove and M., [25]). *For all integers $n \geq 1$ and $q \geq 2$ we have*

$$\mathrm{xt}_q(\mathbb{S}^n_1) = \mathrm{xt}_q(\mathbb{S}^1_1) = \mathrm{xt}_q([0, \pi]). \tag{8.2}$$

Those points of a q-extender, which do not appear in antipodal pairs will all lie on a great circle in such a way that they realize an $\mathbb{S}^1$- extender on that circle. In particular, if q is even, then every q-extender consists of pairs of antipodal points.

THEOREM 20 (Grove and M., [25]). *There exists a positive function $\varepsilon(n)$ such that every Riemannian manifold M^n with $\mathrm{curv}(M^n) \geq 1$ and $\mathrm{xt}_{n+1}(M^n) \geq \frac{\pi}{2} - \varepsilon(n)$ is homeomorphic to the sphere $\mathbb{S}^n$ or diffeomorphic to the real projective space $\mathbb{R}P^n = \mathbb{S}^n/\mathbb{Z}_2$. If $\mathrm{xt}_{n+1}(M^n) = \frac{\pi}{2}$ and $\mathrm{diam}(M) = \frac{\pi}{2}$, then M is isometric to $\mathbb{R}P^n_1$.*

For other recent geometric topological applications of the q-extents see e.g. [48], [32].

9. The Real Projective Plane Revisited

In order to illustrate a very powerful alternative way of "counting" the Euler characteristic and hence the total curvature of a surface, we will briefly consider an immersion of the real projective plane into $\mathbb{R}^3$, which is due to W. Boy, see [7] and [2].

According to the Morse theory applied to compact surfaces (see e.g. [13], we only need to count the maxima (*max*), minima (*min*) and saddle points (*saddle*) for a height function (with only nondegenerate critical points) on the surface. The Euler characteristic of the surface is then $\chi =$ *min - saddle + max.*

We do this counting by computer for the case at hand, simply by displaying the level curves for the canonical height function of Figure 8.

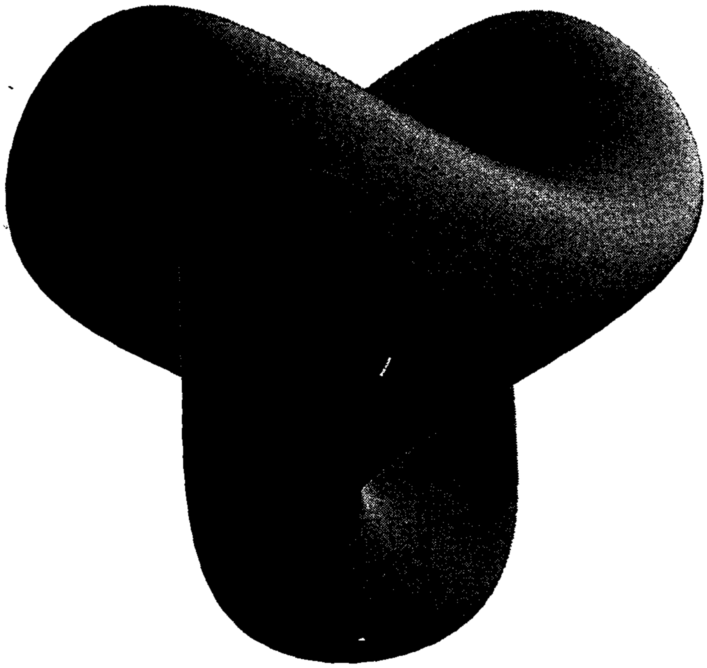

Figure 8. Boy's surface immersed into 3-space

Although the surface is quite convoluted, as seen in Figure 9, the array of 16 consecutive level curves shown in Figure 10 reveal quite clearly that the height function of the surface has: 1 minimum, 2 saddle points (cf. also Figure 11) and 2 maxima.

Hence the Euler characteristic is $1-2+2 = 1$, which shows that Boy's surface is indeed topologically equivalent to $\mathbb{R}P^2$.

10. A Cosmological Application of Comparison Geometry

In this final section we wish to show how the principles of comparison constructions a'la Alexandrov–Toponogov may also be applied to obtain large scale information about the structure of Lorentzian manifolds. Such manifolds are general relativistic models of our universe considered as a space-time entity.

