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THE SPECTRAL GEOMETRY OF RIEMANNIAN SUBMERSIONS

Peter B. Gilkey†, John V. Leahy and **Jeong Hyeong Park‡**

Abstract. We study the spectral geometry of a Riemannian submersion $\pi : Z \to Y$. We give necessary and sufficient conditions that π^* preserve the eigenforms of the Laplacian. We show that if the pull-back of an eigenform is an eigenform, then the eigenvalue can only increase. If G is a compact, connected Lie group with $H^1(G;\mathbb{R}) \neq 0$, we give examples of principal G bundles over homogeneous manifolds where the pull-back of an eigenform from the base is an eigenform on the total space with different eigenvalue.

§0 Introduction

0.1 Mathematical Physics. Bérard Bergery and Bourguignon [2] discuss the Laplacian of a Riemannian submersion and provide an application to quantum physics. They note: "Recently, there has been a renewed interest in classical physics for non-bijective canonical transformations (see [3, 26]). This very general expression should not be taken literally, but more in the sense that certain interesting maps between configuration spaces turn out to be non-linear and non-bijective. From a mathematician's point of view, these maps are in fact extremely nice (namely, coverings or Hopf fibrations in the examples that we detail later). When going to the quantum level, one has to describe how the spectrum of the quantum operators are related. Once more, the quantum operators are not the most general operators, but very natural ones related to the Riemannian geometry of the situation (for example the Laplace operator of a Riemannian metric plus a potential for the energy)."

M. Boiteux [3] studies the Coulumb potential in two and three dimensions and shows that nonbijective transformations require a fiber bundle formulation of mechanics. He shows that the Hopf fibration leads to an inverse harmonic oscillator

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problem. Boiteux notes "in quantum mechanics, those transformations connect operators with different spectra which as such can not be deduced from one another by unitary transformations." Kibler and Négada [23] extend the work of Boiteux to discuss the Kustaanheimo-Stiefel transformation for the Hydrogen atom and to discuss the Stark and Zeeman effects in hydrogen ions. For other related work on non-bijective canonical transformations, we refer to Asorey, Carinena, and Ibort [1], Cerdeira [5], Dehghani and Sobouti [6], Gracia-Bondía [19], Kibler Ronveaux and Négadi [24], and to Kibler and Winternitz [25]. We also refer to Euguchi et al [7] for a general discussion of gauge theory from the point of view of mathematical physics.

In these brief notes, we will present some results concerning the spectral geometry of Riemannian submersions. Despite the fact that much of the motivation is from mathematical physics, we shall not assume any knowledge of the relevant physics and shall concentrate on the mathematical underpinnings of the theory.

0.2 Notational conventions. We assume that all manifolds are compact, connected, smooth, without boundary, and Riemannian. Let $\pi : Z \to Y$ be a smooth surjective map where $\dim(Y) < \dim(Z)$. We say that π is a *submersion* if pushforward $\pi_* : T_zZ \to T_{\pi z}Y$ is surjective for all $z \in Z$. Let $X := \pi^{-1}(y_0)$ be the *fiber* of π over the basepoint y_0 of Y. If π is a submersion, then we can find a neighborhood $\mathcal{O}_y$ of any point $y \in Y$ so that $\pi^{-1}(\mathcal{O}_y) = X \times \mathcal{O}_y$ and so that $\pi(x, y) = y$ is projection on the second factor; this means that π defines a *fiber bundle.* Let $\mathcal{V} := \ker \pi_*$ and $\mathcal{H} := \mathcal{V}^\perp$. If π is a submersion, then $\mathcal{V}$ and $\mathcal{H}$ are smooth complementary distributions of TZ called the vertical and horizontal distributions. Furthermore $\pi_* : \mathcal{H}_z \to T_{\pi z}Y$ is an isomorphism. If $\pi_* : \mathcal{H}_z \to T_{\pi z}Y$ is an isometry for all z, then π_* is called a *Riemannian submersion.*

We shall use capital letters for tensors on Y and lower case letters for tensors on Z. Let $\rho_\mathcal{V}$ and $\rho_\mathcal{H}$ be orthogonal projection on $\mathcal{V}$ and $\mathcal{H}$. Let $\mathcal{V}^*$ and $\mathcal{H}^*$ be the dual co-distributions of T^*Z. We shall use indices i, j, and k to index local orthonormal frames $\{e_i\}$, and $\{e^i\}$ for $\mathcal{V}$ and $\mathcal{V}^*$; we shall use indices a, b, and c to index local orthonormal frames $\{f_a\}$, $\{f^a\}$, $\{F_a\}$, and $\{F^a\}$ for $\mathcal{H}$, $\mathcal{H}^*$, TY, and T^*Y. We shall adopt the Einstein convention and sum over repeated indices.

Let $\operatorname{ext}(\xi)$ and $\operatorname{int}(\xi)$ be exterior and interior multiplication by a covector ξ. We extend int and ext to be ring-homomorphisms; thus $\operatorname{int}(\xi_1 \wedge \xi_2) := \operatorname{int}(\xi_1)\operatorname{int}(\xi_2)$. Let Γ be the Christoffel symbols of the Levi-Civita connection. Let

$$\begin{aligned} &\theta := -g_Z([e_i, f_a], e_i) f^a = {}^Z\Gamma_{iia} f^a, \\ (0.3) \qquad &\omega_{abi} := g_Z(e_i, [f_a, f_b])/2 = ({}^Z\Gamma_{abi} - {}^Z\Gamma_{bai})/2, \\ &\mathcal{E} := \omega_{abi} \operatorname{ext}_Z(e^i) \operatorname{int}_Z(f^a) \operatorname{int}_Z(f^b); \end{aligned}$$

θ is the unnormalized mean curvature co-vector of the fibers of π, ω is the curvature of the horizontal distribution, and $\mathcal{E}$ is an endomorphism of the exterior algebra. The fibers of π are minimal $\iff \theta = 0 \iff \pi$ is a harmonic map. The horizontal distribution $\mathcal{H}$ is integrable $\iff \omega = 0$. Pullback π^* defines a natural map from

$C^\infty Y$ to $C^\infty Z$. Let $\Delta^p = d\delta + \delta d$ be the Laplacian on the space of smooth p forms. The eigenvalues of Δ^p are non-negative. For $\lambda \geq 0$ and $\mu \geq 0$, let $E(\lambda, \Delta_Y^p)$ and $E(\mu, \Delta_Z^p)$ denote the eigenspaces of the Laplacians.

0.4 THEOREM. *Let $\pi : Z \to Y$ be a Riemannian submersion.*

(1) $\Delta_Z^p \pi^* - \pi^* \Delta_Y^p = \{d_Z(\text{int}_Z(\theta) + \mathcal{E}) + (\text{int}_Z(\theta) + \mathcal{E}) d_Z\}\pi^*$.

(2) *If $0 \neq \Phi \in E(\lambda, \Delta_Y^p)$ and if $\pi^*\Phi \in E(\mu, \Delta_Z^p)$, then $\lambda \leq \mu$.*

(3) *Let $p = 0$. If $0 \neq \Phi \in E(\lambda, \Delta_Y^0)$ and if $\pi^*\Phi \in E(\mu, \Delta_Z^0)$, then $\lambda = \mu$. The following conditions are equivalent*

i) $\Delta_Z^0 \pi^* = \pi^* \Delta_Y^0$.

ii) *The fibers of π are minimal.*

(4) *If $1 \leq p \leq \dim Y$, then the following conditions are equivalent:*

i) $\Delta_Z^p \pi^* = \pi^* \Delta_Y^p$.

ii) $\forall \lambda \geq 0, \exists \mu(\lambda) \geq 0$ *so* $\pi^* E(\lambda, \Delta_Y^p) \subset E(\mu(\lambda), \Delta_Z^p)$.

iii) *The fibers of π are minimal and $\mathcal{H}$ is integrable.*

Theorem 0.4 (1) for $p = 0$ and Theorem 0.4 (3) was proved by Watson [32]; Theorem 0.4 (1) for $p > 0$ and the equivalence of (i) and (iii) in Theorem 0.4 (4) was proved by Goldberg and Ishihara [18]. The remaining assertions of Theorem 0.4 were proved by Gilkey, Leahy, and Park [12] and by Gilkey and Park [14]. We note that Furutani [10, 11] proved that maps which intertwine elliptic pseudo-differential operators are necessarily Riemannian submersions. Fuglede [8, 9], and Ishihara [22] have characterized maps which preserve local harmonic functions. Bergery and Bourguignon [2] gave a careful discussion of the relationship between the complete spectrum of Δ_Y^0 and Δ_Z^0 if the fibers of π are totally geodesic. We also refer to Burstall [4], Gudmundsson [20], Park [29], and Watson [33] for related work on this subject; Gudmundsson [21] has compiled an excellent bibliography of harmonic morphisms which contains additional related references.

0.5 The Bochner Laplacian. Let V_Y be a smooth vector bundle over Y which is equipped with a positive definite fiber metric and a Riemannian connection ∇. Use the connection ∇ on V_Y and the Levi-Civita connection on the tangent bundle to define the second covariant derivative

$$\nabla^2 : C^\infty(V_Y) \to C^\infty(V_Y \otimes T^*M \otimes T^*M).$$

The *Bochner Laplacian* is defined by $D_Y := -\operatorname{Tr}(\nabla^2)$. It is a non-negative operator which arises in many contexts. For $p > 0$, the Weitzenböch formulas show that the difference between the Bochner Laplacian and the Laplacian Δ^p on the space of smooth p forms is given by curvature; the Lichnerowicz formula shows that the difference between the Bochner Laplacian and the spin Laplacian on the space of smooth spinors is given by the scalar curvature.

If $\pi : Z \to Y$ is a Riemannian submersion, we can give the pull-back bundle V_Z the pull-back connection and pull-back metric; we use these structures to define the Bochner Laplacian D_Z. Pull-back induces a natural map $\pi^* : C^\infty(V_Y) \to C^\infty(V_Z)$. The following result was proved by Gilkey and Park [17]:

0.6 THEOREM. *Let $\pi : Z \to Y$ be a Riemannian submersion. Let D_Y be the Bochner Laplacian over Y and let D_Z be the induced Bochner Laplacian over Z. If $0 \neq \Phi \in E(\lambda, D_Y)$ and if $\pi^*\Phi \in E(\mu, D_Z)$, then $\lambda = \mu$. The following assertions are equivalent*

(1) $D_Z\pi^* = \pi^* D_Y$.
(2) *The fibers of π are minimal.*

The scalar Laplacian Δ_M^0 is defined by the flat connection on the trivial line bundle over M. Thus Theorem 0.4 (3) is a special case of Theorem 0.6.

0.7 Principal bundles, sphere bundles, and flat SL submersions. We say $\pi : P \to Y$ is a *principal G bundle* if G is a compact Lie group which acts on P without fixed points and if π is the projection to the quotient P/G. We choose a bi-invariant metric on G. If the metric on P is G invariant, if the fibers of π are isometric to G, and if π is a Riemannian submersion, then $\pi : P \to Y$ is a Riemannian principal G bundle. This implies the fibers of π are totally geodesic.

Let $\pi : V \to Y$ be a real vector bundle over Y. We assume V is equipped with a smooth positive definite fiber metric and let $S(V)$ be the associated unit sphere bundle. Let ${}^V\nabla$ be a Riemannian connection on V. We use ${}^V\nabla$ to split the tangent space of the total space of V into horizontal and vertical distributions and to define a Riemannian metric on V. We restrict this metric to the sphere bundle $S(V)$; the induced projection $\pi_S : S(V) \to Y$ is a Riemannian submersion with totally geodesic fibers.

We say that a submersion $\pi : Z \to Y$ is *flat* if the horizontal distribution is integrable. This means that the transition functions of the fibration can be chosen to be locally constant, i.e. are flat. We say a flat submersion π has structure group SL if the transition functions can be chosen to preserve some volume element on the fibers or equivalently if there exists a measure ν on the fibers so the Lie derivative $\mathcal{L}_H\nu = 0$ for every horizontal lift.

0.8 Rigidity of eigenvalues. Generically, the pull back of an eigenform will not be an eigenform; it is quite a special situation when $0 \neq \Phi \in E(\lambda, \Delta_Y^p)$ and $\pi^*\Phi \in E(\mu, \Delta_Z^p)$. We say *eigenvalues change* if $\lambda \neq \mu$. Theorem 0.4 shows that eigenvalues can not change if $p = 0$. Furthermore, if $p > 0$ and if all the eigenforms are preserved, then eigenvalues can not change. There are other circumstances under which even a single eigenvalue can not change; the eigenvalu es are rigid. Assertion (1) of the following theorem was proved in Gilkey, Leahy, a nd Park [13], assertion (2) was proved in Gilkey, Leahy, and Park [12], assertion (3) was proved in Gilkey and Park [15], and assertion (4) was proved in Park [30]. Let $H^k(M;\mathbb{R})$ be the de Rham cohomology groups of a manifold M.

0.9 THEOREM. *Let $\pi : Z \to Y$ be a Riemannian submersion, let $0 \neq \Phi \in E(\lambda, \Delta_Y^p)$ and let $\pi^*\Phi \in E(\mu, \Delta_Z^p)$.*

(1) *If $\pi : Z \to Y$ is flat with structure group SL, then $\lambda = \mu$*
(2) *If $\pi : Z \to Y$ is a principal G bundle with $H^1(G; \mathbb{R}) = 0$, then $\lambda = \mu$.*
(3) *If $\pi : Z \to Y$ is a sphere bundle of fiber dimension $r \geq 2$, then $\lambda = \mu$.*
(4) *If the fibers of π are minimal and if $p = 1$, then $\lambda = \mu$.*

The eigenvalue $\lambda = 0$ is distinguished; the Hodge de Rham theorem permits us to identify $E(0, \Delta_M^p)$ with the cohomology groups $H^p(M; \mathbb{R})$. Assertion (1) under the additional assumption that the fiber was connected in the following theorem was proved in Park [30]; assertion (2) was proved in Gilkey and Park [16]. The crucial point is that there is no condition on the metric; we do *not* assume the submersion is Riemannian in the following theorem.

0.10 THEOREM. *Let $\pi : Z \to Y$ be a submersion, let $0 \neq \Phi \in E(0, \Delta_Y^p)$ and let $\pi^*\Phi \in E(\mu, \Delta_Z^p)$.*

(1) *If $p = 1$, then $\mu = 0$.*
(2) *If the fiber of π is an even dimensional sphere, then $\mu = 0$.*

0.11 Submersions where eigenvalues change. Muto [27, 28] gave examples of Riemannian principal S^1 bundles where eigenvalues change. The following essentially follows from calculations of Muto.

0.12 THEOREM. *Let ${}^L\nabla$ be a unitary connection on a complex line bundle over Y with associated curvature 2-form $\mathcal{F} = \mathcal{F}({}^L\nabla)$ and associated principal circle bundle $S = S(L)$. Let $\Phi \in E(\lambda, \Delta_Y^p)$. Assume that $d_Y\Phi = 0$, that $d_Y \operatorname{int}_Y(\mathcal{F})\Phi = 0$, and that $-\operatorname{ext}_Y(\mathcal{F}) \operatorname{int}_Y(\mathcal{F})\Phi = \epsilon\Phi$ for ϵ constant. Then $\pi^*\Phi \in E(\lambda + \epsilon, \Delta_S^p)$.*

We can use this Theorem to construct examples where eigenvalues change.

0.13 THEOREM. *Let Y be a homogeneous manifold with $H^2(Y; \mathbb{R}) \neq 0$. There exists a complex line bundle L over Y and a unitary connection ${}^L\nabla$ on L such that $\epsilon := |\mathcal{F}|^2$ is constant, such that $0 \neq \mathcal{F} \in E(0, \Delta_Y^2)$, and such that $\pi_S^*\mathcal{F} \in E(\epsilon, \Delta_S^2)$.*

This phenomena of changing eigenvalues also appears in complex geometry. Let L be a holomorphic line bundle over a complex manifold Y. We say that L is **positive** if the curvature $\mathcal{F}$ of L is the Kaehler form of a Hermitian metric on Y; there is a possible sign convention which plays no role in our development. For example, the hyperplane bundle H is a positive line bundle over complex projective space $\mathbb{CP}^n$ and the associated metric is the Fubini-Study metric. More generally, if Y is a holomorphic submanifold of $\mathbb{CP}^\nu$ for some ν, then the restriction of H to Y is a positive line bundle over Y and the metric on Y is the restriction of the Fubini-Study metric to Y. Conversely, if Y admits a positive line bundle L, then there exists a holomorphic embedding $\alpha : Y \to \mathbb{CP}^\nu$ for some ν and a positive integer k so that $L^{\otimes k} = \alpha^* H$. We say Y is a Hodge manifold if Y admits a positive line bundle; we may identify the set of Hodge manifolds with the set of smooth algebraic varieties.

0.14 THEOREM. *Let $\mathcal{F}$ be the curvature of a positive line bundle L over a complex manifold Y of real dimension $2n$. Let $\pi : S \to Y$ be the associated S^1 principal bundle. Then $0 \neq \mathcal{F}^p \in E(0, \Delta_Y^{2p})$ and $\pi^*\mathcal{F}^p \in E(p(n+1-p), \Delta_S^{2p})$ for $1 \leq p \leq n$.*

The following result shows that the hypothesis $H^1(G;\mathbb{R}) \neq 0$ in Theorem 0.9 (2) was essential, a special case was proved in Gilkey, Leahy, and Park [12].

0.15 THEOREM. *Let $\Phi \in E(\lambda, \Delta_Y^2)$ for $\lambda \neq 0$ satisfy $|\Phi|^2 = a$ is constant and $d_Y\Phi = 0$. Let G be a compact connected Lie group with $H^1(G;\mathbb{R}) \neq 0$, and let π be projection on the second factor of $P := G \times Y$. For any $\epsilon > 0$, there exists a metric $g_P(\epsilon)$ on P so that $\pi : P \to Y$ is a Riemannian principal G bundle and so that $\pi^*(\Phi) \in E(\lambda + \epsilon^2 a, \Delta_P^2)$.*

0.16 Remark. To see that Theorem 0.15 is non-trivial, we could take Y to be the unit quaternions S^3. If e is any non-vanishing left invariant covector on Y, then $\Phi = de \in E(4, \Delta_{S^3}^2)$ and $|\Phi|^2$ is constant.

0.17 Remark. We can construct examples where eigenvalues change for $p > 2$ by taking appropriate Riemannian products in Theorems 0.12 and 0.15 above. We do not know of any examples where eigenvalues change if $p = 1$; if eigenvalues change for $p = 1$, the fibers can not be minimal by Theorem 0.9 (4). By Theorem 0.4 (3), eigenvalues do not change if $p = 0$.

0.18 Contents. Here is a brief guide to the paper.

§1 The equations of structure (Thm. 0.4 (1)).
§2 Fiber products (Thm. 0.4 (2)).
§3 The Bochner Laplacian (Thms. 0.4 (3) and 0.6).
§4 The Laplacian on p forms (Thm. 0.4 (4)).
§5 Riemannian submersions where eigenvalues are rigid (Thm. 0.9).
§6 Topological considerations (Thm. 0.10).
§7 Riemannian submersions where eigenvalues do change (Thms. 0.12-0.16).

§1 Equations of structure

In this section, we prove Theorem 0.4 (1). Let $\pi : Z \to Y$ be a Riemannian submersion. Let $\{F_a\}$ be a local orthonormal frame for TY and let $\{f_a\}$ be the horizontal lift; $\{f_a\}$ is a local orthonormal frame for $\mathcal{H}$. Let e_i be a local orthonormal frame for $\mathcal{V}$. Let ${}^Y\Gamma$ and ${}^Z\Gamma$ be the Christoffel symbols of the Levi-Civita connections ${}^Y\nabla$ and ${}^Z\nabla$.