A *line* in a Riemannian manifold is a complete geodesic with the property that it minimizes the distance between *any* pair of its points. In other words, a line γ may be thought of as a minimal connection between two "antipodal points" $\gamma(\infty)$ and $\gamma(-\infty)$ in the (necessarily) non compact manifold.

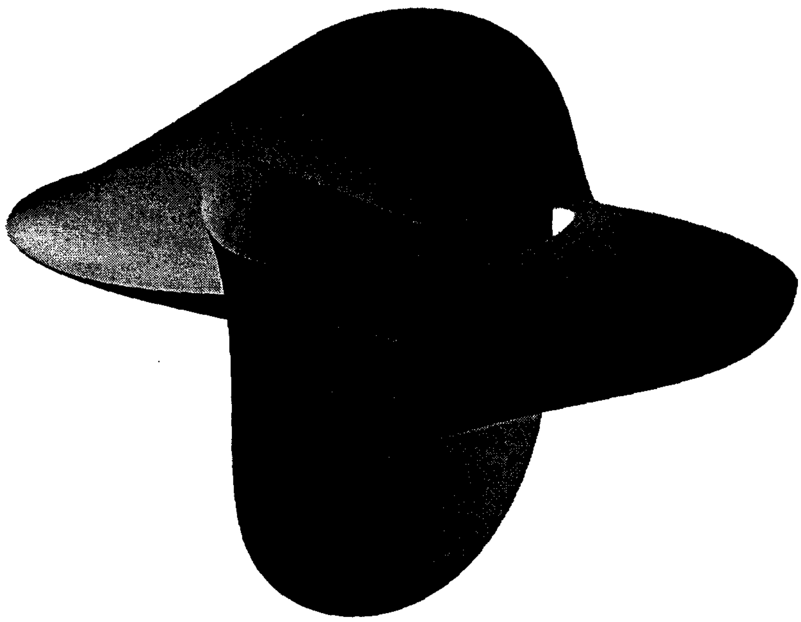

Figure 9. A look into Boy's surface "from behind" the previous figure

The following classical result may now be proved along the very same line of reasoning as was in use for the Maximal Diameter Theorem.

THEOREM 21. (Cohn-Vossen). *Let F be a surface which satisfies the following conditions:*
(o) F is geodesically complete.
(i) F has nonnegative Gauss curvature everywhere.
(ii) F contains a geodesic line.

Then F is a generalized cylinder.

J. Cheeger and D. Gromoll proved in 1971 the generalization of this result to arbitrary dimension and with the assumption of nonnegative Ricci curvature. It was then conjectured by S. T. Yau in 1982, that the corresponding result should hold for general relativistic space–times, i.e., for Lorentzian manifolds. The conjecture was proved by R. Newman in 1990, see [41]. We also refer to [3, Chapter 14], for a nice account of the full history of this interesting theorem.