1.1 LEMMA. *If f and $\tilde{f}$ are the horizontal lifts of vector fields F and $\tilde{F}$ from Y to Z, then $\pi_*[f, \tilde{f}] = [F, \tilde{F}]$. We have ${}^Z\Gamma_{abc} = \pi^*({}^Y\Gamma_{abc})$, $d_Y = \text{ext}_Y(F^a){}^Y\nabla_{F_a}$, and $\delta_Y = -\text{int}_Y(F^a){}^Y\nabla_{F_a}$.*

Proof. Let $\psi(t)$, $\tilde{\psi}(t)$, $\Psi(t)$, and $\tilde{\Psi}(t)$ be the flows of the vector fields f, $\tilde{f}$, F, and $\tilde{F}$. Since $\pi_* f = F$ and $\pi_*\tilde{f} = \tilde{F}$, $\pi\psi(t) = \Psi(t)\pi$ and $\pi\tilde{\psi}(t) = \tilde{\Psi}(t)\pi$. Let

$$h(t) := \psi(-\sqrt{t})\tilde{\psi}(-\sqrt{t})\psi(\sqrt{t})\tilde{\psi}(\sqrt{t})(z_0)$$
$$H(t) := \pi h(t) = \Psi(-\sqrt{t})\tilde{\Psi}(-\sqrt{t})\Psi(\sqrt{t})\tilde{\Psi}(\sqrt{t})(\pi z_0)$$

for $z_0 \in Z$. As $\dot{h}(0) = [f, \tilde{f}](z_0)$ and $\pi_*\dot{h}(0) = \dot{H}(0) = [F, \tilde{F}](\pi z_0)$; the first assertion now follows. Since π is a Riemannian submersion and since the f_a are the horizontal lifts of the F_a, we may use the first assertion to prove the second by computing:

$$\begin{aligned}
{}^Z\Gamma_{abc} &= \{g_Z([f_a, f_b], f_c) - g_Z([f_b, f_c], f_a) + g_Z([f_c, f_a], f_b)\}/2 \\
&= \{g_Y(\pi_*[f_a, f_b], \pi_* f_c) - g_Y(\pi_*[f_b, f_c], \pi_* f_a) + g_Y(\pi_*[f_c, f_a], \pi_* f_b)\}/2 \\
&= \{g_Y([F_a, F_b], F_c) - g_Y([F_b, F_c], F_a) + g_Y([F_c, F_a], F_b)\}/2 = {}^Y\Gamma_{abc}.
\end{aligned}$$

The following operators are invariantly defined:

$$\begin{aligned}
\tilde{d}_Y &:= \operatorname{ext}_Y(F_a)^Y\nabla_{F_a} : C^\infty(\Lambda^p Y) \to C^\infty(\Lambda^{p+1} Y), \\
\tilde{\delta}_Y &:= -\operatorname{int}_Y(F_a)^Y\nabla_{F_a} : C^\infty(\Lambda^p Y) \to C^\infty(\Lambda^{p-1} Y).
\end{aligned}$$

Furthermore, $d_Y = \tilde{d}_Y$ and $\delta_Y = \tilde{\delta}_Y$ if the metric on Y is flat; these two operators have the same leading symbol. Thus $d_Y - \tilde{d}_Y$ and $\delta_Y - \tilde{\delta}_Y$ are invariantly defined 0^{th} order operators which are linear in the Christoffel symbols; this means that they are endomorphisms of the exterior algebra. Since we can always choose a frame so ${}^Y\Gamma(y_0)$ vanishes at a single point, these 0^{th} differences must vanish identically. □

Since $d_Z\pi^* = \pi^* d_Y$, Theorem 0.4 (1) will follow from the following Lemma.

1.2 LEMMA. *If $\pi : Z \to Y$ is a Riemannian submersion, then*

$$\delta_Z\pi^* - \pi^*\delta_Y = (\operatorname{int}_Z(\theta) + \mathcal{E})\pi^*.$$

Proof. Let $F^A := F^{a_1} \wedge \dots \wedge F^{a_p}$ where $A = (1 \le a_1 < \dots < a_p \le \dim(Y))$ is a multi-index. The $\{F^A\}$ for $|A| = p$ form a local orthonormal frame for $\Lambda^p Y$. Expand $\Phi \in C^\infty \Lambda^p Y$ in the form $\Phi = \Sigma_{|A|=p} \Phi_A F^A$. Since π is a Riemannian submersion, $\operatorname{ext}_Z(f^a)\pi^* = \pi^* \operatorname{ext}_Y(F^a)$ and $\operatorname{int}_Z(f^a)\pi^* = \pi^* \operatorname{int}_Y(F^a)$. Then

$${}^Y\nabla_{F_a}\Phi = F_a(\Phi_A)F^A + {}^Y\Gamma_{abc} \operatorname{ext}_Y(F^c) \operatorname{int}_Y(F^b)\Phi;$$

we may expand ${}^Z\nabla\Phi$ similarly. We use Lemma 1.1 to see:

$$\delta_Z\pi^*\Phi = \delta_Z\pi^*(\Phi_A)f^A$$

$$= -\operatorname{int}_Z(e^i)e_i(\pi^*\Phi_A)f^A \tag{1.3}$$

$$- \operatorname{int}_Z(e^i)^Z\Gamma_{iab} \operatorname{ext}_Z(f^b) \operatorname{int}_Z(f^a)\pi^*\Phi \tag{1.4}$$

$$- \operatorname{int}_Z(e^i)^Z\Gamma_{iaj} \operatorname{ext}_Z(e^j) \operatorname{int}_Z(f^a)\pi^*\Phi \tag{1.5}$$

$$- \operatorname{int}_Z(f^a)f_a(\pi^*\Phi_A)f^A \tag{1.6}$$

$$- \operatorname{int}_Z(f^a)^Z\Gamma_{abc} \operatorname{ext}_Z(f^c) \operatorname{int}_Z(f^b)\pi^*\Phi \tag{1.7}$$

$$- \operatorname{int}_Z(f^a)^Z\Gamma_{abi} \operatorname{ext}_Z(e^i) \operatorname{int}_Z(f^b)\pi^*\Phi. \tag{1.8}$$

Since horizontal covector fields are annihilated by $\text{int}_Z(e^i)$, the terms in (1.3) and in (1.4) vanish. Furthermore in (1.5) we must have $i = j$. By definition, $\theta = -{}^Z\Gamma_{iai} f^a$ so the terms in (1.5) yield $\text{int}_Z(\theta)$. Since ${}^Y\Gamma_{abc} = {}^Z\Gamma_{abc}$, the terms in (1.6) and (1.7) yield $\pi^*\delta_Y$. Note that

$$\begin{aligned} -\text{int}_Z(f^a)\,\text{ext}_Z(e^i)\,\text{int}_Z(f^b) &= \text{ext}_Z(e^i)\,\text{int}_Z(f^a)\,\text{int}_Z(f^b) \\ &= -\,\text{ext}_Z(e^i)\,\text{int}_Z(f^b)\,\text{int}_Z(f^a). \end{aligned}$$

Thus we may anti-symmetrize to see the terms in (1.8) yield

$$\text{ext}_Z(e^i)({}^Z\Gamma_{abi} - {}^Z\Gamma_{bai})/2\,\text{int}_Z(f^a)\,\text{int}_Z(f^b)\pi^* = \mathcal{E}. \quad \square$$

§2 Fiber products

In this section, we use fiber products to prove Theorem 0.4 (2). We adopt the following notational conventions. Let $\pi_U : U \to Y$ and $\pi_V : V \to Y$ be Riemannian submersions with horizontal and vertical distributions $\mathcal{H}_U$, $\mathcal{H}_V$, $\mathcal{V}_U$, and $\mathcal{V}_V$. Let

$$W := \{w = (u, v) \in U \times V : \pi_U(u) = \pi_V(v)\}.$$

We pull back $T(U)$ and $T(V)$ from U and V to $U \times V$ and identify $T(U \times V)$ with $T(U) \oplus T(V)$. We embed $T(W)$ in $T(U \times V)$. Let

$$\begin{aligned} &\pi_W(w) := \pi_U(u) = \pi_V(v) : W \to Y, \\ &\mathcal{V}_W(w) := \mathcal{V}_U(u) \oplus \mathcal{V}_V(v), \\ &\mathcal{H}_W(w) := \{(\xi, \eta) \in \mathcal{H}_U(u) \oplus \mathcal{H}_V(v) : (\pi_U)_*\xi = (\pi_V)_*\eta\}. \end{aligned}$$

We define a new metric on W by requiring that $\mathcal{H}_W$, $\mathcal{V}_U$, and $\mathcal{V}_V$ are orthogonal, that the metrics on $\mathcal{V}_U$ and $\mathcal{V}_V$ are induced from the metrics on U and on V, and that $(\pi_W(w))_* : \mathcal{H}_W(w) \to TY(\pi(w))$ is an isometry. The metric on $\mathcal{H}_W$ differs from the subspace metric by a factor of $1/\sqrt{2}$; the diagonal in a right equilateral triangle has length $\sqrt{2}$. Let $\pi_1(u, v) = u$ and $\pi_2(u, v) = v$. Then $\pi_1 : W \to U$, $\pi_2 : W \to V$, and $\pi_W : W \to Y$ are Riemannian submersions.

Let $f_{a,U}$ and $f_{a,V}$ be the horizontal lifts of F_a with respect to the submersions π_U and π_V. Then $f_{a,W} := f_{a,U} + f_{a,V}$ is the horizontal lift of F_a with respect to π_W; $f_{a,W}$ is also the horizontal lift of $f_{a,U}$ with respect to π_1 and of $f_{a,V}$ with respect to π_2. Let $\{e_{i,U}\}$ and $\{e_{\alpha,V}\}$ be local orthonormal frames for the vertical distributions of π_U and π_V and let $\{e_{i,W}, e_{\alpha,W}\}$ be natural extensions to W. Then $e_{i,W}$ is the horizontal lift of $e_{i,U}$ under π_1 and $e_{\alpha,W}$ is the horizontal lift of $e_{\alpha,V}$ under π_2. We have that $\{e_{i,W},\ e_{\alpha,W}\}$ is a local orthonormal frame for the vertical distribution of π_W, that $\{e_{i,W}\}$ is a local orthonormal frame for the vertical distribution of π_2, and that $\{e_{\alpha,W}\}$ is local orthonormal frames for the vertical distribution of π_1.

2.1 LEMMA.

(1) *We have* $\theta_W = \pi_1^*\theta_U + \pi_2^*\theta_V$.

(2) *We have* $\mathcal{E}_W\pi_W^* = \pi_1^*\mathcal{E}_U\pi_U^* + \pi_2^*\mathcal{E}_V\pi_V^*$.

(3) *If* $\Phi \in E(\lambda, \Delta_Y^p)$, *if* $\pi_U^*\Phi \in E(\lambda + \epsilon_U, \Delta_U^p)$, *and if* $\pi_V^*\Phi \in E(\lambda + \epsilon_V, \Delta_V^p)$, *then* $\pi_W^*\Phi \in E(\lambda + \epsilon_U + \epsilon_V, \Delta_W^p)$.

Proof. We use Lemma 1.1 to prove assertions (1) and (2) by computing:

$$\begin{aligned}
\theta_W &= -\{g_W(e_{i,W}, [e_{i,W}, f_{a,W}]) + g_W(e_{\alpha,W}, [e_{\alpha,W}, f_{a,W}])\}\pi_W^*(F^a)\\
&= -\{g_U(e_{i,U}, [e_{i,U}, f_{a,U}]) + g_V(e_{\alpha,V}, [e_{\alpha,V}, f_{a,V}])\}\pi_W^*(F^a)\\
&= \pi_1^*\theta_U + \pi_2^*\theta_V,\\
\omega_{Wiab} &= g_W(e_{i,W}, [f_{a,W}, f_{b,W}])/2 = g_U(e_{i,U}, [f_{a,U}, f_{b,U}])/2 = \omega_{Uiab},\\
\omega_{W\alpha ab} &= g_W(e_{\alpha,W}, [f_{a,W}, f_{b,W}])/2 = g_V(e_{\alpha,V}, [f_{a,V}, f_{b,V}])/2 = \omega_{V\alpha ab}.
\end{aligned}$$

We use assertions (1) and (2) and Theorem 0.4 (1) to compute:

$$\begin{aligned}
(\epsilon_U + \epsilon_V)\pi_W^*\Phi =& \pi_1^*\{\Delta_U^p\pi_U^* - \pi_U^*\Delta_Y^p\}\Phi + \pi_2^*\{\Delta_V^p\pi_V^* - \pi_V^*\Delta_Y^p\}\Phi\\
=& \pi_1^*\{(\mathrm{int}_U(\theta_U) + \mathcal{E}_U)d_U + d_U(\mathrm{int}_U(\theta_U) + \mathcal{E}_U)\}\pi_U^*\Phi\\
&+ \pi_2^*\{(\mathrm{int}_V(\theta_V) + \mathcal{E}_V)d_V + d_V(\mathrm{int}_V(\theta_V) + \mathcal{E}_V)\}\pi_V^*\Phi\\
=& \{(\mathrm{int}_W(\theta_W) + \mathcal{E}_W)d_W + d_W(\mathrm{int}_W(\theta_W) + \mathcal{E}_W)\}\pi_W^*\Phi\\
=& (\Delta_W^p\pi_W^* - \pi_W^*\Delta_Y^p)\Phi. \quad \square
\end{aligned}$$

We can now complete the proof of Theorem 0.4 (2). Let $\pi : Z \to Y$ be a Riemannian submersion. Let $0 \neq \Phi \in E(\lambda, \Delta_Y^p)$ and let $\pi^*\Phi \in E(\lambda + \epsilon, \Delta_Z^p)$. Let $Z_0 = Z$ and inductively let $Z_n = W(Z_{n-1}, Z_{n-1})$ be the fiber product of Z_{n-1} with itself. Let $\pi_n : Z_n \to Y$ be the associated projection. By Lemma 2.1, $\pi_n^*\Phi \in E(\lambda + 2^n\epsilon, \Delta_{Z_n}^p)$. Since the Laplacian on Z_n is a non-negative operator, $\lambda + 2^n\epsilon \geq 0$. Since this holds for all n, $\epsilon \geq 0$ as desired. $\square$

§3 The Bochner Laplacian

In this section, we prove Theorem 0.4(3) and Theorem 0.6. Let $\pi : Z \to Y$ be a Riemannian submersion. Let $0 \neq \Phi \in E(\lambda, \Delta_Y^0)$ and let $\phi = \pi^*\Phi$. Suppose $\phi \in E(\mu, \Delta_Z^0)$. By Theorem 0.4 (1), $(\mu - \lambda)\phi = \mathrm{int}_Z(\theta)\pi^*d_Y\Phi$. Choose y_0 so $\Phi(y_0)$ is maximal. Then $d_Y\Phi(y_0) = 0$ so $(\mu - \lambda)\phi(z_0) = 0$ where $\pi z_0 = y_0$. By replacing Φ by $-\Phi$ if need be, we may assume the maximal value of Φ is positive and conclude $\lambda = \mu$. If $\theta = 0$, then Theorem 0.4 (1) implies $\Delta_Z^0\pi^* = \pi^*\Delta_Y^0$. Conversely, if this identity holds, then $\mathrm{int}_Z(\theta)\pi^*d_Y = 0$. Since θ is a horizontal co-vector, this implies $\theta = 0$. This completes the proof of Theorem 0.4 (3). $\square$

We generalize this argument to prove Theorem 0.6. We first generalize Theorem 0.4 (1) to the context of the Bochner Laplacian:

3.1 LEMMA. *We have* $D_Z\pi^* - \pi^* D_Y = \text{int}_Z(\theta)\pi^*\nabla_Y$.

Proof. Since the calculations are local, we may assume V is trivial. Decompose $\Phi \in C^\infty(V)$ into its components; thus we have $\Phi = (\Phi_1, ..., \Phi_r)$. Let $\{F_a\}$ be a local orthonormal frame for TY. We define the vector valued derivative of Φ by differentiating the components; this means that $F_a(\Phi) := (F_a(\Phi_1), ..., F_a(\Phi_r))$. Let ${}^Y\mathcal{A}$ be the connection 1-form of ∇_Y; we may expand ${}^Y\mathcal{A} := {}^Y\mathcal{A}_a F^a$ where ${}^Y\mathcal{A}_a$ is an $r \times r$ matrix. We then expand

$$\begin{aligned}
&\nabla_Y\Phi := \Phi_{;a} \otimes F^a, \ \nabla_Y^2\Phi := \Phi_{;ab} \otimes F^a \otimes F^b, \\
&\Phi_{;a} := F_a\Phi + {}^Y\mathcal{A}_a\Phi, \ \Phi_{;ab} := F_b\Phi_{;a} + {}^Y\mathcal{A}_b\Phi_{;a} + {}^Y\Gamma_{bca}\Phi_{;c}, \\
&D_Y\Phi := -(F_a\Phi_{;a} + {}^Y\mathcal{A}_a\Phi_{;a} + {}^Y\Gamma_{aca}\Phi_{;c}).
\end{aligned}$$

Let $\phi = \pi^*\Phi$. Note that ${}^Z\mathcal{A} = \pi^{*Y}\mathcal{A}$. Thus $\nabla_Z\phi = \pi^*\nabla_Y\Phi$. Since the vertical covariant derivatives $\phi_{;i} = 0$ and since by Lemma 1.1, ${}^Z\Gamma_{abc} = \pi^{*Y}\Gamma_{abc}$, we compute

$$\begin{aligned}
&D_Z\phi = -(f_a\phi_{;a} + {}^Z\mathcal{A}_a\phi_{;a} + {}^Z\Gamma_{aca}\phi_{;c} + {}^Z\Gamma_{ici}\phi_{;c}), \\
&D_Z\pi^* - \pi^*D_Y = -{}^Z\Gamma_{ici}\phi_{;c} = \text{int}_Z(\theta)(\nabla_Z\phi) = \text{int}_Z(\theta)\pi^*\nabla_Y\Phi. \quad \square
\end{aligned}$$

We can now prove Theorem 0.6 using the same argument used to prove Theorem 0.4 (3). Let $0 \neq \Phi \in E(\lambda, D_Y)$ and $\phi \in E(\mu, D_Z)$. Then $(\mu - \lambda)\phi = \text{int}_Z(\theta)\pi^*\nabla_Y\Phi$. Since ∇ is a Riemannian connection, we may take the inner product with ϕ to see

$$(\mu - \lambda)|\phi|^2 = \text{int}_Z(\theta)\pi^*(\nabla_Y\Phi, \Phi) = \text{int}_Z(\theta)\pi^* d|\Phi|^2/2 = (\theta, \pi^* d|\Phi|^2)/2.$$

Choose y_0 so $|\Phi|^2(y_0)$ is maximal. Then $d|\Phi|^2(y_0) = 0$ so $(\mu - \lambda)|\phi|^2(z_0) = 0$ where $\pi z_0 = y_0$. Since $|\phi|^2(z_0) = |\Phi|^2(y_0) \neq 0$, we conclude $\lambda = \mu$.