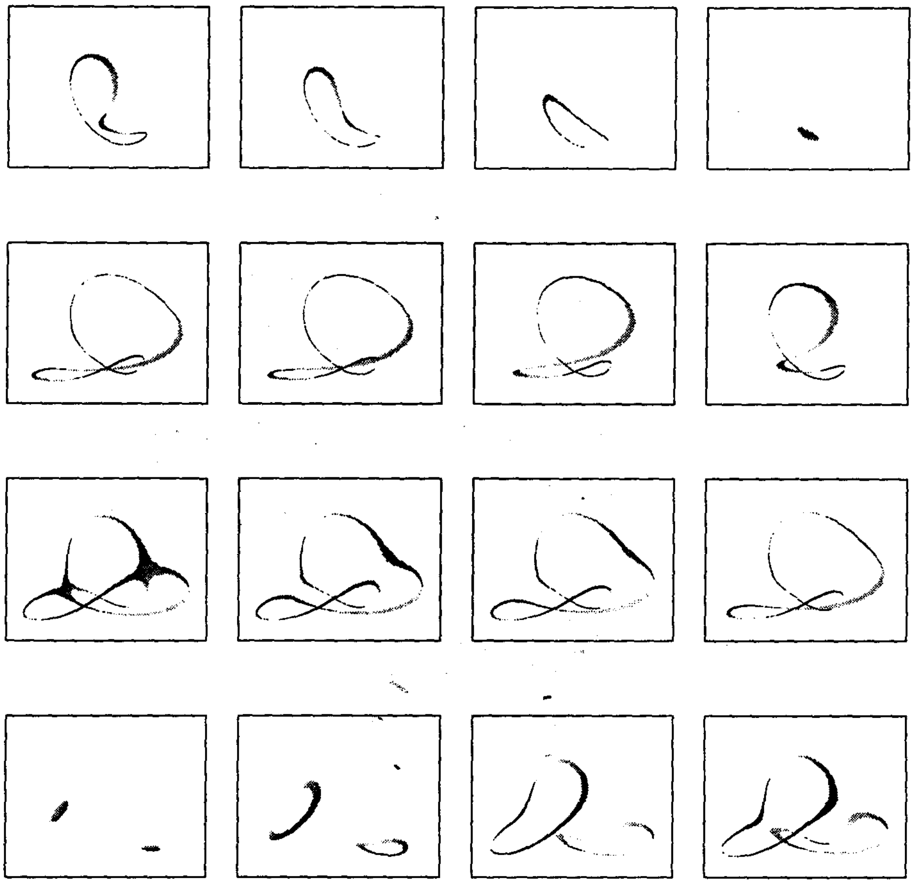

Figure 10. Consecutive level curves for the height function

THEOREM 22 (A "Cosmological" Corollary). *Let M be a space time which satisfies the following conditions:*
(o) M is timelike geodesically complete.
(i) M has nonnegative timelike Ricci curvature everywhere.
(ii) M contains a timelike line.

Then M is a generalized cylinder, i.e., a static space-time.

10.1. A final remark. Since our present universe is *definitely not static* (according to the Hubble Expansion Law or just by plain Observation), the Cosmological Corollary may be considered as a singularity theorem (taking the existence of a line and the nonnegative Ricci curvature for granted): It determines the existence of at least one geodesic which is incomplete in the sense that it cannot

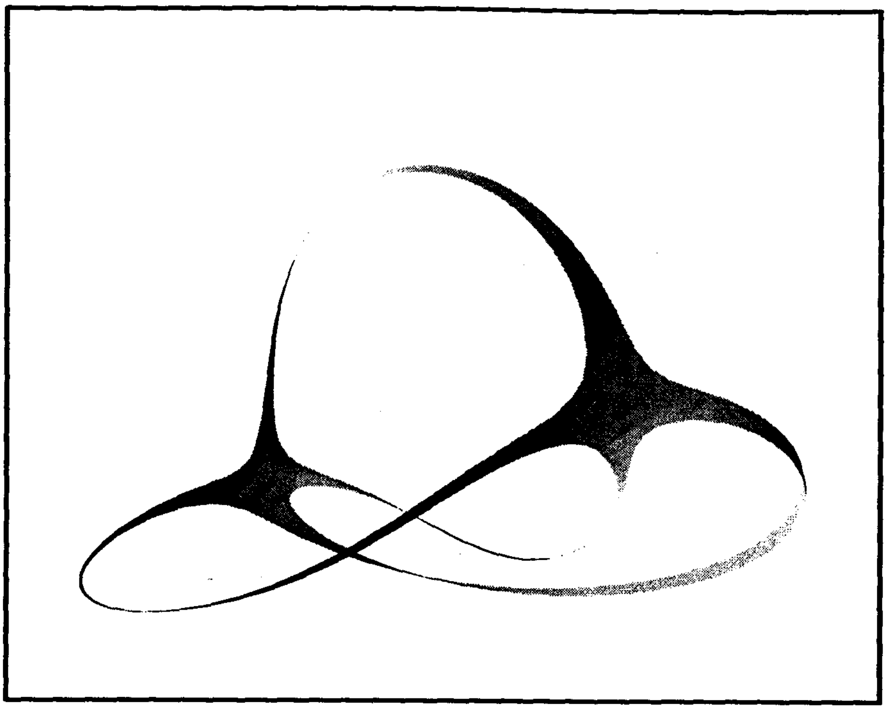

Figure 11. The two saddle points of the height function on Boy's surface

be extended past some finite value of its arclength (eigen time). In other words it must hit into a singularity. Such a singularity could e.g. be a black hole. So, in order to end the present geometric tour on a note which is both prosaic and poetic, let us note that such a black hole seems indeed to be hiding inside the galaxy M87 (in Virgo !) as observed by the Hubble Space Telescope, see [37].

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