If $\theta = 0$, Lemma 3.1 implies $D_Z\pi^* = \pi^*D_Y$. Conversely, suppose $D_Z\pi^* = \pi^*D_Y$. Then $\text{int}_Z(\theta)\pi^*\nabla_Y = 0$. Since θ is a horizontal differential form, $\theta = 0$. $\square$

§4 The Laplacian on p forms

In this section, we prove Theorem 0.4 (4). Let $\pi : Z \to Y$ be a Riemannian submersion and let $p > 0$. Theorem 0.4 (1) shows that (4-iii) implies (4-i); it is immediate that (4-i) implies (4-ii). We suppose that (4-ii) holds. This means that for any $\lambda \in \mathbb{R}$ there exists $\epsilon(\lambda)$ so that $\pi^*E(\lambda, \Delta_Y^p) \subset E(\lambda + \epsilon(\lambda), \Delta_Z^p)$. Let $\rho_{\mathcal{H}}$ be orthogonal projection from $\Lambda^p Z$ to $\Lambda^p\mathcal{H}$. If $\Phi \in E(\lambda, \Delta_Y^p)$, we use Theorem 0.4 (1) to see

$$0 = \epsilon(\lambda)(1 - \rho_{\mathcal{H}})\pi^*\Phi = (1 - \rho_{\mathcal{H}})(d_Z(\text{int}_Z(\theta) + \mathcal{E}) + (\text{int}_Z(\theta) + \mathcal{E})d_Z)\pi^*\Phi.$$

Since the span of the eigenspaces $E(\lambda, \Delta_Y^p)$ is dense in $C^\infty \Lambda^p Y$, we have

$$(4.1) \qquad (1-\rho_{\mathcal{H}})(d_Z(\text{int}_Z(\theta)+\mathcal{E})+(\text{int}_Z(\theta)+\mathcal{E})d_Z)\pi^* = 0 \text{ on } C^\infty \Lambda^p Y.$$

Fix a point $z_0 \in Z$ and let $y_0 = \pi z_0$. Choose $F \in C^\infty Y$ so that $F(y_0) = 0$. Let $\xi := dF(y_0)$. Since $\text{int}_Z(\theta) + \mathcal{E}$ is a 0^{th} order operator, we apply equation (4.1) to $F\Phi$ and evaluate at z_0 to see

$$0 = (1-\rho_{\mathcal{H}})\{\text{ext}_Z(\pi^*\xi)(\text{int}_Z(\theta)+\mathcal{E}) + (\text{int}_Z(\theta)+\mathcal{E})\,\text{ext}_Z(\pi^*\xi)\}\pi^*\Phi(y_0).$$

Since $0 = (1-\rho_{\mathcal{H}})\{\text{ext}_Z(\pi^*\xi)\,\text{int}_Z(\theta) + \text{int}_Z(\theta)\,\text{ext}_Z(\pi^*\xi)\}\pi^*$, and since $\mathcal{E}$ always introduces a vertical covector, we conclude

$$0 = \{\text{ext}_Z(\pi^*\xi)\mathcal{E} + \mathcal{E}\,\text{ext}_Z(\pi^*\xi)\}\pi^*.$$

We set $\pi^*\xi = f^c$ to see that for any c we have

$$\begin{aligned}
0 =&\omega_{abi}\{\text{ext}_Z(f^c)\,\text{ext}_Z(e^i)\,\text{int}_Z(f^a)\,\text{int}_Z(f^b) + \text{ext}_Z(e^i)\,\text{int}_Z(f^a)\,\text{int}_Z(f^b)\,\text{ext}_Z(f^c)\}\\
=&\omega_{abi}\,\text{ext}_Z(e^i)\{-\text{ext}_Z(f^c)\,\text{int}_Z(f^a)\,\text{int}_Z(f^b) + \text{int}_Z(f^a)\,\text{int}_Z(f^b)\,\text{ext}_Z(f^c)\}\\
=&\omega_{abi}\,\text{ext}_Z(e^i)\{\text{int}_Z(f^a)\,\text{ext}_Z(f^c)\,\text{int}_Z(f^b) + \text{int}_Z(f^a)\,\text{int}_Z(f^b)\,\text{ext}_Z(f^c)\\
&\quad -\delta_{ac}\,\text{int}_Z(f^b)\}\\
=&\omega_{abi}\,\text{ext}_Z(e^i)\{-\text{int}_Z(f^a)\,\text{int}_Z(f^b)\,\text{ext}_Z(f^c) + \text{int}_Z(f^a)\,\text{int}_Z(f^b)\,\text{ext}_Z(f^c)\\
&\quad -\delta_{ac}\,\text{int}_Z(f^b) + \delta_{bc}\,\text{int}_Z(f^a)\}\\
=&-2\omega_{cbi}\,\text{ext}_Z(e^i)\,\text{int}_Z(f^b).
\end{aligned}$$

Since $p \geq 1$, this implies $\omega = 0$ so $\mathcal{H}$ is integrable.

We now recall a bit of the geometry of Riemannian submersions with integrable horizontal distributions.

4.2 Lemma. *Let X be the fiber of a Riemannian submersion $\pi : Z \to Y$. Assume the horizontal distribution of π is integrable. Then we can find local coordinates $z = (x, y)$ on Z so $\pi(x, y) = y$ and so $ds_Z^2 = g_{ij}(x,y)dx^i \circ dx^j + h_{ab}(y)dy^a \circ dy^b$. If we set $g_X := \det(g_{ij})^{1/2}$, then $\theta = -d_Y \ln(g_X)$.*

Proof. Choose local coordinates $y = (y^a)$ defined on a neighborhood $\mathcal{O}$ of some point y_0 in Y. Let $\tilde{f}_a$ be the horizontal lift of the coordinate vector fields ∂_a^y from Y to Z over $\pi^{-1}(\mathcal{O})$; this is not an orthonormal frame. We use Lemma 1.1 to see $\pi_*[\tilde{f}_a, \tilde{f}_b] = [\partial_a^y, \partial_b^y] = 0$. Since $\mathcal{H}$ is integrable, $[\tilde{f}_a, \tilde{f}_b] \in \mathcal{H}$ so we see $[\tilde{f}_a, \tilde{f}_b] = 0$. Choose local coordinates $x = (x^i)$ for the fiber $X = \pi^{-1}(y_0)$ near z_0 and use the Frobenius theorem to extend x to a system of coordinates $z = (x, w)$ on a neighborhood of z_0 so that $\tilde{f}_a = \partial_a^w$. The projection of the integral curves of the vector fields ∂_a^w are the integral curves of the vector fields ∂_a^y. Therefore $y = \pi(x, w) = w$ and π is projection on the second factor. Since

$\mathcal{V} = TX = \text{span}\{\partial_i^x\}$ is perpendicular to $\mathcal{H} = TY = \text{span}\{\partial_a^w\}$ and since π_* is an isometry from $\mathcal{H}$ to TY, the metric locally has the form given in the first assertion.

If M is a Riemannian manifold with $ds_M^2 = g_{rs}du^r \circ du^s$, let $g_M := \det(g_{rs})^{1/2}$. Then $\Delta_M^0 = -g_M^{-1}\partial_r g_M g^{rs}\partial_s$. Express the metric on Z locally as in the first assertion. Then $\text{int}_Z(\theta)d_Z\pi^*\Phi = \Delta_Z^0\pi^* - \pi^*\Delta_Y^0 = -\text{int}_Z(g_X^{-1}d_Y g_X)d_Z\pi^*\Phi$ for any $\Phi \in C^\infty(Y)$ so $\theta = -d_Y \ln(g_X)$. □

We continue the proof that Theorem 0.4 (4-ii) implies (4-iii). Let d_X denote exterior differentiation along the fiber. We set $\mathcal{E} = 0$ and use equation (4.1) to see

$$0 = d_X\,\text{int}_Z(\theta)\pi^* \text{ on } C^\infty\Lambda^p Y. \tag{4.3}$$

This implies θ is constant on the fibers so $\theta = \pi^*\Theta$ is the pull back of a globally defined 1-form on the base. Since $\mathcal{H}$ is integrable, we use Lemma 4.2 to give a local decomposition of Z so that we have $\theta = \pi^*\Theta = -d_Y\ln(g_X)$. Let $\psi(y)$ be the volume of the fibers. Let $d\nu_x^e$ be the Euclidean measure. Then

$$\begin{aligned} d_Y\psi(y) &= d_Y \textstyle\int_X g_X(x,y)d\nu_x^e = \int_X (g_X g_X^{-1} d_Y g_X)(x,y)d\nu_x^e \\ &= -\textstyle\int_X g_X(x,y)\theta(x,y)d\nu_x^e = -\Theta(y)\int_X g_X(x,y)d\nu_x^e \\ &= -\,\Theta(y)\psi(y). \end{aligned}$$

Thus $\theta = -\pi^* d_Y \ln\psi$ where $\psi \in C^\infty(Y)$ is globally defined.

Let $g(t)_Z = \psi^{2t}ds_{\mathcal{V}}^2 + ds_{\mathcal{H}}^2$ define a conformal variation of the metric on the vertical distribution and leave the metric on the horizontal distribution unchanged. Then $\pi : Z(t) \to Y$ is a Riemannian submersion with integrable horizontal distribution. We use Lemma 4.2 to see $\theta(t) = (1 + t\dim(X))\theta$ and thus

$$\begin{aligned} &\Delta_{Z(t)}^p\pi^* - \pi^*\Delta_Y^p = (1 + t\dim(X))(d_Z\,\text{int}_Z(\theta) + \text{int}_Z(\theta)d_Z)\pi^* \\ &\quad = (1 + t\dim(X))(\Delta_Z^p\pi^* - \pi^*\Delta_Y^p), \\ &\pi^*E(\lambda, \Delta_Y^p) \subset E(\lambda + (1 + t\dim(X))\epsilon(\lambda), \Delta_{Z(t)}^p). \end{aligned}$$

Since the Laplacian is a non-negative operator, $\lambda + (1 + t\dim(X))\epsilon(\lambda) \geq 0$. Since t is arbitrary, $\epsilon(\lambda) = 0$. Thus $(d_Z\,\text{int}_Z(\theta) + \text{int}_Z(\theta)d_Z)\pi^* = 0$ so $\theta = 0$. This completes the proof of Theorem 0.4 (4). □

§5 Riemannian submersions where eigenvalues are rigid

In §5.1, we will prove Theorem 0.9 (1), in §5.2, we will prove Theorem 0.9 (2), in §5.3, we will prove Theorem 0.9 (3), and in §5.8, we will prove Theorem 0.9 (4).

5.1 Flat Riemannian submersions with structure group SL. Let $\pi : Z \to Y$ be a Riemannian submersion with integrable horizontal distribution. Assume there exists a measure ν on the fibers so the Lie derivative $\mathcal{L}_H\nu = 0$ for all horizontal lifts H. Expand $dvol_Z = e^\alpha \nu \pi^* dvol_Y$ to define a smooth function $\alpha \in C^\infty(Z)$. We apply Lemma 4.2 to choose a local decomposition of Z so that

$$ds_Z^2 = g_{ij}(x,y)dx^i \circ dx^j + h_{ab}(y)dy^a \circ dy^b.$$

Expand $\nu = e^{\beta(x,y)}\nu_x^e$ where ν_x^e is Euclidean measure. Since $\mathcal{L}_{\partial_a^y}\nu = 0$ and since $\mathcal{L}_{\partial_a^y}\nu_x^e = 0$, β is independent of y. We expand

$$dvol_Z = g_X \nu_x^e \pi^* dvol_Y = e^{\beta+\alpha}\nu_x^e \pi^* dvol_Y.$$

This shows $g_X = e^{\beta+\alpha}$ so $\theta = -d_Y \ln(g_X) = -d_Y(\beta+\alpha) = -d_Y(\alpha)$. Suppose $0 \neq \Phi \in E(\lambda, \Delta_Y^p)$ and $\pi^*\Phi \in E(\lambda+\epsilon, \Delta_Z^p)$. As in the proof of Theorem 0.4 (4) given in §4, we consider the canonical variation $ds^2_{Z(t)} = e^{2t\alpha}ds^2_{\mathcal{V}} + ds^2_{\mathcal{H}}$ and see $\pi^*\Phi \in E(\lambda + \epsilon(1+t\dim X), \Delta^p_{Z(t)})$. This shows $\epsilon = 0$ and completes the proof of Theorem 0.9 (1). □

5.2 Principal G bundles. Let G be a compact Lie group with a bi-invariant metric. Assume $H^1(G;\mathbb{R}) = 0$. Let $\pi : P \to Y$ be a principal Riemannian G bundle. Let $\xi \in T_e(G)$ and let $g(t)$ be the 1-parameter subgroup of G with $\dot g(0) = \xi$. Multiplication by $g(t)$ defines a flow on P which is an isometry. Let ξ be the associated Killing vector field; ξ has constant length since the fibers have the bi-invariant metric. Consequently the integral curves of ξ are geodesics; this implies that the fibers of π are totally geodesic so $\theta = 0$ and only the curvature enters. Let $0 \neq \Phi \in E(\lambda, \Delta_Y^p)$ and let $\pi^*\Phi \in E(\mu, \Delta_P^p)$. By replacing Φ by $d_Y\Phi$ if necessary, we may assume without loss of generality that $d_Y\Phi = 0$. Expand $\Phi = \Sigma_{|A|=p}\Phi_A F^A$ and $\mathcal{E}\pi^*\Phi = \Sigma_{|A|=p-2}\operatorname{ext}_P(\beta_A)f^A$. The β_A are vertical co-vectors which are G invariant. Since $d_Y\Phi = 0$ and $\theta = 0$,

$$(\mu-\lambda)\pi^*\Phi = \Delta_P^p\pi^* - \pi^*\Delta_Y^p = d_P\mathcal{E}\pi^*\Phi$$

has no vertical dependence. Thus the vertical derivative of the restriction of β_A to the fibers vanishes. Since $H^1(G;\mathbb{R}) = 0$ and since the β_A are G invariant, this implies the restriction of the β_A to the fibers vanishes and hence $\beta_A = 0$ for all multi-indexes A. This implies $\lambda = \mu$ and completes the proof of Theorem 0.9 (2). □

5.3 The geometry of sphere bundles. Let V be a real vector bundle of rank $r \geq 3$ over Y. We assume V is equipped with a fiber metric and let $S(V)$ be the unit sphere bundle. We use the Riemannian connection ${}^V\nabla$ on the bundle V to split $T(V) = \mathcal{V} \oplus \mathcal{H}$. We use this splitting to define a Riemannian metric g_V so $\pi : V \to Y$ is a Riemannian submersion. The restriction of π to $S(V)$ defines a projection $\pi_S : S(V) \to Y$ which is a Riemannian submersion. We begin the proof of Theorem 0.9 (3) by developing some of the Riemannian geometry of this situation. Let $y = (y^a)$ be local coordinates on Y and let $s = (s_i)$ be a local orthonormal frame for V. The map $(x,y) \to s_i(y)x^i$ introduces local coordinates on V. Let ${}^V\nabla s_i = A_{aij}(y)dy^a s_j$; A_{aij} is the connection 1-form of ${}^V\nabla$ relative to the given local frame. Let ∂_i^y and ∂_a^x be the coordinate frames for the tangent bundle of the total space V.

5.4 LEMMA.

(1) $g_V(\partial_a^y, \partial_b^y) = g_Y(\partial_a, \partial_b) + x^i x^j A_{aik} A_{bjk}$, $g_V(\partial_a^y, \partial_i^x) = A_{aji} x^j$, *and* $g_V(\partial_i^x, \partial_j^x) = \delta_{ij}$.

(2) $\pi_S : S(V) \to Y$ *is a Riemannian submersion with totally geodesic fibers.*

(3) *If* $\tilde{f}_a := \partial_a^y - x^j A_{aji} \partial_i^x$, *then* $\tilde{f}_a$ *is the horizontal lift of* ∂_a^y.

(4) *Let* R_{abij} *be the curvature of* ${}^V\nabla$. *Then* $2\omega_{abi} = -R_{abji} x^j$.

Proof. Let $f : \mathcal{O} \to O(r)$ define a local gauge transformation $\tilde{s} = f(y)s$. Fix $y_0 \in Y$ and choose f so that $f(y_0) = I$ and $df(y_0) = -A(y_0)$. Then $\tilde{s}(y_0) = s(y_0)$ and ${}^V\nabla\tilde{s}(y_0) = 0$. Let $(\tilde{x}, \tilde{y})$ be the new system of local coordinates; the identity $x^i s^i = \tilde{x}^i \tilde{s}^i$ implies $y^a = \tilde{y}^a$ and $x^j = \tilde{x}^i f_{ij}$. Thus at a point $z_0 \in \pi^{-1}(y_0)$ we have:

$$\partial_i^{\tilde{x}} = \partial_i^x \text{ and } \partial_a^{\tilde{y}} = \partial_a^y - A_{aij} x^i \partial_j^x. \tag{5.5}$$

Since ${}^V\nabla\tilde{s}(y_0) = 0$, $\mathcal{H}(z_0) = \text{span}\{\partial_a^{\tilde{y}}\}$ and $\mathcal{V}(z_0) = \text{span}\{\partial_i^{\tilde{x}}\}$. Therefore

$$\begin{aligned} &g_V(\partial_a^{\tilde{y}}, \partial_b^{\tilde{y}})(z_0) = g_Y(\partial_a^y, \partial_b^y)(y_0), \\ &g_V(\partial_a^{\tilde{y}}, \partial_i^x)(z_0) = 0, \text{ and } g_V(\partial_i^x, \partial_j^x)(z_0) = \delta_{ij}. \end{aligned} \tag{5.6}$$

Assertion (1) now follows from equations (5.5) and (5.6). To prove assertion (2), normalize the choice of coordinates on Y so the 1-jets of the metric vanish at y_0 and choose the frame so ${}^V\nabla s(y_0) = 0$. Then the 1-jets of metric on V vanish at (x, y_0). Thus the curve $\gamma(t) = (x + t\bar{x}, y_0)$ is a geodesic in V and the fibers of V are totally geodesic. Let $\xi \in S_0$. Since orthogonal projection of $T_\xi V_0$ on $T_\xi S$ is contained in S_0, S_0 is a totally geodesic submanifold of S so (2) follows. Since $g_V(\tilde{f}_a, \partial_k^x) = 0$ and since $\{\partial_k^x\}$ span the vertical space, $\tilde{f}_a$ is the horizontal lift of ∂_a^y and (3) follows. We choose s so $A(y_0) = 0$ and compute

$$\begin{aligned} 2\omega_{abi}(x, y_0) &= g_V([\tilde{f}_a, \tilde{f}_b], e_i)(x, y_0) = (\partial_b^y A_{aji} - \partial_a^y A_{bji})(y_0) x^j \\ &= R_{baji}(y_0) x^j. \end{aligned}$$

Since ω and R are tensorial, (4) follows. □

We can now complete the proof Theorem 0.9 (3). Let $0 \neq \Phi \in E(\lambda, \Delta_Y^p)$ and let $\pi^*\Phi \in E(\mu, \Delta_S^p)$. Since $\theta = 0$, $(\lambda - \mu)\pi^*\Phi = (d_Z\mathcal{E} + \mathcal{E}d_Z)\pi^*\Phi$. By replacing Φ by $d_Y\Phi$ if necessary, we may assume without loss of generality that $d_Y\Phi = 0$. Let $d_\mathcal{V} := \rho_\mathcal{V} d_S$ be the vertical exterior derivative. We apply $(1 - \rho_\mathcal{H})$ to see

$$\text{ext}_S(d_\mathcal{V}(x^j dx^i)) R_{abij} \, \text{int}_S(f^a) \, \text{int}_S(f^b) \pi^*\Phi = 0. \tag{5.7}$$

If we normalize the coordinates so $A(y_0) = 0$, then $d_\mathcal{V}(x^j dx^i) = dx^j \wedge dx^i$. Since the fiber dimension of V is at least 3, the dimension of the fiber spheres is at least 2 and equation (5.7) implies $R_{abij} \, \text{int}_S(f^a) \, \text{int}_S(f^b)\pi^*\Phi = 0$ for all i, j. This shows $\mathcal{E}\pi^*\Phi = 0$ and hence $\lambda = \mu$. This completes the proof of Theorem 0.9 (3). □

5.8 One forms. Let $0 \neq \Phi \in E(\lambda, \Delta_Y^1)$, and let $\pi^*\Phi \in E(\mu, \Delta_Z^1)$. Suppose that $\theta = 0$. Then

$$(\lambda - \mu)\pi^*\Phi = (d_Z\mathcal{E} + \mathcal{E}d_Z)\pi^*\Phi = \mathcal{E}d_Z\pi^*\Phi.$$

Since $\mathcal{E}d_Z\pi^*\Phi$ has vertical dependence, it must vanish. This shows $\lambda = \mu$ and completes the proof of Theorem 0.9 (4). □

§6 Topological constructions

In this section, we use methods of algebraic topology to prove Theorem 0.10; we refer to Spanier [31] for details concerning the results which we will use.

6.1 The pull back of harmonic 1-forms. Let M be a connected manifold. The Abelianization of the fundamental group $\pi_1(Y)$ is first integer homology group $H_1(M;\mathbb{Z})$. By the universal coefficient theorem, $H^1(M;\mathbb{R}) = Hom(H_1(M;\mathbb{Z});\mathbb{R})$. Thus $H^1(M;\mathbb{R}) = Hom(\pi_1(M);\mathbb{R})$. We can interpret this isomorphism geometrically as follows. Let $[\Phi] \in H^1(M;\mathbb{R})$ represent a de Rham cohomology class and let $[\alpha] \in \pi_1(M)$ represent an element of the fundamental group. Then $(\Phi,\alpha) \rightarrow \int_\alpha \Phi$ extends to a well defined map $\mathcal{I} : H^1(M;\mathbb{R}) \times \pi_1(Y) \rightarrow \mathbb{R}$; the map $[\Phi] \rightarrow \mathcal{I}([\Phi],\cdot)$ provides the isomorphism from $H^1(M;\mathbb{R})$ to $Hom(\pi_1(M);\mathbb{R})$.

Let $\pi : Z \rightarrow Y$ be a fiber bundle; we impose no restrictions on the metric. Suppose that $0 \neq \Phi \in E(0,\Delta_Y^1)$; by the Hodge decomposition theorem, $[\Phi]$ is a non-trivial element in $H^1(Y;\mathbb{R})$. If $\int_\alpha \Phi = 0$ for all closed paths α, we can define the potential function $F(y) := \int_\beta \Phi$ where β is *any* path from y_0 to y. Then $dF = \Phi$ so $[\Phi]$ is trivial in $H^1(Y;\mathbb{R})$. This contradiction shows that there exists a closed curve α in Y so that $\int_\alpha \Phi \neq 0$. For $x \in X$, let $\mathcal{H}_x\alpha$ be the horizontal lift of α to Z with $\mathcal{H}_x\alpha(0) = x$. Let $[x] \in \pi_0(X)$ denote the arc component of X to which x belongs. The map $x \rightarrow [\mathcal{H}_x\alpha(1)]$ extends to a well defined bijective map $\delta_{[\alpha]}$ of $\pi_0(X)$ which only depends on the class $[\alpha]$ in $\pi_1(Y)$. Since Z is compact, $\pi_0(X)$ is finite. Thus there exists an integer ν so $\delta_{[\alpha]}^\nu$ is the identity. Let $[\bar\alpha] = \nu[\alpha]$ in $\pi_1(Y)$. Since $\delta_{[\bar\alpha]} = \delta_{[\alpha]}^\nu = id$, x and $\mathcal{H}_x\bar\alpha(1)$ are in the same arc component of X. Choose a curve β from $\mathcal{H}_x\bar\alpha(1)$ to x. Let γ be the concatenation of $\mathcal{H}_x\bar\alpha$ and some curve β from $\mathcal{H}_x\bar\alpha(1)$ to x which lies in X. Then γ is a closed curve in Z and $[\pi\gamma] = [\alpha]$ in $\nu\alpha$ in $\pi_1(Y)$. Then $[\pi^*\Phi]$ is a non trivial cohomology class in $H^1(Z;\mathbb{R})$ since

$$\int_\gamma \pi^*\Phi = \int_{\pi\gamma} \Phi = \nu \int_\alpha \Phi \neq 0.$$

Suppose $\pi^*\Phi \in E(\mu,\Delta_Z^1)$. Since Φ is harmonic, $d_Y\Phi = 0$ so $d_Z\pi^*\Phi = 0$. Suppose $\mu \neq 0$. Since $\Delta_Z\pi^*\Phi = \mu\pi^*\Phi$ and since $d_Z\pi^*\Phi = 0$, $\pi^*\Phi = \mu^{-1}d_Z\delta_Z\pi^*\Phi$ belongs to the image of d so $[\pi^*\Phi]$ is trivial in the de Rham cohomology group $H^1(Z;\mathbb{R})$. This contradiction completes the proof of Theorem 0.10 (1). □

6.2 Sphere bundles. Let $\pi : S \rightarrow Y$ be a sphere bundle with fiber dimension $\nu - 1$ where ν is odd. Let $0 \neq \Phi \in E(0,\Delta_Y^p)$ and $\pi^*\Phi \in E(\mu,\Delta_Z^p)$. As in the proof given in §6.1 of assertion (1) of Theorem 0.10, we suppose $\mu \neq 0$ so $\pi^* : H^p(Y;\mathbb{R}) \rightarrow H^p(S;\mathbb{R})$ is not injective. By passing to a suitable double cover $\mathbb{Z}_2 \rightarrow \tilde Y \rightarrow Y$ and by considering the associated sphere bundle $\pi : \tilde S \rightarrow \tilde Y$ if necessary, we may assume without loss of generality that the bundle S is orientable. The Gysin sequence gives rise to a long exact sequence

$$\ldots H^{p-\nu}(Y;\mathbb{R}) \xrightarrow{\cup e} H^p(Y;\mathbb{R}) \xrightarrow{\pi^*} H^p(S;\mathbb{R}) \xrightarrow{\epsilon} H^{p-\nu+1}(Y;\mathbb{R}) \ldots$$

In this equation, $\cup e$ denotes cup product with the Euler form, π^* is the pull back, and ϵ is the connecting homomorphism. The crucial point here is that the Euler form e vanishes with $\mathbb{R}$ coefficients since ν is odd. Thus π^* is injective; this contradiction completes the proof of Theorem 0.10 (2). □

§7 Riemannian submersions where eigenvalues do change

7.1 The geometry of principal S^1 bundles. Let L be a complex line bundle over Y. We suppose that L is equipped with a smooth fiber metric and a unitary connection ${}^L\nabla$. Let $\pi : S(L) \to Y$. Then π defines a Riemannian principal S^1 bundle; this is also the circle bundle of the underlying real 2-plane bundle.

7.2 LEMMA. *Let s be a local orthonormal section to L. Let ${}^L\nabla s = \sqrt{-1}\mathcal{A}_s s$ define the normalized connection 1-form $\mathcal{A}_s$. Let $(t, y) \mapsto e^{\sqrt{-1}t}s(y)$ give local coordinates (t, y) to $S = S(L)$.*

(1) *The fibers of π are totally geodesic.*
(2) *We have ∂_t is an invariantly defined unit tangent vector spanning $\mathcal{V}$.*
(3) *If $\tilde{s} = e^{\sqrt{-1}\Phi}s$, then $\partial_t = \partial_{\tilde{t}}$, $\partial_a^y = \partial_a^{\tilde{y}} - \partial_a^y\Phi\partial_t$, and $\tilde{\mathcal{A}}_s = \mathcal{A}_s + d_Y\Phi$.*
(4) *The horizontal lift of a vector field Ψ on Y is given by $\mathcal{H}\Psi := \Psi - \mathcal{A}_s(\Psi)\partial_t$.*
(5) *We have $\chi := dt + \pi^*\mathcal{A}_s$ is dual to ∂_t and spans $\mathcal{V}^*$.*
(6) *The normalized curvature $\mathcal{F} := d_Y\mathcal{A}_s$ is invariantly defined.*
(7) *We have $d_S\chi = \pi^*\mathcal{F}$ and $\mathcal{E} = -\operatorname{ext}_S(\chi)\pi^*\operatorname{int}_Y(\mathcal{F})$.*

Proof. The flow $v \to e^{\sqrt{-1}t}v$ for $v \in S(L)$ and $t \in \mathbb{R}$ is invariantly defined; ∂_t is the associated unit vertical Killing vector field. Assertions (1) and (2) now follow; (1) also follows from Lemma 5.4. Since ${}^L\nabla$ is unitary, $\mathcal{A}_s$ is a real 1-form. If $\tilde{s} = e^{\sqrt{-1}\Phi}s$, then $(\tilde{y}, \tilde{t}) = (y, t - \Phi)$; assertion (3) now follows. We show that $\mathcal{H}$ is invariantly defined by computing:

$$\partial_a^y - \mathcal{A}_s(\partial_a^y)\partial_t = \partial_a^{\tilde{y}} - \partial_a^y\Phi\partial_t - \mathcal{A}_s(\partial_a^y)\partial_t = \partial_a^{\tilde{y}} - \mathcal{A}_{\tilde{s}}(\partial_a^y)\partial_t.$$

Fix $y_0 \in Y$ and choose Φ so $(\mathcal{A}_s + d_Y\Phi)(y_0) = 0$. Since $\mathcal{A}_{\tilde{s}}(y_0) = 0$, the $\partial_a^{\tilde{y}}$ are horizontal. Thus at y_0, $\mathcal{H}\partial_a^y = \partial_a^{\tilde{y}}$ is horizontal. Since $\mathcal{H}\Psi$ is invariantly defined, $\mathcal{H}\Psi$ is the horizontal lift. Since $\chi(\mathcal{H}\Psi) = 0$ for all Ψ and since $\chi(dt) = 1$, χ is the vertical projection of dt and is invariantly defined. By (3), $d_Y\mathcal{A}_{\tilde{s}} = d_Y\mathcal{A}_s$ so the curvature $\mathcal{F}$ is invariantly defined. Clearly $d\chi = \pi^*\mathcal{F}$. We compute:

$$\begin{aligned}\mathcal{E} :=& \operatorname{ext}_S(\chi)g_S([\mathcal{H}\partial_a^y, \mathcal{H}\partial_b^y], \partial_t)\pi^*\operatorname{int}_Y(dy^a)\operatorname{int}_Y(dy^b)/2\\ =& \operatorname{ext}_S(\chi)\pi^*\{-\partial_a^y\mathcal{A}_b + \partial_b^y\mathcal{A}_a\}\operatorname{int}_Y(dy^a)\operatorname{int}_Y(dy^b)/2\\ =& -\operatorname{ext}_S(\chi)\pi^*\operatorname{int}_Y(\mathcal{F}). \quad \square\end{aligned}$$

Proof of Theorem 0.12. We apply the previous Lemma and Theorem 0.4 (1). Since $\theta = 0$ and since $d_Y\Phi = 0$,

$$\begin{aligned}\Delta_S^p\pi^*\Phi - \pi^*\Delta_Y^p\Phi =& d_S\mathcal{E}\pi^*\Phi = -d_S\{\chi \wedge \pi^*(\operatorname{int}_Y(\mathcal{F})\Phi)\}\\ =& -\pi^*\{\operatorname{ext}_Y(\mathcal{F})\operatorname{int}_Y(\mathcal{F})\Phi\} = \epsilon\pi^*\Phi. \quad \square\end{aligned}$$

7.3 The Chern class. If ${}^L\nabla$ is an arbitrary connection on a complex line bundle L over Y, let $c_1({}^L\nabla) := -\mathcal{F}({}^L\nabla)/2\pi$; this is an invariantly defined real closed 2 form on Y. If ${}^L\tilde{\nabla}$ is another connection on L, there exists a 1 form Ψ on M so that ${}^L\nabla = {}^L\tilde{\nabla} + \sqrt{-1}\Psi$. Thus $\mathcal{F}({}^L\nabla) = \mathcal{F}({}^L\tilde{\nabla}) + d\Psi$ so the cohomology class $[c_1({}^L\nabla)]$ is independent of the connection chosen.

Let $H^2(Y;\mathbb{Z})$ be the integer cohomology groups of a compact manifold Y. We use the universal coefficient theorem to see $H^2(Y;\mathbb{R}) = H^2(Y;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$; see, for example, Spanier [31]. This gives a natural map $\kappa : H^2(Y;\mathbb{Z}) \to H^2(Y;\mathbb{R})$. One can define $c_1(L) \in H^2(Y;\mathbb{Z})$ topologically so that $[c_1({}^L\nabla)] = \kappa c_1(L)$. Let $\text{Vect}^1(Y)$ be the set of isomorphism classes of complex line bundles over Y. We use tensor product to give $\text{Vect}^1(Y)$ the structure of an Abelian group. Then c_1 is an isomorphism from $\text{Vect}^1(Y)$ to $H^2(Y;\mathbb{Z})$.

Suppose that $H^1(Y;\mathbb{R}) \neq 0$. Choose $L \in \text{Vect}^1(Y)$ so that $0 \neq \kappa(c_1(L))$ in $H^2(Y;\mathbb{R})$. We use the Hodge decomposition theorem to find $0 \neq \Phi \in E(0, \Delta_Y^2)$ so that $[-\Phi/2\pi] = \kappa(c_1(L))$ in $H^2(Y;\mathbb{R})$. Let ${}^L\tilde{\nabla}$ be any unitary connection on L. Then $[\mathcal{F}({}^L\tilde{\nabla})] = [\Phi]$ in $H^2(Y;\mathbb{R})$. Thus we can find a smooth 1-form Ψ on Y so that $\Phi = \mathcal{F}({}^L\tilde{\nabla}) + d\Psi$. Let ${}^L\nabla := {}^L\tilde{\nabla} + \sqrt{-1}\Psi$ be a new unitary connection on L. Then $\mathcal{F} = \mathcal{F}({}^L\nabla) = \Phi$ is harmonic.

Proof of Theorem 0.13. Let Y be homogeneous with $H^2(Y;\mathbb{R}) \neq 0$. We argue as above to find a unitary connection ${}^L\nabla$ on a complex line bundle L over Y such that $0 \neq \Phi := \mathcal{F}({}^L\nabla) \in E(0, \Delta_Y^2)$. Let $0 \neq \epsilon = |\Phi|^2 = -\text{int}_Y(\mathcal{F})\Phi$; $-\text{ext}_Y(\mathcal{F})\,\text{int}_Y(\mathcal{F})\Phi = \epsilon\Phi$. Since Φ is harmonic, $d_Y\Phi = 0$. By Theorem 0.12, it suffices to show $d_Y\epsilon = 0$ or equivalently that ϵ is constant to show that $\pi^*\Phi \in E(\epsilon, \Delta_S^2)$. Let $\mathcal{G}$ be the connected component of the identity in the isometry group of Y. Pullback induces a natural action of $\mathcal{G}$ on the de Rham cohomology groups $H^2(Y;\mathbb{R})$. Since $\mathcal{G}$ is connected, the homotopy axiom for cohomology implies this action is trivial. We identify $H^2(Y;\mathbb{R})$ with $E(0, \Delta_Y^2)$ to see $E(0, \Delta_Y^2)$ is fixed by this action of $\mathcal{G}$. Thus $\Phi \circ \psi = \Phi$ for any $\psi \in \mathcal{G}$ and thus $|\Phi|^2$ is fixed by $\mathcal{G}$. Since Y is a homogeneous space, $\mathcal{G}$ acts transitively on Y. This shows $|\Phi|^2$ is constant. □

Proof of Theorem 0.14. Let L be a positive line bundle over a holomorphic manifold Y. Let $\mathcal{F}$ be the curvature of L. Then $\mathcal{F}$ is the Kaehler form of the metric on Y so $\mathcal{F}^p$ is harmonic for $1 \leq p \leq n := \dim_{\mathbb{C}} Y$. We have $\text{int}_Y(\mathcal{F})\mathcal{F}^p = -(n+1-p)\mathcal{F}^{p-1}$ is harmonic and $-\text{ext}_Y(\mathcal{F})\,\text{int}_Y(\mathcal{F})\mathcal{F}^p = p(n+1-p)\mathcal{F}^p$. The Theorem now follows from Theorem 0.12. □

7.4 Example: the Hopf fibration. We can illustrate both Theorems 0.13 and 0.14 as follows. The circle S^1 acts by complex multiplication without fixed points on the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$; this defines a Riemannian principal S^1 bundle $\pi : S^{2n+1} \to \mathbb{CP}^n$ called the Hopf fibration; S^{2n+1} is identified with the sphere of radius 2 in $\mathbb{R}^{2n}$ to ensure the fibration is a Riemannian submersion. The unitary group $U(n+1)$ acts transitively on S^{2n+1}. Since this action commutes with the circle action, it descends to define a transitive action of $U(n+1)$ on $\mathbb{CP}^n$. The associated line bundle L is called the canonical line bundle. It inherits a natural unitary connection ∇. The curvature $\mathcal{F}$ of ∇ is modulo a possible sign convention the Kaehler form of the Fubini-Study metric on $\mathbb{CP}^n$. Note that $\mathcal{F}^p$ generates $E(0, \Delta^{2p}_{\mathbb{CP}^n})$ for $0 < p \leq n$ so

$$\pi^* E(0, \Delta^{2p}_{\mathbb{CP}^n}) \subset E(p(n+1-p), \Delta^{2p}_{S^{2n+1}}).$$

We note that by Theorem 0.9, this phenomena of changing eigenvalues does not generalize to the Hopf fibrations $S^{4n+3} \to \mathbb{QP}^n$ which have fiber S^3.

Proof of Theorem 0.15. Assume that $\Phi \in E(\lambda, \Delta^2_Y)$ with $\lambda \neq 0$ and that $d_Y\Phi = 0$. Let $\chi := \lambda^{-1}\delta_Y\Phi$; $d_Y\chi = \Phi$. We assume $|\Phi|^2 = a$ is constant. Then we have that $\text{int}_Y(\Phi)\Phi = -|\Phi|^2 = -a$. Let G be a compact connected Lie group. Let $\{e_i\}$ and $\{e^i\}$ be orthonormal bases for the Lie algebra and the Lie co-algebra of G. Since $H^1(G;\mathbb{R}) \neq 0$, we may assume $e^1 \in E(0, \Delta^1_G)$. Let π be projection on the second factor of $P := G \times Y$. We define a metric $ds^2_P(\epsilon)$ by requiring that $\mathcal{H}(\Psi) := \Psi - \epsilon\chi(\Psi)e_1$ be the horizontal lift of Ψ and by requiring that $\{e_i\}$ is an orthonormal frame for $\mathcal{V}$. The splitting is G equivariant and with this metric, π defines a Riemannian principal G bundle. Note that $\{e^1 + \epsilon\chi, e^2, ...\}$ is the corresponding dual orthonormal frame for $\mathcal{V}^*$. We compute

$$\begin{aligned}[F_a, F_b] =& [f_a, f_b] - \epsilon f_a(\chi(f_b)) + \epsilon f_b(\chi(f_a)) \\ =& ([f_a, f_b] - \epsilon\chi([f_a, f_b])e_1) - \epsilon d_P\chi(f_a, f_b)e_1.\end{aligned}$$

Consequently $\mathcal{E}\pi^*\Phi = -\text{ext}_P(e^1 + \epsilon\chi)\pi^*\text{int}_P(\epsilon\Phi)$. Since $d_Y\Phi = 0$ and since the fibers of π are minimal, we see

$$(\Delta^2_P\pi^* - \pi^*\Delta^2_Y)(\Phi) = \epsilon^2 d_P(\chi|\Phi|^2) = a\epsilon^2\Phi. \quad \square$$

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Mathematics Department, University of Oregon, Eugene Or 97403 USA
gilkey@math.uoregon.edu

Mathematics Department, University of Oregon, Eugene Or 97403 USA
leahy@math.uoregon.edu

Department of Mathematics, Honam University, Seobongdong 59, Kwangsanku, Kwangju, 506-090 South Korea
jhpark@honam.honam.ac.kr